ESAIM: COCV 14 (2008) 561–574
DOI: 10.1051/cocv:2007066

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

A CARLEMAN ESTIMATES BASED APPROACH FOR THE STABILIZATION OF SOME LOCALLY DAMPED SEMILINEAR HYPERBOLIC EQUATIONS

Louis Tebou[1]

Abstract. First, we consider a semilinear hyperbolic equation with a locally distributed damping in a bounded domain. The damping is located on a neighborhood of a suitable portion of the boundary. Using a Carleman estimate [Duyckaerts, Zhang and Zuazua, *Ann. Inst. H. Poincaré Anal. Non Linéaire* (to appear); Fu, Yong and Zhang, *SIAM J. Contr. Opt.* **46** (2007) 1578–1614], we prove that the energy of this system decays exponentially to zero as the time variable goes to infinity. Second, relying on another Carleman estimate [Ruiz, *J. Math. Pures Appl.* **71** (1992) 455–467], we address the same type of problem in an exterior domain for a locally damped semilinear wave equation. For both problems, our method of proof is constructive, and much simpler than those found in the literature. In particular, we improve in some way on earlier results by Dafermos, Haraux, Nakao, Slemrod and Zuazua.

Mathematics Subject Classification. 93D15, 35L05, 35L70.

Received August 18, 2006.
Published online December 21, 2007.

1. Problem formulation and statements of main results

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with boundary of class C^2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function with

$$\begin{aligned} & f(0) = 0, \ sf(s) \geq 0, \quad \forall s \in \mathbb{R}, \\ & \exists C_0 > 0 : |f'(s)| \leq C_0 \left(1 + |s|^q\right), \quad \forall s \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where $q \geq 0$, $(N-2)q \leq 2$. Let $a \in L^\infty(\Omega)$ be a nonnegative function satisfying

$$\exists a_0 > 0 : a(x) \geq a_0, \quad \forall x \in \omega, \tag{1.2}$$

the subset ω being a neighbourhood of Γ_0, that is to say, the intersection of Ω and a neighborhood of Γ_0, where Γ_0 is a suitable portion of the boundary that will be defined later. Throughout the paper ∂_i stands for $\partial/\partial x_i$,

Keywords and phrases. Hyperbolic equation, exponential decay, localized damping, Carleman estimates.

[1] Department of Mathematics, Florida International University, Miami FL 33199, USA; teboul@fiu.edu

and we use the Einstein summation convention on repeated indices. Consider the damped hyperbolic equation

$$\begin{cases} y_{tt} - \partial_i(b_{ij}(x)\partial_j y) + p(x)y + f(y) + ag(y_t, \nabla y) = 0 \text{ in } \Omega \times (0,\infty) \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,\infty) \\ y(0) = y^0 \text{ in } \Omega \\ y_t(0) = y^1 \text{ in } \Omega, \end{cases} \tag{1.3}$$

where $p \in L^m_+(\Omega)$, ($m = 2$ for $N = 1$, $m > 2$ for $N = 2$, and $m \geq N$ for $N \geq 3$), and $g : \mathbb{R}^{N+1} \to \mathbb{R}$ is a globally Lipschitz function satisfying

$$\begin{aligned} & g(0,q) = 0, \quad \forall q \in \mathbb{R}^N, \\ & \quad \exists L > 0 : |g(r,w) - g(r',w')| \leq L\left(|r - r'| + |w - w'|\right), \quad \forall w,\ w' \in \mathbb{R}^N,\ \forall r,\ r' \in \mathbb{R}, \\ & \quad \exists b > 0 : g(r,w)r \geq br^2, \quad \forall w \in \mathbb{R}^N,\ \forall r \in \mathbb{R}. \end{aligned} \tag{1.4}$$

It follows from (1.4.1), and (1.4.2) that

$$|g(r,w)| \leq L|r|, \quad \forall w \in \mathbb{R}^N,\ \forall r \in \mathbb{R}. \tag{1.5}$$

As for the coefficients $(b_{ij})_{i,j}$, they satisfy:

$$b_{ij} \in C^1(\bar{\Omega}); \quad b_{ij} = b_{ji}, \quad \forall i,\ j = 1, 2, ..., N, \tag{1.6}$$

and

$$\exists b_0 > 0 : b_{ij}(x) z_i z_j \geq b_0 z_i z_i, \quad \forall (x,z) \in \bar{\Omega} \times \mathbb{R}^N. \tag{1.7}$$

Now let $\{y^0, y^1\} \in H^1_0(\Omega) \times L^2(\Omega)$. System (1.3) is then well-posed in the space $H^1_0(\Omega) \times L^2(\Omega)$; this result is well-known in the case where either g is independent of q [3,13,20] or $f = 0$ [35,43]. To our knowledge, the well-posedness of the general system (1.3) is yet to be established. Therefore our main purpose in this paper is twofold:

(i) to prove that under the above hypotheses on the data, system (1.3) has a unique weak solution

$$y \in C([0,\infty); H^1_0(\Omega)) \cap C^1([0,\infty); L^2(\Omega)); \tag{1.8}$$

(ii) to prove that for every weak solution, the energy given by

$$E(t) = \frac{1}{2}\int_\Omega \{|y_t(x,t)|^2 + b_{ij}(x)\partial_j y(x,t)\partial_i y(x,t) + p(x)|y(x,t)|^2\}\,\mathrm{d}x + \int_\Omega F(y(x,t))\,\mathrm{d}x, \tag{1.9}$$

where $F(s) = \int_0^s f(r)\mathrm{d}r$, decays exponentially to zero as the time $t \to \infty$. The energy E is a nonincreasing function of the time variable t, as we have the dissipation law:

$$E(t) + \int_\Omega a g(y_t, \nabla y) y_t\,\mathrm{d}x = E(s), \quad \forall 0 \leq s < t < \infty. \tag{1.10}$$

Our well-posedness result states as follows:

Theorem 1.1 (well-posedness)**.** *Let* $\{y^0, y^1\} \in H^1_0(\Omega) \times L^2(\Omega)$*, and assume that the functions* b_{ij}*,* a*,* f*,* p*, and* g *satisfy the hypotheses given above. Then system* (1.3) *has a unique weak solution satisfying* (1.8).

Before stating the stabilization result, some additional notations are needed. Following [10,11], we introduce a function $d \in C^2(\bar{\Omega})$ satisfying for some $m_0 \geq 4$:

$$
\begin{aligned}
&\text{(i)}\ \ (2b_{il}(b_{kj}d_{x_k})_{x_l} - b_{ij,x_l}b_{kl}d_{x_k})\, z_i z_j \geq m_0 b_{ij} z_i z_j, \quad \forall (x,z) \in \bar{\Omega} \times \mathbb{R}^N.\\
&\text{(ii)}\ \ \min\left\{|\nabla d(x)|; x \in \bar{\Omega}\right\} > 0.\\
&\text{(iii)}\ \ \frac{1}{4} b_{ij}(x) d_{x_i}(x) d_{x_j}(x) \geq R_1^2 \geq R_0^2 > 0, \quad \forall x \in \bar{\Omega},
\end{aligned}
\tag{1.11}
$$

where $R_0 = \min\left\{\sqrt{d(x)}; x \in \bar{\Omega}\right\}$, and $R_1 = \max\left\{\sqrt{d(x)}; x \in \bar{\Omega}\right\}$. We now define Γ_0; let ν be the unit normal pointing into the exterior of Ω, and set

$$
\Gamma_0 = \left\{x \in \partial\Omega; b_{ij}\nu_i d_{x_j}(x) > 0\right\}. \tag{1.12}
$$

It is easy to check that if $b_{ij} = \gamma\delta_{ij}$, ($\delta$ denotes the Kronecker symbol), setting $d(x) = |x - x_0|^2$ for any $x_0 \in \mathbb{R}^N \setminus \bar{\Omega}$, then eventually modifying d as in [11], we see that (1.11) is satisfied; in this case,

$$
\Gamma_0 = \{x \in \partial\Omega; (x - x_0)\cdot\nu > 0\}, \tag{1.13}
$$

which is the usual portion of the boundary that arises in the framework of the multiplier method [17,21,34].

