EXISTENCE AND L_∞ ESTIMATES OF SOME MOUNTAIN-PASS TYPE SOLUTIONS

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Abstract. We prove the existence of a positive solution to the BVP

$$(\Phi(t)u'(t))' = f(t, u(t)), \qquad u'(0) = u(1) = 0,$$

imposing some conditions on Φ and f. In particular, we assume $\Phi(t)f(t, u)$ to be decreasing in t. Our method combines variational and topological arguments and can be applied to some elliptic problems in annular domains. An L_∞ bound for the solution is provided by the L_∞ norm of any test function with negative energy.

1. Introduction

Early since its publication in 1973, the Mountain Pass Theorem of Ambrosetti and Rabinowitz [10] has provided existence and multiplicity results in Differential Equations as well as a comprehensive perspective of variational methods. The characterization of Mountain Pass type solutions became itself a subject of interest. In [5], the multi-peak shape of the solutions to the Dirichlet problem

$$\epsilon^2 \Delta u - u + f(u) = 0 \quad \text{in } \Omega; \quad u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ in } \Omega,$$

is established as ϵ tends to zero. In the same spirit, Bonheure *et al.* [3] showed that, for a superlinear elliptic problem with sign-changing weight, the major contribution of volume of mountain pass type solutions should concentrate in prescribed regions of the domain as a certain coefficient μ affecting the negative part of the non-linearity tends to $+\infty$. In the above examples the role played by a parameter as it approaches some limit is crucial. In [6] the author proved the existence of positive Mountain Pass type solutions to a class of singular differential equations with an increasing friction term and Dirichlet boundary conditions. In addition, a bound for the L_∞ norm of the solution was provided by the L_∞ norm of any regular function with negative energy. Our method combined arguments in the direct calculus of variations with phase plane techniques. In fact, pursuing the nature of the optimal min-max path connecting the origin to some function where the underlying functional is negative, we were lead to consider a family of minimizers of truncated functionals containing,

as a particular element, a classical solution to our boundary value problem (BVP). In this work we approach with similar arguments a more general class of equations that include some elliptic problems in an annulus. More precisely, we will be interested in positive solutions to

$$(\Phi(t)u'(t))' + f(t, u(t)) = 0, \tag{1.1}$$

$$u'(0) = u(1) = 0. \tag{1.2}$$

By positive solution we mean a C^2 function u verifying the above equalities and such that $u(t) > 0$ for all $t \in [0, 1[$. Similar problems have been considered in [1,2,8,9].

2. VARIATIONAL SETTING AND RESULTS

We begin by listing the assumptions on the terms of equations (1.1)–(1.2).
$\Phi \in C^1([0, 1])$ is strictly positive and we choose $m, \overline{m} > 0$ such that, for all $t \in]0, 1[$,

$$0 < m \le \Phi(t) \le \overline{m}. \tag{2.1}$$

We assume that

$$f(t, u)\Phi(t) \text{ is decreasing in } t \text{ for every } u \ge 0, \tag{2.2}$$

$$f : [0, 1] \times [0, +\infty[\mapsto \mathbb{R} \quad \text{is locally Lipschitz in the variable } u, \tag{2.3}$$

and, for some $\delta > 0$,

$$f(t, u) = 0 \ \forall (t, u) \in [0, 1] \times [0, \delta] \text{ and } f(t, u) > 0 \text{ in } [0, 1] \times]\delta, +\infty[. \tag{2.4}$$

The technical assumption (2.4) will be relaxed subsequently to a sub-linear growth near zero. Since we are looking for positive solutions we assume throughout the paper that f is extended by zero in $[0, 1] \times]-\infty, 0[$. The reader may easily verify that any non-trivial solution to (1.1)–(1.2) with this extension – which we will still denote by f – should be positive in $[0, 1[$ therefore being a solution of the initial problem. We shall consider the Sobolev space $H \subset H^1(]0, 1[)$ consisting in absolutely continuous functions u such that

$$\|u\|^2 := \int_0^1 u'^2(t)\,\mathrm{d}t \ < \infty, \ u(1) = 0.$$

In the sequel we will also refer

$$\|u\|_\infty := \sup\{u(t) \ : \ t \in [0, 1]\},$$

the natural norm on the space of continuous functions $C([0, 1])$.

