CONTROLLED FUNCTIONAL DIFFERENTIAL EQUATIONS:
APPROXIMATE AND EXACT ASYMPTOTIC TRACKING
WITH PRESCRIBED TRANSIENT PERFORMANCE

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Abstract. A tracking problem is considered in the context of a class $S$ of multi-input, multi-output, nonlinear systems modelled by controlled functional differential equations. The class contains, as a prototype, all finite-dimensional, linear, $m$-input, $m$-output, minimum-phase systems with sign-definite “high-frequency gain”. The first control objective is tracking of reference signals $r$ by the output $y$ of any system in $S$: given $\lambda \geq 0$, construct a feedback strategy which ensures that, for every $r$ (assumed bounded with essentially bounded derivative) and every system of class $S$, the tracking error $e = y - r$ is such that, in the case $\lambda > 0$, $\limsup_{t \to \infty} \|e(t)\| < \lambda$ or, in the case $\lambda = 0$, $\lim_{t \to \infty} \|e(t)\| = 0$. The second objective is guaranteed output transient performance: the error is required to evolve within a prescribed performance funnel $F_\varphi$ (determined by a function $\varphi$). For suitably chosen functions $\alpha$, $\nu$ and $\theta$, both objectives are achieved via a control structure of the form $u(t) = -\nu(k(t))\theta(e(t))$ with $k(t) = \alpha(\varphi(t)\|e(t)\|)$, whilst maintaining boundedness of the control and gain functions $u$ and $k$. In the case $\lambda = 0$, the feedback strategy may be discontinuous: to accommodate this feature, a unifying framework of differential inclusions is adopted in the analysis of the general case $\lambda \geq 0$.

Mathematics Subject Classification. 93D15, 93C30, 34K20, 34A60.

Received June 25, 2007. Revised March 26, 2008. Published online July 19, 2008.

1. Introduction

In precursors [6–8] to the present paper, an approximate tracking problem is addressed for various classes of systems. Let $S$ be some given system class and let $R$ be a class of reference signals. By approximate tracking, we mean attainment of the following: for any prescribed $\lambda > 0$, determine a continuous output feedback strategy which ensures that, for every system (with output $y$) in $S$ and every reference signal $r \in R$, (i) the tracking error $e = y - r$ is ultimately contained in the ball of radius $\lambda$ centred at 0 (equivalently, $\limsup_{t \to \infty} \|e(t)\| < \lambda$), and (ii) the error $e$ exhibits prescribed transient behaviour (that is, for some suitable prescribed function $\varphi$ with $0 < \liminf_{t \to \infty} \varphi(t) < \infty$, we have $\|e(t)\| < 1/\varphi(t)$ for all $t > 0$). The present paper encompasses not only approximate tracking but also the problem of asymptotic tracking with prescribed transient behaviour: in the latter case, an output feedback strategy (possibly discontinuous) is sought which ensures that, for every system of class $S$, every reference signal $r \in R$ and some suitable prescribed function $\varphi$, with $\varphi(t) \to \infty$ as $t \to \infty$,
we have $\|e(t)\| < 1/\varphi(t)$ for all $t > 0$ (and so $e(t) \to 0$ as $t \to \infty$). Both cases (approximate and asymptotic tracking) are analysed within a unified framework of functional differential inclusions.

The focus of our study will be nonlinear systems (akin to those considered in [6]), with control input $t \mapsto u(t) \in \mathbb{R}^m$, modelled by functional differential equations of the form

$$\dot{y}(t) = f(d(t), (Ty)(t), u(t)), \quad y|_{[-h,0]} = y^0 \in C([-h,0], \mathbb{R}^m), \quad h \geq 0,$$

where $f$ is continuous, $T$ is a causal operator, $d$ may be thought of as a continuous and bounded perturbation, and $h \geq 0$ quantifies the “memory” of the system. As in [6–8], the class $\mathcal{R}$ of reference signals is taken to be the space $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ of bounded locally absolutely continuous functions $r: \mathbb{R}_+ \to \mathbb{R}^m$ with essentially bounded derivative $\dot{r} \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$.

The paper is structured as follows. Section 2 formulates the control objectives and, in Section 3, a full description of the system class $\mathcal{S}$ is provided. Section 4 details the feedback structure, the potentially discontinuous nature of which leads to an interpretation of the closed-loop system in the form of a functional differential inclusion. An existence theory (which may be of independent interest) for functional differential inclusions of sufficient generality to encompass the closed-loop system is developed in Section 5. The main results on transient behaviour and asymptotic tracking for the closed-loop system are given in Section 6.

### 2. CONTROL OBJECTIVES AND THE PERFORMANCE FUNNEL

The two control objectives are:

(i) tracking of any reference signal $r \in \mathcal{R} := W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ by the output $y$, that is, for arbitrary $\lambda \geq 0$, we seek an output feedback strategy which ensures that, for every $r \in \mathcal{R}$, every solution of the closed-loop system is bounded and the tracking error $e = y - r$ is such that either $\limsup_{t \to \infty} \|e(t)\| < \lambda$ if $\lambda > 0$ or $\lim_{t \to \infty} \|e(t)\| = 0$ if $\lambda = 0$;

(ii) prescribed transient behaviour of the tracking error.

Both objectives are captured in the concept of a performance funnel

$$\mathcal{F}_\varphi := \{(t,e) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \varphi(t) \|e\| < 1\}$$

associated with a function $\varphi$ (the reciprocal of which determines the funnel boundary) in

$$\Phi_\lambda := \left\{ \varphi \in AC_{loc}(\mathbb{R}_+, \mathbb{R}) \mid \varphi(0) = 0, \ \varphi(s) > 0 \ \forall \ s > 0, \ \liminf_{s \to \infty} \varphi(s) = 1/\lambda, \ \exists \ c > 0 : \ \varphi(s) \leq c[1 + \varphi(s)] \ \text{for a.a.} \ s > 0 \right\},$$

with the convention that, if $\lambda = 0$, then $1/\lambda := \infty$ (and so $\varphi(t) \to \infty$ as $t \to \infty$). Here, $AC_{loc}(\mathbb{R}_+, \mathbb{R})$ denotes the space of locally absolutely continuous functions $\mathbb{R}_+ \to \mathbb{R}$.

If a feedback structure can be devised which ensures that, for every system of the underlying class and every $r \in \mathcal{R}$, the graph of the tracking error $e = y - r$ is properly contained in $\mathcal{F}_\varphi$ in the sense that $\sup_{t \in \mathbb{R}_+} \|e(t)\| < 1$ then the tracking objective (i) is attained, and (ii) transient behaviour is governed by the choice of $\varphi$:
for example, if $\lambda > 0$ and $\phi$ is chosen as the function $t \mapsto \min\{t/\tau, 1\}/\lambda$, then the prescribed tracking accuracy $\lambda > 0$ is achieved within the prescribed time $\tau > 0$.

The intuition underpinning the feedback structure proposed below is an intrinsic high-gain property of the system class which ensures that, if $(t, e(t))$ approaches the funnel boundary, then the control input attains values sufficiently large to preclude boundary contact.

3. Class of systems

For $m \in \mathbb{N}$ and an interval $I \subset \mathbb{R}$, $C(I, \mathbb{R}^m)$ denotes the space of continuous functions $I \to \mathbb{R}^m$. If $I$ is an interval of the form $[-h, a]$ or $[-h, a]$, $0 < a < \infty$, and $x \in C(I, \mathbb{R}^m)$, then, for each $\sigma \in J := I \setminus [-h, 0)$, we define the function $x_{\sigma} \in C([-h, \infty), \mathbb{R}^m)$ by

$$x_{\sigma}(t) := \begin{cases} x(t), & t \in [-h, \sigma], \\ x(\sigma), & t > \sigma. \end{cases}$$

For $h, t \in \mathbb{R}^+$, $w \in C([-h, t], \mathbb{R}^m)$, $\tau > t$ and $\delta > 0$, define

$$C(w; h, t, \tau, \delta) := \{ v \in C([-h, \tau], \mathbb{R}^m) \mid v|_{[-h, h]} = w, \|v(s) - w(t)\| \leq \delta \quad \forall s \in [t, \tau]\},$$

that is, the space of all continuous extensions $v$ of $w \in C([-h, t], \mathbb{R}^m)$ to the interval $[-h, \tau]$ with the property that $\|v(s) - w(t)\| \leq \delta$ for all $s \in [t, \tau]$.

We first define a class of operators $\mathcal{T}_h$, parameterized by $h \geq 0$.

**Definition 3.1** (operator class $\mathcal{T}_h$). An operator $T$ is said to be of class $\mathcal{T}_h$ if, and only if, the following hold:

(i) For some $q \in \mathbb{N}$, $T: C([-h, \infty), \mathbb{R}^m) \to L_{\text{loc}}^\infty(J, \mathbb{R}^q)$.