We also note that the constraints on the coefficients b_{ij} are almost necessary in order to establish the Carleman estimates needed in the development of our proof method; without these constraints, establishing those estimates would in most cases be impossible as shown in [24]. Indeed in [24], the authors, using a Gaussian beam approach, show that observability estimates, which are weaker than Carleman estimates, may fail in the absence of such constraints.

Our stabilization result reads:

Theorem 1.2 (stabilization)**.** *Let $\{y^0, y^1\} \in H_0^1(\Omega) \times L^2(\Omega)$. Let ω be a neighborhood of Γ_0. Assume that the functions (b_{ij}), d, a, f, p, and g satisfy the hypotheses given above. Then there exist positive constants M and α, possibly depending on $E(0)$ such that the energy E of each solution of* (1.3) *satisfies:*

$$
E(t) \leq M[\exp(-\alpha t)]E(0), \quad \forall t \geq 0. \tag{1.14}
$$

Remark 1.1. The possible dependence on $E(0)$ of the constants M and α will be given explicitly as part of the proof of Theorem 1.2.

Remark 1.2. It will follow from our proof that the constants M and α do not depend on the initial data when f is globally Lipschitz. Thus, in this case, it will be proven that the exponential decay of the energy is uniform in the energy space. However, in general, our method leads to a decay estimate that is uniform only on every ball in the energy space.

Literature. The stability of the wave equation with locally distributed damping has a history of about three decades that begins with a result of Dafermos [6]. In [6], the author considers the wave equation

$$
\begin{cases}
y_{tt} - \Delta y + a g(y_t) = 0 \text{ in } \Omega \times (0,\infty)\\
y = 0 \text{ on } \Sigma = \partial\Omega \times (0,\infty)\\
y(0) = y^0; \quad y_t(0) = y^1 \text{ in } \Omega,
\end{cases}
\tag{1.15}
$$

where $a \in L^\infty(\Omega)$, $a \geq 0$ almost everywhere in Ω. Assuming that $meas(suppa) > 0$, and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is continuously differentiable and strictly increasing, Dafermos shows that for any weak solution of (1.11), one has

$$
(y, y_t) \to (0,0) \quad \text{strongly in } H_0^1(\Omega) \times L^2(\Omega), \text{ as } t \to \infty. \tag{1.16}
$$

Then Haraux [12] generalized Dafermos' result to include functions g that have a monotone graph but are neither strictly increasing nor smooth. Later on Slemrod [35] got rid of the monotonicity hypothesis while allowing g to have its graph in the first and third quadrants, weakened the smoothness assumption by taking only globally Lipschitz functions g, and he proved that

$$(y, y_t) \to (0, 0) \quad \text{weakly in } H_0^1(\Omega) \times L^2(\Omega), \text{ as } t \to \infty. \tag{1.17}$$

In the same paper [35], the author also studied for the first time the stability of a system like (1.3) with $b_{ij} = \delta_{ij}$, $p \equiv 0$, and $f \equiv 0$; he proved (1.17) for this new system by assuming that $g(r, q)$ is globally Lipschitz, and satisfies "$g(r, q)r \geq 0,\ \forall r,\ q$". Slemrod's results were later generalized to allow for more general nonlinearities g, and other distributed systems (*e.g.* Petrowski, coupled systems, hybrid systems) by Vancostenoble [42,43]. It is also worth mentioning Haraux's paper [15] where the author proposes a simplified approach to the weak stability of (1.3) and (1.15). In all of the aforementioned works, the authors are interested in finding a class of feedback controls as large as possible that would yield a strong or weak stability of the system considered. Under the mild conditions on the nonlinearity g in those earlier works [6,12,15,35,42,43], no decay rates are known. Another approach to the stability problem for locally damped semilinear wave equations is to prescribe sufficient conditions on g and the damping location, that allow to obtain decay rates; in this direction Zuazua [45,46] was a pioneer. In [45], the author considers system (1.3) with $b_{ij} = \delta_{ij}$ and "$g(s, q) = g(s) = s$", and provides two different proofs of the exponential decay of its energy:

- one for globally Lipschitz nonlinearities f satisfying either $\lim_{s\to-\infty} f'(s)$ and $\lim_{s\to\infty} f'(s)$ both exist, or $\lim_{|s|\to\infty} \frac{f(s)}{s}$ exists; it is easy to check that this condition excludes functions such as $f(s) = \beta s \sin^2\left(\ln(1+s^2)\right)$, $(\beta > 0)$, that are globally Lipschitz but for which none of the aforementioned limits exist;
- one for superlinear functions f that further satisfy $sf(s) \geq (2+\delta)F(s)$ for some $\delta > 0$.

All the proofs in [45,46] are based on the unique continuation property of Ruiz [33], and they lead to uniform exponential decay estimates of the energy. Later on, Dehman [7] reduced the two proofs in [45] to a single proof, but for initial data that are bounded in the energy space. Subsequently, using Strichartz dispersive inequalities, the results of [7,45,46] were improved to include all the subcritical nonlinearities f in the three dimensional setting, meaning that $q < 4$ in (1.1), by Dehman, Lebeau and Zuazua in [8]. It is also of interest to mention Nakao's papers [29,30], where the authors discuss the same type of questions for systems involving nonlinearities of the form $f(x, s)$ – that are bounded in x – and nonlinear damping locally distributed on a neighborhood of a suitable subset of the boundary; they establish polynomial and exponential energy decay estimates for small enough initial data. All the proofs in [7,8,29,30,45,46] are based on a compactness-uniqueness argument which relies on the unique continuation property of either [33] or [36], and consequently do not lead to explicit decay rates. Our constructive approach, to be developed below, enables us to provide, in a single proof, uniform exponential decay estimates of the energy for globally Lipschitz nonlinearities f and local stabilization for all other functions f satisfying (1.1), and it leads to explicit decay rates.

Concerning "linear" problems ($f(s) = ms,\ m \geq 0,\ g(r, q) = g(r)$) with linear or nonlinear dissipations, we refer the reader to *e.g.* [1,5,14,18,23,25–27,37–40].

The rest of the paper is organized as follows: Section 2 is devoted to the proofs of Theorem 1.1, and Theorem 1.2, while in Section 3, we discuss a similar problem in an exterior domain. Section 4 wraps up the paper with a few open problems.

