

A CONVERSE TO THE LIONS-STAMPACCHIA THEOREM*

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Abstract. In this paper we show that a linear variational inequality over an infinite dimensional real Hilbert space admits solutions for every nonempty bounded closed and convex set, if and only if the linear operator involved in the variational inequality is pseudo-monotone in the sense of Brezis.

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1. INTRODUCTION AND NOTATION

Let us consider an infinite dimensional real Hilbert space X with scalar product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. We assume given a linear and continuous operator $A : X \rightarrow X$, (in short, $A \in \mathcal{L}(X)$), a closed and convex subset K of X and a fixed element $f \in X$. We begin by recalling some preliminary definitions. By a variational inequality we mean the problem $\mathcal{V}(A, K, f)$ of finding $u \in K$ such that $\langle Au - f, v - u \rangle \geq 0$ for each $v \in K$. This concept was introduced by Fichera [3] in his analysis of Signorini's problem. In their celebrated 1967 paper, Lions and Stampacchia [5] used variational inequalities associated to bilinear forms which are coercive or simply non negative in real Hilbert spaces as a tool for the study of partial differential elliptic and parabolic equations. They had in view applications to problems with unilateral constraints in mechanics (we refer to Duvaut and Lions [2] for details). The theory has since been expanded to include various applications in different areas such as economics, finance, optimization and game theory.

Precisely, the Lions-Stampacchia Theorem says that the linear variational inequality $\mathcal{V}(A, K, f)$ admits at least one solution for every closed and convex set K which is also nonempty and bounded, and every $f \in X$ provided that A is *coercive*, that is

$$\langle Au, u \rangle \geq a\|u\|^2 \text{ for every } u \in X \text{ and some } a > 0.$$

This landmark result has given rise to a rapidly growing new field. For an up-to-date overview of this topic, we recommend the papers by Lions, Magenes, Mancino and Mazzone published in the *Proceedings of the School of*

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Mathematics “Stampacchia” held in the memory of Lions and Stampacchia in Erice [6], as well as the monograph by Goeleven and Motreanu [4].

An important notion in the study of variational inequalities was provided by Brezis [1], who proved [1], Theorem 24, that the Lions-Stampacchia Theorem actually holds within the setting of reflexive Banach spaces, and for a very large class of (non-linear) operators, called *pseudo-monotone* operators. Precisely, let X be a reflexive Banach space with continuous dual X^* . Let us denote by $\langle \cdot, \cdot \rangle$ the duality product between X and X^* , and by the symbol \rightharpoonup the weak convergence on X . We say that the operator $A : X \rightarrow X^*$ is pseudo-monotone, if it is bounded and if $\{u_n\}_{n \in \mathbb{N}^*}$ is a sequence in X such that

$$u_n \rightharpoonup u \quad \text{and} \quad \limsup_n \langle Au_n, u_n - u \rangle \leq 0,$$

then

$$\langle Au, u - v \rangle \leq \liminf_n \langle Au_n, u_n - v \rangle \quad \forall v \in X. \tag{1.1}$$

The class of pseudo-monotone operators contains monotone operators which are hemicontinuous, compact operators, as well as various combinations of these two classes.

It is well known that, as long as non-linear operators are concerned, problem $\mathcal{V}(A, K, f)$ may admit solutions for every nonempty bounded closed and convex set K , even if the operator A is not pseudo-monotone (Example A.1 in the Appendix provides such an operator which is continuous and positively homogeneous). The aim of this note is to establish that, in the original *linear* setting of the Lions-Stampacchia Theorem, the pseudo-monotonicity of the operator A , which, in general, is only a sufficient condition for the existence of solutions for every bounded convex set K , becomes also a necessary one.

More precisely, we prove (Thm. 3.1, Sect. 3) that, given an infinite dimensional real Hilbert space X and $A \in \mathcal{L}(X)$, the variational inequality $\mathcal{V}(A, K, f)$ has solutions for every nonempty bounded closed and convex set K and $f \in X$ if and only if A is pseudo-monotone in the sense of Brezis.

The validity of a similar statement when X is a reflexive Banach space remains an open problem.

2. A TECHNICAL PROPOSITION

Our main result hardly relies on the following technical result.

