LIPSCHITZ STABILITY IN THE DETERMINATION OF THE PRINCIPAL PART OF A PARABOLIC EQUATION

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Abstract. Let \(y(h)(t,x)\) be one solution to

\[
\partial_t y(t,x) - \sum_{i,j=1}^{n} \partial_j (a_{ij}(x) \partial_i y(t,x)) = h(t,x), \quad 0 < t < T, \quad x \in \Omega
\]

with a non-homogeneous term \(h\), and \(y|_{(0,T) \times \partial \Omega} = 0\), where \(\Omega \subset \mathbb{R}^n\) is a bounded domain. We discuss an inverse problem of determining \(n(n+1)/2\) unknown functions \(a_{ij}\) by \(\{\partial_\nu y(h_\ell)\}_{\ell=0}^{\ell_0}\), where \(\Gamma_0\) is an arbitrary subboundary, \(\partial_\nu\) denotes the normal derivative, \(0 < \theta < T\), and \(\ell_0 \in \mathbb{N}\). In the case of \(\ell_0 = (n+1)^2/2\), we prove the Lipschitz stability in the inverse problem by choosing \((h_1, \ldots, h_{\ell_0})\) from a set \(\mathcal{H} \subset \{C^\infty((-\infty, T) \times \omega)\}_{\ell_0}\) with an arbitrarily fixed subdomain \(\omega \subset \Omega\). Moreover, we can take \(\ell_0 = (n+3)n/2\) by making special choices for \(h_\ell\), \(1 \leq \ell \leq \ell_0\), the proof is based on a Carleman estimate.

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1. INTRODUCTION AND MAIN RESULTS

In this paper we consider the following parabolic equation:

\[
\partial_t y(t,x) - \sum_{i,j=1}^{n} \partial_j (a_{ij}(x) \partial_i y(t,x)) = h(t,x), \quad (t,x) \in Q \equiv (0,T) \times \Omega
\]

\(
(1.1)
\)

\[
y(t,x) = 0, \quad (t,x) \in \Sigma \equiv (0,T) \times \partial \Omega, \quad y(0,\cdot) \in L^2(\Omega).
\]

(1.2)

Here \(\Omega \subset \mathbb{R}^n\) is a bounded domain whose boundary \(\partial \Omega\) is sufficiently smooth, and \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\), \(\partial_i = \frac{\partial}{\partial x_i}\), \(\partial_j = \frac{\partial}{\partial x_j}\), \(\nabla = (\partial_1, \ldots, \partial_n)\), \(h \in C^\infty((-\infty, T) \times \omega)\), and \(\omega\) is an arbitrarily fixed subdomain of \(\Omega\). Let \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) be a multi-index with \(\alpha_j \in \mathbb{N} \cup \{0\}\). We set \(\partial_\nu^\alpha = \partial_\nu^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n}\), \(|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n\), and \(\nu = \nu(x) = (\nu_1(x), \ldots, \nu_n(x))\) is the external unit normal vector to \(\partial \Omega\) at \(x\). Let \(\partial_\nu = \nu \cdot \nabla\).

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Assume that
\[ a_{ij} \in C^1(\overline{\Omega}), \quad a_{ij} = a_{ji}, \quad 1 \leq i, j \leq n, \]
and that the coefficients \( \{a_{ij}\}_{1 \leq i, j \leq n} \) satisfy the uniform ellipticity: there exists a constant \( r > 0 \) such that
\[ \sum_{i,j=1}^n a_{ij}(x)\zeta_i \zeta_j \geq r|\zeta|^2, \quad \zeta \in \mathbb{R}^n, \quad x \in \overline{\Omega}. \]

For \( y(0, \cdot) \in L^2(\Omega) \), we can prove (e.g., Pazy [37]) that \( y(\{a_{ij}\}, h) \in C([0, T]; L^2(\Omega)) \cap C([0, T]; H^2(\Omega) \cap H^1_0(\Omega)) \cap C^1((0, T); L^2(\Omega)) \) and see also (1.6) below. By \( y(\{a_{ij}\}, h)(t, x) \) we denote one function satisfying (1.1)–(1.2). We note that \( y(\{a_{ij}\}, h) \) is uniquely determined upon specification of an initial value in \( L^2(\Omega) \).

We consider the following inverse problem:

**Inverse problem.** Let \( \theta \in (0, T) \) be arbitrarily fixed and \( \Gamma_0 \neq \emptyset \) be an arbitrary relatively open subset of \( \partial \Omega \). Select \( \ell_0 \in \mathbb{N}, h_{\ell} \in C^0_{\infty}((0, T) \times \omega), 1 \leq \ell \leq \ell_0 \) suitably and determine \( a_{ij}(x), x \in \Omega, 1 \leq i, j \leq n \) by observation data \( \partial_{\nu} y(\{a_{ij}\}, h_{\ell})|_{\partial \Omega} \) and \( y(\{a_{ij}\}, h_{\ell})(\theta, x), x \in \Omega, 1 \leq \ell \leq \ell_0 \).

We can consider a more general parabolic equation with lower-order terms:
\[
\partial_t y(t, x) = \sum_{i,j=1}^n \partial_j(a_{ij}(x)\partial_i y(t, x)) + \sum_{i=1}^n b_i(x)\partial_i y(t, x) + c(x)y(t, x) + h(t, x), \quad (t, x) \in Q
\]
and discuss the determination of \( a_{ij}, b_i, c, 1 \leq i, j \leq n \) by similar observations. The method is same because a basic estimate (Thm. 2.1) is insensitive to such lower-order terms. However for simplicity, we consider only the determination of the principal part.

In the formulation of the inverse problem, the initial values are also unknown. The non-homogeneous terms \( h_{\ell} \), \( 1 \leq \ell \leq \ell_0 \), are considered as input sources to system (1.1)–(1.2) and are spatially restricted to a small subdomain \( \omega \subset \Omega \). Then we determine \( a_{ij}(x), x \in \Omega \) by observation data \( \partial_{\nu} y(\{a_{ij}\}, h_{\ell})|_{\partial \Omega} \) and \( y(\{a_{ij}\}, h_{\ell})(\theta, x), x \in \Omega, 1 \leq \ell \leq \ell_0 \), which are regarded as outputs.

We shall determine \( a_{ij} \) in the neighbourhood of some known set of coefficients \( a_{ij}^{(2)} \). We shall denote by \( a_{ij}^{(1)} \) the unknown set of coefficients. Solutions associated to \( a_{ij}^{(2)} \) will thus be known. The full knowledge of \( a_{ij}^{(2)} \) allows for instance to approximately control the solutions to (1.1)–(1.2) associated to \( a_{ij}^{(2)} \) with the function \( h \) as the control function. In other words, in order to determine \( \frac{n(n+1)}{2} \) coefficients \( a_{ij}^{(1)} \), we are assumed to be able to operate the heat processes associated to \( a_{ij}^{(2)} \) by suitably changing input source \( h_{\ell} \).

We note that we need not know initial data in repeating the processes associated to \( a_{ij}^{(1)} \), and initial values for the both parabolic equations with \( a_{ij}^{(1)} \) and \( a_{ij}^{(2)} \), may be arbitrarily changed during the repeated processes. Our main concern is the stability estimate for the inverse problem: Estimate \( \sum_{i,j=1}^n \|a_{ij}^{(1)} - a_{ij}^{(2)}\|_{H^1(\Omega)} \) by suitable norms of \( \partial_{\nu} y \left( \{a_{ij}^{(1)}\}; h_{\ell}\right) - \partial_{\nu} y \left( \{a_{ij}^{(2)}\}; h_{\ell}\right) \) and \( y \left( \{a_{ij}^{(1)}\}; h_{\ell}\right)(\theta, \cdot) - y \left( \{a_{ij}^{(2)}\}; h_{\ell}\right)(\theta, \cdot), 1 \leq \ell \leq \ell_0 \). The stability is a fundamental mathematical subject in the inverse problem and immediately yields the uniqueness. Stability estimates for inverse problems are not only important from the theoretical viewpoint, but also useful for numerical algorithms. In particular, by Cheng and Yamamoto [10] for example, a stability estimate gives convergence rates of Tikhonov regularized solutions, which are widely used as approximating solutions to the inverse problems.
We can consider an inverse problem for a usual initial value/boundary value problem by setting \( \theta = 0 \). In the case where \( \theta = 0 \) and \( \Gamma_0 \) is an arbitrary subboundary of \( \Omega \), the corresponding inverse problem is open (e.g., Chap. 9, Sect. 2 in Isakov [25]) even for the inverse problem of determining a single coefficient in a parabolic equation. In the case of \( \theta = 0 \), if \( \Gamma_0 \subset \partial \Omega \) is a sufficiently large portion and unknown coefficients \( a_{ij} \) satisfy some additional conditions, then applying an argument in Theorem 4.7 in Klibanov [30], we can prove the stability provided that initial values satisfy some non-degeneracy condition similar to (1.7) below. The observations, and the application of the method in [8] needs independent consideration. Moreover, since we aim to steer systems by choosing controls which can be limited in any small part \( \Omega \), we have to establish a relevant Carleman estimate (Thm. 2.1 below). Here we assume that initial data are also unknown to be determined. If we can estimate \( \sum_{i,j=1}^{n} \|a_{ij}^{(1)} - a_{ij}^{(2)}\|_{H^1(\Omega)} \), then we can apply the argument in Yamamoto and Zou [41] (pp. 1187–1188), and we can estimate \( y((a_{ij}^{(1)}), h_\ell)(0, \cdot) - y((a_{ij}^{(2)}), h_\ell)(0, \cdot) \). The argument is concerned with the parabolic equation backward in time.

As for the backward heat equation, see the monographs Ames and Straughan [2], Payne [36] and Klibanov [31] as a recent paper. Our main concern is the determination of coefficients and so we will omit the estimation of initial values.
Let $\Omega$ and $G$. YUAN AND M. YAMAMOTO

Let us fix constants $M > 0$, we can improve the regularity in the proof, we have to assume that $y \in C^{\infty}(\Omega)$, which means that the unknown initial values are bounded with an \textit{a priori} bound $M > 0$.

Henceforth, for an arbitrarily fixed $M > 0$, we assume that

$$\|y(t, \cdot, \cdot)\|_{L^2(\Omega)} \leq M, \quad j = 1, 2,$$

which means that the unknown initial values are bounded with an \textit{a priori} bound $M > 0$.

Now we are ready to state our main results.

Next we introduce an admissible set of unknown coefficients $\{a_{ij}\}$. We choose $m \in \mathbb{N}$ such that

$$m > \frac{n}{2} + 3.$$

Let us fix constants $M_0 > 0$, $r > 0$ and smooth functions $\eta_{ij} = \eta_{ij}(x)$, $1 \leq i, j \leq n$ on $\overline{\Omega}$. Let $\omega_1 = \{x \in \Omega; \text{dist}(x, \partial\Omega) < r_0\}$ with sufficiently small $r_0 > 0$. Then we note that $\partial \omega_1 \supset \partial \Omega$. Henceforth $[\gamma]$ denotes the greatest integer not exceeding $\gamma \in \mathbb{R}$. We set

$$\mathcal{U} = \{\{a_{ij}\}; \|a_{ij}\|_{C^{m-1,1}(\overline{\Omega})} \leq M_0, \quad a_{ij} = \eta_{ij} \text{ in } \omega_1, \text{ and } (1.3) \text{ and } (1.4) \text{ are satisfied with fixed } r > 0\}.$$ (1.5)

For $\ell \in \mathbb{N} \cup \{0\}$ and $m_0 \in \{0, 1, \ldots, 2 \left[\frac{m+1}{2}\right]\}$, and $0 < \tau_1 < \tau_2 < T$, we can prove

$$\|y(\{a_{ij}\}, h)\|_{C^{\ell}(\tau_1, \tau_2; H^{m_0}(\omega))} \leq C_0(\|y(\{a_{ij}\}, h)(0, \cdot)\|_{L^2(\Omega)} + \|h\|_{W^{\ell+1,1}(0, T; H^{m_0}(\omega))}).$$ (1.6)

Here $C_0 > 0$ depends only $\ell$, $\tau_1$, $\tau_2$, and $\mathcal{U}$, and $\|h\|_{W^{\ell+1,1}(0, T; H^{m_0}(\omega))} = \sum_{j=0}^{\ell} \|\partial_t^j h\|_{L^1(0, T; H^{m_0}(\omega))}$. The proof relies on semigroup theory (e.g., [37]) and is given in Appendix B.

\textbf{Remark 1.1.} In (1.5), we assume that $a_{ij} \in C^{m-1,1}(\overline{\Omega})$. This can be partly relaxed. However, for the proof, we have to assume that $y(\{a_{ij}\}, h)(t, \cdot) \in C^3(\overline{\Omega})$ and by semigroup theory we discuss the approximate controllability in $H^m(\Omega) \subset C^3(\overline{\Omega})$ (by the Sobolev embedding theorem, e.g., Thm. 5.4 in Adams [1]), so that with the Sobolev space we have to relate the regularity of functions in the domain of the operator $A^{\frac{m+1}{2}}$ where the operator $A$ is defined by (1.9) below. For it, we need the regularity in $H^m(\Omega)$ for an elliptic equation

$$\sum_{i,j=1}^{n} \partial_j (a_{ij}(x) \partial_i u(x)) = b(x), \quad x \in \Omega \quad \text{(e.g., Chap. 8 in Gilbarg and Trudinger [16]) and } a_{ij} \in C^{m-1,1}(\overline{\Omega}) \text{ is a required regularity condition (e.g., Thm. 8.13, p. 187, in [16]).}$$

Moreover we assume that unknown coefficients $\{a_{ij}\}$ are given near the boundary $\partial\Omega$, that is, $a_{ij} = \eta_{ij} \text{ in } \omega_1$. This means that we are interested in the determination of coefficients in a compact subset of $\Omega$ away from $\partial\Omega$ with some distance. As is seen from the proof, condition (1.7) below is necessary and the homogeneous Dirichlet boundary condition (1.2) implies that (1.7) does not hold on $\partial\Omega$, because there exist zero column vectors of $D(y(\{a_{ij}^{(2)}\}, h(t)))$. This technically motivates that we discuss the determination of $a_{ij}^{(1)} \text{ on } \overline{\Omega \setminus \partial\Omega}$, and that we assume $a_{ij}^{(1)} = \eta_{ij} \text{ in } \omega_1$. We further note that since we consider solutions in a time interval away from $t = 0$, we can improve the regularity in $t \in (\tau_1, \tau_2)$ as we wish (see the proof of (1.6) in Appendix B), while the $x$-regularity in $H^m(\Omega)$ with $m > \frac{n}{2} + 3$, is necessary for our argument.