2. Proofs of Theorems 1.1 and 1.2

2.1. **Proof of Theorem 1.1**

We may rewrite the first equation of (1.3) in the form

$$\begin{cases} y_t - z = 0 \text{ in } \Omega \times (0,\infty), \\ z_t - \partial_i(b_{ij}(x)\partial_j y) + py = -f(y) - ag(z,\nabla y) \text{ in } \Omega \times (0,\infty). \end{cases} \tag{2.1}$$

Setting $Z = \begin{pmatrix} y \\ z \end{pmatrix}$, (2.1) becomes $Z' + \mathcal{A}Z = \mathcal{G}(Z)$, so that (1.3) is equivalent to

$$\begin{cases} Z' + \mathcal{A}Z = \mathcal{G}(Z) \text{ in } (0,\infty), \\ Z(0) = \begin{pmatrix} y^0 \\ y^1 \end{pmatrix}, \end{cases} \tag{2.2}$$

where the unbounded operator $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & -I \\ -\partial_i(b_{ij}(x)\partial_j) + pI & 0 \end{pmatrix} \tag{2.3}$$

with $D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. Set $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$. The nonlinear operator $\mathcal{G} : \mathcal{H} \longrightarrow \mathcal{H}$, is given by $\mathcal{G}(Z) = \begin{pmatrix} 0 \\ -f(y) - ag(z,\nabla y) \end{pmatrix}$.

We now equip the Hilbert space $\mathcal{H}$ with the norm

$$||Z||_{\mathcal{H}}^2 = \int_\Omega \left\{ b_{ij}(x)\partial_j y \partial_i y + p|y|^2 \right\} \mathrm{d}x + \int_\Omega |z|^2 \mathrm{d}x. \tag{2.4}$$

Let us show that the operator $\mathcal{A}$ is maximal monotone. This amounts to proving that:

(i) $(\mathcal{A}Z, Z) \geq 0, \quad \forall Z = \begin{pmatrix} y \\ z \end{pmatrix} \in D(\mathcal{A})$,

(ii) $\mathcal{A} + \mathcal{I}$ is surjective, ($\mathcal{I}$ is the identity operator)

where in (i), $(.,.)$ denotes the scalar product induced by the norm defined in (2.4).

Proof of (i). Since for all $Z \in D(\mathcal{A})$, we have

$$\mathcal{A}Z = \begin{pmatrix} -z \\ -\partial_i(b_{ij}(x)\partial_j y) + py \end{pmatrix},$$

it follows that

$$(\mathcal{A}Z, Z) = -\int_\Omega b_{ij}(x)\partial_j y \partial_i z \,\mathrm{d}x - \int_\Omega pzy \,\mathrm{d}x + \int_\Omega b_{ij}(x)\partial_j z \partial_i y \,\mathrm{d}x + \int_\Omega pzy \,\mathrm{d}x = 0, \text{ as} \tag{2.5}$$

which establishes (i). □

Proof of (ii). We shall prove that for all $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathcal{H}$, there exists Z in $D(\mathcal{A})$ such that

$$\mathcal{A}Z + Z = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{2.6}$$

We may rewrite equation (2.6) as

$$\begin{cases} -z + y = u \text{ in } \Omega, \\ -\partial_i(b_{ij}(x)\partial_j y) + py = v \text{ in } \Omega. \end{cases} \tag{2.7}$$

Since v belongs to $L^2(\Omega)$, the application of the theory of elliptic problems [4,22] gives the existence and uniqueness of y in $H^2(\Omega) \cap H_0^1(\Omega)$. The existence of z follows immediately, and (ii) is proven.

On the other hand, one easily checks that the nonlinear operator $\mathcal{G}$ is locally Lipschitz on $\mathcal{H}$, and we have the dissipation law (1.10). The application of [31], Theorem 1.4, p. 185, shows that (3.2) has a unique weak solution

$$Z \in \mathcal{C}([0,\infty);\mathcal{H}), \tag{2.8}$$

and Theorem 1.1 is proven. □

2.2. **Proof of Theorem 1.2**

For the sequel we need the following notations: let d be given as above, and let $T > 2R_1$, where $R_1 = \max\left\{\sqrt{d(x)}; x \in \bar{\Omega}\right\}$. Choose a constant $\mu \in (0,1)$ such that

$$(2R_1/T)^2 < \mu < 2R_1/T. \tag{2.9}$$

Set $\varphi(x,t) = d(x) - \mu(t - T/2)^2$. Define a differential operator $\mathcal{P}$ by $\mathcal{P}u = u_{tt} - (b_{ij}(x)u_{x_j})_{x_i}$. Also set $Q = \Omega \times (0,T)$. The proof of Theorem 1.2 is based on the following Carleman estimate due to Duyckaerts, Zhang and Zuazua [10], Theorem 2.4 (see also [11], Th. 7.1, for the special case $V \equiv 0$):

Lemma 2.1. *Let b_{ij} satisfy* (1.6)–(1.7), *and $V \in L^\infty(0,T;L^m(\Omega))$ with $m \in [N,\infty]$. Assume that* (1.11) *holds, and that there exists some $\delta > 0$ such that $\omega = \mathcal{O}_\delta(\Gamma_0) \cap \Omega$, where $\mathcal{O}_\delta(\Gamma_0) = \left\{x \in \mathbb{R}^N; |x - x'| < \delta, \text{ for some } x' \in \Gamma_0\right\}$. Then there exists $\lambda_0 > 1$ and a positive constant $C = C(\Omega,T)$, such that for all $\lambda \geq \lambda_0$ and any $u \in C([0,T];L^2(\Omega))$ satisfying $u(x,0) = u(x,T) = 0$ for $x \in \Omega$, $\mathcal{P}u \in H^{-1}(Q)$, and*

$$(u, \mathcal{P}\eta) = \langle \mathcal{P}u, \eta\rangle_{H^{-1}(Q),H_0^1(Q)}, \quad \forall \eta \in H_0^1(Q) \text{ with } \mathcal{P}\eta \in L^2(Q), \tag{2.10}$$

it holds

$$\lambda||e^{\lambda\varphi}u||^2_{L^2(Q)} \leq C\left(||e^{\lambda\varphi}(\mathcal{P}u - Vu)||^2_{H^{-1}(Q)} + \frac{1}{\lambda^{(2-2N/m)}}||e^{\lambda\varphi}Vu||^2_{L^2(0,T;H^{-N/m}(\Omega))} + \lambda^2||e^{\lambda\varphi}u||^2_{L^2(0,T;L^2(\omega))}\right). \tag{2.11}$$

In order to prove Theorem 1.2, it suffices to show that there exists $C_0 > 0$ such that

$$E(T) \leq C_0 \int_0^T \int_\Omega a g(y_t, \nabla y) y_t \,\mathrm{d}x\,\mathrm{d}t. \tag{2.12}$$

The possible dependence of C_0 on the initial data will be given as part of the proof of (2.12). After proving (2.12), we will use the semigroup property to derive the claimed exponential decay estimate. We now proceed to prove (2.12).

