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ASYMPTOTIC BEHAVIOR OF NONLINEAR SYSTEMS IN VARYING DOMAINS WITH BOUNDARY CONDITIONS ON VARYING SETS

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Abstract. For a fixed bounded open set $\Omega \subset \mathbb{R}^N$, a sequence of open sets $\Omega_n \subset \Omega$ and a sequence of sets $\Gamma_n \subset \partial\Omega \cap \partial\Omega_n$, we study the asymptotic behavior of the solution of a nonlinear elliptic system posed on Ω_n, satisfying Neumann boundary conditions on Γ_n and Dirichlet boundary conditions on $\partial\Omega_n \setminus \Gamma_n$. We obtain a representation of the limit problem which is stable by homogenization and we prove that this representation depends on Ω_n and Γ_n locally.

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1. Introduction

For a given Lipschitz bounded open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a sequence of open sets $\Omega_n \subset \Omega$ and a sequence of sets $\Gamma_n \subset \partial\Omega \cap \partial\Omega_n$, we study the asymptotic behavior of the solution u_n of the nonlinear elliptic system

$$\begin{cases} -\mathrm{div}\,(a(x, Du_n) - G_n) = g_n & \text{in } \Omega_n \\ u_n = 0 \quad \text{on } \partial\Omega_n \setminus \Gamma_n \\ (a(x, Du_n) - G_n)\,\nu = 0 \quad \text{on } \Gamma_n, \end{cases} \tag{1.1}$$

where $a : \Omega \times \mathbb{R}^{M\times N} \to \mathbb{R}^{M\times N}$, $M \geq 1$, is a Carathéodory function which satisfies standard assumptions so that the operator $v \in W_0^{1,p}(\Omega)^M \mapsto -\mathrm{div}\, a(x, Dv) \in W^{-1,p'}(\Omega)^M$, $p \geq 2$, defines a monotone operator in the sense of Leray and Lions [16] (see Sect. 2 for the precise assumptions on a) and ν denotes the unitary outward normal to Ω. The sequences g_n and G_n are assumed to converge in $L^{p'}(\Omega)^M$ weakly and $L^{p'}(\Omega)^{M\times N}$ strongly to some functions g and G respectively.

Assuming that $\|u_n\|_{W^{1,p}(\Omega_n)^M}$ is bounded (this holds for example if there exists $C > 0$ independent of n with $\|v\|_{W^{1,p}(\Omega_n)} \leq C\|\nabla v\|_{L^p(\Omega_n)^N}$, for every $v \in W^{1,p}(\Omega_n)$, $v = 0$ on $\partial\Omega_n \setminus \Gamma_n$) and extending u_n by zero outside Ω_n, we prove the existence of a nonnegative Borel measure μ in $\overline{\Omega}$ which does not charge sets of p-capacity zero, and a μ-Carathéodory function $F : \overline{\Omega} \times \mathbb{R}^M \to \mathbb{R}^M$ satisfying monotonicity and continuity properties related

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to those imposed to a (see (3.20), (3.21), (3.22)), such that u_n converges weakly in $W^{1,p}(\Omega)^M$ and strongly in $W^{1,q}(\Omega)^M$, $1 \leq q < p$, to the solution u of the problem

$$\left\{ \begin{array}{l} u \in W^{1,p}(\Omega)^M \cap L^p_\mu(\overline{\Omega})^M \\ \displaystyle\int_\Omega a(x, Du) : Dv \,\mathrm{d}x + \int_{\overline{\Omega}} F(x,u)v \,\mathrm{d}\mu = \int_\Omega gv \,\mathrm{d}x + \int_\Omega G : Dv \,\mathrm{d}x \\ \forall v \in W^{1,p}(\Omega)^M \cap L^p_\mu(\overline{\Omega})^M, \end{array} \right. \tag{1.2}$$

which (if μ is smooth) can be written as

$$\left\{ \begin{array}{ll} -\mathrm{div}\,(a(x,Du) - G) + F(x,u)\mu = g & \text{in } \Omega \\ \quad (a(x,Du) - G)\,\nu + F(x,u)\mu = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1.3}$$

The pair (F, μ) does not depend on g_n or G_n, and it depends on Ω_n and Γ_n locally in the sense that if we consider a Lipschitz open set $\omega \subset \Omega$ and we replace in (1.1) Ω_n by $\Omega_n \cap \omega$, Γ_n by $\Gamma_n \cap \overline{\omega}$, then the previous result holds with (F, μ) replaced by $(F_{|\bar{\omega}}, \mu_{|\bar{\omega}})$.

The term $F(x,u)\mu$ in (1.2) is similar to the *strange term* which appears in the homogenization of Dirichlet problems on varying domains (see [1,3–12,19,20]). In fact, if Γ_n is empty, our result follows from [5]. When Γ_n is not empty the main difference is that now μ is defined on $\bar{\Omega}$ and not only on Ω and then the term $F(x,u)\mu$ does not only appears in the equation but also in the boundary conditions of (1.3). Taking $\Omega = \Omega_n$ for every $n \in \mathbb{N}$, the above result proves that the boundary condition corresponding to the limit of a sequence of nonlinear elliptic systems with Dirichlet an Neumann conditions on varying subsets of $\partial\Omega$ is a Fourier-Robin condition. Indeed, the proof of this fact was the origin of the present work. We have preferred to present here the more general case where the open sets Ω_n are variable, in order to show that the homogenization of elliptic Dirichlet problems in varying domains (corresponding to $\Gamma_n = \emptyset$) and the homogenization of elliptic problems with Neumann and Dirichlet conditions imposed on varying sets of the boundary admit a common formulation.

As in the case of Dirichlet problems on varying domains [9], we observe that (1.1) can be written in such way that its structure is similar to (1.2). For this purpose, it is enough to define μ_n as (C_p stands for the p-capacity, see Sect. 2)

$$\mu_n(B) = \left\{ \begin{array}{ll} +\infty & \text{if } C_p(B \cap (\overline{\Omega} \setminus (\Omega_n \cup \Gamma_n))) > 0 \\ 0 & \text{if } C_p(B \cap (\overline{\Omega} \setminus (\Omega_n \cup \Gamma_n))) = 0, \end{array} \right. \qquad \forall B \subset \overline{\Omega} \text{ Borel}, \tag{1.4}$$

and $F_n : \overline{\Omega} \times \mathbb{R}^M \longrightarrow \mathbb{R}^M$ as, for example, $F_n(x,s) = |s|^{p-2}s$. Then (1.1) is equivalent to

$$\left\{ \begin{array}{l} u_n \in W^{1,p}(\Omega)^M \cap L^p_{\mu_n}(\overline{\Omega})^M \\ \displaystyle\int_\Omega a(x, Du_n) : Dv \,\mathrm{d}x + \int_{\overline{\Omega}} F_n(x,u_n)v \,\mathrm{d}\mu_n = \int_\Omega g_n v \,\mathrm{d}x + \int_\Omega G_n : Dv \,\mathrm{d}x \\ \forall v \in W^{1,p}(\Omega)^M \cap L^p_{\mu_n}(\overline{\Omega})^M. \end{array} \right. \tag{1.5}$$

Hence, we can consider (1.1) as a particular case of (1.2). For this reason, better than the homogenization of (1.1), we will study the homogenization of (1.5) for a sequence μ_n of Borel measures in $\overline{\Omega}$ (not necessarily defined from sequences Ω_n, Γ_n as above) which vanish on sets of p-capacity zero and a sequence $F_n : \overline{\Omega} \times \mathbb{R}^M \longrightarrow \mathbb{R}^M$ of monotone μ_n-measurable functions (see Sect. 2 for the precise hypotheses on F_n). We prove that, in this more general form, the problem is stable for homogenization, *i.e.* for every sequences μ_n and F_n there exist μ and F such that, at least for a subsequence, the limit problem of (1.5) is still given by (1.2).

Throughout the paper we just consider the case $p \geq 2$. The case $1 \leq p < 2$ can be treated in a similar way, after proper modification on the growth and coerciveness hypotheses for the functions a and F_n. The case of linear equations and μ_n concentrated on $\partial\Omega$ (which for problem (1.1) means $\Omega_n = \Omega$ for every $n \in \mathbb{N}$) has been studied in [2], see also [13] for related problems.

2. Notations and definitions

The minimum and the maximum of two numbers a, b are respectively denoted by $a \wedge b$, $a \vee b$.

The scalar product of two matrices A, $B \in \mathbb{R}^{M\times N}$ will be denoted by $A : B$.

For a Borel set $B \subset \mathbb{R}^N$ and a Borel measure μ in B, we denote by $L^q_\mu(B)$, $1 \leq q \leq +\infty$, the usual Lebesgue spaces with respect to the measure μ. If μ is the Lebesgue measure, we use the standard notation $L^q(B)$.

For every Lipschitz open set $O \subset \mathbb{R}^N$, we denote by $W^{1,q}(O)$, $1 \leq q \leq +\infty$, the usual Sobolev spaces. We recall that, since we are assuming O Lipschitz, the elements of $W^{1,q}(O)$ have a trace on ∂O and then, they are defined in $\overline{O}$. Moreover, $C^\infty(\overline{O})$ is dense in $W^{1,q}(O)$ if $q < +\infty$. For every subset Υ of ∂O, we define $W^{1,q}_\Upsilon(O)$ as the closure in $W^{1,q}(O)$ of the functions in $C^\infty(\overline{O})$ which vanish in a neighborhood of $\overline{\Upsilon}$. In the case $\Upsilon = \partial O$, we write $W^{1,q}_0(O)$ instead of $W^{1,q}_\Upsilon(O)$.

Along the paper we denote by p a fixed number such that $p \geq 2$. Also we consider a bounded Lipschitz open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and a bounded open set $\hat{\Omega}$, such that $\overline{\Omega} \subset \hat{\Omega}$.

We denote by $P : W^{1,p}(\Omega) \longrightarrow W^{1,p}_0(\hat{\Omega})$ a bounded linear operator such that

$$P(u) = u \text{ in } \Omega, \quad \forall u \in W^{1,p}(\Omega). \tag{2.6}$$

This operator is also chosen bounded from $W^{1,p}(\Omega)\cap L^\infty(\Omega)$ into $W^{1,p}_0(\hat{\Omega})\cap L^\infty(\hat{\Omega})$ and such that it transforms nonnegative functions into nonnegative functions. The existence of this extension operator is guaranteed because Ω is Lipschitz (see *e.g.* [17]).

