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CORRIGENDUM TO: "ON THE CIRCLE CRITERION FOR BOUNDARY CONTROL SYSTEMS IN FACTOR FORM: LYAPUNOV STABILITY AND LUR'E EQUATIONS"

Piotr Grabowski[1] and Frank M. Callier[2]

Abstract. A corrected version of [P. Grabowski and F.M. Callier, *ESAIM: COCV* **12** (2006) 169–197], Theorem 4.1, p. 186, and Example, is given.

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1. Introduction

The authors are deeply indebted to Hartmut Logemann, Department of Mathematics, University of Bath, UK for pointing out a counterexample, repeated below, showing that the statement of [2], Theorem 4.1, p. 186, is wrong.

With the notation of [2] all assumptions of that theorem are met for

$$\mathrm{H} = \mathbb{R}, \quad A = -1 = A^{-1},\ h = -1(\iff c^{\#}x = x),\ d = 1, \quad \delta = 1,\ e = \frac{8}{3},\ q = \frac{16}{3},$$

however the system (3.1) has exactly two solutions $(\mathcal{H}, \mathcal{G}) = (-\frac{8}{3}, 0)$, $(\mathcal{H}, \mathcal{G}) = (-\frac{2}{3}, 2)$ and none of them is such that $\mathcal{H} \geq 0$. This counterexample demonstrates that the assumptions of [2], Theorem 4.1, p. 186, are not enough to ensure non-negativity of $\mathcal{H}$.

The aim of this note is to correct the result by adding reasonable and non-restrictive assumptions which can be verified without solving (3.1) explicitly.

2. Corrigendum of [2], Theorem 4.1 (i), p. 186

Theorem 2.1. *Let assumptions* **(H1)**–**(H5)** *hold. Moreover assume that:*

(H6) *The operator* $A : (D(A) \subset H) \longrightarrow H$ *is such that the semigroup generated by* A^{-1} *is* **AS**.

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[1] Institute of Automatics, AGH University of Science and Technology, avenue A. Mickiewicz 30, B1, rm. 314, 30-059 Cracow, Poland. pgrab@ia.agh.edu.pl

[2] University of Namur (FUNDP), Department of Mathematics, Rempart de la Vierge 8, 5000 Namur, Belgium.

Then:

(i) *The system* (3.1) *has a solution* $(\mathcal{H},\mathcal{G})$, $\mathcal{H}\in\mathbf{L}(H)$, $\mathcal{H}=\mathcal{H}^*\geq 0$, *provided that if* $q>0$ *then, in addition, the assumption* **(A3)** *holds and*

$$\frac{1}{1+\mu_0\hat{g}}\in H^\infty(\mathbb{C}^+)\quad \textit{for}\quad \mu_0:=\frac{k_1+k_2}{2}, \tag{2.1}$$

$\mathcal{G}\in H$, *where in particular:* $\mathcal{G}$ *is the solution of the realization equation* (4.4), *where* ϕ *is the spectral factor of the Popov function* π *(given by* (4.2)*) such that* $\phi(0)=\sqrt{\delta}$, *and both* ϕ *and* $1/\phi$ *are in* $H^\infty(\mathbb{C}^+)$.

Remark 2.1. It should be emphasized that if $q\leq 0$ the statement of [2], Theorem 4.1**(i)**, p. 186, is fully correct, *i.e.*, the assertion holds without **(A3)** and (2.1). The claim [2], Theorem 4.1**(ii)**, p. 186, does not require any correction.

Proof. The whole reasoning of the existing proof remains correct after removing: the sentence starting from the words: "The symbol of the Toeplitz operator ...", the footnote on p. 186 and after dropping the inequality $\mathcal{H}\geq 0$ in the sentence just following (4.17). Having this done, we may correct the proof as follows. Since X is a solution of (4.15) given by (4.10) it is clear that

$$\mathcal{H}=-X=\psi^*\left[(q\mathbb{F}-eI)\mathcal{R}^{-1}(q\mathbb{F}-eI)^*-qI\right]\psi\geq 0 \tag{2.2}$$

if $q\leq 0$, whence the claim of the remark above is met.

