ESAIM: COCV 14 (2008) 494–516
DOI: 10.1051/cocv:2007064

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

A CLASS OF MINIMUM PRINCIPLES FOR CHARACTERIZING THE TRAJECTORIES AND THE RELAXATION OF DISSIPATIVE SYSTEMS

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Abstract. This work is concerned with the reformulation of evolutionary problems in a weak form enabling consideration of solutions that may exhibit *evolving microstructures*. This reformulation is accomplished by expressing the evolutionary problem in variational form, *i.e.*, by identifying a functional whose minimizers represent *entire trajectories* of the system. The particular class of functionals under consideration is derived by first defining a sequence of time-discretized minimum problems and subsequently formally passing to the limit of continuous time. The resulting functionals may be regarded as a *weighted dissipation-energy functional* with a weight decaying with a rate $1/\epsilon$. The corresponding Euler-Lagrange equation leads to an elliptic regularization of the original evolutionary problem. The Γ-limit of these functionals for $\epsilon \to 0$ is highly degenerate and provides limited information regarding the limiting trajectories of the system. Instead we seek to characterize the minimizing trajectories directly. The special class of problems characterized by a *rate-independent* dissipation functional is amenable to a particularly illuminating analysis. For these systems it is possible to derive *a priori* bounds that are independent of the regularizing parameter, whence it is possible to extract convergent subsequences and find the limiting trajectories. Under general assumptions on the functionals, we show that all such limits satisfy the *energetic formulation* (S) & (E) for rate-independent systems. Moreover, we show that the accumulation points of the regularized solutions solve the associated limiting energetic formulation.

Mathematics Subject Classification. 49J40, 49M20, 49S05, 74N10.

Received May 20, 2006. Revised October 31, 2006.
Published online December 21, 2007.

1. Introduction

The formation of microstructure in quasi-static problems and its connection with non-attainment has received considerable attention, particularly following the seminal paper of Ball and James [3]. A vast body of mathematical literature exits at present that makes that connection sharp. The *evolution* of microstructure is a somewhat more complex problem whose systematic study is comparatively less advanced. A fundamental

Keywords and phrases. Weighted energy-dissipation functional, incremental minimization problems, relaxation of evolutionary problems, rate-independent processes, energetic solutions.

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Article published by EDP Sciences

question concerns whether entire dissipative processes involving microstructure evolution can be given a variational characterization as solutions of a *minimum* problem. The present paucity of such minimum principles constitutes a severe impediment to the application of modern tools of the calculus of variations to evolutionary problems.

In their classical formulation, the class of evolutionary problems under consideration here take place in a Banach space Y and their strong form is the doubly nonlinear differential inclusion (*cf.* [8,34])

$$0 \in \partial\Psi(\dot{u}(t)) + \mathrm{D}\mathcal{E}(t, u(t)), \tag{1.1a}$$

$$u(0) = u_0, \tag{1.1b}$$

where $\Psi : Y \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is a convex dissipation potential; $\mathcal{E} : Y \to \mathbb{R}_\infty$ is an energy function; $\partial\Psi$ is the subdifferential of Ψ, representing the system of dissipative forces; $\mathrm{D}\mathcal{E}$ is the Fréchet derivative of $\mathcal{E}$, representing the conservative force system; and time t varies in the interval $[0, T]$. Equation (1.1a) establishes a balance between dissipative forces and conservative forces, and the trajectory $u(t)$ of the system is the result of this balance and of the initial condition (1.1b). We regard (1.1) as a *model* problem furnishing a convenient illustration of the basic strategy proposed in this work. As a model problem, (1.1) is general enough to describe, *e.g.*, the quasistatic viscoelasticity of solids under the assumption of linearized kinematics; heat conduction; viscous drag on a solid immersed in a Stokes' flow; and other cases of interest. However, it bears emphasis that the basic strategy for formulating variational principles for trajectories developed here is applicable to more general dissipative systems, including systems with Newtonian viscosity and finite-deformation viscoplasticity (*cf.*, *e.g.*, [45] for examples). However, these extensions entail a certain degree of added complexity, *e.g.*, in terms of geometrical mechanics (*cf.*, *e.g.*, [30]), and will not be pursued here in the interest of simplicity.

The aim of this paper is to reformulate the evolutionary problem (1.1) in a weaker form enabling the consideration of systems where $\mathcal{E}$ is not differentiable or not even lower semi-continuous, thus allowing for solutions that may exhibit *evolving microstructures*. This reformulation is accomplished by expressing (1.1) in variational form, *i.e.*, by identifying a functional whose minimizers represent *entire trajectories* of the system. A number of variational principles have been proposed for characterizing entire trajectories of dissipative systems, including the *Brézis-Ekeland variational principle* ([5], see also [16,40], Th. 8.93), Gurtin's variational principle for linear viscoelasticity ([18,19]), and others. An obvious – albeit contrived, as it unnaturally doubles the order of the problem – alternative variational characterization of trajectories is to minimize an L^2-norm of the residual of (1.1a). This multiplicity of proposals begs the question of what constitutes a physically and mathematically meaningful variational characterization, if any, of the trajectories of dissipative systems.

In this work, we investigate a possible answer to this question, which builds upon recent work on *time-discretized incremental* variational principles for dissipative systems. It is now widely appreciated that a carefully crafted time discretization of evolutionary problems may result in a *sequence* of minimum problems characterizing the successive states of the system. (*e.g.*, [2,6,9,10,20–23,29,33,35,37–39,41,45]). This sequence of minimum problems must be solved *causally*: the first problem propagates the known initial conditions; the second problem propagates the solution of the first problem; and so on. Instead of this time-stepping solution procedure, we proceed to combine all the incremental functionals under a single functional for the entire trajectory by recourse to the notion of *Pareto optimality* of multi-objective optimization problems (*cf.*, *e.g.*, [7]). Specifically, the resulting functional for the – still time-discretized – trajectories is constructed as a weighted sum of all the incremental functionals. The weights applied to the individual incremental functionals are known as *Pareto weights*. In this context, the effect of causality is to introduce a strict ordering in the set of Pareto weights ensuring that the first incremental problem is accorded disproportionately higher priority over the second, the second over the third, and so on. This is accomplished by introducing a sequence of Pareto weights, parameterized by a small parameter ϵ, with the property that the ratio between successive weights becomes vanishingly small as $\epsilon \to 0$. The last step in the derivation of the trajectory-wise functional is to formally pass to the limit of continuous time. In this limit, the discrete Pareto weights are replaced by a time-dependent Pareto weighting function. For example, for a particular choice of Pareto weighting function of exponential form, the resulting minimum problem is

$$\mathcal{I}_\epsilon(u) = \mathrm{e}^{-T/\epsilon}\mathcal{E}(T, u(T)) + \int_0^T \mathrm{e}^{-t/\epsilon}\left[\Psi(\dot u) + \frac{1}{\epsilon}\mathcal{E}(t,u)\right]\mathrm{d}t. \tag{1.2}$$

We call $\mathcal{I}_\epsilon$ a *weighted dissipation-energy functional.*

Under conditions of sufficient smoothness, the Euler-Lagrange equations of this functional are

$$\mathrm{D}\Psi(\dot u) + \mathrm{D}\mathcal{E}(t,u) - \epsilon\mathrm{D}^2\Psi(\dot u)\ddot u = 0, \tag{1.3a}$$

$$u(0) = u_0, \tag{1.3b}$$

$$\mathrm{D}\Psi(\dot u(T)) + \mathrm{D}\mathcal{E}(T, u(T)) = 0, \tag{1.3c}$$

which reveals that the functional (1.2) defines an *elliptic regularization* of the classical problem (1.1). Under an additional requirement of stability, the classical problem (1.1) is finally replaced by the minimum problem

$$\inf_{u\in\mathbb{Y}} \mathcal{I}_\epsilon(u) \tag{1.4}$$

where $\mathbb{Y}$ represents some suitable space of paths $u : [0,T] \to Y$. This completes the reformulation of the evolutionary problem (1.1) as a minimum problem for trajectories.

This reformulation in terms of weighted dissipation-energy functionals opens the way for the application of the tools of modern calculus of variations to evolutionary problems. Of particular interest is the existence of trajectories for the regularized problem (1.4), and its *causal* limit as $\epsilon \to 0$. We are specifically interested in the case in which $\mathcal{E}$ and hence $\mathcal{I}_\epsilon$ are not lower semi-continuous and, therefore, the infimum of $\mathcal{I}_\epsilon$ is not attained in general. A natural extension of this program is to consider sequences of dissipation and energy functionals, Ψ_k and $\mathcal{E}_k$, respectively, arising as a result of approximation, perturbation, or other modifications of the base functionals. In this case, we become interested in understanding the joint limit of $\epsilon \to 0$ and $k \to \infty$. A natural choice of topology for understanding these limits is the topology of Γ-convergence (*e.g.*, [11,12]). Unfortunately, we find that the Γ-limits of $\mathcal{I}_\epsilon$ are highly degenerate and provide limited information regarding the limiting trajectories of the system. Therefore, we instead attempt to characterize the minimizers directly.

A class of problems that is amenable to effective analysis concerns *rate-independent systems* for which the dissipation potential Ψ is homogeneous of degree 1. A striking first property of rate-independent problems is that all minimizers u^ϵ of $\mathcal{I}_\epsilon$ satisfy the energy balance

$$\mathcal{E}(t, u(t)) + \int_s^t \Psi(\dot u(\tau))\mathrm{d}\tau = \mathcal{E}(s, u(s)) + \int_s^t \partial_\tau\mathcal{E}(\tau, u(\tau))\mathrm{d}\tau,$$

independently of the value of ϵ. (In Sect. 4.2 we replace $\int_s^t \Psi(\dot u(\tau))\mathrm{d}\tau$ by the more general version $\int_s^t \Psi(\mathrm{d}u)$ that holds for BV functions.) Under suitable coercivity assumptions it is then possible to derive *a priori* bounds for u^ϵ which likewise are independent of ϵ, with the result that it is possible to extract convergent subsequences and find limiting functions u. Under very general assumptions we show that all such limits satisfy the *energetic formulation* for rate-independent systems of Mielke *et al.* ([15,26,33–35] and the survey [29]), *i.e.*, they satisfy the *stability condition* (S) and the *energy balance* (E). Moreover, we show that if $(\Psi_k)_{k\in\mathbb{N}}$ continuously converges to Ψ and $\mathcal{E}_k$ Γ-converges to $\mathcal{E}$ in the weak topology of a Banach space, then the accumulation points of the family $(u_{\epsilon,k})_{\epsilon>0,k\in\mathbb{N}}$ for $\epsilon, 1/k \to 0$ solve the associated limiting energetic formulation. Related relaxations and Γ-limits for rate-independent systems are treated in [17,24,28,32,36,44] by considering the sequence of incremental problems. In the latter works, convergence of the time-incremental solutions associated with the $\mathcal{E}_k$ and Ψ_k to a solution of the limit problem is established under conditions similar to those considered here.

2. Formal derivation of the variational principle for trajectories

A common device for reducing evolutionary problems of the form (1.1) to a sequence of variational problems is time discretization (*e.g.*, [2,26,35,38,45]). Specifically, suppose that we are given the state u_0 of the system at time $t_0 = 0$ and wish to approximate the solution u_n at times $t_n = n\triangle t$, $n = 1, \dots, N$. A *sequence* of minimum problems that delivers consistent approximations to the solution of the continuous-time evolutionary problem (1.1) is [38]

$$\inf_{u_{n+1}\in Y} F_{n+1}(u_{n+1}; u_n), \quad n = 0, \dots, N-1, \tag{2.1}$$

where

$$F_{n+1}(u_{n+1}; u_n) = \triangle t \Psi\left(\frac{u_{n+1} - u_n}{\triangle t}\right) + \mathcal{E}(t_{n+1}, u_{n+1}) - \mathcal{E}(t_n, u_n)$$

is an incremental functional that combines energy and kinetics. In addition, in (2.1) it is tacitly understood that the problems are solved *causally*: problem $n = 0$ is solved first with initial conditions u_0 in order to computed u_1; subsequently, problem $n = 1$ is solved to compute u_2, taking the solution u_1 of the preceding problem as initial condition; and so on. We note that the datum $-\mathcal{E}(t_n, u_n)$ is added to (2.1) so that kinetic and energy terms are of the same order in $\triangle t$.