Note that problem (1.1)–(1.2) may be viewed as the Euler-Lagrange equation of the functional $J : H \to \mathbb{R}$ defined by:

$$J(u) := \frac{1}{2}\int_0^1 \Phi(t)u'^2(t)\,\mathrm{d}t - \int_0^1 F(t, u(t))\,\mathrm{d}t,$$

where $F(t, u) = \int_0^u f(t, s)\mathrm{d}s$. We will suppose that J satisfies the fundamental property:

$$\exists\, h \in H \ : J(h) < 0. \tag{2.5}$$

Remark 2.1. Property (2.5) can be easily verified if, for some $\epsilon > 0$, $f(t, u) \ge \epsilon u^\alpha - C$ for all $u \ge 0$ and $t \in [0, 1]$, where $\alpha > 1$ and $C > 0$.

We denote $\overline{M} = \|h\|_\infty$. Since
$$\forall w \in H, \ \|w\|_\infty \leq \delta \ \Rightarrow \ J(w) \geq 0,$$
(where δ was defined in (2.4)) we have $\overline{M} > \delta$. For all $M \in [\delta, \overline{M}]$, consider the subset of H
$$\mathfrak{C}_M = \{u \in H : \max u \geq M\},$$
and the truncated functional $J_M : H \to \mathbb{R}$,
$$J_M(u) = \frac{1}{2} \int_0^1 \Phi(t){u'}^2(t) \, \mathrm{d}t - \int_0^1 F_M(t, u(t)) \, \mathrm{d}t$$
where
$$F_M(u) = \begin{cases} F(t, u) & \text{if } u \leq M \\[2mm] F(t, M) & \text{if } u > M. \end{cases}$$

Remark 2.2. From the compact injection of $H^1(]0, 1[)$ in $C([0, 1])$ we conclude that $\mathfrak{C}_M$ is weakly sequentially closed and that J_M is coercive and weakly lower semi-continuous.

We will be interested in the family of minimizers of J_M in $\mathfrak{C}_M$. By Remark 2.2 we know that a minimizer exists for every $M \in [\delta, \overline{M}]$. We also know that:

Lemma 2.3. *Let u be a minimizer of J_M in $\mathfrak{C}_M$. Then*
$$\max_{[0,1]} u = M \quad and \quad \min_{[0,1]} u = 0.$$

Proof. Given $w \in \mathfrak{C}_M$, let
$$\overline{w}(t) = \max\{0, \min\{w(t), M\}\}.$$
Of course, $\overline{w} \in H \cap \mathfrak{C}_M$. If $\overline{w} \neq w$ we have,
$$\int_0^1 \Phi(t)\overline{w}'^2(t) \, \mathrm{d}t < \int_0^1 \Phi(t){w'}^2(t) \, \mathrm{d}t$$
and
$$\int_0^1 F_M(t, \overline{w}(t)) \, \mathrm{d}t = \int_0^1 F_M(t, w(t)) \, \mathrm{d}t.$$
Then $J_M(\overline{w}) < J_M(w)$ which is absurd and the lemma follows. $\qquad\square$

Given $M \in [\delta, \overline{M}]$, we define two types of minimizers of J_M in $\mathfrak{C}_M$:

Definition 2.4. Let u be a minimizer of J_M in $\mathfrak{C}_M$. We say that u is a minimizer of **type A** if
$$u \in C^1([0, 1]), \ u(0) = M, \ u(t) < M \ \forall t \in \,]0, 1].$$
We say that u is a minimizer of **type B** if, for some $\bar{t} \geq 0$, we have
$$u \in C^1([0, 1]), \ u(t) = M \text{ in } [0, \bar{t}], \ u(t) < M \text{ in }]\bar{t}, 1] \text{ and } u'(0) = 0.$$

