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GAIN-LOSS PRICING UNDER AMBIGUITY OF MEASURE [*]

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Abstract. Motivated by the observation that the gain-loss criterion, while offering economically meaningful prices of contingent claims, is sensitive to the reference measure governing the underlying stock price process (a situation referred to as ambiguity of measure), we propose a gain-loss pricing model robust to shifts in the reference measure. Using a dual representation property of polyhedral risk measures we obtain a one-step, gain-loss criterion based theorem of asset pricing under ambiguity of measure, and illustrate its use.

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1. Introduction and background

The problem of pricing financial instruments called "contingent claims" is one of the fundamental problems of financial mathematics. When a financial market is not complete, and yet does not allow arbitrage opportunities (or free lunches), it is well known that there exists a set of "risk neutral" probability measures that make the (discounted) prices of traded instruments martingales. A noticeable feature of the set of risk neutral measures is that the value of the cheapest portfolio to dominate the pay-off at maturity of a contingent claim coincides with the maximum expected value of the (discounted) pay-off of the claim with respect to this set. This value, called the super-hedging price, allows the seller to assemble a portfolio that achieves a value at least as large as the pay-off to the claim holder at the maturity date of the claim in all non-negligible events. (If the claim is attainable, then the smallest price to super-hedge is the hedging price, and its expected value does not depend on the chosen risk neutral measure, so the previous statement still applies.) The super-hedging price is the natural price to be asked by the writer of a contingent claim and, together with the bid price obtained by considering the analogous problem from the point of view of the buyer, it constitutes an interval which is sometimes called the "no-arbitrage price interval" for the claim in question.

Keywords and phrases. Contingent claim, pricing, gain-loss ratio, hedging, martingales, stochastic programming, risk measures.

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A writer may nevertheless be induced for various reasons not to ask the whole super-hedging price to sell a claim with pay-off F_T; see *e.g.*, chapters 7 and 8 of [16] for a discussion and examples showing that the super-hedging price may be too high. In such a case, he/she will not be able to set up a super-hedging portfolio, which implies that he/she will face a positive probability of "falling short", *i.e.*, his/her portfolio will take values V_T smaller than those of the claim on a non-negligible event. Thus, the writer will need to choose his/her investment strategy according to some optimality criterion to be decided. The gain-loss pricing criterion of Bernardo and Ledoit [3] that we use in the present paper suggests to choose the portfolio which gives the best value of the difference of expected positive final positions and a parameter λ (greater than one) times the expected negative final positions, $\mathbb{E}[(V_T - F_T)_+] - \lambda\mathbb{E}[(V_T - F_T)_-]$, aimed at weighting "losses" more than "gains". This criterion gives rise to a new concept that is more general than the ordinary arbitrage (or free lunch), the "λ gain-loss ratio opportunity", *i.e.*, a portfolio which can be set up at a negative price but yields a positive value for the difference between gains and "λ-losses". As for the maximum and minimum no-arbitrage prices, it is possible to determine the maximum and minimum prices which do not introduce λ gain-loss opportunities in the market. Thus, a new price interval (the "λ gain-loss price interval") is determined, generally contained in the no-arbitrage interval (thus more significant from an economical point of view). Clearly, such a price interval depends on the "real world" reference probability measure P *via* the expected value, *i.e.*, one may obtain completely different pricing intervals using two different, but equivalent reference measures, as we shall illustrate with a simple example from [3,23]. The dependence of gain-loss price bounds on the underlying probability distribution is referred to as "ambiguity of measure" [14]. This kind of sensitivity to the choice of measure is not an issue in the no-arbitrage interval since arbitrage free price bounds in incomplete markets remain unchanged even if one works with another probability measure as long as equivalence between different measures is preserved. In the light of these remarks, the main purpose of the present paper is to give a "robust" version of the gain-loss pricing approach which can deal with ambiguity in the probability distribution by "immunizing" the price bounds against shifts in probability distribution of the assets. This immunization is close in spirit to the classical problem of moments where the reader is referred to [22,30] and to the bibliography therein. For recent applications, in finance and stochastic programming, of ambiguity in probability distribution with moment constraints, we cite [4,5,11,13,29]. Ambiguity in option valuation from a different perspective (ambiguity in the equivalent martingale measures in arbitrage free pricing) is discussed in Cont [9].

In a recent paper that inspired the present one, Korf [20] gave a one-step variant of the fundamental theorem of asset pricing (FTAP) in mathematical finance [17,18] (in the context of arbitrage free pricing) using stochastic programming duality theory and a novel constraint qualification called "direction free feasibility". In Theorem 4.1 of the present paper we give a result in the spirit of Korf's FTAP result for robust Bernardo-Ledoit gain-loss price bounds in the face of shifts in probability measure. As a by-product of our analysis we also obtain a result (Cor. 4.6) analogous to this one-step fundamental asset pricing theorem in the framework of gain-loss pricing of Bernardo and Ledoit (for a fixed measure P) which yields tighter bounds for contingent claim pricing in incomplete markets.

We obtain our results in an infinite state probability setting, and using ideas from stochastic minimax programming [28] and from polyhedral risk measures [12]. For an in-depth treatment of risk measures in finance under discrete-time models, we direct the reader to the book by Föllmer and Schied [16], the recent paper by Ben-Tal and Teboulle [2], and to the papers by Ruszczyński and Shapiro [25,26].

The organization of the paper is as follows. In Section 2, we review the result of Korf [20] and we give some other key observations useful for our purposes. We define the gain-loss pricing problems in Section 3 along with a numerical example to motivate the need to acknowledge dependence of the resulting price bounds on the probability measure. In Section 4, we propose a version of the gain-loss pricing approach that allows an immunization of the bounds to the choice of probability measure, and discuss its consequences. We propose a variant of the pricing scheme based on Kullback-Leibler relative entropy in Section 5, and we illustrate its use for option pricing without intermediate trading. We conclude in Section 6 with a summary and future research directions.

2. The one-step FTAP

Let $Z_0 \in \mathbb{R}_+^{J+1}$ and $Z_T \in \mathcal{L}_+^1(\Omega, \mathcal{F}, P, \mathbb{R}^{J+1})$ denote the vector of asset prices at times $t = 0$ and $t = T$, respectively, where $(\Omega, \mathcal{F}, P)$ is assumed to be a P-complete underlying probability space. It is assumed without loss of generality that the first asset is the risk-free asset and that its price is equal to one at times $t = 0$ and $t = T$. A contingent claim is a contract to pay $F_T \in \mathcal{L}^1(\Omega, \mathcal{F}, P, \mathbb{R})$ in the future at time $t = T$. The writer of the claim collects a price (premium) $F_0 \in \mathbb{R}_+$ in exchange for the claim. With this endowment F_0, the writer can commit to a portfolio $\Theta \in \mathbb{R}^{J+1}$ today at time $t = 0$ so as to cover his/her cash outlay at time $t = T$. The portfolio problem of the risk-neutral writer is the following $\mathcal{P}_f$

$$\begin{array}{ll} \min & -\mathbb{E}[Z_T \cdot \Theta] \\ \text{s.t.} & Z_0 \cdot \Theta \le F_0 \\ & Z_T \cdot \Theta \ge F_T,\ P-\text{a.s.} \end{array}$$

where $Z_T \cdot \Theta$ denotes the usual (Euclidean) inner product of vectors Z_T and Θ. It was assumed in Korf [20] that F_0 and F_T were such that the problem $\mathcal{P}_f$ was feasible.

