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UNIQUENESS OF STABLE MEISSNER STATE SOLUTIONS OF THE CHERN-SIMONS-HIGGS ENERGY*

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Abstract. For external magnetic field $h_{ex} \leq C\varepsilon^{-\alpha}$, we prove that a Meissner state solution for the Chern-Simons-Higgs functional exists. Furthermore, if the solution is stable among all vortexless solutions, then it is unique.

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1. Introduction

In this paper, we study uniqueness of stable Meissner solutions for the following Chern-Simons-Higgs functional

$$G_{csh}\left(u,A\right)=\frac{1}{2}\int_{\Omega}\left|\nabla_{A}u\right|^{2}+\frac{\mu_{\varepsilon}^{2}}{4}\frac{\left|\operatorname{curl}A-h_{ex}\right|}{\left|u\right|^{2}}+\frac{1}{\varepsilon^{2}}\left|u\right|^{2}\left(1-\left|u\right|^{2}\right)^{2}. \tag{1.1}$$

The associated Euler-Lagrange equations for (1.1) are

$$-\frac{\mu_{\varepsilon}^{2}}{4}\frac{\left|\operatorname{curl}A-h_{ex}\right|^{2}}{|u|^{4}}u=\nabla_{A}^{2}u+\frac{1}{\varepsilon^{2}}u\left(1-|u|^{2}\right)\left(3|u|^{2}-1\right) \tag{1.2}$$

$$0=-\frac{\mu_{\varepsilon}^{2}}{4}\operatorname{curl}\left(\frac{\operatorname{curl}A-h_{ex}}{|u|^{2}}\right)+j_{A}(u). \tag{1.3}$$

The paper is motivated by Serfaty's work [9] on Ginzburg-Landau energy where she proved uniqueness of stable Meissner state solutions for $h_{ex} \leq C\varepsilon^{-\alpha}$. In addition, it was proved in the same work that vortexless solution to Ginzburg-Landau equation continue to exists for h_{ex} higher than the critical field (up to $h_{ex} \leq C\varepsilon^{-\alpha}$) and is locally minimizing (for h_{ex} below the first critical field, it is proved by Sandier and Serfaty [8] that the vortexless solution to G-L equation is globally minimizing). The uniqueness of the Meissner state for the Ginzburg-Landau energy has been studied elsewhere, including Ye and Zhou [12] for the case with trivial gauge field and Bonnet *et al.* [3] for the full Ginzburg-Landau energy. In [3] the authors show uniqueness of the

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Meissner solution for small ε and $h_{ex} \approx C\varepsilon^{-1}$ by looking for solutions in a particular function space; whereas in [9] the author showed the uniqueness of the Meissner solution for $h_{ex} \leq C\varepsilon^{-\alpha}$ for solutions in a different function space.

Remark 1.1. The study of uniqueness of solutions to the Ginzburg-Landau energy when vortices are present is much more difficult. Pacard and Riviere [7] proved uniqueness of critical points u_ε of the Ginzburg-Landau energy with trivial gauge field when the singularities of the limiting field are nondegenerate critical points of the renormalized energy.

We follow the approach of [9] to study Meissner solutions of the Chern-Simons-Higgs energy.

Recently, the authors [10] proved existence of vortexless solutions to (1.2)–(1.3) in the case $h_{ex} \leq \frac{2|\log\varepsilon|}{\mu_\varepsilon^2}$, $1 \gg \mu_\varepsilon \gg \mathrm{e}^{-|\log\varepsilon|^\alpha}$ for $0 < \alpha < 1$. The solution obtained in [10] is a minimizer in

$$V = \left\{(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1\left(\Omega, \mathbb{R}^2\right) : |u| = 1 \text{ on } \partial\Omega\right\}.$$

It is also shown in [10] that for h_{ex} higher than critical field, a minimizer in V must have a vortex.