Henceforth, for an arbitrarily fixed $M > 0$, we assume that

$$\|y(\{a_{ij}^{(j)}\}, h)(0, \cdot)\|_{L^2(\Omega)} \leq M, \quad j = 1, 2,$$

which means that the unknown initial values are bounded with an \textit{a priori} bound $M > 0$. Now we are ready to state our main results.
Theorem 1.1. Let $0 < \tau_1 < \theta < \tau_2 < T$, $\Gamma_0 \neq \emptyset$ be an arbitrary relatively open subset of $\partial \Omega$, and let $\{a_{ij}^{(2)}\} \in U$ be arbitrarily fixed. We assume that $h_\ell \in C_0^\infty((0,T) \times \omega)$, $1 \leq \ell \leq \frac{(n+1)^2n}{2}$, satisfy
\[ D((\{a_{ij}^{(2)}\}, h_\ell))(\theta, x) \neq 0 \] for $x \in \Omega \setminus \omega_1$. Then there exists a constant $C_1 = C_1(U, M, \{h_\ell\}) > 0$ such that
\[
\sum_{i,j=1}^n \|a_{ij}^{(1)}(1) - a_{ij}^{(2)}\|_{H^1(\Omega)} \leq C_1 \sum_{\ell=1}^{(n+1)^2n} \|\partial_x y(\{a_{ij}^{(1)}\}, h_\ell) - \partial_x y(\{a_{ij}^{(2)}\}, h_\ell)\|_{H^2(\tau_1, \tau_2; L^2(\Gamma_0))} \\
+ C_1 \sum_{\ell=1}^{(n+1)^2n} \|y(\{a_{ij}^{(1)}\}, h_\ell)(\theta, \cdot) - y(\{a_{ij}^{(2)}\}, h_\ell)(\theta, \cdot)\|_{H^2(\Omega)}
\] (1.7)
for all $\{a_{ij}^{(1)}\} \in U$.

In order to estimate $\{a_{ij}^{(1)}\}$ around a given $\{a_{ij}^{(2)}\}$, we have to choose $h_\ell$, $1 \leq \ell \leq \frac{(n+1)^2n}{2}$ whose supports are restricted to a small set $(0,T) \times \omega$, so that the systems are steered to satisfy (1.7) on $\Omega \setminus \omega_1$ at the time $\theta$. The choice is related to approximate controllability of parabolic equations (e.g., [39]).

Henceforth we define an operator $A$ in $L^2(\Omega)$ by
\[
\begin{cases}
(Ay)(x) = -\sum_{i,j=1}^n \partial_j(a_{ij}(x))\partial_i y(x), & x \in \Omega, \\
D(A) = H^2(\Omega) \cap H^1_0(\Omega),
\end{cases}
\] (1.9)
where $D(A)$ denotes the domain of the operator $A$, and let $y(\{a_{ij}\}, h, \mu)$ denote the solution to (1.1) and (1.2) with $y(0, x) = \mu(x), x \in \Omega$.

Then we can prove:

Proposition 1.1. Let $m_1 = \left\lceil \frac{m+1}{2} \right\rceil$, that is, $m_1 = \frac{m}{2}$ if $m$ is even and $m_1 = \frac{m+1}{2}$ if $m$ is odd. Let $\{a_{ij}\} \in U$. For each $\theta > 0$ and $\mu \in L^2(\Omega)$, the set
\[
\{y(\{a_{ij}\}, h, \mu)(\theta, \cdot); h \in C_0^\infty((0,T) \times \omega)\}
\] is dense in $D(A^{m_1}) = \{y \in H^{2m_1}(\Omega); A^jy|_{\partial \Omega} = 0, 0 \leq j \leq m_1 - 1\}$.

By Proposition 1.1, we can prove the existence of $h_\ell \in C_0^\infty((0,T) \times \omega)$, $1 \leq \ell \leq \frac{(n+1)^2n}{2}$ such that (1.7) holds on $\Omega \setminus \omega_1$, which guarantees the Lipschitz stability in determining $\{a_{ij}^{(1)}\}$. In fact, we arbitrarily choose $\{\rho_\ell\}_{1 \leq \ell \leq \frac{(n+1)^2n}{2}} \subset C_0^\infty(\Omega)$ such that $\det D(\{\rho_\ell\})(x) \neq 0$ for $x \in \Omega \setminus \omega_1$. We note that $\rho_\ell \in D(A^{m_1})$. In terms of Proposition 1.1, for sufficiently small $\varepsilon > 0$, we can fix $h_\ell \in C_0^\infty((0,T) \times \omega)$, $1 \leq \ell \leq \frac{(n+1)^2n}{2}$ satisfying
\[
\|y(\{a_{ij}^{(2)}\}, h_\ell)(\theta, \cdot) - \rho_\ell\|_{H^{2m_1}(\Omega)} < \varepsilon, \quad 1 \leq \ell \leq \frac{(n+1)^2n}{2}.
\]
By the Sobolev embedding theorem, we have
\[
\|y(\{a_{ij}^{(2)}\}, h_\ell)(\theta, \cdot) - \rho_\ell\|_{C^2(\Omega)} < C\varepsilon, \quad 1 \leq \ell \leq \frac{(n+1)^2n}{2}.
\]
By det $D\{ρ_1\}(x) \neq 0$ for $x \in Ω \setminus ω_1$, we obtain (1.7) on $Ω \setminus ω_1$ for sufficiently small $ε > 0$. The control functions $h_ε$ can be constructed in practice by means of the methods in Fabre et al. [14], Glowinski and Lions [17].

Now we discuss the set of such $h_ε$, $1 ≤ ε ≤ \frac{(n+1)^2}{2}$. For simplicity, for the system with known $a^{(2)}_{ij}$, we assume the zero initial value. That is, we let $y(\{a^{(2)}_{ij}\}, h, 0)$ be the unique solution to (1.1) and (1.2) with $y(x, 0) = 0$, $x ∈ Ω$. We set $ε_0 = \frac{(n+1)^2}{2}$ and

$$H = \{(h_1, ..., h_{ε_0}) ∈ \{C^∞_0((0, T) × ω)\}^{ε_0}; D\{y(\{a^{(2)}_{ij}\}, hε, 0)\}(θ, x) \neq 0 \text{ for } x ∈ Ω \setminus ω_1\}.$$ 

From elliptic regularity (e.g., Thm. 8.13 in [16]) and semigroup theory (e.g., [37]), we can prove that there exists a constant $C_2 > 0$ such that

$$\|y(\{a_{ij}\}, h, 0)\|_{C([0,T];C^2(Ω)))} ≤ C_2\|h\|_{L^2(0,T;H^∞(Ω))},$$

where the constant $C_2$ can be taken uniformly for $\{a_{ij}\} ∈ U$. See Appendix B for the proof.

Therefore we can prove that for $(h_1, ..., h_{ε_0}) ∈ H$, there exists $ε = ε(h_1, ..., h_{ε_0}) > 0$ such that if $(\tilde{h}_1, ..., \tilde{h}_{ε_0}) ∈ \{C^∞_0((0, T) × ω)\}^{ε_0}$ and $\max_{1 ≤ i ≤ ε_0}\|h_ε - \tilde{h}_ε\|_{L^2(0,T;H^∞(Ω))} < ε$, then $(\tilde{h}_1, ..., \tilde{h}_{ε_0}) ∈ H$ by the definition of $D\{y(\{a^{(2)}_{ij}\}, hε, 0)\}(θ, x)$. This means the stability of input sources $(h_1, ..., h_{ε_0})$ realizing the Lipschitz stability.

Since $C^∞_0((0, T) × ω)$ is dense in $C^0_0((0, T) × ω)$ with $ε ∈ N$, we can take $C^0_0((0, T) × ω)$ as a class of interior input sources, using parabolic regularity properties (e.g., [37]).

Furthermore we can prove an even better result with smaller $ε_0$ in Theorem 1.1. That is, with arbitrary initial values for system (1.1) associated to the set of coefficients $a^{(2)}_{ij}$, we can choose $h_ε, 1 ≤ ε ≤ \frac{(n+3)^2}{2}$ to establish the Lipschitz stability around $a^{(2)}_{ij}$ by means of $\frac{(n+3)^2}{2}$ data. The choice of such $h_ε$ is different from Theorem 1.1, but Proposition 1.1 guarantees that such a choice is possible.

**Theorem 1.2.** Let $0 < τ_1 < θ < τ_2 < T$, $Γ_0 ≠ ∅$ be an arbitrary relatively open subset of $∂Ω$ and let us fix $\{a^{(2)}_{ij}\} ∈ U$. Then we can choose suitable $h_ε ∈ C^∞_0((0, T) × ω)$, $1 ≤ ε ≤ \frac{(n+3)^2}{2}$ such that there exists a constant $C_2 = C_2(U, M, \{h_ε\}) > 0$ such that

$$\sum_{i,j=1}^{n} \|a_{ij}^{(1)} - a_{ij}^{(2)}\|_{H^1(Ω)} ≤ C_2 \sum_{\ell=1}^{\frac{(n+3)^2}{2}} \|\partial_\gamma y(\{a_{ij}^{(1)}\}, h_ε) - \partial_\gamma y(\{a_{ij}^{(2)}\}, h_ε)\|_{H^2(\tau_1, \tau_2; L^2(Γ_0))}$$

$$+ C_2 \sum_{\ell=1}^{\frac{(n+3)^2}{2}} \|y(\{a_{ij}^{(1)}\}, h_ε)(θ, ·) - y(\{a_{ij}^{(2)}\}, h_ε)(θ, ·)\|_{H^2(Ω)}$$

for all $\{a_{ij}^{(1)}\} ∈ U$.

Since the number of the unknown coefficients is $\frac{n(n+1)}{2}$, it is natural to expect that suitable $\frac{n(n+1)}{2}$-times observations can yield the Lipschitz stability, and even the result in Theorem 1.2 holds with overdetermining observations (i.e., $\frac{n(n+3)}{2}$-times observations). We do not presently know whether we can reduce the number of observations to $\frac{n(n+3)}{2}$. In particular, for the case $a_{ij}(x) = \begin{cases} a(x), & i = j; \\
0, & i \neq j \end{cases}$, we can prove that a single observation by a suitable single input $h_1$ yields the Lipschitz stability. The proof is done similarly to Imanuvilov and Yamamoto [23] where an inverse problem for an acoustic equation $\partial_t^2 u = \text{div}(a(x)\nabla u)$ is discussed. In (1.11), we can replace $\|a_{ij}^{(1)} - a_{ij}^{(2)}\|_{H^1(Ω)}$ by a weaker norm $\|a_{ij}^{(1)} - a_{ij}^{(2)}\|_{L^2(Ω)}$ and can adopt the corresponding weaker norms of observation data. As is stated as Theorem 2.1, our basic tool is an $L^2$-weighted estimate called a Carleman estimate where the right-hand side is estimated by an $L^2$-weighted norm. We can prove a similar Carleman estimate where the right-hand side is estimated in an $H^{-1}$-weighted space (Imanuvilov and
Yamamoto [22,24]. Then \( \| a^{(1)}_{ij} - a^{(2)}_{ij} \|_{L^2(\Omega)} \) can be estimated by such an \( H^{-1} \)-Carleman estimate by a method similar to [23]. However we do not still know whether we can reduce the number of observations in the case of \( \| a^{(1)}_{ij} - a^{(2)}_{ij} \|_{L^2(\Omega)} \).

As for inverse problems of determining coefficients in parabolic equations, we refer to Danilaev [11], Elayyan and Isakov [12], Imanuvilov and Yamamoto [20,22], Isakov [25], Isakov and Kindermann [26], Ivanchov [27], Klibanov [30], Klibanov and Timonov [32], Yamamoto and Zou [41]. In particular, in [12,26,30], determination problems for principal parts are discussed. In those existing papers, the determination of a single coefficient is discussed, while we here consider an inverse problem of determining multiple coefficients of the principal part by a finite set of observations.

Our formulation is with a finite number of observations and this kind of inverse problems was firstly solved by Bukhgeim and Klibanov [8], whose methodology is based on Carleman estimates. For similar inverse problems for other equations, we refer to Baudouin and Puel [3], Bellassoued [4], Bellassoued and Yamamoto [5], Bukhgeim [7], Imanuvilov and Yamamoto [21,23], Isakov [25], Kha˘ıdarov [28], Klibanov [29,30], Klibanov and Timonov [32], Klibanov and Yamamoto [33], Yamamoto [40].

For proving Theorems 1.1 and 1.2, we establish a Carleman estimate (Thm. 2.1) for functions with non-compact support, and we apply a modification of arguments in [8,23].

For proving Theorems 1.1 and 1.2, we establish a Carleman estimate (Thm. 2.1) for functions with non-compact support, and we apply a modification of arguments in [8,23].

This paper is composed of four sections and three appendices. In Section 2 we present Carleman estimates and the proof is given in Appendix A. In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, we prove Proposition 1.1. In Appendix B, we prove estimates (1.6) and (1.10). Appendix C is devoted to the proof of the existence of a suitable weight function for our Carleman estimate.

2. CARLEMAN ESTIMATES

In this section we will prove Carleman estimates for the parabolic equation. The results in this section may have independent interests.