Now, consider the case $q>0$ ($\Longrightarrow \mu_0\neq 0$) where, in addition **(A3)** (*i.e.*, d is an admissible factor control vector) and (2.1) hold. Observe that

$$1-\mu_0\underbrace{c^\# d}_{=-\hat{g}(0)}\neq 0,$$

for if not, by (4.2), we would have $\pi(0)=\delta=\left(1-\frac{k_1}{\mu_0}\right)\left(1-\frac{k_2}{\mu_0}\right)=-\left(\frac{k_2-k_1}{k_1+k_2}\right)^2<0$, which contradicts (4.3). Since the LHS of (2.2) satisfies the Riccati equation

$$(A^{-1})^*\mathcal{H}+\mathcal{H}A^{-1}+\underbrace{\left[\frac{1}{\sqrt{\delta}}(-\mathcal{H}d+eh)\right]}_{=-\mathcal{G}}\left[\frac{1}{\sqrt{\delta}}(-\mathcal{H}d+eh)\right]^*-qhh^*=0 \tag{2.3}$$

then, adding $\dfrac{\mu_0}{1-\mu_0c^\# d}hd^*\mathcal{H}+\dfrac{\mu_0}{1-\mu_0c^\# d}\mathcal{H}dh^*$ to both sides of (2.3), we conclude that $\mathcal{H}$ satisfies the Lyapunov operator equation

$$\left[A^{-1}+\frac{\mu_0}{1-\mu_0c^\# d}dh^*\right]^*\mathcal{H}+\mathcal{H}\left[A^{-1}+\frac{\mu_0}{1-\mu_0c^\# d}dh^*\right]=-(\mathcal{G}-q_1h)(\mathcal{G}-q_1h)^*-q_0hh^*$$

with

$$q_1:=\frac{\mu_0\sqrt{\delta}}{1-\mu_0c^\# d},\qquad q_0=\frac{(k_2-k_1)^2}{4(1-\mu_0c^\# d)^2}>0,$$

or equivalently,

$$\langle A_0x,\mathcal{H}x\rangle_{\mathrm{H}}+\langle\mathcal{H}x,A_0x\rangle_{\mathrm{H}}=-\left[(\mathcal{G}-q_1h)^*A_0x\right]^2-q_0\left[h^*A_0x\right]^2\qquad\forall x\in D(A_0), \tag{2.4}$$

where

$$A_0x:=A(x-\mu_0dc^\# x),\qquad D(A_0)=\left\{x\in D(d^*)\colon x-\mu_0dc^\# x\in D(A)\right\}\cdot$$

This is because $A_0^{-1} = A^{-1} + \dfrac{\mu_0}{1-\mu_0 c^{\#}d}\mathrm{d}h^* \in \mathbf{L}(\mathrm{H})$. The operator A_0 arises by applying negative linear feedback $u = -\mu_0 y$ to

$$\left\{\begin{array}{lcl} \dot{x} & = & A(x+ud) \\ y & = & c^{\#}x \end{array}\right\} \tag{2.5}$$

and it corresponds to the Lur'e control system of [2], Figure 1.1, p. 170, with $f(y) = \mu_0 y$. Since $c^{\#}$ is admissible and $\hat{g} \in \mathrm{H}^\infty(\mathbb{C}^+)$, for $\mathrm{L}^2(0,\infty)$-controls the output is given by

$$y = \overline{P}x_0 + \overline{\mathbb{F}}u$$

where $\overline{P}$ and $\overline{\mathbb{F}}$ stand for the extended observability map and the extended input-output operator, both associated with (2.5). Thus, for the closed-loop system, by the Paley-Wiener theory, one has

$$(I + \mu_0\overline{\mathbb{F}})y = \overline{P}x_0 \iff (1+\mu_0\hat{g})y = \widehat{\overline{P}x_0},$$

and, due to (2.1), the last equation has a unique solution $\hat{y} \in \mathrm{H}^2(\mathbb{C}^+)$. *Via* the feedback law equation $u = -\mu_0 y$ this implies that for any x_0: $u \in \mathrm{L}^2(0,\infty)$. Now [2], Lemma 2.11, p. 177, implies that for every initial condition x_0 the first equation of (2.5) has a unique weak solution, whence, by Ball's theorem [1], p. 371 (see also [4], p. 259), the operator A_0 generates a C_0-semigroup $\{S_0(t)\}_{t\geq 0}$ on H which is **AS**.

Now, for every $x_0 \in D(A_0)$ and each $t \geq 0$, (2.4) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle S_0(t)x_0, \mathcal{H}S_0(t)x_0\rangle_{\mathrm{H}} = -\left[(\mathcal{G}-q_1h)^*A_0S_0(t)x_0\right]^2 - q_0\left[h^*A_0S_0(t)x_0\right]^2.$$

Integrating both sides from 0 to t and employing **AS** we obtain

$$\langle x_0, \mathcal{H}x_0\rangle_{\mathrm{H}} = \int_0^\infty \left\{\left[(\mathcal{G}-q_1h)^*A_0S_0(t)x_0\right]^2 + q_0\left[h^*A_0S_0(t)x_0\right]^2\right\}\mathrm{d}t \geq 0 \qquad \forall x_0 \in D(A_0).$$

Since $D(A_0)$ is dense in H as a C_0-semigroup generator and $\mathcal{H} = \mathcal{H}^* \in \mathbf{L}(\mathrm{H})$ we get $\mathcal{H} \geq 0$. □