Proposition 2.1. *Let X be a real Hilbert space and suppose that $A \in \mathcal{L}(X)$ is an operator which is not pseudo-monotone. Then we can construct an infinite-dimensional and separable closed subspace H of X such that the restriction of A to H is both symmetric,*

$$\langle Au, v \rangle = \langle Av, u \rangle \quad \forall u, v \in H, \tag{2.1}$$

and negatively defined,

$$\langle Au, u \rangle \leq -\alpha \|u\|^2, \tag{2.2}$$

for some $\alpha > 0$.

Proof. As A is not pseudo-monotone it is well known (see for instance Lem. A.2 in the Appendix) that there is a sequence $\{x_n\}_{n \in \mathbb{N}^*}$ such that $x_n \rightarrow 0$ and a real $a > 0$ satisfying the relation:

$$\langle Ax_n, x_n \rangle < -a \quad \forall n \in \mathbb{N}^*.$$

Throughout the proof we denote by $\|A\| = \sup_{u \in X, \|u\|=1} \|Au\|$ the norm of A and by $m = \sup_{n \in \mathbb{N}^*} \|x_n\|$. In order to define the desired subspace H , let us first recursively construct two sequences $\{y_n\}_{n \in \mathbb{N}^*} \subset X$, and $\{k_n\}_{n \in \mathbb{N}^*} \subset \mathbb{N}^*$,

such that, for every $j \in \mathbb{N}^*$, the following relations hold:

$$\langle Ay_i, y_j \rangle = \langle Ay_j, y_i \rangle = \langle y_i, y_j \rangle = 0 \quad \text{if } 1 \leq i < j, \tag{2.3}$$

$$k_i \leq k_j \quad \text{if } 1 \leq i < j, \tag{2.4}$$

$$\|y_j - x_{k_j}\| \leq \min \left(\frac{1}{2} \sqrt{\frac{a}{\|A\|}}, \frac{a}{8m\|A\|} \right). \tag{2.5}$$

We start by setting $y_1 = x_1$ and $k_1 = 1$, and we suppose that we have already defined the elements y_j, k_j fulfilling relations (2.3)–(2.5) for $1 \leq j < n$.

In order to define y_n and k_n , recall that if as usual we denote by $d(x, S) = \inf_{y \in S} \|x - y\|$ the distance between an element x and a set S , then for any sequence $\{z_i\}_{i \in \mathbb{N}^*}$ and any closed subspace S of X of finite co-dimension we have

$$[z_i \rightarrow 0] \implies [d(z_i, S) \rightarrow 0].$$

Applying this observation to the sequence $\{x_i\}_{i \in \mathbb{N}^*}$ and to the subspace

$$T_n = \{y \in X : \langle y, y_j \rangle = \langle y, Ay_j \rangle = \langle y, A^*y_j \rangle = 0, 1 \leq j < n\},$$

it results that the distance between the elements of the sequence $\{x_i\}_{i \in \mathbb{N}^*}$ and T_n goes to zero. Hence it is possible to pick $k_n \in \mathbb{N}^*$, such that both $k_n > k_{n-1}$ (which ensure us that the element k_n fulfills relation (2.4)) and

$$d(x_{k_n}, T_n) \leq \min \left(\frac{1}{2} \sqrt{\frac{a}{\|A\|}}, \frac{a}{8m\|A\|} \right).$$

Let us now define y_n , as being the projection of x_{k_n} on the closed linear subspace T_n . Then, $y_n \in T_n$, and therefore satisfies relation (2.3) as well as

$$\|y_n - x_{k_n}\| \leq \min \left(\frac{1}{2} \sqrt{\frac{a}{\|A\|}}, \frac{a}{8m\|A\|} \right).$$

Since the newly defined elements $y_n \in X$ and $k_n \in \mathbb{N}^*$ fulfill relations (2.3)–(2.5), this completes our recursive construction.

Define now H as the closure of the linear span of the sequence $\{y_i\}_{i \in \mathbb{N}^*}$. Recall that $\langle Ax_n, x_n \rangle < -a$, to deduce that

$$a < |\langle Ax_n, x_n \rangle| \leq \|A\| \|x_n\|^2 \quad \forall n \in \mathbb{N}^*,$$

and therefore that $\|x_n\| \geq \sqrt{\frac{a}{\|A\|}}$. By using relation (2.5) and the previous inequality, we prove that

$$\|y_n\| \geq \|x_{k_n}\| - \|y_n - x_{k_n}\| \geq \sqrt{\frac{a}{\|A\|}} - \frac{1}{2} \sqrt{\frac{a}{\|A\|}} = \frac{1}{2} \sqrt{\frac{a}{\|A\|}} \quad \forall n \in \mathbb{N}^*.$$

Thus all the elements y_n are non-null; by taking into account also the fact that $\langle y_i, y_j \rangle = 0 \quad \forall i \neq j$ (see relation (2.3)), it follows that the sequence $\{y_n\}_{n \in \mathbb{N}^*}$ is composed from linearly independent vectors. Accordingly, H is an infinite-dimensional and separable subspace of X and moreover, if we set $b_n = \frac{y_n}{\|y_n\|}$ we obtain a Hilbert basis of H .