(ii) $T$ is a causal operator: for all $x, y \in C([-h, \infty), \mathbb{R}^m)$ and all $\tau > 0$

$$x(t) = y(t) \quad \forall t \in [-h, \tau] \quad \implies \quad (Tx)(t) = (Ty)(t) \quad \forall t \in [0, \tau].$$

(iii) For each $t \geq 0$ and each $w \in C([-h, t], \mathbb{R}^m)$, there exist $\tau > t$, $\delta > 0$ and $c_0 > 0$ such that

$$\text{ess-sup}_{s \in [t, \tau]} \| (Tx Whispering) - (Ty Whispering) \| \leq c_0 \text{ sup}_{s \in [t, \tau]} \| x(s) - y(s) \| \quad \forall x, y \in C(w; h, t, \tau, \delta).$$

(iv) For every $c_1 > 0$, there exists $c_2 > 0$ such that, for all $y \in C([-h, \infty), \mathbb{R}^m)$,

$$\text{sup}_{t \in [-h, \infty)} \| y(t) \| \leq c_1 \quad \implies \quad \| (Ty)(t) \| \leq c_2 \quad \text{for a.a. } t \geq 0.$$

**Remark 3.2.** Property (iii) is a technical assumption of local Lipschitz type which is used in establishing well-posedness of the closed-loop system (defined later in Sect. 4.1). We will have occasion to give meaning to $Tx$, for a function $x \in C(I, \mathbb{R}^m)$ on a bounded interval $I$ of the form $[-h, a]$ or $[-h, a]$, where $0 < a < \infty$. This we do by showing that $T$ “localizes”, in a natural way, to an operator $\hat{T}: C(I, \mathbb{R}^m) \to L_{\text{loc}}^\infty(J, \mathbb{R}^q)$, where $J := I \setminus [-h, 0)$. In particular, and invoking causality, we may define $\hat{T}x \in L_{\text{loc}}^\infty(J, \mathbb{R}^q)$ by the property

$$\hat{T}x|_{[0, \sigma]} = Tx_{\sigma}|_{[0, \sigma]} \quad \forall \sigma \in J.$$  

Henceforth, we will not distinguish notationally an operator $T$ and its “localisation” $\hat{T}$: the correct interpretation being clear from context. For example, with this convention in place, we may reinterpret the lefthand side of the displayed inequality in property (iii) above as $\text{ess-sup}_{s \in [t, \tau]} \| (Tx Whispering) - (Ty Whispering) \|$, where $T = \hat{T}: C([-h, \tau], \mathbb{R}^m) \to L_{\text{loc}}^\infty([0, \tau], \mathbb{R}^q)$ now represents a “localization” of the original causal operator $T: C([-h, \infty), \mathbb{R}^m) \to L_{\text{loc}}^\infty(\mathbb{R}^+, \mathbb{R}^q)$. 

CONTROLLED FUNCTIONAL DIFFERENTIAL EQUATIONS
We are now in a position to define the system class.

**Definition 3.3** (system class \( \mathcal{S} \)). The class \( \mathcal{S} \) is comprised of \( m \)-input \((u(t) \in \mathbb{R}^m)\), \( m \)-output \((y(t) \in \mathbb{R}^m)\), nonlinear systems \((f, d, T)\) of the form (1.1), satisfying the following assumptions.

(A1) The function \( f : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^m \) is continuous.

(A2) For each compact set \( \mathcal{K} \subset \mathbb{R}^p \times \mathbb{R}^q \), the continuous function \( \gamma_{\mathcal{K}} : \mathbb{R} \rightarrow \mathbb{R} \), given by

\[
\gamma_{\mathcal{K}}(s) := \min \{ (v, f(l, w, sv)) \mid (l, w) \in \mathcal{K}, \|v\| = 1 \}, \quad (3.1)
\]

is such that either (i) \( \limsup_{s \to -\infty} \gamma_{\mathcal{K}}(s) = \infty \), or (ii) \( \limsup_{s \to -\infty} \gamma_{\mathcal{K}}(s) = \infty \).

(A3) \( d \in C(\mathbb{R}_+, \mathbb{R}^p) \) is bounded.

(A4) \( T : C([-h, \infty), \mathbb{R}^m) \rightarrow L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^q) \) is of class \( \mathcal{T}_h \).

### 3.1. Prototypical subclasses of \( \mathcal{S} \)

#### 3.1.1. Linear prototype

With reference to Figure 2, a system (1.1) of class \( \mathcal{S} \) can be thought of as an interconnection of two subsystems. The dynamical subsystem \( \Lambda_1 \), which can be influenced directly by the control input \( u \), is also driven by a disturbance \( d \) and by the output \( w \) from the subsystem \( \Lambda_2 \), formulated as a causal operator mapping the the signal \( y \) to \( w \) (an internal quantity, unavailable for feedback purposes).

To illustrate this, consider the prototype class \( \mathcal{L} \) of finite-dimensional, minimum-phase, \( m \)-input \((u(t) \in \mathbb{R}^m)\), \( m \)-output \((y(t) \in \mathbb{R}^m)\) linear systems \((A, B, C)\) with sign-definite high-frequency gain, in the sense that either \( CB \) or \( -CB \) is positive definite (symmetry of \( CB \) is not assumed). The minimum-phase property is characterized by

\[
\det \begin{bmatrix}
  sI - A \\
  C & B
\end{bmatrix} \neq 0 \quad \text{for all } s \in \mathbb{C}_+ := \{ s \in \mathbb{C} \mid \text{Re}(s) \geq 0 \}. \quad (3.2)
\]

Specifically,

\[
\mathcal{L} = \{ (A, B, C) \mid A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \quad C \in \mathbb{R}^{m \times n}, \quad m, n \in \mathbb{N}, \quad m \leq n, \quad CB \text{ sign definite, } (3.2) \text{ holds} \}.
\]

It is well known (see for example [4], Lem. 2.1.3) that, for each \((A, B, C) \in \mathcal{L}\) (and assuming \( m < n \)), there exists a similarity transformation which takes the system into the form

\[
\begin{align*}
\dot{y}(t) &= A_1 y(t) + A_2 z(t) + CBu(t), \quad y(0) = y^0, \\
\dot{z}(t) &= A_3 y(t) + A_4 z(t), \quad z(0) = z^0,
\end{align*} \quad (3.3)
\]

where, by the minimum-phase property, \( A_4 \) is a Hurwitz matrix. Defining the function \( d \) (continuous and bounded) and operator \( T \) (linear) by

\[
d(t) := A_2(\exp(A_4 t))z^0, \quad (Ty)(t) := A_1 y(t) + A_2 \int_0^t (\exp A_4(t - s))A_3y(s)ds, \quad (4.4)
\]
we see that the original system $(A, B, C) \in \mathcal{L}$ can be recast in the form of the (linear) functional differential equation

$$\dot{y}(t) = d(t) + (T y)(t) + C B u(t), \quad y(0) = y^0 \in \mathbb{R}^m,$$

which is of the form (1.1) with $h = 0$ and $f: \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, $(l, w, v) \mapsto l + w + C B v$. Clearly, Assumption (A1) holds. Since $A_1$ is Hurwitz, we see that (A3) and (A4) (with $h = 0$) are valid. It remains to show that (A2) also holds. Recall that $CB$ is sign definite and so either (i) $CB$ is positive definite, which we write symbolically as $CB > 0$, or (ii) $-CB > 0$. Let $\mathcal{K} \subset \mathbb{R}^m \times \mathbb{R}^m$ be compact and define

$$c_{\mathcal{K}} := \min \{v, l + w\} \mid (l, w) \in \mathcal{K}, \|v\| = 1\}.$$

Now, observe that

$$CB > 0 \quad \Rightarrow \quad \min \{v, CBv\} \|v\| = 1 = \frac{1}{2} \|CB + B^T C^T\|^{-1}$$

$$-CB > 0 \quad \Rightarrow \quad \min \{v, CBv\} \|v\| = 1 = -\frac{1}{2} \|CB + B^T C^T\|$$

Therefore,

(i) $CB > 0, \ s \geq 0 \quad \Rightarrow \quad \gamma_{\mathcal{K}}(s) = c_{\mathcal{K}} + \frac{s}{2} \|CB + B^T C^T\|^{-1}$ and so (A2)(i) holds,

(ii) $-CB > 0, \ s \leq 0 \quad \Rightarrow \quad \gamma_{\mathcal{K}}(s) = c_{\mathcal{K}} - \frac{s}{2} \|CB + B^T C^T\|$ and so (A2)(ii) holds.

### 3.1.2. Systems with input nonlinearity

To illustrate the generality afforded by Assumption (A2), consider a single-input, single-output ($m = 1$) system (3.3) of class $\mathcal{L}$ with a nonlinearity $g$ in the input channel

$$\dot{y}(t) = A_1 y(t) + A_2 z(t) + \beta g(u(t)), \quad y(0) = y^0, \quad z(0) = z^0,$$

where $\beta := CB$ is now a non-zero real number. We assume only that $g: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous unbounded function with bounded even part, for example, $g: v \mapsto (1 + v) \cos v$. Such a function can influence/reverse the polarity of an input signal $u(\cdot)$ in a manner unpredictable by a controller. Defining $d$ and $T$ as in (3.4), system (3.5) can be expressed as

$$\dot{y}(t) = d(t) + (T y)(t) + \beta g(u(t)), \quad y(0) = y^0 \in \mathbb{R},$$

which again is of form (1.1). Assumptions (A1), (A3) and (A4) clearly hold. Define $g_o$ and $g_e$ to be the odd and even parts, respectively, of the function $\beta g$. To see that (A2) holds, let $\mathcal{K} \subset \mathbb{R} \times \mathbb{R}$ be compact, define $c_{\mathcal{K}}$ as above, and observe that, since $v g_o(sv) = g_o(s)$ for all $|v| = 1$ and all $s \in \mathbb{R}$,

$$\gamma_{\mathcal{K}}(s) \leq \min \{v(l + w + g_e(sv)) \mid (l, w) \in \mathcal{K}, \|v\| = 1\} + g_o(s) \geq c_{\mathcal{K}} - |g_e(s)| + g_o(s) \quad \forall \ s.$$  (3.6)

Since the function $g_o$ is odd and unbounded, there must exist an unbounded monotone sequence $(s_n)$ (either strictly increasing or strictly decreasing) such that $g_o(s_n) \rightarrow \infty$ as $n \rightarrow \infty$ which, together with boundedness of $g_e$ and (3.6), ensures $\gamma_{\mathcal{K}}(s_n) \rightarrow \infty$ as $n \rightarrow \infty$.

### 3.1.3. Nonlinear systems

Now consider a further generalization of systems of form (3.5) to nonlinear systems of the form

$$\dot{y}(t) = f_1(y(t), z(t)) + g(u(t)), \quad y(0) = y^0 \in \mathbb{R},$$

$$\dot{z}(t) = f_2(y(t), z(t)), \quad z(0) = z^0 \in \mathbb{R}^p,$$

where $f_1$ and $g$ are nonlinear functions.
with \( f_1 \) continuous, \( f_2 \) locally Lipschitz, and (as above) \( g \) continuous and unbounded with bounded even part (we have absorbed the parameter \( \beta \neq 0 \) in \( g \)). Temporarily regarding \( y \) as an independent input to the second subsystem in (3.7), denote the unique solution of the initial-value problem \( \dot{z} = f_2(y, z), \quad z(0) = z_0^0 \), by \( z(\cdot; z_0^0, y) \).

