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A CHARACTERIZATION OF GRADIENT YOUNG-CONCENTRATION MEASURES GENERATED BY SOLUTIONS OF DIRICHLET-TYPE PROBLEMS WITH LARGE SOURCES *

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Abstract. We show how to capture the gradient concentration of the solutions of Dirichlet-type problems subjected to large sources of order $\frac{1}{\sqrt{\varepsilon}}$ concentrated on an ε-neighborhood of a hypersurface of the domain. To this end we define the gradient Young-concentration measures generated by sequences of finite energy and establish a very simple characterization of these measures.

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1. Introduction

Let us consider the one-dimensional Dirichlet problem in $(-1, 1)$ with a second member measure

$$\begin{cases} (\sigma_\varepsilon(x)u')' = \dfrac{1}{\sqrt{\varepsilon}} \displaystyle\sum_{k=0}^{m} a_k \delta_{t_k^\varepsilon} \\ u(-1) = u(1) = 0, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$, $m \in \mathbb{N}$, $a_k \in \mathbb{R}^*$, $(t_k^\varepsilon)_{k=0,\ldots,m}$ is a non decreasing family of numbers in $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ with $t_0^\varepsilon = -\frac{\varepsilon}{2}$, $t_m^\varepsilon = \frac{\varepsilon}{2}$, $\lim_{\varepsilon\to 0} \frac{t_{k+1}^\varepsilon - t_k^\varepsilon}{\varepsilon} = \frac{1}{m} := l$ for $k = 0, \ldots, m-1$, and σ_ε is given by

$$\sigma_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}, & \text{if } x \in (-1, 1) \setminus (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \\ 1, & \text{if } x \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}). \end{cases}$$

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Let $\bar{u}_\varepsilon$ be the solution of (1.1); then clearly $\bar{u}'_\varepsilon$ is of the form

$$\bar{u}'_\varepsilon(x) = \begin{cases} \varepsilon c_\varepsilon, & \text{if } x \in (-1, -\frac{\varepsilon}{2}) \\ c_\varepsilon + \frac{s_k}{\sqrt{\varepsilon}}, & \text{if } x \in (t^\varepsilon_k, t^\varepsilon_{k+1}),\ k = 0, \dots, m-1 \\ \varepsilon(c_\varepsilon + \frac{s_m}{\sqrt{\varepsilon}}), & \text{if } x \in (\frac{\varepsilon}{2}, 1), \end{cases}$$

where $s_k := \sum_{i=0}^k a_i$ and c_ε is a constant which can be computed from the boundary conditions. Setting $\bar{v}_\varepsilon = \frac{\bar{u}_\varepsilon}{\sqrt{\varepsilon}}$ one can show that the measure $\mathbb{1}_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})}\varepsilon|\bar{v}'_\varepsilon|^2 \mathrm{d}x$ weakly converges to $l\sum_{k=0}^{m-1}(L + s_k)^2\delta_0$, where L is the limit of $\sqrt{\varepsilon}c_\varepsilon$ (see Sect. 4 for the computations). Consequently the gradient of the solution $\bar{u}_\varepsilon$ presents a concentration phenomenon in $L^2((-1, 1))$. The knowledge of the limit measure $l\sum_{k=0}^{m-1}(L + s_k)^2\delta_0$ is not completely satisfactory to describe the concentration phenomenon. Indeed $\bar{v}'_\varepsilon$ clearly oscillates near $x = 0$ from the observation that not all the coefficients s_k are necessarily of the same sign. A natural question to ask is how to capture these oscillations at the limit. To this end, we compute the weak limit $\bar{\mu} = \bar{\mu}_x \otimes \bar{\pi}$ of the measure

$$\bar{\mu}_\varepsilon := \delta_{\frac{\bar{v}'_\varepsilon}{|\bar{v}'_\varepsilon|}(x)} \otimes \mathbb{1}_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})}\varepsilon|\bar{v}'_\varepsilon|^2 \mathrm{d}x \in \mathbb{M}([-1, 1] \times \{-1, 1\})$$

in the sense that

$$\varepsilon \int_{-1}^1 \mathbb{1}_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})}\varphi(\bar{v}'_\varepsilon)\theta(x)\ \mathrm{d}x \to \int_{-1}^1 \theta(x) \left(\int_{\{-1,1\}} \varphi(\zeta)\ \mathrm{d}\bar{\mu}_x \right) \mathrm{d}\bar{\pi}$$

for all 2-homogeneous and continuous functions φ on $\mathbb{R}$ and all functions θ in $\mathcal{C}([-1, 1])$. We write $\bar{\mu}_x \otimes \bar{\pi}$ to denote the slicing decomposition of the measure $\bar{\mu}$ with respect to its projection measure $\bar{\pi}$ on $[-1, 1]$. We will call $\bar{\mu}$ the *gradient Young-concentration measure generated by* $(\bar{v}_{\varepsilon>0})$. Note that $(\bar{\mu}_x)_{x\in(-1,1)}$ is a family of probability measures on the unit sphere $\mathbb{S}^0 := \{-1, 1\}$ of $\mathbb{R}$. Why is $\bar{\mu}$ useful to capture the oscillations of $\bar{v}'_\varepsilon$ at the limit? It is easy to show that $\bar{\pi}$ is nothing but the weak limit $l\sum_{k=0}^{m-1}(L + s_k)^2\delta_0$ of the measure $\mathbb{1}_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})}\ \varepsilon|\bar{v}'_\varepsilon|^2\ \mathrm{d}x$. Moreover, we show that (see Sect. 4 for the details)

$$\bar{\mu}_x = a\delta_1 + b\delta_{-1}$$

where, if we assume for instance $L \geq 0$,

$$a = \frac{\displaystyle\sum_{k=0,\dots,m-1: s_k \geq 0} (L + s_k)^2}{\displaystyle\sum_{k=0}^{m-1}(L + s_k)^2}, \qquad b = 1 - a = \frac{\displaystyle\sum_{k=0,\dots,m-1: s_k < 0} (L + s_k)^2}{\displaystyle\sum_{k=0}^{m-1}(L + s_k)^2}$$

(if $L \leq 0$, take the condition $s_k > 0$ in the sum defining a, and $s_k \leq 0$ in the sum defining b). One sees that whereas the measure $\bar{\pi}$ provides the intensity $l\sum_{k=0}^{m-1}(L + s_k)^2$ and the support $\{0\}$ of the concentration, the coefficients a and b in $[0, 1]$ provide the desired additional information, *i.e.*, the proportion of change of sign of $\bar{v}'_\varepsilon$ captured at the limit, with respect to the total mass of the limit measure $\bar{\pi}$.

In this article we want to generalize the previous notion of gradient Young-concentration measure when $\bar{u}_\varepsilon$ is the solution of a Dirichlet problem on a domain Ω of $\mathbb{R}^N$, $N > 1$, with a source of magnitude order $\frac{1}{\sqrt{\varepsilon}}$. We assume that a $(N - 1)$-dimensional manifold Σ, possessing an unit normal vector ν at $\mathcal{H}^{N-1}$ a.e. x, splits Ω into two subdomains and that the conductivity outside the ε-neighborhood Σ_ε of Σ grows as $\frac{1}{\varepsilon}$. The simplest model is given by

$$\bar{u}_\varepsilon \in \ \operatorname{argmin} \left\{ \frac{1}{\varepsilon} \int_{\Omega\setminus\Sigma_\varepsilon} |\nabla u|^2\ \mathrm{d}x + \int_{\Sigma_\varepsilon} |\nabla u|^2\ \mathrm{d}x - \frac{1}{\sqrt{\varepsilon}}\langle \mathcal{F}_\varepsilon, u\rangle : u \in W^{1,2}_0(\Omega) \right\} \tag{1.2}$$

where $\mathcal{F}_\varepsilon \in H^{-1}(\Omega)$; setting $v := \frac{u}{\sqrt{\varepsilon}}$, (1.2) is equivalent to

$$\bar{v}_\varepsilon \in \ \text{argmin} \ \left\{ \int_{\Omega\setminus\Sigma_\varepsilon} |\nabla v|^2 \ \mathrm{d}x + \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2 \ \mathrm{d}x - \langle \mathcal{F}_\varepsilon, v\rangle : v \in W_0^{1,2}(\Omega) \right\}.$$

By analogy with the one-dimensional case, the estimation

$$\sup_{\varepsilon>0} \left(\int_{\Omega\setminus\Sigma_\varepsilon} |\nabla v_\varepsilon|^2 \ \mathrm{d}x + \varepsilon \int_{\Sigma_\varepsilon} \left|\frac{\partial v_\varepsilon}{\partial \nu}\right|^2 \ \mathrm{d}x \right) < +\infty$$

implies that $\frac{\partial \bar{v}_\varepsilon}{\partial \nu}$ presents a concentration phenomenon on $S \subset\subset \Omega$. To describe it, we will characterize the weak cluster points $\bar{\mu}$ of the sequence $\delta_{\frac{\partial \bar{v}_\varepsilon}{\partial \nu}/|\frac{\partial \bar{v}_\varepsilon}{\partial \nu}|\ (x)} \otimes \varepsilon \mathbb{1}_{B_\varepsilon} |\frac{\partial \bar{v}_\varepsilon}{\partial \nu}|^2 \ \mathrm{d}x$ in the set of non-negative bounded Borel measures $\mathbb{M}^+(\bar{\Omega}\times\mathbb{S}^0)$ on $\bar{\Omega}\times\mathbb{S}^0$, where B_ε denotes the ε-neighborhood of S. We will call $\bar{\mu}$ the gradient Young-concentration measure generated by $(\bar{v}_\varepsilon)_{\varepsilon>0}$. In contrast to the one-dimensional case, it is difficult to express the measure $\bar{\mu}$ and we adopt the following strategy: under some additional hypotheses on $\mathcal{F}_\varepsilon$, one can prove that a subsequence of $\bar{v}_\varepsilon$ strongly converges in $L^2(\Omega)$ to some function $\bar{v} \in W^{1,2}(\Omega\setminus\Sigma)$; our objective is to provide an intrinsic characterization of $\bar{\mu}$ (*i.e.*, independent of the sequence) with respect to the Sobolev function $\bar{v}$, in the spirit of [3,4]. Actually, we characterize the limits (v,μ) generated by sequences $(v_\varepsilon)_\varepsilon$ satisfying the partial finite energy condition

$$\sup_{\varepsilon>0} \left(\int_{\Omega\setminus\Sigma_\varepsilon} |\nabla v_\varepsilon|^2 \ \mathrm{d}x + \varepsilon \int_{\Sigma_\varepsilon} \left|\frac{\partial v_\varepsilon}{\partial \nu}\right|^2 \ \mathrm{d}x \right) < +\infty.$$

We show that v and $\mu = \mu_x \otimes \pi$ with $\mu = (a(x)\delta_1 + b(x)\delta_{-1}) \otimes \bar{\pi}$, are linked by the necessary and sufficient conditions:

$$\begin{array}{l} v \in W^{1,2}(\Omega\setminus\Sigma),\ \ v = 0 \text{ on } \partial\Omega; \\ \operatorname{spt}(\pi) \subset \bar{S}; \\ \dfrac{\mathrm{d}\pi}{\mathrm{d}\mathcal{H}^{N-1}\lfloor_S}(x)(a(x)c + b(x)d) \geq \varphi([v](x)) \end{array} \tag{1.3}$$

for $\mathcal{H}^{N-1}\lfloor_S$ *a.e.* x and for all $(c,d) \in \mathbb{R}^+ \times \mathbb{R}^+$ where

$$\varphi(\xi) = \begin{cases} c\xi^2, & \text{if } \xi \geq 0 \\ d\xi^2, & \text{if } \xi \leq 0. \end{cases}$$

In addition, in Example 3.2 we will exhibit a large class of non trivial pairs (v,μ) satisfying (1.3) and generated by sequences $(\bar{v}_\varepsilon)_{\varepsilon>0}$ of solutions to Dirichlet problems with large sources concentrated on Σ_ε. Moreover the probability measure μ_x can be completely expressed.

It is worth noticing that the internal energy functional of (1.2) or (1.4), (1.5) below, rewritten in terms of the rescaled function v, is the type of functionals considered in [5,6] in order to model adhesive bounded joints through the computation of a variational limit problem. In this paper, it is not our purpose to describe a variational limit of (1.2), (1.4) or (1.5), even if this program would be interesting because of the additional difficulty due to the source. Among the physical motivations of (1.2), or of more general Dirichlet problems of the form

$$\begin{cases} -\operatorname{div}(\sigma_\varepsilon(x)\partial f(\nabla u)) = \frac{1}{\sqrt{\varepsilon}}\mathcal{F}_\varepsilon \text{ on } \Omega \\ u = 0 \qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

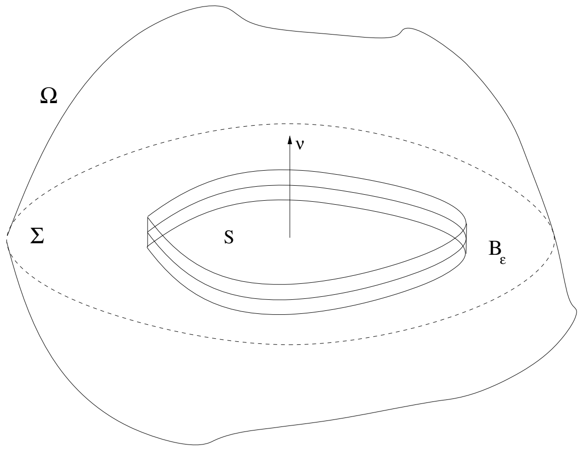

FIGURE 1. The physical configuration.

where $\sigma_\varepsilon := \frac{1}{\varepsilon}\mathbb{1}_{\Omega\setminus\Sigma_\varepsilon} + \mathbb{1}_{\Sigma_\varepsilon}$ and $f : \mathbb{R}^N \to \mathbb{R}$ is a positively 2-homogeneous convex function, one may mention various applications to:

- heat conduction or electrostatic problems subjected to large concentrated sources, with high conductivity outside the support of the sources;
- membrane problems with large concentrated exterior loading and high stiffness outside the support of the loading.