We wish instead to collect the sequence (2.1) of incremental problems into a single minimum problem for the entire trajectory $u = \{u_1, \dots, u_N\}$. In the theory of optimization, a standard device for combining multiple objective functions is supplied by the concept of *Pareto optimality* (*cf.*, *e.g.*, [7]). In this spirit, a candidate functional on trajectories is

$$\widetilde{\mathcal{I}}(u; \lambda) = \sum_{n=0}^{N-1} \lambda_{n+1} F_{n+1}(u_{n+1}; u_n)$$

where $\lambda = \{\lambda_1, \dots, \lambda_N\}$ are positive *Pareto weights*. However, it order to ensure causality it is necessary to choose the weights in such a way that the minimization of the single functional $\mathcal{I}(u; \lambda)$ with respect to the entire trajectory u is equivalent to the sequential solution of the incremental problems (2.1). This is accomplished by introducing the ordering: $\lambda_1 \gg \lambda_2 \gg \dots$, which accords disproportionately larger importance to the first incremental problem relative to the second; to the second incremental problem relative to the third, and so on. More specifically, we may accomplish this causal ordering by considering a sequence of positive weights $\lambda_1^\epsilon > \lambda_2^\epsilon > \dots$ parameterized by a real parameter $\epsilon \geq 0$ and such that

$$\lim_{\epsilon\to 0} \frac{\lambda_{n+1}^\epsilon}{\lambda_n^\epsilon} = 0. \tag{2.2}$$

Inserting these weights into $\mathcal{I}(u; \lambda)$ gives

$$\widetilde{\mathcal{I}}(u; \lambda^\epsilon) = \sum_{n=0}^{N-1} \lambda_{n+1}^\epsilon \left\{ \Psi\left(\frac{u_{n+1} - u_n}{\triangle t}\right) + \frac{\mathcal{E}(t_{n+1}, u_{n+1}) - \mathcal{E}(t_n, u_n)}{\triangle t} \right\} \triangle t. \tag{2.3}$$

Suppose, in addition, that there is a function $\lambda_\epsilon > 0$ such that

$$\lambda_n^\epsilon = \lambda_\epsilon(t_n). \tag{2.4}$$

Then, causality requires λ_ϵ be monotonically decreasing, and the limiting condition (2.2) requires that

$$\lim_{\epsilon\to 0} \frac{\lambda_\epsilon(b)}{\lambda_\epsilon(a)} = 0, \quad \forall a, b \in [0, T],\ a < b. \tag{2.5}$$

Inserting (2.4) into (2.3) we obtain

$$\widetilde{\mathcal{I}}(u; \lambda^\epsilon) = \sum_{n=0}^{N-1} \lambda_\epsilon(t_{n+1}) \left\{ \Psi\left(\frac{u_{n+1} - u_n}{\triangle t}\right) + \frac{\mathcal{E}(t_{n+1}, u_{n+1}) - \mathcal{E}(t_n, u_n)}{\triangle t} \right\} \triangle t.$$

This functional may be regarded as a time discretization of the continuous-time functional

$$\widetilde{\mathcal{I}}(u;\lambda_\epsilon) = \int_0^T \lambda_\epsilon \Big[\Psi(\dot u) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u)\Big]\mathrm{d}t.$$

Alternatively, an integration by parts gives the functional in the from

$$\widetilde{\mathcal{I}}(u;\lambda_\epsilon) = \lambda_\epsilon(T)\mathcal{E}(T,u(T)) - \lambda_\epsilon(0)\mathcal{E}(0,u(0)) + \int_0^T \Big[\lambda_\epsilon\Psi(\dot u) - \dot\lambda_\epsilon\mathcal{E}(t,u)\Big]\mathrm{d}t. \tag{2.6}$$

Given sufficient smoothness, the stationarity condition for $\widetilde{\mathcal{I}}(u;\lambda_\epsilon)$ with prescribed initial condition $u(0) = u_0$ is

$$\delta\widetilde{\mathcal{I}}(u;\lambda_\epsilon) = \lambda_\epsilon(T)\mathrm{D}\mathcal{E}(T,u(T))v(T) + \int_0^T \Big[\lambda_\epsilon\mathrm{D}\Psi(\dot u)\dot v - \dot\lambda_\epsilon\mathrm{D}\mathcal{E}(t,u)v\Big]\mathrm{d}t = 0,$$

where the variations v can be taken, *e.g.*, to be smooth and have compact support within the interval $(0,T]$. An integration by parts further gives

$$\delta\widetilde{\mathcal{I}}(u;\lambda_\epsilon) = \lambda_\epsilon(T)\big[\mathrm{D}\Psi(\dot u(T)) + \mathrm{D}\mathcal{E}(T,u(T))\big]v(T) + \int_0^T \Big[-\frac{\mathrm{d}}{\mathrm{d}t}\big(\lambda_\epsilon\mathrm{D}\Psi(\dot u)\big) - \dot\lambda_\epsilon\mathrm{D}\mathcal{E}(t,u)\Big]v\,\mathrm{d}t = 0$$

and the corresponding Euler-Lagrange equations follow as

$$\lambda_\epsilon(t)\mathrm{D}^2\Psi(\dot u)\ddot u + \dot\lambda_\epsilon(t)[\mathrm{D}\Psi(\dot u) + \mathrm{D}\mathcal{E}(t,u)] = 0, \tag{2.7a}$$

$$u(0) = u_0, \tag{2.7b}$$

$$\mathrm{D}\Psi(\dot u(T)) + \mathrm{D}\mathcal{E}(T,u(T)) = 0. \tag{2.7c}$$

We assume that $\dot\lambda_\epsilon/\lambda_\epsilon \downarrow -\infty$ uniformly on $[0,T]$, which implies (2.5), and thus (2.7) may be regarded as an elliptic regularization of the evolutionary problem (1.1). An admissible and particularly simple choice of causal weights is obtained by assuming that

$$\frac{\dot\lambda_\epsilon(t)}{\lambda_\epsilon(t)} = -1/\epsilon,$$

which gives

$$\lambda_\epsilon(t) = \mathrm{e}^{-t/\epsilon}, \tag{2.8}$$

where, for definiteness, we have set $\lambda_\epsilon(0) = 1$. For this particular choice of weights, the functional (2.6) becomes

$$\widetilde{\mathcal{I}}_\epsilon(u) = \mathcal{I}_\epsilon(u) - \mathcal{E}(0,u(0)) \tag{2.9}$$

with

$$\mathcal{I}_\epsilon(u) = \mathrm{e}^{-T/\epsilon}\mathcal{E}(T,u(T)) + \int_0^T \mathrm{e}^{-t/\epsilon}\Big[\Psi(\dot u) + \frac{1}{\epsilon}\mathcal{E}(t,u)\Big]\mathrm{d}t.$$

Since $u(0) = u_0$ is a given initial datum, $\widetilde{\mathcal{I}}_\epsilon$ differs from $\mathcal{I}_\epsilon$ by an inconsequential additive constant. For definiteness, henceforth we choose to work with the functional $\mathcal{I}_\epsilon$ and call it the *weighted dissipation-energy functional.* The corresponding Euler-Lagrange equations (2.7) reduce to

$$-\epsilon\mathrm{D}^2\Psi(\dot u)\ddot u + \mathrm{D}\Psi(\dot u) + \mathrm{D}\mathcal{E}(t,u) = 0, \tag{2.10a}$$

$$u(0) = u_0, \tag{2.10b}$$

$$\mathrm{D}\Psi(\dot u(T)) + \mathrm{D}\mathcal{E}(T,u(T)) = 0, \tag{2.10c}$$

whence the elliptic character of the regularization (2.10a) is particularly apparent.

We now may regard the one-parameter family of minimum problems:

$$\inf_{u\in\mathbb{Y},\, u(0)=u_0} \mathcal{I}_\epsilon(u), \tag{2.11}$$

where $\mathbb{Y}$ is some suitable space of trajectories $u:[0,T]\to Y$, as a continuous-time version of the sequence (2.1) of incremental problems. In particular, one would expect that the limit of $\triangle t\to 0$ of (2.1) and the limit of $\epsilon\to 0$ of (2.11) characterize the same trajectories, and that these trajectories satisfy (1.1) in some appropriate sense. Establishing this connection rigorously is beyond the scope of this paper. Instead, we shall simply postulate (2.11) as the fundamental physical principle of interest and proceed to elucidate its properties and behavior in the strict causal limit of $\epsilon\to 0$. A partial justification is obtained from the analysis of the examples discussed in the following section.

3. Three illustrative examples

The following examples illuminate the connections between the minimizers u^ϵ of the weighted dissipation-energy functional $\mathcal{I}_\epsilon$ and the solution u of the original problem (1.1).

3.1. A scalar viscous example

As a prototypical double-well potential we consider the tri-quadratic energy function

$$F_{\rm tq}(u)=\begin{cases}\frac{1}{2}(u{+}1)^2 & \text{for } u\le -\frac{1}{2},\\ \frac{1}{4}-\frac{1}{2}u^2 & \text{for } |u|\le \frac{1}{2},\\ \frac{1}{2}(u{-}1)^2 & \text{for } u\ge \frac{1}{2},\end{cases} \tag{3.1}$$

and the associated evolutionary problem

$$\begin{aligned}&\dot u + F_{\rm tq}'(u)-\delta=0,\\ &u(0)=-1,\end{aligned}$$

with $\delta\in(1/2,1)$. This is a gradient flow for the potential $\mathcal{E}:u\mapsto F_{\rm tq}(u)-\delta u$; and since $F_{\rm tq}'$ is piecewise linear, it is possible to calculate the exact solution u. This solution is strictly monotone and for $t\to\infty$ it converges to the unique steady state $u=1+\delta$. We additionally fix T such that $u(T)=1$. The weighted dissipation-energy functional (2.9) for $u^\epsilon:[0,T]\to\mathbb{R}$ takes the form

$$\mathcal{I}_\epsilon(u)=\mathrm{e}^{-T/\epsilon}\mathcal{E}(u(T))+\int_0^T \mathrm{e}^{-t/\epsilon}\Big(\frac{1}{2}\dot u^2+\frac{1}{\epsilon}\big(F_{\rm tq}(u)-\delta u\big)\Big)\mathrm{d}t$$

and the associated Euler-Lagrange equations (1.3) reduce to

$$\begin{aligned}&-\epsilon\ddot u+\dot u+F_{\rm tq}'(u)-\delta=0,\\ &u(0)=-1,\\ &u'(T)+F_{\rm tq}'(u(T))-\delta=0.\end{aligned} \tag{3.2}$$

This problem can conveniently be analyzed in the $(u,\dot u)$ phase plane, *cf.* Figure 1. The phase portrait follows readily from the piecewise linear structure of (3.2). Thus, the region $u\le -1/2$ follows from the identity $-\epsilon\ddot u+\dot u+u+1-\delta=0$ and corresponds to a linear saddle at $(u,\dot u)=(\delta{-}1,0)$, which is outside the domain $u\le -1/2$. Similarly, the region $|u|\le 1/2$ is governed by the identity $-\epsilon\ddot u+\dot u-u-\delta=0$, which corresponds to a the source point at $(-\delta,0)$, again outside the domain. For $u\ge 1/2$ we again have a linear saddle,

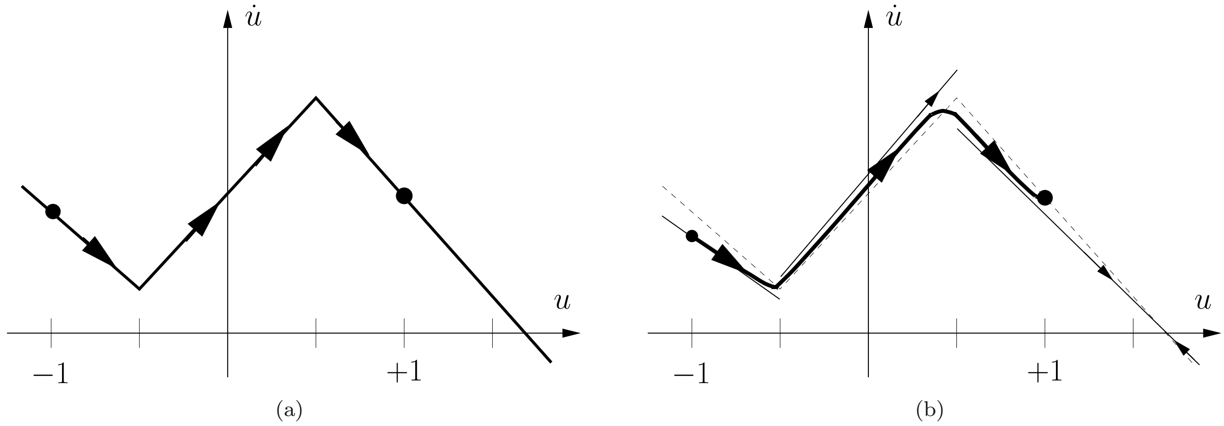

FIGURE 1. (a) Orbit for $\epsilon = 0$. (b) Orbit for $\epsilon > 0$ and three invariant straight lines.

now located inside the domain at $(1+\delta, 0)$. A closer analysis of the invariant manifolds associated with the three fixed points, including those which are not in the correct domains, shows that any solution u^ϵ satisfying (3.2) has to stay ϵ-close to the graph $\dot u = \delta - F'_{\rm tq}(u)$.