If we now assume that the second subsystem in (3.7) is input-to-state stable (ISS) (see [13]), then, for each \( z_0^0 \in \mathbb{R}^p \), we may define an operator \( C(\mathbb{R}_+, \mathbb{R}) \rightarrow C(\mathbb{R}_+, \mathbb{R} \times \mathbb{R}^p) \) by

\[
(Ty)(t) := (y(t), z(t; z_0^0, y)) \quad \forall t \in \mathbb{R}_+.
\]

This operator \( T \) is of class \( \mathcal{T}_0 \) (Assumption (A4) holds with \( h = 0, m = 1 \) and \( q = p + 1 \)). System (3.7) may be expressed as the functional differential equation

\[
\dot{y}(t) = f_1((Ty)(t)) + g(u(t)), \quad y(0) = y_0^0,
\]

which is of the form (1.1) with \( h = 0 \) and \( f: (l, w, v) \mapsto f_1(w) + g(v) \). Evidently, Assumption (A1) holds, Assumption (A3) is vacuous, and Assumption (A2) holds by the argument (mutatis mutandis) used in Section 3.1.2.

3.1.4. Systems with delays and hysteresis

Finally, we remark that nonlinear delay elements are incorporated in the operator class \( \mathcal{T}_h \), see for example [12], whilst the class \( \mathcal{T}_0 \) encompasses a wide range of hysteresis operators, including many physically motivated effects: as observed in [5], examples such as relay hysteresis, elastic-plastic hysteresis, backlash hysteresis, Prandtl and Preisach operators (for background, see [2,10]) are of class \( \mathcal{T}_0 \).

4. Feedback control

We proceed to make precise the proposed output feedback structure. Let \( \lambda \geq 0 \) and \( \varphi \in \Phi_\lambda \). Let \( \nu: \mathbb{R} \rightarrow \mathbb{R} \) be any continuous function with the properties

\[
\limsup_{k \to -\infty} \nu(k) = +\infty \quad \text{and} \quad \liminf_{k \to -\infty} \nu(k) = -\infty,
\]

for example, \( \nu: k \mapsto k \cos k \). Let \( \alpha: [0, 1) \rightarrow \mathbb{R}_+ \) be a continuous unbounded injection, for example, \( \alpha: s \mapsto s/(1 - s) \). Define

\[
\mu := \begin{cases} 
1 & \text{if } \varphi \text{ is bounded}, \\
\frac{1}{2\sup_{t < -1} \varphi(t)} & \text{otherwise}.
\end{cases}
\]

If \( \mu > 0 \), let \( \text{sat}_\mu: \mathbb{R}^m \rightarrow \mathcal{B} := \{ v \in \mathbb{R}^m | \| v \| \leq 1 \} \) be any continuous function with the property that \( \text{sat}_\mu(e) = \| e \|^{-1} e \) for all \( \| e \| > \mu \), in which case the control strategy takes the form

\[
u(t) = -\nu(k(t))\text{sat}_\mu(y(t) - r(t)), \quad k(t) = \alpha(\varphi(t)\| y(t) - r(t) \|).
\]

In the case \( \mu = 0 \), the control strategy is given formally by

\[
u(t) = -\nu(k(t))\| y(t) - r(t) \|^{-1}(y(t) - r(t)), \quad k(t) = \alpha(\varphi(t)\| y(t) - r(t) \|).
\]

We accommodate each case and the (potential) discontinuity in (4.2) by embedding the control in a set-valued map \( \theta_\mu \), defined as follows:

\[
\theta_\mu(e) = \begin{cases} 
\{ \| e \|^{-1} \} & \text{if } \| e \| > \mu, \\
\mathcal{B} & \text{if } \| e \| \leq \mu,
\end{cases}
\]
The central issue is to establish that $\varphi$ is discontinuous. However, this need not always be the case. For example, with $\lambda = 0$ is warranted. of the main result in Theorem 6.1. Before proceeding to establish these facts, some commentary on the case

4.1. Closed-loop system

Let $\lambda \geq 0$, $\varphi \in \Phi_\lambda$, $r \in \mathcal{R}$ and let $\mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R}^m$ denote the set

$$\{(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \varphi(t)\|\xi - r(t)\| < 1\}.$$  

Let $(f, d, T) \in \mathcal{S}$. The conjunction of (1.1) with (4.3) yields the following closed-loop initial-value problem

$$\dot{y}(t) \in F(t, y(t), (Ty)(t)), \quad y|_{[-h, 0]} = y^0 \in C([-h, 0], \mathbb{R}^m),$$

where the set-valued map $(t, y, u) \mapsto F(t, y, u) \subset \mathbb{R}^m$, given by

$$F(t, y, u) := \{f(d(t), w, u) \mid u \in -\nu(\alpha(\varphi(t)\|y - r(t)\|))\theta_\mu(y - r(t))\},$$

is upper semicontinuous on $\mathcal{D} \times \mathbb{R}^3$ with non-empty, convex, compact values. By a solution of (4.4) we mean a function $y \in C(\bar{I}, \mathbb{R}^m)$ on some interval $I$ of the form $[-h, \rho]$, $0 < \rho < \infty$ or $[-h, \omega)$, $0 < \omega \leq \infty$, such that $y|_{[-h, 0]} = y^0$, $y|_I$ is locally absolutely continuous, with $(t, y(t)) \in \mathcal{D}$ for all $t \in J$ and $\dot{y}(t) \in F(t, y(t), (Ty)(t))$ for almost all $t \in J$, where $J := I \setminus [-h, 0)$. A solution is said to be maximal if it has no proper right extension that is also a solution. A solution defined on $[-h, \infty)$ is said to be global. We will demonstrate that the control objectives are achieved by establishing that: (i) the initial-value problem (4.4) has a solution; (ii) every solution can be extended to a maximal solution; (iii) every maximal solution is global. Facts (i) and (ii) are a consequence (Cor. 5.2) of the existence theory (Thm. 5.1) developed in Section 5 below; fact (iii) is the essence of the main result in Theorem 6.1. Before proceeding to establish these facts, some commentary on the case $\lambda = 0$ is warranted.

4.1.1. Commentary on the asymptotic tracking problem

Assume that $\lambda = 0$, in which case we have $\mu = 0$, and so the formal control structure (4.2) is potentially discontinuous. However, this need not always be the case. For example, with

$$\nu: k \mapsto k \cos(ak) \quad \text{and} \quad \alpha: s \mapsto \frac{s}{1 - s},$$

where $a > 0$, the feedback (4.2) is, in fact, continuous on the domain $\mathcal{D}$: in particular, the control takes the form

$$u(t) = \psi(t, y(t) - r(t)),$$

with $\psi \in C(\mathcal{D}, \mathbb{R}^m)$ given by

$$\psi(t, \xi) := -\cos \left(\frac{a\varphi(t)\|\xi\|}{1 - \varphi(t)\|\xi\|}\right) \left(\frac{\varphi(t)\xi}{1 - \varphi(t)\|\xi\|}\right) \forall (t, \xi) \in \mathcal{D},$$

in which case the map $F$ in (4.4) is singleton valued.
Example. Consider a single-input, single-output system (3.7) of the nonlinear prototype class, with \( f_1, f_2 : \mathbb{R}^2 \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) given by

\[
\begin{align*}
    f_1(y, z) &= z \sin y, \\
    f_2(y, z) &= -z |z| + y, \\
    g(u) &= u^{1/3}.
\end{align*}
\] (4.8)

As reference signal \( r \in \mathbb{R} \), we take \( r = \zeta_1/2 \), where \( \zeta_1 \) is the first component of the (chaotic) solution of the following Lorenz system of equations:

\[
\begin{align*}
    \dot{\zeta}_1(t) &= \zeta_2(t) - \zeta_1(t), & \quad \zeta_1(0) &= 1, \\
    \dot{\zeta}_2(t) &= c_0 \zeta_1(t) - c_1 \zeta_2(t) - \zeta_1(t) \zeta_3(t), & \quad \zeta_2(0) &= 0, \\
    \dot{\zeta}_3(t) &= \zeta_1(t) \zeta_2(t) - c_2 \zeta_3(t), & \quad \zeta_3(0) &= 3.
\end{align*}
\] (4.9)

with parameter values \( c_0 = 28/10, c_1 = 1/10 \) and \( c_2 = 8/30 \). It is well known that the unique global solution of (4.9) is bounded with bounded derivative, see for example [15].

Adopting control parameters \( a = 1/4 \) and \( \varphi : t \mapsto 2t \), Figures 3–5 depict the behaviour of the closed-loop system with zero initial state.