For the sequel, we need some additional notations. Set

$$\begin{aligned} &T_i = (T/2) - \varepsilon_i T, \quad T_i' = (T/2) + \varepsilon_i T, \quad i = 0,1 \\ &R_0 = \min\left\{\sqrt{d(x)}; x \in \bar{\Omega}\right\}, \quad \tilde{Q} = \Omega \times [(0,T_1) \cup (T_1',T)], \quad Q_0 = \Omega \times (T_0, T_0') \end{aligned} \tag{2.13}$$

where $0 < \varepsilon_0 < \varepsilon_1 < 1/2$ will be specified later on. Now, we proceed as in [10,11]. Thanks to (2.9), and the definition of φ, we have

$$\varphi(x,0) = \varphi(x,T) = d(x) - \mu(T^2/4) \leq R_1^2 - \mu(T^2/4) < 0, \quad \forall x \in \bar{\Omega}. \tag{2.14}$$

Therefore there exists $\varepsilon_1 \in (0, 1/2)$, close to $1/2$ such that

$$\varphi(x,t) \leq (R_1^2 - \mu(T^2/4))/2 < 0, \quad \forall (x,t) \in \tilde{Q}. \tag{2.15}$$

Similarly, we have

$$\varphi(x, T/2) = d(x) \geq R_0^2 > 0, \quad \forall x \in \bar{\Omega}. \tag{2.16}$$

Consequently, there exists some $\varepsilon_0 \in (0, 1/2)$, close to zero, such that

$$\varphi(x,t) \geq R_0^2/2, \quad \forall (x,t) \in \bar{Q}_0. \tag{2.17}$$

We now have all the ingredients necessary for a clear proof of (2.12). From now on, C denotes various positive constants independent of the initial data, and λ.

Let $r \in C^2([0,T])$ be a nonnegative function such that

$$\begin{aligned} r(0) = r(T) = 0,& \\ r \equiv 1 \text{ in } [T_1, T_1'].& \end{aligned} \tag{2.18}$$

If we set $u(x,t) = r(t) y_t(x,t)$ for $(x,t) \in \Omega \times (0,T)$, then u satisfies all the requirements of Lemma 2.1 with $V = 0$, and

$$\mathcal{P}u = r'' y_t + 2r' y_{tt} - f'(y)u - p(x)u - ra(g(y_t, \nabla y))_t. \tag{2.19}$$

Accordingly,

$$\lambda ||\mathrm{e}^{\lambda\varphi} u||_{L^2(Q)}^2 \leq C\Big(||\mathrm{e}^{\lambda\varphi} \mathcal{P}u||_{H^{-1}(Q)}^2 + \lambda^2 ||\mathrm{e}^{\lambda\varphi} u||_{L^2(0,T;L^2(\omega))}^2 \Big). \tag{2.20}$$

Now using Hahn-Banach theorem, Hölder inequality, and possibly Sobolev embedding theorem, we find

$$\begin{aligned}
&||\mathrm{e}^{\lambda\varphi} r'' y_t||_{H^{-1}(Q)} \leq C ||\mathrm{e}^{\lambda\varphi} y_t||_{L^2(\tilde{Q})}, \\
&||2\mathrm{e}^{\lambda\varphi} r' y_{tt}||_{H^{-1}(Q)} \leq C(1+\lambda) ||\mathrm{e}^{\lambda\varphi} y_t||_{L^2(\tilde{Q})}, \\
&||\mathrm{e}^{\lambda\varphi} pu||_{H^{-1}(Q)} \leq C ||\mathrm{e}^{\lambda\varphi} u||_{L^2(Q)}, \\
&||\mathrm{e}^{\lambda\varphi} f'(y)u||_{H^{-1}(Q)} \leq C ||\mathrm{e}^{\lambda\varphi} u||_{L^2(Q)} \left(1 + E(0)^q\right), \\
&||\mathrm{e}^{\lambda\varphi} ra(g(y_t, \nabla y))_t||_{H^{-1}(Q)} \leq C(1+\lambda) ||\mathrm{e}^{\lambda\varphi} a(g(y_t, \nabla y))||_{L^2(Q)}.
\end{aligned} \tag{2.21}$$

It should be noted that in order to obtain the second and last inequality in (2.21), one has to perform an integration by parts after the application of the Hahn-Banach theorem before proceeding to use Hölder inequality, and Sobolev embedding theorem.

Combining (2.21) and (2.19), and reporting the result in (2.20), we get

$$\begin{aligned}
\lambda ||\mathrm{e}^{\lambda\varphi} u||_{L^2(Q)}^2 \leq{}& C\lambda^2 ||\mathrm{e}^{\lambda\varphi} y_t||_{L^2(\tilde{Q})}^2 + C(1 + E(0)^q) ||\mathrm{e}^{\lambda\varphi} u||_{L^2(Q)}^2 \\
&+ C\lambda^2 ||\mathrm{e}^{\lambda\varphi} a(g(y_t, \nabla y))||_{L^2(Q)}^2 + C\lambda^2 ||\mathrm{e}^{\lambda\varphi} u||_{L^2(0,T;L^2(\omega))}^2.
\end{aligned} \tag{2.22}$$

Choosing $\lambda \geq \lambda_1 = 2C(1 + E(0)^q)$, where C is the constant in (2.22), we derive from (2.22) that

$$\lambda ||\mathrm{e}^{\lambda\varphi} u||_{L^2(Q)}^2 \leq C(1+\lambda^2) ||\mathrm{e}^{\lambda\varphi} y_t||_{L^2(\tilde{Q})}^2 + C(1+\lambda^2) \int_0^T \int_\Omega \mathrm{e}^{2\lambda\varphi} a g(y_t, \nabla y) y_t \,\mathrm{d}x\,\mathrm{d}t. \tag{2.23}$$

We note that in order to obtain (2.23), we also made use of (1.4), (1.5), (1.2) and the fact that $u = ry_t$.

Now by (2.18), we have

$$||\mathrm{e}^{\lambda\varphi} u||_{L^2(Q)}^2 \geq \int_{T_1}^{T_1'} \int_\Omega \mathrm{e}^{2\lambda\varphi} y_t^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{2.24}$$

so that

$$||\mathrm{e}^{\lambda\varphi} y_t||^2_{L^2(Q)} \leq ||\mathrm{e}^{\lambda\varphi} u||^2_{L^2(Q)} + ||\mathrm{e}^{\lambda\varphi} y_t||^2_{L^2(\tilde{Q})}. \tag{2.25}$$

Combining (2.23) and (2.25), we find

$$\lambda||\mathrm{e}^{\lambda\varphi} y_t||^2_{L^2(Q)} \leq C\lambda(1+\lambda^2)||\mathrm{e}^{\lambda\varphi} y_t||^2_{L^2(\tilde{Q})} + C\lambda(1+\lambda^2)\int_0^T\int_\Omega \mathrm{e}^{2\lambda\varphi} ag(y_t,\nabla y)y_t\,\mathrm{d}x\,\mathrm{d}t. \tag{2.26}$$

Thanks to (2.17), we have

$$||\mathrm{e}^{\lambda\varphi} y_t||^2_{L^2(Q)} \geq \mathrm{e}^{\lambda R_0^2}||y_t||^2_{L^2(Q_0)}. \tag{2.27}$$