3.2. A scalar rate-independent problem

Rate-independent systems are in sharp contrast to systems with viscosity in that the latter possess an intrinsic time scale of viscous relaxation, whereas the former lack a time-scale and can react instantaneously *via* jumps. As a simple illustrative example we consider the weighted dissipation-energy functional

$$\mathcal{I}_\epsilon(u) = \mathrm{e}^{-T/\epsilon}\mathcal{E}(T, u(T)) + \int_0^T \mathrm{e}^{-t/\epsilon}\Big(\rho|\dot u(t)| + \tfrac{1}{\epsilon}\mathcal{E}\big(t, u(t)\big)\Big)\mathrm{d}t,$$
$$\text{with } \mathcal{E}(t,u) = F_{\rm tq}(u(t)) - \ell(t)u(t),$$

with $\rho \in (0,1)$, $u : [0,T] \to \mathbb{R}$ and $\ell : [0,T] \to \mathbb{R}$.

The problem is now nonsmooth, because the dissipation function $\Psi(v) = \rho|v|$, while convex, is not differentiable at the origin. We have $\partial\Psi(v) = \rho\,\mathrm{Sign}(v)$, where

$$\begin{aligned}&\mathrm{Sign}(v) = \{-1\} && \text{for } v \in [-\infty, 0),\\ &\mathrm{Sign}(0) = [-1, 1], &&\\ &\mathrm{Sign}(v) = \{1\} && \text{for } v \in (0, \infty],\end{aligned}$$

is the multi-valued *signum* function, and the Euler-Lagrange equation is given by [1]

$$\begin{aligned}&s(t) \in \rho\,\mathrm{Sign}(\dot u(t)),\\ &-\epsilon\dot s(t) + s(t) + F'_{\rm tq}(u(t)) = \ell(t), \quad \text{a.e. in } [0,T],\end{aligned} \tag{3.3}$$

where we assume that $u \in \mathrm{BV}([0,T])$ and $s \in \mathrm{W}^{1,1}([0,T])$. The first of these equations tells us that $\pm\,\dot u > 0$ implies $s = \pm\,\rho$. Hence, multiplying the second equation by $\dot u$ we obtain

$$\rho|\dot u| + F'_{\rm tq}(u)\dot u = \ell(t)\dot u. \tag{3.4}$$

Integration over time gives the energy balance

$$F_{\rm tq}(u(t)) - \ell(t)u(t) + \int_0^t \rho|\dot u(t)|\mathrm{d}t = F_{\rm tq}(u(0)) - \ell(0)u(0) - \int_0^t \dot\ell(t)\,u(s)\mathrm{d}s.$$

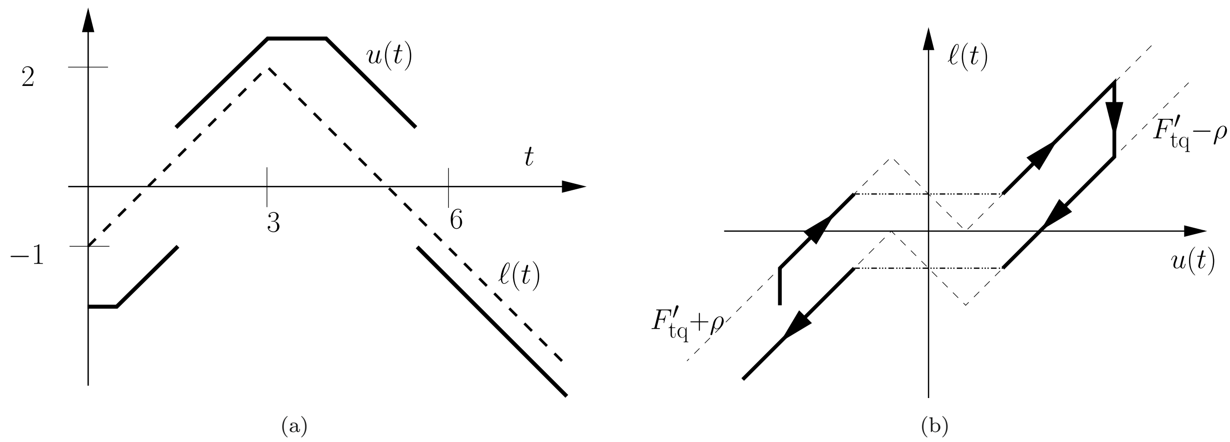

FIGURE 2. (a) $u(t)$ (full line with jumps) and $\ell(t)$ (dashed line) as functions of t. (b) Hysteretic behavior with jumps in the (u,ℓ)-plane.

Surprisingly, this equation is independent of ϵ, which is a general feature of the rate-independent case, see (4.4) and Proposition 4.1. In the scalar case it is easy to construct the solution from energy balance. At points where u is differentiable (3.4) together with $s \in \operatorname{Sign}(\dot u)$ provide the differential inclusion

$$0 \in \rho \operatorname{Sign}(\dot u) + F'_{\mathrm{tq}}(u) - \ell(t).$$

In addition, we have the jump condition

$$F_{\mathrm{tq}}(u(t{+}0)) - \ell(t)u(t{+}0) + \rho\big|u(t{+}0){-}u(t{-}0)\big| = F_{\mathrm{tq}}(u(t{-}0)) - \ell(t)u(t{-}0).$$

In the special case $\ell(t) = \min\{t{-}1, 5{-}t\}$ on the interval $[0,8]$ and $u(0) = -2$ we obtain the solution

$$u(t) = \begin{cases} -2 & \text{for } t \in [0,\rho], \\ t-\rho-2 & \text{for } t \in [\rho, 1{+}\rho), \\ t-\rho & \text{for } t \in (1{+}\rho, 3], \\ 3-\rho & \text{for } t \in [3, 3{+}2\rho], \\ 6-t+\rho & \text{for } t \in [3{+}3\rho, 5{+}\rho), \\ 4-t+\rho & \text{for } t \in (5{+}\rho, 8]. \end{cases}$$

This is also the solution of the global (S) & (E) energetic formulation discussed subsequently.

3.3. **Linear parabolic problems**

On the Hilbert space H the linear evolutionary problem

$$\begin{aligned} &\dot u + Au = \ell(t), \\ &u(0) = u_0, \end{aligned} \tag{3.5}$$

defines an abstract parabolic problem when $A : D(A) \to H$ is self-adjoint and positive definite, *i.e.*, $\langle Av, v\rangle \geq \alpha\|v\|^2$ with $\alpha > 0$. For $s \geq 0$ we set $X_s = D(A^{s/2})$ which is again a Hilbert space when equipped with the graph norm. For $s < 0$ we let $X_s = X^*_{-s}$, the dual of X_{-s}. We will use the well-known fact, that the unique solution u of (3.5) satisfies the estimate (see *e.g.*, [42])

$$\|u\|_{\mathrm{L}^2([0,T],X_3)} + \|\dot u\|_{\mathrm{L}^2([0,T],X_1)} + \|\ddot u\|_{\mathrm{L}^2([0,T],X_{-1})} \leq C\Big(\|\ell\|_{\mathrm{L}^2([0,T],X_0)} + \|\dot\ell\|_{\mathrm{L}^2([0,T],X_{-1})} + \|u_0\|_{X_2}\Big). \tag{3.6}$$

Next we consider the associated minimization problem for

$$\mathcal{I}_\epsilon(u) = \mathrm{e}^{-T/\epsilon}\mathcal{E}(T,u(T)) + \int_0^T \mathrm{e}^{-t/\epsilon}\Big(\tfrac12\langle \dot u,\dot u\rangle + \tfrac{1}{\epsilon}\mathcal{E}(t,u)\Big)\mathrm{d}t,$$
$$\text{with } \mathcal{E}(t,u) = \tfrac12\langle Au,u\rangle - \langle \ell(t),u\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $H = X_0$. The corresponding Euler-Lagrange equation is

$$\begin{aligned} &-\epsilon\ddot u + \dot u + Au = \ell(t),\\ &u(0) = u_0,\\ &\dot u(T) + Au(T) = \ell(T). \end{aligned} \tag{3.7}$$

In [25], Chapter 6, this equation is called the *elliptic regularization* of (3.5). It is shown there that (3.7) has a unique solution which satisfies similar estimates to those given for u in (3.6). Let u_* and u_ϵ be the unique solutions of (3.5) and (3.7), then the estimate

$$\big\|u_* - u_\epsilon\big\|_{\mathrm{L}^2((0,T),X_0)} \leq \frac{\epsilon}{\alpha}\|\ddot u_*\|_{\mathrm{L}^2((0,T),X_{-1})} \tag{3.8}$$

holds, which again shows that the weighted dissipation-energy functional $\mathcal{I}_\epsilon$ is useful to construct approximate solutions to (3.5). For the proof of this result, we define the difference $w = u_\epsilon - u_*$ which satisfies the problem

$$-\epsilon\ddot w + \dot w + Aw = \epsilon\ddot u_*, \tag{3.9a}$$
$$w(0) = 0, \tag{3.9b}$$
$$\dot w(T) + Aw(T) = 0. \tag{3.9c}$$

We begin by noting that $\|\ddot u_*\|_{\mathrm{L}^2([0,T],X_{-1})}$ is finite, see (3.6). Next we set

$$\rho(t) = \frac12\langle w,w\rangle = \frac12\|w\|^2_{X_0}$$

and multiply (3.9a) by w to obtain

$$\alpha\rho + \frac{\epsilon^2}{2\alpha}\|\ddot u\|^2_{X_{-1}} \;\geq\; \epsilon\|\ddot u_*\|_{X_{-1}}\|w\|_{X_1} \;\geq\; \langle \epsilon\ddot u_*, w\rangle = \; -\epsilon\ddot\rho + \epsilon\langle\dot w,\dot w\rangle + \dot\rho + \langle Aw,w\rangle \;\geq\; -\epsilon\ddot\rho + \dot\rho + 2\alpha\rho.$$

Using the boundary conditions (3.9b) and (3.9c) for w we find the differential estimate

$$-\epsilon\ddot\rho + \dot\rho + \alpha\rho \leq \frac{\epsilon^2}{2\alpha}\|\ddot u\|^2_{X_{-1}}, \tag{3.10a}$$
$$\rho(0) = \dot\rho(0) = 0, \tag{3.10b}$$
$$\rho(T) \geq 0, \quad \dot\rho(T) = \langle w(T),\dot w(T)\rangle = -\langle Aw(T),w(T)\rangle \leq 0. \tag{3.10c}$$

Integrating (3.10a) over $[0,T]$ we find the desired result (3.8):

$$\frac{\alpha}{2}\|u_\epsilon - u_*\|^2_{\mathrm{L}^2((0,T),X_0)} = \alpha\int_0^T \rho(t)\mathrm{d}t$$
$$\leq \int_0^T \frac{\epsilon^2}{2\alpha}\|\ddot{u}_*(s)\|^2_{X_{-1}}\mathrm{d}t + \epsilon\big(\dot\rho(T)-\dot\rho(0)\big) - \big(\rho(T)-\rho(0)\big) \ \leq\ \frac{\epsilon^2}{2\alpha}\|\ddot{u}_*\|^2_{\mathrm{L}^2((0,T),X_{-1})},$$

where the last estimate used the boundary conditions (3.10b) and (3.10c).

4. Abstract results for rate-independent problems

The time-rescaling invariance of rate-independent systems make them special. In particular, the minimizers are allowed to have jumps, which is not possible if the dissipation functional Ψ has superlinear growth.

4.1. The abstract assumptions

We start by formulating the problem precisely in a separable, reflexive Banach space X that is compactly embedded into the larger Banach space Z. In continuum mechanical applications, typical choices for the spaces X and Z are $X = \mathrm{W}^{k,p}(\Omega)$, $k\in\mathbb{N}$, and $Z = \mathrm{L}^1(\Omega)$, respectively, *cf.* [26,33].

In this section the general assumptions on Ψ and $\mathcal{E}$ are the following. The dissipation potential Ψ is convex, 1-homogeneous and lower semi-continuous on X. Moreover, it is coercive in Z, *i.e.*,

$$\exists\, c>0 \ \ \forall\, v\in X: \quad \Psi(v)\geq c\|v\|_Z. \tag{4.1}$$

For the energy-storage functional $\mathcal{E}:[0,T]\times X\to\mathbb{R}_\infty$ we assume that it is weakly lower semi-continuous and satisfies the coercivity

$$\exists\, \alpha,c,C>0 \ \ \forall\,(t,u)\in[0,T]\times X: \quad \mathcal{E}(t,u)\geq c\|u\|_X^\alpha - C. \tag{4.2}$$

We additionally assume that $t\mapsto\mathcal{E}(t,u)$ is differentiable whenever $\mathcal{E}(t,u)<\infty$, namely,

$$\exists\, c_1^E, c_0^E>0 \ \ \forall\,(t,u)\in[0,T]\times X: \quad |\partial_t\mathcal{E}(t,u)|\leq c_1^E\big(\mathcal{E}(t,u)+c_0^E\big). \tag{4.3}$$

This type of *control of power* of the external loading was first introduced in [15,29] and proves very useful for obtaining *a priori* bounds, see below.