There are, of course, practical issues relating to the synthesis of the control strategy (4.6)–(4.7). Whilst later analysis will establish the fact that \( \sup_{t \in \mathbb{R}^+} \varphi(t) \|y(t) - r(t)\| < 1 \), and so boundedness of the control function \( u(t) \) for large \( t \) may encounter numerical ill-conditioning insofar as it involves the product of “large” and “small” quantities (since \( \varphi(t) \to \infty \) and \( \|y(t) - r(t)\| \to 0 \) as \( t \to \infty \)). These practical issues are not addressed in this paper (the purpose of which is to highlight those performance characteristics that are attainable in principle): however, we remark that the ill-conditioning associated with the case \( \mu = 0 \) may be circumvented (at the expense of some degradation in performance) on setting \( \lambda > 0 \) and replacing unbounded \( \varphi \) by a bounded function \( \varphi \in \Phi_\lambda \) with \( \lim \inf_{t \in \mathbb{R}^+} \varphi(t) = 1/\lambda \), in which case, the guaranteed performance is weakened to that of approximate tracking, as quantified by \( \lim \sup_{t \to \infty} \|y(t) - r(t)\| < \lambda \).
5. Existence theory

Here, we present an existence theory of sufficient generality to encompass (4.4). Let \( D \) be a domain in \( \mathbb{R}_+ \times \mathbb{R}^m \), that is, a non-empty, connected, relatively open subset of \( \mathbb{R}_+ \times \mathbb{R}^m \). Let \( (t, y, w) \mapsto G(t, y, w) \subseteq \mathbb{R}^m \) be upper semicontinuous on \( G := D \times \mathbb{R}^q \), with non-empty, convex and compact values. Let \( h \geq 0 \) and \( T: C([-h, \infty), \mathbb{R}^m) \rightarrow L^\infty_0(\mathbb{R}_+, \mathbb{R}^q) \) be a causal operator of class \( T_h \). For \( t_0 \geq 0 \), consider the initial-value problem

\[
\dot{y}(t) \in G(t, y(t), (Ty)(t)), \quad y|_{[-h, t_0]} = y^0 \in C([-h, t_0], \mathbb{R}^m), \quad (t_0, y^0(t_0)) \in D. \tag{5.1}
\]

We emphasize that, for reasons which will become apparent in the proof of Theorem 5.1 below, the parameter \( t_0 \geq 0 \) has been incorporated in (5.1): this necessitates the obvious generalization of the earlier concept of a solution introduced in the context of (4.4) wherein \( t_0 = 0 \). Specifically, by a solution of (5.1) we mean a function \( y \in C(I, \mathbb{R}^m) \) for some interval \( I \) of the form \([-h, \rho]\), \( t_0 < \rho < \infty \) or \([-h, \omega]\), \( t_0 < \omega \leq \infty \), such that \( y|_{[-h, t_0]} = y^0 \), \( y\) is locally absolutely continuous, \( \dot{y}(t) \in G(t, y(t), (Ty)(t)) \) for almost all \( t \in J \), and \( (t, y(t)) \in D \) for all \( t \in J \), where \( J := I \setminus [-h, t_0] \). Again, a solution is said to be maximal if it has no proper right extension that is also a solution.

**Theorem 5.1.** For each \( t_0 \geq 0 \) and \( y^0 \in C([-h, t_0], \mathbb{R}^m) \) with \((t_0, y^0(t_0)) \in D\),

(i) the initial-value problem (5.1) has a solution;

(ii) every solution can be extended to a maximal solution \( y: [-h, \omega) \rightarrow \mathbb{R}^m \);

(iii) if \( y: [-h, \omega) \rightarrow \mathbb{R}^m \) is a maximal solution of (5.1) and \( \omega < \infty \), then, for every \( \sigma \in [t_0, \omega) \) and every compact set \( K \subset D \), there exists \( t \in [\sigma, \omega) \) such that \((t, y(t)) \notin K\).

A proof of this result can be found in the Appendix.

**Corollary 5.2.** Let \( (f, d, T) \in S, \lambda \geq 0 \) and \( \phi \in \Phi_\lambda \). Then, for every reference signal \( r \in \mathbb{R} \) and all initial data \( y^0 \in C([-h, 0], \mathbb{R}^m) \), application of the feedback (4.3) to the system (1.1) yields the initial-value problem (4.4)–(4.5) which has a solution and every solution can be extended to a maximal solution \( y: [-h, \omega) \rightarrow \mathbb{R}^m \), \( 0 < \omega \leq \infty \). Furthermore, if \( y: [-h, \omega) \rightarrow \mathbb{R}^m \) is a maximal solution and there exists a compact set \( K \subset D \) such that \((t, y(t)) \in K \) for all \( t \in [\sigma, \omega) \), then \( \omega = \infty \).

**Proof.** Defining the domain \( D := \{(t, y) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \varphi(t)\|y - r(t)\| < 1 \} \), we identify the initial-value problem (4.4)–(4.5) as a particular case of (5.1) (with \( G = F \) and \( t_0 = 0 \)):

\[
\dot{y}(t) \in F(t, y(t), (Ty)(t)), \quad y|_{[-h, 0]} = y^0 \in C([-h, 0], \mathbb{R}^m), \quad (0, y^0(0)) \in D, \tag{5.2}
\]

where \( F(t, y, w) = \{f(d(t), w, u) \mid u \in -\nu(\alpha(\varphi(t)\|y - r(t)\|))\theta_\nu(y - r(t))\} \).

An application of Theorem 5.1 completes the proof. \( \square \)
6. Main Result

We now arrive at the main result, statement (ii) of which asserts that the output of the closed-loop system evolves within the performance funnel and is bounded away from the funnel boundary.

**Theorem 6.1.** Let \( (f,d,T,h) \in \mathcal{S}, \lambda \geq 0 \) and \( \varphi \in \Phi_\lambda \). Then for every reference signal \( r \in \mathcal{R} \) and all initial data \( y^0 \in C([-h,0],\mathbb{R}^m) \), application of the feedback \( (4.3) \) to the system \( (1.1) \) yields the closed-loop initial-value problem \( (4.4)-(4.5) \) which has a solution and each solution can be extended to a maximal solution \( y: [-h, \omega] \to \mathbb{R}^m \). Every maximal solution \( y: [-h, \omega] \to \mathbb{R}^m \) has the properties:

(i) \( \omega = \infty \);
(ii) \( \sup_{t \in \mathbb{R}^+} \varphi(t)\|y(t) - r(t)\| < 1 \);
(iii) the function \( k: t \mapsto \alpha(\varphi(t)\|y(t) - r(t)\|) \) is bounded.

**Remark 6.2.** The conjunction of assertions (i) and (ii) ensures that both control objectives are attained. Assertion (iii) implies boundedness of the control. In the case where \( \varphi(t) \to \infty \) as \( t \to \infty \), assertion (ii) implies asymptotic tracking: \( \|y(t) - r(t)\| \to 0 \) as \( t \to \infty \).

**Proof.** Let \( r \in \mathcal{R} \) and \( y^0 \in C([-h,0],\mathbb{R}^m) \). By Corollary 5.2, the closed-loop initial-value problem \( (4.4)-(4.5) \) has a solution and every solution can be maximally extended. Let \( y: [-h, \omega] \to \mathbb{R}^m \) be a maximal solution of \( (4.4) \). Defining \( e(t) = y(t) - r(t) \) for all \( t \in [0, \omega) \), we have

\[
e(t) + \dot{r}(t) = F(t,e(t) + r(t),(Ty)(t)) \quad \text{for a.a. } t \in [0, \omega).
\]

(6.1)

Since \( (t,y(t)) \in \mathcal{D} \) for all \( t \in [0, \omega) \), it follows that \( \varphi(t)\|e(t)\| < 1 \) for all \( t \in [0, \omega) \). By properties of \( \varphi \in \Phi_\lambda \), we may infer boundedness of the function \( e \). Furthermore, since \( r \in \mathcal{R} \) is bounded, we may conclude that \( y \) is bounded. Invoking Assumptions (A3) and (A4) (in particular, property (iv) of the operator class \( \mathcal{T}_h \)), we deduce the existence of a non-empty, compact set \( K \subset \mathbb{R}^p \times \mathbb{R}^q \) such that \( (d(t),(Ty)(t)) \in K \) for almost all \( t \in [0, \omega) \). With this set, we associate the function \( \gamma_K \), defined as in \( (3.1) \). Writing

\[
\Sigma := \{ t \in [0, \omega] | \|e(t)\| > \mu \}, \quad \text{and} \quad k(t) := \alpha(\varphi(t)\|e(t)\|) \quad \forall \ t \in [0, \omega),
\]

we have

\[
t \in \Sigma \implies (e(t),f(d(t),(Ty)(t),-\nu(k(t))\|e(t)\|^{-1}e(t))) \leq \|e(t)\| \min\{\|u,f(v,w,\nu(k(t))u)\| (v,w) \in K, \|u\| = 1\}
\]

\[
\leq -\|e(t)\| \gamma_K(\nu(k(t))).
\]

(6.2)

Noting that

\[
t \in \Sigma \implies F(t,e(t) + r(t),(Ty)(t)) = \{f(d(t),(Ty)(t),-\nu(k(t))\|e(t)\|^{-1}e(t))\},
\]

we may infer from \( (6.2) \) that

\[
\langle e(t),v \rangle \leq -\gamma_K(\nu(k(t))\|e(t)\|) \quad \forall \ v \in F(t,e(t) + r(t),(Ty)(t)), \quad \forall \ t \in \Sigma.
\]

Therefore, by \( (6.1) \) and essential boundedness of \( \dot{r} \), there exists \( c_0 > 0 \) such that

\[
\langle e(t),\dot{e}(t) \rangle \leq [c_0 - \gamma_K(\nu(k(t)))]\|e(t)\| \quad \text{for a.a. } t \in \Sigma.
\]

(6.3)

By Assumption A2, either (i) \( \limsup_{s \to +\infty} \gamma_K(s) = \infty \), or (ii) \( \limsup_{s \to -\infty} \gamma_K(s) = \infty \). Therefore, there exists an unbounded sequence \( (s_n) \subset \mathbb{R} \), which is either strictly increasing (in case (i)) or strictly decreasing (in case (ii)), such that the sequence \( (\gamma_K(s_n)) \) is unbounded and strictly increasing, with \( \gamma_K(s_n) > 0 \) for
all \( n \in \mathbb{N} \). By properties (4.1) and continuity of \( \nu \), for every \( a, b \in \mathbb{R} \) the set \( \{ \kappa > a \mid \nu(\kappa) = b \} \) is non-empty. Let \( k_1 \in \{ \kappa > \alpha(4) \mid \nu(\kappa) = s_1 \} \) be arbitrary and define the strictly-increasing unbounded sequence \((k_n)\) in \((\alpha(4), \infty)\) by the recursion \( k_{n+1} := \inf\{\kappa > k_n \mid \nu(\kappa) = s_n + 1\} \), and so \( \gamma(\nu(\kappa(n))) = \gamma(\kappa(n)) \to \infty \) as \( n \to \infty \).

