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# A VARIATIONAL APPROACH TO IMPLICIT ODES AND DIFFERENTIAL INCLUSIONS

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**Abstract.** An alternative approach for the analysis and the numerical approximation of ODEs, using a variational framework, is presented. It is based on the natural and elementary idea of minimizing the residual of the differential equation measured in a usual $L^p$ norm. Typical existence results for Cauchy problems can thus be recovered, and finer sets of assumptions for existence are made explicit. We treat, in particular, the cases of an explicit ODE and a differential inclusion. This approach also allows for a whole strategy to approximate numerically the solution. It is briefly indicated here as it will be pursued systematically and in a much more broad fashion in a subsequent paper.

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## 1. Introduction

We explore a treatment of a typical Cauchy problem for a system of implicit ODEs or differential inclusions based on a minimization principle in which we seek to minimize the departure of feasible fields from being a solution of the problem. This is an elementary point of view which we systematically apply both to show existence of solutions and to set up numerical approximations of exact solutions. Because of its simplicity and straightforwardness, it has been used in some other contexts (see for instance [1]). Our goal is to provide a unified perspective for the analysis and numerical approximation of this kind of problems. More precisely, instead of searching directly for a solution of the problem

$$F(t, y(t), y'(t)) = 0, \quad t \in (0, T), \qquad y(0) = y_0, \tag{1.1}$$

we seek to minimize a certain functional $I(y)$ over a suitable class of functions $y$ complying with initial conditions $y(0) = y_0$, in such a way that

$$I(y) \geq 0, \quad \text{but } I(y) = 0 \iff y \text{ is a solution of (1.1)}.$$

*Keywords and phrases.* Variational methods, convexity, coercivity, value function.

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The map

$$F(t,y,z):(0,T)\times\mathbf{R}^N\times\mathbf{R}^N\to\mathbf{R}^m,$$

is assumed to be a Carathéodory map, *i.e.* measurable in $t$ and continuous in $(y,z)$. The functional $I$ may be understood and designed as a measure of the departure of a feasible $y$ from being a solution of (1.1). As such, there are infinitely many possibilities, and we stick to the simple choice

$$I_p(y)=\frac{1}{p}\int_0^T |F(t,y(t),y'(t))|^p\,\mathrm{d}x,\quad p\ge 1,$$

and the suitable extension for $p=\infty$. For the numerical implementation we will pay more attention to the case $p=2$. One may be tempted to discard this approach as too naive to be fruitful, yet we have found that it leads to quite general and flexible results and approximation strategies that at the same time can be implemented very efficiently.

From the point of view of the Calculus of Variations [3], the analysis of the problem

$$\text{Minimize in } y\in W^{1,p}(0,T):\quad I_p(y),$$

subject to $y(0)=y_0$, requires typical properties of convexity of $F$ with respect to $z$, and coercivity which we will assume in the mild form

$$|F(t,y,z)|\ge C|z|-M_1|y|-M_0, \tag{1.2}$$

for positive constants $C$, $M_1$ and $M_0$. Notice that "true" coercivity is out of the question because the functional $I_p$ vanishes precisely in the solutions we seek. Thus, in principle, the classical Tonelli theorem [3] cannot be directly applied. Yet there is something very special about the structure of the integrands

$$F_p(t,y,z)=|F(t,y,z)|^p,$$

because under the main hypothesis (in addition to the convexity already referred to above)

$$\frac{1}{h}\min_{z:F(s,y,z)=0}\int_s^{s+h}|F(t,y+(t-s)z,z)|\,\mathrm{d}t\to 0,\ \text{as } h\to 0, \tag{1.3}$$

for all $s$ and $y$, we can show, in a rather elementary way, that for the variational problem

$$\text{Minimize in } y\in W^{1,p}(0,T):\quad I_p(y)=\frac{1}{p}\int_0^T|F(t,y(t),y'(t))|^p\,\mathrm{d}t,$$

we have existence of minimizers $\overline{y}$ and, in addition, the value of the minimum vanishes so that $\overline{y}$ is in fact a solution of our Cauchy problem (1.1) (Th. 2.1). The proof relies on a close analysis of the value function (the value of the minimum) as a function of the upper limit $T$.

The condition (1.3) is a replacement of lipschitziannity. Indeed, it allows to prove existence results without Lipschitz hypotheses (see Th. 3.1, and Cor. 3.1).