Note that we treat this study in a vectorial environment, that is, we consider the following problems defined over $W_0^{1,2}(\Omega,\mathbb{R}^m)$, with $m \geq 1$ by

$$\bar{u}_\varepsilon \in \operatorname{argmin}\left\{\frac{1}{\varepsilon^{p-1}}\int_{\Omega\setminus\Sigma_\varepsilon} f(\nabla u)\,\mathrm{d}x + \int_{\Sigma_\varepsilon} g(\nabla u)\,\mathrm{d}x - \frac{1}{\varepsilon^{1-1/p}}\langle \mathcal{F}_\varepsilon, u\rangle : u \in W_0^{1,p}(\Omega,\mathbb{R}^m)\right\},\ p>1 \tag{1.5}$$

where $f, g : \mathbb{R}^{N\times m} \to \mathbb{R}$ are two positively p-homogeneous and quasiconvex functions, and $\frac{1}{\varepsilon^{1-1/p}}\mathcal{F}_\varepsilon$ is the exterior loading. To shorten the proofs, we only consider the case $p = 2$.

2. Towards the definition of gradient Young-concentration measures

Let N, m be two positive integers. Throughout this paper $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ denote the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure on Ω, respectively. We will sometimes write dx for the measure $\mathcal{L}^N$ and $|B|$ for the N-dimensional Lebesgue measure or the $(N-1)$-dimensional Hausdorff measure of any Borel subset B of Ω, if there is no ambiguity. In the N-dimensional Euclidean space $\mathbb{R}^N$ with respect to the orthonormal frame $(O, e_1, \ldots, e_N)$, we consider a $(N-1)$-hypersurface Σ which splits Ω into two subdomains Ω^+ and Ω^-, *i.e.* $\Omega = \Omega^+ \cup \Sigma \cup \Omega^-$. To avoid certain technical complications, we will additionally assume that Σ is included in the hyperplane vect$(e_1, \ldots, e_{N-1})$ generated by $\{e_1, \ldots, e_{N-1}\}$ and orthogonal to the unit vector $\nu = e_N$, but we point out that we could treat the problem in the more general case where Σ is the graph of a Lipschitz function. Such a general geometry leads to technical complications in the proofs, but the overall strategy remains the same. For any point x in Ω, we write $\hat{x}$ for its projection on vect$(e_1, \ldots, e_{N-1})$, so that $x = (\hat{x}, x_N)$. We set $\Sigma_\varepsilon := \{x + t\nu : x \in \Sigma,\ -\frac{\varepsilon}{2} < t < \frac{\varepsilon}{2}\}$ and, for $S \subset\subset \Sigma$, $B_\varepsilon := \{x + t\nu : x \in S,\ -\frac{\varepsilon}{2} < t < \frac{\varepsilon}{2}\}$ (see Fig. 1).

We assume that $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ is a bounded sequence in the topological dual of $W_0^{1,1}(\Omega,\mathbb{R}^m)$, and we define on $W_0^{1,2}(\Omega,\mathbb{R}^m)$ the family of functionals

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_{\Omega\setminus\Sigma_\varepsilon} f(\nabla u)\ \mathrm{d}x + \int_{\Sigma_\varepsilon} g(\nabla u)\ \mathrm{d}x - \frac{1}{\sqrt{\varepsilon}}\langle \mathcal{F}_\varepsilon, u\rangle$$

where $f, g : \mathbb{R}^{N\times m} \to \mathbb{R}$ are two positively 2-homogeneous and quasiconvex functions satisfying the standard growth conditions: there exist two constants $0 < \alpha \leq \beta$ such that

$$\alpha|\zeta|^2 \leq f(\zeta) \leq \beta(1+|\zeta|^2) \quad \forall \zeta \in \mathbb{R}^{N\times m}, \tag{2.1}$$

same for g. We consider the functional G_ε defined in $W_0^{1,2}(\Omega,\mathbb{R}^m)$ by

$$G_\varepsilon(v) = F_\varepsilon(\sqrt{\varepsilon}v) = \int_{\Omega\setminus\Sigma_\varepsilon} f(\nabla v)\ \mathrm{d}x + \varepsilon\int_{\Sigma_\varepsilon} g(\nabla v)\ \mathrm{d}x - \langle \mathcal{F}_\varepsilon, v\rangle$$

and the problem $\bar{v}_\varepsilon \in \mathrm{argmin}_{v\in W_0^{1,2}(\Omega,\mathbb{R}^m)} G_\varepsilon$. Note that, by using the direct methods of the Calculus of Variations, one can easily establish that $\mathrm{argmin}_{v\in W_0^{1,2}(\Omega,\mathbb{R}^m)} G_\varepsilon \neq \emptyset$. We could also treat the problem with various other boundary conditions. Throughout this paper C will denote various constants independent of ε and we do not relabel the subsequences considered.

Lemma 2.1. *Let $(v_\varepsilon)_{\varepsilon>0}$ be a sequence in $W_0^{1,2}(\Omega,\mathbb{R}^m)$ satisfying $\sup_{\varepsilon>0} G_\varepsilon(v_\varepsilon) < \infty$. Then*

$$\sup_{\varepsilon>0}\left(\varepsilon\int_{\Sigma_\varepsilon} |\nabla v_\varepsilon|^2\ \mathrm{d}x + \int_{\Omega\setminus\Sigma_\varepsilon} |\nabla v_\varepsilon|^2\ \mathrm{d}x\right) < +\infty. \tag{2.2}$$

Proof. Estimate (2.2) is a straightforward consequence of estimates (a), (b) and (c) below: there exists a non-negative constant C such that

(a) $\displaystyle\int_{\Sigma_\varepsilon} \varepsilon|\nabla v_\varepsilon|^2\ \mathrm{d}x \leq C\left(1+\int_\Omega |\nabla v_\varepsilon|\ \mathrm{d}x\right)$;

(b) $\displaystyle\int_{\Omega\setminus\Sigma_\varepsilon} |\nabla v_\varepsilon|^2\ \mathrm{d}x \leq C\left(1+\int_\Omega |\nabla v_\varepsilon|\ \mathrm{d}x\right)$;

(c) $\displaystyle\sup_{\varepsilon>0}\int_\Omega |\nabla v_\varepsilon|\ \mathrm{d}x < +\infty$.

Proof of (a). Since $\sup_{\varepsilon>0} G_\varepsilon(v_\varepsilon) < +\infty$, using (2.1), and Poincaré's inequality, we have

$$\begin{aligned}\alpha\int_{\Sigma_\varepsilon} \varepsilon|\nabla v_\varepsilon|^2\ \mathrm{d}x \leq \int_{\Sigma_\varepsilon} \varepsilon g(\nabla v_\varepsilon)\ \mathrm{d}x &\leq C + \|\mathcal{F}_\varepsilon\|\|v_\varepsilon\|_{W_0^{1,1}(\Omega,\mathbb{R}^m)} \\ &\leq C\left(1+\int_\Omega |\nabla v_\varepsilon|\ \mathrm{d}x\right).\end{aligned}$$

Proof of (b). Since $\alpha\int_{\Omega\setminus\Sigma_\varepsilon} |\nabla v_\varepsilon|^2\ \mathrm{d}x \leq \int_{\Omega\setminus\Sigma_\varepsilon} f(\nabla v_\varepsilon)\ \mathrm{d}x$, reproducing the proof of (a), one has

$$\int_{\Omega\setminus\Sigma_\varepsilon} |\nabla v_\varepsilon|^2\ \mathrm{d}x \leq C\left(1+\int_\Omega |\nabla v_\varepsilon|\ \mathrm{d}x\right).$$

Proof of (c). Using the Cauchy-Schwartz inequality and (a), one can show that

$$\int_{\Sigma_\varepsilon} |\nabla v_\varepsilon|\ \mathrm{d}x \leq C\left(1+\int_\Omega |\nabla v_\varepsilon|\ \mathrm{d}x\right)^{\frac{1}{2}}.$$

Using the Cauchy-Schwartz inequality and (b), one can show that

$$\int_{\Omega\setminus\Sigma_\varepsilon} |\nabla v_\varepsilon| \,\mathrm{d}x \le C\left(1+\int_\Omega |\nabla v_\varepsilon| \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

From the previous estimates one has

$$\int_\Omega |\nabla v_\varepsilon| \,\mathrm{d}x = \int_{\Sigma_\varepsilon} |\nabla v_\varepsilon| \,\mathrm{d}x + \int_{\Omega\setminus\Sigma_\varepsilon} |\nabla v_\varepsilon| \,\mathrm{d}x \le C\left(1+\int_\Omega |\nabla v_\varepsilon| \,\mathrm{d}x\right)^{\frac{1}{2}},$$

thus $\sup\limits_{\varepsilon>0}\int_\Omega |\nabla v_\varepsilon| \,\mathrm{d}x < +\infty$. □

Obviously $\bar v_\varepsilon \in \operatorname{argmin}_{v\in W_0^{1,2}(\Omega,\mathbb{R}^m)} G_\varepsilon(v)$ satisfies $\sup\limits_{\varepsilon>0} G_\varepsilon(\bar v_\varepsilon) < +\infty$; consequently, thanks to Lemma 2.1, $\bar v_\varepsilon$ satisfies (2.2), thus the weaker condition

$$\sup_{\varepsilon>0}\left(\varepsilon\int_{\Sigma_\varepsilon}\left|\frac{\partial v_\varepsilon}{\partial\nu}\right|^2 \mathrm{d}x + \int_{\Omega\setminus\Sigma_\varepsilon} |\nabla v_\varepsilon|^2 \,\mathrm{d}x\right) < +\infty.$$

We will consider this condition in the definition of gradient Young-concentration measures. Let us introduce the space

$$W^{1,2}_{\partial\Omega}(\Omega\setminus\Sigma,\mathbb{R}^m) := \left\{v\in W^{1,2}(\Omega^+,\mathbb{R}^m)\cap W^{1,2}(\Omega^-,\mathbb{R}^m) : v=0 \text{ on } \partial\Omega \text{ in the sense of traces}\right\}.$$

Then we have:

Lemma 2.2. *Let $(v_\varepsilon)_{\varepsilon>0}$ be a sequence in $W_0^{1,2}(\Omega,\mathbb{R}^m)$ such that*

$$\sup_{\varepsilon>0}\left(\varepsilon\int_{\Sigma_\varepsilon}\left|\frac{\partial v_\varepsilon}{\partial\nu}\right|^2 \mathrm{d}x + \int_{\Omega\setminus\Sigma_\varepsilon} |\nabla v_\varepsilon|^2 \,\mathrm{d}x\right) < +\infty. \tag{2.3}$$

Then there exist a not relabeled subsequence and $v\in W^{1,2}_{\partial\Omega}(\Omega\setminus\Sigma,\mathbb{R}^m)$ such that $v_\varepsilon\to v$ in $L^2(\Omega,\mathbb{R}^m)$.

Proof. For every function $w\in W_0^{1,2}(\Omega,\mathbb{R}^m)$ we define its ε-translate $T_\varepsilon w$ by

$$T_\varepsilon w(\hat x, x_N) = \begin{cases} w(\hat x, x_N+\varepsilon/2), & \text{if } x\in\Omega^+,\\ w(\hat x, x_N-\varepsilon/2), & \text{if } x\in\Omega^-.\end{cases}$$

First step. We claim that there exist $z\in W^{1,2}_{\partial\Omega}(\Omega\setminus\Sigma,\mathbb{R}^m)$ and a subsequence v_ε such that $T_\varepsilon v_\varepsilon \rightharpoonup z$ in $W^{1,2}_{\partial\Omega}(\Omega\setminus\Sigma,\mathbb{R}^m)$ and strongly in $L^2(\Omega,\mathbb{R}^m)$. Indeed because of the boundary condition, by extending all the considered functions by zero outside of Ω, one may assume that Ω^+ and Ω^- are cubes; clearly $\nabla T_\varepsilon v_\varepsilon = T_\varepsilon\nabla v_\varepsilon$ so that $\sup_\varepsilon\int_{\Omega\setminus\Sigma}|\nabla T_\varepsilon v_\varepsilon|^2\,\mathrm{d}x = \sup_\varepsilon\int_{\Omega\setminus\Sigma_\varepsilon}|\nabla v_\varepsilon|^2\,\mathrm{d}x < +\infty$. Poincaré's inequality then yields $\|T_\varepsilon v_\varepsilon\|_{W^{1,2}_{\partial\Omega}(\Omega\setminus\Sigma,\mathbb{R}^m)} \le C$ and the claim follows immediately. We denote by z^+ and z^- the traces of z considered as a Sobolev function on Ω^+ and Ω^- respectively.