Using once more relation (2.3) we observe that

$$\langle Ab_i, b_j \rangle = \frac{\langle Ay_i, y_j \rangle}{\|y_i\|^2} = 0 \quad \forall i \neq j \in \mathbb{N}^*. \tag{2.6}$$

This relation yields $\langle Au, v \rangle = \langle Av, u \rangle \quad \forall u, v \in H$ (that is relation (2.1) holds), as well as

$$\sup_{u \in H, \|u\|=1} \langle Au, u \rangle = \sup_{n \in \mathbb{N}^*} \langle Ab_n, b_n \rangle. \tag{2.7}$$

In order to prove relation (2.2), remark that obviously,

$$\begin{aligned} \langle Ay_n, y_n \rangle &= \langle Ax_{k_n}, x_{k_n} \rangle + \langle Ax_{k_n}, (y_n - x_{k_n}) \rangle \\ &\quad + \langle A(y_n - x_{k_n}), x_{k_n} \rangle + \langle A(y_n - x_{k_n}), A(y_n - x_{k_n}) \rangle. \end{aligned}$$

So

$$\begin{aligned} \langle Ay_n, y_n \rangle &\leq \langle Ax_{k_n}, x_{k_n} \rangle \\ &\quad + 2\|A\| \|x_{k_n}\| \|y_n - x_{k_n}\| + \|A\| \|y_n - x_{k_n}\|^2. \end{aligned} \tag{2.8}$$

Recall that

$$\langle Ax_{k_n}, x_{k_n} \rangle < -a. \tag{2.9}$$

As $\|y_n - x_{k_n}\| \leq \frac{a}{8m\|A\|}$ and $\|x_n\| \leq m$, it results that

$$2\|A\| \|x_{k_n}\| \|y_n - x_{k_n}\| \leq \frac{a}{4}. \tag{2.10}$$

Finally, since $\|y_n - x_{k_n}\| \leq \frac{1}{2} \sqrt{\frac{a}{\|A\|}}$, it holds

$$\|A\| \|y_n - x_{k_n}\|^2 \leq \frac{a}{4}. \tag{2.11}$$

Combining relations (2.8)–(2.11) we deduce that

$$\langle Ay_n, y_n \rangle < -\frac{a}{2} \quad \forall n \in \mathbb{N}^*. \tag{2.12}$$

In a similar manner we deduce that

$$\begin{aligned} \|y_n\|^2 &= \|x_{k_n}\|^2 + \|y_n - x_{k_n}\|^2 + 2\langle x_{k_n}, y_n - x_{k_n} \rangle \\ &\leq \|x_{k_n}\|^2 + \|y_n - x_{k_n}\|^2 + 2\|x_{k_n}\| \|y_n - x_{k_n}\|. \end{aligned}$$

Recall that $\|x_{k_n}\|^2 \leq m^2$; relation $\|x_{k_n} - y_n\| \leq \frac{1}{2} \sqrt{\frac{a}{\|A\|}}$, implies that $\|x_{k_n} - y_n\|^2 \leq \frac{a}{4\|A\|}$, while combining the facts that $\|x_{k_n}\| \leq m$ and that $\|x_{k_n} - y_n\| \leq \frac{a}{8m\|A\|}$, we deduce that $2\|x_{k_n}\| \|x_{k_n} - y_n\| \leq \frac{a}{4\|A\|}$. Finally, we obtain

$$\|y_n\|^2 \leq m^2 + \frac{a}{2\|A\|} \quad \forall n \in \mathbb{N}^*. \tag{2.13}$$

From relations (2.12) and (2.13) we infer that

$$\langle Ab_n, b_n \rangle \leq -\frac{a\|A\|}{a + 2m^2\|A\|}. \tag{2.14}$$

Relation (2.2) yields from relations (2.7) and (2.14). □

3. THE MAIN RESULT

We can now establish the main result of this note:

Theorem 3.1. *Let X be an infinite dimensional real Hilbert space and A a linear and continuous operator. The following statements are equivalent.*

- (i) A is pseudo-monotone.
- (ii) The variational inequality $\mathcal{V}(A, K, f)$ admits at least a solution for every nonempty bounded closed and convex set K and $f \in X$.