Reporting (2.27) in (2.26), using (2.14), and simplifying, we get

$$||y_t||^2_{L^2(Q_0)} \leq C(1+\lambda^2)\mathrm{e}^{\lambda(R_1^2-R_0^2-\mu(T^2/4))}||y_t||^2_{L^2(\tilde{Q})} + C(1+\lambda^2)\mathrm{e}^{2\lambda R_1^2}\int_0^T\int_\Omega ag(y_t,\nabla y)y_t\,\mathrm{d}x\,\mathrm{d}t. \tag{2.28}$$

Since the energy E is nonincreasing, we have

$$||y_t||^2_{L^2(\tilde{Q})} \leq 2TE(0) = 2TE(T) + 2T\int_0^T\int_\Omega ag(y_t,\nabla y)y_t\,\mathrm{d}x\,\mathrm{d}t. \tag{2.29}$$

Hence

$$||y_t||^2_{L^2(Q_0)} \leq C(1+\lambda^2)\mathrm{e}^{\lambda(R_1^2-R_0^2-\mu(T^2/4))}E(T) + C(1+\lambda^2)\mathrm{e}^{2\lambda R_1^2}\int_0^T\int_\Omega ag(y_t,\nabla y)y_t\,\mathrm{d}x\,\mathrm{d}t. \tag{2.30}$$

At this stage, we note that we will be done with the proof of (2.12) once we have proven that

$$E(T) \leq C_0\int_{Q_0}|y_t|^2\,\mathrm{d}x\,\mathrm{d}t + C_0\int_0^T\int_\Omega ag(y_t,\nabla y)y_t\,\mathrm{d}x\,\mathrm{d}t, \tag{2.31}$$

for some C_0 that may depend on the initial data. To this end, let $h \in \mathcal{C}^1([T_0,T_0'])$ with $h(T_0) = h(T_0') = 0$. Multiplying the first equation in (1.3) by hy and integrating by parts over Q_0, we get

$$2\int_{T_0}^{T_0'} h(t)\mathcal{E}(t)\,\mathrm{d}t + \int_{Q_0} hyf(y)\,\mathrm{d}x\,\mathrm{d}t = 2\int_{Q_0} h|y_t|^2\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_0} h'y_ty\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_0} hag(y_t,\nabla y)y\,\mathrm{d}x\,\mathrm{d}t, \tag{2.32}$$

where

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega\{|y_t(x,t)|^2 + b_{ij}(x)\partial_j y(x,t)\partial_i y(x,t) + p(x)|y(x,t)|^2\}\,\mathrm{d}x. \tag{2.33}$$

We note that the energies E and $\mathcal{E}$ satisfy the estimates

$$\mathcal{E}(t) \leq E(t) \leq C\left(1+E(0)^{\frac{q}{2}}\right)\mathcal{E}(t). \tag{2.34}$$

Some simple calculations in (2.32) show that

$$\int_{T_0}^{T_0'} h(t)\mathcal{E}(t)\,\mathrm{d}t \leq C\int_{Q_0}|y_t|^2\,\mathrm{d}x\,\mathrm{d}t + C\int_{Q_0} ag(y_t,\nabla y)y_t\,\mathrm{d}x\,\mathrm{d}t, \tag{2.35}$$

from which one first derive, thanks to (2.34),

$$\int_{T_0}^{T_0'} h(t)E(t)\,\mathrm{d}t \leq C\left(1+E(0)^{\frac{q}{2}}\right)\left[\int_{Q_0}|y_t|^2\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_0} ag(y_t,\nabla y)y_t\,\mathrm{d}x\,\mathrm{d}t\right], \tag{2.36}$$

then

$$E(T) \le C\left(1+E(0)^{\frac{q}{2}}\right)\left[\int_{Q_0}|y_t|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_0} ag(y_t,\nabla y)y_t\,\mathrm{d}x\,\mathrm{d}t\right], \tag{2.37}$$

as E is nonincreasing.

Combining (2.30) and (2.37), we find

$$E(T) \le C\left(1+E(0)^{\frac{q}{2}}\right)\mathrm{e}^{\lambda(R_1^2-R_0^2-\mu(T^2/4))}E(T)+C\left(1+E(0)^{\frac{q}{2}}\right)(1+\lambda)\mathrm{e}^{2\lambda R_1^2}\int_0^T\int_\Omega ag(y_t,\nabla y)y_t\,\mathrm{d}x\,\mathrm{d}t. \tag{2.38}$$

We may choose λ so large that

$$C\left(1+E(0)^{\frac{q}{2}}\right)\mathrm{e}^{\lambda(R_1^2-R_0^2-\mu(T^2/4))} \le 1/2. \tag{2.39}$$

In this case, (2.38) becomes

$$E(T) \le C\left(1+E(0)^{\frac{q}{2}}\right)(1+\lambda)\mathrm{e}^{2\lambda R_1^2}\int_0^T\int_\Omega ag(y_t,\nabla y)y_t\,\mathrm{d}x\,\mathrm{d}t, \tag{2.40}$$

from which one easily derives

$$E(T) \le C_1\mathrm{e}^{C_2(1+E(0)^q)}\int_0^T\int_\Omega ag(y_t,\nabla y)y_t\,\mathrm{d}x\,\mathrm{d}t, \tag{2.41}$$

where C_1 and C_2 are positive constants that are independent of the initial data.

Using (1.10) in (2.40) we get

$$E(T) \le \frac{C_1\mathrm{e}^{C_2(1+E(0)^q)}}{1+C_1\mathrm{e}^{C_2(1+E(0)^q)}}E(0). \tag{2.42}$$

If we set

$$\gamma = C_1\mathrm{e}^{C_2(1+E(0)^q)}/(1+C_1\mathrm{e}^{C_2(1+E(0)^q)}),$$

and we apply the semigroup property combined with one of the techniques devised in [2,32], we get

$$E(t) \le \frac{1}{\gamma}\mathrm{e}^{(-t\log(1/\gamma))/T}E(0) \le 2\mathrm{e}^{(-t\log(1/\gamma))/T}E(0), \quad \forall t \ge 0, \tag{2.43}$$

which completes the proof Theorem 1.2.

Remark 2.1. As one can see, the advantage of our method is that it provides a much simpler proof than those found in the literature [7,8,29,45,46]. Our constructive approach is much more interesting for globally Lipschitz nonlinearities f because in this case, it enables us to improve in some way on all the earlier results. One of the drawbacks of this method though, is that it does not enable us to obtain uniform exponential decay for nonlinearities f that are not globally Lipschitz; we are able to obtain uniform exponential decay on every ball in the energy space only. Also, it does not seem to work for more general nonlinear dampings such as those found in, say, [19,25,28,29,39]. However we note that one may allow the product $sf(s)$ to be negative; namely "$f(s)s > -\lambda_1 s^2$, $\forall s \ne 0$", where λ_1 is the first eigenvalue for the negative Laplace operator with Dirichlet boundary conditions.

Remark 2.2. The constructive method that we have developed for the proof of Theorem 1.2 critically relies on the Carleman estimate in [10,11] (see Lem. 2.1 above). The merit of this Carleman estimate[10] in the study of control problems is that it applies to general hyperbolic operators, and it allows for more flexibility on the location of the control as compared to the Carleman estimate in [33], which applies to the ordinary wave equation, and for which the control shall be located on a neighborhood of the whole boundary (*cf.* [41,45,46]). We are going to use the Carleman estimate in [33] to prove a result similar to Theorem 1.2 for the wave equation in exterior domains. The reason why we make this choice may be found in Remark 3.4 following this proof.