Remark 2.5. If u is a minimizer of J_M in $\mathfrak{C}_M$ then u satisfies equation (1.1) in the open set $U := u^{-1}(]-\infty, M[)$. In fact, let v be a regular function with support strictly contained in U. Then, for sufficiently small s, we have,
$$u + sv \in \mathfrak{C}_M \quad \text{and} \quad u(t) + sv(t) < M \ \forall t \in \operatorname{supp}(v).$$

Since u is a minimizer, we conclude

$$\lim_{s \to 0} \frac{J_M(u + sv) - J_M(u)}{s} = \lim_{s \to 0} \frac{J(u + sv) - J(u)}{s} = \int_0^1 \Phi(t)u'(t)v'(t)\,\mathrm{d}t - \int_0^1 f(t, u(t))v(t)\,\mathrm{d}t = 0,$$

and the assertion follows. In particular, if u is a minimizer of type A (B), then u satisfies (1.1) in $]0, 1[$ ($]\bar{t}, 1[$). If u is simultaneously of type A and B then it is a classical solution to (1.1)–(1.2).

Lemma 2.6. *Let u be a minimizer of J_M in $\mathfrak{C}_M$. Then u is of type A or B (possibly both).*

Proof. Let

$$\bar{t} := \sup\{t \in [0, 1] : u(t) = M\}.$$

Since $H \subset C([0, 1])$, we have $u(\bar{t}) = M$ and we may consider $w \in H$

$$w(t) := \begin{cases} M & \text{if } t \leq \bar{t} \\ \\ u(t) & \text{if } t > \bar{t}. \end{cases}$$

Moreover,

$$\int_0^1 F_M(t, w(t))\,\mathrm{d}t \geq \int_0^1 F_M(t, u(t))\,\mathrm{d}t,$$

and

$$\int_0^1 \Phi(t){w'}^2(t)\,\mathrm{d}t \leq \int_0^1 \Phi(t){u'}^2(t)\,\mathrm{d}t,$$

the last inequality being strict if $w \neq u$ in $[0, \bar{t}]$. Since $J_M(u) \leq J_M(w)$, we conclude $u \equiv w$. If $\bar{t} = 0$, by Remark 2.5, integrating equation (1.1) between t and a fixed $t_0 \in]0, 1[$, we conclude $u \in C^1([0, 1])$, being therefore of type A.

Claim. *If $\bar{t} > 0$ then u is of type B.*

By Remark 2.5, $u \in C^1([\bar{t}, 1])$ and

$$u'(\bar{t}^+) := \lim_{t \to \bar{t}^+} u'(t)$$

is well defined, necessarily non-positive. Suppose, in view of a contradiction, that $u'(\bar{t}^+) < 0$. Choose $\theta, \epsilon > 0$ such that $u'(t) \leq -\theta$ for every $t \in]\bar{t}, \bar{t} + \epsilon[$ and, for $\epsilon < \bar{t}/2$, define the perturbation

$$v_\epsilon(t) = -(|t - \bar{t}| - \epsilon)_-. \tag{2.6}$$

We assert that, for sufficiently small ϵ,

$$\lim_{s \to 0} \frac{J_M(u + sv_\epsilon) - J_M(u)}{s} < 0. \tag{2.7}$$

If (2.7) holds, for some $s^* > 0$ close to zero, we have $u + s^* v_\epsilon \in \mathfrak{C}_M$ (recall that, by our choice of ϵ, $(u + s^* v_\epsilon)(0) = M$) and $J_M(u + s^* v_\epsilon) < J_M(u)$, a contradiction. In fact, Lemma 2.3 and (2.6) imply $u + s^* v_\epsilon \leq M$. Therefore

$$\lim_{s \to 0} \frac{J_M(u + sv_\epsilon) - J_M(u)}{s} = \int_0^1 \Phi(t)u'(t)v_\epsilon'(t)\,\mathrm{d}t - \int_0^1 f(t, u(t))v_\epsilon(t)\,\mathrm{d}t$$

$$\leq -\theta \int_{\bar{t}}^{\bar{t}+\epsilon} \Phi(t)\,\mathrm{d}t - \int_{\bar{t}-\epsilon}^{\bar{t}+\epsilon} f(t, u(t))v_\epsilon(t)\,\mathrm{d}t.$$