For a Lipschitz open set $\omega \subset \Omega$, we denote

$$\mathcal{S}_\omega = \{\varphi : \ \varphi \in W^{1,\infty}(\omega), \quad \varphi = 0 \text{ in a neighborhood of } \overline{\partial\omega \cap \Omega}\}.$$

Also, we define the bounded linear operators $Z_\omega : W^{1,p}_{\partial\omega\cap\Omega}(\omega) \to W^{1,p}(\Omega)$, $Q_\omega : W^{1,p}_{\partial\omega\cap\Omega}(\omega) \to W^{1,p}_0(\hat{\Omega})$ as

$$Z_\omega(u) = \begin{cases} u & \text{in } \overline{\omega} \\ 0 & \text{in } \overline{\Omega}\setminus\overline{\omega}, \end{cases} \qquad Q_\omega = P \circ Z_\omega.$$

When $u = (u_1, \ldots, u_M)$ is vectorial, we denote

$$P(u) = (P(u_1), \ldots, P(u_M)), \quad Z_\omega(u) = (Z_\omega(u_1), \ldots, Z_\omega(u_M)),$$
$$Q_\omega(u) = (Q_\omega(u_1), \ldots, Q_\omega(u_M)).$$

For $E \subset \hat{\Omega}$ and $1 < p < +\infty$, the p-capacity of E in $\hat{\Omega}$, denoted by $C_p(E)$, is defined by

$$C_p(E) = \inf\left\{ \int_{\hat{\Omega}} |\nabla u|^p \,\mathrm{d}x : \ u \in W^{1,p}_0(\hat{\Omega}), u \geq 1 \text{ a.e. in a neighborhood of } E \right\}.$$

This definition depends on $\hat{\Omega}$, however the sets of p-capacity zero are independent of $\hat{\Omega}$.

We say that a property $\mathcal{P}(x)$ holds quasi everywhere (abbreviated as q.e.) in a set $B \subset \hat{\Omega}$ if it holds for all $x \in B \setminus N$, with $C_p(N) = 0$.

A function $u : \hat{\Omega} \longrightarrow \mathbb{R}$ is said to be quasi continuous if for every $\varepsilon > 0$ there exists a set $B \subset \hat{\Omega}$, with $C_p(B) < \varepsilon$, such that the restriction of u to $\hat{\Omega} \setminus B$ is continuous. It is well known (see *e.g.* [14,15,21]) that every $u \in W^{1,p}(\hat{\Omega})$ has a quasi continuous representative. We shall always identify $u \in W^{1,p}(\hat{\Omega})$ with this quasi continuous representative.

A subset O of $\hat{\Omega}$ is said to be quasi open if for every $\varepsilon > 0$ there exists $B \subset \hat{\Omega}$, with $C_p(B) < \varepsilon$, such that $O \cup B$ is open.

Following [8,9], for every Borel subset B of $\hat{\Omega}$, we denote by $\mathcal{M}_0^p(B)$ the class of all non negative Borel measures μ in B which vanish on Borel sets of p-capacity zero and satisfy the following condition

$$\mu(E) = \inf\{\mu(O \cap B) : \ O \text{ quasi open}, \quad E \subset O \subset \hat{\Omega}\}, \qquad \forall E \subset B \text{ Borel}. \tag{2.7}$$

We will denote by $a : \Omega \times \mathbb{R}^{M\times N} \longrightarrow \mathbb{R}^{M\times N}$ a Carathéodory function such that there exist two positive constants α, γ, and $r \in L^{\frac{p}{p-2}}(\Omega)$ satisfying

$$a(x, 0) = 0 \ \text{ a.e. } x \in \Omega, \tag{2.8}$$

$$(a(x, \xi_1) - a(x, \xi_2)) : (\xi_1 - \xi_2) \geq \alpha|\xi_1 - \xi_2|^p, \quad \forall \xi_1, \, \xi_2 \in \mathbb{R}^{M\times N}, \ \text{ a.e. } x \in \Omega, \tag{2.9}$$

$$|a(x, \xi_1) - a(x, \xi_2)| \leq \left(r(x) + \gamma(|\xi_1| + |\xi_2|)^{p-2}\right) |\xi_1 - \xi_2|, \quad \forall \xi_1, \, \xi_2 \in \mathbb{R}^{M\times N}, \text{ a.e. } x \in \Omega. \tag{2.10}$$

Observe that these hypotheses imply in particular that there exist $\beta > 0$ and $h \in L^{p'}(\Omega)$ such that

$$a(x, \xi) : \xi \geq \alpha|\xi|^p, \quad \forall \xi \in \mathbb{R}^{M\times N}, \text{ a.e. } x \in \Omega, \tag{2.11}$$

$$|a(x, \xi)| \leq h(x) + \beta|\xi|^{p-1}, \quad \forall \xi \in \mathbb{R}^{M\times N}, \text{ a.e. } x \in \Omega. \tag{2.12}$$

For every $n \in \mathbb{N}$, we will also consider $\mu_n \in \mathcal{M}_0^p(\overline{\Omega})$ and $F_n : \overline{\Omega} \times \mathbb{R}^M \longrightarrow \mathbb{R}^M$ such that

$$F_n(\cdot, s) \ \mu_n\text{-measurable}, \forall s \in \mathbb{R}^M, \tag{2.13}$$

$$F_n(x, 0) = 0, \quad \mu_n\text{-a.e. } x \in \overline{\Omega}, \tag{2.14}$$

$$(F_n(x, s_1) - F_n(x, s_2))(s_1 - s_2) \geq \alpha|s_1 - s_2|^p, \quad \forall s_1, \, s_2 \in \mathbb{R}^M, \quad \mu_n\text{-a.e. } x \in \overline{\Omega}, \tag{2.15}$$

$$|F_n(x, s_1) - F_n(x, s_2)| \leq \gamma(|s_1| + |s_2|)^{p-2}|s_1 - s_2|, \quad \forall s_1, \, s_2 \in \mathbb{R}^M, \quad \mu_n\text{-a.e. } x \in \overline{\Omega}. \tag{2.16}$$

Thus, for $\beta > 0$ as above, we have

$$F_n(x, s)s \geq \alpha|s|^p, \quad \forall s \in \mathbb{R}^M, \quad \mu_n\text{-a.e. } x \in \overline{\Omega}, \tag{2.17}$$

$$|F_n(x, s)| \leq \beta|s|^{p-1}, \quad \forall s \in \mathbb{R}^M, \quad \mu_n\text{-a.e. } x \in \overline{\Omega}. \tag{2.18}$$

Remark 2.1. To simplify the exposition we have considered $p \geq 2$. The case $1 < p < 2$ can be studied similarly after proper modification on the hypotheses on a and F_n. We can also consider hypotheses less restrictive than (2.10) and (2.16), assuming that $a(x, \xi)$ and $F_n(x, s)$ are locally Hölder continuous with respect to ξ and s respectively.

We denote by C a generic constant which does not depend on n and can change from line to line.

We denote by $O_{m,n}$ and O_n generic sequences of real numbers which can change from line to line and satisfy

$$\lim_{m\to\infty} \lim_{n\to\infty} |O_{m,n}| = 0, \quad \lim_{n\to\infty} O_n = 0.$$

3. Homogenization result

In this section we state the main results of the paper, relative to the homogenization problem (1.5).

Theorem 3.1. *Let a, μ_n and F_n be in the conditions of Section 2. Then, there exist a subsequence of n, still denoted by n, a measure $\mu \in \mathcal{M}_0^p(\overline{\Omega})$ and a function $F : \overline{\Omega} \times \mathbb{R}^M \longrightarrow \mathbb{R}^M$, with*

$$F(\cdot, s) \ \mu\text{-measurable}, \ \forall s \in \mathbb{R}^M, \tag{3.19}$$

$$F(x, 0) = 0, \quad \mu\text{-a.e. } x \in \overline{\Omega}, \tag{3.20}$$

$$|F(x, s_2) - F(x, s_1)| \le C_1(|s_1| + |s_2|)^{\frac{p(p-2)}{p-1}} |s_2 - s_1|^{\frac{1}{p-1}}, \quad \forall s_1, s_2 \in \mathbb{R}^M, \ \mu\text{-a.e. } x \in \overline{\Omega}, \tag{3.21}$$

$$(F(x, s_2) - F(x, s_1))(s_2 - s_1) \ge C_2 |s_2 - s_1|^p, \quad \forall s_1, s_2 \in \mathbb{R}^M, \ \mu\text{-a.e. } x \in \overline{\Omega}, \tag{3.22}$$

such that the following homogenization result holds: Let $\omega \subset \Omega$ be a Lipschitz open set and consider a sequence $g_n \in L^{p'}(\omega)^M$ which converges weakly in $L^{p'}(\omega)^M$ to a function g, a sequence $G_n \in L^{p'}(\omega)^{M\times N}$ which converges strongly in $L^{p'}(\omega)^{M\times N}$ to a function G and a sequence $u_n \in W^{1,p}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})$ which satisfies

$$\|u_n\|_{W^{1,p}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega} \setminus \overline{\partial\omega\cap\Omega})^M} \le C, \tag{3.23}$$

and

$$\begin{cases} \displaystyle\int_\omega a(x, Du_n) : Dv \,\mathrm{d}x + \int_{\overline{\omega}} F_n(x, u_n) v \,\mathrm{d}\mu_n = \int_\omega g_n v \,\mathrm{d}x + \int_\omega G_n : Dv \,\mathrm{d}x \\ \forall v \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega})^M. \end{cases} \tag{3.24}$$

Then, every cluster point u of u_n in the weak topology of $W^{1,p}(\omega)^M$ satisfies

$$\begin{cases} u \in W^{1,p}(\omega)^M \cap L^p_\mu(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M \\ \displaystyle\int_\omega a(x, Du) : Dv \,\mathrm{d}x + \int_{\overline{\omega}} F(x, u) v \,\mathrm{d}\mu = \int_\omega g v \,\mathrm{d}x + \int_\omega G : Dv \,\mathrm{d}x \\ \forall v \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_\mu(\overline{\omega})^M. \end{cases} \tag{3.25}$$

Moreover, the measure μ can be taken independently of a.