Proof. (i) \Rightarrow (ii). [1], Theorem 24.

(ii) \Rightarrow (i). Let $A \in \mathcal{L}(X)$ be an operator which is not pseudo-monotone. We prove that the variational inequality $\mathcal{V}(A, K, f)$ does not admit a solution for some bounded convex set K and some f . Indeed, according to Proposition 2.1 the bilinear and continuous form $\Theta : H \times H \rightarrow \mathbb{R}$, $\Theta(x, y) = -\langle Ax, y \rangle$ is symmetric and positively defined on some infinite-dimensional separable closed subspace H of X .

Endow the vector space H with the inner product $[x, y] = \Theta(x, y)$, and consider $B = \{b_i : i \in \mathbb{N}^*\}$ a Hilbert basis of $(H, [\cdot, \cdot])$. As usually, if $x \in H$, let x_i denote the i -th coordinate of x with respect to B , $x_i = [x, b_i]$ for every $x \in H$, $i \in \mathbb{N}^*$.

We claim that the set K

$$K = \left\{ x \in H : x_i \geq \frac{1}{2^i} \text{ and } \sum_{i=1}^{\infty} \left(1 + \frac{1}{2^i}\right) x_i^2 \leq 2 \right\},$$

is a bounded closed and convex subset of X such that the variational inequality $\mathcal{V}(A, K, 0)$ does not have solutions.

We observe first that K is a nonempty convex set, which is closed and bounded in $(H, [\cdot, \cdot])$. As the image of H through the injection $\iota : (H, [\cdot, \cdot]) \rightarrow (X, \langle \cdot, \cdot \rangle)$ is closed, it follows that ι is bounded; hence, we conclude that K is closed and bounded also with respect to $(X, \langle \cdot, \cdot \rangle)$.

Remark that, for every $x, y \in H$ it holds that

$$\langle Ax, y - x \rangle = [x, y - x].$$

Accordingly, in order to show that $\mathcal{V}(A, K, 0)$ does not have solutions, it suffices to prove that, for any element $x \in K$, there is some $y_x \in K$ such that

$$[x, y_x - x] > 0. \tag{3.1}$$

For $x = \sum_{i=1}^{\infty} \frac{1}{2^i} b_i$, this is an easy task, since $y_x = 2x$ obviously does the job. Consider now $x \in K$, $x \neq \sum_{i=1}^{\infty} \frac{1}{2^i} b_i$ and set

$$i(x) = \min \left\{ j \in \mathbb{N}^* : x_j > \frac{1}{2^j} \right\}.$$

Define

$$\begin{aligned} y_x(\varepsilon) &= x + \left(\sqrt{x_{i(x)}^2 - \frac{2^{i(x)}\varepsilon}{2^{i(x)} + 1}} - x_{i(x)} \right) b_{i(x)} \\ &\quad + \left(\sqrt{x_{i(x)+1}^2 + \frac{2^{i(x)+1}\varepsilon}{2^{i(x)+1} + 1}} - x_{i(x)+1} \right) b_{i(x)+1}; \end{aligned}$$

in other words,

$$\begin{aligned}
 y_x(\varepsilon) &= \sum_{i=1}^{i=i(x)-1} x_i b_i + \left(\sqrt{x_{i(x)}^2 - \frac{2^{i(x)}\varepsilon}{2^{i(x)}+1}} \right) b_{i(x)} \\
 &+ \left(\sqrt{x_{i(x)+1}^2 + \frac{2^{i(x)+1}\varepsilon}{2^{i(x)+1}+1}} \right) b_{i(x)+1} + \sum_{i=i(x)+2}^{\infty} x_i b_i.
 \end{aligned}$$

It is straightforward to prove that

$$\sum_{i=1}^{\infty} \left(1 + \frac{1}{2^i} \right) (y_x(\varepsilon))_i^2 = \sum_{i=1}^{\infty} \left(1 + \frac{1}{2^i} \right) x_i^2.$$

Since for every ε greater than or equal to zero and less than or equal to $(1 + \frac{1}{2^{i(x)}}) \left(x_{i(x)}^2 - \frac{1}{2^{2i(x)}} \right)$, it holds that

$$\sqrt{x_{i(x)}^2 - \frac{2^{i(x)}\varepsilon}{2^{i(x)}+1}} \geq \frac{1}{2^{i(x)}},$$

we obtain that $y_x(\varepsilon) \in K$.