Korf [20] defined a free lunch portfolio (that can be interpreted as essentially a form of arbitrage opportunity) as a portfolio $\tilde{\Theta} \in \mathbb{R}^{J+1}$ satisfying the conditions

$$Z_0 \cdot \tilde{\Theta} \le 0$$

$$Z_T \cdot \tilde{\Theta} \ge 0,\ P-\text{a.s.}$$

$$P(Z_T \cdot \tilde{\Theta} > 0) > 0.$$

The market is said to offer no free lunches if no such $\tilde{\Theta}$ exists. The Lagrangian dual problem $\mathcal{D}_f$ to $\mathcal{P}_f$ is computed to be the problem[1]

$$\begin{array}{ll} \max & \mathbb{E}[F_T y_T] - F_0 y_0 \\ \text{s.t.} & \mathbb{E}[Z_T(1 + y_T)] = y_0 Z_0 \\ & y_0 \ge 0, \\ & y_T \ge 0,\ P-\text{a.s.} \end{array}$$

where $y_T \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P, \mathbb{R})$. Korf proved that if the market has no free lunches, then the problem $\mathcal{P}_f$ satisfies the direction-free feasibility constraint qualification, and gave the following theorem (see also Shapiro [27] for a more general reference on duality in convex semi-infinite optimization).

Theorem 2.1 (Korf [20])**.** *The following are equivalent*

(a) $\mathcal{P}_f$ *is bounded;*
(b) $\mathcal{D}_f$ *is feasible;*
(c) *the market admits no free lunches;*
(d) *there exists an equivalent martingale measure for the market price process.*

The feasibility of the dual problem $\mathcal{D}_f$ is equivalent to the price process being a martingale under some measure Q equivalent to P given by

$$Q(E) = \int_E \frac{1 + y_T(\omega)}{y_0} P(\mathrm{d}\omega)$$

for a dual feasible pair (y_0, y_T).

[1] Korf's dual has the signs in the objective function reversed, which must be a typographical error in [20].

The above theorem established that boundedness of $\mathcal{P}_f$ corresponds to feasibility of $\mathcal{D}_f$. We also need to deal with the feasibility of $\mathcal{P}_f$, which corresponds to an initial payment F_0 that is at least as large as the optimal value of the following problem $\mathcal{P}_{\text{feas}}$

$$\begin{array}{ll} \min & \nu \\ \text{s.t.} & Z_0 \cdot \Theta \leq \nu \\ & Z_T \cdot \Theta \geq F_T,\ P-\text{a.s.} \end{array}$$

In other words, the optimal value in $\mathcal{P}_{\text{feas}}$ gives the super-hedging price. As a result of weak duality, the optimal value in $\mathcal{P}_{\text{feas}}$ is at least as large as the optimal value of the dual problem $\mathcal{D}_{\text{feas}}$

$$\begin{array}{ll} \max & \mathbb{E}[F_T\zeta_T] \\ \text{s.t.} & \mathbb{E}[Z_T\zeta_T] = Z_0 \\ & \zeta_T \geq 0,\ P-\text{a.s.} \end{array}$$

where $\zeta_T \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P, \mathbb{R})$. We can equivalently express this observation as saying that F_0 should be at least as large as the optimal value in $\mathcal{D}_m$

$$\begin{array}{ll} \max & \mathbb{E}^Q[F_T] \\ \text{s.t.} & \mathbb{E}^Q[Z_T] = Z_0 \\ & \dfrac{\mathrm{d}Q}{\mathrm{d}P} \geq 0. \end{array}$$

We can also show that if $\mathcal{P}_{\text{feas}}$ has an optimal solution, the optimal values $\text{val}(\mathcal{P}_{\text{feas}}), \text{val}(\mathcal{D}_{\text{feas}})$ of the respective problems $\mathcal{P}_{\text{feas}}$ and $\mathcal{D}_{\text{feas}}$ are equal under the assumption of no free lunches using Proposition 3.4 of [27]. According to this result, under two technical requirements (satisfied in our setting, see [27]) if the set $\text{Sol}(\mathcal{P}_{\text{feas}})$ of optimal solutions to $\mathcal{P}_{\text{feas}}$ is of the form

$$\text{Sol}(\mathcal{P}_{\text{feas}}) = A + L,$$

where A is a non-empty bounded subset of $\mathbb{R}^{J+1}$ and L is a linear subspace of $\mathbb{R}^{J+1}$, we have $\text{val}(\mathcal{P}_{\text{feas}}) = \text{val}(\mathcal{D}_{\text{feas}})$.

Proposition 2.2. *If $\mathcal{P}_{\text{feas}}$ has an optimal solution, and the market admits no free lunches then* $\text{val}(\mathcal{P}_{\text{feas}}) = \text{val}(\mathcal{D}_{\text{feas}})$.

Proof. We verify that $\text{Sol}(\mathcal{P}_{\text{feas}})$ has indeed the desired form. Let Θ^* be an optimal solution to $\mathcal{P}_{\text{feas}}$. Define $A = \{\Theta^*\}$, a bounded subset of $\mathbb{R}^{J+1}$ and $L = \{\Theta | Z_0 \cdot \Theta = 0, Z_T \cdot \Theta = 0,\ P-\text{a.s.}\}$, a linear subspace of $\mathbb{R}^{J+1}$. It is clear that $A + L \subseteq \text{Sol}(\mathcal{P}_{\text{feas}})$. Now, assume Θ_2 is another optimal solution to $\mathcal{P}_{\text{feas}}$, which implies that $Z_0 \cdot (\Theta_2 - \Theta^*) = 0$. If $Z_T \cdot (\Theta_2 - \Theta^*) \geq 0,\ P-\text{a.s.}$ with $P(Z_T \cdot (\Theta_2 - \Theta^*) > 0) > 0$, then $\Theta_2 - \Theta^*$ is a free lunch portfolio, which is a contradiction. By the same token, if $Z_T \cdot (\Theta_2 - \Theta^*) \leq 0,\ P-\text{a.s.}$ with $P(Z_T \cdot (\Theta_2 - \Theta^*) < 0) > 0$, then $\Theta^* - \Theta_2$ is a free lunch portfolio, which is again a contradiction. Hence, we must have $Z_T \cdot (\Theta_2 - \Theta^*) = 0,\ P-\text{a.s.}$, which implies $\Theta_2 - \Theta^* \in L$. Therefore, $\text{Sol}(\mathcal{P}_{\text{feas}}) \subseteq A + L$, which completes the verification. Now, the result is a consequence of Proposition 3.4 [27]. □

A similar development can be given for the risk-neutral buyer's portfolio problem

$$\begin{array}{ll} \min & \mathbb{E}[Z_T \cdot \Theta] \\ \text{s.t.} & Z_0 \cdot \Theta \geq F_0 \\ & Z_T \cdot \Theta \leq F_T,\ P-\text{a.s.} \end{array}$$

which is the opposite of that of the risk-neutral writer.