4.2. The energy balance and *a priori* bounds

It is shown in [34] (see also [31]) that under suitable assumptions any absolutely continuous solution u of the rate-independent differential inclusion (1.1) satisfies the energy balance

$$\mathcal{E}(t,u(t)) + \int_{[0,t]}\Psi(\mathrm{d}u) - \int_0^t \partial_s\mathcal{E}(s,u(s))\mathrm{d}s = C \quad \text{for all } t\in[0,T], \tag{4.4}$$

where $C=\mathcal{E}(0,u(0))$. As in [26,34,35] we will deal with BV functions that are defined everywhere. For non-increasing and positive $a\in\mathrm{C}([0,T],\mathbb{R})$, an interval $J\subset[0,T]$ and $u\in\mathrm{BV}([0,T],Z)$ we use the BV notation

$$\int_J a(s)\Psi(\mathrm{d}u) = \sup\Big\{\ \sum_{j=1}^N a(s_j)\Psi(u(s_j){-}u(s_{j-1}))\ \Big|\ N\in\mathbb{N},\quad s_0,s_N\in J,\ s_0{<}s_1{<}\cdots{<}s_{N-1}{<}s_N\ \Big\}. \tag{4.5}$$

Note that it is important here, to distinguish the case where the boundary points of J are included into J or not. Moreover, the supremum definition is only suitable for nondecreasing and positive a, as in this case the approximating sums on the right-hand side of (4.5) behaves monotonous under refining the partition. For general $a \in \mathrm{C}([0,T],\mathbb{R})$ one has to proceed as for the Riemann-Stieltjes integral and replace the supremum by a limit for the fineness of the partition going to 0.

The weighted dissipation-energy functional $\mathcal{I}_\epsilon$ is now written as

$$\mathcal{I}_\epsilon(u) = \mathrm{e}^{-T/\epsilon}\mathcal{E}(T,u(T)) + \int_{[0,T]} \mathrm{e}^{-t/\epsilon}\Psi(\mathrm{d}u) + \int_0^T \frac{\mathrm{e}^{-t/\epsilon}}{\epsilon}\,\mathcal{E}(t,u(t))\mathrm{d}t.$$

A surprising fact is that the energy balance (4.4), which is independent of ϵ, holds for all minimizers of $\mathcal{I}_\epsilon$.

Proposition 4.1. *Let Ψ and $\mathcal{E}$ satisfy the assumptions of Section 4.1. If u_* is a minimizer of $\mathcal{I}_\epsilon$ under the constraint $u(0) = u_0$, then u_* satisfies the energy balance (4.4) for almost all $t \in [0,T]$. If additionally, the initial condition u_0 satisfies the stability condition*

$$\forall\, \widetilde{u} \in X: \quad \mathcal{E}(0,u_0) \le \mathcal{E}(0,\widetilde{u}) + \Psi(\widetilde{u}-u_0), \tag{4.6}$$

then the energy balance (4.4) holds with $C = \mathcal{E}(0,u_0)$.

Proof. We compare the energy of u with that of rescaled functions $\widetilde{u}$. For this, choose an increasing diffeomorphism $\beta : [0,T] \to [0,T]$ (*i.e.*, $\beta(0) = 0$ and $\beta(T) = T$) and define $t = \beta(s)$ as well as $\widetilde{u}$ *via* $u_*(s) = \widetilde{u}(\beta(s))$. Using the transformation rule for integrals with $\mathrm{d}t = \dot{\beta}(s)\mathrm{d}s$ and $\mathrm{d}\widetilde{u}|_{t=\beta(s)} = \mathrm{d}u_*|_s$ we may express $\mathcal{I}_\epsilon(\widetilde{u})$ in terms of u_* again and find

$$\begin{aligned}\mathcal{I}_\epsilon(\widetilde{u}) - \mathrm{e}^{-T/\epsilon}\mathcal{E}(T,u_*(T)) &= \int_{[0,T]} \mathrm{e}^{-t/\epsilon}\Psi(\mathrm{d}\widetilde{u}) + \int_0^T \frac{\mathrm{e}^{-t/\epsilon}}{\epsilon}\,\mathcal{E}(t,\widetilde{u}(t))\mathrm{d}t \\ &= \int_{[0,T]} \mathrm{e}^{-\beta(s)/\epsilon}\Psi(\mathrm{d}u_*) + \int_0^T \frac{\mathrm{e}^{-\beta(s)/\epsilon}}{\epsilon}\,\dot{\beta}(s)\,\mathcal{E}(\beta(s),u_*(s))\mathrm{d}s.\end{aligned}$$

With $\rho \in \mathrm{C}_\mathrm{c}^\infty((0,T))$ we choose β *via* $\mathrm{e}^{-\beta(s)/\epsilon} - \mathrm{e}^{-s/\epsilon} = \delta\rho(s)$, where $0 < \delta \ll 1$. This leads to the expansion $\beta(s) = s - \epsilon\delta\rho(s)\mathrm{e}^{s/\epsilon} + O(\delta^2)$. Since u_* is a minimizer we have

$$\begin{aligned}0 &\le \mathcal{I}_\epsilon(\widetilde{u}) - \mathcal{I}_\epsilon(u_*) \\ &= \int_{[0,T]} \big(\mathrm{e}^{-\beta(s)/\epsilon} - \mathrm{e}^{-s/\epsilon}\big)\Psi(\mathrm{d}u_*) + \int_0^T \frac{1}{\epsilon}\big(\mathrm{e}^{-\beta(s)/\epsilon}\dot{\beta}(s)\mathcal{E}(\beta(s),u_*(s)) - \mathrm{e}^{-s/\epsilon}\mathcal{E}(s,u_*(s))\big)\mathrm{d}s.\end{aligned}$$

Using $\frac{\dot{\beta}}{\epsilon}\mathrm{e}^{-\beta/\epsilon} = \frac{1}{\epsilon}\mathrm{e}^{-s/\epsilon} - \delta\dot{\rho}$, dividing by $\delta > 0$ and taking the limit $\delta \searrow 0$ we obtain

$$0 \le \int_0^T \rho(s)\Psi(\mathrm{d}u_*) + \int_0^T \big(-\rho(s)\partial_s\mathcal{E}(s,u_*(s)) - \dot{\rho}(s)\mathcal{E}(s,u_*(s))\big)\mathrm{d}s.$$

Since $\rho \in \mathrm{C}_\mathrm{c}^\infty((0,T))$ is arbitrary, the almost everywhere validity of the energy balance (4.4) follows from the lemma of Du Bois-Reymond.

Finally we assume that the stability condition (4.6) holds. We let $e(t) = \mathcal{E}(t,u_*(t))$, $\Delta(t) = \int_{[0,t]}\Psi(\mathrm{d}u_*)$ and $w(t) = \int_0^t \partial_s\mathcal{E}(s,u_*(s))\mathrm{d}s$, then the energy balance reads $e(t)+\Delta(t)-w(t) = C$, where Δ is monotone and hence in $\mathrm{BV}([0,T])$ and $w \in \mathrm{W}^{1,\infty}([0,T])$ (by using (4.3)). Hence, we may define $\Delta^{+0} = \lim_{0<t\to 0}\Delta(t) = \Psi(u^{+0}-u_0)$, where $u^{+0} = \lim_{0<t\to 0} u(t)$, which exists in Z as $\int_{(0,t]}\Psi(\mathrm{d}u) \to 0$ for $t \to 0$. Setting $e^{+0} = \lim_{0<t\to 0} e(t) = C - \Delta^{+0}$ we find by weak lower semi-continuity of $\mathcal{E}$ that $\mathcal{E}(0,u^{+0}) \le e^{+0}$. Using the stability condition (4.6) this provides

$$\mathcal{E}(0,u_0) \le \mathcal{E}(0,u^{+0}) + \Psi(u^{+0}-u_0) \le e^{+0} + \Delta^{+0} = C.$$

For the opposite estimate we use that u_* minimizes. For $\kappa \in (0,T)$ let $u_\kappa(t) = u_0$ for $t \in [0,\kappa)$ and $u_*(t)$ otherwise. Using the (almost everywhere) energy balance for u_* gives

$$\begin{aligned} 0 \le\ & \mathcal{I}_\epsilon(u_\kappa) - \mathcal{I}_\epsilon(u_*) \\ &= \mathrm{e}^{-\kappa/\epsilon}\Psi(u_*(\kappa)-u_0) - \int_{[0,\kappa]} \mathrm{e}^{-t/\epsilon}\Psi(\mathrm{d}u_*) + \int_0^\kappa \frac{\mathrm{e}^{-t/\epsilon}}{\epsilon}\big(\mathcal{E}(t,u_0) - C - w(t) + \Delta(t)\big)\mathrm{d}t \\ &= \mathrm{e}^{-\kappa/\epsilon}\big(\Psi(u_*(\kappa)-u_0) - \int_{[0,\kappa]} \Psi(\mathrm{d}u_*) + \int_0^\kappa \frac{\mathrm{e}^{-t/\epsilon}}{\epsilon}\big(\mathcal{E}(0,u_0) - C + o(1)_{t\to 0}\big)\mathrm{d}t, \end{aligned}$$

where the last step used a cancellation occurring after integration by parts of $\int_0^\kappa \frac{\mathrm{e}^{-t/\epsilon}}{\epsilon}\Delta(t)\mathrm{d}t$ with $\Delta(t) = \int_{[0,t]} \Psi(\mathrm{d}u_*)$. Since the first term in the last equation is non-positive, we conclude the desired result $\mathcal{E}(0,u_0) \ge C$ by making κ sufficiently small. □

It is remarkable that the energy balance (4.4) holds exactly for the minimizers u^ϵ of $\mathcal{I}_\epsilon$, despite their dependence on ϵ. For the subsequent analysis we always assume that the stability condition (4.6) holds for the initial condition u_0. Then, the minimizers satisfy useful *energetic a priori estimates*, namely

$$\mathcal{E}(t,u(t)) + c_0^E \le \Big(\mathcal{E}(0,u(0)) + c_0^E\Big)\,\mathrm{e}^{c_1^E\,t} \tag{4.7a}$$

$$\int_{[0,t]} \Psi(\mathrm{d}u) \le \Big(\mathcal{E}(0,u(0)) + c_0^E\Big)\,\mathrm{e}^{c_1^E\,t}. \tag{4.7b}$$

These estimates holds for any function $u : [0,T] \to Z$ satisfying the energy balance (4.4), which implicitly means that $t \mapsto \partial_t\mathcal{E}(t,u(t))$ is measurable. (At this point, we need not assume measurability of $t \mapsto u(t) \in X$, see [13,29].) Writing $e(t) = \mathcal{E}(t,u(t))$ again, the first is obtained by inserting (4.3) into the energy balance (4.4) and neglecting the dissipation, namely $e(t) \le e(0) + \int_0^t c_1^E\big(e(s)+c_0^E\big)\mathrm{d}s$. Adding c_0^E on both sides and using Gronwall's lemma the estimate (4.7a) is established. Now using the energy balance once again the dissipation can be estimated *via*

$$\begin{aligned} \int_{[0,t]} \Psi(\mathrm{d}u) &\le e(0) - e(t) + \int_0^t c_1^E\big(\mathcal{E}(s,u(s)) + c_0^E\big)\mathrm{d}s \\ &\le e(0) - e(t) + \int_0^t c_1^E\big(e(0)+c_0^E\big)\mathrm{e}^{c_1^E\,s}\big)\mathrm{d}s \\ &= e(0) - e(t) + \big(e(0)+c_0^E\big)\big(\mathrm{e}^{c_1^E\,t}-1\big) \ \le\ \big(e(0)+c_0^E\big)\mathrm{e}^{c_1^E\,t}. \end{aligned}$$

The energetic *a priori* estimates (4.7) and the coercivity assumptions (4.1) and (4.2) imply that all minimizers u of $\mathcal{I}_\epsilon$ satisfy ϵ-independent *a priori* bounds:

$$\|u\|_{\mathrm{L}^\infty([0,T],X)} \le C_1 \quad \text{and} \quad \int_{[0,T]} \|\mathrm{d}u\|_Z \le C_2. \tag{4.8}$$

These bounds suggest defining the weighted dissipation-energy functional $\mathcal{I}_\epsilon$ on the Banach space

$$\mathbb{Y} := \mathrm{L}^\infty([0,T],X) \cap \mathrm{BV}([0,T],Z),$$

despite the lack of equi-coercivity of the sequence $\mathcal{I}_\epsilon$ in this space. In fact, owing to the exponential weight the bound $\mathcal{I}_\epsilon(u) \le C$ results in the *very weak a priori* estimates

$$\|u\|^\alpha_{\mathrm{L}^\alpha([0,T],X)} \le C\,\frac{\mathrm{e}^{T/\epsilon}}{\epsilon} \quad \text{and} \quad \int_{[0,T]} \|\mathrm{d}u\|_Z \le C\mathrm{e}^{T/\epsilon}.$$

Only for minimizers we obtain the much better estimates (4.7) and hence (4.8) that are independent of ϵ.