We proceed to prove boundedness of \( k \). Seeking a contradiction, suppose \( k \) is unbounded (in which case, \( \text{im}(k) = \text{im}(\alpha) = [\alpha(0), \infty) \)). For each \( n \in \mathbb{N} \), define

\[
\tau_n := \inf\{t \in [0, \omega] \mid k(t) = k_{n+1}\} \quad \text{and} \quad \sigma_n := \sup\{t \in [0, \tau_n] \mid \gamma(\nu(\kappa(t))) = \gamma(\nu(\kappa(n)))\}.
\]

We briefly digress to assemble some useful facts.

**Proposition 6.3.** (a) \( \sigma_n < \tau_n \forall n \in \mathbb{N} \). (b) \( k(\sigma_n) < k(\tau_n) \forall n \in \mathbb{N} \). (c) \( k(t) \geq k_n \forall t \in [\sigma_n, \tau_n] \forall n \in \mathbb{N} \). (d) \( \gamma(\nu(\kappa(t))) \geq \gamma(\nu(\kappa(n))) \) for some \( n \in \mathbb{N} \).

**Proof.** (a) Suppose, for contradiction, that \( \sigma_n = \tau_n \) for some \( n \in \mathbb{N} \). Then,

\[
\gamma(\kappa(\sigma(n+1))) = \gamma(\nu(\kappa(n+1))) = \gamma(\nu(\kappa(\tau(n)))) = \gamma(\nu(\kappa(n))) = \gamma(\kappa(n)),
\]

which contradicts strict monotonicity of the sequence \((\gamma(\kappa(s_n)))\).

(b) Suppose, for contradiction, that \( k(\sigma(n)) \geq k(\tau(n)) = k_{n+1} \) for some \( n \in \mathbb{N} \). Then, since \( k(0) = \alpha(0) < \alpha(1/2) < k_{n+1} \), there exists \( s \leq \sigma(n) \leq \tau(n) \) such that \( k(s) = k_{n+1} \), whence the contradiction: \( \tau_n = \inf\{t \in [0, \omega] \mid k(t) = k_{n+1}\} \leq s < \tau_n \).

(c) Suppose, for contradiction, that, for some \( n \in \mathbb{N} \) and \( t \in [\sigma(n), \tau(n)] \), \( k(t) < k_n \). Then, since \( k(\tau(n)) = k_{n+1} \), there exists \( s \in (\sigma(n), \tau(n)] \) such that \( k(s) = k_n \). Invoking the definition of \( \sigma(n) \), we arrive at a contradiction:

\[
\sigma(n) < s \leq \sigma(n),
\]

(d) Suppose, for contradiction, that, for some \( n \in \mathbb{N} \) and \( t \in [\sigma(n), \tau(n)] \), \( \gamma(\nu(\kappa(t))) < \gamma(\nu(\kappa(n))) \). Since

\[
\gamma(\nu(\kappa(n))) = \gamma(\kappa(n)) \leq \gamma(\nu(\kappa(n+1))) = \gamma(\nu(\kappa(\tau(n)))) = \gamma(\nu(\kappa(n))),
\]

it follows that, for some \( s \in (\sigma(n), \tau(n)] \), \( \gamma(\nu(\kappa(s))) = \gamma(\nu(\kappa(n))) \), which contradicts the definition of \( \sigma(n) \).

(e) Suppose, for contradiction, that, for some \( n \in \mathbb{N} \), there exists \( t \in [\sigma(n), \tau(n)] \) such that \( t \not\in \Sigma \), then \( \|e(t)\| \leq \mu \). Note that \( \alpha(0) < \alpha(1/2) \) and, if \( \mu > 0 \), then \( \alpha(\mu \varphi(t)) \leq \alpha(1/2) \). Therefore, we arrive at a contradiction.

\[
\alpha(1/2) < k_n < k(t) = \alpha(\varphi(t)||e(t)||) \leq \alpha(1/2).
\]

We now return to the proof of Theorem 6.1. From assertions (c) and (d) of Proposition 6.3, we may infer that

\[
\frac{1}{2} < \alpha^{-1}(k_n) \leq \alpha^{-1}(k(t)) \leq \varphi(t)||e(t)|| < 1 \quad \forall t \in [\sigma(n), \tau(n)] \forall n \in \mathbb{N},
\]

\[
\alpha^{-1} : [\alpha(0), \infty) \to [0, 1) \text{ is the inverse of the bijection } \alpha : [0, 1) \to \text{im}(\alpha), \text{ and}
\]

\[
-2\varphi^2(t)||e(t)||\gamma(\nu(\kappa(t))) \leq -\varphi(t)\gamma(\nu(\kappa(t))) \quad \forall t \in [\sigma(n), \tau(n)] \forall n \in \mathbb{N}.
\]

By properties of \( \varphi \in \Phi_\chi \), there exists \( c_1 > 0 \) such that \( \varphi(t) \leq c_1[1 + \varphi(t)] \) for almost all \( t \) which, together with (6.3), yields, for almost all \( t \in \Sigma \),

\[
\frac{d}{dt}||e(t)||^2 = 2\varphi(t)\dot{\varphi}(t)||e(t)||^2 + 2\varphi^2(t)(\epsilon(t), \dot{e}(t)) 
\]

\[
\leq 2c_1\varphi(t)[1 + \varphi(t)]||e(t)||^2 + 2\varphi^2(t)||e(t)||\||e(t)|| \leq 2c_2 - \gamma(\nu(\kappa(t)))\}
\]

Invoking (6.4), (6.5) and boundedness of \( e \), we may conclude the existence of \( c_2 > 0 \) such that

\[
\frac{d}{dt}||e(t)||^2 \leq \varphi(t)[c_2 - \gamma(\nu(\kappa(t)))] \quad \text{for a.a. } t \in [\sigma(n), \tau(n)] \forall n \in \mathbb{N}.
\]
Fix $n \in \mathbb{N}$ sufficiently large so that $c_2 - \gamma \LipW \nu (k_n) < 0$. Recalling that $\gamma \LipW \nu (k(t)) \geq \gamma \LipW \nu (k_n)$ for all $t \in [\sigma_n, \tau_n]$, we have
\[ \frac{d}{dt} [\varphi(t) \| e(t) \|]^2 < 0 \quad \text{for a.a. } t \in [\sigma_n, \tau_n] \]
and so $\varphi(\tau_n) \| e(\tau_n) \| < \varphi(\sigma_n) \| e(\sigma_n) \|$. Therefore,
\[ k(\tau_n) = \alpha (\varphi(\tau_n) \| e(\tau_n) \|) < \alpha (\varphi(\sigma_n) \| e(\sigma_n) \|) = k(\sigma_n), \]
which contradicts assertion (b) of Proposition 6.3. This proves boundedness of $k$ (and so $\nu \circ k : t \mapsto \nu (\alpha (\varphi(t) \| y(t) - r(t) \|))$ is also bounded). By boundedness of $t \mapsto k(t) = \alpha (\varphi(t) \| e(t) \|)$, it follows that $\sup_{t \in [0, \omega]} \varphi(t) \| y(t) - r(t) \| < 1$, equivalently, there exists $\varepsilon \in (0, 1)$ such that $\varphi(t) \| y(t) - r(t) \| \leq 1 - \varepsilon$ for all $t \in [0, \omega)$.

Finally, we show that $\omega = \infty$. By boundedness of $y$, there exists $c_3 > 0$ such that $\| y(t) \| \leq c_3$ for all $t \in [0, \omega)$. Suppose $\omega < \infty$. Then
\[ \bar{K} := \{(t, v) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \varphi(t) \| v - r(t) \| \leq 1 - \varepsilon, \| v \| \leq c_3, \ t \in [0, \omega)\} \]
is a compact subset of $\mathcal{D}$ with the property $(t, y(t)) \in \bar{K}$ for all $t \in [0, \omega)$, which contradicts assertion (iii) of Theorem 5.1. Therefore, $\omega = \infty$. This completes the proof. \hfill \Box

Remark 6.4. To paraphrase Wonham [17], p. 210, the internal model principle states that every “good” regulator must incorporate a model of the outside world (in the sense that the feedback loop incorporates a suitably reduplicated model of the dynamic structure of the exogenous signals which the closed-loop system is required to track). In the context of linear systems with linear regulators (see [16,17]), “good” means “structurally stable”; in a more general context of smooth nonlinear systems (see [14]), “good” amounts to a “signal detection” property. In effect, “good” implies some robustness property of the closed loop. The feedback structure proposed in the present paper ensures tracking of any signal of class $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$, yet it does not contain a model capable of replicating this class of signals. For consistency with the internal model principle, one must therefore conclude that the closed-loop system of the present paper lacks certain robustness properties. This perceived lack of robustness may stem from the potential singularity introduced via the injection $\alpha$ in the closed loop or from the unbounded nature of the funnel function $\varphi$. It is not unreasonable to expect that the adoption of a bounded function $\varphi$ (with attendant reduction in performance from asymptotic to approximate tracking) might induce some robustness in the closed loop. However, in the absence of a rigorous robustness analysis, the results of the paper are mainly of a theoretical nature, serving to illustrate those performance characteristics that are attainable, in principle, under weak assumptions on the plant data.

A. APPENDIX: PROOF OF THEOREM 5.1

Let $X$ be a normed vector space. The open ball of radius $\varepsilon > 0$ centred at $x \in X$ is denoted by $B_\varepsilon(x)$ (the ambient space $X$ being clear from context), $\overline{B}_\varepsilon(x)$ denotes the closure of $B_\varepsilon(x)$: if $x = 0$, then, for simplicity, we write $\overline{B}_\varepsilon$ in place of $\overline{B}_\varepsilon(0)$.