Remark 1.2. When $\mu_\varepsilon \to \mu \in (0, +\infty]$ the critical magnetic field was shown to be asymptotically $h_{c_1} = H_1(\mu, \Omega)\,|\log\varepsilon|$, where the constant $H_1(\mu, \Omega)$ is calculated in terms of a scaled London equation, see [5,6]. A straightforward modification of the analysis of [10] shows that this critical field strength is in fact sharp and that $|u_\varepsilon|$ is strictly bounded away from zero.

It is a natural question to ask whether vortexless solutions continue to exist for h_{ex} higher than critical field and whether it is unique. In this paper, we prove the existence of stable vortexless solutions to (1.2)–(1.3) for $h_{ex} \leq C\varepsilon^{-\alpha}$ and $\limsup_\varepsilon \mu_\varepsilon < \infty$. Under the additional assumption that $\mu_\varepsilon \geq \varepsilon^{\frac{1}{9}}$, the stable vortexless solution obtained is unique. In our setting, we define solution (u, A) of (1.2)–(1.3) to be vortexless if it satisfies $|u| \geq \frac{9}{10}$ in Ω.

Our main results are the following theorems. We again concentrate on the technically interesting $\mu_\varepsilon \to 0$ case.

Theorem 1.3. *There exists $\alpha_0 \in (0, 1/24)$ such that for $\alpha < \alpha_0$, if $h_{ex} \leq C\varepsilon^{-\alpha}$, and $\limsup_\varepsilon \mu_\varepsilon < \infty$, there exists a vortexless solution to (1.2)–(1.3) which is stable under perturbations among vortexless mappings.*

Theorem 1.4. *Assuming $\mu_\varepsilon \geq \varepsilon^{\frac{1}{9}}$, $\limsup_\varepsilon \mu_\varepsilon < \infty$. There exists $\alpha \in (0, 1/24)$ and ε_0 such that, if $\varepsilon < \varepsilon_0$, and $h_{ex} \leq C\varepsilon^{-\alpha}$, a vortexless solution of (1.2)–(1.3) that is stable under perturbation among vortexless functions and satisfies $\int_\Omega |\nabla u|^2 \leq o\left(\varepsilon^\beta\right)$ for some $\beta > 0$ is unique. Let $E_0 = \left\{(u, A) \in D : \ |u| \geq \frac{9}{10}\right\}$. For $\varepsilon < \varepsilon_0$, there exists a unique solution of (1.2)–(1.3) that minimizes G_{csh} over E_0, and its energy is $G_0 + o(1)$ where*

$$G_0 = G_{csh}(1, h_{ex}\nabla^\perp\xi_0)$$

and ξ_0 solves the London equation (2.1).

For $h_{ex} \leq \frac{2|\log\varepsilon|}{\mu_\varepsilon^2}$, $1 \gg \mu_\varepsilon \gg \mathrm{e}^{-|\log\varepsilon|^\alpha}$ for $0 < \alpha < 1$, existence of solutions to (1.2)–(1.3) which satisfy $|u_\varepsilon| \geq \frac{1}{4}$ in Ω was obtained in [10] for all $\varepsilon < \varepsilon_0$. The solution obtained in [10] is a minimizer in V. From there it is not hard to show that $|u| \geq \frac{9}{10}$ in Ω for a smaller choice of ε_0. For h_{ex} higher than the critical field (up to $C\varepsilon^{-\alpha}$), we will prove that vortexless solution continue to exist and is locally minimizing in V.

Remark 1.5. Uniqueness of periodic topological-type vortex solution has been established in the Chern-Simons-Higgs model in the self-dual case, $\mu = \varepsilon$ and $h_{ex} = 0$, see [4,11].