It is well-known that the super-hedging price, while offering full protection, may be too high from a practical point of view; see *e.g.*, [16], Example 7.21. It is also pointed out in Cont [9] that the price bounds obtained by solving the writer and buyer problems (the arbitrage free price bounds) can be quite wide, and very different from the prices quoted for buying and selling liquidly traded options in the market, which makes these bounds of questionable value in pricing options that are not liquidly traded in the market, *e.g.*, exotic options for which the value has to be computed using a pricing model.

3. Gain-loss pricing

Bernardo and Ledoit [3] proposed a pricing framework where attractiveness of an investment portfolio is measured by an expected gain to expected loss ratio, and high gain-loss ratio opportunities are ruled out. The limitation imposed on the gain-loss ratio translates in the pricing space into a limitation on the pricing kernels and leads to narrower pricing bounds in incomplete markets. In this section we prepare the ground to obtain an analogue of Theorem 2.1 using gain-loss ratios as a special case of a slightly more general result under ambiguity of measure.

Let us fix $\lambda > 1$. We will say that a portfolio $\Theta^* \in \mathbb{R}^{J+1}$ is a λ gain-loss opportunity if

$$Z_0 \cdot \Theta^* \leq 0$$

$$\mathbb{E}[(Z_T \cdot \Theta^*)_+] - \lambda \mathbb{E}[(Z_T \cdot \Theta^*)_-] > 0,$$

where $(.)_+ = \max(.,0)$ and $(.)_- = -\min(.,0)$. Note that every free lunch portfolio is also a λ gain-loss opportunity, and for $\lambda = +\infty$ every gain-loss opportunity portfolio is a free lunch portfolio. We say that the market admits no λ gain-loss ratio opportunity if Θ^* satisfying the above conditions cannot be found, or equivalently if the following optimization problem (over Θ)

$$\begin{array}{ll} \max & \mathbb{E}[(Z_T \cdot \Theta)_+] - \lambda \mathbb{E}[(Z_T \cdot \Theta)_-] \\ \text{s.t.} & Z_0 \cdot \Theta \leq 0 \end{array}$$

has optimal value zero.

Now, the portfolio problem of the gain-loss ratio writer is the following $\mathcal{P}_g(\lambda)$

$$\begin{array}{ll} \min & \lambda \mathbb{E}[(Z_T \cdot \Theta - F_T)_-] - \mathbb{E}[(Z_T \cdot \Theta - F_T)_+] \\ \text{s.t.} & Z_0 \cdot \Theta \leq F_0. \end{array}$$

The gain-loss ratio writer does no longer guarantee that he/she will meet the obligations from the contingent claim P-almost surely. However, he/she wants to minimize the potential losses that would be incurred should a shortfall occur using a gain-loss ratio of λ. The economic interest in this problem lies in the fact that the writer may settle for a smaller premium F_0 in selling the claim than in Section 2. The magnitude of F_0 depends of course on the risk aversion of the writer that he/she controls *via* his/her choice of λ. Therefore, the indifference price of the writer is obtained at those values of F_0 where the problem $\mathcal{P}_g(\lambda)$ is bounded by zero from below.

Similarly, the buyer's portfolio problem, which is the opposite of the writer is

$$\begin{array}{ll} \max & \mathbb{E}[(F_T - Z_T \cdot \Theta)_+] - \lambda \mathbb{E}[(F_T - Z_T \cdot \Theta)_-] \\ \text{s.t.} & Z_0 \cdot \Theta \geq F_0. \end{array}$$

As the writer's problem allows an economically more appealing and intuitive interpretation we will focus in the rest of the paper on the writer's problem. Needless to say, all the developments of the paper also apply to the problem of the buyer.

We will derive the dual problem $\mathcal{D}_g(\lambda)$ corresponding to $\mathcal{P}_g(\lambda)$ as a special case of a more general problem in Section 4. Here we state directly the dual problem $\mathcal{D}_g(\lambda)$ as

$$\begin{array}{rl} \max & \mathbb{E}[\zeta F_T] - y_0 F_0 \\ \text{s.t.} & \mathbb{E}[\zeta Z_T] = y_0 Z_0 \\ & y_0 \geq 0 \\ & \zeta \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P, \mathbb{R}) \\ & \zeta \in [1, \lambda]\ P - \text{a.s.} \end{array}$$

We observe that when $\lambda = +\infty$, an optimal solution to $\mathcal{D}_f$ is feasible in the above problem using the transformation $\zeta = y_T + 1$, and the objective function values of the two problems are equal up to a constant, which is equal to $\mathbb{E}[F_T]$. Using the above dual problem we will obtain a result similar to Theorem 4.1 in Section 4 as a corollary to a more general result. Note that feasibility of the dual problem $\mathcal{D}_g(\lambda)$ is equivalent to the price process being a martingale under some measure Q equivalent to probability measure P given by

$$Q(E) = \int_E \frac{\zeta(\omega)}{y_0} P(\mathrm{d}\omega)$$

for a dual feasible couple (y_0, ζ).