We record the following properties of $G$:

\begin{align*}
& \text{(a) } \text{graph}(G) := \{(z, \zeta) \mid \zeta \in G(z), \ z \in \mathcal{G}\} \text{ is closed;} \\
& \text{(b) } \text{if } \mathcal{K} \subset \mathcal{G} \text{ is compact, then } G(\mathcal{K}) := \bigcup_{z \in \mathcal{K}} G(z) \text{ is compact;} \\
& \text{(c) } \text{for each } \varepsilon > 0, \text{ there exists a locally Lipschitz function } g : \mathcal{G} \to \mathbb{R}^m \text{ such that graph}(g) \subset \text{graph}(G) + B_\varepsilon. \tag{A.1}
\end{align*}

For (a) see [1], Proposition 2, p. 41, for (b) see [1], Proposition 3, p. 42, for (c) see [1], Theorem 1, p. 84.

To facilitate the proof of the general result in Theorem 5.1, we first establish a variant in the restricted context wherein $G$ is a singleton-valued map $G : (t, y, w) \mapsto \{g(t, y, w)\}$ and $g : \mathcal{G} \to \mathbb{R}^m$ is locally Lipschitz.
Lemma A.1. Let $g: \mathcal{G} \to \mathbb{R}^m$ be a locally Lipschitz function. For $t_0 \geq 0$ and $y^0 \in C([-h, t_0], \mathbb{R}^m)$, the initial-value problem

$$
\dot{y}(t) = g(t, y(t), (Ty)(t)), \quad y([-h, t_0]) = y^0 \in C([-h, t_0], \mathbb{R}^m), \quad (t_0, y^0(t_0)) \in \mathcal{D},
$$

has a unique maximal solution, $y: [-h, \omega) \to \mathbb{R}^m$. Furthermore, if $\omega < \infty$, then, for every $\sigma \in [t_0, \omega)$ and every compact set $K \subset \mathcal{D}$, there exists $t \in [\sigma, \omega)$ such that $(t, y(t)) \notin K$.

Proof. Step 1: Existence of a unique solution on a small interval.

By property (iii) of $T \in \mathcal{T}_h$, there exist $\delta > 0$, $c_0 > 0$ and $\tau > t_0$ such that

$$
\| (Ty)(t) - (Tz)(t) \| \leq c_0 \max_{s \in [t_0, \tau]} \| y(s) - z(s) \| \text{ for a.a. } t \in [t_0, \tau] \text{ and all } y, z \in C(y^0; h, t_0, \tau, \delta).
$$

(A.3)

Without loss of generality, we may assume that $\delta \in (0, 1)$ and $\tau - t_0 > 0$ are sufficiently small so that $[t_0, \tau] \times \mathcal{B}_\delta(y^0(t_0)) \subset \mathcal{D}$. For each $\rho \in (t_0, \tau]$, define $C_\rho := C(y^0; h, t_0, \rho, \delta)$ which, equipped with the metric

$$(y, z) \mapsto \beta_\rho(y, z) := \sup_{t \in [-h, \rho]} \| y(t) - z(t) \|,
$$

is a complete metric space. Observe that, if $y \in C_\rho$, then $(t, y(t)) \in \mathcal{D}$ for all $t \in [t_0, \rho]$. For each $\rho \in (t_0, \tau]$, define the operator $Z_\rho$ on $C_\rho$ by

$$
(Z_\rho y)(t) := \begin{cases} 
    y^0(t), & t \in [-h, t_0], \\
    y^0(t_0) + \int_{t_0}^t g(s, y(s), (Ty)(s))ds, & t \in (t_0, \rho).
\end{cases}
$$

We proceed to show that $Z_\rho$ is a contraction. Define $c_1 := \max_{s \in [-h, t_0]} \| y^0(s) \| + \delta$. By property (iv) of $T$, there exists $c_2 > 0$ such that

$$
\sup_{t \in [-h, \tau]} \| y(t) \| < c_1 \implies \| (Ty)(t) \| < c_2 \text{ for a.a. } t \in [t_0, \tau].
$$

By the local Lipschitz property of $g$, there exists a constant $c_3 > 0$ such that, for all $t \in [t_0, \tau]$,

$$
\| g(t, y, w) - g(t, z, w) \| \leq c_3 \left( \| y - z \| + \| w - x \| \right) \quad \forall \ y, z \in \mathcal{B}_{c_1}, \quad \forall \ w, x \in \mathcal{B}_{c_2}.
$$

Write

$$
\rho^* := \max\{ \| g(t, y, w) \| \mid (t, y, w) \in [t_0, \tau] \times \mathcal{B}_\delta(y^0(t_0)) \times \mathcal{B}_{c_2} \}.
$$

Fix $\rho^* \in (t_0, \tau]$ sufficiently close to $t_0$ so that

$$
(\rho^* - t_0)(\rho^* + (c_0 + 1)c_3) < \delta.
$$

Let $\rho \in (t_0, \rho^*)$ and $y \in C_\rho$. By definition, $(Z_\rho y)[-h, t_0] = y^0$ and

$$
\| (Z_\rho y)(t) - y^0(t) \| = \left\| \int_{t_0}^t g(s, y(s), (Ty)(s))ds \right\|
\leq \int_{t_0}^\rho \| g(s, y(s), (Ty)(s)) \|ds \leq (\rho - t_0)g^* < \delta \quad \forall \ t \in [t_0, \rho].
$$
Therefore \((Z_\rho y)(\cdot) \in C_\rho\). Furthermore,

\[
\beta_\rho(Z_\rho y, Z_\rho z) = \sup_{t \in [t_0, \rho]} \left\| \int_{t_0}^{\rho} [g(s, y(s), (Ty)(s)) - g(s, z(s), (Tz)(s))] ds \right\|
\]

\[
\leq \int_{t_0}^{\rho} \|g(s, y(s), (Ty)(s)) - g(s, z(s), (Tz)(s))\| ds
\]

\[
\leq (\rho - t_0)c_3 \left[ \text{ess-sup}_{s \in [t_0, \rho]} \| (Ty)(s) - (Tz)(s) \| + \beta_\rho(y, z) \right]
\]

\[
\leq (c_0 + 1)(\rho - t_0)c_3 \beta_\rho(y, z) \quad \forall y, z \in C_\rho,
\]

wherein the last inequality follows by (A.3). Since \((c_0 + 1)(\rho - t_0)c_3 < \delta < 1\), we may infer that \(Z_\rho: C_\rho \rightarrow C_\rho\) is a contraction. By the contraction mapping theorem, \(Z_\rho\) has a unique fixed point. Thus we have shown that, for each \(\rho \in (t_0, \rho^*)\), the initial-value problem (A.2) has a unique solution \(y \in C_\rho\). We stress that the uniqueness property of \(y\) holds only in relation to solutions in the restricted class \(C_\rho\): there may exist another solution on the interval \([-h, \rho]\) which is not contained in the space \(C_\rho\). However, the following argument establishes uniqueness of the solution on a sufficiently small interval. Let \(y^*\) (not necessarily in \(C_\rho\)) be a solution on \([-h, \rho^*]\). Define

\[
\Delta := \{t \in [t_0, \rho^*] \mid \| y^*(t) - y^0(t_0) \| = \delta \}, \quad \rho := \left\{ \begin{array}{ll} \inf \Delta, & \Delta \neq \emptyset, \\
\rho^*, & \Delta = \emptyset. \end{array} \right.
\]

Clearly \(\rho > t_0\) and \(y := y^*|_{[-h, \rho]}\) is in \(C_\rho\). Therefore, \(y\) is the unique solution of (A.2) on the interval \([-h, \rho]\).

**Step 2:** Extended uniqueness: any two solutions must coincide on the intersection of their domains.

Let \(y_1: I_1 \rightarrow \mathbb{R}^m\) and \(y_2: I_2 \rightarrow \mathbb{R}^m\) be solutions of (A.2) and, without loss of generality, assume \(I_2 \subset I_1\). For contradiction, suppose that \(y_1|_{I_2} \neq y_2\). Let \(t^* := \inf\{t \in I_2 \mid y_1(t) \neq y_2(t)\}\). By the result in Step 1, the solutions \(y_1\) and \(y_2\) must coincide on some interval \([-h, \rho]\), with \(\rho > t_0\). Therefore, \(t^* > t_0\). An application of the result of Step 1 in the context of an initial-value problem of the form (A.2), with \(t^*\) replacing \(t_0\) and initial function \(y_1|_{[-h, t^*]} \in C([-h, t^*], \mathbb{R}^m)\) replacing \(y^0\), yields the existence of a unique solution \(y \in C([-h, \rho], \mathbb{R}^m)\) for some \(\rho > t^*\). It follows that \(y_1(t) = y_2(t) = y(t)\) for all \(t \in [-h, \rho]\), contradicting the definition of \(t^*\).

**Step 3:** Existence of a unique maximal solution.

Let \(P\) be the set of all \(\rho > t_0\) such that there exists a solution \(y_\rho\) of (A.2) on the interval \([-h, \rho]\). By Step 1, we know that \(P \neq \emptyset\). Let \(\omega := \sup P\) and define \(y: [-h, \omega) \rightarrow \mathbb{R}^m\) by the property

\[
y|_{[-h, \rho]} = y_\rho \quad \forall \rho \in P.
\]

The function \(y\) is well-defined since, by Step 2, for all \(\rho_1, \rho_2 \in P\), we have \(y_{\rho_2} = y_{\rho_1}|_{[-h, \rho_1]}\) whenever \(\rho_2 \leq \rho_1\). Clearly \(y\) is a maximal solution and uniqueness follows by Step 2.