The proof of this result is carried on in Section 6. To do it, in Section 5 we consider a bounded open set $\hat{\Omega}$ with $\overline{\Omega} \subset \hat{\Omega}$ and then, for u_n and u as in the statement of Theorem 3.1, we estimate the difference between u_n and the corrector with limit u relative to the homogenization problem for the operator $v \to -\mathrm{div}\left(|\nabla v|^{p-2}\nabla v\right) + |v|^{p-2} v \,\mathrm{d}\mu_n$ in $\hat{\Omega}$ with Dirichlet conditions. The properties of this corrector will be recalled in Section 4. As a consequence we obtain some estimates for Du_n (Lems. 5.2 and 5.3) which allow us to prove (Prop. 5.4) the existence of $\mu \in \mathcal{M}_0^p(\overline{\Omega})$ and $T \in L^{p'}_\mu(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$ such that u belongs to $L^p_\mu(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$ and

$$\int_\omega a(x, Du) : Dv \,\mathrm{d}x + \int_{\overline{\omega}} T v \,\mathrm{d}\mu = \int_\omega g v \,\mathrm{d}x + \int_\omega G : Dv \,\mathrm{d}x,$$

for every $v \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_\mu(\overline{\omega})^M$ (see [8,10]). The estimates obtained in Section 5 prove that T is of the form $F(x, u(x))$ (estimate (5.60)), but for a function F only defined on the set of pairs $(x, v(x))$ such that v is the limit of some sequence v_n in the conditions of the sequence u_n which appears in the statement of Theorem 3.1. We will prove in Lemma 6.1 that the set of such functions v is large enough to allow us to define F in the whole of $\overline{\Omega} \times \mathbb{R}^M$ and then to conclude Theorem 3.1.

We will also prove in Section 6 the following consequence of Theorem 3.1.

Theorem 3.2. *Under the same assumptions that Theorem 3.1, the following results hold:*
(i) For every $\lambda > 0$, the unique solution u_n of

$$\begin{cases} u_n \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega})^M \\ \displaystyle\int_\omega a(x, Du_n) : Dv \,\mathrm{d}x + \lambda \int_\omega |u_n|^{p-2} u_n v \,\mathrm{d}x + \int_{\overline{\omega}} F_n(x, u_n) v \,\mathrm{d}\mu_n = \\ \qquad\qquad\qquad\qquad \displaystyle\int_\omega g_n v \,\mathrm{d}x + \int_\omega G_n : Dv \,\mathrm{d}x \\ \forall v \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega})^M, \end{cases} \tag{3.26}$$

converges weakly in $W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M$ and strongly in $W^{1,q}_{\partial\omega\cap\Omega}(\omega)^M$, $1 \le q < p$, to the unique solution u of

$$\begin{cases} u \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_{\mu}(\overline{\omega})^M \\ \displaystyle\int_\omega a(x, Du) : Dv \,\mathrm{d}x + \lambda \int_\omega |u|^{p-2} u v \,\mathrm{d}x + \int_{\overline{\omega}} F(x, u) v \,\mathrm{d}\mu = \int_\omega g v \,\mathrm{d}x + \int_\omega G : Dv \,\mathrm{d}x \\ \forall v \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_{\mu}(\overline{\omega})^M. \end{cases} \tag{3.27}$$

(ii) Assume that there exists (a Poincaré's constant) $C_P > 0$ such that

$$\|v\|_{L^p(\omega)} \le C_P \left(\|\nabla v\|^p_{L^p(\omega)^N} + \|v\|^p_{L^p_{\mu_n}(\overline{\omega})} \right)^{\frac{1}{p}}, \quad \forall v \in W^{1,p}_{\partial\omega\cap\Omega}(\omega) \cap L^p_{\mu_n}(\overline{\omega}),\ \forall n \in \mathbb{N}. \tag{3.28}$$

Then, the unique solution u_n of

$$\begin{cases} u_n \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega})^M \\ \displaystyle\int_\omega a(x, Du_n) : Dv \,\mathrm{d}x + \int_{\overline{\omega}} F_n(x, u_n) v \,\mathrm{d}\mu_n = \int_\omega g_n v \,\mathrm{d}x + \int_\omega G_n : Dv \,\mathrm{d}x \\ \forall v \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega})^M, \end{cases} \tag{3.29}$$

converges weakly in $W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M$ and strongly in $W^{1,q}_{\partial\omega\cap\Omega}(\omega)^M$, $1 \le q < p$, to the unique solution u of

$$\begin{cases} u \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_{\mu}(\overline{\omega})^M \\ \displaystyle\int_\omega a(x, Du) : Dv \,\mathrm{d}x + \int_{\overline{\omega}} F(x, u) v \,\mathrm{d}\mu = \int_\omega g v \,\mathrm{d}x + \int_\omega G : Dv \,\mathrm{d}x \\ \forall v \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_{\mu}(\overline{\omega})^M. \end{cases} \tag{3.30}$$

Remark 3.3. As we said in the Introduction, the homogenization of problem (3.29) gives in particular the homogenization of problem (1.1). For this, given a sequence of Lipschitz open sets $\Omega_n \subset \Omega$ and a sequence $\Gamma_n \subset \partial\Omega \cap \Omega_n$, we define a sequence of measures $\mu_n \in \mathcal{M}^p_0(\overline{\Omega})$ by (1.4). Then, problem (1.1), understood in the variational form

$$\begin{cases} u_n \in W^{1,p}(\Omega_n)^M, \quad u_n = 0 \text{ q.e. on } \partial\Omega_n \setminus \Gamma_n \\ \displaystyle\int_{\Omega_n} a(x, Du_n) : Dv \,\mathrm{d}x = \int_{\Omega_n} g_n v \,\mathrm{d}x + \int_{\Omega_n} G_n : Dv \,\mathrm{d}x \\ \forall v \in W^{1,p}(\Omega_n)^M, \quad v = 0 \text{ q.e. on } \partial\Omega_n \setminus \Gamma_n, \end{cases}$$

is equivalent to problem (3.29) with $\omega = \Omega$ and (for example) $F_n(x, s) = |s|^{p-2}s$. An interesting particular case is when $\Omega_n = \Omega$ for every $n \in \mathbb{N}$, *i.e.* when we have a nonlinear homogenization problem in a fixed bounded open set $\Omega \subset \mathbb{R}^N$, where we impose Dirichlet and Neumann conditions in varying subsets of the boundary.

In this case, by Proposition 4.4 the measure μ is supported on $\partial\Omega$, and thus, if μ is sufficiently smooth, the limit problem (3.30) is equivalent to the Fourier-Robin problem

$$\begin{cases} -\mathrm{div}\,(a(x,Du)-G)=g \quad \text{in } \Omega \\ (a(x,Du)-G)\,\nu+F(x,u)\mu=0 \quad \text{on } \partial\Omega. \end{cases}$$

Remark 3.4. Similarly to the homogenization of nonlinear Dirichlet problems in varying domains (see *e.g.* [5]), some properties on a and F_n are inherited by F. Namely, we have:

(i) If $a(x,\xi)$ is linear with respect to ξ and $F_n(x,s)$ is linear with respect to s, for every $n\in\mathbb{N}$ (so, $p=2$), then, $F(x,s)$ is linear with respect to s.

(ii) If a and F_n, $n\in\mathbb{N}$, satisfy the homogeneity assumption

$$a(x,\lambda\xi)=|\lambda|^{p-2}\lambda a(x,\xi),\quad \forall\lambda\in\mathbb{R},\ \forall\xi\in\mathbb{R}^{M\times N},\ \text{a.e. } x\in\Omega,$$

$$F_n(x,\lambda s)=|\lambda|^{p-2}\lambda F_n(x,s),\quad \forall\lambda\in\mathbb{R},\ \forall s\in\mathbb{R}^{M},\ \mu_n\text{-a.e. } x\in\overline{\Omega},$$

then F also satisfies

$$F(x,\lambda s)=|\lambda|^{p-2}\lambda F(x,s),\quad \forall\lambda\in\mathbb{R},\ \forall s\in\mathbb{R}^{M},\ \mu\text{-a.e. } x\in\overline{\Omega}.$$

The proof of these results is analogous to the corresponding one of Theorems 8.1 and 8.5 in [5] and follows from the fact that the functions q_n^m of Lemma 6.1 satisfy

$$(\lambda q_1+\tau q_2)_n^m=\lambda(q_1)_n^m+\tau(q_2)_n^m,\quad \forall q_1,\,q_2\in\mathbb{Q}^M,\ \forall\lambda,\,\tau\in\mathbb{R},$$

if we assume (i), and

$$(\lambda q)_n^m=\lambda(q)_n^m,\quad \forall q\in\mathbb{Q}^M,\ \forall\lambda\in\mathbb{R},$$

if we assume (ii). Thus the functions T_q defined by Lemma 6.1 satisfy

$$T_{\lambda q_1+\tau q_2}=\lambda T_{q_1}+\tau T_{q_2},\quad \forall q_1,\,q_2\in\mathbb{Q}^M,\ \forall\lambda,\,\tau\in\mathbb{R},$$

if we assume (i), and

$$T_{\lambda q}=|\lambda|^{p-2}\lambda T_q,\quad \forall q\in\mathbb{Q}^M,\ \forall\lambda\in\mathbb{R},$$

if we assume (ii).

4. Preliminaries

In this section we recall some results related to the homogenization of the p-Laplace operator with Dirichlet boundary conditions in varying domains. From them we will obtain other results we will use later.

Throughout this section, we consider a sequence $\hat{\mu}_n\in\mathcal{M}_0^p(\hat{\Omega})$ and we denote by w_n the solution of the problem

$$\begin{cases} w_n\in W_0^{1,p}(\hat{\Omega})\cap L^p_{\hat{\mu}_n}(\hat{\Omega}) \\ \displaystyle\int_{\hat{\Omega}}|\nabla w_n|^{p-2}\nabla w_n\nabla v\,\mathrm{d}x+\int_{\hat{\Omega}}|w_n|^{p-2}w_n v\,\mathrm{d}\hat{\mu}_n=\int_{\hat{\Omega}}v\,\mathrm{d}x \\ \forall v\in W_0^{1,p}(\hat{\Omega})\cap L^p_{\hat{\mu}_n}(\hat{\Omega}). \end{cases} \tag{4.31}$$

The following result has been proved in [8,10].