Second step. We claim that there exists $v\in L^2(\Omega,\mathbb{R}^m)$ such that we can extract from the previous subsequence $(v_\varepsilon)_{\varepsilon>0}$ a subsequence which weakly converges to v in $L^2(\Omega,\mathbb{R}^m)$. Indeed

$$\int_\Omega v_\varepsilon^2\,\mathrm{d}x = \int_{\Omega\setminus\Sigma_\varepsilon} v_\varepsilon^2\,\mathrm{d}x + \int_{\Sigma_\varepsilon} v_\varepsilon^2\,\mathrm{d}x = \int_{\Omega^+\cup\Omega^-}|T_\varepsilon v_\varepsilon|^2\,\mathrm{d}x + \int_{\Sigma_\varepsilon} v_\varepsilon^2\,\mathrm{d}x. \tag{2.4}$$

We are going to show that $\int_{\Sigma_\varepsilon} v_\varepsilon^2 \mathrm{d}x \to 0$ as $\varepsilon \to 0$. An integration with respect to x_N gives

$$v_\varepsilon(\hat{x}, x_N) = v_\varepsilon(\hat{x}, \varepsilon/2) + \int_{\varepsilon/2}^{x_N} \frac{\partial v_\varepsilon}{\partial \nu}(\hat{x}, s)\mathrm{d}s.$$

It is not difficult to show, setting $\Sigma_\varepsilon^+ = \Sigma_\varepsilon \cap \Omega^+$, that this implies

$$\int_{\Sigma_\varepsilon^+} |v_\varepsilon|^2 \leq C\varepsilon \left(\int_{S+\varepsilon/2\nu} |v_\varepsilon|^2 \mathrm{d}\hat{x} + \varepsilon \int_{\Sigma_\varepsilon^+} \left| \frac{\partial v_\varepsilon}{\partial \nu} \right|^2 \mathrm{d}s \right).$$

The same calculation on $\Sigma_\varepsilon^- = \Sigma_\varepsilon \cap \Omega^-$ gives

$$\int_{\Sigma_\varepsilon^-} |v_\varepsilon|^2 \leq C\varepsilon \left(\int_{S-\varepsilon/2\nu} |v_\varepsilon|^2 \mathrm{d}\hat{x} + \varepsilon \int_{\Sigma_\varepsilon^-} \left| \frac{\partial v_\varepsilon}{\partial \nu} \right|^2 \mathrm{d}s \right).$$

Summing up, we obtain

$$\int_{\Sigma_\varepsilon} |v_\varepsilon|^2 \, \mathrm{d}x \leq C\varepsilon \left(\int_{S+\varepsilon/2\nu} |v_\varepsilon|^2 \, \mathrm{d}\hat{x} + \int_{S-\varepsilon/2\nu} |v_\varepsilon|^2 \, \mathrm{d}\hat{x} + \varepsilon \int_{\Sigma_\varepsilon} \left| \frac{\partial v_\varepsilon}{\partial \nu} \right|^2 \, \mathrm{d}x \right).$$

By using (2.3) and the fact that

$$\lim_{\varepsilon \to 0} \left(\int_{S+\varepsilon/2\nu} |v_\varepsilon|^2 \, \mathrm{d}\hat{x} + \int_{S-\varepsilon/2\nu} |v_\varepsilon|^2 \, \mathrm{d}\hat{x} \right) = \int_S |z^+|^2 \, \mathrm{d}\hat{x} + \int_S |z^-|^2 \, \mathrm{d}\hat{x},$$

we have

$$\lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} |v_\varepsilon|^2 \, \mathrm{d}x = 0. \tag{2.5}$$

Estimate (2.5) and the fact that $T_\varepsilon v_\varepsilon$ converges to z in $L^2(\Omega, \mathbb{R}^m)$ imply

$$\lim_{\varepsilon \to 0} \int_\Omega |v_\varepsilon|^2 \, \mathrm{d}x = \int_\Omega |z|^2 \, \mathrm{d}x \tag{2.6}$$

from (2.4). In particular we deduce that v_ε is bounded in $L^2(\Omega, \mathbb{R})$ and the conclusion follows immediately.
Last step. To conclude, we establish that $v = z$. Indeed, for every $\varphi \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^m)$ we have

$$\int_\Omega v \cdot \varphi \, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon \cdot \varphi \, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega T_\varepsilon v_\varepsilon \cdot T_\varepsilon \varphi \, \mathrm{d}x = \int_\Omega z \cdot \varphi \, \mathrm{d}x.$$

This completes the proof because $v_\varepsilon \rightharpoonup v$ in $L^2(\Omega, \mathbb{R}^m)$ and, from (2.6), $\|v_\varepsilon\|_{L^2(\Omega,\mathbb{R})} \to \|v\|_{L^2(\Omega,\mathbb{R})}$. □

We denote by $\mathbb{S}^{m-1}$ the unit sphere of $\mathbb{R}^m$ and by $\mathbb{M}^+(\bar{\Omega} \times \mathbb{S}^{m-1})$ the set of non negative bounded Borel measures on $\bar{\Omega} \times \mathbb{S}^{m-1}$. For every measure μ in $\mathbb{M}^+(\bar{\Omega} \times \mathbb{S}^{m-1})$, $\mu = \mu_x \otimes \pi$ denotes its slicing decomposition. We recall that μ_x is a probability measure on $\mathbb{S}^{m-1}$ and π is the projection measure of μ on $\bar{\Omega}$.

Recall that the 2-homogeneous extension $\tilde{\varphi} : \mathbb{R}^m \to \mathbb{R}$ of a continuous function φ on $\mathbb{S}^{m-1}$ is defined for all $\zeta \in \mathbb{R}^m$ by

$$\tilde{\varphi}(\zeta) = \begin{cases} |\zeta|^2 \varphi(\frac{\zeta}{|\zeta|}), & \text{if } \zeta \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is well known (see [3], p. 741) that the 2-homogeneous extension of a Lipschitz function φ in $\mathbb{S}^{m-1}$ fulfills the following locally Lipschitz property: there exists $c = c(\varphi)$ such that

$$\forall \zeta,\ \zeta' \in \mathbb{R}^m,\ |\tilde{\varphi}(\zeta) - \tilde{\varphi}(\zeta')| \le c|\zeta - \zeta'|\,(|\zeta| + |\zeta'|)\,. \tag{2.7}$$

To shorten notation we will not distinguish φ from its 2-homogeneous extension $\tilde{\varphi}$.

Definition 2.3. We say that a pair (v, μ) belongs to the set $\mathcal{E} \subset L^2(\Omega, \mathbb{R}^m) \times \mathbb{M}^+(\bar{\Omega} \times \mathbb{S}^{m-1})$ of *elementary gradient Young-concentration measures* if and only if $v \in W_0^{1,2}(\Omega, \mathbb{R}^m)$ and $\mu = \delta_{\frac{\partial v}{\partial \nu}/|\frac{\partial v}{\partial \nu}|\ (x)} \otimes |\frac{\partial v}{\partial \nu}|^2\ \mathrm{d}x$.

We introduce the convergence of a sequence $((v_\varepsilon, \mu_\varepsilon))_{\varepsilon>0}$ of $\mathcal{E}$ to a pair (v, μ) of $L^2(\Omega, \mathbb{R}^m) \times \mathbb{M}^+(\bar{\Omega} \times \mathbb{S}^{m-1})$ *in the sense of concentration measures* as follows:

$$(v_\varepsilon, \mu_\varepsilon) \rightharpoonup (v, \mu) \iff \begin{cases} v_\varepsilon \to v \text{ strongly in } L^2(\Omega, \mathbb{R}^m) \\ \varepsilon \mathbb{1}_{B_\varepsilon} \mu_\varepsilon \overset{*}{\rightharpoonup} \mu \text{ in } \mathbb{M}^+(\bar{\Omega} \times \mathbb{S}^{m-1}). \end{cases}$$

Definition 2.4. Let $\mathcal{B}$ be the set of pairs (v, μ) of $\mathcal{E}$ such that (2.3) holds. The set $\mathcal{YC}$ of gradient Young-concentration measures is the sequential closure of $\mathcal{B}$ for the convergence in the sense of concentration measures defined above, *i.e.*

$$(v, \mu) \in \mathcal{YC} \iff \exists\ v_\varepsilon \in W_0^{1,2}(\Omega, \mathbb{R}^m) \text{ s.t. } \begin{cases} v_\varepsilon \to v \text{ strongly in } L^2(\Omega, \mathbb{R}^m), \\ \sup\limits_{\varepsilon>0} \left(\varepsilon \displaystyle\int_{\Sigma_\varepsilon} \left| \frac{\partial v_\varepsilon}{\partial \nu} \right|^2 \mathrm{d}x + \int_{\Omega \setminus \Sigma_\varepsilon} |\nabla v_\varepsilon|^2\ \mathrm{d}x \right) < +\infty, \\ \delta_{\frac{\partial v_\varepsilon}{\partial \nu}/\left|\frac{\partial v_\varepsilon}{\partial \nu}\right|\ (x)} \otimes \varepsilon \mathbb{1}_{B_\varepsilon} |\frac{\partial v_\varepsilon}{\partial \nu}|^2\ \mathrm{d}x \overset{*}{\rightharpoonup} \mu \end{cases} \tag{2.8}$$

when $\varepsilon \to 0$. We say that the sequence $(v_\varepsilon)_{\varepsilon>0}$ *generates the gradient Young-concentration measure* (v, μ) or, in short, that the sequence $(v_\varepsilon)_{\varepsilon>0}$ generates the Young-concentration measure μ. Recall that the weak convergence $\overset{*}{\rightharpoonup}$ above is defined by

$$\int_{B_\varepsilon} \varepsilon \theta(x) \varphi\left(\frac{\partial v_\varepsilon}{\partial \nu}\right)\ \mathrm{d}x \to \int_{\bar{\Omega}} \int_{\mathbb{S}^{m-1}} \theta(x) \varphi(\zeta)\ \mathrm{d}\mu$$

for all $\theta \in \mathcal{C}(\bar{\Omega})$ and all $\varphi \in \mathcal{C}(\mathbb{S}^{m-1})$. The set $\mathcal{YC}_{abs}$ is the subset of the elements (v, μ) of $\mathcal{YC}$ such that the projection measure π of μ on $\bar{\Omega}$ is absolutely continuous with respect to the measure $\mathcal{H}^{N-1}\lfloor S$.

Remark 2.5. Let $(v_\varepsilon)_{\varepsilon>0}$ be a sequence of $W_0^{1,2}(\Omega, \mathbb{R}^m)$ which satisfies (2.3). Clearly $\delta_{\frac{\partial v_\varepsilon}{\partial \nu}/\left|\frac{\partial v_\varepsilon}{\partial \nu}\right|\ (x)} \otimes \varepsilon \mathbb{1}_{B_\varepsilon} |\frac{\partial v_\varepsilon}{\partial \nu}|^2\ \mathrm{d}x$ is bounded in $\mathbb{M}^+(\bar{\Omega} \times \mathbb{S}^{m-1})$. On the other hand, from Lemma 2.2, one can extract a subsequence of $(v_\varepsilon)_{\varepsilon>0}$ which converges in $L^2(\Omega, \mathbb{R}^m)$ to some $v \in W_{\partial\Omega}^{1,2}(\Omega \setminus \Sigma, \mathbb{R}^m)$. Then one can extract a subsequence such that $(v_\varepsilon)_{\varepsilon>0}$ generates some $(v, \mu) \in \mathcal{YC}$, that is $(v_\varepsilon, \delta_{\frac{\partial v_\varepsilon}{\partial \nu}/\left|\frac{\partial v_\varepsilon}{\partial \nu}\right|\ (x)} \otimes \varepsilon \mathbb{1}_{B_\varepsilon} |\frac{\partial v_\varepsilon}{\partial \nu}|^2\ \mathrm{d}x) \rightharpoonup (v, \mu) \in \mathcal{YC}$. Note that, according to Lemma 2.1, the sequence $(\bar{v}_\varepsilon)_{\varepsilon>0}$ of solutions of Dirichlet problems $\bar{v}_\varepsilon \in \operatorname{argmin}_{v \in W_0^{1,2}(\Omega, \mathbb{R}^m)} G_\varepsilon$ satisfies (2.3), then generates a gradient Young-concentration measure (v, μ).

3. Characterization of the set of gradient Young-concentration measures

This section is devoted to the proof of the following characterization of $\mathcal{YC}$:

Theorem 3.1 (characterization)**.** *A pair $(v, \mu = \mu_x \otimes \pi)$ belongs to $\mathcal{YC}$ if and only if $v \in W^{1,2}_{\partial\Omega}(\Omega \setminus \Sigma, \mathbb{R}^m)$, π is concentrated on $\bar{S}$ and, for every $\varphi \in \mathcal{C}(\mathbb{S}^{m-1})$ such that $\varphi^{**} > -\infty$,*

$$\begin{aligned} &\frac{\mathrm{d}\pi}{\mathrm{d}\mathcal{H}^{N-1}\lfloor S}(x) \int_{\mathbb{S}^{m-1}} \varphi(\zeta)\ \mathrm{d}\mu_x \geq \varphi^{**}([v](x)) \quad \textit{for } \mathcal{H}^{N-1} \textit{ a.e. } x \in S \\ &\int_{\mathbb{S}^{m-1}} \varphi(\zeta)\ \mathrm{d}\mu_x \geq 0 \quad \textit{for } \pi_s \textit{ a.e. } x \in \bar{S} \end{aligned} \tag{3.1}$$

where $\pi = \frac{\mathrm{d}\pi}{\mathrm{d}\mathcal{H}^{N-1}\lfloor S}\mathcal{H}^{N-1}\lfloor S + \pi_s$ is the Radon-Nikodym decomposition of π with respect to the measure $\mathcal{H}^{N-1}\lfloor S$.

By analogy with the gradient Young measures (see [4]), the Sobolev function v will be refered to as *the underlying deformation* of the measure μ. Note that for each $v \in W^{1,2}_{\partial\Omega}(\Omega \setminus \Sigma, \mathbb{R}^m)$, the pair $(v, \delta_{\frac{[v]}{|[v]|}(x)} \otimes |[v](x)|^2 \mathcal{H}^{N-1}\lfloor S)$ belongs to $\mathcal{YC}$ (more precisely to $\mathcal{YC}_{\text{abs}}$) and v is its underlying deformation. We call these pairs *elementary limit gradient Young-concentration measures*.