Observe that, by (2.1),

$$-\theta \int_{\bar{t}}^{\bar{t}+\epsilon} \Phi(t)\,\mathrm{d}t \leq -m\theta\epsilon \tag{2.8}$$

and, for some $C > 0$ depending only on f,

$$\int_{\bar{t}-\epsilon}^{\bar{t}+\epsilon} f(t, u(t))v_\epsilon(t)\,\mathrm{d}t \geq -C\epsilon^2. \tag{2.9}$$

Therefore, by (2.8) and (2.9), we have

$$\lim_{s \to 0} \frac{J_M(u + sv_\epsilon) - J_M(u)}{s} \leq -m\theta\epsilon + C\epsilon^2,$$

and the claim follows by taking ϵ sufficiently small. $\qquad\square$

In the next lemma we prove a necessary ordering relation between type A and type B minimizers of J_M in $\mathfrak{C}_M$.

Lemma 2.7. *Suppose that for a certain $M \in\,]0, \overline{M}\,]$ there exist minimizers u and v of J_M in $\mathfrak{C}_M$ such that u is of type A and v is of type B. Then $u(t) < v(t)$ for all $t \in\,]0, 1[$ or else u is a classical solution to (1.1)–(1.2).*

Proof. Suppose that u is not a solution to (1.1)–(1.2). By Remark 2.5, we have $u'(0) < 0$. Then $u(t) < v(t)$ for all $t \in\,]0, \epsilon[$ provided ϵ is sufficiently small. Assume, in view of a contradiction, the existence of $t^* \in\,]0, 1[$ such that

$$u(t^*) = v(t^*) < M \quad \text{and} \quad u'(t^*) > v'(t^*),$$

(the case $u'(t^*) = v'(t^*)$ is excluded by the Existence and Uniqueness Theorem using (2.3)). Moreover, suppose that

$$\frac{1}{2} \int_{t^*}^1 \Phi(t){u'}^2(t)\,\mathrm{d}t - \int_{t^*}^1 F_M(t, u(t))\,\mathrm{d}t \leq \frac{1}{2} \int_{t^*}^1 \Phi(t){v'}^2(t)\,\mathrm{d}t - \int_{t^*}^1 F_M(t, v(t))\,\mathrm{d}t, \tag{2.10}$$

and let

$$v^*(t) = \begin{cases} v(t) & \text{if } 0 \leq t \leq t^* \\[2mm] u(t) & \text{if } t^* < t \leq 1. \end{cases}$$

Then $v^* \in H$ and

$$J_M(v^*) \leq J_M(v),$$

i.e. v^* is also a minimizer of J_M in $\mathfrak{C}_M$. But this is absurd since v^* is not differentiable at t^*. In case where, instead of (2.10), we had the reversed inequality we would get the same contradiction by considering:

$$u^*(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq t^* \\[2mm] v(t) & \text{if } t^* < t \leq 1. \end{cases} \qquad\square$$

In the next lemma we establish an important fact concerning the coexistence of type A and type B minimizers at a same truncating level.

Lemma 2.8. *Assume that conditions (2.1), (2.2) and (2.3) hold. Suppose that for a certain $M \in\,]0, \overline{M}\,]$ there exist minimizers u and v of J_M in $\mathfrak{C}_M$ such that u is of type A and v is of type B. Then the minimizer u is a classical solution to (1.1)–(1.2).*

Proof. Let us prove that u admits an inverse function. By Remark 2.5, we may write

$$u'(t) = \frac{1}{\phi(t)} \left(\phi(0)u'(0) - \int_0^t f(s, u(s)) \, \mathrm{d}s \right).$$