Proposition 4.1. *Let w_n be the sequence defined by (4.31). Then w_n is nonnegative q.e. in $\hat{\Omega}$ and its norm in $W_0^{1,p}(\hat{\Omega}) \cap L^\infty(\hat{\Omega}) \cap L^p_{\hat{\mu}_n}(\hat{\Omega})$ is bounded. Up to a subsequence, there exists a nonnegative function $w \in W_0^{1,p}(\hat{\Omega}) \cap L^\infty(\hat{\Omega})$, such that w_n converges weakly to w in $W_0^{1,p}(\hat{\Omega})$, strongly in $W_0^{1,q}(\hat{\Omega})$, $1 \le q < p$, and weakly-$*$ in $L^\infty(\hat{\Omega})$. Moreover, there exists a unique measure $\hat{\mu} \in \mathcal{M}_0^p(\hat{\Omega})$ such that w is the solution of*

$$\left\{\begin{array}{l} w \in W_0^{1,p}(\hat{\Omega}) \cap L^p_{\hat{\mu}}(\hat{\Omega}) \\ \displaystyle\int_{\hat{\Omega}} |\nabla w|^{p-2}\nabla w \nabla v \, \mathrm{d}x + \int_{\hat{\Omega}} |w|^{p-2} w v \, \mathrm{d}\hat{\mu} = \int_{\hat{\Omega}} v \, \mathrm{d}x \\ \forall v \in W_0^{1,p}(\hat{\Omega}) \cap L^p_{\hat{\mu}}(\hat{\Omega}). \end{array}\right. \tag{4.32}$$

The interest of w_n is that for a function ψ smooth enough, the sequence $w_n\psi$ provides a corrector with limit $w\psi$ relative to the homogenization problem for the operator $v \to -\mathrm{div}\left(|\nabla v|^{p-2}\nabla v\right) + |v|^{p-2} v \, \mathrm{d}\mu_n$ in $\hat{\Omega}$ with Dirichlet conditions [8,10]. The following properties of w_n, w and $\hat{\mu}$ have been proved in [8,10] (see also [5]).

Proposition 4.2. *The sequence of solutions w_n of (4.31), the function w and the measure $\hat{\mu}$ given by Proposition 4.1 satisfy*

(a) *For every Borel set $B \subset \hat{\Omega}$ with $C_p(B \cap \{w = 0\}) > 0$, it holds $\hat{\mu}(B) = +\infty$.*

(b) *The set $\{w\varphi : \ \varphi \in C_c^\infty(\hat{\Omega})\}$ is dense in $W_0^{1,p}(\hat{\Omega}) \cap L^p_{\hat{\mu}}(\hat{\Omega})$.*

(c) *For every $\psi, \varphi \in W^{1,p}(\hat{\Omega}) \cap L^\infty(\hat{\Omega})$, we have*

$$\lim_{n\to\infty} \left(\int_{\hat{\Omega}} |\nabla((w_n - w)\psi)|^p \varphi \, \mathrm{d}x + \int_{\hat{\Omega}} |w_n\psi|^p \varphi \, \mathrm{d}\hat{\mu}_n \right) = \int_{\hat{\Omega}} |w\psi|^p \varphi \, \mathrm{d}\hat{\mu}. \tag{4.33}$$

(d) *If v_n is a sequence in $W^{1,p}(\hat{\Omega})$ which converges weakly in $W^{1,p}(\hat{\Omega})$ to a function v, then it holds*

$$\liminf_{n\to\infty} \left(\int_{\hat{\Omega}} |\nabla(v_n - v)|^p \, \mathrm{d}x + \int_{\hat{\Omega}} |v_n|^p \, \mathrm{d}\hat{\mu}_n \right) \ge \int_{\hat{\Omega}} |v|^p \, \mathrm{d}\hat{\mu}. \tag{4.34}$$

In particular, if $\|v_n\|_{L^p_{\hat{\mu}_n}(\hat{\Omega})}$ is bounded, the function v is in $L^p_{\hat{\mu}}(\hat{\Omega})$.

Remark 4.3. We will apply the previous results to the sequence $\hat{\mu}_n$ defined by (see Sect. 2)

$$\hat{\mu}_n(B) = \mu_n(B \cap \overline{\Omega}), \quad \forall B \subset \hat{\Omega} \text{ Borel.} \tag{4.35}$$

From Proposition 4.2, every function $v \in L^p_{\hat{\mu}_n}(\hat{\Omega})$ vanishes q.e. on $\{w_n = 0\}$. So, although in Section 2 we have considered F_n defined on $\hat{\Omega} \times \mathbb{R}^M$, only its values in $\{w_n > 0\} \times \mathbb{R}^M$ are relevant.

In the present paper, we are interested in a sequence of measures $\hat{\mu}_n$ having their supports contained in a fixed closed set (see (4.35)). We will use the following result.

Proposition 4.4. *If there exists a compact set $K \subset \hat{\Omega}$ such that* $\mathrm{supp}\,(\hat{\mu}_n) \subset K$*, for every $n \in \mathbb{N}$, then the measure $\hat{\mu}$ given by Proposition 4.1 also satisfies* $\mathrm{supp}\,(\hat{\mu}) \subset K$.

Proof. Since $-\mathrm{div}(|\nabla w_n|^{p-2}\nabla w_n) = 1$ in $\hat{\Omega} \setminus K$, and w_n converges weakly to w in $W_0^{1,p}(\hat{\Omega})$ and strongly in $W_0^{1,q}(\hat{\Omega})$, $1 \le q < p$, we deduce that

$$-\,\mathrm{div}(|\nabla w|^{p-2}\nabla w) = 1 \ \text{ in } \hat{\Omega} \setminus K. \tag{4.36}$$

Then, taking in (4.32) $v = w\varphi$, with $\varphi \in C_c^\infty(\hat{\Omega} \setminus K)$, we get

$$\int_{\hat{\Omega}} |w|^p \varphi \, \mathrm{d}\hat{\mu} = 0, \quad \forall \varphi \in C_c^\infty(\hat{\Omega} \setminus K). \tag{4.37}$$

On the other hand, from $w \geq 0$ q.e. in $\hat{\Omega}$, (4.36) and the strong maximum principle for the p-Laplace operator (see *e.g.* [18]), we deduce $w > 0$ in $\hat{\Omega} \setminus K$. Together with (4.37), this implies that the support of $\hat{\mu}$ is contained in K. □

Better than Proposition 4.2 (b) and (d), we will use the following results.

Proposition 4.5. *Assume that the support of the measure $\hat{\mu}$ given by Proposition 4.1 is contained in $\overline{\Omega}$ and let ω be a Lipschitz open subset of Ω. Then, the set*

$$D_\omega = \{w\varphi : \ \varphi \in \mathcal{S}_\omega \cap C^\infty(\overline{\varpi})\}, \tag{4.38}$$

is dense in $W^{1,p}_{\partial\omega\cap\Omega}(\omega) \cap L^p_{\hat{\mu}}(\overline{\varpi})$.

Proof. First of all, we remark that for every u in $W^{1,p}_{\partial\omega\cap\Omega}(\omega) \cap L^p_{\hat{\mu}}(\overline{\varpi})$, there exist a sequence $u_n \in W^{1,p}_{\partial\omega\cap\Omega}(\omega) \cap L^p_{\hat{\mu}}(\overline{\varpi})$ and $O_n \subset \hat{\Omega}$ open, with $\overline{\partial\omega \cap \Omega} \subset O_n$, such that $u_n = 0$ q.e. in $O_n \cap \overline{\varpi}$ and u_n converges to u in $W^{1,p}_{\partial\omega\cap\Omega}(\omega) \cap L^p_{\hat{\mu}}(\overline{\varpi})$. For this purpose, we use that by definition of $W^{1,p}_{\partial\omega\cap\Omega}(\omega)$, there exists a sequence $O_n \subset \hat{\Omega}$ open, with $\overline{\partial\omega \cap \Omega} \subset O_n$ and $\psi_n \in C^\infty(\overline{\varpi})$, with $\psi_n = 0$ in $O_n \cap \overline{\varpi}$, such that ψ_n converges to u in $W^{1,p}(\omega)$. Then we take

$$u_n = (\psi_n \wedge u^+)^+ - ((-\psi_n) \wedge u^-)^+.$$

In order to prove Proposition 4.5, it is then enough to show that for every $u \in W^{1,p}_{\partial\omega\cap\Omega}(\omega) \cap L^p_{\hat{\mu}}(\overline{\varpi})$ such that there exits an open set $O \subset \hat{\Omega}$ with $\overline{\partial\omega \cap \Omega} \subset O$, $u = 0$ q.e. in $O \cap \overline{\varpi}$, and for every $\varepsilon > 0$, there exists $\varphi \in C^\infty_c(\hat{\Omega})$ which vanishes in a neighborhood of $\overline{\partial\omega \cap \Omega}$ such that

$$\|u - w\varphi\|_{W^{1,p}(\omega)\cap L^p_{\hat{\mu}}(\overline{\varpi})} < \varepsilon. \tag{4.39}$$

Using a regularization by convolution, it is enough to prove (4.39) for φ in $W^{1,p}_0(\hat{\Omega}) \cap L^\infty(\hat{\Omega})$ which vanishes on a neighborhood of $\overline{\partial\omega \cap \Omega}$.

Given u, O and ε as above, we observe that $Z_\omega(u)$ is in $W^{1,p}(\Omega) \cap L^p_{\hat{\mu}}(\overline{\Omega})$ and, since $\text{supp}(\hat{\mu}) \subset \overline{\Omega}$, we have that $Q_\omega(u) \in W^{1,p}_0(\hat{\Omega}) \cap L^p_{\hat{\mu}}(\hat{\Omega})$ (P, Z_ω and Q_ω are defined in Sect. 2). Thus, taking an open set O' with $\overline{\partial\omega \cap \Omega} \subset O'$, $\overline{O'} \subset O$ and a function $\psi \in C^\infty(\hat{\Omega})$, with $\psi = 1$ in $\hat{\Omega} \setminus O$, $\psi = 0$ in O', the function $u^* = Q_\omega(u)\psi$ is in $W^{1,p}_0(\hat{\Omega}) \cap L^p_{\hat{\mu}}(\hat{\Omega})$, vanishes in O' and is equal to u q.e. in $\overline{\varpi}$.