If, in addition, we strengthen the hypotheses, then uniqueness follows. Specifically, if

$$C|z|-M|y|\le|F(t,y,z)|,\quad C,M>0$$

for all $t$, $y$, $z$, and

$$|F(t,y_1-y_2,z_1-z_2)|\le D\left(|F(t,y_1,z_1)|+|F(t,y_2,z_2)|+|y_1-y_2|\right),\quad D>0$$

for all $t$, $y_1$, $y_2$, $z_1$, $z_2$, then there is a unique solution of the Cauchy problem (Th. 2.2). This uniqueness is a direct consequence of the main estimate

$$\|y_1 - y_2\|_{W^{1,1}(0,T)} \leq C\left(I_1(y_1) + I_1(y_2)\right), \tag{1.4}$$

valid for all pairs $(y_1, y_2)$ of feasible functions, where the constant $C$ is independent of $y_i$. This estimate is also the starting point of the numerical approximation. If we put $y_1 = y$ and $y_2 = \overline{y}$, the solution of (1.1), then we have

$$\|y - \overline{y}\|_{W^{1,1}(0,T)} \leq C I_1(y),$$

which shows that indeed $I_1(y)$ is a measure of the departure of $y$ from $\overline{y}$, the error. In particular, if $y$ is such that $I_1(y) \leq \epsilon$, then $y$ will be a uniformly good approximation of the exact solution $\overline{y}$ all over the time interval considered $(0, T)$. There cannot be propagation of errors as time proceeds. By choosing an appropriate approximation scheme in terms of a finite number of parameters, and optimizing on these, we can have alternative approximation procedures for the solution of the Cauchy problem (1.1). There are many possibilities for the choice of the different ingredients of these numerical schemes. We will explore some of these in a more detailed way in a subsequent work.

The paper is organized as follows. The next section is the main part and focuses on providing the main results on existence and uniqueness. Section 3 is an attempt to look more explicitly at two particular cases which are especially relevant: that of an explicit ODE (system) [2], and a differential inclusion as such [6]. In Section 4, we indicate a few general ideas on how this variational approach can also be utilized to setup suitable numerical approximations [5]. Finally, Section 5 is devoted to emphasizing some conclusions.

## 2. The main existence theorem

As indicated in the Introduction, we pretend to examine the variational problem

$$\text{Minimize in } y: \quad I_p(y) = \frac{1}{p}\int_0^T |F(t, y(t), y'(t))|^p \,\mathrm{d}t \tag{2.1}$$

where $y$ runs through all of the absolutely continuous mappings and is subjected to $y(0) = y_0$. We will restrict attention to the case $p = 1$, and focus on the variational problem

$$\text{Minimize in } y: \quad I(y) = \int_0^T |F(t, y(t), y'(t))| \,\mathrm{d}t. \tag{2.2}$$

Our hypotheses on $F$ are:

(1) $F$ is measurable in $t$ and continuous in $(y, z)$.
(2) $F$ is convex in $z$.
(3) Coercivity: for all $t \in (0, T)$

$$F(t, y, z) \geq C|z| - M_1|y| - M_0, \quad C, M_1, M_0 > 0.$$

(4) For every $s$ and $y$,

$$\frac{1}{h}\min_{z: F(s,y,z)=0} \int_s^{s+h} |F(t, y + (t-s)z, z)| \,\mathrm{d}t \to 0, \text{ as } h \to 0. \tag{2.3}$$

Our main result is the following.

**Theorem 2.1.** *Under the hypotheses indicated for $F$, the Cauchy problem*

$$F(t, y(t), y'(t)) = 0 \text{ in } (0, T), \quad y(0) = y_0,$$

*admits at least one global, Lipschitz solution.*

Before proving this result, we examine an easy lemma which shows the type of ideas involved in this approach.

**Lemma 2.1.** *Suppose that $F$ is as above, and let $y$ be an admissible map for our variational problem. Then $y \in L^\infty(0, T)$, and for every $t \in (0, T)$, we have*

$$|y(t)| \leq \left( \frac{TM_0}{C} + |y_0| + \frac{1}{C} I(y) \right) \mathrm{e}^{M_1 T/C}.$$

*Proof.* The proof is elementary. Put

$$|y(t) - y_0| \leq \int_0^t |y'(s)| \,\mathrm{d}s \leq \frac{1}{C} I(y) + \frac{M_1}{C} \int_0^t |y(s)| \,\mathrm{d}s + \frac{TM_0}{C}.$$

Use Gronwall's lemma to conclude. □

*Proof of Theorem 2.1.* As remarked before, our strategy consists in considering the variational problem (2.2), and showing these two facts:

(1) the infimum $m$ is indeed a minimum; and
(2) it vanishes.