For numerical computation, one may resort to the case when Ω is a finite set, *i.e.*, $\Omega = \{\omega_1, \ldots, \omega_n\}$, after a suitable discretization. In this case, the measure P is identified with a n-vector $p = (p_1, \ldots, p_n)$, and random variable F_T can be viewed as an n-vector $F = (F_1, \ldots, F_n)$ while each component Z_T^i of Z_T is now identified with an n-vector $Z^i = (Z_1^i, \ldots, Z_n^i)$ for $i = 1, \ldots, J+1$. By our assumption, the first component Z_T^1 corresponds to the vector with all the components equal to one. In this case, it is easy to see that we are facing a linear programming problem over the variables y_0 and $y_\ell, \ell = 1, \ldots, n$

$$\begin{array}{rl} \max & \displaystyle\sum_{\ell=1}^n y_\ell F_\ell - y_0 F_0 \\ \text{s.t.} & \displaystyle\sum_{\ell=1}^n Z_\ell^i \, y_\ell = y_0 Z_0^i, \quad \forall i = 1, \ldots, J+1 \\ & y_0 \geq 0 \\ & p_\ell \leq y_\ell \leq \lambda p_\ell, \quad \forall \ell = 1, \ldots, n. \end{array}$$

Example 3.1. Consider the following discrete state-space two-period example from [3] where an incomplete market with a stock and a bond is given. Both trade at some price today while the stock is priced in three equally probable states the next period, *i.e.*, $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Trading takes place only today. Both the stock and the bond are valued at one today. The stock price in the next period is given by $(2, 1, 0)$ in states 1, 2, and 3 while the bond is valued at one at all states in the next period. We are interested in pricing a call option written on the stock with strike price equal to one, and maturing at the next period. The no-arbitrage pricing interval obtained by solving the writer and buyer portfolio problems is the interval $(0, 1/2)$. In particular, the upper value $1/2$ is the smallest value of F_0 such that the problem $\mathcal{P}_f$ has a feasible solution.

Now, turning to problem $\mathcal{P}_g$ with $\lambda = 2$ for the writer, we can compute the "no λ gain-loss ratio opportunity" interval to be $(1/4, 2/5)$. In particular, the value of $2/5$ is the indifference price of the writer at level $\lambda = 2$ which results in an optimal value of zero in both the primal and dual problems $\mathcal{P}_g(\lambda)$ and $\mathcal{D}_g(\lambda)$. The dual variable (measure) ζ (which is now a three-dimensional vector) has optimal values $(2, 1, 2)$ with $y_0 = 5/3$, which corresponds to a martingale measure Q with $(Q(\omega_1), Q(\omega_2), Q(\omega_3))$ equal to $(2/5, 1/5, 2/5)$.

4. Ambiguity of measure

In the illustrative example of the previous section, it was assumed that all states of nature were equally likely, which can be considered as a reference measure. If we pass to an equivalent, but different measure and assign probabilities $(1/6, 2/3, 1/6)$ to the three states of nature, the "no λ gain-loss ratio opportunity" interval shifts to $(1/10, 1/4)$, which is different from the previous interval while the no-arbitrage bounds $(0, 1/2)$ remain the same since they do not depend on the underlying measure. This assertion that no-arbitrage bounds do not change remains valid provided that the new underlying measure is equivalent to the reference measure (as long as the super-hedging requirement is that the portfolio value dominates the claim almost surely), as is the case in this small example. In the light of this simple example, it is a worthwhile undertaking to investigate a pricing model which alleviates this sensitivity of the "no λ gain-loss ratio opportunity" interval to the underlying measure. In this section we define an extension of the gain-loss pricing framework which remains robust in the face of shifts in the reference measure P using a min-max approach. In our subsequent analysis, we adopt the notation of Shapiro and Ahmed [28].

Let us denote by $\mathcal{X}$ the (linear) space of all finite signed measures on $(\Omega, \mathcal{F})$. We say that a measure $\mu \in \mathcal{X}$ is non-negative and write $\mu \succeq 0$, if $\mu(A) \geq 0$ for any $A \in \mathcal{F}$. For two measures $\mu_1, \mu_2 \in \mathcal{X}$ we write $\mu_2 \succeq \mu_1$ if $\mu_2 - \mu_1 \succeq 0$. That is, $\mu_2(A) \geq \mu_1(A)$ for any $A \in \mathcal{F}$. We say that μ is a probability measure if $\mu \succeq 0$ and $\mu(\Omega) = 1$. Consider now the set

$$\mathcal{M} = \{\mu \in \mathcal{X} : (1 - \varepsilon_1)P^* \preceq \mu \preceq (1 + \varepsilon_2)P^*\} \tag{4.1}$$

for some reference probability measure P^*, and numbers $\varepsilon_1 \in [0, 1]$ and $\varepsilon_2 \geq 0$. Let $\mathcal{P}$ denote the set of all probability measures from the set $\mathcal{M}$.

Let us again consider a fixed $\lambda > 1$. We will say that a portfolio $\Theta^* \in \mathbb{R}^{J+1}$ implies a λ gain-loss ratio for reference measure $P \in \mathcal{P}$ if

$$Z_0 \cdot \Theta^* \leq 0$$
$$\mathbb{E}^P[(Z_T \cdot \Theta^*)_+] - \lambda \mathbb{E}^P[(Z_T \cdot \Theta^*)_-] > 0.$$

We say that the market admits no "λ gain-loss ratio opportunity under measure ambiguity" if Θ^* satisfying the above conditions cannot be found for any of the probability measures in $\mathcal{P}$, or equivalently if the following optimization problem $\mathcal{P}_1$

$$\begin{array}{ll} \max & \min_{P \in \mathcal{P}} \mathbb{E}^P[(Z_T \cdot \Theta)_+] - \lambda \mathbb{E}^P[(Z_T \cdot \Theta)_-] \\ \text{s.t.} & Z_0 \cdot \Theta \leq 0 \end{array}$$

has optimal value zero.

We are now focusing on the writer's portfolio problem $\mathcal{P}_{\text{gam}}(\lambda)$ over the variables $\Theta \in \mathbb{R}^{J+1}$

$$\begin{array}{ll} \min & \psi(\Theta) \\ \text{s.t.} & Z_0 \cdot \Theta \leq F_0 \end{array}$$

where the function ψ is defined by

$$\psi(\Theta) = \max_{P \in \mathcal{P}} \lambda \mathbb{E}^P[(Z_T \cdot \Theta - F_T)_-] - \mathbb{E}^p[(Z_T \cdot \Theta - F_T)_+]. \tag{4.2}$$

Notice that ψ is a convex function of Θ. Following [28] we can pose $\mathcal{P}_{\text{gam}}(\lambda)$ as

$$\begin{array}{ll} \min & \mathbb{E}^{P^*}[\lambda(Z_T \cdot \Theta - F_T)_- - (Z_T \cdot \Theta - F_T)_+ + L_{\varepsilon_1,\varepsilon_2}(\lambda(Z_T \cdot \Theta - F_T)_- - (Z_T \cdot \Theta - F_T)_+ - \gamma)] \\ \text{s.t.} & Z_0 \cdot \Theta \leq F_0 \end{array}$$

over the variables Θ and $\gamma \in \mathbb{R}$ and where the function $L_{\varepsilon_1,\varepsilon_2}$ is defined as

$$L_{\varepsilon_1,\varepsilon_2}(a) = \begin{cases} -\varepsilon_1 a & \text{if } a \le 0 \\ \varepsilon_2 a & \text{if } a > 0. \end{cases}$$

However, this approach complicates the passage to a dual problem. We will obtain a Lagrangian dual problem for $\mathcal{P}_{\text{gam}}(\lambda)$ that is useful for our purposes using Lemma 4.1 below, which gives a dual representation of the objective function; see [24] for background on convex duality in infinite dimensional spaces.