We define $\mu^* \in \mathcal{M}^p_0(\hat{\Omega})$ by

$$\mu^*(B) = \begin{cases} \hat{\mu}(B) & \text{if } C_p(B \cap O') = 0 \\ +\infty & \text{if } C_p(B \cap O') > 0, \end{cases} \qquad \forall B \subset \hat{\Omega} \text{ Borel},$$

and we observe that since $u^* = 0$ in O' then $u^* \in L^p_{\mu^*}(\hat{\Omega})$. From Proposition 4.2(b) applied to μ^* we derive that taking w^* as the solution of

$$\begin{cases} w^* \in W^{1,p}_0(\hat{\Omega}) \cap L^p_{\mu^*}(\hat{\Omega}) \\ \displaystyle\int_{\hat{\Omega}} |\nabla w^*|^{p-2}\nabla w^* \nabla v \,\mathrm{d}x + \int_{\hat{\Omega}} |w^*|^{p-2} w^* v \,\mathrm{d}\mu^* = \int_{\hat{\Omega}} v \,\mathrm{d}x \\ \forall v \in W^{1,p}_0(\hat{\Omega}) \cap L^p_{\mu^*}(\hat{\Omega}), \end{cases} \tag{4.40}$$

there exists $\varphi^* \in C^\infty_c(\hat{\Omega})$ such that

$$\|u^* - w^*\varphi^*\|_{W^{1,p}_0(\hat{\Omega})\cap L^p_{\mu^*}(\hat{\Omega})} < \varepsilon.$$

Using then that $(w^*-\eta)^+\varphi^*$ converges to $w^*\varphi^*$ in $W_0^{1,p}(\hat\Omega)\cap L^p_{\mu^*}(\hat\Omega)$ when η tends to zero from the right, we also have, for $\eta>0$ small enough

$$\|u^*-(w^*-\eta)^+\varphi^*\|_{W_0^{1,p}(\hat\Omega)\cap L^p_{\mu^*}(\hat\Omega)}<\varepsilon. \tag{4.41}$$

Since u^* and $(w^*-\eta)^+$ vanish in O', this also holds replacing μ^* by $\hat\mu$.

From $\hat\mu\le\mu^*$, the comparison principle (see Prop. 1.5 in [8]) proves that $0\le w^*\le w$ and thus,

$$w\frac{(w^*-\eta)^+}{w\vee\eta}\varphi^*=(w^*-\eta)^+\varphi^*.$$

Taking then

$$\varphi=\frac{(w^*-\eta)^+}{w\vee\eta}\varphi^*\in W^{1,p}_{\partial\omega\cap\Omega}(\omega)\cap L^\infty(\omega),$$

and using that w^* and then φ vanishes in O', we conclude from (4.41) the proof of Proposition 4.5. □

Proposition 4.6. *Assume that in Proposition 4.1* $\operatorname{supp}(\hat\mu_n)\subset\overline{\Omega}$*, for every* $n\in\mathbb{N}$*, and consider an open Lipschitz subset* ω *of* Ω*. Then for every sequence* $v_n\in W^{1,p}(\omega)$*, which converges weakly in* $W^{1,p}(\omega)$ *to a function* v *and every* $\psi\in\mathcal{S}_\omega$*, we have*

$$\liminf_{n\to\infty}\left(\int_\omega|\nabla(v_n-v)|^p|\psi|^p\,\mathrm{d}x+\int_{\overline{\omega}}|v_n|^p|\psi|^p\,\mathrm{d}\hat\mu_n\right)\ge\frac{1}{\|P\|}\int_{\overline{\omega}}|v|^p|\psi|^p\,\mathrm{d}\hat\mu. \tag{4.42}$$

Proof. Similarly to the proof of Proposition 4.5, we remark that thanks to $\operatorname{supp}(\hat\mu_n)\subset\overline{\Omega}$, we have that

$$\int_{\overline{\omega}}|v_n|^p|\psi|^p\,\mathrm{d}\hat\mu_n=\int_{\hat\Omega}|Q_\omega(v_n\psi)|^p\,\mathrm{d}\hat\mu_n.$$

So, using the convexity of the function $\xi\in\mathbb{R}^N\to|\xi|^p\in\mathbb{R}$, Rellich-Kondrachov's compactness theorem, $\|P\|\ge 1$, $\|Z_\omega\|=1$ and (4.34), we obtain

$$\begin{aligned}
&\int_\omega|\nabla(v_n-v)|^p|\psi|^p\,\mathrm{d}x+\int_{\overline{\omega}}|v_n|^p|\psi|^p\,\mathrm{d}\hat\mu_n\ge\\
&\|(v_n-v)\psi\|^p_{W^{1,p}(\omega)}-p\int_\omega|\nabla((v_n-v)\psi)|^{p-2}\nabla((v_n-v)\psi)\nabla\psi\,(v_n-v)\,\mathrm{d}x\\
&-\int_\omega|(v_n-v)\psi|^p\,\mathrm{d}x+\int_{\overline{\omega}}|v_n|^p|\psi|^p\,\mathrm{d}\hat\mu_n\\
&=\|(v_n-v)\psi\|^p_{W^{1,p}(\omega)}+\int_{\overline{\omega}}|v_n|^p|\psi|^p\,\mathrm{d}\hat\mu_n+O_n\\
&\ge\frac{1}{\|P\|}\int_{\hat\Omega}|\nabla Q_\omega((v_n-v)\psi)|^p\,\mathrm{d}x+\int_{\hat\Omega}|Q_\omega(v_n\psi)|^p\,\mathrm{d}\hat\mu_n+O_n\\
&\ge\frac{1}{\|P\|}\left(\int_{\hat\Omega}|\nabla Q_\omega((v_n-v)\psi)|^p\,\mathrm{d}x+\int_{\hat\Omega}|Q_\omega(v_n\psi)|^p\,\mathrm{d}\hat\mu_n\right)+O_n\\
&\ge\frac{1}{\|P\|}\int_{\hat\Omega}|Q_\omega(v\psi)|^p\,\mathrm{d}\hat\mu+O_n=\frac{1}{\|P\|}\int_{\overline{\omega}}|v|^p|\psi|^p\,\mathrm{d}\hat\mu+O_n.
\end{aligned}$$

This proves (4.42). □

5. Estimates and a local first representation of the limit problem

For Ω, a, μ_n and F_n as in Section 2 and a Lipschitz open subset ω of Ω, we will consider along this section a sequence $u_n \in W^{1,p}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$ which satisfies (3.23) and (3.24), with g_n converging weakly in $L^{p'}(\omega)^M$ to a function g and G_n converging strongly in $L^{p'}(\omega)^{M\times N}$ to a function G. Thanks to Proposition 4.1 applied to the sequence $\hat{\mu}_n$ defined as

$$\hat{\mu}_n(B) = \mu_n(B \cap \overline{\Omega}), \qquad \forall B \subset \hat{\Omega} \text{ Borel}, \tag{5.43}$$

we can also assume that there exist w and $\hat{\mu}$ in the conditions of this proposition. By Proposition 4.4, the support of this measure $\hat{\mu}$ is contained in $\overline{\Omega}$. The restriction of $\hat{\mu}$ to $\overline{\Omega}$ will be denoted by μ.

Thanks to (3.23), we can also assume that there exists $u \in W^{1,p}(\omega)^M$ such that u_n converges weakly to u in $W^{1,p}(\omega)^M$. Since by (3.23) and (4.42), the function u satisfies

$$\int_{\hat{\omega}} |u|^p |\psi|^p \,\mathrm{d}\mu \le C\|\psi\|_{L^\infty(\omega)}, \quad \forall \psi \in S_\omega,$$

we also get from the monotone convergence theorem that u satisfies

$$u \in L^p_\mu(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M. \tag{5.44}$$

Our purpose in the present section is to obtain some estimates for the sequence u_n. As a consequence, we will obtain a first representation for the problem satisfied by the function u (limit problem). The fact of considering ω instead of Ω will allow us to prove the local character of the limit problem.

In order to study the asymptotic behavior of u_n, we start with the following result, which follows from Proposition 5.4 in [5].

Proposition 5.1. *The sequence u_n considered above converges to u strongly in $W^{1,q}(\omega)^M$, $1 \le q < p$ and therefore Du_n converges in measure in ω.*

As a consequence, we have the following lemma.

Lemma 5.2. *The following convergences hold*

$$a(x, Du_n) \to a(x, Du) \;\; \textit{strongly in } L^r(\omega)^{M\times N},\ 1 \le r < p',\ \textit{and weakly in } L^{p'}(\omega)^{M\times N}, \tag{5.45}$$

$$|a(x, Du_n) - a(x, D(u_n - u))| \to |a(x, Du)| \;\textit{strongly in } L^{p'}(\omega). \tag{5.46}$$

Proof. From Proposition 5.1, (2.10) and (2.12), we easily derive (5.45).

To prove (5.46) we use that Proposition 5.1 and the inequality

$$|a(x, Du_n) - a(x, D(u_n - u))| \le \gamma(r + |Du_n| + |D(u_n - u)|)^{p-2}|Du| \;\text{ a.e. in } \Omega,$$

show that $|a(x, Du_n) - a(x, D(u_n - u))|$ converges in measure to $|a(x, Du)|$ and its p'-th power is equiintegrable. This implies (5.46). □

For u_n and z_n satisfying similar conditions to u_n, the following lemma provides an estimate for $D(u_n - z_n)$. The idea is to study the difference $u_n - z_n - w_n\psi_m$, where ψ_m is such that $w\psi_m$ is close to $u - z$ (z is the limit of z_n). Recall that $w_n\psi_m$ is the corrector with limit $w\psi_m \sim u - z$ relative to the homogenization problem for the operator $v \to -\mathrm{div}\left(|\nabla v|^{p-2}\nabla v\right) + |v|^{p-2}v\,\mathrm{d}\mu_n$ in $\hat{\Omega}$ with Dirichlet conditions.