Notice that none of these two facts is straightforward from general results.

We will start by showing (2). To this aim write

$$I(y; s, t) = \int_s^t |F(r, y(r), y'(r))| \,\mathrm{d}r.$$

We will examine the associated value function as in dynamic programming [4].

We will consider the infimum $m$ as a function of $s \in (0, T)$, *i.e.*,

$$m(s) = \inf_{y \in W^{1,1}(0,s), y(0) = y_0} I(y; 0, s).$$

It is obvious that $m(0) = 0$. Let us look at $m(s + h)$ for $h > 0$, and try to relate it to $m(s)$

$$m(s + h) = \inf_{y \in W^{1,1}(0,s+h), y(0) = y_0} I(y; 0, s + h).$$

It is clear that

$$m(s + h) = \inf_{w \in \mathbf{R}^N} \left( \inf_{y \in W^{1,1}(0,s), y(0) = y_0, y(s) = w} I(y; 0, s) + \inf_{y \in W^{1,1}(s,s+h), y(s) = w} I(y; s, s + h) \right).$$

We claim that the first infimum,

$$\inf_{w \in \mathbf{R}^N} \inf_{y \in W^{1,1}(0,s), y(0) = y_0, y(s) = w} I(y; 0, s), \tag{2.4}$$

is indeed a minimum. This is in fact an easy consequence of Lemma 2.1: in seeking the infimum in the interval $(0, s)$, feasible mappings cannot escape at infinity at the end point.

Let $w$ be the optimal final value in (2.4). Then

$$m(s+h) \leq m(s) + \inf_{y \in W^{1,1}(s,s+h), y(s)=w} I(y; s, s+h).$$

By choosing $y(t) = w + (t-s)z$ for $z$ such that $F(s, w, z) = 0$, we have that

$$m(s+h) \leq m(s) + \int_s^{s+h} |F(t, w + (t-s)z, z)| \, \mathrm{d}t.$$

By (2.3), conclude that

$$0 \leq \frac{m(s+h) - m(s)}{h} \leq \frac{1}{h} \min_{z: F(s,y,z)=0} \int_s^{s+h} |F(t, y + (t-s)z, z)| \, \mathrm{d}t \to 0, \text{ as } h \to 0.$$

Hence, the right derivative of $m$ vanishes for all time $s$, and therefore $m(s) \equiv 0$ for $s \in (0, T)$.

We now show that the vanishing infimum is in fact a minimum. Let $\{y_j\}$ be a minimizing sequence, and let $s \in (0, T)$ be given. Then, it is easy to write

$$C \int_s^{s+h} |y_j'(r)| \, \mathrm{d}r \leq \int_0^T |F(r, y_j(r), y_j'(r))| \, \mathrm{d}r + M_1 \int_s^{s+h} |y_j(r)| \, \mathrm{d}r + M_0 h.$$

We can further estimate

$$C \int_s^{s+h} |y_j'(r)| \, \mathrm{d}r \leq \epsilon_j + MRh$$

where

$$\epsilon_j = \int_0^T |F(r, y_j(r), y_j'(r))| \, \mathrm{d}r \to 0,$$

and $M$, and $R$ are uniform constants according to Lemma 2.1. Likewise

$$C \int_0^T |y_j'(s)| \, \mathrm{d}s \leq \epsilon_j + MRT.$$

Therefore, we conclude that (possibly for a subsequence), there is a feasible $y$ such that $y_j \rightharpoonup y$ in $W^{1,1}(0, T; \mathbf{R}^N)$, because, due to the preceding estimate on each subinterval $(s, s+h)$, there can be no jump on the derivatives. By the convexity of $F(t, y, z)$ with respect to $z$, and also of $|F(x, y, z)|$, we conclude that $I(y) = 0$ and so $y$ is a solution of our Cauchy problem. That each solution is Lipschitz comes directly from the fact that $F(t, y(t), y'(t)) = 0$ in $(0, T)$ together with the coercivity on $F$ and the uniform bound on $y$, imply a uniform bound on $y'$ as well. □

If we strengthen a bit more the hypotheses on $F$, then uniqueness of solutions can be shown. In addition, one can obtain global pointwise error estimates of approximations to the solution in terms of the integral error.