Lemma 4.1. *Let* $Z \in \mathcal{L}^1(\Omega, \mathcal{F}, P, \mathbb{R})$ *and* $\lambda > 1$. *Then, we have*

$$\lambda\mathbb{E}[(Z)_-] - \mathbb{E}[(Z)_+] = \sup\{-\mathbb{E}[\eta Z] \mid \eta \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P, \mathbb{R}), \eta \in [1,\lambda] \text{ a.s.}\}. \tag{4.3}$$

Proof. Using Definition 2.2 of [12] we can write $\lambda\mathbb{E}[(Z)_-] - \mathbb{E}[(Z)_+]$ as

$$\begin{aligned}&\lambda\mathbb{E}[(Z)_-] - \mathbb{E}[(Z)_+] = \\ &\quad \inf\{0 \cdot y_1 + \mathbb{E}[\lambda y_2^{(2)} - y_2^{(1)}] : y_1 \in \{0\}, y_1 + y_2^{(1)} - y_2^{(2)} = Z \text{ a.s., } y_2 \in \mathbb{R}_+ \times \mathbb{R}_+ \text{ a.s.,} \\ &\quad y_2 \in \mathcal{L}^1(\Omega, \mathcal{F}, P, \mathbb{R})\}.\end{aligned}$$

Here, in the notation of [12] we have $D_1 = \mathbb{R}$ and D_2 is the interval $[1,\lambda]$ by a simple computation. Then doing calculations similar to those in Example 2.10 of [12], we have that the dual representation (2.3) of Theorem 2.4 of [12] holds. □

Using the result of this lemma we rewrite the problem of the writer as

$$\min_{\Theta} \max_{\substack{\eta \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P, \mathbb{R}), \eta \in [1,\lambda] \text{ a.s.} \\ P \in \mathcal{P}, y_0 \ge 0}} -\mathbb{E}[\eta(Z_T \cdot \Theta - F_T)] + y_0(Z_0 \cdot \Theta - F_0).$$

Interchanging the min and the max the Lagrangian dual of the problem $\mathcal{P}_{\text{gam}}(\lambda)$ is the following

$$\max_{\substack{\eta \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P, \mathbb{R}), \eta \in [1,\lambda] \text{ a.s.} \\ P \in \mathcal{P}, y_0 \ge 0}} \min_{\Theta} -\mathbb{E}[\eta(Z_T \cdot \Theta - F_T)] + y_0(Z_0 \cdot \Theta - F_0).$$

By evaluating the inner minimization over Θ we obtain the dual problem $\mathcal{D}_{\text{gam}}(\lambda)$ expressed as

$$\begin{array}{rl} \max & \mathbb{E}^P[\eta F_T] - y_0 F_0 \\ \text{s.t.} & \mathbb{E}^P[\eta Z_T] = y_0 Z_0 \\ & y_0 \ge 0 \\ & \eta \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P, \mathbb{R}) \\ & \eta \in [1,\lambda] \; P-\text{a.s.} \\ & P \in \mathcal{P}. \end{array}$$

It is easy to see that we can equivalently rewrite the dual as the convex programming problem using measure variables Y:

$$\begin{array}{rl} \max & \displaystyle\int_\Omega F_T(\omega) Y(\mathrm{d}\omega) - y_0 F_0 \\ \text{s.t.} & \displaystyle\int_\Omega Z_T(\omega) Y(\mathrm{d}\omega) = y_0 Z_0 \\ & y_0 \ge 0 \\ & P \preceq Y \preceq \lambda P \\ & P \in \mathcal{P}. \end{array}$$

Now, we have the following result.

Theorem 4.2. *The following are equivalent:*

(a) $\mathcal{P}_{\rm gam}(\lambda)$ *is bounded;*
(b) *the dual* $\mathcal{D}_{\rm gam}(\lambda)$ *is feasible;*
(c) *the market admits no* λ *gain-loss ratio opportunities under measure ambiguity;*
(d) *there exists an equivalent martingale measure for the market price process, satisfying the side conditions*

$$\frac{\mathrm{d}Q}{\mathrm{d}P} \in [V, V\lambda] \; P - \text{a.s.}$$

with $V > 0$ *for some* $P \in \mathcal{P}$.

Proof. We first prove that (b) and (c) are equivalent. Consider the following pair of primal-dual problems $\mathcal{P}_1$ and $\mathcal{D}_1$, where $\mathcal{P}_1$ was already defined above and $\mathcal{D}_1$ is the feasibility problem:

$$\begin{array}{ll} \min & 0 \\ \text{s.t.} & \mathbb{E}^P[\eta Z_T] = y_0 Z_0 \\ & y_0 \geq 0 \\ & \eta \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P, \mathbb{R}) \\ & \eta \in [1, \lambda] \; P - \text{a.s.} \\ & P \in \mathcal{P}. \end{array}$$

Problems $\mathcal{D}_1$ and $\mathcal{D}_{\rm gam}(\lambda)$ have identical feasible sets. Hence (b) implies that $\mathcal{D}_1$ is feasible. By weak duality, we have that this implies (c). If (c) holds, *i.e.*, if $\sup \mathcal{P}_1$ is equal to zero, using the fact that $\mathcal{P}_1$ is a convex programming problem (maximization of a concave function over a linear constraint) which is strictly feasible, by Theorem 3.2.2 of [1] we have $\min \mathcal{D}_1 = 0$, *i.e.*, the minimum value of zero is attained in $\mathcal{D}_1$ by some feasible point. Hence, we have that $\mathcal{D}_1$ is feasible, which implies that $\mathcal{D}_{\rm gam}(\lambda)$ is feasible.

We have that (b) implies (a) since it is always true that $\inf \mathcal{P}_{\rm gam}(\lambda) \geq \sup \mathcal{D}_{\rm gam}(\lambda)$, and if $\mathcal{D}_{\rm gam}(\lambda)$ is feasible, we have $\sup \mathcal{D}_{\rm gam}(\lambda) > -\infty$, which implies that $\mathcal{P}_{\rm gam}(\lambda)$ is bounded. On the other hand if (a) holds, due to the strict feasibility of $\mathcal{P}_{\rm gam}(\lambda)$ and by Theorem 3.2.2 of [1] we have that $\inf \mathcal{P}_{\rm gam}(\lambda) = \sup \mathcal{D}_{\rm gam}(\lambda)$. Moreover, boundedness of $\mathcal{P}_{\rm gam}(\lambda)$ implies $\inf \mathcal{P}_{\rm gam}(\lambda) > -\infty$, hence $\sup \mathcal{D}_{\rm gam}(\lambda)$ is finite, which in turn implies that $\mathcal{D}_{\rm gam}(\lambda)$ is feasible. Therefore, (a) implies (b).