Gradient Young-concentration measures which are not elementary limit gradient Young-concentration measures and which are generated by a sequence of solutions of Dirichlet problems of the type (1.4) exist in abundance, as seen in the next example.

Example 3.2. Fix an arbitrary open interval I in $\mathbb{R}$, a function $\phi : I \to \mathbb{R}^m$ in $W^{1,2}_0(I, \mathbb{R}^m)$ and $v \in W^{1,2}_{\partial\Omega}(\Omega \setminus \Sigma, \mathbb{R}^m)$. For $\mathcal{H}^{N-1}$ almost every $x \in \Sigma$, define now the probability measure μ_x on $\mathbb{S}^{m-1}$ by $\mu_x = \tilde{\mu}_x / \|\tilde{\mu}_x\|$, where for all $\varphi \in \mathcal{C}(\mathbb{S}^{m-1})$

$$\langle \tilde{\mu}_x, \varphi \rangle := \fint_I \varphi\big([v](x) + \frac{\mathrm{d}\phi}{\mathrm{d}y}(y)\big)\ \mathrm{d}y.$$

Consider the measure $\mu = \mu_x \otimes \|\tilde{\mu}_x\| \mathcal{H}^{N-1}\lfloor S$. According to Jensen's inequality, clearly the pair (v, μ) fulfills all the conditions of Theorem 3.1, but it is a non elementary limit concentration measure when $\phi \not\equiv 0$ (the elementary limit concentration measures correspond to the case when $\phi \equiv 0$). In the scalar case $m = 1$ where $\mathbb{S}^{m-1} = \{-1, 1\}$ and $\mu_x = a(x)\delta_1 + b(x)\delta_{-1}$, we can express a and b completely thanks to Theorem 3.1. Indeed $\varphi \in \mathcal{C}(\mathbb{S}^{m-1})$ fulfills the condition $\varphi^{**} > -\infty$, if and only if its extension is of the form

$$\varphi(\zeta) = \begin{cases} c\zeta^2, & \text{if } \zeta \geq 0 \\ \mathrm{d}\zeta^2, & \text{if } \zeta \leq 0 \end{cases}$$

where c and d are two non-negative real numbers (note that in this case $\varphi^{**} = \varphi$). With these considerations taking successively $c > 0$, $d = 0$ and $c = 0$, $d > 0$, we easily deduce

$$a(x) = \frac{\int_{[[v](x) > -\frac{\mathrm{d}\phi}{\mathrm{d}y}]} \left|[v](x) + \frac{\mathrm{d}\phi}{\mathrm{d}y}(y)\right|^2\ \mathrm{d}y}{\int_I \left|[v](x) + \frac{\mathrm{d}\phi}{\mathrm{d}y}(y)\right|^2\ \mathrm{d}y}, \quad b(x) = \frac{\int_{[[v](x) < -\frac{\mathrm{d}\phi}{\mathrm{d}y}]} \left|[v](x) + \frac{\mathrm{d}\phi}{\mathrm{d}y}(y)\right|^2\ \mathrm{d}y}{\int_I \left|[v](x) + \frac{\mathrm{d}\phi}{\mathrm{d}y}(y)\right|^2\ \mathrm{d}y}.$$

Let us now exhibit a sequence $(v_\varepsilon)_{\varepsilon>0}$ generating (v, μ). To this end we define $R_\varepsilon : W^{1,2}(\Omega \setminus \Sigma, \mathbb{R}^m) \to W^{1,2}(\Omega, \mathbb{R}^m)$ by

$$R_\varepsilon v(\hat{x}, x_N) = \begin{cases} \frac{x_N}{\varepsilon}\left[v(\hat{x}, \varepsilon/2) - v(\hat{x}, -\varepsilon/2)\right] + \frac{1}{2}\left[v(\hat{x}, \varepsilon/2) + v(\hat{x}, -\varepsilon/2)\right] & \text{if } x \in \Sigma_\varepsilon \\ v & \text{if } x \in \Omega \setminus \Sigma_\varepsilon, \end{cases} \tag{3.2}$$

where $v_\varepsilon(\hat{x}, \varepsilon/2)$ and $v_\varepsilon(\hat{x}, -\varepsilon/2)$ should be taken within the meaning of traces on $\Sigma + \frac{\varepsilon}{2}e_N$ and $\Sigma - \frac{\varepsilon}{2}e_N$ respectively. Take $I = (-\frac{1}{2}, \frac{1}{2})$ and set $v_\varepsilon(x) := R_\varepsilon v(x) + \phi(\frac{x_N}{\varepsilon})$, $B = S \times I$. For all $\theta \in \mathcal{C}(\bar{\Omega})$ and all $\varphi \in \mathcal{C}(\mathbb{S}^{m-1})$

we have

$$\begin{aligned}
\varepsilon \int_{B_\varepsilon} \theta(x)\varphi\left(\frac{\partial v_\varepsilon}{\partial \nu}\right)\,\mathrm{d}x &= \varepsilon \int_{B_\varepsilon} \theta(x)\varphi\left(\frac{1}{\varepsilon}\left(v(\hat{x},\varepsilon/2) - v(\hat{x},-\varepsilon/2) + \frac{\mathrm{d}\phi}{\mathrm{d}\nu}\left(\frac{x_N}{\varepsilon}\right)\right)\right)\,\mathrm{d}x \\
&= \frac{1}{\varepsilon}\int_{B_\varepsilon} \theta(x)\varphi\left(v(\hat{x},\varepsilon/2) - v(\hat{x},-\varepsilon/2) + \frac{\mathrm{d}\phi}{\mathrm{d}\nu}\left(\frac{x_N}{\varepsilon}\right)\right)\,\mathrm{d}x \\
&= \int_{B} \theta(\hat{x},\varepsilon x_N)\varphi\left(v(\hat{x},\varepsilon/2) - v(\hat{x},-\varepsilon/2) + \frac{\mathrm{d}\phi}{\mathrm{d}\nu}(x_N)\right)\,\mathrm{d}x
\end{aligned}$$

which tends to

$$\int_S \theta(\hat{x},0)\left(\int_I \varphi\left([v](\hat{x}) + \frac{\mathrm{d}\phi}{\mathrm{d}y}(y)\right)\,\mathrm{d}y\right)\,\mathrm{d}\hat{x} = \langle \tilde{\mu}_x \otimes \mathcal{H}^{N-1}\lfloor S, \theta\otimes\varphi\rangle.$$

Since moreover clearly $v_\varepsilon \to v$ in $L^2(\Omega,\mathbb{R}^m)$, the sequence $(v_\varepsilon)_{\varepsilon>0}$ generates (v,μ).

We claim that $u_\varepsilon = \sqrt{\varepsilon} v_\varepsilon$, where v_ε is the function constructed above, is the solution to a Dirichlet problem with large source. In order to shorten the calculation we prove the claim in the scalar case ($m=1$) and we begin by the one dimensional case ($N=1$).

Take $\phi \in \mathcal{C}_0^2(I)$, $v \in \mathcal{C}^2((-1,0)\cup(0,1))$ with $v(-1)=v(1)=0$ and denote by $[v]_\varepsilon$ the approximate jump $v(\varepsilon/2) - v(-\varepsilon/2)$. Then by an elementary computation, it is readily seen that u_ε is solution to

$$\begin{cases} (\sigma_\varepsilon u_\varepsilon')' = \frac{1}{\sqrt{\varepsilon}}\mathcal{F}_\varepsilon \text{ in } (-1,1) \\ u_\varepsilon(-1) = u_\varepsilon(1) = 0, \end{cases}$$

where $\mathcal{F}_\varepsilon$ is the measure

$$v''\lfloor\left(-1,-\frac{\varepsilon}{2}\right)\cup\left(\frac{\varepsilon}{2},1\right) + \frac{1}{\varepsilon}\phi''\left(\frac{t}{\varepsilon}\right)\lfloor\left(-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right) + \left([v]_\varepsilon - v'\left(-\frac{\varepsilon}{2}\right)\right)\delta_{-\frac{\varepsilon}{2}} + \left(v'\left(-\frac{\varepsilon}{2}\right) - [v]_\varepsilon\right)\delta_{\frac{\varepsilon}{2}}.$$

Note that, choosing v such that $v''=0$, *i.e.* v of the form $v(t) = a^-(t+1)$ on $(-1,0)$ and $v(t) = a^+(t-1)$ on $(0,1)$, the measure $\mathcal{F}_\varepsilon$ is concentrated on $(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})$.

We treat now the case $N=2$ (the calculation in the general case is similar). We set $\Omega = (-1,1)^2$ and $\Sigma = (-1,1)\times\{0\}$. Take $\phi\in\mathcal{C}_0^2(I)$ and $v\in\mathcal{C}^2(\Omega\setminus\Sigma)$ satisfying $v=0$ on $\partial\Omega$. One can easily show that u_ε is the solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma_\varepsilon \nabla u_\varepsilon) = \frac{1}{\sqrt{\varepsilon}}\mathcal{F}_\varepsilon & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\sigma_\varepsilon := \frac{1}{\varepsilon}\mathbb{1}_{\Omega\setminus\Sigma_\varepsilon} + \mathbb{1}_{\Sigma_\varepsilon}$, and $\mathcal{F}_\varepsilon$ is given by

$$\begin{aligned}
\mathcal{F}_\varepsilon = {} & \Delta u_\varepsilon\lfloor(\Omega\setminus\Sigma_\varepsilon) \\
& + \left[[v]_\varepsilon(x_1) - \frac{\partial v}{\partial x_2}\left(x_1,-\frac{\varepsilon}{2}\right)\right]\mathcal{H}^1\lfloor\left(S - \frac{\varepsilon}{2}e_2\right) + \left[\frac{\partial v}{\partial x_2}\left(x_1,\frac{\varepsilon}{2}\right) - [v]_\varepsilon(x_1)\right]\mathcal{H}^1\lfloor\left(S + \frac{\varepsilon}{2}e_2\right) \\
& + \left[x_2\left(\frac{\partial^2 v}{\partial^2 x_1}\left(x_1,\frac{\varepsilon}{2}\right) - \frac{\partial^2 v}{\partial^2 x_1}\left(x_1,-\frac{\varepsilon}{2}\right)\right) + \frac{\varepsilon}{2}\left(\frac{\partial^2 v}{\partial^2 x_1}\left(x_1,\frac{\varepsilon}{2}\right) + \frac{\partial^2 v}{\partial^2 x_1}\left(x_1,-\frac{\varepsilon}{2}\right)\right)\right]\lfloor\Sigma_\varepsilon + \frac{1}{\varepsilon}\frac{\mathrm{d}^2\phi}{\mathrm{d}^2 x_2}\left(\frac{x_2}{\varepsilon}\right)\lfloor\Sigma_\varepsilon.
\end{aligned}$$

The approximated jump $[v]_\varepsilon$ is given now by $[v]_\varepsilon(x_1) := v(x_1, \frac{\varepsilon}{2}) - v(x_1, -\frac{\varepsilon}{2})$. Note that, taking v solution of the Dirichlet-Neumann problem

$$\begin{cases} \Delta v = 0 \text{ in } \Omega^+ \\ v = 0 \text{ on } \partial\Omega^+ \setminus \Sigma \\ \frac{\partial v}{\partial x_2} \text{ prescribed on } \Sigma, \end{cases} \qquad \begin{cases} \Delta v = 0 \text{ in } \Omega^- \\ v = 0 \text{ on } \partial\Omega^- \setminus \Sigma \\ \frac{\partial v}{\partial x_2} \text{ prescribed on } \Sigma, \end{cases}$$

the source $\mathcal{F}_\varepsilon$ is concentrated on Σ_ε.

The proof of Theorem 3.1 is divided into two propositions, the necessary condition (Prop. 3.3) and the sufficient condition (Prop. 3.5).

Proposition 3.3 (necessary condition)**.** *Assume that* $(v, \mu = \mu_x \otimes \pi)$ *belongs to* $\mathcal{YC}$*. Then* $v \in W^{1,2}_{\partial\Omega}(\Omega\backslash\Sigma, \mathbb{R}^m)$*,* π *is concentrated on* $\bar{S}$ *and, for every* $\varphi \in \mathcal{C}(\mathbb{S}^{m-1})$ *such that* $\varphi^{**} > -\infty$*,* (3.1) *holds.*

The proof of Proposition 3.3 is based on next Lemma 3.4.