Lemma 2.1. Let \( \Gamma_0 \neq \emptyset \subset \partial \Omega \) be an arbitrary relatively open subset. Then there exists a function \( d \in C^2(\overline{\Omega}) \) such that

\[
d(x) > 0 \quad \text{for } x \in \Omega, \quad |\nabla d(x)| > 0 \quad \text{for } x \in \overline{\Omega}
\]

and

\[
\sum_{i,j=1}^{n} a_{ij}(x) \partial_i d(x) \nu_j(x) \leq 0, \quad x \in \partial \Omega \setminus \Gamma_0
\]

for all \( a_{ij} \in C^1(\overline{\Omega}) \), \( a_{ij} = a_{ji} \), \( 1 \leq i, j \leq n \) satisfying (1.4).

Lemma 2.1 can be derived directly from Lemma 1.2 in [19] where \( d(x) > 0 \) is not stated, and for convenience we prove this lemma in Appendix C.

Example 2.1. Let us consider a special case where \( a_{ij} = 0 \) if \( i \neq j \) and \( a_{ii} = 1 \) and

\[
\Omega = \{ x \in \mathbb{R}^n; |x| < R \}, \quad \Gamma_0 = \{ x \in \partial \Omega; (x - x_0, \nu(x)) \geq 0 \}
\]

with an arbitrarily fixed \( x_0 \in \mathbb{R}^n \setminus \overline{\Omega} \). Here \( (\cdot, \cdot) \) denotes the scalar product in \( \mathbb{R}^n \). Then we can take \( d(x) = |x - x_0|^2 \).

We present Carleman estimates for an operator \( L \):

\[
(Ly)(t, x) = \partial_t y(t, x) - \sum_{i,j=1}^{n} \partial_j (a_{ij}(x) \partial_i y(t, x)).
\]
Theorem 2.1. Assume that (1.4) holds and that $a_{ij} \in C^1(\Omega)$, $a_{ij} = a_{ji}$, $1 \leq i, j \leq n$. Let $d \in C^2(\overline{\Omega})$ be a function satisfying (2.1) and (2.2), and let $0 \leq \theta_1 < \theta < \theta_2$ be fixed.

(1) Let $\varphi(t, x) = e^{s d(t, x) - \beta(t-\theta)^2}$, where $\beta > 0$ is a constant. Then there exists a number $\lambda_0 > 0$ such that for an arbitrary $\lambda \geq \lambda_0$, we can choose a constant $s_0(\lambda) \geq 0$ satisfying: there exists a constant $C_1 = C_1(s_0, \lambda) > 0$ such that

$$
\int_{(\tau_1, \tau_2) \times \Omega} \left\{ \frac{1}{s} \left( |\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i \partial_j v|^2 \right) + s |\nabla v|^2 + s^3 |v|^2 \right\} e^{2s\varphi} \, dx \, dt
\leq C_1 \int_{(\tau_1, \tau_2) \times \Omega} |L v|^2 e^{2s\varphi} \, dx \, dt + C_1 s \int_{\tau_1}^{\tau_2} |\partial_\nu v|^2 e^{2s\varphi} \, d\Sigma
$$

for all $s > s_0$ and all $v$ satisfying

$$
\begin{aligned}
Lv &\in L^2((\tau_1, \tau_2) \times \Omega), \quad v \in L^2(\tau_1, \tau_2; H^2(\Omega) \cap H_0^1(\Omega)),

\varphi(\tau_1, \cdot) &= \varphi(\tau_2, \cdot) = 0.
\end{aligned}
$$

Moreover the constants $s_0$ and $C_1$ continuously depend on $\lambda$ and $\sum_{i,j=1}^n \|a_{ij}\|_{C^1(\overline{\Omega})}$, $\tau_1$, $\tau_2$, $\Omega$, $\rho$, where $\lambda_0$ continuously depends on $\sum_{i,j=1}^n \|a_{ij}\|_{C^1(\overline{\Omega})}$, $\tau_1$, $\tau_2$, $\Omega$, $\rho$.

(2) Let $\varphi(t, x) = e^{s d(t, x) - \beta(t-\theta)^2 + M_1}$, where $M_1 > \sup_{t \leq \theta} \beta(t-\theta)^2$. Then there exist positive constants $\lambda_0$, $s_0$ and $C_2 = C_2(\lambda_0, s_0)$ such that

$$
\int_{(\tau_1, \tau_2) \times \Omega} \left\{ \frac{1}{s} \left( |\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i \partial_j v|^2 \right) + s \lambda_0^2 |\nabla v|^2 + s^3 \lambda_0^4 |v|^2 \right\} e^{2s\varphi} \, dx \, dt
\leq C_2 \int_{(\tau_1, \tau_2) \times \Omega} |L v|^2 e^{2s\varphi} \, dx \, dt + C_2 s \lambda_0 \int_{\tau_1}^{\tau_2} \varphi |\partial_\nu v|^2 e^{2s\varphi} \, d\Sigma
$$

for all $s > s_0$, $\lambda > \lambda_0$ and all $v$ satisfying (2.5). The constants $\lambda_0$, $s_0$ and $C_2$ continuously depend on $\sum_{i,j=1}^n \|a_{ij}\|_{C^1(\overline{\Omega})}$, $\tau_1$, $\tau_2$, $\Omega$, $\rho$.

We prove the theorem in Appendix A.

As for Carleman estimates with regular weight function $\varphi(t, x)$, see Eller and Isakov [13], Hörmander [18], Isakov [25], Khaidarov [28], Klibanov and Timonov [32], Lavrent’ev, Romanov and Shishat-skii [34]. With these Carleman estimates for parabolic equations, we often have to change independent variables to address the case of an arbitrary subboundary $\Gamma_0$ of the boundary $\partial \Omega$. As a result, it becomes much more complicated to obtain a Lipschitz stability estimate over $\Omega \setminus \omega_1$, for the coefficients which one tries to identify. As for Carleman estimates for parabolic equations with singular weight function $\varphi(t, x)$, we can refer to Fursikov and Imanuvilov [15], Imanuvilov [19], Imanuvilov and Yamamoto [22,24], and such Carleman estimates hold for a function $v$ not satisfying $v(\tau_1, \cdot) = v(\tau_2, \cdot) = 0$.

Inequality (2.6) is a Carleman estimate for functions with non-compact support, and estimates the left-hand side with the weighted $L^2$-norms of $Lv$ in $(\tau_1, \tau_2) \times \Omega$ and $\partial_\nu v$ on $(\tau_1, \tau_2) \times \Gamma_0$. Once we can prove a Carleman estimate for functions with compact support, we can immediately estimate functions with non-compact support by means of a cut-off function, but the norm of the boundary value is stronger than the weighted $L^2$-norm, and any Carleman estimates for functions with compact support, does not give a better estimate for our inverse problem.

Thanks to two large parameters $\lambda$, $s$ and the form of the weight function, Carleman estimate (2.6) can be applied to inverse problems for a coupling system of parabolic and hyperbolic equations and thermoelastic plate
3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. By $0 < \tau_1 < \tau_2 < T$, we choose and fix $\tau_3, \tau_4 > 0$ such that

$$0 < \tau_3 < \tau_1 < \tau_2 < \tau_4 < T.$$  

It is sufficient to prove (1.8) with the norm in $H^2(\tau_3, \tau_4; L^2(\Gamma_0))$ of the first term on the right-hand side. Let $d \in C^2(\Omega)$ satisfy (2.1) and (2.2). We choose $\beta > 0$ such that $\sup_{x \in \Omega} d(x) < \beta \min\{|\tau_1 - \theta|^2, |\tau_2 - \theta|^2\}$. We set

$$\varphi(t, x) = \exp\{\lambda d(x) - \beta|t - \theta|^2\}.$$  

Let $d_0 = \inf_{x \in \Omega} \exp\{\lambda d(x)\} \geq 1$. Then, by the choice of $\beta > 0$, we have

$$\varphi(\theta, x) \geq d_0, \quad \varphi(\tau_1, x), \varphi(\tau_2, x) \leq 1 \leq d_0, \quad x \in \Omega.$$  

Thus for a sufficiently small $\varepsilon > 0$, we can choose a small $\delta = \delta(\varepsilon) > 0$ such that $\tau_1 < \tau_1 + 2\delta < \theta - \delta < \theta + \delta < \tau_2 - 2\delta < \tau_2$, 

$$\varphi(t, x) \geq d_0 - \varepsilon, \quad (t, x) \in [\theta - \delta, \theta + \delta] \times \Omega,$$  

and

$$\varphi(t, x) \leq d_0 - 2\varepsilon, \quad (t, x) \in ([\tau_1, \tau_1 + 2\delta] \cup [\tau_2 - 2\delta, \tau_2]) \times \Omega.$$  

We introduce a cut-off function $\chi$ satisfying $0 \leq \chi \leq 1$, $\chi \in C^\infty_0(0, T)$ and

$$\chi(t) = \begin{cases} 
0, & t \in [\tau_1, \tau_1 + \delta] \cup [\tau_2 - \delta, \tau_2], \\
1, & t \in [\tau_1 + 2\delta, \tau_2 - 2\delta].
\end{cases}$$  

(3.1)

Let us set

$$f_{ij}(x) = a_{ij}^{(1)}(x) - a_{ij}^{(2)}(x), \quad R_e(t, x) = y([a_{ij}^{(2)}(x), h_e(t, x)],$$  

(3.2)

$$(L^{(1)}y)(t, x) \equiv \partial_t y - \sum_{i,j=1}^n \partial_j(a_{ij}^{(1)}(x)\partial_i y).$$

By (1.1) and (1.2), we can see that the differences

$$\tilde{y}_e(t, x) = y([a_{ij}^{(1)}(x), h_e(t, x)] - y([a_{ij}^{(2)}(x), h_e(t, x)]$$

satisfy

$$L^{(1)}\tilde{y}_e(t, x) = \sum_{i,j=1}^n \partial_j (f_{ij}(x)\partial_i R_e(t, x)), \quad (t, x) \in (0, T) \times \Omega,$$

(3.3)

$$\tilde{y}_e(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega, \quad 1 \leq t \leq \frac{(n + 1)^2 n}{2}.$$  

(3.4)

We set

$$z_\ell(t, x) = \partial_t \tilde{y}_e(t, x), \quad \Phi = \sup_{(t, x) \in (\tau_1, \tau_2) \times \Omega} \varphi(t, x)$$  

(3.5)
Then by (3.1), (3.3) and (3.5), we have

\[
(\|z\|_{L^2(\tau_1, \tau_2; L^2(\Omega))}^2 + \|\nabla z\|_{L^2(\tau_1, \tau_2; L^2(\Omega))}^2 + \|\partial_t z\|_{L^2(\tau_1, \tau_2; L^2(\Omega))}^2)^{\frac{1}{2}}.
\]

Furthermore, by (3.1), we have

\[
\frac{(n+1)^2 \sum_{\ell = 1}^{(n+1)^2 \ell}}{n \sum_{i,j = 1}^n} \|\partial_j f_{ij}(x)\|_{L^2(\tau_1, \tau_2; L^2(\Omega))}^2 \leq C_i\sum_{i,j = 1}^n \int_{Q_1} \int_{\Gamma_0} \int_{\Gamma_0} \|\partial_j f_{ij}(x)\|_{L^2(\tau_1, \tau_2; L^2(\Omega))}^2 e^{2s\varphi} dx dt + C_1 U^2 e^{2s(d_0 - 2c)} + C_1 s \int_{\tau_1}^{\tau_2} \int_{\Gamma_0} \|\partial_t z\|_{L^2(\tau_1, \tau_2; L^2(\Omega))}^2 e^{2s\varphi} d\Sigma, \quad s \geq s_0
\]
Here we have used that \( \partial \chi \neq 0 \) only if \( \varphi(t, x) \leq d_0 - 2 \varepsilon \). On the other hand, we have
\[
\int_\Omega |\partial_t \tilde{\eta}_t(\theta, x)|^2 e^{2s\varphi(\theta, x)}dx = \int_\Omega |\chi(\theta) \partial_t \tilde{\eta}_t(\theta, x)|^2 e^{2s\varphi(\theta, x)}dx
= \int_{\tau_1}^\theta \left( \int_\Omega |\chi(t) \partial_t \tilde{\eta}_t(t, x)|^2 e^{2s\varphi(t, x)}dx \right) dt
\leq \int_{Q_1} 2(\partial_t^2 \tilde{\eta}_t || \partial_t \tilde{\eta}_t |^2 + s |\partial_t \varphi|| \chi |\partial_t \tilde{\eta}_t |^2 e^{2s\varphi} dx dt + \int_{Q_1} 2(\partial_t \tilde{\eta}_t |^2 |\partial_t \chi | e^{2s\varphi} dx dt
\leq C_2 \int_{Q_1} |\chi \partial_t z| ^2 e^{2s\varphi(t, x)}dx dt + C_2(s + 1) \int_{Q_1} |\chi z| ^2 e^{2s\varphi(t, x)}dx dt + C_2 U^2 e^{2s(d_0 - 2\varepsilon)}.
\] (3.10)