Finally, the equivalence of (b) and (d) comes from the dual problem $\mathcal{D}_{\rm gam}(\lambda)$. The feasibility of the dual problem $\mathcal{D}_{\rm gam}(\lambda)$ implies by the Radon-Nikodým theorem that the price process is a martingale under some measure Q equivalent to some feasible probability measure $\tilde{P}$, given by

$$Q(E) = \int_E \frac{\eta(\omega)}{y_0} \tilde{P}(\mathrm{d}\omega)$$

for a dual feasible triple $(y_0, \eta, \tilde{P})$. Let V be the inverse of y_0. The variable V is positive since y_0 should assume a positive value in $\mathcal{D}_{\rm gam}(\lambda)$ as a result of our assumption that the first component Z_0^1 of Z_0 and the first component Z_T^1 of Z_T are equal to one, and that $\eta \in [1, \lambda]$, P-almost surely for any feasible P.

Conversely, if there exists an equivalent martingale measure Q for the market price process (*i.e.*, such that one has $\mathbb{E}^Q[Z_T] = Z_0$), satisfying the side conditions

$$\frac{\mathrm{d}Q}{\mathrm{d}P} \in [V, V\lambda] \; P - \text{a.s.}$$

where $V > 0$, for some $P \in \mathcal{P}$, then this is equivalent to having a feasible solution to $\mathcal{D}_{\rm gam}(\lambda)$. □

Remark 4.3. As a result of the theorem, the indifference price F_0^a (or, the "no λ gain-loss price under measure ambiguity") for the writer is given by

$$\begin{array}{rl} \max & \mathbb{E}^Q[F_T] \\ \text{s.t.} & \mathbb{E}^Q[Z_T] = Z_0 \\ & \dfrac{\mathrm{d}Q}{\mathrm{d}P} \in [V, V\lambda] \ P-\text{a.s.} \\ & V \geq 0 \\ & P \in \mathcal{P} \end{array}$$

where the Radon-Nikodỳm derivative of the equivalent martingale measure lies in $\mathcal{L}^\infty(\Omega, \mathcal{F}, P, \mathbb{R})$. As already observed in the last part of the proof the variable V takes on only positive feasible values.

Remark 4.4. Without having to repeat all the previous developments for the buyer problem under ambiguity

$$\begin{array}{rl} \max & \min\limits_{P \in \mathcal{P}} \mathbb{E}[(F_T - Z_T \cdot \Theta)_+] - \lambda \mathbb{E}[(F_T - Z_T \cdot \Theta)_-] \\ \text{s.t.} & Z_0 \cdot \Theta \geq F_0 \end{array}$$

we can assert that the indifference price F_0^b (the "no λ gain-loss price under measure ambiguity") for the buyer is computed by solving

$$\begin{array}{rl} \min & \mathbb{E}^Q[F_T] \\ \text{s.t.} & \mathbb{E}^Q[Z_T] = Z_0 \\ & \dfrac{\mathrm{d}Q}{\mathrm{d}P} \in [V, V\lambda] \ P-\text{a.s.} \\ & V \geq 0 \\ & P \in \mathcal{P}. \end{array}$$

Obviously, we have $F_0^b \leq F_0^a$.

Remark 4.5. The dual problem $\mathcal{D}_g(\lambda)$ for a fixed reference measure (that we can suppress to avoid encumbering the notation unnecessarily) is obtained as

$$\begin{array}{rl} \max & \mathbb{E}[\eta F_T] - y_0 F_0 \\ \text{s.t.} & \mathbb{E}[\eta Z_T] = y_0 Z_0 \\ & y_0 \geq 0 \\ & \eta \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P, \mathbb{R}) \\ & \eta \in [1, \lambda] \ P-\text{a.s.} \end{array}$$

by shrinking the set of allowed probability measures to a singleton, namely the reference measure P^*. This problem is equivalently expressed as

$$\begin{array}{rl} \max & \displaystyle\int_\Omega F_T(\omega) Y(\mathrm{d}\omega) - y_0 F_0 \\ \text{s.t.} & \displaystyle\int_\Omega Z_T(\omega) Y(\mathrm{d}\omega) = y_0 Z_0 \\ & y_0 \geq 0 \\ & P^* \preceq Y \preceq \lambda P^*. \end{array}$$

Now, the following result which is in the spirit of Theorem 1 from [20], is obtained as a corollary of the previous theorem.

Corollary 4.6. *The following are equivalent*

(a) $\mathcal{P}_g(\lambda)$ *is bounded;*
(b) $\mathcal{D}_g(\lambda)$ *is feasible;*
(c) *the market admits no* λ *gain-loss ratio opportunities;*
(d) *there exists an equivalent martingale measure for the market price process, satisfying the side conditions*

$$\frac{\mathrm{d}Q}{\mathrm{d}P^*} \in [V, V\lambda]\ P^* - \text{a.s.}$$

with $V > 0$.

The indifference price, referred to as F_0^f (or, the "no λ gain-loss price") is given this time by

$$\begin{array}{ll} \max & \mathbb{E}^Q[F_T] \\ \text{s.t.} & \mathbb{E}^Q[Z_T] = Z_0 \\ & \dfrac{\mathrm{d}Q}{\mathrm{d}P} \in [V, V\lambda]\ P - \text{a.s.} \\ & V \geq 0 \end{array}$$

where we take P to be the reference measure P^*, and the variable V is positive as it is the inverse of y_0, which should assume a positive value by the same reasoning as in $\mathcal{D}_{\text{gam}}(\lambda)$. Note that F_0^f is at most equal to the no-free-lunch price F_0 as every feasible solution to the above problem in Q is also feasible for $\mathcal{D}_m$. On the other hand, a similar relationship about F_0^a and F_0 cannot be guaranteed.

Concerning numerical computation when Ω consists of a finite number of atoms, we can calculate the indifference price F_0^f above by solving the linear fractional programming problem with quasi-linear (both quasi-convex and quasi-concave) objective function in y_0, and y_ℓ, $\ell = 1, \ldots, n$, and linear constraints in all the variables (y_0, y_ℓ, $\ell = 1, \ldots, n$)

$$\begin{array}{ll} \max & \dfrac{\sum_{\ell=1}^n y_\ell F_\ell}{y_0} \\ \text{s.t.} & \displaystyle\sum_{\ell=1}^n Z_\ell^i\, y_\ell = y_0 Z_0^i, \quad \forall i = 1, \ldots, J+1 \\ & y_0 \geq 0 \\ & p_\ell \leq y_\ell \leq \lambda p_\ell, \quad \forall \ell = 1, \ldots, n \end{array}$$

which is equivalent to the nonlinear program in variables $q_\ell = y_\ell / y_0, \ell = 1, \ldots, n$ and y_0

$$\begin{array}{ll} \max & \displaystyle\sum_{\ell=1}^n q_\ell F_\ell \\ \text{s.t.} & \displaystyle\sum_{\ell=1}^n Z_\ell^i\, q_\ell = Z_0^i, \quad \forall i = 1, \ldots, J+1 \\ & y_0 \geq 0 \\ & p_\ell \leq q_\ell y_0 \leq \lambda p_\ell, \quad \forall \ell = 1, \ldots, n. \end{array}$$