**Theorem 2.2.** *Suppose, in addition to the previous hypotheses on $F$, that there is a constant $D > 0$ such that*

$$|F(t, y_1 - y_2, z_1 - z_2)| \leq D \left( |F(t, y_1, z_1)| + |F(t, y_2, z_2)| + |y_1 - y_2| \right)$$

*for all $t$, $y_1$, $y_2$, $z_1$, $z_2$, and that $M_0$ can be taken to vanish ($M = M_1$). Then the solution of the Cauchy problem is unique. Even more, if $y_1(t)$ and $y_2(t)$ are two feasible mappings, then*

$$\|y_1 - y_2\|_{L^\infty(0,T)} \le \frac{D}{C}(I(y_1) + I(y_2))\mathrm{e}^{T(D+M)/C},$$
$$\|y_1' - y_2'\|_{L^1(0,T)} \le \left(\frac{D}{C} + \frac{TD(D+M)}{C^2}\mathrm{e}^{T(D+M)/C}\right)(I(y_1) + I(y_2)).$$

*Proof.* Let $y_1(t)$ and $y_2(t)$ be admissible mappings for our variational principle, and put $y = y_1 - y_2$. Then

$$|y(t)| \le \int_0^t |y'(r)|\,\mathrm{d}r \le \frac{1}{C}\int_0^T |F(r, y(r), y'(r))|\,\mathrm{d}r + \frac{M}{C}\int_0^t |y(r)|\,\mathrm{d}r. \tag{2.5}$$

By our new assumption

$$|y(t)| \le \frac{D}{C}(I(y_1) + I(y_2)) + \frac{D+M}{C}\int_0^t |y(r)|\,\mathrm{d}r.$$

Conclude by Gronwall's lemma that

$$|y(t)| \le \frac{D}{C}(I(y_1) + I(y_2))\mathrm{e}^{T(D+M)/C}.$$

If we take this estimate back to (2.5), we immediately obtain

$$\|y'\|_{L^1(0,T)} \le \left(\frac{D}{C} + \frac{TD(D+M)}{C^2}\mathrm{e}^{T(D+M)/C}\right)(I(y_1) + I(y_2)). \qquad \square$$

By taking $y_2 = y$ the unique solution of the Cauchy problem, we have global pointwise estimates as well as estimates of the error in $L^1$ for the derivatives.

**Corollary 2.1.** *Let $F$ be as in the previous theorem, and let $y$ be the unique solution for our problem. Then if $z$ is admissible, there exists a constant $R$ (independent of $z$) such that*

$$\|z - y\|_{L^\infty(0,T)} \le RI(z),$$
$$\|z' - y'\|_{L^1(0,T)} \le RI(z).$$

By using the same sort of ideas, similar results are also valid for exponents $p > 1$. We will explicitly refer to this, for the case $p = 2$, in Section 4.

## 3. Particular cases

It is interesting to notice that in the particular case

$$F(t, y, z) = z - f(t, y)$$

then a sufficient condition for all of our hypotheses to hold for $F$ is the typical uniform Lipschitz property for $f(t, y)$ in $y$, and continuity with respect to $t$ [2]. This is in fact elementary to check.

(1) $F$ is certainly measurable in $t$ and continuous in $(y, z)$.
(2) $F$ is linear in $z$.
(3) Coercivity is a direct consequence of the lipschitziannity. Notice that then we can have at most linear growth in $y$ for $f$.

(4) The only $z$ we can take in the minimum in condition (2.3) is obviously $z = f(s, y)$, thus

$$\begin{aligned} |F(t, y + (t-s)z, z)| &= |z - f(t, y + (t-s)z)| \\ &= |f(s, y) - f(t, y + (t-s)f(s, y))|, \end{aligned}$$

and

$$\begin{aligned} |F(t, y + (t-s)z, z)| &\le |f(s, y) - f(t, y)| + |f(t, y) - f(t, y + (t-s)f(s, y))| \\ &\le |f(s, y) - f(t, y)| + M|f(s, y)|(t-s) \end{aligned}$$

if $M$ is the Lipschitz constant. Then

$$\frac{1}{h}\int_s^{s+h} |F(t, y + (t-s)z, z)| \, \mathrm{d}t \le \sup_{t\in(s,s+h)} |f(s, y) - f(t, y)| + M|f(s, y)|\frac{h}{2},$$

and conclude that this quantity goes to zero as $h \to 0$.