Let us set

$$\begin{aligned}
 f_x(\varepsilon) &= [x, y_x(\varepsilon)] = \langle x, x \rangle + x_{i(x)} \left(\sqrt{x_{i(x)}^2 - \frac{2^{i(x)}\varepsilon}{2^{i(x)}+1}} - x_{i(x)} \right) \\
 &+ x_{i(x)+1} \left(\sqrt{x_{i(x)+1}^2 + \frac{2^{i(x)+1}\varepsilon}{2^{i(x)+1}+1}} - x_{i(x)+1} \right);
 \end{aligned}$$

we easily deduce that $f_x(0) = [x, x]$ and that

$$f'_x(0) = \frac{2^{i(x)}}{(2^{i(x)+1}+1)(2^{i(x)}+1)} > 0.$$

Accordingly, $f_x(\bar{\varepsilon}) > f_x(0)$ for some value $\bar{\varepsilon}$ greater than zero and less than or equal to the real number $(1 + \frac{1}{2^{i(x)}}) \left(x_{i(x)}^2 - \frac{1}{2^{2i(x)}} \right)$:

$$[x, x] = f_x(0) < f_x(\bar{\varepsilon}) = [x, y_x(\bar{\varepsilon})]$$

and relation (3.1) is fulfilled, when $x \neq \sum_{i=1}^{\infty} \frac{1}{2^i} b_i$, by setting $y_x = y_x(\bar{\varepsilon})$. The proof of Theorem 3.1 is thus completed. □

Since the pseudo-monotonicity of $A \in \mathcal{L}(X)$ is equivalent to the pseudo-monotonicity of its adjoint $A^* \in \mathcal{L}(X)$, the following consequence of Theorem 3.1 holds true.

Corollary 3.2. *Let A be a linear and continuous operator defined on an infinite dimensional Hilbert space X . The variational inequality $VI(A, K, f)$ has solutions for every nonempty bounded closed and convex set K and every $f \in X$, if and only if the same holds for $VI(A^*, K, f)$.*

A. APPENDIX

In the first part of this section we observe that in a real Hilbert setting, there exists a continuous and positively homogeneous operator which is not pseudo-monotone but for which the variational inequality $\mathcal{V}(A, K, f)$ has solutions provided that K is a nonempty closed and convex bounded set.

Example A.1. Let X be a separable Hilbert space with basis $\{b_i : i \in \mathbb{N}^*\}$. As customary, for every real number a , let us set $a_+ = \max(a, 0)$ for the positive part of a . For every $i \in \mathbb{N}^*$, let us define

$$A_i : X \rightarrow X, \quad A_i(x) = -(3\langle x, b_i \rangle - 2\|x\|)_+ b_i,$$

and set $A(x) = \sum_{i=1}^{\infty} A_i(x)$. Then A is a continuous and positively homogeneous mapping which fails to be pseudo-monotone, while the variational inequality $\mathcal{V}(A, K, 0)$ admits solutions for every bounded closed and convex set K .

Indeed, remark that any two sets from the family of open convex cones

$$K_i = \{x \in X : 3\langle x, b_i \rangle > 2\|x\|\}, \quad i \in \mathbb{N}^*$$

are disjoint. This fact proves that the definition of the operator A is meaningful, as at any point x , at most one among the values $A_i(x)$, $i \in \mathbb{N}^*$, may be non-null.

On one hand, it is easy to see that this operator is continuous and positively homogeneous, as is each of the operators A_i . On the other, $A(b_i) = -b_i$, so

$$0 = \langle A0, 0 - 0 \rangle > \liminf_i \langle Ab_i, b_i - 0 \rangle = -1;$$

this inequality proves that relation (1.1) does not hold for b_i instead of u_i , and 0 instead of u and v . Finally remark that

$$b_i \rightharpoonup 0 \text{ and } \limsup_i \langle Ab_i, b_i - 0 \rangle = -1 \leq 0,$$

to infer that the operator A is not pseudo-monotone.