On the other hand, the practical computation of F_0^a can be accomplished by solving the linear-fractional programming problem

$$\begin{aligned} \max \quad & \frac{\sum_{\ell=1}^n y_\ell F_\ell}{y_0} \\ \text{s.t.} \quad & \sum_{\ell=1}^n Z_\ell^i \, y_\ell = y_0 Z_0^i, \quad \forall i = 1, \ldots, J+1 \\ & y_0 \geq 0 \\ & p_\ell \leq y_\ell \leq \lambda p_\ell, \quad \forall \ell = 1, \ldots, n \\ & (1-\varepsilon_1) p_\ell^* \leq p_\ell \leq (1+\varepsilon_2) p_\ell^*, \quad \forall \ell = 1, \ldots, n \end{aligned}$$

while we note again the relationships $V = 1/y_0$ and $q_\ell = y_\ell / y_0$, for all $\ell = 1, \ldots, n$. Linear-fractional programming problems are almost as routinely solved as linear programming problems by state-of-the art optimization software. In fact, the discretized version

$$\begin{aligned} \max \quad & \sum_{\ell=1}^n y_\ell F_\ell - F_0 y_0 \\ \text{s.t.} \quad & \sum_{\ell=1}^n Z_\ell^i \, y_\ell = y_0 Z_0^i, \quad \forall i = 1, \ldots, J+1 \\ & y_0 \geq 0 \\ & p_\ell \leq y_\ell \leq \lambda p_\ell, \quad \forall \ell = 1, \ldots, n \\ & (1-\varepsilon_1) p_\ell^* \leq p_\ell \leq (1+\varepsilon_2) p_\ell^*, \quad \forall \ell = 1, \ldots, n \end{aligned}$$

of $\mathcal{D}_{\text{gam}}(\lambda)$ is nothing other than a device to solve the above linear-fractional program by solving a sequence of linear programming problems for different trial values of F_0 until zero is achieved as optimal value.

Example 4.7. Returning to the previous example in discrete time, choosing the reference measure as $P^* = (1/3, 1/3, 1/3)$, $\varepsilon_1 = 1/2$, and $\varepsilon_2 = 1$, we obtain the pricing interval as $[0.10, 0.4545]$. The worst-case measure for the writer is the measure $(0.417, 0.167, 0.417)$, while for the buyer it is $(1/6, 2/3, 1/6)$. The pricing bounds that address ambiguity are still not as wide as the no-arbitrage bounds, *i.e.*, we achieve a writer's price smaller than 0.5, and a buyer price greater than zero.

A final note to close this section is on the choice of the set $\mathcal{M}$ (and, hence of $\mathcal{P}$) which did not play any role in our main result. While it is true that our result holds for other specifications of $\mathcal{M}$ as we shall see in Section 5 below, it is important to keep in mind that convexity of $\mathcal{P}$ plays an important role in numerical computation.

5. Gain-loss bounds based on ϕ-divergence

In this section we develop a version of the gain-loss bounds based on the ϕ-divergence functional introduced by Csiszar [10]. Our reference on ϕ-divergence is the paper by Ben-Tal and Teboulle [2]. A recent example of the use of ϕ-divergence functionals in portfolio optimization is [8].

Let $\phi : \mathbb{R} \mapsto (-\infty, +\infty]$ be a proper closed convex function such that $\operatorname{dom} \phi$ is an interval with endpoints $\alpha < \beta$. Since ϕ is closed, then

$$\lim_{t \to \alpha_+} \phi(t) = \phi(\alpha)$$

if α is finite, and

$$\lim_{t \to \beta_-} \phi(t) = \phi(\beta)$$

if β is finite. We assume that $1 \in \text{int dom}\ \phi$ and that the minimum of ϕ is zero and attained at the point $t^* = 1 \in \text{int dom}\ \phi$. The class of such functions is denoted by Φ. Now, given $\phi \in \Phi$, the ϕ-divergence of the probability measure Q with respect to P is

$$I_\phi(Q,P) = \begin{cases} \int_\Omega \phi(\frac{\mathrm{d}Q}{\mathrm{d}P})\mathrm{d}P & \text{if } Q \ll P \\ +\infty & \text{otherwise,} \end{cases} \tag{5.1}$$

where the notation $Q \ll P$ is meant to be understood as "Q is absolutely continuous with respect to P". For the choice of $\phi(t) = t\ln t - t + 1$ we obtain the well-known Kullback-Leibler relative entropy [21] with which we shall work in the rest of this section.

Whenever I_ϕ is finite, *i.e.*, $Q \ll P$, the ϕ-divergence can be expressed as

$$I_\phi(Q,P) = \mathbb{E}^P\left[\phi\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right].$$

For a fixed reference measure P^*, we propose to use in this section the set $\mathcal{P}$ given by the set of all probability measures P such that $I_\phi(P,P^*) \le d$ from some $d \ge 0$. *I.e.*, we consider the set

$$\mathcal{K}(P^*,d) = \{P \in \pi : I_\phi(P,P^*) \le d\}$$

where we use $\phi(t) = t\ln t - t + 1$ and $\pi = \{\mu \in \mathcal{X} : \mu \succeq 0, \mu(\Omega) = 1\}$.

For practical computation we obtain the problem $\mathcal{D}_{\text{gam}}(\lambda)$ in the form

$$\begin{aligned} \max \quad & \sum_{\ell=1}^n y_\ell F_\ell - F_0 y_0 \\ \text{s.t.} \quad & \sum_{\ell=1}^n Z_\ell^i\, y_\ell = y_0 Z_0^i, \quad \forall i = 1,\dots,J+1 \\ & y_0 \ge 0 \\ & p_\ell \le y_\ell \le \lambda p_\ell, \quad \forall \ell = 1,\dots,n \\ & \sum_{\ell=1}^n p_\ell \ln \frac{p_\ell}{p_\ell^*} \le d \end{aligned}$$

which is a convex programming problem with a single non-linear (convex) constraint.

We illustrate the use of bounds obtained from Kullback-Leibler relative entropy-based pricing on an example from [3,23].

5.1. **Relative entropy and option pricing bounds**

Assume we have a stock and a bond, and we are trying to price a European Call option written on the stock with strike price equal to 100, and one year to expiration. The one year continuously compounded rate of return of the bond is 4.88%, and the annual volatility of the compounded rate of the return on the stock is 14.09%. We assume that there is no intermediate trading between now and expiration. The Black-Scholes price of this option [6] is equal to 5.22.