Lemma 3.4. *Let* $(v_\varepsilon)_{\varepsilon>0}$ *be a sequence in* $W^{1,2}_0(\Omega, \mathbb{R}^m)$ *satisfying* (2.3) *and converging to* v *in* $L^2(\Omega, \mathbb{R}^m)$*. Then* $v \in W^{1,2}_{\partial\Omega}(\Omega \setminus \Sigma, \mathbb{R}^m)$*; moreover for all continuous functions* φ *on* $\mathbb{S}^{m-1}$ *and all non-negative functions* θ *in* $\mathcal{C}(\bar{S})$*, we have*

$$\liminf_{\varepsilon\to 0} \varepsilon \int_{B_\varepsilon} \theta(\hat{x})\varphi\left(\frac{\partial v_\varepsilon}{\partial \nu}\right)\, \mathrm{d}x \geq \int_S \theta(\hat{x})\varphi^{**}([v])\, \mathrm{d}\hat{x}. \tag{3.3}$$

Proof. According to Lemma 2.2, one has $v \in W^{1,2}_{\partial\Omega}(\Omega \setminus \Sigma, \mathbb{R}^m)$. Consider the Moreau-Yosida proximal approximation φ_p of the continuous function $\varphi : \mathbb{S}^{m-1} \to \mathbb{R}$ defined by

$$\varphi_p(\zeta) = \inf_{\xi\in\mathbb{S}^{m-1}} \left(\varphi(\xi) + p|\xi - \zeta|^2\right).$$

It is well known that φ_p is a Lipschitz function and that the sequence $(\varphi_p)_{p\in\mathbb{N}}$ converges increasing to φ (for a proof consult [1], Thm. 2.6.4). Consequently, according to Dini's theorem $(\varphi_p)_{p\in\mathbb{N}}$ converges uniformly to φ on $\mathbb{S}^{m-1}$. Let $\eta > 0$ and $p(\eta) \in \mathbb{N}$ be such that

$$\sup_{\xi\in\mathbb{S}^{m-1}} |\varphi(\xi) - \varphi_{p(\eta)}(\xi)| < \eta. \tag{3.4}$$

We establish the existence of a non-negative constant C independent on ε and η such that

$$\liminf_{\varepsilon\to 0} \varepsilon \int_{B_\varepsilon} \theta(\hat{x})\varphi_{p(\eta)}\left(\frac{\partial v_\varepsilon}{\partial \nu}\right)\, \mathrm{d}x \geq \int_S \theta(\hat{x})\varphi^{**}([v])\, \mathrm{d}\hat{x} - C\eta. \tag{3.5}$$

Consider the function $w_\varepsilon := v_\varepsilon - R_\varepsilon v$, where $R_\varepsilon : W^{1,2}(\Omega \setminus \Sigma, \mathbb{R}^m) \to W^{1,2}(\Omega, \mathbb{R}^m)$ is the map defined in (3.2). It is readily seen that w_ε converges to 0 in $L^2(\Omega, \mathbb{R}^m)$, since $v_\varepsilon \to v$ in $L^2(\Omega, \mathbb{R}^m)$. By using the locally Lipschitz property (2.7) satisfied by the 2-homogeneous extension of $\varphi_{p(\eta)}$ (recall that $\varphi_{p(\eta)}$ is Lipschitz), we have

$$\begin{aligned} \liminf_{\varepsilon\to 0} \varepsilon \int_{B_\varepsilon} \theta(\hat{x})\, \varphi_{p(\eta)}\left(\frac{\partial v_\varepsilon}{\partial \nu}\right)\, \mathrm{d}x &= \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{B_\varepsilon} \theta(\hat{x})\, \varphi_{p(\eta)}\left(\varepsilon\frac{\partial v_\varepsilon}{\partial \nu}\right)\, \mathrm{d}x \\ &= \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{B_\varepsilon} \theta(\hat{x})\, \varphi_{p(\eta)}\left(\varepsilon\frac{\partial w_\varepsilon}{\partial \nu} + v(\hat{x}, \varepsilon/2) - v(\hat{x}, -\varepsilon/2)\right)\, \mathrm{d}x \\ &= \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{B_\varepsilon} \theta(\hat{x})\, \varphi_{p(\eta)}\left([v] + \varepsilon\frac{\partial w_\varepsilon}{\partial \nu}\right)\, \mathrm{d}x. \end{aligned} \tag{3.6}$$

Indeed using the Cauchy-Schwartz inequality and (2.3), it is easy to establish

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B_\varepsilon}|v(\hat{x},\varepsilon/2)-v(\hat{x},-\varepsilon/2)-[v]|\left[|v(\hat{x},\varepsilon/2)-v(\hat{x},-\varepsilon/2)|+|[v]|+2\left|\varepsilon\frac{\partial w_\varepsilon}{\partial\nu}\right|\right]\ \mathrm{d}x=0.$$

Since $w_\varepsilon\to 0$ in $L^2(\Omega,\mathbb{R}^m)$, by using a truncation argument, one can modify w_ε into a Sobolev function $\tilde{w}_\varepsilon$ vanishing at the boundary of $S\pm\frac{\varepsilon'}{2}\nu$ of B_ε where $\varepsilon':=\varepsilon(1+t(\varepsilon))$ and $t(\varepsilon)>0$ is suitably chosen so that $\lim_{\varepsilon\to 0}t(\varepsilon)=0$ and

$$\liminf_{\varepsilon\to 0}⨍_{B_\varepsilon}\theta(\hat{x})\varphi_{p(\eta)}\left([v]+\varepsilon\frac{\partial w_\varepsilon}{\partial\nu}\right)\ \mathrm{d}x=\liminf_{\varepsilon\to 0}⨍_{B_{\varepsilon'}}\theta(\hat{x})\varphi_{p(\eta)}\left([v]+\varepsilon\frac{\partial\tilde{w}_\varepsilon}{\partial\nu}\right)\ \mathrm{d}x \tag{3.7}$$

(see Lem. 4.5 in [5]). According to (3.7), (3.4) and Jensen's inequality, we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}⨍_{B_\varepsilon}\theta(\hat{x})\,\varphi_{p(\eta)}\left([v]+\varepsilon\frac{\partial w_\varepsilon}{\partial\nu}\right)\ \mathrm{d}x=\liminf_{\varepsilon\to 0}⨍_{B_{\varepsilon'}}\theta(\hat{x})\,\varphi_{p(\eta)}\left([v]+\varepsilon\frac{\partial\tilde{w}_\varepsilon}{\partial\nu}\right)\mathrm{d}x\geq\\
&\liminf_{\varepsilon\to 0}⨍_{B_{\varepsilon'}}\theta(\hat{x})\,\varphi\left([v]+\varepsilon\frac{\partial\tilde{w}_\varepsilon}{\partial\nu}\right)\mathrm{d}x-\eta\frac{\|\theta\|_\infty}{|S|}\sup_{\varepsilon>0}\frac{1}{\varepsilon}\int_{B_{\varepsilon'}}\left|[v]+\varepsilon\frac{\partial\tilde{w}_\varepsilon}{\partial\nu}\right|^2\ \mathrm{d}x\geq\\
&\liminf_{\varepsilon\to 0}⨍_{S\times\left(-\frac{\varepsilon'}{2},\frac{\varepsilon'}{2}\right)}\theta(\hat{x})\,\varphi^{**}\left([v]+\varepsilon\frac{\partial\tilde{w}_\varepsilon}{\partial\nu}\right)\mathrm{d}x-C\eta\geq\\
&\int_S\theta(\hat{x})\,\varphi^{**}([v])\ \mathrm{d}\hat{x}-C\eta.
\end{aligned}\tag{3.8}$$

Notice that in the second inequality, we have used the bound $\sup_{\varepsilon>0}\frac{1}{\varepsilon}\int_{B_\varepsilon}|[v]+\frac{\partial\tilde{w}_\varepsilon}{\partial\nu}|^2\ \mathrm{d}x<+\infty$: this is a straightforward consequence of (3.7) applied with $\varphi=|.|^2$ and (2.3). We conclude the proof of (3.5) by combining (3.6) and (3.8). Estimate (3.3) follows letting $\eta\to 0$ and noticing that $\varphi\geq\varphi_{p(\eta)}$. $\square$

Proof of Proposition 3.3. Let (v,μ) in $\mathcal{YC}$. Then, by definition of $\mathcal{YC}$, there exists a sequence $(v_\varepsilon)_{\varepsilon>0}$ in $W_0^{1,2}(\Omega,\mathbb{R}^m)$ satisfying (2.3) such that

$$\begin{aligned}
&v_\varepsilon\to v\text{ strongly in }L^2(\Omega,\mathbb{R}^m),\\
&\mu_\varepsilon:=\delta_{\frac{\partial v_\varepsilon}{\partial\nu}/\left|\frac{\partial v_\varepsilon}{\partial\nu}\right|\ (x)}\otimes\varepsilon\mathbb{1}_{B_\varepsilon}|\tfrac{\partial v_\varepsilon}{\partial\nu}|^2\ \mathrm{d}x\overset{*}{\rightharpoonup}\mu
\end{aligned}$$

when $\varepsilon\to 0$. According to Lemma 2.2, v belongs to $W^{1,2}_{\partial\Omega}(\Omega\setminus\Sigma,\mathbb{R}^m)$. The fact that π is concentrated on $\bar{S}$ is easy to see. We are going to prove (3.1). Let θ be a non-negative function in $\mathcal{C}(\bar{S})$ and φ in $\mathcal{C}(\mathbb{S}^{m-1})$ such that $\varphi^{**}>-\infty$. Since $v_\varepsilon\to v$ in $L^2(\Omega,\mathbb{R}^m)$, by using Lemma 3.4, one has

$$\liminf_{\varepsilon\to 0}\int_{B_\varepsilon}\varepsilon\theta(\hat{x})\,\varphi\left(\frac{\partial v_\varepsilon}{\partial\nu}\right)\ \mathrm{d}x\geq\int_S\theta(\hat{x})\,\varphi^{**}([v])\ \mathrm{d}\hat{x}. \tag{3.9}$$

On the other hand, writing $\displaystyle\int_{B_\varepsilon}\varepsilon\theta(\hat{x})\,\varphi\left(\frac{\partial v_\varepsilon}{\partial\nu}\right)\ \mathrm{d}x$ as $\displaystyle\int_{\bar{\Omega}}\theta(\hat{x})\left(\int_{\mathbb{S}^{m-1}}\varphi(\zeta)\mathrm{d}\delta_{\frac{\partial v_\varepsilon}{\partial\nu}/|\frac{\partial v_\varepsilon}{\partial\nu}|(x)}\right)\varepsilon\mathbb{1}_{B_\varepsilon}\left|\frac{\partial v_\varepsilon}{\partial\nu}\right|^2\mathrm{d}x$, we obtain

$$\liminf_{\varepsilon\to 0}\int_{B_\varepsilon}\varepsilon\theta(\hat{x})\varphi\left(\frac{\partial v_\varepsilon}{\partial\nu}\right)\ \mathrm{d}x=\int_{\bar{S}}\theta(\hat{x})\left(\int_{\mathbb{S}^{m-1}}\varphi(\zeta)\ \mathrm{d}\mu_x\right)\ \mathrm{d}\pi, \tag{3.10}$$

where $\mu = \mu_x \otimes \pi$ is the slicing decomposition of μ with respect to its projection π on $\bar{\Omega}$. Combining (3.9) and (3.10), it follows that

$$\frac{\mathrm{d}\pi}{\mathrm{d}\mathcal{H}^{N-1}}(x)\int_{\mathbb{S}^{m-1}} \varphi(\zeta)\ \mathrm{d}\mu_x \geq \varphi^{**}([u](x)) \quad \text{for } \mathcal{H}^{N-1}\lfloor S \text{ a.e. } x,$$
$$\int_{\mathbb{S}^{m-1}} \varphi(\zeta)\ \mathrm{d}\mu_x \geq 0 \quad \text{for } \pi_{\mathbf{s}} \text{ a.e. } x \in \bar{S}. \qquad \square$$

Proposition 3.5 (sufficient conditions)**.** *Assume that* $(v, \mu = \mu_x \otimes \pi)$ *belongs to* $W^{1,2}_{\partial\Omega}(\Omega \setminus \Sigma, \mathbb{R}^m) \times \mathbb{M}^+(\bar{\Omega} \times \mathbb{S}^{m-1})$*, satisfies* (3.1)*, and that* π *is concentrated on* $\bar{S}$*. Then* (v, μ) *belongs to* $\mathcal{YC}$*.*

Proof. We divide the proof into three steps. In Step 1, using the Hahn-Banach separation theorem, we approximate the homogeneous components μ of $\mathcal{YC}$ (*i.e.* μ_x does not depend of x) by the same type of measures as in Example 3.2. In Step 2, we construct a sequence of Sobolev functions generating these measures and conclude by a standard diagonalization argument. In Step 3, we build a sequence of Sobolev functions generating the components μ of $\mathcal{YC}$: according to a standard covering argument, we localize the construction above and stick together the various generating functions obtained in Step 2. We show that this sequence generates the pair (v, μ). In order to make easier the reading of the proof, we begin by treating the case $N = m = 1$.

Proof in the case $N = m = 1$. Let (v, μ) in $W^{1,2}_{\partial\Omega}((-1/2, 0) \cup (0, 1/2)) \otimes \mathbb{M}^+([-1/2, 1/2] \times \mathbb{S}^0)$, satisfying (3.1) with π concentrated in $\{0\}$. The measure $\mu = \mu_0 \otimes \pi(0)\delta_0$ can be written as $\pi(0)\mu_0 \otimes \delta_0$, where $\pi(0)\mu_0 \in \mathbb{M}^+(\mathbb{S}^0)$ satisfies

$$\int_{\mathbb{S}^0} \varphi(\zeta)\ \mathrm{d}(\pi(0)\mu_0) \geq \varphi^{**}([v](0)) \tag{3.11}$$

for all $\varphi \in \mathcal{C}(\mathbb{S}^0)$ such that $\varphi^{**} > -\infty$. Note that such functions φ, (actually their extensions) are of the form

$$\varphi(\zeta) = \begin{cases} c\zeta^2, & \text{if } \zeta \leq 0 \\ d\zeta^2, & \text{if } \zeta \geq 0 \end{cases}$$

where c and d are two nonnegative real numbers. Note also that since μ is homogeneous, Step 3 described above is not necessary.