It is interesting to contrast the preceding results with the case of a general dissipation functional. Thus, suppose that $\mathcal{I}_\epsilon(u) = \int_0^T \mathrm{e}^{-t/\epsilon}\big(\Psi(u,\dot u) + \frac{1}{\epsilon}\mathcal{E}(t,u)\big)\mathrm{d}t$, where Ψ is additionally allowed to depend on u, but still $v \mapsto \Psi(u,v)$ is convex for each u. Moreover, assume Ψ to be smooth. The corresponding Euler-Lagrange equation is

$$-\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\Big(\partial_v\Psi(u,\dot u)\Big) + \epsilon\partial_u\Psi(u,\dot u) - \partial_v\Psi(u,\dot u) + \partial_u\mathcal{E}(t,u) = 0.$$

Defining $E(t) = \mathcal{E}(t,u) - \epsilon R(u,\dot u)$ with

$$R(u,v) = \partial_v\Psi(u,v)[v] - \Psi(u,v) = \int_{\theta=0}^1 \theta\mathrm{D}^2\Psi(u,\theta v)[v,v]\mathrm{d}\theta \geq 0$$

(by convexity) we easily find the identity $\frac{\mathrm{d}}{\mathrm{d}t}E = \partial_t\mathcal{E}(t,u) - \partial_v\Psi(u,\dot u)[\dot u]$, which in turn gives the energy balance

$$\mathcal{E}(t,u(t)) - \epsilon R(u(t),\dot u(t)) + \int_s^t \partial_v\Psi(u(\tau),\dot u(\tau))[\dot u(\tau)]\mathrm{d}\tau = \mathcal{E}(s,u(s)) - \epsilon R(u(s),\dot u(s)) + \int_s^t \partial_\tau\mathcal{E}(\tau,u(\tau))\mathrm{d}s.$$

In the rate-independent case we have $R(u,v) = 0$ and $\partial_v\Psi(u,v)[v] = \Psi(u,v)$, which returns the energy balance from above. In the rate-dependent case we may write $\widehat\Psi(u,v) = \partial_v\Psi(u,v)[v]$ and assume $c_1^\Psi\widehat\Psi(u,v) \leq R(u,v) \leq c_2^\Psi\widehat\Psi(u,v)$, which holds with $c_1 = c_2 = \frac{p-1}{p}$ for $\Psi(u,v) = \|v\|_{\mathrm{L}^p}^p$. Let $\widetilde e(t) = \mathcal{E}(t,u(t)) + c_0^E - \epsilon R(u(t),\dot u(t))$ and $\psi(t) = \widehat\Psi(u(t),\dot u(t))$. Then, for $0 \leq s < t \leq T$ we have the estimate

$$\widetilde e(t) + (1 - \epsilon c_1^E c_2^\Psi)\int_s^t \psi(\tau)\mathrm{d}\tau \leq \widetilde e(s) + \int_s^t c_1^E\widetilde e(\tau)\mathrm{d}\tau.$$

Thus, for $\epsilon \leq 1/(c_1^E c_2^\Psi)$ Gronwall's lemma and $\psi \geq 0$ give $\widetilde e(t) \leq \mathrm{e}^{c_1^E(t-s)}\widetilde e(s)$. Inserting this into the integral on the right-hand side we obtain an estimate in terms of $\widetilde e(s)$ alone. However, it is not clear how the boundary condition at $t = T$ can be used to derive *a priori* bounds for $\mathcal{E}(t,u(t))$ and for $\int_{[0,T]}\Psi(\mathrm{d}u)$ which are independent of ϵ.

4.3. **Convergence to energetic solutions**

As already noted, a central objective of the analysis is to ascertain the *strict causal* limit $\epsilon \to 0$. Unfortunately, the Γ-convergence machinery fails to deliver useful results. Under suitable continuity assumptions an easy calculation gives

$$\Gamma\text{-}\lim_{\epsilon\to 0}\mathcal{I}_\epsilon(u) = \Psi\Big(\lim_{s\searrow 0}u(s) - u(0)\Big) + \mathcal{E}(0, \lim_{s\searrow 0}u(s)).$$

This limit supplies scant information regarding the limiting trajectories, namely, whether a jump point $u(0{+}0) = \lim_{s\searrow 0}u(s)$ minimizes energy plus dissipation from $u(0)$. By contrast, by virtue of the *a priori* estimates (4.7) or (4.8), the minimizers u_ϵ of $\mathcal{I}_\epsilon$ are well-behaved and we are able to extract convergent subsequences.

We regard the weighted dissipation-energy functionals $\mathcal{I}_\epsilon$ to be defined in the space

$$\mathbb{Y} = \mathrm{L}^\infty([0,T],X) \cap \mathrm{BV}([0,T],Z),$$

and we let $\overset{\mathbb{Y}}{\rightharpoonup}$ denote weak* convergence in this space, namely,

$$u_k \overset{\mathbb{Y}}{\rightharpoonup} u \quad \overset{\text{def}}{\Longleftrightarrow} \quad \begin{cases} \sup_{k\in\mathbb{N}} \int_{[0,T]}\|\mathrm{d}u_k\|_Z < \infty \text{ and} \\ \forall\, w \in \mathrm{L}^1([0,T],X^*): \\ \int_0^T\langle u_k(t),w(t)\rangle_X\mathrm{d}t \to \int_0^T\langle u(t),w(t)\rangle_X\mathrm{d}t. \end{cases} \tag{4.9}$$

The supremum condition for the sequence is motivated by the *a priori* estimates (4.8), which are valid for minimizers of $\mathcal{I}_\epsilon$. Under the additional assumption that the reflexive space X is compactly embedded into Z the convergence in $\mathbb{Y}$ implies in fact pointwise convergence almost everywhere, namely if $u_k \overset{\mathbb{Y}}{\rightharpoonup} u$, then

$$\text{For } a.e.\ t \in [0,T]\colon \quad u_k(t) \to u(t) \text{ in } Z \quad \text{and } u_k(t) \rightharpoonup u(t) \text{ in } X. \tag{4.10}$$

In fact, to see this, we employ Helly's selection principle (*cf.* [26,35]) to find a subsequence $(u_{k_n})_{n\in N}$ that converges for all $t \in [0,T]$ to a limit $\widetilde{u}(t)$ strongly in Z and hence weakly in X. However, since the limit function $\widetilde{u}$ must be equal to u almost everywhere, we conclude the convergence of the whole sequence by the usual contradiction argument.

The following result is a first special case of Theorem 4.4 for constant sequences $\Psi_k = \Psi$ and $\mathcal{E}_k = \mathcal{E}$.

Theorem 4.2. *If X, Z, Ψ and $\mathcal{E}$ satisfy the assumptions of Section 4.1, then any family $(u_\epsilon)_{\epsilon\in(0,1)}$ of minimizers for the family $\mathcal{I}_\epsilon$ is weakly precompact in $\mathbb{Y}$. Moreover, any limit point $u \in \mathbb{Y}$ obtained for $\epsilon \to 0$ is a solution of the* energetic formulation, i.e., *for each $t \in [0,T]$ we have stability* **(S)** *and energy balance* **(E)***:*

$$\begin{array}{ll} \textbf{(S)} & \forall\, y \in X\colon \mathcal{E}(t,u(t)) \le \mathcal{E}(t,y) + \Psi(y - u(t)), \\ \textbf{(E)} & \mathcal{E}(t,u(t)) + \int_{[0,t]} \Psi(\mathrm{d}u) = \mathcal{E}(0,u(0)) + \int_0^t \partial_s \mathcal{E}(s,u(s))\mathrm{d}s. \end{array} \tag{4.11}$$

This theorem shows that, under natural assumptions, all possible limit points are *energetic solutions*, *i.e.*, solutions of the energetic formulation (4.11) of the rate-independent problem associated with Ψ and $\mathcal{E}$.

4.4. **Relaxation**

Next we consider the case of sequences of Ψ_k, $k \in \mathbb{N}$, of dissipation functionals as well as sequences of energy functionals $\mathcal{E}_k$, $k \in \mathbb{N}$, with properties as in the foregoing except for the lower semi-continuity of the $\mathcal{E}_k(t,\cdot)$. The sequences $(\Psi_k)_{k\in\mathbb{N}}$ and $(\mathcal{E}_k)_{k\in\mathbb{N}}$ in turn define the sequence of weighted dissipation-energy functionals

$$\mathcal{I}_\epsilon^k(u) = \mathrm{e}^{-T/\epsilon}\mathcal{E}_k(T,u(T)) + \int_{[0,T]} \mathrm{e}^{-t/\epsilon}\,\Psi_k(\mathrm{d}u) + \int_0^T \frac{\mathrm{e}^{-t/\epsilon}}{\epsilon}\,\mathcal{E}_k(t,u(t))\mathrm{d}t.$$

Such sequences may occur in several contexts, including: numerical approximations; penalty formulations of side conditions or constraints; singular perturbations such as occurring in sharp-interface models; and others. A particular case of interest is the constant sequence $\Psi_k = \Psi$ and $\mathcal{E}_k = \mathcal{E}$ considered in Section 4.3. In this case, the Γ-limit of $\mathcal{E}_k$ coincides the relaxation with its lower semi-continuous envelop, or *relaxation*, in the sense of the direct method of the calculus of variations (*cf.*, *e.g.*, [11,12]).

We work in a reflexive Banach space X and denote by $\rightharpoonup$ and $\to$ weak and strong convergence, respectively. Our assumptions on the sequences $(\Psi_k)_{k\in\mathbb{N}}$ and $(\mathcal{E}_k)_{k\in\mathbb{N}}$ and their limits Ψ and $\mathcal{E}$, respectively, are the following:

(A1) **Weak continuous convergence of Ψ_k:**

$$v_k \rightharpoonup v \quad \Longrightarrow \quad \Psi_k(v_k) \to \Psi(v). \tag{4.12}$$

(A2) **Weak Γ-convergence of $\mathcal{E}_k$:**

$$\begin{array}{l} \mathcal{E}_k(t,\cdot) \xrightarrow{\ \Gamma\ } \mathcal{E}(t,\cdot),\ i.e. \\ \text{(i)} \quad u_k \rightharpoonup u \quad \Longrightarrow \quad \liminf_{k\to\infty} \mathcal{E}_k(t,u_k) \ge \mathcal{E}(t,u), \\ \text{(ii)} \quad \forall\,(t,u)\,\exists\,\widetilde{u}_k \text{ with } \widetilde{u}_k \rightharpoonup u:\ \mathcal{E}_k(t,\widetilde{u}_k) \to \mathcal{E}(t,u). \end{array} \tag{4.13}$$

The sequence $(\widetilde{u}_k)_{k\in\mathbb{N}}$ is called a *recovery sequence* for u. In addition we assume that for each $r > 0$ there exists $R > 0$ such that for all (t,u) with $\|u\|_X \le r$ the recovery sequence $\widetilde{u}_k$ can be chosen such that $\|\widetilde{u}_k\|_X \le R$.

(A3) **Energetic weak continuity of the power of external forces** $\partial_t\mathcal{E}_k$:

$$\left.\begin{array}{l} u_k \rightharpoonup u \\ \mathcal{E}_k(t,u_k) \to \mathcal{E}(t,u) \end{array}\right\} \quad \Longrightarrow \quad \partial_t\mathcal{E}_k(t,u_k) \to \partial_t\mathcal{E}(t,u). \tag{4.14}$$

In [15,29] it is shown that (4.14) is a reasonable assumption in many applications, including finite-strain elasticity and plasticity. We also refer to [17,24,32,36] for related relaxations and Γ-limits in the rate-independent setting based on similar assumptions.