(5) The condition for uniqueness is again straightforward from the Lipschitz property and the linear growth of $f$ on $y$.

Moreover, we have the estimates of Corollary 2.1 in this case: if $y$ is the unique solution of the Cauchy problem, then

$$\|z - y\|_{L^\infty(0,T)} \le R \int_0^T |z'(r) - f(r, z(r))| \, \mathrm{d}r,$$
$$\|z' - y'\|_{L^1(0,T)} \le R \int_0^T |z'(r) - f(r, z(r))| \, \mathrm{d}r.$$

We would like to focus next in a typical, autonomous differential inclusion [6]

$$y'(t) \in f(y(t)) \text{ in } (0, T), \quad y(0) = y_0.$$

Our main assumption on the set-valued map $f$ is

$$f(y) \subset B(M_1|y| + M_0)$$

is convex, for all $y \in \mathbf{R}^N$ where $M_1$ and $M_0$ are non-negative constants and $B(r)$ is the ball of radius $r$ centered at the origin in $\mathbf{R}^N$. Notice that no lipschitziannity is assumed on $f$.

**Theorem 3.1.** *Under this hypothesis on $f$, there are global, Lipschitz solutions to the differential inclusion*

$$y'(t) \in f(y(t)) \text{ in } (0, T), \quad y(0) = y_0.$$

*Proof.* Let $K \subset \mathbf{R}^N \times \mathbf{R}^N$ be the graph of $f$, and put

$$F(y, z) = C_z \operatorname{dist}((y, z), K),$$

where $C_z$ indicates the convexification with respect to the variable $z$. It is clear, because of the convexity of each set $f(y)$, that our differential inclusion is equivalent to the implicit differential equation

$$F(y(t), y'(t)) = 0 \text{ in } (0, T), \quad y(0) = y_0.$$

In fact, we have $\{F = 0\} = K$, because the convexification with respect to $z$ cannot enlarge the zero set if each $z$-slice of $K$ is already convex. It suffices to check the hypotheses to apply Theorem 2.1. The first three requirements are a direct consequence of our hypothesis on the set-valued map $f$. Concerning the fourth condition, notice that if $(y, z) \in K$, then

$$F(y + (t - s)z, z) \leq \mathrm{dist}\,((y + (t - s)z, z), K) \leq (t - s)|z|.$$

Condition (2.3) follows immediately. □

A similar result can be obtained for non-autonomous differential inclusions, by assuming our main assumption on $f$ uniformly for all $t$, and adding the continuity with respect to $t$.

It is interesting that in revisiting the case of an equation

$$y'(t) = f(y(t)) \text{ in } (0, T), \quad y(0) = y_0,$$

we can have an existence theorem (not uniqueness) without lipschitziannity.

**Corollary 3.1.** *Suppose that $f : (0, T) \times \mathbf{R}^N \to \mathbf{R}^N$ is continuous and*

$$|f(t, y)| \leq M_1|y| + M_0, \textit{ uniformly in } t \in (0, T).$$

*Then there are global, Lipschitz solutions for the corresponding Cauchy problem.*

Some controlled, appropriate dependence of the constants $M_i$ on $t$ can be allowed as well.

## 4. Some ideas on the numerical approximation

For simplicity we will focus on the problem

$$y'(t) = f(t, y(t)) \text{ in } (0, T), \quad y(0) = y_0, \tag{4.1}$$

although more general situations can also be considered as in the previous sections. We will assume uniform lipschitziannity in order to ensure uniqueness, and for practical reasons, we will deal with the equivalent variational problem

$$\text{Minimize in } y \in H^1(0, T): \quad I_2(y) = \frac{1}{2}\int_0^T |y'(t) - f(t, y(t))|^2 \,\mathrm{d}t \tag{4.2}$$

subject to $y(0) = y_0$.

We want to approximate the unique minimizer of this problem by calculating suboptimal solutions in appropriate finite dimensional subspaces of $H^1(0, T)$. In the same vein as in Section 2, we have the following lemma. Let $z$ be the solution of problem (4.1).