**Step 4:** Assume that \(y: [-h, \omega) \rightarrow \mathbb{R}^m\) is a maximal solution with \(\omega < \infty\). Seeking a contradiction, suppose there exist \(\sigma \in [t_0, \omega]\) and a compact set \(K \subset D\) such that \((t, y(t)) \in K\) for all \(t \in [\sigma, \omega]\). Then \(y\) is bounded and, by property (iv) of \(T \in T_h, Ty\) is essentially bounded. Therefore, the function \(t \mapsto (t, y(t), (Ty)(t))\) is essentially bounded and so, by continuity of \(g\), it follows that \(\dot{y}\) is essentially bounded on the interval \([t_0, \omega]\). Therefore \(y\) is uniformly continuous on \([-h, \omega]\) and so extends to \(y^* \in C([-h, \omega], \mathbb{R}^m)\). By compactness of \(K\), we have \((\omega, y^*(\omega)) \in K \subset D\). An application of the result of Step 1 in the context of an initial-value problem of the form (A.2), with \(\omega\) replacing \(t_0\) and \(y^*\) replacing \(y^0\), yields the existence of a unique solution \(y^* \in C([-h, \rho], \mathbb{R}^m)\) for some \(\rho > \omega\), with \(y^*|_{[-h, \omega]} = y\). This contradicts maximality of \(y\).

\[\blacksquare\]

We are now in a position to prove the existence of a solution to the problem (5.1).
Proof of Theorem 5.1

Proof. (i) Let \((\varepsilon_n) \subset (0,1)\) be a monotonically decreasing sequence with \(\varepsilon_n \to 0\) as \(n \to \infty\). By property (A.1c), for each \(n \in \mathbb{N}\), there exists a locally Lipschitz function \(g_n : \mathcal{G} \to \mathbb{R}^m\) with

\[
\text{graph}(g_n) \subset \text{graph}(G) + B_{\varepsilon_n}.
\]  

By Lemma A.1, for each \(n \in \mathbb{N}\), the initial-value problem

\[
\dot{y}(t) = g_n(t,y(t),(Ty)(t)), \quad y|_{[-h,t_0]} = y^0 \in C([-h,t_0],\mathbb{R}^m), \quad (t_0,y^0(t_0)) \in \mathcal{D},
\]

has a unique maximal solution which we denote by \(y_n : [-h,\omega_n) \to \mathbb{R}^m\).

Recalling that \(\mathcal{D}\) is a relatively open subset of \(\mathbb{R}_+ \times \mathbb{R}^m\) and invoking property (iii) of \(T \in \mathcal{T}_h\), we may choose \(\delta > 0\) sufficiently small and \(\omega^* > t_0\) sufficiently close to \(t_0\) so that

\[
[t_0,\omega^*) \times \overline{B}_\delta(y^0(t_0)) =: \mathcal{K}_0 \subset \mathcal{D},
\]

and there exists \(c_0 > 0\) such that

\[
\text{ess-sup}_{t \in [t_0,\omega^*)} \| (Ty(t)) - (Tz(t)) \| \leq c_0 \max_{t \in [t_0,\omega^*)} \| y(t) - z(t) \| \quad \forall y,z \in C([y_0;h,t_0,\omega^*,\delta]).
\]  

(A.5)

For each \(n \in \mathbb{N}\), define

\[
\omega_n^* := \min\{\omega^*,\omega_n\}, \quad \Delta_n := \{ t \in [t_0,\omega_n^*) \mid \| y_n(t) - y^0(t_0) \| = \delta \}, \quad \rho_n := \left\{ \begin{array}{ll}
\inf \Delta_n, & \text{if } \Delta_n \neq \emptyset, \\
\omega_n^*, & \text{if } \Delta_n = \emptyset.
\end{array} \right.
\]

We claim that \(\rho_n < \omega_n\) for all \(n \in \mathbb{N}\). Suppose otherwise, then there exists \(n \in \mathbb{N}\) such that \(\rho_n = \omega_n\). It follows that \(\Delta_n = \emptyset\) and so \(\omega_n = \omega_n^* \leq \omega^*\). Therefore, \((t,y_n(t)) \in \mathcal{K}_0 \subset \mathcal{D}\) for all \(t \in [t_0,\omega_n)\), contradicting the final assertion of Lemma A.1. Therefore, \(\rho_n < \omega_n\) for all \(n \in \mathbb{N}\). Furthermore, for each \(n \in \mathbb{N}\), \(y_n(t) \in \overline{B}_\delta(y^0(t_0))\) for all \(t \in [t_0,\rho_n]\) and so

\[
\|y_n(t)\| \leq c_1 := \max_{s \in [-h,t_0]} \|y^0(s)\| + \delta \quad \text{for all } t \in [-h,\rho_n] \text{ and all } n \in \mathbb{N}.
\]

By property (iv) of \(T \in \mathcal{T}_h\), there exists \(c_2 > 0\) such that

\[
\|(Ty_n)(t)\| \leq c_2 \quad \text{for a.a. } t \in [t_0,\rho_n] \text{ and all } n \in \mathbb{N}.
\]

Write \(\mathcal{K}_1 := \mathcal{K}_0 \times \mathbb{R}_{c_2}\) and observe

\[
(t,y_n(t), (Ty_n)(t)) \in \mathcal{K}_1 \quad \text{for a.a. } t \in [t_0,\rho_n] \text{ and all } n \in \mathbb{N}.
\]

By property (A.1b) of \(G\), the set \(\mathcal{K}_2 := G(\mathcal{K}_1)\) is compact. Let \(c_3 := 1 + \max_{v \in \mathcal{K}_2} \|v\|\). Then, in view of (A.4),

\[
\|g_n(t,y,w)\| < c_3 \quad \text{for all } (t,y,w) \in \mathcal{K}_1 \text{ and all } n \in \mathbb{N}.
\]

(A.6)

Therefore,

\[
\|y_n(\rho_n) - y^0(t_0)\| \leq \int_{t_0}^{\rho_n} \|\dot{y}_n(t)\| \, dt = \int_{t_0}^{\rho_n} \|g_n(t,y_n(t), (Ty_n)(t))\| \, dt < c_3|\rho_n - t_0| \quad \forall n \in \mathbb{N}.
\]  

(A.7)
Next, define \( \rho := \inf_{n \in \mathbb{N}} \rho_n \geq t_0 \). Seeking a contradiction, suppose \( \rho = t_0 \). Fix \( n \in \mathbb{N} \) sufficiently large so that \( c_3|\rho_n - t_0| < \delta \) and \( \rho_n < \omega^* \). Recalling that \( \rho_n < \omega_n \), we have \( \rho_n < \min\{\omega^*, \omega_n\} = \omega_n^* \) and so \( \Delta_n \neq \emptyset \) and we arrive at a contradiction:
\[
\delta = \|y_n(\rho_n) - y^0(t_0)\| < c_3|\rho_n - t_0| < \delta.
\]
Therefore \( \rho \in (t_0, \omega^*) \). For each \( n \in \mathbb{N} \), define
\[
z_n := y_n|_{[t_0, \rho]} \quad \text{and} \quad w_n := (Ty_n)|_{[t_0, \rho]}.
\]
For all \( t \in [t_0, \rho] \), \( (z_n(t)) \subset \mathbb{R}_0(y^0(t_0)) \) and by (A.6),
\[
\|z_n(t)\| < c_3 \quad \text{for a.a. } t \in [t_0, \rho] \text{ and all } n \in \mathbb{N}.
\]
Therefore, the sequence \( (z_n) \subset C([t_0, \rho], \mathbb{R}^m) \) is uniformly bounded and equicontinuous. By the Arzelà-Ascoli theorem, and extracting a subsequence if necessary, we may assume that \( (z_n) \) converges uniformly to \( z \in C([t_0, \rho], \mathbb{R}^m) \).

To complete the proof of assertion (i), we adopt an argument akin to that used in the proof of [3], Theorem 3.1.7 and [9], Theorem 2D.5.

By weak* compactness of the unit ball in \( L^\infty([t_0, \rho], \mathbb{R}^m) \) (Alaoglu’s theorem), together with (A.8), the sequence \( (z_n) \subset L^\infty([t_0, \rho], \mathbb{R}^m) \subset L^1([t_0, \rho], \mathbb{R}^m) \) has a subsequence (which we do not relabel) with weak* limit \( v \in L^\infty([t_0, \rho], \mathbb{R}^m) \), that is,
\[
\lim_{n \to \infty} \int_{t_0}^{\rho} \langle p(t), z_n(t) \rangle \, dt = \int_{t_0}^{\rho} \langle p(t), v(t) \rangle \, dt \quad \forall \, p \in L^1([t_0, \rho], \mathbb{R}^m)
\]
and so, a fortiori, the sequence \( (z_n) \) converges weakly in \( L^1([t_0, \rho], \mathbb{R}^m) \) to \( v \). Let \( \{e_1, \ldots, e_m\} \) be a basis for \( \mathbb{R}^m \).

For \( k = 1, \ldots, m \) and \( t \in [t_0, \rho] \), define \( p_{k,t} \in L^1([t_0, \rho], \mathbb{R}^m) \) by
\[
p_{k,t}(s) := \begin{cases} 
  e_k, & s \in [t_0, t] \\
  0, & \text{otherwise}.
\end{cases}
\]
Setting \( p = p_{k,t} \) \((k = 1, \ldots, m \text{ and } t \in [t_0, \rho])\) in (A.9) and integrating, we may now conclude that
\[
z(t) = \lim_{n \to \infty} z_n(t) = y^0(t_0) + \int_{t_0}^{t} v(s) \, ds \quad \forall \, t \in [t_0, \rho].
\]
Therefore, \( z \in AC([t_0, \rho], \mathbb{R}^m) \) (the space of absolutely continuous functions \([t_0, \rho] \to \mathbb{R}^m\)) and \( \dot{z}(t) = v(t) \) for almost all \( t \in [t_0, \rho] \).

Let \( y \in C([-h, t_0], \mathbb{R}^m) \) denote the concatenation of \( y^0 \) and \( z \), and write \( w := (Ty)|_{[t_0, \rho]} \). Therefore, \( y|_{[-h,t_0]} = y^0, y|_{[t_0, \rho]} = z \in AC([t_0, \rho], \mathbb{R}^m) \) and, to conclude that \( y \) is a solution of the initial-value problem (5.1), it suffices to show that \( \dot{z}(t) \in G(t, z(t), w(t)) \) for almost all \( t \in [t_0, \rho] \).