The uniqueness proof is motivated by an idea of Serfaty [9] for Ginzburg-Landau energy, G_{gl}: assuming there are two solutions (u_1, A_1) and (u_2, A_2), she proved, through explicit computations, that

$$G_{gl}\left(\frac{u_1 + u_2}{2}, \frac{A_1 + A_2}{2}\right) < \frac{G_{gl}(u_1, A_1) + G_{gl}(u_2, A_2)}{2}. \tag{1.4}$$

It then follows that for all $t \in (0,1)$, $G_{gl}\left((1-t)\,u_1 + tu_2, (1-t)\,A_1 + tA_2\right) \leq \max\left(G_{gl}\left(u_1, A_1\right), G_{gl}\left(u_2, A_2\right)\right)$, which contradicts the assumed stability of solutions. The idea of Serfaty is the following: for vortexless solutions, we can write $u = \eta e^{i\varphi}$ and (u, A) is gauge equivalent to $(\eta, A - d\varphi) = (\eta, A')$. The Ginzburg-Landau energy becomes

$$G_{gl}\left(u, A\right) = \frac{1}{2}\int_\Omega \left|\eta\right|^2 \left|A'\right|^2 + \left|\nabla\eta\right|^2 + \frac{1}{2\varepsilon^2}\left(1-\eta^2\right)^2 + \left|dA' - h_{ex}\right|^2.$$

The term $I(\eta) = \int_\Omega \frac{1}{2\varepsilon^2}\left(1-\eta^2\right)^2$ is convex for vortexless solutions $\left(\eta \geq \frac{3}{4}\right)$; it follows that

$$\frac{I\left(\eta_1\right) + I\left(\eta_2\right)}{2} - I\left(\frac{\eta_1+\eta_2}{2}\right) \geq \frac{C}{\varepsilon^2}\int_\Omega \left(\eta_1 - \eta_2\right)^2. \tag{1.5}$$

On the other hand for $K\left(\eta, A'\right) = \int_\Omega \left|\eta\right|^2 \left|A'\right|^2$, direct calculation shows

$$\left|\frac{K\left(\eta_1, A_1'\right) + K\left(\eta_2, A_2'\right)}{2} - K\left(\frac{\eta_1+\eta_2}{2}, \frac{A_1'+A_2'}{2}\right)\right| \leq C\left(\max\left(\left|A_1'\right|_{L^\infty}, \left|A_2'\right|_{L^\infty}\right)\right)^2 \int_\Omega \left(\eta_1-\eta_2\right)^2. \tag{1.6}$$

Since $\left|A_i'\right|_{L^\infty} = o\left(\frac{1}{\varepsilon}\right)$, the convex term from $\int_\Omega \frac{1}{2\varepsilon^2}\left(1-\eta^2\right)^2$ dominates over $\int_\Omega \left|\eta\right|^2\left|A\right|^2$ and (1.4) follows from (1.5), (1.6) and the convexity of the rest of the terms.

In our case, under the same gauge choice, the Chern-Simons-Higgs energy becomes

$$G_{csh}\left(u, A\right) = \frac{1}{2}\int_\Omega \eta^2\left|A'\right|^2 + \left|\nabla\eta\right|^2 + \frac{1}{\varepsilon^2}\eta^2\left(1-\eta^2\right)^2 + \frac{\mu_\varepsilon^2}{4}\frac{\left|\operatorname{curl} A' - h_{ex}\right|^2}{\eta^2}.$$

The term $\int_\Omega \frac{1}{\varepsilon^2}\eta^2\left(1-\eta^2\right)^2$ is convex for vortexless solutions $\left(\eta \geq \frac{9}{10}\right)$ with a similar bound from below as (1.5) and the term $\int_\Omega \eta^2\left|A'\right|^2$ is controlled above by (1.6). Finally for term $L\left(\eta, A'\right) = \int_\Omega \frac{\mu_\varepsilon^2}{4}\frac{\left|\operatorname{curl} A' - h_{ex}\right|^2}{\eta^2}$, we have

$$\left|\frac{L\left(\eta_1, A_1'\right) + L\left(\eta_2, A_2'\right)}{2} - L\left(\frac{\eta_1+\eta_2}{2}, \frac{A_1'+A_2'}{2}\right)\right| \leq C\left(\max\left(\left|\operatorname{curl} A_1'\right|_{L^\infty}, \left|\operatorname{curl} A_2'\right|_{L^\infty}\right)\right)^2 \int_\Omega \left(\eta_1-\eta_2\right)^2.$$

Since $\left|A'\right|_{L^\infty} = o\left(\frac{1}{\varepsilon}\right)$, $\left|\operatorname{curl} A'\right|_{L^\infty} = o\left(\frac{1}{\varepsilon}\right)$ (Lem. 3.3), we obtain the same conclusion.