Consider now a discretization Z^2 of the stock price at expiration of the option (Z^1 is reserved for the bond price which is equal to one). We assume the possible realizations of Z^2 are given by $Z_1^2 = 41, Z_2^2 = 42,\dots,Z_{120}^2 = 160$. The riskless asset, *i.e.*, the bond has value equal to one in all 120 states of nature, while the option value at expiration F_ℓ takes the value 0 for $\ell = 1,\dots,60$ and then evolves as $F_{61} = 1, F_{62} = 2,\dots,F_{120} = 60$. Assume the stock price is currently equal to 95. The no-arbitrage bounds for this incomplete market example are quite far apart: 0 and 28.21. We use the Black-Scholes risk-neutral probabilities as benchmark,

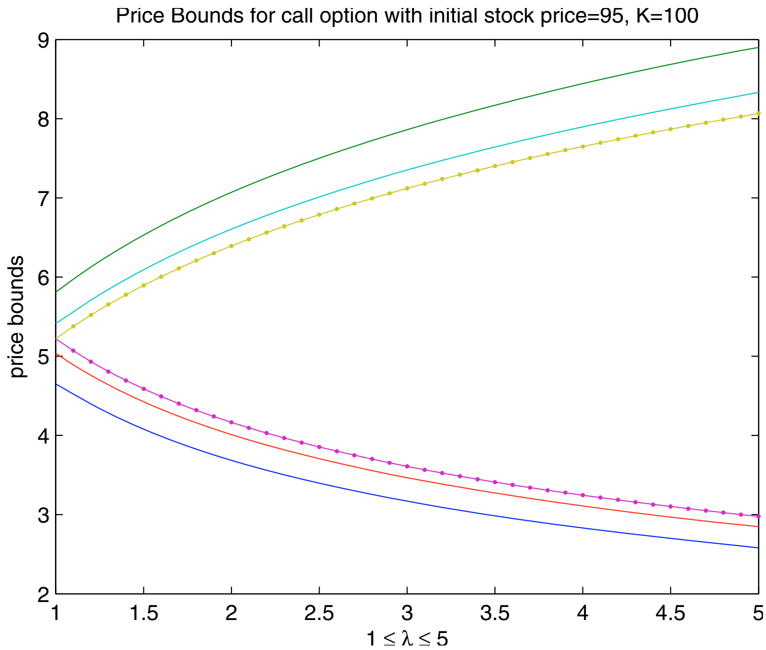

FIGURE 1. Bounds on call option price where the initial price of the underlying is equal to 95. The option expires in one year and has strike $K = 100$, the one year continuously compounded risk free rate of return is 4.88%, and and the standard deviation of the continuously compounded rate of return on the stock is $\sigma = 0.1409$ per year. The solid curves represent the bounds computed with the relative entropy based gain-loss approach under ambiguity; the outer smooth curves correspond to $d = 0.01$, while the dotted curves correspond to the original gain-loss bounds under the risk-neutral reference measure. The smooth curves sandwiched between the outermost smooth and the inner dotted curves correspond to the choice $d = 0.001$ in the relative entropy distance constraint.

hence we compute $p_1^* = \Pr\{\gamma \le \ln 41.5\}$, $p_2^* = \Pr\{\gamma \in (\ln 41.5, \ln 42.5]\}$, and so on until $p_{119}^* = \Pr\{\gamma \in (\ln 158.5, \ln 159.5]\}$ and $p_{120}^* = \Pr\{\gamma > \ln 159.5\}$, where γ is a normally distributed random variable with mean $\ln 95 + 0.0488 - \frac{0.1409^2}{2}$ and standard deviation 0.1409. For further details about this example, we refer the reader to Section 3 of [23] and Section V of [3].

Now, let us take the view-point of the investor whose confidence in the risk-neutral probability measure is limited and compute gain-loss price bounds using the Kullback-Leibler relative entropy constraint with different values of d. We solve the resulting convex programming problems through GAMS/PATHNLP [7,15]. The results of this experiment are summarized in Figure 1 where the solid curves represent the price bounds from the ambiguous gain-loss approach while the dotted curves represent the gain-loss approach without ambiguity of measure as advocated in [3,23]. With $d = 0.01$, and solving the discretized version of problem $\mathcal{D}_{\text{gam}}(\lambda)$ for the writer and buyer, for $\lambda = 5$, we obtain the price bounds 2.58 for the buyer *versus* 8.9 for the writer, which are wider than the corresponding gain-loss bounds obtained under the risk-neutral measure 2.97 and 8.09, as expected and already observed in Example 4.7. For $\lambda = 2$, the numbers are 3.68 and 7.06, respectively while for $\lambda = 1.5$ they are 4.39 and 6.12. Finally, for λ arbitrarily close to one, we obtain the bounds 4.65 and 5.81. Notice that these bounds form an interval containing the Black-Scholes value of the option.

If we took $d = 0$ we would obtain the Black-Scholes value 5.22 as common bound for $\lambda = 1$ by solving the associated buyer and writer problems, as done in [3].

We observe from the results that the price to pay to acknowledge an ambiguity in the reference measure measured by a information-theoretic distance of at most 0.01 translates into approximately 10% wider price bounds in this example. It is reasonable to expect that the bound intervals will widen with increasing d, and *vice versa*, *i.e.*, will narrow down with decreasing d. This is indeed the case in practical computation as we observe in Figure 1 with the smooth curves (for $d = 0.001$) sandwiched between the outer smooth curve corresponding to $d = 0.01$ and the dotted curve corresponding to the original Bernardo-Ledoit gain-loss approach.

The experiment can be repeated using different values for the stock price. However, the reader is warned that the results depend on the choice of initial stock price. For the initial stock price equal to the strike price, *i.e.*, for an at-the-money call, we may not be able to decrease λ to 1. We obtain infeasible problems for values below $\lambda = 1.8$. This leads to the observation as in [3] that bounds are looser for near-the-money options.

6. Conclusions and future work

Departing from the observation that the price bounds on contingent claims obtained from the gain-loss criterion of Bernardo and Ledoit [3] are sensitive to the reference measure, we developed a result akin to the one-step fundamental theorem of asset pricing using the gain-loss based criterion of Bernardo-Ledoit acknowledging a certain ambiguity of measure. The ambiguity of measure can be specified in different ways, as long as it leads to computationally tractable optimization problems. We have illustrated the computation of the bounds in option pricing without intermediate trading in an incomplete market setting where an investor questions the validity of the risk-neutral probabilities in a Black-Scholes framework.

In the future, we intend to develop multiperiod extensions of the present work. However, this extension will require a more complicated duality setup than required for the present paper, *e.g.*, the duality approaches used in [19,31].

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