By (3.8)–(3.10), we obtain
\[
\int_\Omega |\partial_t \tilde{\eta}_t(\theta, x)|^2 e^{2s\varphi(\theta, x)}dx
\leq C_3 \left\{ \sum_{i,j=1}^{n} \sum_{|\alpha| \leq 1} \int_{Q_1} \chi^2 |\partial^\alpha f_{ij}(x)|^2 e^{2s\varphi} dx dt + U^2 e^{2s(d_0 - 2\varepsilon)} + sU^2 \Phi V^2 \right\}
\] (3.11)

for sufficiently large \( s > 0 \). Similarly we have
\[
\int_\Omega |\nabla \partial_t \tilde{\eta}_t(\theta, x)|^2 e^{2s\varphi(\theta, x)}dx
\leq C_4 \left\{ \sum_{i,j=1}^{n} \sum_{|\alpha| \leq 1} \int_{Q_1} \chi^2 |\partial^\alpha f_{ij}(x)|^2 e^{2s\varphi} dx dt + U^2 e^{2s(d_0 - 2\varepsilon)} + sU^2 \Phi V^2 \right\}
\] (3.12)

for sufficiently large \( s > 0 \). By (3.3), we have
\[
L^{(1)} \tilde{\eta}_t(\theta, x) = \sum_{i,j=1}^{n} (\partial_j f_{ij}(x)) \partial_t R_t(\theta, x) + \sum_{i,j=1}^{n} f_{ij}(x) \partial_t \partial_j R_t(\theta, x), \quad x \in \Omega
\] (3.13)

for \( 1 \leq \ell \leq \frac{(n+1)^2 n}{2} \). Let us consider the above equations for \( 1 \leq \ell \leq n + 1 \). Then we have
\[
\begin{pmatrix}
\partial_t R_1(\theta, x) \\
\partial_t R_2(\theta, x) \\
\vdots \\
\partial_t R_{n+1}(\theta, x)
\end{pmatrix} = \begin{pmatrix}
\sum_{j=1}^{n} \partial_j f_{1j}(x) \\
\sum_{j=1}^{n} \partial_j f_{2j}(x) \\
\vdots \\
\sum_{j=1}^{n} \partial_j f_{n+1j}(x)
\end{pmatrix}
\] (3.14)
Because linear system (3.14) is composed of \((n + 1)\) equations with respect to \(n\) unknowns and possesses a solution \(\left(\sum_{j=1}^{n} \partial_j f_{ij}(x), \sum_{j=1}^{n} \partial_j f_{3j}(x), \ldots, \sum_{j=1}^{n} \partial_j f_{nj}(x)\right)\), the coefficients matrix must satisfy

\[
\begin{vmatrix}
L^{(1)} \tilde{y}(\theta, x) - \sum_{j=1}^{n} f_{ij} \partial_j R_1(\theta, x) & \partial_1 R_1(\theta, x) & \cdots & \partial_n R_1(\theta, x) \\
L^{(1)} \tilde{y}_2(\theta, x) - \sum_{j=1}^{n} f_{ij} \partial_j R_2(\theta, x) & \partial_1 R_2(\theta, x) & \cdots & \partial_n R_2(\theta, x) \\
& \ddots & \ddots & \cdots \\
L^{(1)} \tilde{y}_{n+1}(\theta, x) - \sum_{j=1}^{n} f_{ij} \partial_j R_{n+1}(\theta, x) & \partial_1 R_{n+1}(\theta, x) & \cdots & \partial_n R_{n+1}(\theta, x)
\end{vmatrix} = 0.
\]

Let us set \(D^{(k)}_{ij}(x) \equiv D^{(k)}_{ij}(\{y(\{a_{ij}^{(2)}\}, h_\ell)\})\). Then we have

\[
\sum_{j=1}^{n} D^{(k)}_{ij}(x) f_{ij}(x) + 2 \sum_{i<j} D^{(k)}_{ij}(x) f_{ij}(x) = Y_1(x), \quad x \in \Omega \setminus \omega_i,
\]

where

\[
Y_1(x) = \det
\begin{vmatrix}
L^{(1)} \tilde{y}_1(\theta, x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_1)(\theta, x) & \cdots & \partial_n y(\{a_{ij}^{(2)}\}, h_1)(\theta, x) \\
L^{(1)} \tilde{y}_2(\theta, x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_2)(\theta, x) & \cdots & \partial_n y(\{a_{ij}^{(2)}\}, h_2)(\theta, x) \\
& \ddots & \ddots & \ddots \\
L^{(1)} \tilde{y}_{n+1}(\theta, x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_{n+1})(\theta, x) & \cdots & \partial_n y(\{a_{ij}^{(2)}\}, h_{n+1})(\theta, x)
\end{vmatrix}.
\]

We set

\[
Y_2(x) = \det
\begin{vmatrix}
L^{(1)} \tilde{y}_{n+2}(\theta, x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_{n+2})(\theta, x) & \cdots & \partial_n y(\{a_{ij}^{(2)}\}, h_{n+2})(\theta, x) \\
L^{(1)} \tilde{y}_{n+3}(\theta, x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_{n+3})(\theta, x) & \cdots & \partial_n y(\{a_{ij}^{(2)}\}, h_{n+3})(\theta, x) \\
& \ddots & \ddots & \ddots \\
L^{(1)} \tilde{y}_{2n+2}(\theta, x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_{2n+2})(\theta, x) & \cdots & \partial_n y(\{a_{ij}^{(2)}\}, h_{2n+2})(\theta, x)
\end{vmatrix}.
\]

Similarly to (3.15), we can obtain

\[
\sum_{j=1}^{n} D^{(k)}_{ij}(x) f_{ij}(x) + 2 \sum_{i<j} D^{(k)}_{ij}(x) f_{ij}(x) = Y_k(x), \quad x \in \Omega \setminus \omega_i,
\]
for $1 \leq k \leq \frac{n(n+1)}{2}$. Equation (3.16) is a linear system with respect to $\frac{n(n+1)}{2}$ unknown $f_{ij}$. Condition (1.7) implies that the determinant of the coefficient matrix does not vanish on $\Omega \setminus \omega_1$. Applying the Cramer formula, we can solve (3.16) uniquely with respect to $\frac{n(n+1)}{2}$ unknowns $f_{ij}$. Therefore, taking into consideration the definition of $Y_k(x)$, we can represent the solution $f_{ij}$ by

$$f_{ij}(x) = \sum_{i,j=1}^{n} c_{ij}^l(x)L^{(1)}_i(\theta, x), \quad x \in \Omega \setminus \omega_1, \quad 1 \leq i, j \leq n \quad (3.17)$$

with some $c_{ij}^l(x)$, $1 \leq i, j \leq n$, $1 \leq \ell \leq \frac{(n+1)^2n}{2}$. By the Sobolev embedding theorem (e.g., Thm. 5.4 in [1], Cor. 9.1, p. 46, in Vol. 1 of [35]), we see that $H^m(\Omega) \subset C^4(\overline{\Omega})$. Hence $y((a_{ij}^{(2)}), h \ell) \in C^3([\tau_1, \tau_2]; H^m(\Omega)) \subset C^3([\tau_1, \tau_2]; C^4(\overline{\Omega}))$, and so $c_{ij}^l \in C^4(\Omega \setminus \omega_1)$, $1 \leq i, j \leq n$, $1 \leq \ell \leq \frac{(n+1)^2n}{2}$. Moreover, since $c_{ij}^l$ are given by values (not including the derivatives) of $D_{ij}^l$, we see that $\|c_{ij}^l\|_{C^3(\Omega \setminus \omega_1)} \leq C_5$.

By noting also that $f_{ij}(x) = 0$, $x \in \omega_1$, $1 \leq i, j \leq n$, by means of (3.17) and $c_{ij}^l \in C^4(\Omega \setminus \omega_1)$, we have

$$\int_{\Omega} \sum_{|\alpha| \leq 1} |\partial_{\alpha} f_{ij}(x)|^2 e^{2s\varphi(\theta, x)} dx \leq C_5 \sum_{|\alpha| \leq 1} \sum_{\ell = 1}^{\frac{(n+1)^2n}{2}} \int_{\Omega} |\partial_{\alpha} \partial_\ell g(\theta, x)|^2 e^{2s\varphi(\theta, x)} dx$$

$$+ C_5 \sum_{\ell = 1}^{\frac{(n+1)^2n}{2}} \sum_{|\alpha| \leq 3} \int_{\Omega} |\partial_{\alpha} \partial_\ell g(\theta, x)|^2 e^{2s\varphi(\theta, x)} dx, \quad 1 \leq i, j \leq n, \quad 1 \leq \ell \leq \frac{(n+1)^2n}{2}. \quad (3.18)$$

By (3.11) and (3.12), we have

$$\sum_{\ell = 1}^{\frac{(n+1)^2n}{2}} \sum_{|\alpha| \leq 1} \int_{\Omega} |\partial_{\alpha} \partial_\ell g(\theta, x)|^2 e^{2s\varphi(\theta, x)} dx$$

$$\leq C_6 \sum_{\ell = 1}^{\frac{(n+1)^2n}{2}} \sum_{|\alpha| \leq 1} \int_{\Omega} |\partial_{\alpha} f_{ij}(x)|^2 e^{2s\varphi(\theta, x)} \left( \int_{\tau_1}^{\tau_2} e^{2s(\varphi(t, x) - \varphi(\theta, x))} dt \right) dx$$

$$+ C_6 U^2 e^{2s(d_0 - 2c)} + C_6 e^{2\varphi V^2} \quad (3.19)$$

for all large $s > 0$. By (3.18) and (3.19), we obtain

$$\sum_{\ell = 1}^{\frac{(n+1)^2n}{2}} \sum_{|\alpha| \leq 1} \int_{\Omega} |\partial_{\alpha} f_{ij}(x)|^2 e^{2s\varphi(\theta, x)} dx$$

$$\leq C_7 \sum_{\ell = 1}^{\frac{(n+1)^2n}{2}} \sum_{|\alpha| \leq 1} \int_{\Omega} |\partial_{\alpha} f_{ij}(x)|^2 e^{2s\varphi(\theta, x)} \left( \int_{\tau_1}^{\tau_2} e^{2s(\varphi(t, x) - \varphi(\theta, x))} dt \right) dx$$

$$+ C_7 \sum_{\ell = 1}^{\frac{(n+1)^2n}{2}} \sum_{|\alpha| \leq 3} \int_{\Omega} |\partial_{\alpha} \partial_\ell g(\theta, x)|^2 e^{2s\varphi(\theta, x)} dx + C_7 U^2 e^{2s(d_0 - 2c)} + C_7 e^{2\varphi V^2} \quad (3.20)$$
for large $s > 0$. Applying the Lebesgue theorem, we have

$$\sup_{x \in \Omega} \left| \int_{\tau_1}^{\tau_2} e^{2s(\varphi(t,x) - \varphi(\theta,x))} dt \right| = \sup_{x \in \Omega} \left| \int_{\tau_1}^{\tau_2} \exp \left( 2se^{-\lambda_1(t-\theta)} \right) dt \right|$$

$$\leq \int_{\tau_1}^{\tau_2} \exp \left( 2se^{-\lambda_1(t-\theta)} \right) dt = o(1) \quad \text{as} \quad s \to \infty,$$

where $d_1 = \inf_{x \in \Omega} d(x)$. Then

$$\sum_{i,j=1}^{n} \sum_{|\alpha| \leq 1} \int_{\Omega} \left| \partial_{x}^{\alpha} f_{ij}(x) \right|^2 e^{2s\varphi(\theta,x)} \left( \int_{\tau_1}^{\tau_2} e^{2s(\varphi(t,x) - \varphi(\theta,x))} dt \right) dx$$

$$= o(1) \sum_{i,j=1}^{n} \sum_{|\alpha| \leq 1} \int_{\Omega} \left| \partial_{x}^{\alpha} f_{ij}(x) \right|^2 e^{2s\varphi(\theta,x)} dx \quad \text{as} \quad s \to \infty.$$

Hence, from (3.20) we have

$$(1 - o(1)) \sum_{i,j=1}^{n} \sum_{|\alpha| \leq 1} \int_{\Omega} \left| \partial_{x}^{\alpha} f_{ij}(x) \right|^2 e^{2s\varphi(\theta,x)} dx \leq C_8 U^2 e^{2s(d_0 - 2\epsilon)} + C_8 \kappa^2 \Phi^2$$

$$+ C_8 \sum_{\ell=1}^{n} \sum_{|\alpha| \leq 1} \int_{\Omega} \left| \partial_{x}^{\alpha} \widetilde{y}_\ell(\theta, x) \right|^2 e^{2s\varphi(\theta,x)} dx \quad \text{as} \quad s \to \infty.$$

By $\varphi(\theta, x) \geq d_0$ for $x \in \Omega$, we obtain

$$(1 - o(1)) \sum_{i,j=1}^{n} \sum_{|\alpha| \leq 1} \int_{\Omega} \left| \partial_{x}^{\alpha} f_{ij}(x) \right|^2 dx \leq C_9 U^2 e^{2s(d_0 - 2\epsilon)} + C_9 \kappa^2 \Phi^2$$

$$+ C_9 \sum_{\ell=1}^{n} \sum_{|\alpha| \leq 1} \int_{\Omega} \left| \partial_{x}^{\alpha} \widetilde{y}_\ell(\theta, x) \right|^2 dx \quad \text{as} \quad s \to \infty,$$

that is,

$$(1 - o(1)) \sum_{i,j=1}^{n} \sum_{|\alpha| \leq 1} \int_{\Omega} \left| \partial_{x}^{\alpha} f_{ij}(x) \right|^2 dx \leq C_9 U^2 e^{4s\epsilon} + C_9 \kappa^2 \epsilon \Phi^2$$

$$+ C_9 \sum_{\ell=1}^{n} \sum_{|\alpha| \leq 1} \int_{\Omega} \left| \partial_{x}^{\alpha} \widetilde{y}_\ell(\theta, x) \right|^2 dx \quad \text{as} \quad s \to \infty. \quad (3.21)$$

On the other hand, we can prove the following estimate:

$$U^2 \leq C_{10} V^2 + C_{10} \sum_{i,j=1}^{n} \sum_{|\alpha| \leq 1} \int_{\Omega} \left| \partial_{x}^{\alpha} f_{ij}(x) \right|^2 dx. \quad (3.22)$$
In fact, by (3.3) and (3.4) we have

\[
\begin{dcases}
L^{(1)} \partial_t \tilde{y}_\ell(t,x) = \sum_{i,j=1}^{n} \partial_j (f_{ij}(x) \partial_i R_\ell(t,x)), \quad (t, x) \in (0, T) \times \Omega, \\
\partial_t \tilde{y}_\ell(t,x) = 0, \quad (t, x) \in (0, T) \times \partial \Omega, \quad 1 \leq \ell \leq \frac{(n+1)^2 n}{2}
\end{dcases}
\tag{3.23}
\]