Theorem 4.3. *Let X be a separable, reflexive Banach space and let the assumptions* (4.1), (4.2), *and* (4.3) *hold uniformly in $k \in \mathbb{N}$ and let* (A1) *to* (A3) *be satisfied. Then, for each $\epsilon > 0$ the weighted dissipation-energy functional $\mathcal{I}_\epsilon : \mathbb{Y} \to \mathbb{R}_\infty$ defined by*

$$\mathcal{I}_\epsilon(u) = \mathrm{e}^{-T/\epsilon}\mathcal{E}(T,u(T)) + \int_{[0,T]} \mathrm{e}^{-t/\epsilon}\Psi(\mathrm{d}u) + \int_0^T \frac{1}{\epsilon}\mathrm{e}^{-t/\epsilon}\mathcal{E}(t,u(t))\mathrm{d}t \tag{4.15}$$

is the Γ-limit of $(\mathcal{I}_\epsilon^k)_{k\in\mathbb{N}}$ with respect to weak convergence in $\mathbb{Y}$.*

Proof. ad (i) Lower semicontinuity:

Choose any $u \in \mathbb{Y}$ and an arbitrary sequence (u_k) in $\mathbb{Y}$ with $u_k \overset{\mathbb{Y}}{\rightharpoonup} u$. Let $\beta = \liminf_{k\to\infty} \mathcal{I}_\epsilon^k(u_k)$. Then we have to show that $\alpha = \mathcal{I}_\epsilon(u) \le \beta$. For $\beta = \infty$ nothing is to be shown, whence we take $\beta < \infty$ and may assume $\mathcal{I}_\epsilon^k(u_k) \le \beta+1$. After extracting a subsequence if necessary (*cf.* (4.10)) we may assume that

$$\forall\, t \in [0,T]: \; u_k(t) \rightharpoonup u(t) \text{ in } X. \tag{4.16}$$

Using (4.12) for any finite partition $0 \le s_0 < s_1 < \cdots < s_{N-1} < s_N \le T$ we obtain

$$\sum_{j=1}^N \mathrm{e}^{s_j/\epsilon}\Psi_k(u_k(s_j) - u_k(s_{j-1})) \overset{k\to\infty}{\longrightarrow} \sum_{j=1}^N \mathrm{e}^{s_j/\epsilon}\Psi(u(s_j) - u(s_{j-1})).$$

The right-hand side can be made larger than $\int_{[0,T]} \mathrm{e}^{-t/\epsilon}\Psi(\mathrm{d}u) - \delta$ for any $\delta > 0$. Hence, there exists k_0 such that the left-hand side is larger than $\int_{[0,T]} \mathrm{e}^{-t/\epsilon}\Psi(\mathrm{d}u) - 2\delta$ for all $k \ge k_0$. Taking the supremum on the left-hand side gives $\int_{[0,T]} \mathrm{e}^{-t/\epsilon}\Psi_k(\mathrm{d}u_k) \ge \int_{[0,T]} \mathrm{e}^{-t/\epsilon}\Psi(\mathrm{d}u) - 2\delta$ for $k \ge k_0$. Since $\delta > 0$ is arbitrary we obtain

$$\liminf_{k\to\infty} \int_{[0,T]} \mathrm{e}^{-t/\epsilon}\Psi_k(\mathrm{d}u_k) \ge \int_{[0,T]} \mathrm{e}^{-t/\epsilon}\Psi(\mathrm{d}u). \tag{4.17}$$

Now we estimate the stored energy. Using (4.16) and part (i) of (4.13) gives $\mathcal{E}(t,u(t)) \le \liminf_{k\to\infty} \mathcal{E}_k(t,u_k(t))$ for all $t \in [0,T]$. Now, (4.3) implies $\mathcal{E}_k(t,u) \ge -c_0^E$, and Fatou's Lemma yields

$$\begin{aligned} \int_0^T \mathrm{e}^{-t/\epsilon}\mathcal{E}(t,u(t))\mathrm{d}t \quad &\le \quad \int_0^T \mathrm{e}^{-t/\epsilon} \liminf_{k\to\infty} \mathcal{E}_k(t,u_k(t))\mathrm{d}t \\ &\overset{\text{Fatou}}{\le} \quad \liminf_{k\to\infty} \int_0^T \mathrm{e}^{-t/\epsilon}\mathcal{E}_k(t,u_k(t))\mathrm{d}t. \end{aligned}$$

Adding this inequality to (4.17) we obtain the desired assertion $\mathcal{I}_\epsilon(u) \le \liminf_{k\to\infty} \mathcal{I}_\epsilon^k(u_k)$.

ad (ii) Recovery sequence:

First we note that on bounded sets of $\mathbb{Y}$ the weak* topology defined in (4.9) is metrizable, since the predual $\mathrm{L}^1([0,T],X^*)$ is separable. We denote such a metric by $d_{\mathbb{Y}}$.

We fix any $u \in \mathbb{Y}$, and without loss of generality we may assume $\alpha = \mathcal{I}_\epsilon(u) < \infty$. From [13] we know for each $u \in \mathbb{Y}$ there exists a sequences of partitions $0 \leq s_0^n < s_1^n < \cdots < s_{N_n-1}^n < s_{N_n}^n \leq T$ with $\Phi_n = \max\{s_j^n - s_{j-1}^n \mid j = 1, \ldots, N_n\} \to 0$ such that the sequence of piecewise constant interpolants $U_n \in \mathbb{Y}$ with

$$\begin{aligned} &U_n(t) = u(s_j^n) \text{ for } t \in (s_{j-1}^n, s_j^n], \quad j = 1, \ldots, N_n, \\ &U_n(t) = u(s_0^n) \text{ for } t \in [0, s_0^n], \quad U(t) = u(s_{N_n}^n) \text{ for } t \in (s_{N_n}^n, T], \end{aligned}$$

satisfies

$$\rho_n := \big|\mathcal{I}_\epsilon(u) - \mathcal{I}_\epsilon(U_n)\big| \to 0 \quad \text{and } \big\|u - U_n\|_{\mathrm{L}^1((0,T),X)} \to 0.$$

We note that $\mathcal{I}_\epsilon(U_n)$ is a Riemann-Stieltjes sum arising by approximation of the integral $\mathcal{I}_\epsilon(u)$ by step functions. Clearly we also have $r_n = d_\mathbb{Y}(u, U_n) \to 0$.

Now we fix $n \in \mathbb{N}$. According to part (ii) of (4.13) all $u(s_j^n)$, $j = 0, \ldots, N_n$, have a recovery sequence $(\widetilde{u}_k^{n,j})_{k\in\mathbb{N}}$ for $\mathcal{E}(s_j^n, \cdot)$, *i.e.*,

$$\widetilde{u}_k^{n,j} \overset{k\to\infty}{\rightharpoonup} u(s_j^n) \text{ in } X \quad \text{and} \quad \mathcal{E}_k(s_j^n, \widetilde{u}_k^{n,j}) \overset{k\to\infty}{\to} \mathcal{E}(s_j^n, u(s_j^n)).$$

Again we define the associated piecewise interpolants $\widetilde{u}_{n,k} \in \mathbb{Y}$ *via*

$$\begin{aligned} &\widetilde{u}_{n,k}(t) = \widetilde{u}_k^{n,j} \text{ for } t \in (s_{j-1}^n, s_j^n], \quad j = 1, \ldots, N_n, \\ &\widetilde{u}_{n,k}(t) = u_0^{n,j} \text{ for } t \in [0, s_0^n], \quad \widetilde{u}_{n,k}(t) = u_{N_n}^{n,j} \text{ for } t \in (s_{N_n}^n, T]. \end{aligned}$$

By construction and using (4.12) we conclude, for each fixed $n \in \mathbb{N}$,

$$\widetilde{\rho}_{n,k} = \big|\mathcal{I}_\epsilon^k(\widetilde{u}_{n,k}) - \mathcal{I}_\epsilon(U_n)\big| \to 0 \quad \text{ and } \quad \widetilde{r}_{n,k} = d_\mathbb{Y}(\widetilde{u}_{n,k}, U_n) \to 0 \quad \text{for } k \to \infty.$$

For the latter statement we use (A2) to conclude that all $\widetilde{u}_{n,k}$, $n, k \in \mathbb{N}$, lie in a bounded set of $\mathbb{Y}$, where $d_\mathbb{Y}$ provides the weak topology.

Now let $K_0 = 1$ and for $n \in \mathbb{N}$ choose $K_n > K_{n-1}$ such that $\widetilde{\rho}_{n,k}$, $\widetilde{r}_{n,k} \leq 1/n$ for all $k \geq K_n$. Then, define $\widetilde{n}(k)$ such that $\widetilde{n}(k) = m$ for $K_m \leq k < K_{m+1}$ and set $\widehat{u}_k = \widetilde{u}_{\widetilde{n}(k),k} \in \mathbb{Y}$. Then, $\widetilde{n}(k) \to \infty$ for $k \to \infty$ and

$$\begin{aligned} d_\mathbb{Y}(\widehat{u}_k, u) &\leq d_\mathbb{Y}(\widetilde{u}_{\widetilde{n}(k),k}, U_{\widetilde{n}(k)}) + d_\mathbb{Y}(U_{\widetilde{n}(k)}, u) \\ &\leq \widetilde{r}_{\widetilde{n}(k),k} + r_{\widetilde{n}(k)} \ \leq \ 1/\widetilde{n}(k) + r_{\widetilde{n}(k)} \to 0 \quad \text{for } k \to \infty. \end{aligned}$$

Moreover, for the functional $\mathcal{I}_\epsilon$ we obtain similarly

$$\big|\mathcal{I}_\epsilon(u) - \mathcal{I}_\epsilon^k(\widehat{u}_k)\big| \ \leq \ \widetilde{\rho}_{\widetilde{n}(k),k} + \rho_{\widetilde{n}(k)} \ \leq \ 1/\widetilde{n}(k) + r_{\widetilde{n}(k)} \to 0 \quad \text{for } k \to \infty.$$

This proves that $(\widehat{u}_k)_{k\in\mathbb{N}}$ is a recovery sequence. □

4.5. Joint limit $\epsilon \to 0$ and $k \to \infty$

The question now naturally arises as to whether the joint limit of $\epsilon \to 0$ and $k \to \infty$, corresponding to simultaneously enforcing strict causality and relaxing the weighted dissipation-energy functionals, is well behaved. Again we note that Γ-convergence returns trivial functionals that control $u(0)$ and $u(0+0) = \lim_{s\searrow 0} u(s)$ only. This difficulty notwithstanding, next we show that the minimizers $u_{\epsilon,k}$ of $\mathcal{I}_\epsilon^k$ are well behaved and each of their accumulation point satisfies a rate-independent problem, namely the *energetic formulation* associated with the limits Ψ and $\mathcal{E}$.

Theorem 4.4. *Let the assumptions of Section 4.1 hold uniformly in $k \in \mathbb{N}$ and let* (A1) *to* (A3) *(see* (4.12)*–*(4.14)*) be satisfied. Then, any family $(u_{\epsilon,k})_{\epsilon>0,k\in\mathbb{N}}$ of minimizers for the family $\mathcal{I}_\epsilon^k$ is weakly precompact in $\mathbb{Y}$. Moreover, any limit point $u \in \mathbb{Y}$ obtained for $(\epsilon, 1/k) \to (0,0)$ is a solution of the* energetic formulation *for the* *limit dissipation* Ψ *and the limit energy* $\mathcal{E}$*,* i.e., *for each* $t \in [0,T]$ *we have stability* **(S)** *and energy balance* **(E)** *as defined in* (4.11)*. Moreover, the convergent subsequence* $(u_{\epsilon_l,k_l})_{l\in\mathbb{N}}$ *can be chosen such that additionally the convergence*

$$\mathcal{E}_{k_l}(t,u_{\epsilon_l,k_l}(t)) \to \mathcal{E}(t,u(t)) \quad \textit{and} \quad \int_{[r,t]} \Psi_{k_l}(\mathrm{d}u_{\epsilon_l,k_l}) \to \int_{[r,t]} \Psi(\mathrm{d}u) \tag{4.18}$$

holds for all $0 \le r \le t \le T$*.*

Proof. (Compactness and limit points). The uniform bounds on Ψ_k and $\mathcal{E}_k$ show that the minimizers $u_{\epsilon,k}$ satisfy the uniform *a priori* bounds

$$\|u_{\epsilon,k}\|_{\mathrm{L}^\infty((0,T),X)} + \int_{[0,T]} \|\mathrm{d}u\|_Z \le C.$$

Now, Helly's selection principle is applicable and provides a subsequence which converges for each $t \in [0,T]$ weakly in X and strongly in Z. Together with the *a priori* bound in $\mathrm{L}^\infty((0,T),X)$ this shows that the sequence also converges in $\mathbb{Y}$.

ad (S): To establish stability, we consider a subsequence such that $u_{\epsilon_l,k_l}(t_*) \rightharpoonup u(t_*)$ in X. We have to show that $u(t_*)$ is stable, *i.e.*,

$$\text{(S)} \qquad \forall\, \widetilde{u} \in X: \; \mathcal{E}(t_*,u(t_*)) \le \mathcal{E}(t_*,\widetilde{u}) + \Psi(\widetilde{u}-u(t_*)). \tag{4.19}$$

By the definition of Γ-limit, there exists a recovery sequence $\widehat{u}_l$ such that $\widehat{u}_l \rightharpoonup \widetilde{u}$ and $\mathcal{E}_{k_l}(t_*,\widehat{u}_l) \to \mathcal{E}(t_*,\widetilde{u})$. Hence, we define the comparison functions

$$\widetilde{u}_{\epsilon_l,k_l}(t) = \begin{cases} u_{\epsilon_l,k_l}(t) & \text{for } t \le t_*, \\ \widehat{u}_l & \text{for } t > t_*. \end{cases}$$