Note that, if $u(0) \leq \delta$, then $u'(0) < 0$ (in case $u'(0) = 0$ we would conclude from (2.4), Lem. 2.3 and Rem. 2.5 that $u \equiv u(0)$, contradicting the assumption that $u \in H$). If $u(0) > \delta$, (2.4) and Remark 2.5 imply $u'(t) < 0$ for all $t \in \,]0, 1]$. In both cases we conclude that $u'(t) < 0$ in $\,]0, 1]$. We may therefore define

$$t_A : \ [0, M] \to [0, 1], \quad u \circ t_A = \mathrm{I}, \quad t_A \in C([0, M]) \cap C^1([0, M[),$$

where I is the identity function. Using similar arguments we may define an inverse function for $v(t)$

$$t_B : \ [0, M] \to [\bar{t}, 1], \quad v \circ t_B = \mathrm{I}, \quad t_B \in C([0, M]) \cap C^1([0, M[).$$

We suppose, in view of a contradiction, that u is not a solution to (1.1)–(1.2). By Lemma 2.7 we have

$$t_A(u) \leq t_B(u) \quad \forall u \in [0, M].$$

Consider[1]

$$Z_A : \ [0, M] \to \mathbb{R}, \quad u \mapsto \Phi(t_A(u))u'(t_A(u)),$$

and

$$Z_B : \ [0, M] \to \mathbb{R}, \quad v \mapsto \Phi(t_B(v))u'(t_B(v)).$$

Since $u'(0) < 0$, (2.1) implies

$$Z_A(M) < 0 = Z_B(M). \tag{2.11}$$

Note that Z_A and Z_B are negative in $[0, M[$. By (1.1), we may write, for $u, v \in [0, M[$,

$$-\frac{\mathrm{d}Z_A}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t_A} = f(t_A(u), u) \quad \text{and} \quad -\frac{\mathrm{d}Z_B}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}t_B} = f(t_B(v), v)$$

or

$$\frac{\mathrm{d}Z_A}{\mathrm{d}u} = -\frac{\Phi(t_A(u))}{Z_A}f(t_A(u), u) \quad \text{and} \quad \frac{\mathrm{d}Z_B}{\mathrm{d}v} = -\frac{\Phi(t_B(v))}{Z_B}f(t_B(v), v). \tag{2.12}$$

Claim. *Assumption* (2.11) *implies that* $Z_A(u) < Z_B(u)$ *for all* $u \in [0, M]$.

We have $Z_A(0) \neq Z_B(0)$. In fact, if $Z_A(0) = Z_B(0)$, then $u'(1) = v'(1)$. Since $u, v \in H$, the Existence and Uniqueness Theorem implies $u(t) = v(t)$ for all $t \in [\bar{t}, 1]$. In particular, (2.11) fails for $u = M$.

Suppose that, for some $u^* \in \,]0, M[$, we had

$$Z_A(u^*) = Z_B(u^*).$$

We choose u^* to be the maximum point satisfying the previous equality. Then

$$\frac{\mathrm{d}Z_A}{\mathrm{d}u}(u^*) \leq \frac{\mathrm{d}Z_B}{\mathrm{d}u}(u^*). \tag{2.13}$$

The equality of the derivatives is excluded by the Existence and Uniqueness Theorem applied to (2.12) and the fact that $Z_A(0) \neq Z_B(0)$. In view of (2.12) and assumption (2.2) (recalling $Z_A(u^*) = Z_B(u^*) < 0$ and $t_B(u^*) \geq t_A(u^*)$) we have

$$\frac{\mathrm{d}Z_A}{\mathrm{d}u}(u^*) > \frac{\mathrm{d}Z_B}{\mathrm{d}u}(u^*),$$

contradicting (2.13) and the claim is proved.

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[1] This change of variables is adapted from [7].

In particular, if u is not a classical solution to (1.1)–(1.2), then $Z_A(0) < Z_B(0)$ or

$$u'(1) < v'(1) < 0.$$

We conclude the existence of $t^* < 1$ such that $u(t^*) > v(t^*)$, a contradiction with Lemma 2.7. The proof is complete. □

We are now in a position to prove:

Proposition 2.9. *Assume that conditions* (2.1), (2.2), (2.3), (2.4) *and* (2.5) *hold. Then there exists a positive solution u to* (1.1)–(1.2) *such that*

$$\|u\|_\infty \leq \|h\|_\infty$$

where h was defined in (2.5).