Apply Lemma 1.1 in [19] (cf. Lem. 2.4 in [20]) to (3.23). Then we can see that there exist \( \psi_0 \in C^2(\Omega) \) and a constant \( \sigma_0 > 0 \) such that for a constant \( \sigma \geq \sigma_0 \) we can choose \( \eta_0(\sigma) > 0 \) such that for each \( \eta > \eta_0(\sigma) \), we have

\[
\int_{(\tau_3, \tau_4) \times \Omega} \left( \frac{\eta \sigma \psi(x)}{(t - \tau_3)(\tau_4 - t)} |\nabla \partial_t \tilde{y}_\ell|^2 + \frac{\eta^3 e^{3 \sigma \psi(x)}}{(t - \tau_3)^3 (\tau_4 - t)^3} |\partial_t \tilde{y}_\ell|^2 \right) J(t,x) \, dx \, dt 
\leq C_{11} \sum_{i,j=1}^{n} \sum_{|\alpha| \leq 1} \int_{(\tau_3, \tau_4) \times \Omega} |\partial^\alpha f_{ij}(x)|^2 J(t,x) \, dx \, dt 
\]

\[
+ C_{11} n \int_{\tau_3}^{\tau_4} \int_{\Gamma_0} |\partial_t \tilde{y}_\ell|^2 \frac{\eta \sigma \psi(x)}{(t - \tau_3)(\tau_4 - t)} J(t,x) \, d\Sigma.
\]

Here we set

\[
J(t,x) = \exp \left\{ \frac{2\eta \left( e^{\sigma \psi(x)} - e^{2\sigma \|\psi\|_{C,\Omega}} \right)}{(t - \tau_3)(\tau_4 - t)} \right\}.
\]

By the proof of Lemma 1.1 in [19], we see that the constant \( C_{11} > 0 \) can be taken uniformly in \( a_{ij} \in U \), and see also [9,15] as for the proof. We note that \( C_{11} \) is dependent on \( \sigma \), but independent of \( \eta \), and the constant \( \sigma_0 \) depends on \( U \). We fix \( \sigma > \sigma_0 \) and \( \eta > \eta_0(\sigma) \). Then

\[
0 < C_{11} \leq J(t,x)
\]

for \( x \in \overline{\Omega} \) and \( \tau_1 < t < \tau_2 \) and

\[
J(t, x), \quad \frac{\eta \sigma \psi(x)}{(t - \tau_3)(\tau_4 - t)} J(t,x) \leq C_{12}
\]

for \( x \in \overline{\Omega} \) and \( \tau_3 < t < \tau_4 \). Hence we have

\[
\int_{Q_1} \left( |\nabla \partial_t \tilde{y}_\ell|^2 + |\partial_t \tilde{y}_\ell|^2 \right) \, dx \, dt 
\leq C_{13} \sum_{i,j=1}^{n} \sum_{|\alpha| \leq 1} \int_{(\tau_3, \tau_4) \times \Omega} |\partial^\alpha f_{ij}(x)|^2 \, dx \, dt + C_{13} \int_{\tau_3}^{\tau_4} \int_{\Gamma_0} |\partial_t \tilde{y}_\ell|^2 \, d\Sigma.
\tag{3.24}
\]

Similarly, we can obtain

\[
\int_{Q_1} \left( |\nabla \partial_t^2 \tilde{y}_\ell|^2 + |\partial_t^2 \tilde{y}_\ell|^2 \right) \, dx \, dt 
\leq C_{13} \sum_{i,j=1}^{n} \sum_{|\alpha| \leq 1} \int_{(\tau_3, \tau_4) \times \Omega} |\partial^\alpha f_{ij}(x)|^2 \, dx \, dt + C_{13} \int_{\tau_3}^{\tau_4} \int_{\Gamma_0} |\partial_t^2 \tilde{y}_\ell|^2 \, d\Sigma.
\tag{3.25}
\]

By (3.24) and (3.25), we complete the proof of (3.22).

We can obtain (1.6) by substituting (3.22) into (3.21) and taking \( s \) large enough. Thus the proof of Theorem 1.1 is completed.
Proof of Theorem 1.2. Let $B = (b_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix such that $b_{ij} \in \mathbb{R}$ and $\det B > 0$. We set

$$\tilde{g}_i(x) = \sum_{j=1}^{n} b_{ij} x_j, \quad 1 \leq i \leq n$$

and

$$\begin{align*}
\tilde{g}_1(x) &= x_1^2, \quad \tilde{g}_2(x) = 2x_1x_2, \quad \tilde{g}_3(x) = 2x_1x_3, \ldots, \quad \tilde{g}_n(x) = 2x_1x_n, \\
\tilde{g}_{n+1}(x) &= x_2^2, \quad \tilde{g}_{n+2}(x) = 2x_2x_3, \ldots, \quad \tilde{g}_{2n-1}(x) = 2x_2x_n, \\
& \quad \vdots \\
\tilde{g}_{(n+1)n}(x) &= x_{n-1}^2, \quad \tilde{g}_{(n+1)n-1}(x) = 2x_{n-1}x_n, \\
\tilde{g}_{(n+1)n}(x) &= x_n^2.
\end{align*}$$

Let us define an $(n+1)^2$-dimensional vector by

$$\begin{pmatrix}
g_1(x), g_2(x), \ldots, g_{n+1}(x), g_{n+2}(x), g_{n+3}(x), \ldots, g_{2n+2}(x), \\
g_{n+3+2n^2-n}(x), g_{n+3+2n^2-n+1}(x), \ldots, g_{n+3+2n^2-n+n}(x)
\end{pmatrix} = \begin{pmatrix}
\tilde{g}_1(x), \tilde{g}_1(x), \ldots, \tilde{g}_n(x), \tilde{g}_2(x), \tilde{g}_1(x), \ldots, \tilde{g}_n(x), \\
\tilde{g}_{2(n+1)}(x), \tilde{g}_1(x), \ldots, \tilde{g}_n(x)
\end{pmatrix}.$$  \hfill (3.26)

Therefore, noting that $\partial_i \partial_j \tilde{g}_k = 0$, we obtain

$$D_2^k({\{g_i\}})(x) = \det \begin{pmatrix}
\partial_i \partial_j \tilde{g}_k & * \\
0 & B
\end{pmatrix} = (\partial_i \partial_j \tilde{g}_k) \det B, \quad 1 \leq k \leq \frac{n(n+1)}{2}, 1 \leq i,j \leq n.$$  

Hence

$$D({\{g_i\}})(x) = \begin{pmatrix}
2\det B & 0 & 0 & \cdots & 0 & 0 \\
0 & 2\det B & 0 & \cdots & 0 & 0 \\
0 & 0 & 2\det B & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2\det B & 0 \\
0 & 0 & 0 & \cdots & 0 & 2\det B
\end{pmatrix}.$$  

Consequently we have $\det D({\{g_i\}})(x) = (2\det B)^{\frac{n(n+1)}{2}} > 0$. We introduce a cut-off function $\chi_1 \in C_0^\infty(\Omega)$ such that $\chi_1 \equiv 1$ on $\Omega \setminus \omega_1$. Then we have

$$\chi_1 g_i \in D \left( A^{\frac{n+1}{2}} \right) \quad \text{and} \quad D({\{\chi_1 g_i\}})(x) > 0, \quad x \in \Omega \setminus \omega_1. \hfill (3.27)$$

Here we recall that $A$ is defined by (1.9).
By Proposition 1.1, for arbitrarily \( \mu_\ell \in L^2(\Omega) \), we can choose \( h_\ell \in C^\infty_0((0, T) \times \omega) \), \( 1 \leq \ell \leq \frac{\left( n+3 \right) n}{2} \), so that for a sufficiently small \( \varepsilon > 0 \) we have

\[
\| y(\{a^{(2)}_ij\}, h_\ell, \mu_\ell)(\theta, \cdot) - \chi_1 \hat{y}_e \|_{H^m(\Omega)} \leq \varepsilon, \quad 1 \leq \ell \leq \frac{(n+1)n}{2}
\]

and

\[
\| y(\{a^{(2)}_ij\}, h_{\ell+1}+k, \mu_{n+1+2}+k)(\theta, \cdot) - \chi_1 \hat{y}_k \|_{H^m(\Omega)} \leq \varepsilon, \quad 1 \leq k \leq n.
\]

Here we note that \( y(\{a^{(2)}_ij\}, h, \mu) \) denotes the solution to (1.1) and (1.2) with \( y(0, \cdot) = \mu \). Since \( m > \frac{n}{2} + 3 \), we have \( H^m(\Omega) \subset C^2(\Omega) \). Then we can obtain

\[
\| y(\{a^{(2)}_ij\}, h_\ell, \mu_\ell)(\theta, \cdot) - \chi_1 \hat{y}_e \|_{C^2(\overline{\Omega \setminus \omega_1})} \leq \varepsilon, \quad 1 \leq \ell \leq \frac{(n+1)n}{2}
\]

and

\[
\| y(\{a^{(2)}_ij\}, h_{\ell+1}+k, \mu_{n+1+2}+k)(\theta, \cdot) - \chi_1 \hat{y}_k \|_{C^2(\overline{\Omega \setminus \omega_1})} \leq \varepsilon, \quad 1 \leq k \leq n.
\]

Let

\[
(\hat{h}_\ell)_{1 \leq \ell \leq \frac{(n+1)^2 n}{2}} = \left( \hat{h}_1, \hat{h}_2, \ldots, \hat{h}_{n+1}, \hat{h}_{n+2}, \hat{h}_{n+3}, \ldots, \hat{h}_{2n+2}, \ldots, \right)
\]

\[
\hat{h}_{n^3+2n^2-n}, \hat{h}_{n^3+2n^2-n+1}, \ldots, \hat{h}_{n^3+2n^2-n+n}
\]

\[
\equiv \left( h_1, h_{n+1}+1, \ldots, h_{n+1}+n, h_2, h_{n+1}+1, \ldots, h_{n+1}+n, \ldots, h_{n+1}, h_{n+1}+1, \ldots, h_{n+1}+n \right)
\]

and

\[
(\hat{\mu}_\ell)_{1 \leq \ell \leq \frac{(n+1)^2 n}{2}} \equiv \left( \hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_{n+1}, \hat{\mu}_{n+2}, \hat{\mu}_{n+3}, \ldots, \hat{\mu}_{2n+2}, \ldots, \right)
\]

\[
\hat{\mu}_{n^3+2n^2-n}, \hat{\mu}_{n^3+2n^2-n+1}, \ldots, \hat{\mu}_{n^3+2n^2-n+n}
\]

\[
\equiv \left( \mu_1, \mu_{n+1}+1, \ldots, \mu_{n+1}+n, \mu_2, \mu_{n+1}+1, \ldots, \mu_{n+1}+n, \ldots, \mu_{n+1}, \mu_{n+1}+1, \ldots, \mu_{n+1}+n \right)
\]

By (3.27)-(3.31), we can obtain

\[
D(y(\{a^{(2)}_ij\}, \hat{h}_\ell, \hat{\mu}_\ell))(\theta, x) > 0, \quad x \in \Omega \setminus \omega_1
\]

by taking \( \varepsilon \) small enough. Thus, by applying Theorem 1.1 to \( \hat{h}_\ell, 1 \leq \ell \leq \frac{(n+1)^2 n}{2} \), the proof of Theorem 1.2 is completed.
4. PROOF OF PROPOSITION 1.1

For the proof, we will use the dual space \( D(A^{−m_1}) \), which is defined as follows. By \( \| \cdot \| \) and \( (\cdot, \cdot) \) we denote the norm and the scalar product in \( L^2(\Omega) \), respectively. We recall that the operator \( A \) in \( L^2(\Omega) \) is defined by (1.9). Henceforth \( [\gamma] \) denotes the greatest integer not exceeding \( \gamma \in \mathbb{R} \). Since \( a_{ij} \in C^{m−1,1}(\overline{\Omega}) \), elliptic regularity results (e.g., Thm. 8.13, p. 187, in [16]) yield

\[
C_1^{-1}\|Au\| \leq \|u\|_{H^2(\Omega)} \leq \|Au\|, \quad u \in D(A).
\]

Next, using \( \|A^{k−1}u\| \leq C'_k\|A^k u\| \) for \( u \in D(A^k) \) and \( k \in \mathbb{N} \), we repeatedly apply Theorem 8.13 in [16] and we see

\[
C_1^{-1}\|A^k u\| \leq \|u\|_{H^{2k}(\Omega)} \leq C_1\|A^k u\|, \quad u \in D(A^k)
\]

(4.1)

where \( k = 0, 1, \ldots, \left[ \frac{m_1+1}{2} \right] \). Here the constant \( C_1 > 0 \) is independent of \( u \in D(A^k) \), and \( \| \cdot \|_{H^{2k}(\Omega)} \) denotes the norm in \( H^{2k}(\Omega) \). In particular,

\[
C_1^{-1}\|A^{m_1} u\| \leq \|u\|_{H^{2m_1}(\Omega)} \leq C_1\|A^{m_1} u\|, \quad u \in D(A^{m_1}).
\]

(4.2)

Moreover it is known that there exists a sequence of eigenvalues \( \{\kappa_j\}_{j \in \mathbb{N}} \) of \( A \):

\[
0 < \kappa_1 \leq \kappa_2 \leq \ldots \longrightarrow \infty,
\]

where \( \kappa_j \) appears the same time as its multiplicity. Then we can form an orthonormal basis \( \{e_j\}_{j \in \mathbb{N}} \) in \( L^2(\Omega) \) such that \( Ae_j = \kappa_j e_j \). We have

\[
\|A^\ell u\| = \left( \sum_{j=1}^{\infty} \kappa_j^{2\ell} (u, e_j)^2 \right)^{\frac{1}{2}}
\]

and \( D(A^\ell), \ell \in \mathbb{N} \cup \{0\} \), is a Hilbert space with the scalar product

\[
(u, v)_{D(A^\ell)} = \sum_{j=1}^{\infty} \kappa_j^{2\ell} (u, e_j)(v, e_j).
\]

In particular, \( D(A^0) = L^2(\Omega) \), and \( D(A^{m_1}) \) is dense in \( L^2(\Omega) \), and the embedding is continuous. Identifying the dual space of \( L^2(\Omega) \) with \( L^2(\Omega) \), we have \( D(A^{m_1}) \subset L^2(\Omega) \subset (D(A^{m_1}))' \) topologically (e.g., [6]). Henceforth we set \( (D(A^{m_1}))' = D(A^{-m_1}) \) and \( D(A^{m_1}) < u, \xi >_{D(A^{-m_1})} \) denotes the value of a linear functional \( \xi \in (D(A^{m_1}))' \) at \( u \). We note that

\[
D(A^{m_1}) < u, \xi >_{D(A^{-m_1})} = (u, \xi)
\]

if \( u \in D(A^{m_1}) \) and \( \xi \in L^2(\Omega) \) (e.g., V.2 in [6]).