**Lemma 4.1.** *Let $y$ be a function in $H^1(0, T)$ complying with the initial condition $y(0) = y_0$. Then*

$$\|y - z\|_{L^\infty(0,T)} \leq \sqrt{2T}\mathrm{e}^{MT}\sqrt{I_2(y)},$$
$$\|y' - z'\|^2_{L^2(0,T)} \leq 2\left(2 + MT\left(\mathrm{e}^{2MT} - 1\right)\right) I_2(y).$$

*In particular, if $\{z_j\}$ is minimizing for (4.2), then*

$$z_j \to z \quad \textit{strong in } H^1(0, T).$$

Here $M$ is the Lipschitz constant. Notice that this lemma says that no matter how you find a good approximation to the solution, the global error can always be measured by the functional $I_2$. This is in fact a good test to ensure the goodness of approximations, and explains why this variational procedure can be understood as a natural adaptivity criterion.

In abstract terms, let us consider a finite dimensional subspace of dimension $n$ of the space $H^1(0,T)$ complying with the boundary condition $y(0) = y_0$. Let us designate such subspace $V_n$, and let $\pi_n$ stand for the natural projection of $H^1(0,T)$ onto $V_n$. We should have the approximation property that for any given $y \in H^1(0,T)$

$$\|\pi_n y - y\|_{H^1(0,T)} \to 0 \text{ as } n \to \infty.$$

Even more explicitly, we have suitable rates of convergence

$$\|\pi_n y - y\|_{H^1(0,T)} \le C(y)C_n, \quad C_n \to 0 \text{ as } n \to \infty, \tag{4.3}$$

where $C(y)$ is a constant depending on the initial $y$ but independent of $n$. Typically, we will have $C_n = 1/n^p$, $p \in \mathbb{N}$.

We focus on the discrete variational problem

$$\text{Minimize in } y \in V_n : \quad I_2(y). \tag{4.4}$$

Let $m_n \ge 0$ be the value of this minimum.

**Lemma 4.2.** *If $z$ is the solution of (4.1), and $M$ is the Lipschitz constant of $f$ with respect to $y$, then*

$$m_n \le MC(z)C_n,$$

*where $C(z)$ and $C_n$ are determined in (4.3).*

The proof is elementary.

The variational problem (4.4) is a finite-dimensional optimization problem. It may or may not have optimal solutions, but we can always find a sub-optimal solution. Let $y_n \in V_n$ be such that

$$I_2(y_n) \le 2m_n.$$

**Lemma 4.3.** *If $y_n$ is a sub-optimal solution for the appropriate discrete version of (4.2), then we have*

$$\|y_n - z\|_{H^1(0,T)} \le CC_n,$$

*where as before $C$ is a constant depending on $z$, $M$ and $T$, but independent of $n$. The constant $C_n$ represents the suitable approximation constant in (4.3). In addition we also get*

$$\|y_n - z\|_{L^\infty(0,T)} \le C\sqrt{C_n}.$$

This lemma can be regarded as the starting point of a whole practical philosophy to numerically approximate the solution of the Cauchy problem. Any specific, appropriate choice of the spaces $V_n$ and the way to find the approximations $z_n$ leads to concrete algorithms, all of which avoid propagation of errors as they are global in nature. Such a detailed and specific analysis exceeds the scope of this note. We will pursue this direction in the near future.

## 5. Conclusions

We have implemented some elementary ideas leading to various existence results for the Cauchy problem

$$F(t, y(t), y'(t)) = 0 \text{ in } (0, T), \quad y(0) = y_0.$$

Aside from the variational approach itself, we have isolated the condition

$$\frac{1}{h} \min_{z:F(s,y,z)=0} \int_s^{s+h} |F(t, y + (t-s)z, z)| \, \mathrm{d}t \to 0, \text{ as } h \to 0$$

for all $s$ and $y$, as a replacement for the typical Lipschitz requirement to have existence of solutions.

This variational approach also allows for a direct treatment of the numerical approximation of solutions. Its flexibility in accommodating additional constraints on the numerical schemes is most remarkable. We have found that procedures with prescribed qualitative properties can be directly implemented. For example, bounded, positive, monotone, convex, periodic schemes can be examined from this perspective. We plan to do so in a forthcoming work. Variable step size algorithms can be constructed as well. In some cases, classical implicit Runge-Kutta schemes can be recovered [5].

Since functional $I_2$ is a measure of the global error over the time interval $(0, T)$, minimizing $I_2(y)$ provides a natural adaptativity scheme that is tailored to find good approximations. It provides a faithful and easy-to-apply criterion to judge how good a given approximation is regardless of how it has been found.

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