3. The wave equation in an exterior domain

In this section we consider the damped wave equation

$$\begin{cases} y_{tt} - \Delta y + p(x)y + f(y) + ay_t = 0 \text{ in } \Omega \times (0,\infty) \\ y = 0 \text{ on } \partial\Omega \times (0,\infty) \\ y(0) = y^0, \quad y_t(0) = y^1 \text{ in } \Omega \end{cases} \tag{3.1}$$

where $\Omega = \mathbb{R}^N \setminus D$, the set D being compact with a smooth boundary, and $y^0 \in H_0^1(\Omega)$, $y^1 \in L^2(\Omega)$. The functions f and a are given as above while now the function p satisfies

$$\begin{aligned} &p \in L_+^m(\Omega), \\ &p \in L_+^1(\Omega) \cap L_+^2(\Omega) \text{ for } N = 1, \quad m > 2 \text{ for } N = 2, \quad \text{ and } m \geq N \text{ for } N \geq 3, \\ &\exists b_0 > 0 : p(x) \geq b_0, \quad \textit{a.e. } x \in V_L = \{x \in \Omega; |x| > L\}, \end{aligned} \tag{3.2}$$

for some $L > 0$, and the location of the feedback control, ω, is the union of V_L and a neighborhood of the boundary of Ω. By neighborhood of $\partial\Omega$, we actually mean the intersection of Ω and a a neighborhood of $\partial\Omega$. Condition (3.2) on the function p ensures the coerciveness of the energy given by

$$\tilde{E}(t) = \frac{1}{2}\int_\Omega \{|y_t(x,t)|^2 + |\nabla y(x,t)|^2 + p(x)|y(x,t)|^2\}\,\mathrm{d}x + \int_\Omega F(y(x,t))\,\mathrm{d}x. \tag{3.3}$$

This energy is also a nonincreasing function of the time variable. More precisely, we have:

Theorem 3.1. *Let $y^0 \in H_0^1(\Omega)$ and $y^1 \in L^2(\Omega)$. Suppose that ω is the union of V_L and a neighborhood of $\partial\Omega$. Assume that f satisfies* (1.1)*, p satisfies* (3.2)*, and that for a, condition* (1.2) *holds. Then there exist positive constants M and α, possibly depending on $\tilde{E}(0)$ such that the energy $\tilde{E}$ of each solution of* (3.1) *satisfies:*

$$\tilde{E}(t) \leq M[\exp(-\alpha t)]\tilde{E}(0), \quad \forall t \geq 0. \tag{3.4}$$

Remark 3.1. As was the case for Theorem 1.2, when f is globally Lipschitz, the exponential decay will be proven to be uniform in the energy space while for other functions the exponential decay will be proven to be uniform only on every ball in the energy space. The possible dependence of M and α on $\tilde{E}(0)$ will be given as part of the proof of Theorem 3.1.

Remark 3.2. We note that the condition imposed on p in the lower line of (3.2) is essential in the proof of the exponential decay of the energy $\tilde{E}$ (no such condition is needed for bounded sets Ω); in fact, this condition enables us to estimate the quantity $\int_\Omega |y|^2\,\mathrm{d}x$ in terms of the energy. It was observed in (*e.g.* [28]) that when there is no potential, the decay of $\tilde{E}$ is polynomial. Whether exponential decay could hold without that condition is, to the best of our knowledge, unknown.

For the sequel we need the following notations: let $\mu > 0$, $\eta > 1$, and for $(x,t) \in \mathbb{R}^{N+1}$, set $\varphi(x,t) = \eta^2 t^2 - |x|^2$, and $D^\mu = \left\{(x,t) \in \mathbb{R}^{N+1}; \varphi(x,t) > \mu\right\}$. The proof of Theorem 3.1 will be based on the following Carleman estimate due to Ruiz [33], Proposition 1:

Lemma 3.1. *Let K be a compact subset of D^μ, then there exists a $\lambda_0 > 0$ and a constant $C = C(K,\mu)$, independent of u and λ such that for any $\lambda > \lambda_0$ and $u \in C_0^\infty(K)$ we have*

$$\lambda||\mathrm{e}^{2\lambda\varphi}u||^2_{L^2(K)} \leq C||\mathrm{e}^{2\lambda\varphi}\Box u||^2_{H^{-1}(K)}, \tag{3.5}$$

where $\Box u = u_{tt} - \Delta u$.

Furthermore estimate (3.5) *holds for all $u \in L^2(K)$ such that $\Box u \in H^{-1}(K)$.*

Proof of Theorem 3.1. Set $T_0 = \mathrm{diam}(\Omega \setminus \omega) + \inf\{|x|; x \in \Omega \setminus \omega\}$. Let $T_1 > T_0$. Then there exists $\mu > 0$ such that $T_1^2 > T_0^2 + \mu$. Let $\eta > 1$. For every $t \geq T_1$, and every $x \in \overline{\Omega \setminus \omega}$, we have $\eta^2 t^2 - |x|^2 > \mu$, so that for each $T > T_1$, the set $K = \overline{\Omega \setminus \omega} \times [T_1, T]$ is a compact subset of D^μ. Further, if we set $u = y_t$, then $u \in L^2(K)$, and $\Box u \in H^{-1}(K)$, since $\Box u = -pu - f'(y)u - au_t$. Applying Lemma 3.1, we obtain

$$\lambda ||\mathrm{e}^{2\lambda\varphi} u||^2_{L^2(K)} \leq C ||\mathrm{e}^{2\lambda\varphi}(-pu - f'(y)u - au_t)||^2_{H^{-1}(K)}. \tag{3.6}$$

Now by Hahn-Banach theorem, Sobolev embedding theorem, and Hölder inequality we find

$$\begin{aligned} &||\mathrm{e}^{2\lambda\varphi} pu||_{H^{-1}(K)} \leq C ||\mathrm{e}^{2\lambda\varphi} u||_{L^2(K)}, \\ &||\mathrm{e}^{2\lambda\varphi} f'(y)u||_{H^{-1}(K)} \leq C ||\mathrm{e}^{2\lambda\varphi} u||_{L^2(K)} \left(1 + \tilde{E}(0)^q\right). \end{aligned} \tag{3.7}$$

On the other hand, applying Hahn-Banach theorem, and integration by parts over K, we get

$$||\mathrm{e}^{2\lambda\varphi} au_t||_{H^{-1}(K)} \leq C(1+\lambda) ||\mathrm{e}^{2\lambda\varphi} au||_{L^2(K)}. \tag{3.8}$$

When f is globally Lipschitz and λ is large enough, a combination of (3.6)–(3.8), and (1.2) yields

$$||y_t||^2_{L^2(\Omega \times (T_1,T))} \leq C(\lambda) \int_{T_1}^T \int_\Omega a|y_t|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{3.9}$$

For other functions f, we have instead

$$||y_t||^2_{L^2(\Omega \times (T_1,T))} \leq C\left(\lambda, \tilde{E}(0)^q\right) \int_{T_1}^T \int_\Omega a|y_t|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{3.10}$$

At this stage, we note that we will be done once we have proven that

$$\tilde{E}(T) \leq C_0(\tilde{E}(0)^q) \int_{T_1}^T \int_\Omega |y_t|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{3.11}$$

for some positive constant C_0 which may or may not depend on the initial data, depending on the values of q. Indeed once (3.11) is proven, the combination of (3.11) and (3.10) yields

$$\tilde{E}(T) \leq C_0(\tilde{E}(0)^q) \int_0^T \int_\Omega a|y_t|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{3.12}$$

from which one derives the claimed exponential decay with

$$M = 1 + (1/C_0(\tilde{E}(0)^q)), \quad \alpha = (\log M)/T, \tag{3.13}$$

thanks to the semigroup property, and one of the techniques devised in [2,32].