Since \( L^2(\Omega) \) is dense in \( D(A^{−m_1}) \), in terms of the choice of the norm on \( D(A) \), we see that \( A^{−m_1} \) is extended uniquely to a bounded operator in \( D(A^{−m_1}) \), and the embedding is continuous. Identifying the dual space of \( L^2(\Omega) \) with \( L^2(\Omega) \), we have \( D(A^{m_1}) \subset L^2(\Omega) \subset (D(A^{m_1}))' \) topologically (e.g., [6]). Henceforth we set \( (D(A^{m_1}))' = D(A^{-m_1}) \) and \( D(A^{m_1}) < u, \xi >_{D(A^{-m_1})} \) denotes the value of a linear functional \( \xi \in (D(A^{m_1}))' \) at \( u \). We note that

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\[
D(A^{m_1}) < u, \xi >_{D(A^{-m_1})} = (u, \xi)
\]

if \( u \in D(A^{m_1}) \) and \( \xi \in L^2(\Omega) \) (e.g., V.2 in [6]).

Now we proceed to the proof of the proposition. Without loss of generality, we can suppose that \( y(0) = \mu = 0 \), because the parabolic equation (1.1) is linear. First we consider

\[
\begin{align*}
\frac{\partial z}{\partial t} + Az(t, x) &= 0, \quad (t, x) \in Q, \\
z &= 0, \quad (t, x) \in \Sigma, \\
z(T, x) &= \xi(x), \quad x \in \Omega,
\end{align*}
\]

(4.3)
where $\xi \in \mathcal{D}(A^{-m_1})$. We can verify (e.g., [37]) that for every $\xi \in \mathcal{D}(A^{-m_1})$, there exists a unique solution $z \in C([0,T];\mathcal{D}(A^{-m_1}))$ such that

$$\|z\|_{C([0,T];\mathcal{D}(A^{-m_1}))} \leq C\|\xi\|_{\mathcal{D}(A^{-m_1})}.$$ 

Recall that $g(\{a_{ij}\},h,0)(t,x)$ is the solution to (1.1) and (1.2) with $g(0) = 0$ where $h \in C_0^\infty((0,T) \times \omega)$. We will prove

$$\mathcal{D}(A^{m_1}) \langle g(\{a_{ij}\},h,0)(T,\cdot),\xi \rangle_{\mathcal{D}(A^{-m_1})} = L^2(0,T;\mathcal{D}(A^{m_1})) \langle h,z \rangle_{L^2(0,T;\mathcal{D}(A^{-m_1}))}. \tag{4.4}$$

In fact, by the density of $C_0^\infty(\Omega)$ in $\mathcal{D}(A^{m_1})$, there exists a sequence $\xi_k \in C_0^\infty(\Omega)$, $k \in \mathbb{N}$ such that $\xi_k \rightarrow \xi$ in $\mathcal{D}(A^{m_1})$. By $z_k$ we denote the solution to (4.3) with the final value $\xi_k$ at $t = T$. Then $z_k, g(\{a_{ij}\},h,0) \in C^4([0,T];L^2(\Omega)) \cap C([0,T];\mathcal{D}(A))$ (e.g., Thm. 3.5, p. 114, in [37]). Therefore we can multiply (1.1) with $z_k(t,x)$, so that by integrating by parts, we have

$$(g(\{a_{ij}\},h,0)(T,\cdot),\xi_k)_{L^2(\Omega)} = (h,z_k)_{L^2(0,T \times \Omega)}.$$ 

Noting that $h \in C_0^\infty((0,T) \times \omega)$, we can further rewrite it as

$$\mathcal{D}(A^{m_1}) \langle g(\{a_{ij}\},h,0)(T,\cdot),\xi_k \rangle_{\mathcal{D}(A^{-m_1})} = L^2(0,T;\mathcal{D}(A^{m_1})) \langle h,z_k \rangle_{L^2(0,T;\mathcal{D}(A^{-m_1}))}.$$ 

Since $g(\{a_{ij}\},h,0)(t,\cdot) = \int_0^t e^{-(t-s)A}h(s,\cdot)ds$ for $t > 0$ (e.g., [37]) and $h \in C_0^\infty((0,T) \times \omega)$, we directly see that $g(\{a_{ij}\},h,0)(T,\cdot) \in \mathcal{D}(A^{m_1})$. Hence, as $k \rightarrow \infty$, we have

$$\mathcal{D}(A^{m_1}) \langle g(\{a_{ij}\},h,0)(T,\cdot),\xi \rangle_{\mathcal{D}(A^{-m_1})} = L^2(0,T;\mathcal{D}(A^{m_1})) \langle h,z \rangle_{L^2(0,T;\mathcal{D}(A^{-m_1}))}.$$ 

Thus we proved (4.4).

For the proof of the proposition, it is sufficient to verify that if

$$\mathcal{D}(A^{m_1}) \langle g(\{a_{ij}\},h,0)(T,\cdot),\xi \rangle_{\mathcal{D}(A^{-m_1})} = 0 \tag{4.5}$$

for all $h \in C_0^\infty((0,T) \times \omega)$, then $\xi = 0$. Let us assume (4.5). Then for any $\delta \in (0,T)$, by (4.4) we have

$$L^2(0,T-\delta;\mathcal{D}(A^{m_1})) \langle h,z \rangle_{L^2(0,T-\delta;\mathcal{D}(A^{-m_1}))} = 0 \quad \text{for all } h \in C_0^\infty((0,T-\delta) \times \omega).$$

By the smoothing property for the parabolic equation (e.g., [37]), we know that $z \in L^2(0,T-\delta;\mathcal{D}(A)) \subset L^2(0,T-\delta;H^2(\Omega) \cap H^1_0(\Omega))$. Therefore

$$L^2(0,T-\delta;\mathcal{D}(A^{m_1})) \langle h,z \rangle_{L^2(0,T-\delta;\mathcal{D}(A^{-m_1}))} = (h,z)_{L^2(0,T-\delta;L^2(\Omega))}$$

for all $h \in C_0^\infty((0,T-\delta) \times \omega)$. Hence we have $z = 0$ in $(0,T-\delta) \times \omega$. By the unique continuation for parabolic equations (e.g., Saut and Scheurer [38]), we can see that $z = 0$ in $(0,T-\delta) \times \Omega$. We note that a Carleman estimate yields the unique continuation by an argument similar to the one in obtaining (3.22). Since $\delta$ is arbitrary and $z \in C([0,T];\mathcal{D}(A^{-m_1}))$, we can obtain $\xi = 0$. Thus the proof of Proposition 1.1 is completed.
Appendix A. Proof of Theorem 2.1

The proof is adapted from the proofs in [15,19], where the authors treat the case of the weight function containing a singular function.

Henceforth we take $\lambda > 1$ and by $C_j$ we denote generic constants which do not depend on $s$ and $\lambda$, and continuously depend on $\sum_{i,j=1}^n |a_{ij}|_{C^1(\Omega)}$, $\tau_1, \tau_2, \Omega, r$. It suffices to prove (2.6) for the operator

$$\tilde{L} v = \partial_t v - \sum_{i,j=1}^n a_{ij}(x) \partial_i \partial_j v.$$ 

In fact, we have

$$Lv = \tilde{L} v - \sum_{i,j=1}^n (\partial_j a_{ij}(x)) \partial_i v(t, x)$$

and $\partial_j a_{ij} \in L^\infty(\Omega)$. Therefore

$$C_1 \left( \int_{(\tau_1, \tau_2) \times \Omega} |L v|^2 e^{2s\varphi} dx \, dt + \int_{(\tau_1, \tau_2) \times \Omega} |\nabla v|^2 e^{2s\varphi} dx \, dt \right)$$

$$\geq \int_{(\tau_1, \tau_2) \times \Omega} |\tilde{L} v|^2 e^{2s\varphi} dx \, dt.$$

Hence in (2.6) with $\tilde{L}$, we further choose $s_0 > 0$ sufficiently large and we absorb the term $C_1 \int_{(\tau_1, \tau_2) \times \Omega} |\nabla v|^2 e^{2s\varphi} dx \, dt$ into the left-hand side. Moreover, fixing $\lambda$ in (2.6), we obtain (2.4).

Henceforth we set

$$Q_1 = (\tau_1, \tau_2) \times \Omega, \quad \Sigma_1 = (\tau_1, \tau_2) \times \partial \Omega$$

and

$$a(x, \zeta, \xi) \equiv \sum_{i,j=1}^n a_{ij}(x) \zeta_i \xi_j, \quad \zeta = (\zeta_1, ..., \zeta_n), \quad \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n, \quad (t, x) \in Q_1.$$ 

Let $w(t, x) = e^{s\varphi} v(t, x)$. By (2.5) we have

$$w(\tau_1, \cdot) = w(\tau_2, \cdot) = 0 \quad \text{in} \ \Omega. \quad (A.1)$$

Let

$$Pw \equiv e^{s\varphi} \tilde{L} e^{-s\varphi} w = e^{s\varphi} \tilde{L} v \quad \text{in} \ Q_1.$$ 

It is easy to see that the operator $P$ has the form

$$Pw = \partial_t w - \sum_{i,j=1}^n a_{ij} \partial_i \partial_j w + 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(\partial_i d)\partial_j w$$

$$+ s\lambda^2 \varphi w \sum_{i,j=1}^n a_{ij}(\partial_i d)\partial_j d - s^2 \lambda^2 \varphi^2 w \sum_{i,j=1}^n a_{ij}(\partial_i d)\partial_j d$$

$$+ s\lambda \varphi w \sum_{i,j=1}^n a_{ij} \partial_i \partial_j d - sw \partial_t \varphi. \quad (A.2)$$
We set
\[ P_1w + P_2w = Pw - s\lambda^2w^2 \sum_{i,j=1}^n a_{ij}(\partial_i d)\partial_j d \]
\[ -s\lambda \varphi w \sum_{i,j=1}^n a_{ij}\partial_i \partial_j d + sw\partial_i \varphi \equiv f_s \text{ in } Q_1, \]  
(A.3)
where
\[ P_1w = -\sum_{i,j=1}^n a_{ij}\partial_i \partial_j w - s^2\lambda^2\varphi^2 a(x, \nabla d, \nabla d)w, \]  
(A.4)
\[ P_2w = \partial_t w + 2s\lambda \varphi \sum_{i,j=1}^n a_{ij}(\partial_i d)\partial_j w. \]  
(A.5)
Equation (A.3) implies
\[ \|f_s\|_{L^2(Q_1)}^2 = \|P_1w\|_{L^2(Q_1)}^2 + \|P_2w\|_{L^2(Q_1)}^2 + 2\langle P_1w, P_2w \rangle_{L^2(Q_1)}. \]  
(A.6)
By virtue of (A.4) and (A.5) we have
\[ (P_1w, P_2w)_{L^2(Q_1)} = \left( -\sum_{i,j=1}^n a_{ij}\partial_i \partial_j w - s^2\lambda^2\varphi^2 a(x, \nabla d, \nabla d), \partial_t w \right)_{L^2(Q_1)} \]
\[ -\int_{Q_1} 2s^3\lambda^3w^2\varphi^3 a(x, \nabla d, \nabla d)a(x, \nabla d, \nabla \omega) \, dx \, dt - \int_{Q_1} 2s\lambda \varphi \sum_{i,j=1}^n a_{ij}\partial_i \partial_j w \sum_{k,l=1}^n a_{kl}(\partial_k d)\partial_l \omega \, dx \, dt \]
\[ \equiv I_1 + I_2 + I_3. \]  
(A.7)
We note
\[ \nabla w = (\partial_t w)\nu \quad \text{on } \Sigma_1, \]  
(A.8)
because \( v \in L^2(\tau_1, \tau_2; H^2(\Omega) \cap H_0^1(\Omega)) \) implies \( w|_{\Sigma_1} = 0. \)
Noting also that \( a_{ij} = a_{ji} \) and \( w(\tau_1, \cdot) = w(\tau_2, \cdot) = 0 \), we transform \( I_1, I_2 \) and \( I_3 \) by integrating by parts respectively:
\[ I_1 = \int_{Q_1} \left[ \partial_t w \sum_{i,j=1}^n (\partial_i a_{ij})\partial_j w + \sum_{i,j=1}^n a_{ij}(\partial_j w)\partial_i \partial_j w - \frac{s^2\lambda^2}{2}\varphi^2 a(x, \nabla d, \nabla d)\partial_\nu(w^2) \right] \, dx \, dt \]
\[ = \int_{Q_1} \left[ \partial_t w \sum_{i,j=1}^n (\partial_i a_{ij})\partial_j w + w^2 \frac{s^2\lambda^2}{2}(\varphi^2 a(x, \nabla d, \nabla d)) \right] \, dx \, dt, \]  
(A.9)
\[ I_2 = -\int_{Q_1} s^3\lambda^3\varphi^3 \sum_{i,j=1}^n a_{ij}a(x, \nabla d, \nabla d)(\partial_\nu d)\partial_j(w^2) \, dx \, dt \]
\[ = \int_{Q_1} \left[ 3s^3\lambda^4w^2\varphi^3 a(x, \nabla d, \nabla d)^2 + s^3\lambda^3w^2\varphi^3 \sum_{i,j=1}^n \partial_j(a_{ij}a(x, \nabla d, \nabla d)\partial_\nu d) \right] \, dx \, dt \]  
(A.10)
Integrating by parts, we have

\[ I_3 = \int_{Q_1} - \left( \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j w \right) \left( 2s \lambda \varphi \sum_{k,\ell = 1}^{n} a_{k\ell} (\partial_k d) \partial_{\ell w} \right) \, dx \, dt \]

\[ = \int_{Q_1} \left\{ \sum_{i,j=1}^{n} 2s \lambda \varphi (\partial_j a_{ij}) (\partial_i w) \sum_{k,\ell = 1}^{n} a_{k\ell} (\partial_k d) \partial_{\ell w} + 2s \lambda^2 \varphi \sum_{i,j=1}^{n} a_{ij} (\partial_i w) (\partial_j d) \sum_{k,\ell = 1}^{n} a_{k\ell} (\partial_k d) \partial_{\ell w} \right. \]

\[ + 2s \lambda \varphi \sum_{i,j=1}^{n} \left[ a_{ij} \partial_i w \sum_{k,\ell = 1}^{n} \partial_j (a_{k\ell} \partial_k d) \partial_{\ell w} \right] + 2s \lambda \varphi \sum_{i,j=1}^{n} \left( a_{ij} \partial_i w \sum_{k,\ell = 1}^{n} a_{k\ell} (\partial_k d) \partial_{\ell w} \right) \right\} \, dx \, dt \]

\[ - 2 \int_{\Sigma_1} \left( \sum_{i,j=1}^{n} a_{ij} \nu_i \partial_j w \right) \left( s \lambda \varphi \sum_{k,\ell = 1}^{n} a_{k\ell} (\partial_k d) \partial_{\ell w} \right) \, d\Sigma. \]