Now assume $t_* < T$. Since u_{ϵ_l,k_l} minimizes $\mathcal{I}_{\epsilon_l}^{k_l}$ we have

$$\begin{aligned} 0 &\le \mathrm{e}^{t_*/\epsilon_l}\Big(\mathcal{I}_{\epsilon_l}^{k_l}(\widetilde{u}_{\epsilon_l,k_l}) - \mathcal{I}_{\epsilon_l}^{k_l}(u_{\epsilon_l,k_l})\Big) = \widetilde{g}_l - g_l \quad \text{where} \\ g_l &= \int_{[t_*,T]} \mathrm{e}^{(t_*-t)/\epsilon_l}\Psi_{k_l}(\mathrm{d}u_{\epsilon_l,k_l}) + \int_{t_*}^T \mathrm{e}^{(t_*-t)/\epsilon_l}\frac{1}{\epsilon_l}\mathcal{E}_{k_l}(t,u_{\epsilon_l,k_l}(t))\mathrm{d}t \quad \text{and} \\ \widetilde{g}_l &= \int_{[t_*,T]} \mathrm{e}^{(t_*-t)/\epsilon_l}\Psi_{k_l}(\mathrm{d}\widetilde{u}_{\epsilon_l,k_l}) + \int_{t_*}^T \mathrm{e}^{(t_*-t)/\epsilon_l}\frac{1}{\epsilon_l}\mathcal{E}_{k_l}(t,\widetilde{u}_{\epsilon_l,k_l}(t))\mathrm{d}t, \end{aligned}$$

where we have neglected the boundary terms at $t = T$, since they disappear in the limit $\epsilon_l \to 0$. The limit $l \to \infty$ of $\widetilde{g}_l$ is readily obtained, since $\widetilde{u}_{\epsilon_l,k_l}(t)$ is constant for $t > t_*$. We find

$$\lim_{l\to\infty} \widetilde{g}_l = \lim_{l\to\infty} \Psi_{k_l}(\widehat{u}_l - u_{\epsilon_l,k_l}(t_*)) + \lim_{l\to\infty} \mathcal{E}_{k_l}(t_*,\widehat{u}_l) = \Psi(\widetilde{u}-u(t_*)) + \mathcal{E}(t_*,\widetilde{u}), \tag{4.20}$$

where we used the weak continuous convergence (4.12) for Ψ_k. To calculate the limit of g_l we use the fact that $u_{\epsilon,k}$ minimizes $\mathcal{I}_\epsilon^k$ and hence satisfies the energy balance (4.4). Moreover, we employ the integration by parts formula for the BV function $t \mapsto \int_{t_*}^t \Psi_{k_l}(\mathrm{d}u_{\epsilon_l,k_l})$ and the smooth function $t \mapsto \mathrm{e}^{(t_*-t)/\epsilon_l}$ to obtain

$$\begin{aligned} g_l &= \int_{[t_*,T]} \mathrm{e}^{(t_*-t)/\epsilon_l}\Psi_{k_l}(\mathrm{d}u_{\epsilon_l,k_l}) - \int_{t_*}^T \mathrm{e}^{(t_*-t)/\epsilon_l}\int_{[t_*,t]}\Psi_{k_l}(\mathrm{d}u_{\epsilon_l,k_l})\mathrm{d}t \\ &\quad + \int_{t_*}^T \mathrm{e}^{(t_*-t)/\epsilon_l}\frac{1}{\epsilon_l}\Big(\mathcal{E}(t_*,u_{\epsilon_l,k_l}(t_*)) + \int_{t_*}^t \partial_s\mathcal{E}_{k_l}(s,u_{\epsilon_l,k_l}(s))\mathrm{d}s\Big)\mathrm{d}t \\ &= \Big[\mathrm{e}^{(t_*-t)/\epsilon_l}\int_{[t_*,t]}\Psi_{k_l}(\mathrm{d}u_{\epsilon_l,k_l})\Big]\Big|_{t=t_*}^{t=T} \\ &\quad + \mathcal{E}_{k_l}(t_*,u_{\epsilon_l,k_l}(t_*))\big(1-\mathrm{e}^{(t_*-T)/\epsilon_l}\big) + \int_{t_*}^T \mathrm{e}^{(t_*-t)/\epsilon_l}\frac{1}{\epsilon_l}O((t-t_*))\mathrm{d}t. \end{aligned}$$

Hence, we find $\liminf_{l\to\infty} g_l = \liminf_{l\to\infty}\mathcal{E}_{k_l}(t_*,u_{\epsilon_l,k_l}(t_*)) \geq \mathcal{E}(t_*,u(t_*))$. Moreover, $\widetilde{g}_l \geq g_l$ implies $\lim_{l\to\infty}\widetilde{g}_l \geq \lim_{l\to\infty} g_l$, and, together with (4.20), the desired stability (S) is established in the case $t_* < T$.

For the case $t_* = T$, we simply observe that $\mathcal{I}_{\epsilon_l}^{k_l}$ takes the form

$$\mathcal{I}_{\epsilon_l}^{k_l}(u) = \mathcal{J}_l(u) + \mathrm{e}^{-T/\epsilon}\big(\Psi_{k_l}(u(T)-u(T-0)) + \mathcal{E}_{k_l}(T,u(T))\big)$$

$$\text{where } \mathcal{J}_l(u) = \int_{[0,T)}\mathrm{e}^{-t/\epsilon}\Psi_{k_l}(\mathrm{d}u) + \int_0^T \mathrm{e}^{-t/\epsilon}\mathcal{E}_{k_l}(t,u(t))\mathrm{d}t.$$

As above we may now compare u_{k_l,ϵ_l} with $\widetilde{u}_{k_l,\epsilon_l}$, which is identical to u_{k_l,ϵ_l} on $[0,T)$ and equals $\widehat{u}_l$ at $t = T$, where $\widehat{u}_l$ is a recovery sequence for $\mathcal{E}(T,\widetilde{u})$. With $u_l^{-0} = u_{k_l,\epsilon_l}(T-0)$ and $u_l = u_{k_l,\epsilon_l}(T)$ we find

$$\begin{aligned} 0 &\leq \mathrm{e}^{T/\epsilon}\big(\mathcal{I}_{\epsilon_l}^{k_l}(\widetilde{u}) - \mathcal{I}_{\epsilon_l}^{k_l}(u)\big) \\ &= \big(\Psi_{k_l}(\widehat{u}_l - u_l^{-0}) + \mathcal{E}_{k_l}(T,\widehat{u}_l)\big) - \big(\Psi_{k_l}(u_l - u_l^{-0}) + \mathcal{E}_{k_l}(T,u_l)\big) \\ &\leq \mathcal{E}_{k_l}(T,\widehat{u}_l) - \mathcal{E}_{k_l}(T,u_l) + \Psi_{k_l}(\widehat{u}_l - u_l), \end{aligned}$$

where we used the triangle inequality for the last estimate. We conclude by passing to the limit $l \to \infty$ and find

$$\mathcal{E}(T,u(T)) \leq \liminf_{l\to\infty}\mathcal{E}_{k_l}(T,u_l) \leq \lim_{l\to\infty}\mathcal{E}_{k_l}(T,\widehat{u}_l) + \Psi_{k_l}(\widehat{u}_l - u_l) = \mathcal{E}(T,\widetilde{u}) + \Psi(\widetilde{u} - u(T)).$$

ad (E): For the upper energy estimate we first show that we also have convergence of the energies. By Γ-convergence we have $\mathcal{E}(t,u(t)) \leq \liminf_{l\to\infty}\mathcal{E}_{k_l}(t,u_{\epsilon_l,k_l}(t))$, since $u_{\epsilon_l,k_l}(t) \rightharpoonup u(t)$. Since each $u_{\epsilon_l,k_l}(t)$ is stable we also have

$$\mathcal{E}_{k_l}(t,u_{\epsilon_l,k_l}(t)) \leq \mathcal{E}_{k_l}(t,\widetilde{u}_l) + \Psi_{k_l}(\widetilde{u}_l - u_{\epsilon_l,k_l}(t))$$

for any $\widetilde{u}_l$. According to (4.13) there exists a recovery sequence with $\widetilde{u}_l \rightharpoonup u(t)$ and $\mathcal{E}_{k_l}(t,\widetilde{u}_l) \to \mathcal{E}(t,u(t))$. Invoking the weak continuous convergence of Ψ_k, the lower estimate $\liminf_{l\to\infty}\mathcal{E}_{k_l}(t,u_{\epsilon_l,k_l}(t)) \leq \mathcal{E}(t,u(t)) + 0$ is established and we conclude the convergence of energies as stated in (4.18). For each $l \in \mathbb{N}$ the energy balance for u_{ϵ_l,k_l} holds, namely for $0 \leq r < t \leq T$ we have

$$\mathcal{E}_{k_l}(t,u_{\epsilon_l,k_l}(t)) + \int_{[r,t]}\Psi_{k_l}(\mathrm{d}u_{\epsilon_l,k_l}) = \mathcal{E}_{k_l}(r,u_{\epsilon_l,k_l}(r)) + \int_r^t \partial_s\mathcal{E}_{k_l}(s,u_{\epsilon_l,k_l}(s))\mathrm{d}s. \tag{4.21}$$

Taking the limit $l \to \infty$ we use energy convergence of (4.18) and condition (4.14) and see that the first, the third and the fourth term in the above equation converge while the second is lower semicontinuous (see the proof of (4.17), which is based on (4.13)). Hence, the upper energy estimate holds:

$$\mathcal{E}(t,u(t)) + \int_{[r,t]}\Psi(\mathrm{d}u) \leq \mathcal{E}(r,u(r)) + \int_r^t \partial_s\mathcal{E}(s,u(s))\mathrm{d}s.$$

Finally using [29], Proposition 5.7, it is shown that stability of u implies the lower energy estimate $\mathcal{E}(t,u(t)) + \int_r^t \Psi(\mathrm{d}u) \geq \mathcal{E}(r,u(r)) + \int_r^t \partial_s\mathcal{E}(s,u(s))\mathrm{d}s$. Hence, the limit process u satisfies the energy balance. Moreover, the convergence of the dissipation in (4.18) follows by taking the limit in (4.21). ☐

This theorem has the drawback that exact minimizers $u_{\epsilon,k}$ are required. However, often it is desirable to work with quasi-minimizers, which conveniently allows for a certain latitude relative to full minimization. For instance, this additional latitude is especially useful when the functionals $\mathcal{E}_k$ or Ψ_k are not lower-semicontinuous and existence of minimizers cannot established. It would appear possible to generalize the preceding theorem to such situations by a judicious choice of the notion of quasi-minimizer. Firstly, we must be able to obtain an approximate version of the energy balance resulting in *a priori* bounds independent of ϵ and k. However, under these conditions it is no longer sufficient to bring $\mathcal{I}_\epsilon^k(u)$ close to $\inf_{v\in\mathbb{Y}}\mathcal{I}_\epsilon^k(v)$, but instead such closeness must be ensured on each subinterval $[t_*,T]$. We refer to [36] where a similar concept of *approximate solutions* for the time-incremental problem is developed.

5. Two examples of relaxation

We conclude with two illustrative examples for which the relaxation of $\mathcal{I}_\epsilon$ can be ascertained explicitly.

5.1. **A viscous example**

We begin by considering the simple problem on $X = \mathrm{L}^2(\Omega)$ with $\Omega = (0,1) \subset \mathbb{R}$ and

$$\mathcal{E}(t,y) = \int_\Omega F_{\mathrm{tq}}(y(x)) - \ell(t,x)y(x)\mathrm{d}x \quad \text{where } \ell \in \mathrm{C}^0([0,T]\times\overline{\Omega}).$$

Here F_{tq} is the tri-quadratic potential introduced in (3.1) and ℓ is a general loading. The dissipation functional will the defined by the L^2 norm *via* $\Psi(v) = \int_\Omega \frac{1}{2}v(x)^2\mathrm{d}x$. Hence, the weighted dissipation-energy functional to be investigated is

$$\mathcal{I}_\epsilon(y) = \int_{\Omega_T} \mathrm{e}^{-t/\epsilon}\Big(\frac{1}{2}\partial_t y^2 + \frac{1}{\epsilon}(F_{\mathrm{tq}}(y) - \ell(t,x)y)\Big)\mathrm{d}x\,\mathrm{d}t,$$

where $\Omega_T = [0,T]\times\Omega$. Note that $\mathbb{Y} = \mathrm{L}^2([0,T],X) = \mathrm{L}^2(\Omega_T)$. The Euler-Lagrange equations of $\mathcal{I}_\epsilon$ are

$$-\epsilon\partial_t^2 y + \partial_t y + F'_{\mathrm{tq}}(y) - \ell(t,x) = 0, \quad y(0,x) = y_0(x), \quad \partial_t y(T,x) = 0,$$

which is a singularly perturbed problem. The special problem with the functional $\mathcal{I}_\epsilon$ is that it is nonconvex in the variable y. It has some regularizing term through $|\partial_t y^2|$, but there is no term controlling the oscillations in x. We compare the relaxation of $\mathcal{I}_\epsilon$ with the naive convexification $\mathcal{I}_\epsilon^{\mathrm{cvx}} : \mathbb{Y} \to \mathbb{R}$ with

$$\mathcal{I}_\epsilon^{\mathrm{cvx}}(u) = \int_{\Omega_T} \mathrm{e}^{-t/\epsilon}\Big(\frac{1}{2}\partial_t u^2 + \frac{1}{\epsilon}(F_{\mathrm{cvx}}(u) - \ell u)\Big)\mathrm{d}x\,\mathrm{d}t,$$

where

$$F_{\mathrm{cvx}}(u) = \begin{cases} F_{\mathrm{tq}}(u) & \text{for } |u| \geq 1, \\ 0 & \text{for } |u| \leq 1. \end{cases}$$