Lemma 5.3. *There exists $C > 0$ which only depends on α and γ such that for every $\varphi \in \mathcal{S}_\omega$, $\varphi \ge 0$ in ω, we have*

$$\limsup_{n\to\infty} \left(\int_\omega |D(u_n - u)|^p \varphi \,\mathrm{d}x + \int_{\overline{\omega}} |u_n|^p \varphi \,\mathrm{d}\mu_n \right) \le C \int_{\overline{\omega}} |u|^p \varphi \,\mathrm{d}\mu. \tag{5.47}$$

Moreover, if besides u_n, u, g_n, g, G_n and G, we consider $z_n \in W^{1,p}(\omega)^M \cap L^p_{\mu_n}(\varpi \setminus \overline{\partial\omega \cap \Omega})^M$ with $\|z_n\|_{W^{1,p}(\omega)^M \cap L^p_{\mu_n}(\varpi\setminus\overline{\partial\omega\cap\Omega})^M}$ bounded, converging weakly in $W^{1,p}(\omega)^M$ to a function $z \in W^{1,p}(\omega)^M \cap L^p_\mu(\varpi \setminus \overline{\partial\omega \cap \Omega})^M$, such that there exist h_n converging weakly in $L^{p'}(\omega)^M$ to a function h and H_n converging strongly in $L^{p'}(\omega)^{M\times N}$ to a function H with

$$\begin{cases} \displaystyle\int_\omega a(x, Dz_n) : Dv \,\mathrm{d}x + \int_\varpi F_n(x, z_n)v \,\mathrm{d}\mu_n = \int_\omega h_n v \,\mathrm{d}x + \int_\omega H_n : Dv \,\mathrm{d}x \\ \forall v \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_{\mu_n}(\varpi)^M, \end{cases} \tag{5.48}$$

then, for every $\varphi \in \mathcal{S}_\omega$, $\varphi \geq 0$ in ω, we have

$$\limsup_{n\to\infty} \left(\int_\omega |D(u_n - z_n - u + z)|^p \varphi \,\mathrm{d}x + \int_\varpi |u_n - z_n|^p \varphi \,\mathrm{d}\mu_n \right) \leq C \int_\varpi (|u| + |z|)^{\frac{p(p-2)}{p-1}} |u - z|^{\frac{p}{p-1}} \varphi \,\mathrm{d}\mu. \tag{5.49}$$

Proof. For $\varphi \in \mathcal{S}_\omega$, $\varphi \geq 0$ in ω, we consider $\phi \in \mathcal{S}_\omega$, such that $\phi = 1$ in $\mathrm{supp}(\varphi)$. Then, $(u - z)\phi \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_\mu(\varpi)^M$ and so, from Proposition 4.5, there exists $\psi_m \in S^M_\omega$ such that $w\psi_m$ converges to $(u - z)\phi$ in $W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_\mu(\varpi)^M$. Taking $(u_n - z_n - w_n\psi_m)\varphi$ as test function in the difference of (3.24) and (5.48), we get

$$\int_\omega [a(x, Du_n) - a(x, Dz_n)] : D(u_n - z_n - w_n\psi_m)\varphi \,\mathrm{d}x + \int_\omega [a(x, Du_n) - a(x, Dz_n)] : [(u_n - z_n - w_n\psi_m) \otimes \nabla\varphi]\mathrm{d}x$$
$$+ \int_\varpi [F_n(x, u_n) - F_n(x, z_n)](u_n - z_n - w_n\psi_m)\varphi \,\mathrm{d}\mu_n = \int_\omega (g_n - h_n)(u_n - z_n - w_n\psi_m)\varphi \,\mathrm{d}x$$
$$+ \int_\omega (G_n - H_n) : D((u_n - z_n - w_n\psi_m)\varphi)\mathrm{d}x.$$

Applying (5.46) to u_n and z_n and using that $(u_n - z_n - w_n\psi_m)$ converges weakly in $W^{1,p}(\omega)^M$, and then strongly in $L^p(\omega)^M$, to $(u - z)(1 - \phi)$ when n and then m tends to infinity, $g_n - h_n$ converges weakly in $L^{p'}(\omega)^M$, $G_n - H_n$ converges strongly in $L^{p'}(\omega)^{M\times N}$ and $\phi = 1$ in $\mathrm{supp}(\varphi)$, the above equality gives

$$\int_\omega [a(x, D(u_n - u)) - a(x, D(z_n - z))] : D(u_n - z_n - w_n\psi_m)\varphi \,\mathrm{d}x$$
$$+ \int_\varpi [F_n(x, u_n) - F_n(x, z_n)](u_n - z_n - w_n\psi_m)\varphi \,\mathrm{d}\mu_n = O_{m,n},$$

or

$$\int_\omega [a(x, D(u_n - u)) - a(x, D(z_n - z))] : D(u_n - z_n - u + z)\varphi \,\mathrm{d}x + \int_\varpi [F_n(x, u_n) - F_n(x, z_n)](u_n - z_n)\varphi \,\mathrm{d}\mu_n =$$
$$\int_\omega [a(x, D(u_n - u)) - a(x, D(z_n - z))] : D(w_n\psi_m - u + z)\varphi \,\mathrm{d}x + \int_\varpi [F_n(x, u_n) - F_n(x, z_n)]w_n\psi_m\varphi \,\mathrm{d}\mu_n + O_{m,n}.$$

By (2.9), (2.10), $|D(u_n - z_n - u + z)|\varphi$ converging weakly to zero in $L^{p'}(\omega)$ (use Lem. 5.1), (2.15) and (2.16), this gives

$$\alpha \int_\omega |D(u_n - u - z_n + z)|^p \varphi \,\mathrm{d}x + \alpha \int_{\overline{\varpi}} |u_n - z_n|^p \varphi \,\mathrm{d}\mu_n \leq$$
$$\gamma \int_\omega [|D(u_n - u)| + |D(z_n - z)|]^{p-2} |D(u_n - z_n - u + z)||D((w_n - w)\psi_m + w\psi_m - u + z)|\varphi \,\mathrm{d}x$$
$$+ \gamma \int_{\overline{\varpi}} (|u_n| + |z_n|)^{p-2} |u_n - z_n||w_n\psi_m|\varphi \,\mathrm{d}\mu_n + O_{m,n}.$$

Young's and Hölder's inequality allows us to write

$$\int_\omega |D(u_n - u - z_n + z)|^p \varphi \,\mathrm{d}x + \int_{\overline{\varpi}} |u_n - z_n|^p \varphi \,\mathrm{d}\mu_n \leq$$
$$C \left(\int_\omega (|D(u_n - u)| + |D(z_n - z)|)^p \varphi \,\mathrm{d}x + \int_{\overline{\varpi}} (|u_n| + |z_n|)^p \varphi \,\mathrm{d}\mu_n \right)^{\frac{p-2}{p-1}}$$
$$\times \left(\int_\omega |D((w_n - w)\psi_m)|^p \varphi \,\mathrm{d}x + \int_{\overline{\varpi}} |w_n\psi_m|^p \varphi \,\mathrm{d}\hat{\mu}_n \right)^{\frac{1}{p-1}} + O_{m,n}$$
$$\leq C \left(\int_\omega (|D(u_n - u)| + |D(z_n - z)|)^p \varphi \,\mathrm{d}x + \int_{\overline{\varpi}} (|u_n| + |z_n|)^p \varphi \,\mathrm{d}\mu_n \right)^{\frac{p-2}{p-1}}$$
$$\times \left(\int_{\hat{\Omega}} |D\big((w_n - w)Q_\omega(\psi_m)\big)|^p Q_\omega(\varphi) \,\mathrm{d}x + \int_{\hat{\Omega}} |w_n Q_\omega(\psi_m)|^p Q_\omega(\varphi) \mathrm{d}\hat{\mu}_n \right)^{\frac{1}{p-1}} + O_{m,n}. \quad (5.50)$$

Taking $z_n = z = 0$ in (5.50), and using (4.33), $\mathrm{supp}(\hat{\mu}) \subset \overline{\Omega}$ and μ equals to the restriction of $\hat{\mu}$ in $\overline{\Omega}$, we deduce (5.47).

Finally, to obtain (5.49), it is enough to apply (4.33) in (5.50) together with the estimation (5.47) for u_n and z_n. □

Using Lemma 5.3, we will now obtain a first version of the limit problem satisfied by u.

Proposition 5.4. *There exists $T \in L^{p'}_\mu(\overline{\varpi} \setminus \overline{\partial\omega \cap \Omega})^M$ such that u satisfies*

$$\begin{cases} u \in W^{1,p}(\omega)^M \cap L^p_\mu(\overline{\varpi} \setminus \overline{\partial\omega \cap \Omega})^M \\ \displaystyle\int_\omega a(x, Du) : Dv \,\mathrm{d}x + \int_{\overline{\varpi}} Tv \,\mathrm{d}\mu = \int_\omega gv \,\mathrm{d}x + \int_\omega G : Dv \,\mathrm{d}x \\ \forall v \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_\mu(\overline{\varpi})^M. \end{cases} \quad (5.51)$$

The function T is such that for every $\psi \in \mathcal{S}^M_\omega$, we have

$$\int_{\overline{\varpi}} Tw\psi \,\mathrm{d}\mu = \lim_{n\to\infty} \left(\int_\omega a(x, D(u_n - u)) : (\psi \otimes \nabla(w_n - w)) \mathrm{d}x + \int_{\overline{\varpi}} F_n(x, u_n) w_n \psi \,\mathrm{d}\mu_n \right), \quad (5.52)$$

and satisfies that there exists $C > 0$, which only depends on α and γ, such that

$$|T| \leq C|u|^{p-1} \quad \mu\text{-a.e. in } \overline{\varpi} \setminus \overline{\partial\omega \cap \Omega}. \quad (5.53)$$

Proof. Given $\psi \in \mathcal{S}_\omega^M$, we take $w_n\psi$ as test function in (3.24). This gives

$$\int_\omega a(x,Du_n):D\psi\, w_n \mathrm{d}x + \int_\omega a(x,Du_n):(\psi\otimes\nabla w)\,\mathrm{d}x \\ + \int_\omega a(x,Du_n):(\psi\otimes\nabla(w_n-w))\mathrm{d}x + \int_{\overline{\omega}} F_n(x,u_n)w_n\psi\,\mathrm{d}\mu_n = \\ \int_\omega g_n w_n\psi\,\mathrm{d}x + \int_\omega G_n : D(w_n\psi)\mathrm{d}x. \quad (5.54)$$