We now prove (3.11). To this end, let $r \in \mathcal{C}^1([T_1, T])$ with $r(T) = r(T_1) = 0$. Multiplying the first equation in (3.1) by ry and integrating by parts over $\Omega \times [T_1, T]$, we get

$$2\int_{T_1}^T r(t)\tilde{\mathcal{E}}(t)\,\mathrm{d}t + \int_{T_1}^T \int_\Omega ryf(y)\,\mathrm{d}x\,\mathrm{d}t = 2\int_{T_1}^T \int_\Omega r|y_t|^2\,\mathrm{d}x\,\mathrm{d}t + \int_{T_1}^T \int_\Omega r' y_t y\,\mathrm{d}x\,\mathrm{d}t - \int_{T_1}^T \int_\Omega ray_t y\,\mathrm{d}x\,\mathrm{d}t, \tag{3.14}$$

where

$$\tilde{\mathcal{E}}(t) = \frac{1}{2} \int_\Omega \{|y_t(x,t)|^2 + |\nabla y(x,t)|^2 + p(x)|y(x,t)|^2\}\,\mathrm{d}x. \tag{3.15}$$

We observe that the energies $\tilde{E}$ and $\tilde{\mathcal{E}}$ satisfy

$$\tilde{\mathcal{E}}(t) \leq \tilde{E}(t) \leq C\left(1+\tilde{E}(0)^{\frac{q}{2}}\right)\tilde{\mathcal{E}}(t). \tag{3.16}$$

One easily derives (3.11) from (3.14)–(3.16), (1.1) and the fact that the energy $\tilde{E}$ is nonincreasing. This completes the proof of Theorem 3.1. □

Remark 3.3. It should be noted that the constant C_0 appearing in (3.11) has the form $C_1 \mathrm{e}^{C_2(1+\tilde{E}(0)^q)}$, where C_1 and C_2 are independent of the initial data.

Remark 3.4. The Carleman estimates in [10,11], which are provided for bounded domains, may be extended to unbounded domains; this is, surprisingly, easily done using an appropriate cut-off function (see *e.g.* [44] for the case of the plate equation) that transforms the Carleman estimate problem in the unbounded domain into a Carleman estimate problem in a bounded domain. This estimate may then be used to prove Theorem 3.1. Doing this would improve that theorem by allowing for a larger class of damping locations, and hyperbolic equations as in the case of bounded domains. Let us note however that the proof of Theorem 3.1 provided above, not only needs some different ideas, but also has the merit of being shorter than the one we would have gotten if we had used the analogue for unbounded domains of the Carleman estimate in [10].

It is also known that using rotated multipliers [9], one can prove Carleman estimates for the ordinary wave equation; these estimates may be generalized to second order hyperbolic operators where the coefficients of the principal part satisfy conditions similar to (1.11); this can be carried out by combining ideas from [9–11]. Once this is done, applying them to prove Theorem 1.2, or Theorem 3.1 would lead to a larger class of damping locations than the Carleman estimate in [10] could allow.

4. Final remarks and open problems

As the decay rate provided by our approach is not uniform for functions f that are not globally Lipschitz, it is of interest to know how the nonuniform decay rate depends explicitly on the initial data; having this information would help understand how the decay rate is affected by a perturbation of the initial data. This explains why we indicated in the proofs of Theorems 1.2 and 3.1, how to arrive at such an explicit estimate. To our knowledge, the only paper in the literature that does the same is [32].

On a different note, we want to draw the reader's attention on the restrictions on the growth of the nonlinearity f (see (1.1)); these restrictions were critical in the development of our constructive method. It was observed in [8] in the case of the wave equation in 3-D that one could allow for nonlinearities f behaving like $s|s|^p$, with $0 \leq p < 4$ ($p = 4$ being the critical case for Strichartz inequality based methods); it should be noted that the proof provided in [8] is based on Strichartz inequalities, and a compactness-uniqueness argument unlike our proof which is constructive. Whether some of the ideas developed in [8] could be used to build a constructive proof for more general nonlinearities, is to the best of our knowledge unknown.

The constructive method that we have devised above to solve stabilization problems for semilinear hyperbolic equations with locally distributed damping is quite general. Indeed the method applies to a wide varieties of second order evolution equations with locally distributed damping provided for such systems, one has an H^{-1}-type Carleman estimate similar to one of those stated in Lemma 2.1, and Lemma 3.1 above, or to the one established in [16]. We point out that such Carleman estimates are yet to be proven:

– for the general elasticity system

$$\begin{cases} y_{i,tt} - \sigma_{ij,j} = h \text{ in } \Omega \times (0,T) \\ y_i = 0 \text{ on } \partial\Omega \times (0,T) \\ y_i(0) = 0, \quad y_i(T) = 0, \quad i = 1,\ 2,\ \ldots,\ N, \end{cases} \tag{4.1}$$

where $h \in H^{-1}(\Omega \times (0,T))$.

In (1.1) the elasticity stress tensor (σ_{ij}) is given by

$$\sigma_{ij} = \sigma_{ij}(y) = a_{ijkl}\varepsilon_{kl}$$

where (ε_{kl}) defined by

$$\varepsilon_{kl} = \varepsilon_{kl}(y) = \frac{1}{2}(y_{k,l} + y_{l,k})$$

is the strain tensor. The a_{ijkl} are the elasticity coefficients. They satisfy the symmetry properties

$$a_{ijkl} = a_{jilk} = a_{klij}, \quad \forall i, j, k, l.$$

The a_{ijkl} depend on the space variable x but not on time, and that they are continuously differentiable, and satisfy the ellipticity condition

$$\exists a_0 > 0 : a_{ijkl}u_{ij}u_{kl} \geq a_0 u_{ij}u_{kl} \tag{4.2}$$

for all second order symmetric tensors (u_{ij});

– and for the Euler-Bernoulli equations

$$\begin{cases} y_{tt} + \Delta^2 y = h \text{ in } \Omega \times (0,T) \\ y = \frac{\partial y}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0,T) \\ y(0) = y^0, \quad y_t(0) = y^1 \text{ in } \Omega. \end{cases} \tag{4.3}$$

Acknowledgements. The author thanks the referees for their pertinent remarks, and suggestions that helped improve the content and presentation of this paper. He also thanks Professor Zuazua for the useful comments provided. He also thanks Professor Zuazua for the fruitful discussions that they had concerning certain points in this paper.