Theorem 5.1. *Let $\mathcal{I}_\epsilon^{\mathrm{relax}}$ be the relaxation of $\mathcal{I}_\epsilon$ on $\mathbb{Y}$,* i.e.,

$$\mathcal{I}_\epsilon^{\mathrm{relax}}(u) := \inf\Big\{ \liminf_{k\to\infty} \mathcal{I}_\epsilon(u_k) \,\Big|\, u_k \rightharpoonup u \text{ in } \mathbb{Y} \Big\}.$$

Then, for each $\epsilon > 0$ we have $\mathcal{I}_\epsilon > \mathcal{I}_\epsilon^{\mathrm{relax}}$, i.e., *there exists u with $\mathcal{I}_\epsilon(u) > \mathcal{I}_\epsilon^{\mathrm{relax}}(u)$. Moreover, if $\ell \not\equiv 0$, then $\mathcal{I}_\epsilon^{\mathrm{relax}} > \mathcal{I}_\epsilon^{\mathrm{cvx}}$.*

Proof. For simplicity, we omit ϵ, which is fixed throughout the proof. We first show that $\mathcal{I}^{\rm relax}$ is different from $\mathcal{I}$. Consider any sequence u_n for which $\partial_t u_n \equiv 0$, $|u_n(x)| = 1$ *a.e.*, and $u_n \rightharpoonup 0$ in $\mathbb{Y}$. Then, $\mathcal{I}(y_n) \to 0$ but $\mathcal{I}(0) = 1-{\rm e}^{-T/\epsilon} > 0$. This shows that $\mathcal{I}$ is not lower semi-continuous and hence $\mathcal{I} > \mathcal{I}^{\rm relax}$. To prove that $\mathcal{I}^{\rm relax}$ is not identical to $\mathcal{I}^{\rm cvx}$, we minimize $\mathcal{I}$ and $\mathcal{I}^{\rm cvx}$ under the constraint $y(0) = w \in X$, namely

$$J(w) = \inf\{\mathcal{I}(u) \mid u \in \mathbb{Y}, u(0) = w\} \quad \text{and}$$
$$J^{\rm cvx}(w) = \inf\{\mathcal{I}^{\rm cvx}(u) \mid u \in \mathbb{Y}, u(0) = w\}.$$

Since in both functionals there is no coupling between different values of $x \in \Omega$, we obtain functions $j, j^{\rm cvx} : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$J(w) = \int_\Omega j(x, w(x)){\rm d}x \quad \text{and} \quad J^{\rm cvx}(w) = \int_\Omega j^{\rm cvx}(x, w(x)){\rm d}x \quad \text{with}$$
$$j(x, w) = \inf\left\{ \int_0^T {\rm e}^{-t/\epsilon}\big(\frac{1}{2}\dot z^2 + \frac{1}{\epsilon}[F_{\rm tq}(z) - \ell(t,x)z]{\rm d}t \;\Big|\; z \in {\rm H}^1(0,T), z(0) = w \right\}$$

and similarly for $j^{\rm cvx}(x, w)$ where $F_{\rm tq}$ is simply replaced by $F_{\rm cvx}$. Clearly the convexity of $F_{\rm cvx}$ implies convexity of $J^{\rm cvx}$ and of $j^{\rm cvx}$. Because of $F_{\rm tq}(w) > F_{\rm cvx}(w)$ for $w \in (-1, 1)$ we have $j(x, w) > j^{\rm cvx}(x, w)$ for all $x \in \Omega$ and all $w \in (-1, 1)$. We claim that the convexification $j^{**}(x, \cdot)$ of $j(x, \cdot)$ lies strictly above $j^{\rm cxv}(x, \cdot)$ unless $\ell \equiv 0$. To obtain a contradiction we assume $j^{**}(x, \cdot) \equiv j^{\rm cxv}(x, \cdot)$. Hence, $j(x, \cdot)$ lies strictly above its convexification $j^{**}(x, \cdot)$ and thus the convexification j^{**} must be affine for $w \in [-1, 1]$. From $j^{**} = j^{\rm cvx}$ we conclude that $j^{\rm cvx}$ must be affine on $[-1, 1]$ as well. Taking into account the Euler-Lagrange equation $-\epsilon\ddot z + \dot z + F'_{\rm cvx}(z) = \ell(t, x)$ with the boundary conditions $z(0) = w$ and $\dot z(T) = 0$, this can only be possible if $\ell \equiv 0$. Thus, we know that there exists $w \in X$, such that $J^{\rm cvx}(w) < J^{**}(w)$. It is now easy to construct a sequence $(w_n)_{n\in N}$ with $w_n \rightharpoonup w$ in X, such that $J(w_n) \to J^{**}(w)$. Define now $u \in \mathbb{Y}$ such that $u(0) = w$ and $\mathcal{I}^{\rm cvx}(u) = J^{\rm cvx}(w)$. Moreover, let $(u_n)_{n\in\mathbb{N}}$ be any sequence of functions such that $u_n \rightharpoonup u$ in $\mathbb{Y}$ and $\liminf_{n\to\infty} \mathcal{I}(u_n) = \mathcal{I}^{\rm relax}(u)$. Then, $\mathcal{I}(u_n) \le C$ implies a uniform bound on u_n in ${\rm H}^1([0, T], X)$. Hence, the mapping $u_n \mapsto u_n(0) \in X$ is bounded and, thus, weakly continuous. We conclude $u_n(0) \rightharpoonup u(0) = w$ in X and, moreover, $\liminf_{n\to\infty} \mathcal{I}(u_n) \ge \liminf_{n\to\infty} J(w_n) \ge J^{**}(w)$. This implies $\mathcal{I}^{\rm relax}(u) \ge J^{**}(w) > J^{\rm cvx}(w) = \mathcal{I}^{\rm cvx}(u)$, which is the desired result. □

It would be interesting to consider Young-measure relaxations instead of the simple relaxation in the weak topology just considered. In particular, it should be possible to derive a transport equation for Young measures in the spirit of [4,14,27,28,43].

5.2. **A rate-independent example**

The second example has a similar structure but is formulated in the rate-independent setting:

$$0 \in \kappa\,{\rm Sign}(\partial_t u(t, x)) + F'_{\rm tq}(u(t, x)) - \ell(t, x) \quad \text{for } (t, x) \in \Omega_T. \tag{5.1}$$

The energetic formulation based on the *stability condition* (S) and the *energy balance* (E) (*cf.* [29,34,35]) takes the form:

$$\text{(S)}\quad \forall \widehat y \in Y : \mathcal{E}(t, u(t)) \le \mathcal{E}(t, \widehat u) + \kappa\|\widehat u - u(t)\|_{{\rm L}^1(\Omega)},$$
$$\text{(E)}\quad \mathcal{E}(t, u(t)) + \int_{[0,t]} \kappa\|{\rm d}u\|_{{\rm L}^1(\Omega)} = \mathcal{E}(0, u(0)) - \int_{\Omega_t} \dot\ell(s, x)u(s, x){\rm d}x\,{\rm d}s,$$

which must hold for all $t \in [0, T]$. In general, this problem does not have a strong solution, and a question of central interest concerns the precise manner in which the potential barrier at $u \in [-1/2, 1/2]$ is overcome. We consider the weighted dissipation-energy functional

$$\mathcal{I}_\epsilon(u) = \int_{[0,T]} {\rm e}^{-t/\epsilon}\kappa\|{\rm d}u\|_{{\rm L}^1(\Omega)} + \int_{\Omega_T} \frac{{\rm e}^{-t/\epsilon}}{\epsilon}\Big(F_{\rm tq}(u(t, x)) - \ell(t, x)u(t, x)\Big){\rm d}x\,{\rm d}t,$$

with a fixed $\ell \in \mathrm{C}^1([0,T]\times\Omega)$. Owing to the non-smooth character of the problem, the corresponding Euler-Lagrange equations take the form of a differential inclusion. Next we show that the relaxation of the functional follows readily in the space

$$\mathcal{Y} = \mathrm{L}^2(\Omega_T) \cap \mathrm{BV}([0,T], \mathrm{L}^1(\Omega)),$$

where, as before, $\Omega_T = [0,T]\times\Omega$.

Theorem 5.2. *The relaxation of $\mathcal{I}_\epsilon : \mathcal{Y} \to \mathbb{R}$ equipped with the weak topology of $\mathrm{L}^2(\Omega_T)$ is given by the convexification*

$$\mathcal{I}_\epsilon^{\mathrm{cvx}}(u) = \int_{[0,T]} \mathrm{e}^{-t/\epsilon}\kappa\|\mathrm{d}u\|_{\mathrm{L}^1(\Omega)} + \int_{\Omega_T} \frac{\mathrm{e}^{-t/\epsilon}}{\epsilon}\Big(F_{\mathrm{cvx}}(y(t,x)) - \ell(t,x)y(t,x)\Big)\mathrm{d}x\,\mathrm{d}t.$$

Proof. Again we omit the index ϵ and for simplicity we assume that $\Omega = (a,b) \subset \mathbb{R}$. We trivially have $\mathcal{I}^{\mathrm{relax}} \geq \mathcal{I}^{\mathrm{cvx}}$ since $\mathcal{I}^{\mathrm{cvx}}$ is convex and lower semi-continuous. To show the reverse estimate, we have to construct for each $y \in \mathcal{Y}$ a recovery sequence $(u_n)_n$ with

$$u_n \rightharpoonup u \text{ in } \mathrm{L}^2(\Omega_T) \quad \text{and} \quad \liminf_{n\to\infty} \mathcal{I}(u_n) \leq \mathcal{I}^{\mathrm{cvx}}(u).$$

By considering small representative volume elements $x_0+\delta[0,1) \subset \Omega$, introducing the variable $\xi = (x-x_0)/\delta \in [0,1)$ and taking the limit $\delta \to 0$, we find that it suffices to consider the case that the limit function $u \in \mathcal{Y}$ is independent of x. Thus, we assume that $u(t,x) = z(t)$ with $z \in \mathrm{BV}([0,T];\mathbb{R})$. Moreover, we need to find one function $\widetilde{u} \in \mathrm{L}^2([0,T]\times[0,1))$ with

$$\forall\, t \in [0,T]: \ \int_0^1 \widetilde{u}(t,\xi)\mathrm{d}\xi = z(t) \quad \text{and}$$

$$\mathcal{I}_{x_0}(\widetilde{u}) \leq \int_{[0,T]} \mathrm{e}^{-t/\epsilon}\kappa|\mathrm{d}z| + \int_0^T \tfrac{\mathrm{e}^{-t/\epsilon}}{\epsilon}\big(F_{\mathrm{cvx}}(z(t)) - \ell(t,x_0)z(t)\big)\mathrm{d}t,$$

where

$$\mathcal{I}_{x_0}(\widetilde{u}) = \int_{[0,T]} \mathrm{e}^{-t/\epsilon}\kappa\|\mathrm{d}\widetilde{u}\|_{\mathrm{L}^1([0,1))} + \int_0^T\int_0^1 \tfrac{\mathrm{e}^{-t/\epsilon}}{\epsilon}\big(F_{\mathrm{tq}}(\widetilde{u}(t,\xi)) - \ell(t,x_0)\widetilde{u}(t,\xi)\big)\mathrm{d}\xi\,\mathrm{d}t.$$

It is easy to verify that the following function $\widetilde{u}$ satisfies both conditions:

$$\widetilde{u}(t,\xi) = \begin{cases} z(t) & \text{for } (t,\xi) \text{ with } z(t) \leq -1, \\ -1 & \text{for } (t,\xi) \text{ with } 0 \leq \xi \leq (1-z(t))/2 \leq 1, \\ +1 & \text{for } (t,\xi) \text{ with } 0 \leq (1-z(t))/2 < \xi \leq 1, \\ z(t) & \text{for } (t,\xi) \text{ with } z(t) \geq 1. \end{cases}$$

Note that $\widetilde{u}$ does not take values in $(-1,1)$ where F_{tq} is larger than F_{cvx}. Weak convergence is obtained by rescaling the above construction into the interval $[x_0, x_0+\delta)$, and the result follows. □

Thus, we may use the regularized functional to obtain a regularization of the rate-independent evolutionary problem (5.1). It is obtained simply by replacing F by F_{cvx}:

$$0 \in \kappa\,\mathrm{Sign}(\partial_t y(t,x)) + F'_{\mathrm{cvx}}(y(t,x)) - \ell(t,x) \quad \text{for } (t,x) \in \Omega_T. \tag{5.2}$$

Note that this problem is exactly the same that is obtained by solving the global minimization problem (S) & (E), which always has solutions. To see this, just solve the problem for each value of x separately. Each of the solutions obtained in this way also solves the relaxed problem (5.2). This relaxed problem admits solutions that may be mechanically unimportant, but they are nevertheless needed mathematically to make the set of solutions weakly closed.

Acknowledgements. This work was partly carried out during MO's stay at Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart, Germany, under the auspices of the Humboldt Foundation. MO gratefully acknowledges the financial support provided by the Foundation and the hospitality extended by the Institute. AM acknowledges partial support from the Deutsche Forschungsgemeinschaft (DFG) *via* the Research Unit FOR 797 "MicroPlast" under Mie 459/5-1 and *via* MATHEON under Project C18. The authors are grateful to an unknown referee for correcting two errors.

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