2. Proof of existence

Following [10], we introduce the following notation.

$$F\left(u\right) = \frac{1}{2}\int_\Omega \left|\nabla u\right|^2 + \frac{1}{\varepsilon^2}\left|u\right|^2\left(1-\left|u\right|^2\right)^2,$$

and we assume

$$A = d^*\xi, \quad \xi = h_{ex}\xi_0 + \zeta,$$

where

$$\begin{cases} -\frac{\mu_\varepsilon^2}{4}\Delta^2\xi_0 + \Delta\xi_0 = 0 & \text{in } \Omega, \\ \Delta\xi_0 = 1 & \text{on } \partial\Omega, \\ \xi_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

and

$$\zeta = \Delta\zeta = 0 \qquad \text{on } \partial\Omega.$$

We quote the following estimate from [10].

Lemma 2.1. *Suppose* $|\Omega| \leq F$, $G_{csh}(u, A) \leq M_\varepsilon$ *and* $\eta = |u| \geq \frac{1}{2}$ *on* $\partial\Omega$, *then for all* $2 < p < \infty$ *and* $0 < \beta < \frac{2}{p}$, *the following estimates hold*

$$\|\eta\|_{H^1} \leq C\sqrt{M_\varepsilon}, \tag{2.2}$$

$$\left\|1 - \eta^2\right\|_{L^2} \leq C\varepsilon M_\varepsilon, \tag{2.3}$$

$$\|1 - \eta\|_{L^p} \leq C_{p,\beta}\varepsilon^\beta M_\varepsilon^{\frac{1+\beta}{2}}, \tag{2.4}$$

$$\|\eta\|_{L^p} \leq C_{p,\beta}\varepsilon^\beta M_\varepsilon^{\frac{1+\beta}{2}} + |\Omega|\,. \tag{2.5}$$

Moreover, for all $1 \leq \alpha < 2$, $0 < \beta < \frac{2-\alpha}{\alpha}$, we have bounds

$$\|j_A(u)\|_{L^\alpha} \leq \left(C_{\alpha,\beta}\varepsilon^\beta M_\varepsilon^{\frac{1+\beta}{2}} + |\Omega|\right) M_\varepsilon^{\frac{1}{2}}, \tag{2.6}$$

$$\|h - h_{ex}\|_{L^\alpha} \leq \frac{C_{\alpha,\beta}}{\mu_\varepsilon}\sqrt{M_\varepsilon}\left(C_{\alpha,\beta}\varepsilon^\beta M_\varepsilon^{\frac{1+\beta}{2}} + |\Omega|\right), \tag{2.7}$$

where $C_{\alpha,\beta} \longrightarrow \infty$ as $\alpha \longrightarrow 2$. If (u, A) is a weak solution of (1.3), we have

$$\left\|\frac{h - h_{ex}}{\eta^2}\right\|_{W^{1,q}} \leq \frac{C_q}{\mu_\varepsilon^2}\sqrt{M_\varepsilon}\left(C_{q,\beta}\varepsilon^\beta M_\varepsilon^{\frac{1+\beta}{2}} + |\Omega|\right) \tag{2.8}$$

for all $1 \leq q < 2$, $0 < \beta < \frac{2-q}{q}$.

An immediate corollary of Lemma 2.1 is the following lemma.