By using (A.8) and \( a_{ij} = a_{ji} \), we can obtain

\[ I_3 = \int_{Q_1} \left\{ \sum_{i,j=1}^{n} 2s \lambda \varphi (\partial_j a_{ij}) (\partial_i w) \sum_{k,\ell = 1}^{n} a_{k\ell} (\partial_k d) \partial_{\ell w} + 2s \lambda^2 \varphi \sum_{i,j=1}^{n} a_{ij} (\partial_i w) (\partial_j d) \sum_{k,\ell = 1}^{n} a_{k\ell} (\partial_k d) \partial_{\ell w} \right. \]

\[ + 2s \lambda \varphi \sum_{i,j=1}^{n} \left[ a_{ij} \partial_i w \sum_{k,\ell = 1}^{n} \partial_j (a_{k\ell} \partial_k d) \partial_{\ell w} \right] + s \lambda \varphi \sum_{k,\ell = 1}^{n} \left( a_{k\ell} \partial_k d \sum_{i,j=1}^{n} a_{ij} \partial_{\ell w} \right) \left. \right\} \, dx \, dt \]

\[ - 2s \lambda \int_{\Sigma_1} |\partial_{\nu} w|^2 a(x, \nu, \nu) a(x, \nabla d, \nu) \, d\Sigma. \]

Integrating by parts, we have

\[ I_3 = \int_{Q_1} \left\{ \sum_{i,j=1}^{n} 2s \lambda \varphi (\partial_j a_{ij}) (\partial_i w) \sum_{k,\ell = 1}^{n} a_{k\ell} (\partial_k d) \partial_{\ell w} \right. \]

\[ + 2s \lambda^2 \varphi a(x, \nabla d, \nabla w)^2 + 2s \lambda \varphi \sum_{i,j=1}^{n} a_{ij} \partial_i w \sum_{k,\ell = 1}^{n} \partial_j (a_{k\ell} \partial_k d) \partial_{\ell w} \]

\[ - s \lambda^2 \varphi a(x, \nabla d, \nabla d) a(x, \nabla w, \nabla w) - s \lambda \varphi \sum_{k,\ell = 1}^{n} \partial_t (a_{k\ell} \partial_k d) a(x, \nabla w, \nabla w) \]

\[ - s \lambda \varphi \sum_{k,\ell = 1}^{n} \left[ a_{k\ell} \partial_k d \sum_{i,j=1}^{n} (\partial_{\ell a_{ij}}) (\partial_i w) \partial_{\ell w} \right] \right\} \, dx \, dt \]

\[ - s \lambda \int_{\Sigma_1} |\partial_{\nu} w|^2 a(x, \nu, \nu) a(x, \nabla d, \nu) \, d\Sigma. \]  

(A.11)
By (A.9)–(A.11), we have

\[
(P_1w, P_2w)_{L^2(Q_1)} = \int_{Q_1} \left[ 3s^3 \lambda^4 \varphi^3 w^2 a(x, \nabla d, \nabla d)^2 + \left( \sum_{i,j=1}^n (\partial_j a_{ij}) \partial_i w \right) P_2w \right. \\
+ 2s \lambda^2 \varphi a(x, \nabla d, \nabla w)^2 - s \lambda^2 \varphi a(x, \nabla d, \nabla d) a(x, \nabla w, \nabla w) \bigg]\, dx \, dt \\
- s \lambda \int_{\Omega_1} \varphi [\partial_\nu w]^2 a(x, \nu, \nu) a(x, \nabla d, \nu) d\Sigma + X_1,
\]

where

\[
X_1 = \int_{Q_1} \left\{ \frac{w^2 s^2 \lambda^2}{2} \partial_t (\varphi^2 a(x, \nabla d, \nabla d)) + s^3 \lambda^3 w^2 \varphi^3 \sum_{i,j=1}^n \partial_j (a_{ij} a(x, \nabla d, \nabla d) \partial_i d) \right. \\
+ 2s \lambda \varphi \sum_{i,j=1}^n a_{ij} (\partial_i w) \sum_{k,\ell=1}^{n} \partial_j (a_{k\ell} \partial_k d) \partial_\ell w \bigg) - s \lambda \varphi \sum_{k,\ell=1}^{n} \partial_\ell (a_{k\ell} \partial_k d) a(x, \nabla w, \nabla w) \\
- s \lambda \varphi \sum_{k,\ell=1}^{n} \partial_k \partial_k d \sum_{i,j=1}^n (\partial_i a_{ij})(\partial_j w) \bigg]\, dx \, dt.
\]

Then by \( a_{ij} \in C^1(\overline{\Omega}) \), we obtain

\[
|X_1| \leq C_2 \int_{Q_1} \left[ (s \lambda \varphi + 1) |\nabla w|^2 + (s^3 \lambda^3 \varphi^3 + s^2 \lambda^2 \varphi^2) w^2 \right] \, dx \, dt.
\]

Multiply (A.3) by \( s \lambda^2 \varphi w a(x, \nabla d, \nabla d) \) and integrate by parts in \( Q_1 \), and we obtain

\[
\int_{Q_1} s \lambda^2 \varphi f_s a(x, \nabla d, \nabla d) w \, dx \, dt \\
= \int_{Q_1} \left\{ s \lambda^2 \varphi a(x, \nabla d, \nabla d) w P_2 w - s^3 \lambda^4 \varphi^3 a(x, \nabla d, \nabla d)^2 w^2 \right. \\
+ s \lambda^2 \varphi a(x, \nabla w, \nabla w) a(x, \nabla d, \nabla d) + s \lambda^3 \varphi a(x, \nabla d, \nabla d) a(x, \nabla d, \nabla w) \\
+ s \lambda^2 \varphi w \sum_{i,j=1}^n \partial_j (a_{ij} a(x, \nabla d, \nabla d) \partial_i w) \bigg]\, dx \, dt.
\]

Consequently

\[
2 s^3 \lambda^4 \int_{Q_1} \varphi^3 a(x, \nabla d, \nabla d)^2 w^2 \, dx \, dt \\
= 2 \int_{Q_1} s \lambda^2 \varphi a(x, \nabla w, \nabla w) a(x, \nabla d, \nabla d) \, dx \, dt + 2X_2,
\]

(A.14)
where

$$X_2 = \int_{Q_1} \left\{ s\lambda^2 \varphi w \sum_{i,j=1}^n \partial_j (a_{ij} a(x, \nabla d, \nabla d)) \partial_i w + s\lambda^3 \varphi a(x, \nabla d, \nabla d) a(x, \nabla d, \nabla w) w 
+ s\lambda^2 \varphi a(x, \nabla d, \nabla d) w P_2 w - s\lambda^2 f_s \varphi a(x, \nabla d, \nabla d) w \right\} dx dt.$$ 

By $a_{ij} \in C^1(\Omega)$ and the Schwarz inequality, we obtain

$$|X_2| \leq \frac{1}{16} \| P_2 w \|_{L^2(Q_1)} + C_3 \int_{Q_1} \left[ (s^2 \lambda^4 \varphi^2 + s^2 \lambda^4 \varphi)^2 + \lambda^2 \varphi |\nabla w|^2 \right] dx dt
+ \frac{1}{2} \| f_s \|_{L^2(Q_1)}. \quad (A.15)$$

Using $3s^3 \lambda^4 \varphi^2 w^2 a(x, \nabla d, \nabla d)^2 = s^3 \lambda^3 \varphi^2 w^2 a(x, \nabla d, \nabla d)^2 + 2s^3 \lambda^4 \varphi^2 w^2 a(x, \nabla d, \nabla d)^2$ in (A.12) and substituting (A.14) into the above second term, we have

$$(P_1 w, P_2 w)_{L^2(Q_1)} = \int_{Q_1} \left[ s^3 \lambda^4 \varphi^2 w^2 a(x, \nabla d, \nabla d)^2 + \left( \sum_{i,j=1}^n (\partial_j a_{ij}) \partial_i w \right) P_2 w 
+ 2s\lambda^2 \varphi a(x, \nabla d, \nabla w)^2 + s\lambda^2 \varphi a(x, \nabla d, \nabla d) a(x, \nabla w, \nabla w) \right] dx dt
- s\lambda \int_{\Sigma_1} \varphi |\partial_v w|^2 a(x, \nu, \nu) a(x, \nabla d, \nu) d\Sigma + X_1 + 2X_2.$$ 

Therefore we see that

$$2(P_1 w, P_2 w)_{L^2(Q_1)} \geq \int_{Q_1} [s^3 \lambda^4 \varphi^2 w^2 a(x, \nabla d, \nabla d)^2 + 2s\lambda^2 \varphi a(x, \nabla d, \nabla d) a(x, \nabla w, \nabla w)] dx dt
+ \int_{Q_1} 2 \left( \frac{1}{2} P_2 w \right) \left( 2 \sum_{i,j=1}^n (\partial_j a_{ij}) \partial_i w \right) dx dt
- 2s\lambda \int_{\Sigma_1} \varphi |\partial_v w|^2 a(x, \nu, \nu) a(x, \nabla d, \nu) d\Sigma + 2X_1 + 4X_2.$$ 

Applying

$$2 \left| \frac{1}{2} (P_2 w) \left( 2 \sum_{i,j=1}^n (\partial_j a_{ij}) \partial_i w \right) \right| \leq \frac{1}{4} |P_2 w|^2 + 4 \left| \sum_{i,j=1}^n (\partial_j a_{ij}) \partial_i w \right|^2,$$
by virtue of $\lambda > 1$, (A.6), (A.13) and (A.15), we obtain

$$
\| f_a \|_{L^2(\Omega^1)}^2 = \| P_1 w \|_{L^2(\Omega^1)}^2 + \| P_2 w \|_{L^2(\Omega^1)}^2 + 2(P_1 w, P_2 w)_{L^2(\Omega^1)} \\
\geq \| P_1 w \|_{L^2(\Omega^1)}^2 + \frac{1}{2}\| P_2 w \|_{L^2(\Omega^1)}^2 \\
+ \int_{Q_1} 2(s^3 \lambda^4 \varphi^3 w^2 a(x, \nabla d, \nabla d)^2 + s \lambda^2 \varphi a(x, \nabla d) a(x, \nabla w, \nabla w)] dx dt \\
- C_1 \int_{Q_1} [(\lambda^2 \varphi + s \lambda \varphi + 1) |\nabla w|^2 + (s^3 \lambda^3 \varphi^3 + s^2 \lambda^4 \varphi^2 + s^2 \lambda^4 \varphi) w^2] dx dt \\
- 2\| f_a \|_{L^2(\Omega^1)}^2 - 2s \lambda \int_{\Sigma_1} \varphi |\partial_\nu w|^2 a(x, \nu, \nu)a(x, \nabla d, \nu) d\Sigma.
$$

Since $d \in C^2(\overline{\Omega})$ satisfies $|\nabla d(x)| > 0$, $x \in \overline{\Omega}$, by (1.4) we can obtain

$$
\| f_a \|_{L^2(\Omega^1)}^2 \geq \frac{1}{3}\| P_1 w \|_{L^2(\Omega^1)}^2 + \frac{1}{6}\| P_2 w \|_{L^2(\Omega^1)}^2 \\
+ C_5 \int_{Q_1} (s^3 \lambda^4 \varphi^3 w^2 + s \lambda^2 \varphi |\nabla w|^2) dx dt \\
- C_0 \int_{Q_1} [(\lambda^2 \varphi + s \lambda \varphi + 1) |\nabla w|^2 + (s^3 \lambda^3 \varphi^3 + s^2 \lambda^4 \varphi^2 + s^2 \lambda^4 \varphi) w^2] dx dt \\
- \frac{2}{3}s \lambda \int_{\Sigma_1} \varphi |\partial_\nu w|^2 a(x, \nu, \nu)a(x, \nabla d, \nu) d\Sigma.
$$

In terms of the definition of $f_a$ in (A.3), we have

$$
\| f_a \|_{L^2(\Omega^1)}^2 \leq C_7 \int_{Q_1} (s^2 \lambda^4 \varphi^2 w^2 + |Pw|^2) dx dt.
$$