Step 1. Let us consider the following subset of $\mathbb{M}^+(\mathbb{S}^0)$:

$$\mathcal{H} := \left\{\lambda \in \mathbb{M}^+(\mathbb{S}^0) : \forall\varphi \in \mathcal{C}(\mathbb{S}^0),\ \text{s.t. } \varphi^{**} > -\infty,\ \int_{\mathbb{S}^0} \varphi\ \mathrm{d}\lambda \geq \varphi^{**}([v](0))\right\}.$$

$\mathcal{H}$ is non empty, because it contains $\pi(0)\mu(0)$ as (3.11) shows. It is easily seen that $\mathcal{H}$ is convex and closed with respect to the weak convergence in $\mathbb{M}^+(\mathbb{S}^0)$. For all $w \in W^{1,2}_0((-1/2, 1/2))$, consider the measure μ_w in $\mathbb{M}^+(\mathbb{S}^0)$ which acts on all $\varphi \in \mathcal{C}(\mathbb{S}^0)$ as follows:

$$\langle \mu_w, \varphi \rangle := \int_{(-1/2,1/2)} \varphi([v](0) + \dot{w})\ \mathrm{d}x.$$

According to Jensen's inequality we have

$$\begin{aligned} \langle \mu_w, \varphi \rangle &\geq \int_{(-1/2,1/2)} \varphi^{**}([v](0) + \dot{w})\ \mathrm{d}x \\ &\geq \varphi^{**}([v](0)) \end{aligned}$$

so that μ_w belongs to $\mathcal{H}$. Let us introduce the subset $\mathcal{C}$ of $\mathcal{H}$ made up of such measures μ_w, *i.e.*,

$$\mathcal{C} := \left\{ \mu_w : w \in W_0^{1,2}((-1/2, 1/2)) \right\}.$$

In this step we prove that $\mathcal{C}$ is a dense convex subset of $\mathcal{H}$ for the weak convergence of measures. Let us first prove the convexity of $\mathcal{C}$. Let $a,\ b$ in $(0,1)$ be such that $a+b=1$, $w_1,\ w_2$ in $W_0^{1,2}((-1/2,1/2))$. Then

$$\langle a\mu_{w_1} + b\mu_{w_2}, \varphi\rangle = a\int_{-1/2}^{1/2} \varphi([v](0) + w_1')\ \mathrm{d}x + b\int_{-1/2}^{1/2} \varphi([v](0) + w_2')\ \mathrm{d}x.$$

The change of scale $x = \frac{t}{a} - \frac{1}{2} + \frac{a}{2}$ in the first integral and $x = \frac{t}{1-a} - \frac{a}{2(1-a)}$ in the second one give

$$\langle a\mu_{w_1} + b\mu_{w_2}, \varphi\rangle = \langle \mu_w, \varphi\rangle$$

where w is the function of $W_0^{1,2}((-1/2,1/2))$ defined by

$$w(t) = \begin{cases} aw_1\left(\frac{t - \frac{a}{2} + \frac{1}{2}}{a}\right), & \text{if } t \in \left(-\frac{1}{2}, -\frac{1}{2} + a\right), \\ bw_2\left(\frac{t + \frac{b}{2} - \frac{1}{2}}{b}\right), & \text{if } t \in \left(-\frac{1}{2} + a, \frac{1}{2}\right). \end{cases}$$

It remains to establish that $\mathcal{C}$ is a dense subset of $\mathcal{H}$. Indeed otherwise there exists λ_0 in $\mathcal{H}$ which does not belong to $\bar{\mathcal{C}}$. According to the Hahn-Banach theorem, there exist $\varphi_0 \in \mathcal{C}(\mathbb{S}^0)$ and $r \in \mathbb{R}$ such that for all $\mu \in \bar{\mathcal{C}}$, $\langle \mu, \varphi_0\rangle > r > \langle \lambda_0, \varphi_0\rangle$. In particular for all $w \in W_0^{1,2}((-1/2,1/2))$ we have $\langle \mu_w, \varphi_0\rangle > r > \langle \lambda_0, \varphi_0\rangle$; from the characterization of the convexification φ^{**} of φ, this implies

$$\varphi_0^{**}([v](0)) = \inf_{w \in W_0^{1,2}((-1/2,1/2))} \langle \mu_w, \varphi_0\rangle \geq r > \langle \lambda_0, \varphi_0\rangle. \tag{3.12}$$

The first inequality forces φ_0^{**} to be finite; since $\lambda_0 \in \mathcal{H}$ one has $\langle \lambda_0, \varphi_0\rangle \geq \varphi_0^{**}([v](0))$. Therefore (3.12) involves a contradiction.

Step 2. Since $\pi(0)\mu_0$ belongs to $\mathcal{H}$ and, from Step 1, $\mathcal{C}$ is dense in $\mathcal{H}$, there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in $W_0^{1,2}((-1/2,1/2))$ such that

$$\mu_{w_n} \overset{*}{\rightharpoonup} \mu = \pi(0)\mu_0$$

in $\mathbb{M}^+(\mathbb{S}^0)$ when $n \to +\infty$. Consequently

$$\mu_n := \mu_{w_n} \otimes \delta_0 \overset{*}{\rightharpoonup} \mu = \pi(0)\mu_0 \otimes \delta_0 \tag{3.13}$$

for the weak convergence of $\sigma(\mathbb{M}^+([-1/2,1/2] \times \mathbb{S}^0))$. Taking $\theta \equiv 1$ and $\varphi \equiv 1$ (*i.e.* $\tilde{\varphi} = |.|^2$) we also obtain

$$\|\mu_n\| \to \|\mu\| = \pi(0);$$

this implies that for n large enough, say $n \geq n_0$, μ_n belongs to the ball $B(0, \pi(0)+1)$ of $\mathbb{M}^+([-1/2,1/2] \times \mathbb{S}^0)$. We take $n \geq n_0$ and we are going to approximate the measure μ_n by a measure of the form $\delta_{\dot{v}_{n,\varepsilon}/|\dot{v}_{n,\varepsilon}|}(x) \otimes \varepsilon \mathbb{1}_{B_\varepsilon} |\dot{v}_{n,\varepsilon}|^2\ \mathrm{d}x$.

Consider the $W_0^{1,2}((-1/2,1/2))$ function $v_{n,\varepsilon}$ defined by $v_{n,\varepsilon}(x) := R_\varepsilon v + w_n(x/\varepsilon)$, where

$$R_\varepsilon v(x) = \begin{cases} \frac{x}{\varepsilon}\left[v(\frac{\varepsilon}{2}) - v(-\frac{\varepsilon}{2})\right] + \frac{1}{2}\left[v(\frac{\varepsilon}{2}) + v(-\frac{\varepsilon}{2})\right], & \text{if } x \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}), \\ v, & \text{otherwise,} \end{cases}$$

and set $\mu_{n,\varepsilon} := \delta_{\dot{v}_{n,\varepsilon}/|\dot{v}_{n,\varepsilon}|}(x) \otimes \varepsilon \mathbb{1}_{B_\varepsilon} |\dot{v}_{n,\varepsilon}|^2 \, \mathrm{d}x$. A straightforward calculation and a change of scale give, for all $\theta \in \mathcal{C}([-1/2, 1/2])$ and all $\varphi \in \mathcal{C}(\mathbb{S}^0)$,

$$\begin{aligned}
\langle \mu_{n,\varepsilon}, \theta \otimes \varphi \rangle &= \varepsilon \int_{-\varepsilon/2}^{\varepsilon/2} \theta(x) \varphi\left(\frac{1}{\varepsilon}\left(v\left(\frac{\varepsilon}{2}\right) - v\left(\frac{-\varepsilon}{2}\right) + \dot{w}_n\left(\frac{x}{\varepsilon}\right)\right)\right) \, \mathrm{d}x \\
&= \fint_{(-\varepsilon/2, \varepsilon/2)} \theta(x) \varphi\left(v\left(\frac{\varepsilon}{2}\right) - v\left(\frac{-\varepsilon}{2}\right) + \dot{w}_n\left(\frac{x}{\varepsilon}\right)\right) \, \mathrm{d}x \\
&= \int_{-1/2}^{1/2} \theta(\varepsilon y) \varphi\left(v\left(\frac{\varepsilon}{2}\right) - v\left(\frac{-\varepsilon}{2}\right) + \dot{w}_n(y)\right) \, \mathrm{d}y \\
&\to \theta(0) \int_{-1/2}^{1/2} \varphi([v](0) + \dot{w}_n(y)) \, \mathrm{d}y \\
&= \langle \mu_{w_n} \otimes \delta_0, \theta \otimes \varphi \rangle
\end{aligned}$$

when $\varepsilon \to 0$, that is

$$\mu_{n,\varepsilon} \overset{*}{\rightharpoonup} \mu_{w_n} \otimes \delta_0 = \mu_n. \tag{3.14}$$

We are going to show, using a standard diagonalization argument, that a sequence constructed from $v_{n,\varepsilon}$ generates (v, μ). By a straightforward calculation, we get, taking $\theta \equiv 1$ and $\varphi \equiv 1$ as test functions

$$\begin{aligned}
\|\mu_{n,\varepsilon}\| &= \int_{-1/2}^{1/2} \left| v\left(\frac{\varepsilon}{2}\right) - v\left(\frac{-\varepsilon}{2}\right) + \dot{w}_n(y) \right|^2 \, \mathrm{d}y \\
&\leq 2 \int_{-1/2}^{1/2} \left| v\left(\frac{\varepsilon}{2}\right) - v\left(\frac{-\varepsilon}{2}\right) - [v](0) \right|^2 \, \mathrm{d}y + 2\|\mu_n\| \\
&\leq 2 \int_{-1/2}^{1/2} \left| v\left(\frac{\varepsilon}{2}\right) - v\left(\frac{-\varepsilon}{2}\right) - [v](0) \right|^2 \, \mathrm{d}y + 2(1 + \pi(0)).
\end{aligned}$$

This implies, for ε small enough, say $\varepsilon < \varepsilon_0$ where ε_0 does not depend on n, that $\mu_{n,\varepsilon}$ belongs to the closed ball $\overline{B(0, 2(\pi(0) + 2)}$ of $\mathbb{M}^+([-1/2, 1/2] \times \mathbb{S}^0))$. Since the closed balls of $\mathbb{M}^+([-1/2, 1/2] \times \mathbb{S}^0)$ are metrizable, combining (3.13), (3.14) and using a standard diagonalization argument, there exists a map $\varepsilon \mapsto n(\varepsilon)$ such that

$$\mu_\varepsilon := \delta_{\frac{\dot{v}_{n(\varepsilon),\varepsilon}}{|\dot{v}_{n(\varepsilon),\varepsilon}|}(x)} \otimes \varepsilon \mathbb{1}_{B_\varepsilon} |\dot{v}_{n(\varepsilon),\varepsilon}|^2 \, \mathrm{d}x \overset{*}{\rightharpoonup} \mu \tag{3.15}$$

when $\varepsilon \to 0$. Set $v_\varepsilon := v_{n(\varepsilon),\varepsilon}$. Condition (2.3) follows straightforwardly from (3.15) (take $\theta \equiv 1$ and $\varphi \equiv 1$ as test functions). To prove that v_ε generates (v, μ), it remains to establish that $v_\varepsilon \to v$ in $L^2((-1, 1))$. Noticing that $v_\varepsilon = v$ on $(-1, 1) \setminus (-\varepsilon/2, \varepsilon/2)$, and by using Poincaré's inequality and condition (2.3) satisfied by v_ε, we have

$$\begin{aligned}
\int_{(-1,1)} |v_\varepsilon - v|^2 \, \mathrm{d}x &= \int_{(-\varepsilon/2, \varepsilon/2)} |v_\varepsilon - v|^2 \, \mathrm{d}x \\
&\leq \varepsilon^2 \int_{(-\varepsilon/2, \varepsilon/2)} |\dot{v}_\varepsilon - \dot{v}|^2 \, \mathrm{d}x \\
&\leq C\varepsilon
\end{aligned}$$

where C is a non negative constant independent on ε.

Proof of the general case. Let (v,μ) in $W^{1,2}_{\partial\Omega}(\Omega\setminus\Sigma,\mathbb{R}^m)\otimes\mathbb{M}^+(\Omega\times\mathbb{S}^{m-1})$ satisfying (3.1) with π concentrated in $\bar S$. We want to prove that (v,μ) belongs to $\mathcal{YC}$. The proof is very close to the previous one but with an additional step.

Step 1. Let $a\in\mathbb{R}^m$ be fixed, and consider the following subset of $\mathbb{M}^+(\mathbb{S}^{m-1})$:

$$\mathcal{H}(a) := \left\{\lambda\in\mathbb{M}(\mathbb{S}^{m-1}) : \forall\varphi\in\mathcal{C}(\mathbb{S}^{m-1}),\ \text{s.t. } \varphi^{**}>-\infty,\ \int_{\mathbb{S}^{m-1}}\varphi\,\mathrm{d}\lambda\geq\varphi^{**}(a)\right\}. \tag{3.16}$$

For all $w\in W^{1,2}_0(I,\mathbb{R}^m)$ where $I=(-\frac12,\frac12)$, consider the measure μ_w in $\mathbb{M}(\mathbb{S}^{m-1})$ which acts on all $\varphi\in\mathcal{C}(\mathbb{S}^{m-1})$ as follows:

$$\langle\mu_w,\varphi\rangle := \int_I\varphi\left(a+\frac{\mathrm{d}w}{\mathrm{d}x}\right)\,\mathrm{d}x;$$

let us introduce the following subset $\mathcal{C}(a)$ of $\mathbb{M}(\mathbb{S}^{m-1})$:

$$\mathcal{C}(a) := \left\{\mu_w : w\in W^{1,2}_0(I,\mathbb{R}^m)\right\}.$$

Using the same arguments of the first step in the one dimensional case ($N=1$), we obtain that $\mathcal{C}(a)$ is a dense convex subset of $\mathcal{H}(a)$ for the $\sigma(\mathbb{M}(\mathbb{S}^{m-1}),\mathcal{C}(\mathbb{S}^{m-1}))$ topology.