Proof. Recalling our notation $\overline{M} = \|h\|_\infty$, let $I = \left[\delta, \overline{M}\,\right]$ and consider the following subsets I_A and I_B:

$$I_A\,(I_B) = \{M \in \left[\delta, \overline{M}\,\right] : J_M \text{ has a minimizer in } \mathfrak{C}_M \text{ of type } A\,(B)\}.$$

By Lemma 2.6 we have $I = I_A \cup I_B$. We assert that I_A and I_B are non-empty. In fact $\delta \in I_A$ since, as noticed in Lemma 2.8, if $\underline{u}$ is a minimizer of J_δ in $\mathfrak{C}_\delta$ and $\underline{u}'(0) = 0$ then $\underline{u} \equiv \delta$, which is absurd.

Claim 1. *I_B is non-empty.*

Suppose that $\overline{M} \notin I_B$. In this case, let $\overline{u}$ be a type A minimizer of $J_{\overline{M}}$ in $\mathfrak{C}_{\overline{M}}$ with $\overline{u}'(0) < 0$. Let

$$\bar{f}(t, u) := f(t, \min\{u, \overline{u}(t)\}).$$

Define, for $u \in H$,

$$\bar{J}(u) := \frac{1}{2} \int_0^1 \phi(t) {u'}^2(t)\, \mathrm{d}t - \int_0^1 \overline{F}(t, u(t))\, \mathrm{d}t,$$

where $\overline{F}(t, u) = \int_0^u \bar{f}(t, s)\mathrm{d}s$. By (2.5) we have $\bar{J}(\overline{u}) < 0$. Also $\bar{J}$ is coercive and lower semi-continuous in H and therefore attains a minimum at some function $w \in H$ such that $\bar{J}(w) < 0$. In fact

$$0 < w(t) < \overline{u}(t) \quad \forall t \in [0, 1[,$$

(0 and $\overline{u}$ are a pair of well ordered lower and upper solutions respectively) and w is a classical solution to (1.1)–(1.2) (see for instance [4], Chap. 4, for details). In particular, $\|w\|_\infty \in I_B$.

Claim 2. *I_A and I_B are closed subsets of I.*

Let (M_n) be a sequence in $I_A\,(I_B)$ such that $M_n \to M$. Let u_n be a corresponding sequence of type A (B) minimizers of J_{M_n} in $\mathfrak{C}_{M_n}$. Since (u_n) is bounded in H we may extract a weakly convergent subsequence (still denoted by u_n) such that

$$u_n \rightharpoonup u \text{ in } H \quad \text{and} \quad u_n \to u \text{ in } C([0, 1]).$$

We assert that u is a minimizer of J_M in $\mathfrak{C}_M$. In fact, by Lemma 2.3,

$$\lim_{n \to \infty} \int_0^1 F_{M_n}(t, u_n(t))\, \mathrm{d}t = \lim_{n \to \infty} \int_0^1 F(t, u_n(t))\, \mathrm{d}t = \int_0^1 F(t, u(t))\, \mathrm{d}t = \int_0^1 F_M(t, u(t))\, \mathrm{d}t$$

and

$$\int_0^1 \Phi(t) {u'}^2(t)\, \mathrm{d}t \leq \liminf_{n \to \infty} \int_0^1 \Phi(t) {u_n'}^2(t)\, \mathrm{d}t,$$

we conclude

$$J_M(u) \leq \liminf_{n \to \infty} J_{M_n}(u_n).$$

However, if we set $w_n = (M_n/M)u$, we have $w_n \to u$ in H and $w_n \in \mathfrak{C}_{M_n}$, for all $n \in \mathbb{N}$. Therefore