Remark 2.2. The above proof may be slightly, but not essentially, modified by concluding **AS** of the semigroup $\{\mathrm{e}^{tA_0^{-1}}\}_{t\geq 0}$ from the reciprocal system

$$\left\{\begin{array}{lcl} \dot{x} & = & A^{-1}x + ud \\ y & = & -h^*x \end{array}\right\}$$

with the feedback law $u = -\dfrac{\mu_0}{1-\mu_0c^{\#}d}y$, with an aid of [3], Lemma 12, p. 959. This is possible if d is admissible with respect to $\{\mathrm{e}^{tA^{-1}}\}_{t\geq 0}$ and $u \in \mathrm{L}^2(0,\infty)$ for any initial condition $x_0 \in \mathrm{H}$. It is not difficult to see, using duality between observation and control (see [2], p. 173) and the arguments which led to [2], Lemma 2.6, p. 174, that the first condition holds iff d is admissible. Since in the frequency-domain the closed-loop output equation reads as

$$\begin{aligned}\hat{y}(s) &= -h^*\left(sI - A^{-1}\right)^{-1}x_0 - h^*\left(sI-A^{-1}\right)^{-1}d\left[-\frac{\mu_0}{1-\mu_0c^{\#}d}\hat{y}(s)\right] \\ &= \left(U\widehat{\overline{P}x_0}\right)(s) + G(s)\left[-\frac{\mu_0}{1-\mu_0c^{\#}d}\right]\hat{y}(s),\end{aligned}$$

where U is the unitary operator introduced in [2], p. 174, and G is given by [2], (4.12), p. 187, then the second condition holds if $\dfrac{1}{1+\dfrac{\mu_0}{1-\mu_0c^{\#}d}G} \in \mathrm{H}^\infty(\mathbb{C}^+)$. By [2], (4.13), p. 187, the last condition is equivalent to (2.1).

Next, our Lyapunov operator equation

$$\left(A_0^{-1}\right)^* \mathcal{H} + \mathcal{H}A_0^{-1} = -(\mathcal{G} - q_1 h)(\mathcal{G} - q_1 h)^* - q_0 hh^*$$

allows to get directly

$$\langle x_0, \mathcal{H}x_0\rangle_{\mathrm{H}} = \int_0^\infty \left\{ \left[(\mathcal{G} - q_1 h)^* \mathrm{e}^{tA_0^{-1}} x_0\right]^2 + q_0 \left[h^* \mathrm{e}^{tA_0^{-1}} x_0\right]^2 \right\} \mathrm{d}t \geq 0 \qquad \forall x_0 \in \mathrm{H}.$$

3. Correction of [2], Example

Just before the sentence starting from the words ([2], Sect. 5.2, p. 1927): "Thus all assumptions of Theorem 4.1 are met ..." the following text should be inserted[3].

Recall that d is an admissible factor control vector and for $b \in (0,1)$ the assumption (2.1) holds. Indeed, here

$$\frac{1}{1+\mu_0 \hat{g}(s)} = \frac{1}{1 + \dfrac{4b}{a(1+b)} \dfrac{a\mathrm{e}^{-sr}}{1 + b\mathrm{e}^{-2sr}}} = \frac{1 + b\mathrm{e}^{-2sr}}{b\mathrm{e}^{-2sr} + \dfrac{4b}{(1+b)}\mathrm{e}^{-sr} + 1}\cdot$$

The numerator is bounded by $1+b$ on $\overline{\mathbb{C}^+}$, while for the denominator one has

$$b\mathrm{e}^{-2sr} + \frac{4b}{(1+b)}\mathrm{e}^{-sr} + 1 = b\left(z_0 - \mathrm{e}^{-sr}\right)\left(\overline{z_0} - \mathrm{e}^{-sr}\right), \qquad \mathrm{Re}\, z_0 = \frac{-2b}{1+b}, \quad |z_0|^2 = \frac{1}{b},$$

whence

$$\left| b\mathrm{e}^{-2sr} + \frac{4b}{(1+b)}\mathrm{e}^{-sr} + 1 \right| = b\left|z_0 - \mathrm{e}^{-sr}\right| \left|\overline{z_0} - \mathrm{e}^{-sr}\right| \geq b\left(|z_0| - 1\right)^2 = (1 - \sqrt{b})^2,$$

and consequently: $\left\| \dfrac{1}{1+\mu_0\hat{g}} \right\|_{\mathrm{H}^\infty(\mathbb{C}^+)} \leq \dfrac{1+b}{(1-\sqrt{b})^2} < \infty.$

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[3]Since $q = k_1 k_2 < 0$ for $b \in (0, 3 - 2\sqrt{2})$ and sufficiently small ν then, in fact, corrections are needed only for $b \in [3 - 2\sqrt{2}, 1)$.