Lemma 2.2. *Given* $h_{ex} \leq C\varepsilon^{-\alpha}$ *for some* $0 < \alpha < \frac{1}{24}$, $\limsup \mu_\varepsilon < \infty$. *If* $G_{csh}(u, A) \leq M_\varepsilon = C\mu_\varepsilon^2 h_{ex}^2$, *then for any* $2 < p < \infty$,

$$\|\eta\|_{L^p} \leq C_p M_\varepsilon + |\Omega|\,, \tag{2.9}$$

if $2 < p < 22$,

$$\|\eta\|_{L^p} \leq C_p. \tag{2.10}$$

Moreover, if (u, A) *satisfies* (1.3), $A = d^*\xi$, *there exists* $\beta > 0$, *such that*

$$|\nabla\xi|_{L^\infty} \leq \frac{C}{\mu_\varepsilon^2}\sqrt{M_\varepsilon}\left(C_\beta\varepsilon^\beta M_\varepsilon^{\frac{1+\beta}{2}} + |\Omega|\right) + Ch_{ex}; \tag{2.11}$$

in particular, this implies

$$|\nabla\xi|_{L^\infty} \leq \frac{C}{\mu_\varepsilon^2}M_\varepsilon^{\frac{3}{2}} + Ch_{ex}. \tag{2.12}$$

Proof. (2.9) follows directly from (2.5). By (2.5), we have

$$\|\eta\|_{L^p} \leq C\varepsilon^\beta\mu_\varepsilon^{1+\beta}\varepsilon^{-\alpha(1+\beta)} + |\Omega|\,, \tag{2.13}$$

pick β close to $\frac{2}{p}$, for $0 < \alpha < \frac{1}{24}$, $\limsup \mu_\varepsilon < \infty$, (2.10) follows from (2.13) when $2 < p < 22$. To prove (2.11), since

$$\|h - h_{ex}\|_{L^r} \leq \left\|\frac{h - h_{ex}}{\eta^2}\right\|_{L^t}\|\eta\|_{L^{2s}}^2 \tag{2.14}$$

with $\frac{1}{r}=\frac{1}{t}+\frac{1}{s}$. Pick $2<r<s<11$, there exists $q<2$ such that $\frac{2q}{2-q}>t=\frac{rs}{s-r}$. By (2.8) and Sobolev embedding, we deduce

$$\left\|\frac{h-h_{ex}}{\eta^2}\right\|_{L^t}\leq C\left\|\frac{h-h_{ex}}{\eta^2}\right\|_{W^{1,q}}\leq\frac{C_q}{\mu_\varepsilon^2}\sqrt{M_\varepsilon}\left(C_{q,\beta}\varepsilon^\beta M_\varepsilon^{\frac{1+\beta}{2}}+|\Omega|\right)\tag{2.15}$$

for $0<\beta<\frac{2-q}{q}$. (2.11) follows from (2.10), (2.14), (2.15) and Sobolev embedding. Finally (2.12) follows directly from (2.11). □

Following idea of proof of Lemma 2.3 in [10], applying estimates in Lemmas 2.1 and 2.2, we have the following gradient estimate.

Lemma 2.3. *Assume (u,A) is a solution of (1.2)–(1.3) satisfying $\frac{\partial u}{\partial\nu}=0$ on $\partial\Omega$ and $G_{csh}(u,A)\leq M_\varepsilon$, $h_{ex}\leq\frac{\sqrt{M_\varepsilon}}{\mu_\varepsilon}$. If $\varepsilon\frac{M_\varepsilon^2}{\mu_\varepsilon}\leq C$, we have*

$$|\nabla u|\leq\frac{C_0}{\varepsilon},$$

where C_0 is a constant independent of u, A, and ε, μ_ε.