Using that g_n converges weakly in $L^{p'}(\omega)^M$ to g, G_n converges strongly in $L^{p'}(\omega)^{M\times N}$ to G and u_n and w_n converge weakly in $W^{1,p}(\omega)^M$ and $W^{1,p}(\Omega)$ to u and w respectively (and then strongly in $L^p(\omega)^M$ and $L^p(\Omega)$), (5.45) and (5.46), we get

$$\int_\omega a(x,Du):D(w\psi)\mathrm{d}x + \int_\omega a(x,D(u_n-u)):(\psi\otimes\nabla(w_n-w))\mathrm{d}x \\ + \int_{\overline{\omega}} F_n(x,u_n)w_n\psi\,\mathrm{d}\mu_n = \int_\omega gw\psi\,\mathrm{d}x + \int_\omega G:D(w\psi)\mathrm{d}x + O_n. \quad (5.55)$$

Since $\|w_n\|_{W_0^{1,p}(\hat\Omega)\cap L^p_{\mu_n}(\hat\Omega)}$ is bounded, (3.23), (2.12) and (2.18), we have

$$\int_\omega |a(x,D(u_n-u))||\nabla(w_n-w)|\mathrm{d}x + \int_{\overline{\omega}} |F_n(x,u_n)||w_n|\mathrm{d}\mu_n \le C.$$

Thus, there exists a vector Radon measure ρ on $\bar\omega$, such that for every $\varphi \in C^0(\overline{\omega})^M$, it holds

$$\lim_{n\to\infty}\left(\int_\omega a(x,D(u_n-u))(\varphi\otimes\nabla(w_n-w))\mathrm{d}x + \int_{\overline{\omega}} F_n(x,u_n)w_n\varphi\,\mathrm{d}\mu_n\right) = \int_{\overline{\omega}}\varphi\,\mathrm{d}\rho. \quad (5.56)$$

Using the Cauchy-Schwartz inequality, (2.12), (2.18), the weak convergence of $|\nabla(w_n-w)|$ to zero in $L^p(\omega)$, (5.47) and (4.33), we deduce that ρ satisfies

$$\left|\int_{\overline{\omega}}\varphi\,\mathrm{d}\rho\right| \le C\limsup_{n\to+\infty}\left(\int_\omega |D(u_n-u)|^p|\varphi|\mathrm{d}x + \int_{\overline{\omega}}|u_n|^p|\varphi|\mathrm{d}\mu_n\right)^{\frac{p-1}{p}} \\ \times\left(\int_\omega|\nabla(w_n-w)|^p|\varphi|\mathrm{d}x + \int_{\overline{\omega}} w_n^p|\varphi|\mathrm{d}\mu_n\right)^{\frac{1}{p}} \\ \le C\left(\int_{\overline{\omega}}|u|^p|\varphi|\mathrm{d}\mu\right)^{\frac{p-1}{p}}\left(\int_{\overline{\omega}} w^p|\varphi|\mathrm{d}\mu\right)^{\frac{1}{p}}, \quad \forall\varphi\in\mathcal{S}_\omega^M, \quad (5.57)$$

where C only depends on α and γ. From the derivation measures theorem, we deduce that there exists a μ-measurable vector function $L:\overline{\omega}\setminus\overline{\partial\omega\cap\Omega}\to\mathbb{R}^M$ such that

$$\int_{\overline{\omega}}\varphi\,\mathrm{d}\rho = \int_{\overline{\omega}} L\varphi\,\mathrm{d}\mu, \quad \forall\varphi\in\mathcal{S}_\omega^M,$$

and

$$|L| \le C|u|^{p-1}w \quad \mu\text{-a.e. in } \overline{\omega}\setminus\overline{\partial\omega\cap\Omega}.$$

Defining then $T = (\frac{L}{w})\chi_{\{w\neq 0\}} \in L^{p'}_\mu(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$, we have that T satisfies (5.52) and (5.53). Moreover, from (5.55) and (5.56) we obtain

$$\int_\omega a(x, Du) : D(w\psi)\mathrm{d}x + \int_{\overline{\omega}} Tw\psi \,\mathrm{d}\mu = \int_\omega gw\psi \,\mathrm{d}x + \int_\omega G : D(w\psi)\mathrm{d}x,$$

for every $\psi \in \mathcal{S}^M_\omega$, which by the density of the set D_ω given by (4.38) proves that u satisfies (5.51). □

To finish this section, let us obtain an estimate about the dependence of the function T given by Proposition 5.4 with respect to u. For this purpose, as in Lemma 5.3, we consider $z_n \in W^{1,p}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$ with $\|z_n\|_{W^{1,p}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega}\setminus\overline{\partial\omega\cap\Omega})^M}$ bounded, converging weakly in $W^{1,p}(\omega)^M$ to a function $z \in W^{1,p}(\omega)^M \cap L^p_\mu(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$, such that there exist h_n converging weakly in $L^{p'}(\omega)^M$ to a function h and H_n converging strongly in $L^{p'}(\omega)^{M\times N}$ to a function H which satisfy (5.48). From Proposition 5.4, we also know that there exits $T' \in L^{p'}_\mu(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$ such that

$$\int_{\overline{\omega}} T'w\psi \,\mathrm{d}\mu = \lim_{n\to\infty} \left(\int_\omega a(x, D(z_n - z)) : (\psi \otimes \nabla(w_n - w))\mathrm{d}x + \int_{\overline{\omega}} F_n(x, z_n)w_n\psi \,\mathrm{d}\mu_n \right), \tag{5.58}$$

for every $\psi \in \mathcal{S}^M_\omega$, and

$$\int_\omega a(x, Dz) : Dv \,\mathrm{d}x + \int_{\overline{\omega}} T'v \,\mathrm{d}\mu = \int_\omega hv \,\mathrm{d}x + \int_\omega H : Dv \,\mathrm{d}x, \ \forall v \in W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M \cap L^p_\mu(\overline{\omega})^M. \tag{5.59}$$

The following result estimates the difference between T and T'.

Lemma 5.5. *There exist $C_1, C_2 > 0$, such that T and T' satisfy*

$$|T - T'| \leq C_1(|u| + |z|)^{\frac{p(p-2)}{p-1}} |u - z|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \overline{\omega} \setminus \overline{\partial\omega \cap \Omega}, \tag{5.60}$$

and

$$(T - T')(u - z) \geq C_2|u - z|^p \quad \mu\text{-a.e. in } \overline{\omega} \setminus \overline{\partial\omega \cap \Omega}. \tag{5.61}$$

The constant C_1 only depends on α and γ. The constant C_2 only depends on α, Ω and $\hat{\Omega}$.

Proof. By (5.52), (5.58), (2.10), (2.16) and Proposition 5.1, for every $\psi \in \mathcal{S}^M_\omega$, we have

$$\left| \int_{\overline{\omega}} (T - T')w\psi \,\mathrm{d}\mu \right| \leq C \int_\omega (|D(u_n - u)| + |D(z_n - z)|)^{p-2} |D(u_n - z_n - u + z)||\nabla(w_n - w)||\psi|\mathrm{d}x$$
$$+ \int_{\overline{\omega}} (|u_n| + |z_n|)^{p-2} |u_n - z_n||\psi|w_n \,\mathrm{d}\mu_n + O_n,$$

which by Hölder's inequality, (5.47) (applied to u_n and z_n) and (5.49) gives

$$\left| \int_{\overline{\omega}} (T - T')w\psi \,\mathrm{d}\mu \right| \leq C \left(\int_{\overline{\omega}} (|u| + |z|)^p |\psi| \mathrm{d}\mu \right)^{\frac{p-2}{p-1}} \left(\int_{\overline{\omega}} |u - z|^p |\psi| \mathrm{d}\mu \right)^{\frac{1}{p(p-1)}} \left(\int_{\overline{\omega}} |w|^p |\psi| \mathrm{d}\mu \right)^{\frac{1}{p}}.$$

From the measures derivation theorem, we then deduce (5.60).

Let us now prove (5.61). For $\varphi \in S_\omega$, $\varphi \geq 0$, we take $(u_n - z_n)\varphi^p$ as test function in the difference of (3.24) and (5.48). Using Rellich-Kondrachov's compactness theorem, the weak convergence of $g_n - h_n$ in $L^{p'}(\omega)^M$, the strong convergence of $G_n - H_n$ in $L^{p'}(\omega)^{M\times N}$, (5.51) and (5.59), we have

$$\begin{aligned}\int_\omega [a(x,Du_n)-a(x,Dz_n)]:D(u_n-z_n)\varphi^p\,\mathrm{d}x+\int_{\overline{\varpi}}[F_n(x,u_n)-F_n(x,v_n)](u_n-z_n)\varphi^p\,\mathrm{d}\mu_n=\\ \int_\omega (g-h)(u-z)\varphi^p\,\mathrm{d}x\int_\omega (G-H):D((u-z)\varphi^p)\mathrm{d}x+\\ -p\int_\omega [a(x,Du)-a(x,Dz)]:[(u-z)\otimes\nabla\varphi]\varphi^{p-1}\,\mathrm{d}x+O_n\\ =\int_\omega [a(x,Du)-a(x,Dz)]:D(u-z)\varphi^p\,\mathrm{d}x+\int_{\overline{\varpi}}(T-T')(u-z)\varphi^p\,\mathrm{d}\mu+O_n,\end{aligned}\tag{5.62}$$

which by (5.46) (applied to u_n and z_n) implies

$$\begin{aligned}\int_{\overline{\varpi}}(T-T')(u-z)\varphi^p\,\mathrm{d}\mu=\lim_{n\to\infty}\Bigg(\int_\omega [a(x,D(u_n-u))-a(x,D(z_n-z))]:D(u_n-z_n-u+z)\varphi^p\,\mathrm{d}x\\ +\int_{\overline{\varpi}}[F_n(x,u_n)-F_n(x,z_n)](u_n-z_n)\varphi^p\,\mathrm{d}\mu_n\Bigg).\end{aligned}\tag{5.63}$$

Using in the right-hand side of this inequality (2.9), (2.15) and (4.42), we deduce

$$\begin{aligned}\int_{\overline{\varpi}}(T-T')(u-z)\varphi^p\,\mathrm{d}\mu\ &\geq\alpha\left(\int_\omega |D(u_n-u-z_n+z)|^p\varphi^p\,\mathrm{d}x+\int_{\overline{\varpi}}|u_n-z_n|^p\varphi^p\,\mathrm{d}\mu_n\right)\\ &\geq\frac{\alpha}{\|P\|}\int_{\overline{\varpi}}|u-z|^p\varphi^p\,\mathrm{d}\mu+O_n,\qquad\forall\varphi\in S_\omega,\ \varphi\geq 0.\end{aligned}\tag{5.64}$$

From the measures derivation theorem, this proves (5.61). □

6. Proof of the main results

In this section we prove that there exists a μ-Carathéodory function $F:\bar\Omega\times\mathbb{R}^M\to\mathbb{R}^M$ such that the function T given by Proposition 5.4 satisfies $T(x)=F(x,u(x))$ μ-a.e. in $\overline{\varpi}\setminus\overline{\partial\omega\cap\Omega}$. We start with the following lemma. Its proof is completely similar to the one of Theorem 6.9 in [5], and thus we omit it.