$$J_M(u) = \lim_{n \to \infty} J_{M_n}(w_n)$$

and

$$J_{M_n}(w_n) \geq J_{M_n}(u_n),$$

for all $n \in \mathbb{N}$. We conclude

$$J_M(u) \leq \liminf_{n \to \infty} J_{M_n}(u_n) \leq \limsup_{n \to \infty} J_{M_n}(u_n) \leq \lim_{n \to \infty} J_{M_n}(w_n) = J_M(u),$$

or

$$\lim_{n \to \infty} J_{M_n}(u_n) = J_M(u).$$

If, for some u^* in $\mathfrak{C}_M$, we had $J_M(u^*) < J_M(u)$ then, for sufficiently large n, we would obtain

$$J_{M_n}(w_n^*) < J_{M_n}(u_n),$$

where $w_n^* = (M_n/M)u^*$, and the assertion follows.

Note that so far we have just used the fact that u_n is a sequence of minimizers. It remains to prove that the limit function u is of type A (B). If (u_n) is a type A sequence, we may suppose, up to a subsequence that $u_n \to u$ in L_∞ and (1.1) is verified for all u_n in $]0,1[$. We conclude that u satisfies (1.1) in $]0,1[$, being in particular of type A. In case of a type B sequence, the L_∞-convergence argument above insures that

$$u_n' \xrightarrow[H^1]{} u'.$$

Then, up to a subsequence,

$$u_n \xrightarrow[C^1]{} u.$$

In particular we have $u'(\bar{t}) = 0$, where

$$\bar{t} := \max\{t : u(t) = M\},$$

and the claim is proved.

We conclude, since I is connected, that $I_A \cap I_B \neq \varnothing$. By Lemma 2.8 it implies the existence of a classical solution u such that $\max u \in I_A \cap I_B$. $\qquad \square$

In the next result we relax condition (2.4) using a standard approximating technique.

Theorem 2.10. *Suppose that $f(t,u)$ is locally Lipschitz in the variable u and*

$$0 < f(t,u) \leq \rho u + K u^p \quad \text{for } (t,u) \in [0,1] \times \,]0, +\infty[\tag{2.14}$$

for some $p > 1$, $K > 0$ and ρ such that, for all $u \in H$,

$$\rho \int_0^1 u^2(t)\, \mathrm{d}t \leq (m - \epsilon)\|u\|^2,$$

where $m > \epsilon > 0$. Also assume (2.1), (2.2) and that condition (2.5) is fulfilled for some non-negative $h \in H$. Then there exists a positive solution u to (1.1)–(1.2) such that $\max u \leq \|h\|_\infty$.

Proof. We may suppose that f is bounded above by $\|h\|_\infty$. Consider the following translation of the nonlinearity:

$$f_\delta(t, u) = f(t, (u - \delta)_+),$$

where $u_+ := \max\{0, u\}$. Observe that assumption (2.2) is verified by f_δ for all $\delta > 0$ as well as the right hand-side of (2.14) for the same constants K and ρ. Also (2.5) is satisfied for all the functionals

$$J_\delta(u) := \frac{1}{2} \int_0^1 \Phi(t)u'^2(t)\, \mathrm{d}t - \int_0^1 F_\delta(t, u(t))\, \mathrm{d}t,$$

where $F_\delta(t, u) = \int_0^u f_\delta(t, s)\mathrm{d}s$, provided δ is small. We may therefore apply Proposition 2.9 and conclude the existence of a solution u_δ to

$$(\Phi(t)u_\delta'(t))' = f_\delta(t, u_\delta(t)), \quad u_\delta'(0) = u_\delta(1) = 0, \quad \|u_\delta\|_\infty \le \|h\|_\infty. \tag{2.15}$$

Since u_δ is a critical point of J_δ, H is continuously embedded in $L^{p+1}(0, 1)$ with $p > 1$, we have, by (2.14) and classical estimates, for some K_1 independent of δ,

$$m\|u_\delta\|^2 \le \int_0^1 \Phi(t)u_\delta'^2(t)\, \mathrm{d}t = \int_0^1 f_\delta(t, u_\delta(t))u_\delta(t)\, \mathrm{d}t$$

$$\le \rho \int_0^1 u_\delta^2\, \mathrm{d}t + K \int_0^1 |u_\delta|^{p+1}\, \mathrm{d}t \le (m - \epsilon)\|u_\delta\|^2 + K_1\|u_\delta\|^{p+1}. \tag{2.16}$$