By (A.5), we have
\[
\|w_n(t) - w(t)\| \leq c_0 \max_{s \in [t_0, \rho]} \|z_n(s) - z(s)\| \quad \text{for a.a. } t \in [t_0, \rho] \text{ and all } n \in \mathbb{N}.
\]
Therefore, for almost all \( t \in [t_0, \rho] \), \( w_n(t) \to w(t) \) as \( n \to \infty \). Moreover,
\[
\int_{t_0}^{\rho} \|w_n(t) - w(t)\| \, dt \leq c_0 |\rho - t_0| \max_{s \in [t_0, \rho]} \|z_n(s) - z(s)\| \to 0 \quad \text{as } n \to \infty.
\]
Therefore, \((w_n)\) converges (strongly) in \( L^1([t_0, \rho], \mathbb{R}^m) \) to \( w \).
Define the function \( \sigma: K_1 \times \mathbb{R}^m \to \mathbb{R} \) by
\[
\sigma(t, \eta, \xi, q) := \max\{q, \zeta \mid \zeta \in G(t, \eta, \xi)\}.
\]
Observe that, for each \((t, \eta, \xi) \in K_1\), \(q \mapsto \sigma(t, \eta, \xi, q)\) is the support function for the compact and convex set \(G(t, \eta, \xi)\) (and so is globally Lipschitz). Therefore, to establish that \(\dot{z}(t) \in G(t, z(t), w(t))\) for almost all \(t \in [t_0, \rho]\), it suffices to show that
\[
\langle q, \dot{z}(t) \rangle \leq \sigma(t, z(t), w(t), q) \quad \text{for a.a. } t \in [t_0, \rho] \text{ and all } q \in \mathbb{R}^m.
\]  
(A.11)

By continuity of the maps \(q \mapsto \langle q, \zeta \rangle\) and \(q \mapsto \sigma(t, \eta, \xi, q)\) for all \(\zeta \in \mathbb{R}^m\) and all \((t, \eta, \xi) \in K_1\), (A.11) holds if, any only if,
\[
\langle q, \dot{z}(t) \rangle \leq \sigma(t, z(t), w(t), q) \quad \text{for a.a. } t \in [t_0, \rho] \text{ and all } q \in Q^m,
\]  
where \(Q^m \subset \mathbb{R}^m\) is the set of vectors in \(\mathbb{R}^m\) with rational coordinates. We proceed to establish (A.12). First, we show that, for each \(q \in \mathbb{R}^m\), the map \((t, \eta, \xi) \mapsto \sigma(t, \eta, \xi, q)\) is upper semicontinuous on \(G\). Let \(q \in \mathbb{R}^m\) and \((t, \eta, \xi) \in K_1\) be arbitrary and define
\[
\sigma^* := \limsup_{(t', \eta', \xi') \to (t, \eta, \xi)} \sigma(t', \eta', \xi', q).
\]
Let \(((t_k, \eta_k, \xi_k)) \subset K_1\) be a sequence converging to \((t, \eta, \xi)\) such that \(\sigma(t_k, \eta_k, \xi_k, q) \to \sigma^*\) as \(k \to \infty\). For each \(k \in \mathbb{N}\), by compactness of \(G(t_k, \eta_k, \xi_k)\) there exists \(\xi_k \in G(t_k, \eta_k, \xi_k)\) such that \(\langle q, \xi_k \rangle = \sigma(t_k, \eta_k, \xi_k, q)\). The resulting sequence \((\xi_k)\) is contained in the compact set \(K_2 = G(K_1)\) and so has a subsequence converging to \(\zeta \in K_2\). By property (A.1a), the graph of \(G\) is closed and so we may infer that \(\zeta \in G(t, \eta, \xi)\). Therefore,
\[
\limsup_{(t', \eta', \xi') \to (t, \eta, \xi)} \sigma(t', \eta', \xi', q) = \lim_{k \to \infty} \sigma(t_k, \eta_k, \xi_k, q) = \lim_{k \to \infty} \langle q, \xi_k \rangle = \langle q, \zeta \rangle \leq \sigma(t, \eta, \xi, q),
\]
whence upper semicontinuity of \(\sigma(\cdot, \cdot, \cdot, q)\).

For \(p \in L^\infty([t_0, \rho], \mathbb{R}^m)\),
\[
|\sigma(t, z_n(t), w_n(t), p(t))| \leq \max_{v \in K_2} \|v\|\|p(t)\| \leq c_3\|p(t)\| \quad \text{for a.a. } t \in [t_0, \rho] \text{ and all } n \in \mathbb{N}.
\]
Furthermore, in view of (A.4),
\[
\langle p(t), \dot{z}_n(t) \rangle = \langle p(t), g_n(t, z_n(t), w_n(t)) \rangle \\
\leq \sigma(t, z_n(t), w_n(t), p(t)) + \varepsilon_n\|p(t)\| \quad \text{for a.a. } t \in [t_0, \rho] \text{ and all } n \in \mathbb{N},
\]
and so
\[
\int_{t_0}^\rho \left[ \langle p(t), \dot{z}_n(t) \rangle - \varepsilon_n\|p(t)\| \right] dt \leq \int_{t_0}^\rho \sigma(t, z_n(t), w_n(t), p(t)) dt \quad \forall n \in \mathbb{N}.
\]
Taking the limit superior as \(n \to \infty\), invoking Fatou’s lemma and upper semicontinuity of \(\sigma(\cdot, \cdot, \cdot, q)\), we have
\[
\int_{t_0}^\rho \langle p(t), \dot{z}(t) \rangle dt \leq \int_{t_0}^\rho \sigma(t, z(t), w(t), p(t)) dt.
\]  
(A.13)

Let \(q \in Q^m\) and let \(t \in [t_0, \rho]\) be a Lebesgue point for the integrable functions \(\dot{z}\) and \(t \mapsto \sigma(t, z(t), w(t), q)\). For \(\tau > 0\), define \(p \in L^\infty([t_0, \rho], \mathbb{R}^m)\) by
\[
p(s) := \begin{cases} \frac{q}{\tau}, & s \in [t, t + \tau] \cap [t_0, \rho], \\ 0, & \text{otherwise}. \end{cases}
\]
By (A.13), we have
\[ \frac{1}{\tau} \int_0^{t+\tau} [\sigma(s, z(s), w(s), \dot{q}) - \langle q, \dot{z}(s) \rangle] \, ds \geq 0 \quad \forall \tau > 0. \]
Passage to the limit as \( \tau \to 0 \) yields \( \langle q, \dot{z}(t) \rangle \leq \sigma(t, z(t), w(t), q) \), which is valid for all \( t \in [t_0, \rho] \setminus \mathcal{N}(q) \), where \( \mathcal{N}(q) \) is a set of measure zero which may depend on \( q \in \mathbb{Q}^m \). Since \( \mathbb{Q}^m \) is countable, \( \cup_{q \in \mathbb{Q}^m} \mathcal{N}(q) \) has measure zero and so we may conclude that (A.12) (and hence (A.11)) holds. We have now shown that \( y: [-h, \rho] \to \mathbb{R}^m \) is a solution of (5.1), whence assertion (i).

(ii) Let \( y \in C(I_y, \mathbb{R}^m) \) be a solution of (5.1). Define
\[ A := \{ (I, z) \mid I_y \subset I, \ z \in C(I, \mathbb{R}^m) \text{ is a solution of } (5.1) \text{ with } z|_{I_y} = y \} . \]
On this non-empty set define a partial order \( \preceq \) by
\[ (I_1, z_1) \preceq (I_2, z_2) \iff \sup I_1 \leq \sup I_2 \quad \text{and} \quad z_2|_{I_1} = z_1. \]
We proceed to show that \( A \) has a maximal element, that is, an element \( (I^*, z^*) \in A \) such that, for all \( (I, z) \in A \), \( (I^*, z^*) \preceq (I, z) \) implies \( (I, z) = (I^*, z^*) \), in which case \( z^* \in C(I^*, \mathbb{R}^m) \) is a maximal extension of the solution \( y \in C(I_y, \mathbb{R}^m) \) and \( \mathcal{O} \) is a totally ordered subset of \( A \). Let \( \omega := \sup \{ I \mid (I, z) \in \mathcal{O} \} \) and let \( z^*: [-h, \omega) \to \mathbb{R}^m \) be defined by the property that, for every \( (I, z) \in \mathcal{O} \), \( z^*|_{I} = z \). Then \( (\omega, z^*) \in A \) and is an upper bound for \( \mathcal{O} \). By Zorn’s Lemma, it follows that \( A \) contains at least one maximal element. This establishes assertion (ii).

(iii) Assume \( y \in C([-h, \omega), \mathbb{R}^m) \) is a maximal solution of (5.1) and that \( \omega < \infty \). Seeking a contradiction, suppose there exist \( \sigma \in [t_0, \omega) \) and compact \( K \subset \mathcal{D} \) such that \( (t, y(t)) \in K \) for all \( t \in [\sigma, \omega) \). By boundedness of \( y \) and property (iv) of \( T_h \), we conclude that \( T_h y \) is bounded. Therefore, the function \( t \mapsto (t, y(t), (Ty)(t)) \) is essentially bounded and so by property (A.1b) of \( G \), it follows that \( \dot{y} \) is essentially bounded on \( [\sigma, \omega) \). Therefore, \( y \) is uniformly continuous on \( [-h, \omega) \) and so extends to a function \( y^* \in C([-h, \omega], \mathbb{R}^m) \). By compactness of \( K \), we have \( (\omega, y^*(\omega)) \in K \subset \mathcal{D} \). An application of Assertion (i) of the theorem (with \( y \) and \( y^* \) replacing \( t_0 \) and \( y^0, \) respectively) yields the existence of a solution \( y^* \in C([-h, \rho], \mathbb{R}^m) \) for some \( \rho > \omega \), with \( y^*|_{[-h, \omega)} = y \). This contradicts maximality of \( y \).

\[ \square \]

References