We introduce the following regularization of u (similar regularization for Ginzburg-Landau energy is introduced in [1] and used in [9]). Given any $0<\gamma<1$, for any $(u,A)\in V$, u^γ is defined as a minimizer for

$$\inf_{\substack{H^1(\Omega,\mathbb{C})\\|v|=1\text{ on }\partial\Omega}}\frac{1}{2}\int_\Omega|\nabla v|^2+\frac{1}{\varepsilon^2}|v|^2\left(1-|v|^2\right)^2+\frac{|v-u|^2}{\varepsilon^{2\gamma}}.$$

Lemma 2.4. *u^γ is in $H^3(\Omega,\mathbb{C})$ and satisfies*

$$\begin{aligned}-\Delta u^\gamma&=\frac{1}{\varepsilon^2}u^\gamma\left(1-|u^\gamma|^2\right)\left(3|u^\gamma|^2-1\right)+\frac{u-u^\gamma}{\varepsilon^{2\gamma}}\\F\left(u^\gamma\right)&\leq F(u)\\|\nabla u^\gamma|&\leq\frac{C}{\varepsilon}.\end{aligned}$$

Proof. Follow the same proof as in [1,2], where we replace $\frac{1}{\varepsilon^2}u^\gamma\left(1-|u^\gamma|^2\right)$ with $\frac{1}{\varepsilon^2}u^\gamma\left(1-|u^\gamma|^2\right)\left(3|u^\gamma|^2-1\right)$. □

Since $|\nabla u^\gamma|\leq\frac{C}{\varepsilon}$, the vortices of u^γ are well defined. The following ball construction lemma is a variation of the ball construction used in [10].

Proposition 2.5. *There exists $\alpha\in(0,1/24)$, such that if $h_{ex}\leq C\varepsilon^{-\alpha}$, let $u:\Omega\longrightarrow\mathbb{C}$ be such that $|\nabla u|_\infty\leq\frac{C_0}{\varepsilon}$, $|u|=1$ on $\partial\Omega$ and $F(u)\leq C\varepsilon^{-2\alpha}$. Then there exist disjoint balls $\{B_i\}_{i\in I}$ such that for sufficiently small ε*

(1) $\left\{|u(x)|<\frac{10}{11}\right\}\subset\cup_iB_i$.
(2) $\operatorname{card}I\leq C\varepsilon^{-2\alpha}$.
(3) $r_i\leq C\frac{\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|}$.
(4) *If $\overline{B_i}\subset\Omega$, and $d_i=\deg(u,\partial B_i)$, then*

$$F(u,B_i)\geq\pi\frac{|d_i|}{3}|\log\varepsilon|-C.\tag{2.16}$$

Proof. Follow the proof of Proposition 2.13 in [10], choosing $s_1 = \varepsilon^{\frac{2}{3}}$ in the initial step, replacing the assumption $h_{ex} \leq C\frac{|\log \varepsilon|}{\mu_\varepsilon^2}$ by $h_{ex} \leq C\varepsilon^{-\alpha}$ and $\frac{1}{2}$ by $\frac{10}{11}$. □

We recall the definitions

$$V(\xi) = \frac{1}{2}\int |\nabla \xi|^2 + |\Delta \xi|^2 + 2\pi \sum_{i\in I} d_i \xi(a_i) - h_{ex}\int_\Omega \Delta \xi,$$
$$\widetilde{V}(\zeta) = \frac{1}{2}\int_\Omega |\nabla \zeta|^2 + |\Delta \zeta|^2 + 2\pi \sum_{i\in I} d_i \zeta(a_i).$$

Lemma 2.6. *There exists $\alpha \in (0, 1/24)$ such that if $h_{ex} \leq C\varepsilon^{-\alpha}$, $\limsup_{\varepsilon \longrightarrow 0} \mu_\varepsilon < \infty$, given (u, A) satisfying (1.3) and $F(u) \leq C\mu_\varepsilon^2\, h_{ex}^2$, the energy can be split as*

$$\begin{aligned} G_{csh}(u, A) &= F(u) + V(\xi) + o(\varepsilon^\beta) \\ &= G_0 + F(u) + 2\pi h_{ex} \sum_{i\in I} d_i \xi_0(a_i) + \widetilde{V}(\zeta) + o(\varepsilon^\beta), \end{aligned}$$

where (a_i, d_i) denote the vortices of u^γ. $G_0 = \int_\Omega \frac{h_{ex}^2}{2}|\nabla \xi_0|^2 + \frac{\mu_\varepsilon^2}{8}h_{ex}^2|\Delta \xi_0 - 1|^2$, $\beta = \beta(\alpha) > 0$.