References

[1] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control and stabilization from the boundary. *SIAM J. Control Opt.* **30** (1992) 1024–1065.
[2] M.E. Bradley and I. Lasiecka, Global decay rates for the solutions to a von Kármán plate without geometric conditions. *J. Math. Anal. Appl.* **181** (1994) 254–276.
[3] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland (1973).
[4] H. Brezis, *Analyse fonctionnelle. Théorie et Applications*. Masson, Paris (1983).
[5] G. Chen, S.A. Fulling, F.J. Narcowich and S. Sun, Exponential decay of energy of evolution equations with locally distributed damping. *SIAM J. Appl. Math.* **51** (1991) 266–301.
[6] C.M. Dafermos, Asymptotic behaviour of solutions of evolution equations, in *Nonlinear evolution equations*, M.G. Crandall Ed., Academic Press, New-York (1978) 103–123.
[7] B. Dehman, Stabilisation pour l'équation des ondes semi-linéaire. *Asymptotic Anal.* **27** (2001) 171–181.
[8] B. Dehman, G. Lebeau and E. Zuazua, Stabilization and control for the subcritical semilinear wave equation. *Ann. Sci. École Norm. Sup.* **36** (2003) 525–551.
[9] A. Doubova and A. Osses, Rotated weights in global Carleman estimates applied to an inverse problem for the wave equation. *Inverse Problems* **22** (2006) 265–296.
[10] T. Duyckaerts, X. Zhang and E. Zuazua, On the optimality of the observability for parabolic and hyperbolic systems with potentials. *Ann. Inst. H. Poincaré Anal. Non Linéaire* (to appear).
[11] X. Fu, J. Yong and X. Zhang, Exact controllability for multidimensional semilinear hyperbolic equations. *SIAM J. Contr. Opt.* **46** (2007) 1578–1614.
[12] A. Haraux, Stabilization of trajectories for some weakly damped hyperbolic equations. *J. Diff. Eq.* **59** (1985) 145–154.
[13] A. Haraux, Semi-linear hyperbolic problems in bounded domains, in *Mathematical Reports* **3**, Hardwood academic publishers (1987) .
[14] A. Haraux, Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps. *Portugal. Math.* **46** (1989) 245–258.
[15] A. Haraux, Remarks on weak stabilization of semilinear wave equations. *ESAIM: COCV* **6** (2001) 553–560.
[16] O.Yu. Imanuvilov, On Carleman estimates for hyperbolic equations. *Asympt. Anal.* **32** (2002) 185–220.
[17] V. Komornik, *Exact controllability and stabilization. The multiplier method*. RAM, Masson & John Wiley, Paris (1994).

[18] J. Lagnese, Control of wave processes with distributed control supported on a subregion. *SIAM J. Control Opt.* **21** (1983) 68–85.
[19] I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. *Differential Integral Equations* **6** (1993) 507–533.
[20] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod-Gauthier-Villars, Paris (1969).
[21] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation des systèmes distribués*, Vol. **1**, RMA 8. Masson, Paris (1988).
[22] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Vol. **I**. Springer-Verlag, New York-Heidelberg (1973).
[23] K. Liu, Locally distributed control and damping for the conservative systems. *SIAM J. Control Opt.* **35** (1997) 1574–1590.
[24] F. Macià and E. Zuazua, On the lack of observability for wave equations: a gaussian beam approach. *Asymptot. Anal.* **32** (2002) 1–26.
[25] P. Martinez, Ph.D. thesis, University of Strasbourg, France (1998).
[26] P. Martinez, A new method to obtain decay rate estimates for dissipative systems with localized damping. *Rev. Mat. Complut.* **12** (1999) 251–283.
[27] M. Nakao, Decay of solutions of the wave equation with a local degenerate dissipation. *Israel J. Math.* **95** (1996) 25–42.
[28] M. Nakao, Global existence for semilinear wave equations in exterior domains, in *Proceedings of the Third World Congress of Nonlinear Analysts*, Part 4 (Catania, 2000), *Nonlinear Anal.* **47** (2001) 2497–2506.
[29] M. Nakao, Global and periodic solutions for nonlinear wave equations with some localized nonlinear dissipation. *J. Diff. Eq.* **190** (2003) 81–107.
[30] M. Nakao and I.H. Jung, Energy decay for the wave equation in exterior domains with some half-linear dissipation. *Differential Integral Equations* **16** (2003) 927–948.
[31] A. Pazy, *Semigroups of linear operators and applications to partial differential equations, Applied Mathematical Sciences* **44**. Springer-Verlag, New York (1983).
[32] G. Perla Menzala and E. Zuazua, The energy decay rate for the modified von Kármán system of thermoelastic plates: an improvement. *Appl. Math. Lett.* **16** (2003) 531–534.
[33] A. Ruiz, Unique continuation for weak solutions of the wave equation plus a potential. *J. Math. Pures Appl.* **71** (1992) 455–467.
[34] D.L. Russell, Exact boundary value controllability theorems for wave and heat processes in star-complemented regions, in *Differential games and control theory (Proc. NSF—CBMS Regional Res. Conf., Univ. Rhode Island, Kingston, R.I., 1973)* Dekker, New York. *Lect. Notes Pure Appl. Math.* **10**, Dekker, New York (1974) 291–319.
[35] M. Slemrod, Weak asymptotic decay via a relaxed invariance principle for a wave equation with nonlinear, nonmonotone damping. *Proc. Royal Soc. Edinburgh Sect. A* **113** (1989) 87–97.
[36] D. Tataru, The X_θ^s spaces and unique continuation for solutions to the semilinear wave equation. *Comm. Partial Differential Equations* **21** (1996) 841–887.
[37] L.R. Tcheugoué Tébou, Estimations d'énergie pour l'équation des ondes avec un amortissement nonlinéaire localisé. *C. R. Acad. Paris, Sér. I* **325** (1997) 1175–1179.
[38] L.R. Tcheugoué Tébou, On the decay estimates for the wave equation with a local degenerate or nondegenerate dissipation. *Portugal. Math.* **55** (1998) 293–306.
[39] L.R. Tcheugoué Tébou, Stabilization of the wave equation with localized nonlinear damping. *J. Diff. Eq.* **145** (1998) 502–524
[40] L.R. Tcheugoué Tébou, Well-posedness and energy decay estimates for the damped wave equation with L^r localizing coefficient. *Comm. Partial Differential Equations* **23** (1998) 1839–1855.
[41] L.R. Tcheugoué Tébou, A direct method for the stabilization of some locally damped semilinear wave equations. *C. R. Acad. Sci. Paris, Ser. I* **342** (2006) 859–864.
[42] J. Vancostenoble, *Stabilisation non monotone de systèmes vibrants et Contrôlabilité.* Ph.D. thesis, University of Rennes, France (1998).
[43] J. Vancostenoble, Weak asymptotic decay for a wave equation with gradient dependent damping. *Asymptot. Anal.* **26** (2001) 1–20.
[44] X. Zhang and E. Zuazua, On the optimality of the observability inequalities for Kirchhoff plate systems with potentials in unbounded domains, in *Proceedings of the Eleventh International Conference on Hyperbolic Problems: Theory, Numerics and Applications*, Lyon (2006) (to appear).
[45] E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping. *Comm. Partial Differential Equations* **15** (1990) 205–235.
[46] E. Zuazua, Exponential decay for the semilinear wave equation with localized damping in unbounded domains. *J. Math. Pures Appl.* **15** (1990) 205–235.