Step 2. Let x_0 be a fixed element of $\bar S$ and $\hat Q(x_0)$ a fixed open cube of $\mathbb{R}^{N-1}$ centered at x_0. Given $m\in\mathcal{H}(a)$, according to the first step, there exists a sequence $(w_n)_{n\in\mathbb{N}}$ of functions in $W^{1,2}_0(I,\mathbb{R}^m)$ such that

$$\mu_{w_n}\overset{*}{\rightharpoonup} m$$

in $\mathbb{M}^+(\mathbb{S}^{m-1})$, when $n\to+\infty$. Consequently

$$\mu_n := \mu_{w_n}\otimes\frac{\mathcal{H}^{N-1}\lfloor\hat Q(x_0)\cap S}{|\hat Q(x_0)|}\overset{*}{\rightharpoonup} m\otimes\frac{\mathcal{H}^{N-1}\lfloor\hat Q(x_0)\cap S}{|\hat Q(x_0)|} \tag{3.17}$$

in $\mathbb{M}(\bar\Omega\times\mathbb{S}^{m-1})$. For each w_n we define the function $\xi_{n,\varepsilon}$ on B_ε depending only of the variable x_N by

$$\xi_{n,\varepsilon}(x) := |\hat Q(x_0)|^{\frac{-1}{2}}\left[\frac{x_N}{\varepsilon}a+w_n\left(\frac{x_N}{\varepsilon}\right)\right]. \tag{3.18}$$

A straightforward calculation gives

$$\lim_{\varepsilon\to0}\varepsilon\int_{\hat Q(x_0)\times(\frac{-\varepsilon}{2},\frac{\varepsilon}{2})\cap\bar\Omega}\theta(x)\varphi\left(\frac{\partial\xi_{n,\varepsilon}}{\partial\nu}\right)\mathrm{d}x = \left(⨍_{\hat Q(x_0)}\theta(\hat x,0)\mathrm{d}\mathcal{H}^{N-1}\lfloor S\right)\left(\int_I\varphi\left(a+\frac{\mathrm{d}w_n}{\mathrm{d}\nu}\right)\mathrm{d}x\right)$$

for all $\varphi\in\mathcal{C}(\mathbb{S}^{m-1})$ and all $\theta\in\mathcal{C}(\bar\Omega)$, so that

$$\mu_{n,\varepsilon} := \varepsilon\mathbb{1}_{\hat Q(x_0)\times(\frac{-\varepsilon}{2},\frac{\varepsilon}{2})}\delta_{\frac{\partial\xi_{n,\varepsilon}}{\partial\nu}/\left|\frac{\partial\xi_{n,\varepsilon}}{\partial\nu}\right|(x)}\otimes|\frac{\partial\xi_{n,\varepsilon}}{\partial\nu}|^2\mathrm{d}x\overset{*}{\rightharpoonup}\mu_{w_n}\otimes\frac{\mathcal{H}^{N-1}\lfloor\hat Q(x_0)\cap S}{|\hat Q(x_0)|} \tag{3.19}$$

in $\mathbb{M}(\bar\Omega\times\mathbb{S}^{m-1})$ when $\varepsilon\to0$. Combining (3.17), (3.19) and using a diagonalization argument like in Step 2 of the 1-dimensional case $N=1$, there exists a map $n\mapsto n(\varepsilon)$ and ξ_ε such that

$$\begin{aligned}&\xi_\varepsilon := \xi_{n(\varepsilon),\varepsilon} = |\hat Q(x_0)|^{\frac{-1}{2}}\left[\frac{x_N}{\varepsilon}a+w_{n(\varepsilon)}\left(\frac{x_N}{\varepsilon}\right)\right],\ w_{n(\varepsilon)}\in\mathcal{C}^1_0(I,\mathbb{R}^m),\\ &\varepsilon\mathbb{1}_{\hat Q(x_0)\times(\frac{-\varepsilon}{2},\frac{\varepsilon}{2})}\delta_{\frac{\partial\xi_\varepsilon}{\partial\nu}/|\frac{\partial\xi_\varepsilon}{\partial\nu}|(x)}\otimes\left|\frac{\partial\xi_\varepsilon}{\partial\nu}\right|^2\mathrm{d}x\overset{*}{\rightharpoonup} m\otimes\frac{\mathcal{H}^{N-1}\lfloor\hat Q(x_0)\cap S}{|\hat Q(x_0)|}\end{aligned} \tag{3.20}$$

in $\mathbb{M}(\bar\Omega\times\mathbb{S}^{m-1})$.

Step 3. Let (v,μ) in $\mathcal{YC}$: we are going to construct a sequence $(v_\varepsilon)_{\varepsilon>0}$ which generates (v,μ). The sequence $(v_\varepsilon)_{\varepsilon>0}$ is composed by functions in $W_0^{1,2}(\Omega,\mathbb{R}^m)$ whose restriction on $\Omega\setminus\Sigma_\varepsilon$ is equal to v, it converges to v in $L^2(\Omega,\mathbb{R}^m)$, it satisfies (2.3) and

$$\delta_{\frac{\partial v_\varepsilon}{\partial \nu}/\left|\frac{\partial v_\varepsilon}{\partial \nu}\right|(x)}\otimes\varepsilon\mathbb{1}_{B_\varepsilon}\left|\frac{\partial v_\varepsilon}{\partial \nu}\right|^2\,\mathrm{d}x\overset{*}{\rightharpoonup}\mu \tag{3.21}$$

in $\mathbb{M}(\bar\Omega\times\mathbb{S}^{m-1})$.

In what follows, we continue to write the measure $\bar\pi$ on $\mathbb{R}^N$ defined for all Borel set E of $\mathbb{R}^N$ by $\bar\pi(E):=\pi(\bar S\cap E)$ as π. According to the Vitali covering theorem (see [2], Cor. 2.8.15) there exists a finite family $(\hat Q_{i,k})_{i\in I_k}$ of disjoint closed squares in $\mathbb{R}^{N-1}$, centered at $x_{i,k}\in\bar S$, with diameter less than $1/k$, $k\in\mathbb{N}^*$ and satisfying

$$\pi\left(\bar S\setminus\bigcup_{i\in I_k}\hat Q_{i,k}\right)<\frac{1}{k}\cdot \tag{3.22}$$

One may assume that $\hat Q_{i,k}\subset\Sigma$. Define the measure $\mu_{i,k}$ of $\mathbb{M}(\mathbb{S}^{m-1})$ by

$$\mu_{i,k}:=\int_{\hat Q_{i,k}}\mu_x\,\mathrm{d}\pi.$$

From (3.1) and Jensen's inequality, for all $\varphi\in\mathcal{C}(\mathbb{S}^{m-1})$ such that $\varphi^{**}>-\infty$ we have

$$\begin{aligned}\langle\mu_{i,k},\varphi\rangle &= \int_{\hat Q_{i,k}}\int_{\mathbb{S}^{m-1}}\varphi(\zeta)\,\mathrm{d}\mu_x\mathrm{d}\pi\\ &\geq \int_{\hat Q_{i,k}}\varphi^{**}([v](x))\,\mathrm{d}\mathcal{H}^{N-1}\lfloor S\\ &\geq \varphi^{**}\left(|\hat Q_{i,k}|^{\frac12}\fint_{\hat Q_{i,k}}[v](x)\,\mathrm{d}\mathcal{H}^{N-1}\lfloor S\right).\end{aligned} \tag{3.23}$$

Let us set

$$a_{i,k}=|\hat Q_{i,k}|^{\frac12}\fint_{\hat Q_{i,k}}[v]\,\mathrm{d}\mathcal{H}^{N-1}\lfloor S. \tag{3.24}$$

From (3.23), it follows that the measure $\mu_{i,k}$ belongs to $\mathcal{H}(a_{i,k})$ (see definition (3.16)). According to (3.20), there exists a sequence $(\xi_{i,k,\varepsilon})_{\varepsilon>0}$ in $W^{1,2}(B_\varepsilon,\mathbb{R}^m)$, of the form

$$\xi_{i,k,\varepsilon}(x)=\frac{x_N}{\varepsilon}\fint_{\hat Q_{i,k}}[v]\,\mathrm{d}\mathcal{H}^{N-1}\lfloor S+\tilde\xi_{i,k,\varepsilon}(x_N),$$

with $\tilde\xi_{i,k,\varepsilon}$ in $W_0^{1,2}(I,\mathbb{R}^m)$ and with the property that,

$$\mu_{i,k,\varepsilon}:=\varepsilon\mathbb{1}_{\hat Q_{i,k}\times(\frac{-\varepsilon}{2},\frac{\varepsilon}{2})}\delta_{\frac{\partial\xi_{i,k,\varepsilon}}{\partial\nu}/\left|\frac{\partial\xi_{i,k,\varepsilon}}{\partial\nu}\right|(x)}\otimes\left|\frac{\partial\xi_{i,k,\varepsilon}}{\partial\nu}\right|^2\mathrm{d}x\overset{*}{\rightharpoonup}\mu_{i,k}\otimes\frac{\mathcal{H}^{N-1}\lfloor\hat Q_{i,k}\cap S}{|\hat Q_{i,k}|} \tag{3.25}$$

in $\mathbb{M}(\bar\Omega\times\mathbb{S}^{m-1})$ when $\varepsilon\to0$, and $\|\mu_{i,k,\varepsilon}\|\leq2\pi(\hat Q_{i,k})$. For all $\theta\in\mathcal{C}(\bar\Omega)$ and all $\varphi\in\mathcal{C}(\mathbb{S}^{m-1})$, (3.25) yields

$$\begin{aligned}\lim_{\varepsilon\to0}\sum_{i\in I_k}\varepsilon\int_{\hat Q_{i,k}\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})\cap\bar\Omega}\theta(x)\varphi\left(\frac{\partial\xi_{i,k,\varepsilon}}{\partial\nu}\right)\,\mathrm{d}x &= \sum_{i\in I_k}\fint_{\hat Q_{i,k}}\theta(x)\,\mathrm{d}\mathcal{H}^{N-1}\lfloor S\int_{\mathbb{S}^{m-1}}\varphi(\zeta)\,\mathrm{d}\mu_{i,k}\\ &= \sum_{i\in I_k}\fint_{\hat Q_{i,k}}\theta(x)\,\mathrm{d}\mathcal{H}^{N-1}\lfloor S\int_{\hat Q_{i,k}}\int_{\mathbb{S}^{m-1}}\varphi(\zeta)\,\mathrm{d}\mu_x\mathrm{d}\pi.\end{aligned}$$

When θ is a Lipschitz-continuous function, one easily deduces from (3.22) that

$$\lim_{k\to+\infty}\lim_{\varepsilon\to 0}\sum_{i\in I_k}\varepsilon\int_{\hat{Q}_{i,k}\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})\cap\bar{\Omega}}\theta(x)\varphi\left(\frac{\partial\xi_{i,k,\varepsilon}}{\partial\nu}\right)\,\mathrm{d}x=\int_{\bar{S}}\int_{\mathbb{S}^{m-1}}\theta\varphi\,\mathrm{d}\mu. \tag{3.26}$$

In the general case, when $\theta\in\mathcal{C}(\bar{\Omega})$, the same conclusion holds by an approximation argument.

Set $\xi_{k,\varepsilon}(x):=\sum_{i\in I_k}\xi_{i,k,\varepsilon}(x_N)\,\mathbb{1}_{\hat{Q}_{i,k}}(\hat{x})$. In order that the value of $\xi_{k,\varepsilon}$ agrees with that of v on $\Omega\setminus\Sigma_\varepsilon$, we modify $\xi_{i,k,\varepsilon}$ into the function $v_{i,k,\varepsilon}$ given by

$$v_{i,k,\varepsilon}(x):=R_\varepsilon v(x)+\tilde{\xi}_{i,k,\varepsilon}(x_N)\,\mathbb{1}_{\hat{Q}_{i,k}}(\hat{x})$$

where $R_\varepsilon: W^{1,2}(\Omega\setminus\Sigma,\mathbb{R}^m)\to W^{1,2}(\Omega,\mathbb{R}^m)$ is the operator defined in (3.2).