Therefore, using also (2.2), we obtain

$$
C_8 \| Pw \|_{L^2(\Omega^1)}^2 + C_8 s \lambda \int_{\tau_1} \int_{\Gamma_0} \varphi |\partial_\nu w|^2 d\Sigma \\
\geq \frac{1}{3}\| P_1 w \|_{L^2(\Omega^1)}^2 + \frac{1}{6}\| P_2 w \|_{L^2(\Omega^1)}^2 + \int_{Q_1} (C_5 s^3 \lambda^4 \varphi^3 - C_8 s^3 \lambda^3 \varphi^3 - C_8 s^2 \lambda^4 \varphi^2 - C_8 s^2 \lambda^4 \varphi) w^2 dx dt \\
+ \int_{Q_1} (C_5 s \lambda^4 \varphi - C_8 s \lambda \varphi - C_8 \lambda^2 \varphi - C_8) |\nabla w|^2 dx dt.
$$

Noting that $\varphi \geq 1$ on $\overline{Q_1}$, we can find constants $\lambda_0 > 0$ and $s_0 > 0$ which continuously depend on $\sum_{i,j=1}^n |a_{ij}|_{C^1(\overline{\Omega})}$, $\tau_1, \tau_2, \Omega, r$ such that

$$
C_9 s \lambda \int_{\tau_1} \int_{\Gamma_0} \varphi |\partial_\nu w|^2 d\Sigma + C_9 \| Pw \|_{L^2(\Omega^1)}^2 \geq \| P_1 w \|_{L^2(\Omega^1)}^2 + \| P_2 w \|_{L^2(\Omega^1)}^2 \\
+ \int_{Q_1} (s^3 \lambda^4 \varphi^3 w^2 + s \lambda^2 \varphi |\nabla w|^2) dx dt.
$$
for all $s > s_0$ and $\lambda > \lambda_0$. By (A.4) and (A.5), we have

$$|(P_1w)(x,t)|^2 \geq C_{10} \left| \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j w \right|^2 - C_{11} s^4 \lambda^4 \varphi^4 w^2$$

and

$$|(P_2w)(x,t)|^2 \geq C_{10} |\partial_t w|^2 - C_{11} s^2 \lambda^2 \varphi^2 |\nabla w|^2,$$

so that

$$\int_{Q_1} \left\{ \frac{1}{s^2 \varphi} \left[ |\partial_t w|^2 + \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j w \right]^2 \right\} dx \, dt$$

$$\leq C_{12} \int_{Q_1} |Pw|^2 dx \, dt + C_{12} s \lambda \int_{\tau_1}^{\tau_2} \int_{\Gamma_0} \varphi |\partial_\nu w|^2 d\Sigma$$

(A.16)

for all $s > s_0$ and $\lambda > \lambda_0$.

Moreover we have

$$\partial_i \partial_j \left( \frac{w}{\sqrt{\varphi}} \right) = \partial_i \partial_j w \frac{\partial_i \partial_j \varphi}{2 \varphi^{3/2}} - \frac{1}{2 \varphi^{3/2}} \left\{ (\partial_i w)(\partial_j \varphi) + (\partial_j w)(\partial_i \varphi) \right\} + \frac{3}{4 \varphi^{3/2}} (\partial_i \varphi)(\partial_j \varphi) w$$

$$1 \leq i, j \leq n,$$

and

$$\sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j \left( \frac{w}{\sqrt{\varphi}} \right) = \frac{g}{\sqrt{\varphi}} - \frac{1}{\varphi^{3/2}} \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j \varphi$$

$$w = \sum_{i,j=1}^{n} a_{ij} (\partial_i w)(\partial_j \varphi)$$

where we set $g = \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j w$. Since $w(t, \cdot) \in H_1^1(\Omega)$ for almost all $t \in [\tau_1, \tau_2]$, we apply a usual a priori estimate for the Dirichlet problem for the elliptic equation (e.g., [16]), so that

$$\int_{\Omega} \left\{ \sum_{i,j=1}^{n} \left| \partial_i \partial_j \left( \frac{w}{\sqrt{\varphi}} \right) \right|^2 dx \right\} \leq C_{13} \int_{\Omega} \frac{g(t,x)^2}{\varphi} dx + C_{13} \int_{\Omega} \frac{\sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j \varphi^2}{\varphi^{3/2}} |w(t,x)|^2 dx$$

$$+ C_{13} \int_{\Omega} \frac{w(t,x)^2}{\varphi^{3/2}} \left| \sum_{i,j=1}^{n} a_{ij} (\partial_i \varphi)(\partial_j \varphi) \right|^2 dx$$

$$+ C_{13} \int_{\Omega} \frac{1}{\varphi^{3/2}} \left| \sum_{i,j=1}^{n} a_{ij} (\partial_i w)(\partial_j \varphi) \right|^2 dx.$$

(A.18)
On the other hand, (A.17) yields

\[
\int_{\Omega} \frac{1}{w} |\partial_i \partial_j w(t,x)|^2 \, dx \\
\leq C_{14} \int_{\Omega} \left\{ \frac{1}{\varphi} \left( \frac{w}{\sqrt{\varphi}} \right)^2 + \frac{1}{\varphi} \left( \frac{w}{\sqrt{\varphi}} \right)^2 + \frac{1}{\varphi^2} |\partial_j \varphi|^2 w^2 \right\} (t,x) \, dx.
\]

(A.19)

Since \( \partial_i \varphi = \lambda (\partial_i d) \varphi \) and \( \partial_i \partial_j \varphi = \lambda (\partial_i \partial_j d) \varphi + \lambda^2 (\partial_i d)(\partial_j d) \varphi \), we see by \( \lambda > 1 \) that

\[
|\partial_i \varphi(t,x)| \leq C_{15} \lambda \varphi(t,x), \\
|\partial_i \partial_j \varphi(t,x)| \leq C_{15} \lambda^2 \varphi(t,x), \quad 1 \leq i, j \leq n, \ (t,x) \in \overline{Q}_1.
\]

Hence, \( \varphi \geq 1 \), (A.18) and (A.19) yield

\[
\sum_{i,j=1}^n \int_{\Omega} \frac{1}{\varphi(t,x)} |\partial_i \partial_j w(t,x)|^2 \, dx \leq C_{16} \int_{\Omega} g^2(t,x) + C_{16} \int_{\Omega} (\lambda^4 w^2 + \lambda^2 |\nabla w|^2)(t,x) \, dx.
\]

With (A.16), we obtain

\[
\int_{Q_1} \left\{ \frac{1}{s \varphi} \left( |\partial_i w|^2 + \sum_{i,j=1}^n |\partial_i \partial_j w|^2 \right) + s \lambda^2 \varphi |\nabla w|^2 + s^3 \lambda^4 \varphi^3 w^2 \right\} \, dx \, dt \\
\leq C_{17} \int_{Q_1} |Pu|^2 \, dx \, dt + C_{17} \lambda \int_{\Gamma_1} \int_{\Gamma_0} |\partial_i w|^2 \, d\Sigma
\]

for all \( s > s_0 \) and \( \lambda > \lambda_0 \). Substituting \( w = e^{s \varphi} v \) and noting \( v|_{\Sigma} = 0 \) and (A.20), we can complete the proof of (2.6).

In (2.6), fixing \( \lambda > \lambda_0 \) and replacing \( e^{\lambda M_1} s \) by \( s \), we can derive (2.4) from (2.6). Thus the proof of Theorem 2.1 is completed.

**Appendix B. Proof of (1.6) and (1.10)**

For \( \{a_{ij}\} \in \mathcal{U} \), we recall that the operator \( A \) in \( L^2(\Omega) \) is defined by (1.9), and that (4.1) holds. Moreover the fractional power \( A^\gamma \), \( \gamma \in \mathbb{R} \) is defined (e.g., [37]). Applying the interpolation theorem (e.g., [35]) in (4.1), we see that

\[
C_{1}^{-1} \| A^m u \| \leq \| u \|_{H^{m_0}(\Omega)} \leq C_1 \| A^m u \|, \quad m_0 = 0, 1, 2, ..., \left( \frac{m + 1}{2} \right), \ u \in \mathcal{D}(A^m).
\]

(B.1)

On the other hand, \( -A \) generates an analytic semigroup in \( L^2(\Omega) \) (e.g., [37]) and we have

\[
y(t) \equiv y(\{a_{ij}\}, h, \mu)(t, \cdot) = e^{-tA} \mu + \int_0^t e^{-sA} h(t - s) ds, \quad 0 < t < T.
\]

Here and henceforth we regard \( h(t) = h(t, \cdot) \) as an element in \( L^2(0,T; L^2(\Omega)) \). Therefore by \( h \in C_{0}^\infty ((0,T) \times \omega) \), we have

\[
\partial_t^\ell y(t) = (-A)^\ell e^{-tA} \mu + \int_0^t e^{-sA} \partial_t^\ell h(t - s) ds, \quad 0 < t < T.
\]
Furthermore, $h \in C_0^\infty((0,T) \times \omega)$ yields $\|A^{m_0} \partial_t^l h(t)\| \leq C_2 \|\partial_t^l h(t)\|_{H^{m_0}(\omega)}$ by (B.1). On the other hand,\[\|A' e^{-tA}\| \leq \frac{C_3}{t} \] for $t > 0$ (e.g., Sect. 2.6 in [37]), and by the proof in [37], we see that the constant $C_3 > 0$ can be chosen uniformly in $\{a_{ij}\} \in \mathcal{U}$. Consequently, we obtain \[
\begin{align*}
\|A^{m_0} \partial_t^l y(t)\| &\leq \frac{C_3}{\tau_1} \|\mu\| + C_5 \int_0^t \|A^{m_0} \partial_t^l h(t-s)\| ds \\
&\leq \left(\frac{C_3}{\tau_1} + C_3\right) (\|\mu\| + \|h\|_{W^{l+1,1}(0,T;H^{m_0}(\omega))}), \quad \tau_1 \leq t \leq \tau_2.
\end{align*}\] (B.2)

Thus, in terms of (B.1), the proof of (1.6) is completed.

Next we prove (1.10). We see that $H^m(\Omega) \subset C^2(\bar{\Omega})$ from the Sobolev embedding theorem (e.g., [1,35]). Similarly to (B.2), in terms of (B.1) we have
\[
\begin{align*}
\|y(a_{ij},h,0)\|_{C[0,T];C^2(\bar{\Omega})} &\leq C_4 \|y(a_{ij},h,0)\|_{C[0,T];H^m(\Omega)} \\
&\leq C_5 \int_0^t \|A^{m} e^{-sA} h(t-s)\| ds = C_5 \int_0^t \|e^{-sA} A^{m} h(t-s)\| ds \leq C_6 \int_0^t \|h(t-s)\|_{H^m(\Omega)} ds.
\end{align*}
Thus the proof of (1.10) is completed.

**APPENDIX C. PROOF OF LEMMA 2.1**

In Chae, Imanuvilov and Kim [9], Fursikov and Imanuvilov [15], the following lemma is proved. See also Imanuvilov [19].

**Lemma C.1.** Let $\tilde{\Omega} \subset \mathbb{R}^n$ be a bounded domain whose boundary $\partial \tilde{\Omega}$ is of class $C^2$ and $\omega \subset \tilde{\Omega}$ be a subdomain such that $\omega \subset \bar{\Omega}$. Then there exists a function $d \in C^2(\bar{\Omega})$ such that
\[
d(x) > 0, \quad x \in \tilde{\Omega}, \quad d|_{\partial \tilde{\Omega}} = 0, \quad |\nabla d(x)| > 0, \quad x \in \bar{\Omega} \setminus \omega.
\]

Now we proceed to the proof of Lemma 2.1. Let us enlarge the domain $\Omega$ to a domain $\tilde{\Omega}$ which has the following properties:
\[
\Omega \subset \tilde{\Omega} \quad \partial \Omega \setminus \Gamma_0 \subset \partial \tilde{\Omega} \quad \text{Int}(\tilde{\Omega} \setminus \Omega) \neq \emptyset.
\]

Choose a subdomain $\omega$ such that $\omega \subset \text{Int}(\tilde{\Omega} \setminus \Omega)$. Thus, by Lemma C.1, there exists a function $d(x)$ in $\tilde{\Omega}$ which satisfies $d(x) > 0$, $x \in \tilde{\Omega}$, $d|_{\partial \tilde{\Omega}} = 0$ and $|\nabla d(x)| > 0$, $x \in \tilde{\Omega} \setminus \Omega$. Therefore $d(x) > 0$, $x \in \Omega$ and $|\nabla d(x)| > 0$, $x \in \Omega$. Finally, we have to verify \[
\sum_{i,j=1}^n a_{ij}(x) \partial_i d(x) \nu_j(x) \leq 0, \quad x \in \partial \Omega \setminus \Gamma_0.
\] Since $\partial \Omega \setminus \Gamma_0 \subset \partial \tilde{\Omega}$, we have $d|_{\partial \Omega \setminus \Gamma_0} = 0$ from $d|_{\partial \tilde{\Omega}} = 0$, which implies $\nabla d(x) = \partial_n d(x) \nu(x), x \in \partial \Omega \setminus \Gamma_0$, that is, $\nabla d$ is parallel to $\nu$ on $\partial \Omega \setminus \Gamma_0$. Therefore, by $d > 0$ in $\Omega$, we have $\nabla d(x) = -|\nabla d(x)| \nu(x), x \in \partial \Omega \setminus \Gamma_0$. By $|\nabla d(x)| > 0$ on $\tilde{\Omega}$, we obtain $\nu(x) = -\frac{\nabla d(x)}{|\nabla d(x)|}, x \in \partial \Omega \setminus \Gamma_0$. Therefore by (1.4) we see
\[
\sum_{i,j=1}^n a_{ij}(x) \partial_i d(x) \nu_j(x) = \frac{\sum_{i,j=1}^n a_{ij}(x) \partial_i d(x) \partial_j d(x)}{|\nabla d(x)|} \leq 0, \quad x \in \partial \Omega \setminus \Gamma_0.
\]
Thus the proof of Lemma 2.1 is completed.
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