We need now to prove that the variational inequality $\mathcal{V}(A, K, 0)$ has solutions for every bounded closed and convex set K . Let us consider first the case when the domain K of the variational inequality is not entirely contained within one of the cones K_i . As every convex set is also a connected set, and since $\{K_i : i \in \mathbb{N}^*\}$ form a family of disjoint open sets, it follows that K contains some point x which does not belong to any of the cones K_i . Accordingly, $A(x) = 0$, fact which means that x is a solution of the problem $\mathcal{V}(A, K, 0)$.

Consider now the case of a bounded closed and convex set K contained in the cone K_p for some $p \in \mathbb{N}^*$. Remarking that the operators A and A_p coincide on the cone K_p , and thus on K , we deduce that A is pseudo-monotone. Accordingly, the existence of a solution to problem $\mathcal{V}(A, K, 0)$ is guaranteed in this case by Brezis's theorem [1], Theorem 24.

Let us conclude this Appendix by proving the following standard characterization of the class of $\mathcal{L}(X)$ -pseudo-monotone operators, needed in proving our main result.

Lemma A.2. *Let X be a real Hilbert space and $A \in \mathcal{L}(X)$. Then A is pseudo-monotone if and only if*

$$[u_n \rightharpoonup 0] \implies \left[\liminf_n \langle Au_n, u_n \rangle \geq 0 \right]. \quad (\text{A.1})$$

Proof. Let A be a $\mathcal{L}(X)$ -pseudo-monotone operator, and $\{u_n\}_{i \in \mathbb{N}^*}$ be a sequence such that $u_n \rightharpoonup 0$. We only need to prove relation (A.1) when $\liminf_n \langle Au_n, u_n \rangle \leq 0$. In this case we may also suppose, by taking, if necessary, a sub-sequence, that $\limsup_n \langle Au_n, u_n \rangle \leq 0$. Apply definition (1.1) of pseudo-monotonicity to the mapping A ,

and to the sequence $\{u_n\}_{i \in \mathbb{N}^*}$, weakly converging to 0 in order to deduce (by taking $v = 0$) that

$$0 = \langle A0, 0 - 0 \rangle \leq \liminf_n \langle Au_n, u_n \rangle,$$

that is $0 = \liminf_n \langle Au_n, u_n \rangle$. Relation (A.1) holds accordingly for every $\mathcal{L}(X)$ -pseudo-monotone operator.

Let us now consider $A \in \mathcal{L}(X)$ such that relation (A.1) is verified. Pick a sequence $\{u_n\}_{i \in \mathbb{N}^*}$ which converges weakly to u such that

$$\limsup_n \langle Au_n, u_n - u \rangle \leq 0.$$

On one hand, as

$$\lim_n \langle Au, u_n - u \rangle = 0, \tag{A.2}$$

the previous relation implies that

$$\limsup_n \langle A(u_n - u), (u_n - u) \rangle \leq 0. \tag{A.3}$$

On the other hand, applying relation (A.1) to the sequence $\{u_n - u\}_{i \in \mathbb{N}^*}$ (which obviously weakly converges to 0) yields

$$\liminf_n \langle A(u_n - u), (u_n - u) \rangle \geq 0. \tag{A.4}$$

Combining relations (A.2)–(A.4) we deduce that

$$\lim_n \langle Au_n, u_n - u \rangle = 0, \tag{A.5}$$

whenever $u_n \rightharpoonup u$ and $\limsup_n \langle Au_n, u_n - u \rangle \leq 0$.

Recall that any $\mathcal{L}(X)$ -operator is also continuous with respect to the weak topology on X ([7], Lem. 1.2, p. 36). As $\{u_n\}_{i \in \mathbb{N}^*}$ weakly goes to u , we deduce that

$$\lim_n \langle Au_n, w \rangle = \langle Au, w \rangle.$$

When applied for $w = u - v$, the previous relation shows that

$$\langle Au, u - v \rangle = \lim_n \langle Au_n, u - v \rangle = \liminf_n \langle Au_n, u - v \rangle, \tag{A.6}$$

for every sequence $u_n \rightharpoonup u$.

Summing up relations (A.5) and (A.6), we deduce that relation (1.1) holds whenever $u_n \rightharpoonup u$ and $\limsup_n \langle Au_n, u_n - u \rangle \leq 0$; in other words, the operator A is pseudo-monotone, establishing the proof. \square

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