We claim that (3.26) holds when we replace $\xi_{i,k,\varepsilon}$ by $v_{i,k,\varepsilon}$. Indeed one may assume $\varphi\in\mathrm{Lip}(\mathbb{S}^{m-1})$ and, from the Lipschitz property (2.7), it is easily seen that we are reduced to establish

$$\lim_{k\to+\infty}\lim_{\varepsilon\to 0}\sum_{i\in I_k}\varepsilon\int_{\hat{Q}_{i,k}\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})\cap\bar{\Omega}}\left|\frac{1}{\varepsilon}\fint_{\hat{Q}_{i,k}}[v]\mathrm{d}\mathcal{H}^{N-1}\lfloor S-\frac{\partial R_\varepsilon v}{\partial\nu}\right|^2\mathrm{d}x=0:$$

this can be straightforward checked. Thus (3.26) becomes

$$\lim_{k\to+\infty}\lim_{\varepsilon\to 0}\sum_{i\in I_k}\varepsilon\int_{\hat{Q}_{i,k}\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})\cap\bar{\Omega}}\theta(x)\varphi\left(\frac{\partial v_{i,k,\varepsilon}}{\partial\nu}\right)\,\mathrm{d}x=\int_{\bar{S}}\int_{\mathbb{S}^{m-1}}\theta\varphi\,\mathrm{d}\mu. \tag{3.27}$$

It is worth noticing that $v_{k,\varepsilon}:=\sum_{i\in I_k}v_{i,k,\varepsilon}\,\mathbb{1}_{\hat{Q}_{i,k}}$ is not a Sobolev function. Take now a partition of unity $(\varphi_{i,\delta})_{i\in I_k}$ subordinded to the family $(\hat{Q}_{i,k})_{i\in I_k}$, satisfying $\lim_{\delta\to 0}\int_{\hat{Q}_{i,k}}|\varphi_\delta-1|^2\,\mathrm{d}\hat{x}=0$, and set $v_{k,\delta,\varepsilon}:=\sum_{i\in I_k}v_{i,k,\varepsilon}\,\varphi_{i,\delta}$. Taking into account that φ_δ depends only on $\hat{x}$, it is readily seen that (3.27) yields

$$\lim_{k\to+\infty}\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\varepsilon\int_{B_\varepsilon}\theta(x)\varphi(\frac{\partial v_{k,\delta,\varepsilon}}{\partial\nu})\,\mathrm{d}x=\int_{\bar{S}}\int_{\mathbb{S}^{m-1}}\theta\varphi\,\mathrm{d}\mu.$$

Note that $v_{k,\delta,\varepsilon}$ belongs to $W_0^{1,2}(\Omega,\mathbb{R}^m)$. Now, using a standard diagonalization argument (all the considered measures are bounded by $2\pi(\bar{S})$ in $\mathbb{M}(\bar{\Omega}\times\mathbb{S}^{m-1})$), we infer

$$\delta_{\frac{\partial v_\varepsilon}{\partial\nu}/\left|\frac{\partial v_\varepsilon}{\partial\nu}\right|\,(x)}\otimes\varepsilon\mathbb{1}_{B_\varepsilon}|\frac{\partial v_\varepsilon}{\partial\nu}|^2\,\mathrm{d}x\overset{*}{\rightharpoonup}\mu$$

where $v_\varepsilon:=v_{k(\varepsilon),\delta(\varepsilon),\varepsilon}$ for some map $\varepsilon\mapsto(k(\varepsilon),\delta(\varepsilon))$. We have thus proved (3.21). Estimate (2.3) follows straightforward from (3.21) (take $\theta\equiv 1$ and $\varphi\equiv 1$ as a test function). At last we have

$$\begin{aligned}\int_\Omega|v-v_\varepsilon|^2\,\mathrm{d}x&=\int_{\Sigma_\varepsilon}|v-v_\varepsilon|^2\,\mathrm{d}x\\&\leq\varepsilon^2\int_{\Sigma_\varepsilon}\left|\frac{\partial(v-v_\varepsilon)}{\partial\nu}\right|^2\,\mathrm{d}x\\&\leq C\varepsilon\end{aligned}$$

where we have used Poincaré's inequality, and (2.3) in the two last inequalities. $\square$

Corollary 3.6 (characterization of $\mathcal{YC}$ in the scalar case $m=1$)**.** *Assume that* $m=1$. *Then* $(v, \mu = (a(x)\delta_1 + b(x)\delta_{-1}) \otimes \pi)$ *belongs to* $\mathcal{YC}$ *if and only if* $v \in W^{1,2}_{\partial\Omega}(\Omega \setminus \Sigma)$, π *is concentrated on* $\bar{S}$ *and*

$$\frac{\mathrm{d}\pi}{\mathrm{d}\mathcal{H}^{N-1}\lfloor S}(x)\big(a(x)c + b(x)d\big) \geq \varphi([v](x)) \text{ for } \mathcal{H}^{N-1}\lfloor S \text{ a.e. } x \text{ and for all } (c,d) \in \mathbb{R}^+ \times \mathbb{R}^+$$

where $\varphi(\zeta) = \begin{cases} c\zeta^2 & \text{if } \zeta \geq 0, \\ d\zeta^2 & \text{if } \zeta \leq 0. \end{cases}$ *Consequently the following estimates hold:*

$$\begin{aligned} a(x) &\geq \frac{|[v](x)|^2}{\frac{\mathrm{d}\pi}{\mathrm{d}\mathcal{H}^{N-1}\lfloor S}(x)} \text{ for } \mathcal{H}^{N-1}\lfloor S \text{ a.e. } x \text{ such that } [v](x) > 0 \\ b(x) &\geq \frac{|[v](x)|^2}{\frac{\mathrm{d}\pi}{\mathrm{d}\mathcal{H}^{N-1}\lfloor S}(x)} \text{ for } \mathcal{H}^{N-1}\lfloor S \text{ a.e. } x \text{ such that } [v](x) < 0. \end{aligned} \tag{3.28}$$

Proof. Since $m=1$ we have $\mathbb{S}^{m-1} = \{-1, 1\}$, $\mu_x = a(x)\delta_1 + b(x)\delta_{-1}$ with $0 \leq a(x) \leq 1$, $0 \leq b(x) \leq 1$, and $a(x) + b(x) = 1$ for $\mathcal{H}^{N-1}\lfloor S$ a.e. x. Moreover $\varphi \in \mathcal{C}(\mathbb{S}^{m-1})$ fulfils the condition $\varphi^{**} > -\infty$ if and only if its extension is of the form

$$\varphi(\zeta) = \begin{cases} c\zeta^2, & \text{if } \zeta \geq 0 \\ d\zeta^2, & \text{if } \zeta \leq 0 \end{cases}$$

where c and d are two non-negative real numbers. Note that in this case $\varphi^{**} = \varphi$. Then (3.1) is equivalent to

$$\frac{\mathrm{d}\pi}{\mathrm{d}\mathcal{H}^{N-1}\lfloor S}(x)\big(a(x)c + b(x)d\big) \geq \varphi([v](x)) \text{ for } \mathcal{H}^{N-1}\lfloor S \text{ a.e. } x.$$

Estimates (3.28) follow easily, choosing $d=0$ or $c=0$. □

4. Computations in the 1-dimensional case

Let us consider the elementary Dirichlet problem described in the introduction:

$$\begin{cases} (\sigma_\varepsilon(x)u')' = \dfrac{1}{\sqrt{\varepsilon}} \displaystyle\sum_{k=0}^{m} a_k \delta_{t_k^\varepsilon} \\ u(-1) = u(1) = 0, \end{cases}$$

where $a_k \in \mathbb{R}^*$, $(t_k^\varepsilon)_{k=0,\ldots,m}$ is a non decreasing family of numbers in $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ with $t_0^\varepsilon = -\frac{\varepsilon}{2}$, $t_m^\varepsilon = \frac{\varepsilon}{2}$, $\lim_{\varepsilon\to 0} \frac{t_{k+1}^\varepsilon - t_k^\varepsilon}{\varepsilon} = \frac{1}{m} := l$ for $k = 0, \ldots, m-1$, and σ_ε is given by

$$\sigma_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} \text{ if } x \in (-1,1) \setminus (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \\ 1 \text{ if } x \in \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right). \end{cases}$$

Clearly $\bar{u}_\varepsilon'$ is of the form

$$\bar{u}_\varepsilon'(x) = \begin{cases} \varepsilon c_\varepsilon, & \text{if } x \in (-1, -\frac{\varepsilon}{2}) \\ c_\varepsilon + \frac{s_k}{\sqrt{\varepsilon}}, & \text{if } x \in (t_k^\varepsilon, t_{k+1}^\varepsilon),\ k = 0, \ldots, m-1 \\ \varepsilon\left(c_\varepsilon + \frac{s_m}{\sqrt{\varepsilon}}\right), & \text{if } x \in (\frac{\varepsilon}{2}, 1), \end{cases}$$

where $s_k := \sum_{i=0}^{k} a_i$ and c_ε is a constant which can be computed from the boundary conditions.

Let us show that $\bar{u}'_\varepsilon$ has a concentration phenomenon. To this end we set $\bar{v}_\varepsilon = \frac{\bar{u}_\varepsilon}{\sqrt{\varepsilon}}$ and compute the weak limit of the measure $\bar{\pi}_\varepsilon := \mathbb{1}_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})}\varepsilon|\bar{v}'_\varepsilon|^2\ \mathrm{d}x$ in $\mathbb{M}([-1,1])$. Let $\theta \in \mathcal{C}([-1,1])$; we have

$$\begin{aligned}\int_{-1}^{1}\theta(x)\mathbb{1}_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})}\varepsilon|\bar{v}'_\varepsilon|^2\ \mathrm{d}x &= \sum_{k=0}^{m-1}\varepsilon\left|\frac{c_\varepsilon\sqrt{\varepsilon}+s_k}{\varepsilon}\right|^2\int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon}\theta(x)\ \mathrm{d}x\\ &= \sum_{k=0}^{m-1}|c_\varepsilon\sqrt{\varepsilon}+s_k|^2\frac{t_{k+1}^\varepsilon-t_k^\varepsilon}{\varepsilon}\fint_{t_k^\varepsilon}^{t_{k+1}^\varepsilon}\theta(x)\ \mathrm{d}x.\end{aligned}$$

Taking into account that $\bar{v}_\varepsilon$ is a solution of

$$\min\left\{\frac{1}{2}\int_{(-1,1)\setminus(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})}|v'|^2\ \mathrm{d}x+\frac{\varepsilon}{2}\int_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})}|v'|^2\ \mathrm{d}x+\sum_{k=0}^{m}a_k v(t_k^\varepsilon) : v\in W_0^{1,2}((-1,1))\right\},$$

and using point (c) of Lemma 2.1, $c_e\sqrt{\varepsilon}$ is bounded and tends to some constant L. This implies that

$$\lim_{\varepsilon\to 0}\int_{-1}^{1}\theta(x)\mathbb{1}_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})}\varepsilon|\bar{v}'_\varepsilon|^2\ \mathrm{d}x = l\sum_{k=0}^{m-1}(L+s_k)^2\ \theta(0).$$

Thus $\bar{\pi}_\varepsilon$ weakly converges to the measure $\bar{\pi} = l\displaystyle\sum_{k=0}^{m-1}(L+s_k)^2\ \delta_0$ and $\bar{u}'_\varepsilon$ presents a concentration phenomenon of intensity $l\sum_{k=0}^{m-1}(L+s_k)^2$ at 0.

We are going to compute the weak limit $\bar{\mu}$ of the measure $\bar{\mu}_\varepsilon := \delta_{\frac{\bar{v}'_\varepsilon}{|\bar{v}'_\varepsilon|}(x)}\otimes\mathbb{1}_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})}\varepsilon|\bar{v}'_\varepsilon|^2\mathrm{d}x$. We assume for instance that $L\geq 0$. Let φ be any 2-homogeneous continuous function on $\mathbb{R}$ and θ any function in $\mathcal{C}([-1,1])$. We have for small ε

$$\begin{aligned}\varepsilon\int_{-1}^{1}\mathbb{1}_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})}\varphi(\bar{v}'_\varepsilon)\theta(x)\ \mathrm{d}x &= \sum_{k=0,\ldots,m-1:s_k<0}\left|\frac{c_\varepsilon\sqrt{\varepsilon}+s_k}{\varepsilon}\right|^2\varepsilon\varphi(-1)\int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon}\theta(x)\ \mathrm{d}x\\ &\quad+\sum_{k=0,\ldots,m-1:s_k>0}\left|\frac{c_\varepsilon\sqrt{\varepsilon}+s_k}{\varepsilon}\right|^2\varepsilon\varphi(1)\int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon}\theta(x)\ \mathrm{d}x\\ &\quad+\sum_{k=0,\ldots,m-1:s_k=0}\frac{1}{\varepsilon}\varphi(c_\varepsilon\sqrt{\varepsilon})\int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon}\theta(x)\ \mathrm{d}x\\ &= \sum_{k=0,\ldots,m-1:s_k<0}|c_\varepsilon\sqrt{\varepsilon}+s_k|^2\varphi(-1)\frac{t_{k+1}^\varepsilon-t_k^\varepsilon}{\varepsilon}\fint_{t_k^\varepsilon}^{t_{k+1}^\varepsilon}\theta(x)\ \mathrm{d}x\\ &\quad+\sum_{k=0,\ldots,m-1:s_k>0}|c_\varepsilon\sqrt{\varepsilon}+s_k|^2\varphi(1)\frac{t_{k+1}^\varepsilon-t_k^\varepsilon}{\varepsilon}\fint_{t_k^\varepsilon}^{t_{k+1}^\varepsilon}\theta(x)\ \mathrm{d}x\\ &\quad+\sum_{k=0,\ldots,m-1:s_k=0}\varphi(c_\varepsilon\sqrt{\varepsilon})\frac{t_{k+1}^\varepsilon-t_k^\varepsilon}{\varepsilon}\fint_{t_k^\varepsilon}^{t_{k+1}^\varepsilon}\theta(x)\ \mathrm{d}x.\end{aligned}$$

This sequence tends to

$$\left(\sum_{k=0,\ldots,m-1: s_k<0} (L+s_k)^2 \varphi(-1) + \sum_{k=0,\ldots,m-1: s_k\geq 0} (L+s_k)^2 \varphi(1) \right) \; l\theta(0)$$

that we can write as

$$(a\varphi(1) + b\varphi(-1))\, l \sum_{k=0}^{m-1} (L+s_k)^2 \theta(0),$$

where

$$a = \frac{\displaystyle\sum_{k=0,\ldots,m-1: s_k\geq 0} (L+s_k)^2}{\displaystyle\sum_{k=0}^{m-1} (L+s_k)^2}, \qquad b = \frac{\displaystyle\sum_{k=0,\ldots,m-1: s_k<0} (L+s_k)^2}{\displaystyle\sum_{k=0}^{m-1} (L+s_k)^2}\cdot \tag{4.1}$$

This proves that the measure $\bar{\mu}_\varepsilon := \delta_{\frac{\bar{v}'_\varepsilon}{|\bar{v}'_\varepsilon|}(x)} \otimes \mathbb{1}_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})} \varepsilon |\bar{v}'_\varepsilon|^2 \mathrm{d}x$ weakly converges to the measure $\bar{\mu} = (a\delta_1 + b\delta_{-1}) \otimes \bar{\pi}$, where $\bar{\pi} = l \sum_{k=0}^{m-1} (L+s_k)^2 \delta_0$ and a and b are given by (4.1).

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