Lemma 6.1. *We consider a subsequence of n such that there exists the measure μ defined in the beginning of Section 5. Then, up to another subsequence, we have that for every $q\in\mathbb{Q}^M$ and every $m\in\mathbb{N}$, the solution q_n^m of*

$$\left\{\begin{array}{l} q_n^m\in W^{1,p}(\Omega)^M\cap L^p_{\mu_n}(\overline{\Omega})^M\\ \displaystyle\int_\Omega a(x,Dq_n^m):Dv\,\mathrm{d}x+\int_{\overline{\Omega}}F_n(x,q_n^m)v\,\mathrm{d}\mu_n=m\int_\Omega[|w_nq|^{p-2}w_nq-|q_n^m|^{p-2}q_n^m]v\,\mathrm{d}x\\ \forall v\in W^{1,p}(\Omega)^M\cap L^p_{\mu_n}(\overline{\Omega})^M,\end{array}\right.\tag{6.65}$$

converges to a function q^m weakly in $W^{1,p}(\Omega)^M$. This function satisfies that there exists $T_q^m\in L^{p'}(\overline{\Omega})$ such that

$$\left\{\begin{array}{l} q^m\in W^{1,p}(\Omega)^M\cap L^p_{\mu}(\overline{\Omega})^M\\ \displaystyle\int_\Omega a(x,Dq^m):Dv\,\mathrm{d}x+\int_{\overline{\Omega}}T_q^m v\,\mathrm{d}\mu=m\int_\Omega[|wq|^{p-2}wq-|q^m|^{p-2}q^m]v\,\mathrm{d}x\\ \forall v\in W^{1,p}(\Omega)^M\cap L^p_{\mu}(\overline{\Omega})^M.\end{array}\right.\tag{6.66}$$

When m tends to infinity, the sequence q^m converges to wq strongly in $W^{1,p}(\Omega)^M \cap L^p_{\mu_n}(\overline{\Omega})^M$ and the sequence T^m_q converges strongly in $L^{p'}_\mu(\overline{\Omega})^M$ to a function T_q.

Definition 6.2. We consider the subsequence of n given by Lemma 6.1. Then, we define $\mathcal{F} : \overline{\Omega} \times \mathbb{Q}^M \to \mathbb{R}^M$ by

$$\mathcal{F}(x,q) = T_q(x), \quad \forall q \in \mathbb{Q}^M, \ \mu\text{-a.e. } x \in \overline{\Omega}.$$

By Lemma 6.1, (5.53), (5.60) and (5.61), it is easy to show that for every q_1, $q_2 \in \mathbb{Q}^M$ and μ-a.e. $x \in \overline{\Omega}$, we have

$$\mathcal{F}(x,0) = 0, \tag{6.67}$$

$$|\mathcal{F}(x,q_2) - \mathcal{F}(x,q_1)| \leq C_1(|q_1| + |q_2|)^{\frac{p(p-2)}{p-1}} |q_2 - q_1|^{\frac{1}{p-1}} w(x)^{p-1}, \tag{6.68}$$

$$(\mathcal{F}(x,q_2) - \mathcal{F}(x,q_1))(q_2 - q_1) \geq C_2 |q_2 - q_1|^p w(x)^{p-1}. \tag{6.69}$$

Using (6.68), we can extend by continuity $\mathcal{F}$ to $\overline{\Omega} \times \mathbb{R}^M$. Then, we define $F : \overline{\Omega} \times \mathbb{R}^M \to \mathbb{R}^M$ by

$$F(x,s) = \begin{cases} \mathcal{F}\left(x, \dfrac{s}{w(x)}\right) & \text{if } w(x) > 0 \\ |s|^{p-2}s & \text{if } w(x) = 0. \end{cases}$$

Thanks to Lemma 6.1, Proposition 5.4 and estimate (5.60) we can now prove Theorem 3.1.

Proof of Theorem 3.1. We take u_n and u as in the statement of the theorem. By Proposition 5.4, there exists $T \in L^{p'}_\mu(\varpi)^M$ such that u is a solution of (5.51). Applying (5.60), with z replaced by q^m, we have

$$|T - T^m_q| \leq C(|u| + |q^m|)^{\frac{p(p-2)}{p-1}} |u - q^m|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \varpi \setminus \overline{\partial\omega \cap \Omega}, \tag{6.70}$$

and therefore, taking the limit as m tends to infinity, we obtain

$$|T - F(x,wq)| \leq C(|u| + |wq|)^{\frac{p(p-2)}{p-1}} |u - wq|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \varpi \setminus \overline{\partial\omega \cap \Omega}.$$

Thus, for every simple function $\phi(x) = \sum_{i=1}^l s_i \chi_{B_i}(x)$, with $s_i \in \mathbb{R}^M$, B_i Borel, we have

$$|T - F(x,w\phi)| \leq C(|u| + |w\phi|)^{\frac{p(p-2)}{p-1}} |u - w\phi|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \varpi \setminus \overline{\partial\omega \cap \Omega}.$$

Finally, taking in this inequality ϕ as a sequence ϕ_n such that $w\phi_n$ converges μ-a.e. to u in $\varpi \setminus \overline{\partial\omega \cap \Omega}$ (the existence of such sequence is an easy consequence of Prop. 4.5) and passing to the limit in n thanks to the continuity of F with respect to its second variable, we get

$$T = F(x,u) \quad \mu\text{-a.e. in } \varpi \setminus \overline{\partial\omega \cap \Omega}.$$

This proves (3.25) thanks to (5.51) and the fact that the functions of $W^{1,p}_{\partial\omega\cap\Omega}(\omega)$ are zero q.e. on $\overline{\partial\omega \cap \Omega}$. □

Proof of Theorem 3.2. Let us just prove *(ii)*. The proof of *(i)* is much simpler.

Thanks to (3.28) and the assumptions on a and F_n, problem (3.29) has a unique solution. Moreover, taking u_n as test function in (3.29) and using (3.28), (2.11) and (2.17) we get that u_n satisfies (3.23). Thus, from $W^{1,p}_{\partial\omega\cap\Omega}(\omega)$ closed, Theorem 3.1 and Proposition 5.1, we deduce that there exists a subsequence of u_n which converges weakly in $W^{1,p}_{\partial\omega\cap\Omega}(\omega)^M$ and strongly in $W^{1,q}_{\partial\omega\cap\Omega}(\omega)^M$, $1 \leq q < p$, to a solution u of (3.30). If we prove that u is unique, then the whole sequence u_n will converge to u and the proof of *(ii)* will be finished. For this purpose, it is enough to prove that the measure μ also satisfies (3.28) and then, from (2.9) and (3.22), we will get the uniqueness of solution of problem (3.30).

Let v be in $W^{1,p}_{\partial\omega\cap\Omega}(\omega)\cap L^p_\mu(\overline{\omega})$. Using Proposition 4.5, we consider $\psi_m \in \mathcal{S}_\omega$ such that $w\psi_m$ converges to v in $W^{1,p}_{\partial\omega\cap\Omega}(\omega)\cap L^p_\mu(\overline{\omega})$. From (3.28), for every $m\in\mathbb{N}$, we have

$$\begin{aligned}\int_\omega |w_n\psi_m|^p\,\mathrm{d}x &\le C_P^p\left(\int_\omega |\nabla(w_n\psi_m)|^p\,\mathrm{d}x + \int_{\overline{\omega}} |w_n\psi_m|^p\,\mathrm{d}\mu_n\right)\\ &\le C_P^p\left(\int_\omega \left(|\nabla(w_n\psi_m)|^p - |\nabla((w_n-w)\psi_m)|^p\right)\mathrm{d}x\right)\\ &\quad + C_P^p\left(\int_{\hat\Omega}\left|\nabla\big((w_n-w)Q_\omega(\psi_m)\big)\right|^p\mathrm{d}x + \int_{\hat\Omega}|w_nQ_\omega(\psi_m)|^p\,\mathrm{d}\hat\mu_n\right).\end{aligned}$$

Since w_n converges strongly to w in $W^{1,q}(\hat\Omega)$, $1\le q<p$, reasoning similarly to the proof of (5.46), we have that $|\nabla(w_n\psi_m)|^p - |\nabla((w_n-w)\psi_m)|^p$ converges strongly to $|\nabla(w\psi_m)|^p$ in $L^1(\omega)$ and thus

$$\int_\omega \left(|\nabla(w_n\psi_m)|^p - |\nabla((w_n-w)\psi_m)|^p\right)\mathrm{d}x \to \int_\omega |\nabla(w\psi_m)|^p\,\mathrm{d}x,$$

whereas from (4.33) and $\mu=\hat\mu$ in $\operatorname{supp}(\hat\mu)=\overline{\Omega}$ we have

$$\int_{\hat\Omega}|\nabla((w_n-w)Q_\omega(\psi_m))|^p\,\mathrm{d}x + \int_{\hat\Omega}|w_nQ_\omega(\psi_m)|^p\,\mathrm{d}\hat\mu_n \to \int_{\overline{\omega}}|w\psi_m|^p\,\mathrm{d}\mu.$$

Thus, using also the semicontinuity of the norm in $L^p(\omega)$, we get

$$\int_\omega |w\psi_m|^p\,\mathrm{d}x \le C_P^p\left(\int_\omega |\nabla(w\psi_m)|^p\,\mathrm{d}x + \int_{\overline{\omega}}|w\psi_m|^p\,\mathrm{d}\mu\right).$$

Taking then m tending to infinity we derive

$$\int_\omega |v|^p\,\mathrm{d}x \le C_P^p\left(\int_\omega |\nabla v|^p\,\mathrm{d}x + \int_{\overline{\omega}}|v|^p\,\mathrm{d}\mu\right),\qquad \forall v\in W^{1,p}_{\partial\omega\cap\Omega}(\omega)\cap L^p_\mu(\overline{\omega}).$$

This finishes the proof of *(ii)*. □

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