We conclude, for $k^* = (\epsilon/K_1)^{\frac{1}{p-1}}$,

$$\|u_\delta\| \ge k^* > 0,$$

for all sufficiently small δ. Consider a sequence $\delta_n \to 0$ and the corresponding sequence u_n of solutions to (2.15). Noting that $(\|u_n\|)$ is bounded, we may consider a subsequence (still denoted by (u_n)) and $u \in H$ such that

$$u_n \rightharpoonup u \text{ in } H \quad \text{and} \quad u_n \to u \text{ in } C([0, 1]).$$

Then,

$$\int_0^1 f(t, u(t))u(t)\, \mathrm{d}t = \lim_{\delta_n \to 0} \int_0^1 f_{\delta_n}(t, u_n(t))u_n(t)\, \mathrm{d}t = \lim_{\delta_n \to 0} \int_0^1 \Phi(t)u_n'^2(t)\, \mathrm{d}t \ge mk^*,$$

i.e. u is non-trivial. Standard arguments now insure that u is a classical solution to (1.1)–(1.2) with $\|u\|_\infty \le \|h\|_\infty$. □

Remark 2.11. Instead of (1.2) we may consider the more general boundary conditions

$$u'(r) = u(R) = 0 \ \ (r < R),$$

and obtain an equivalent version of Theorem 2.10 with obvious adaptations.

Remark 1. Some type of condition near zero like (2.14) is necessary, as one may deduce from the following example. Consider the existence of a positive solution to the BVP:

$$u'' + (2\lambda - 1)^2 u = 0 \quad u'\left(\frac{\pi}{2}\right) = u(\pi) = 0.$$

As the reader may easily verify, all conditions of Theorem 2.10 are fulfilled except (2.14), provided λ is sufficiently large. If $\lambda \in \mathbb{N}$ there is an infinity of solutions all multiples of $\sin((2\lambda - 1)t)$ functions. If $\lambda \notin \mathbb{N}$ the previous BVP has no solution.

Finally we apply our results to an elliptic problem in an annulus.

Corollary 2.12. *Let $\Omega := B_R\backslash\overline{B_r} \subset \mathbb{R}^N$, where $0 < r < R$ and B_L is the N-dimensional Euclidean ball of center 0 and radius L. Consider the following BVP:*

$$- \Delta u = g(\|x\|, u) \quad in \ \Omega, \tag{2.17}$$

$$u = 0 \ on \ \partial B_R \quad and \quad \frac{\partial u}{\partial n} = 0 \ on \ \partial B_r. \tag{2.18}$$

In view of Remark 2.11, suppose that, for $\Phi(t) = t^{N-1}$, the function

$$f(t,u) : [r,R] \times [0,\infty] \mapsto \mathbb{R}, \qquad (t,u) \to t^{N-1}g(t,u)$$

satisfies (2.2)–(2.5) as well as (2.14). Then there exists a radial symmetric positive solution u to (2.17)–(2.18) such that $\|u\|_\infty \leq \|h\|_\infty$, where h is defined in (2.5).

Proof. Observe that a positive radial symmetric solution to (2.17)–(2.18) can be obtained as a solution of

$$(t^{N-1}u'(t))' + t^{N-1}g(t,u(t)) = 0, \quad u'(r) = u(R) = 0,$$

and apply Theorem 2.10. □

Remark 2.13. We may apply the previous results to the BVP:

$$-\Delta u = \exp(-L\|x\|)u^p \quad in \ \Omega,$$

$$u = 0 \ on \ \partial B_R \ and \ \frac{\partial u}{\partial n} = 0 \ on \ \partial B_r,$$

provided L is large and $p > 1$.

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