Proof. Write

$$|\nabla_A u|^2 = |\nabla u|^2 + |\nabla \xi|^2 + (1-\eta^2)|\nabla \xi|^2 + 2(\mathrm{i}u, \xi_{x_2} u_{x_1} - \xi_{x_1} u_{x_2}),$$
$$\left|\frac{h - h_{ex}}{\eta}\right|^2 = |h - h_{ex}|^2 + \frac{|h - h_{ex}|^2}{|u|^4}|u|^2\left(1 - |u|^2\right).$$

Since (u, A) satisfies (1.3), by (2.3) and (2.12), we conclude

$$\begin{aligned} \int_\Omega (1-\eta^2)|\nabla \xi|^2 &\leq C|\nabla \xi|_{L^\infty}^2 \left\|1-\eta^2\right\|_{L^2} \\ &\leq C\left(\frac{M_\varepsilon^{\frac{3}{2}}}{\mu_\varepsilon^2} + h_{ex}\right)^2 \varepsilon M_\varepsilon \\ &\leq C\left(\mu_\varepsilon h_{ex}^3 + h_{ex}\right)^2 \varepsilon \mu_\varepsilon^2 h_{ex}^2 \\ &\leq C\varepsilon^{1-8\alpha}, \end{aligned}$$

and for $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, by (2.3), (2.9) and (2.15)

$$\begin{aligned} \int_\Omega \frac{|h - h_{ex}|^2}{|u|^4}|u|^2\left(1 - |u|^2\right) &\leq \left\|\frac{|h - h_{ex}|}{|u|^2}\right\|_{L^{2p}}^2 \|\eta\|_{L^{2q}}^2 \left\|1-\eta^2\right\|_{L^2} \\ &\leq C\left(\frac{M_\varepsilon^{\frac{3}{2}}}{\mu_\varepsilon^2}\right)^2 M_\varepsilon^2 \varepsilon M_\varepsilon \\ &\leq C\varepsilon^{1-12\alpha}. \end{aligned}$$

Therefore

$$G_{csh}(u, A) = \frac{1}{2}\int_\Omega |\nabla u|^2 + |\nabla \xi|^2 + \frac{\mu_\varepsilon^2}{4}|h - h_{ex}|^2 + 2(\mathrm{i}u, \xi_{x_2} u_{x_1} - \xi_{x_1} u_{x_2}) + o(\varepsilon^\beta).$$

The rest of the proof follows from similar argument as in Lemmas 4.2 and 4.3 in [9], replacing the assumption $F(u) < M|\log\varepsilon|$ and $h_{ex} \le C|\log\varepsilon|$ by $F(u) \le C\mu_\varepsilon^2 h_{ex}^2$, $h_{ex} \le C\varepsilon^{-\alpha}$. □

Lemma 2.7. *Let α, h_{ex} and μ_ε satisfy the same assumptions as in Lemma 2.6. If (u, A) is a solution of (1.2)–(1.3) such that u^γ has no vortex $\left(|u^\gamma| \ge \frac{9}{10}\right)$ and that $G_{csh}(u, A) \le G_0$ and $F(u) \le C\mu_\varepsilon^2 h_{ex}^2$, then u has no vortex in Ω.*

Proof. From Lemma 2.6 and the assumption, we obtain

$$G_0 \ge G_{csh}(u, A) = G_0 + F(u) + \widetilde{V}(\zeta) + o\left(\varepsilon^\beta\right),$$

therefore

$$F(u) + \widetilde{V}(\zeta) \le o\left(\varepsilon^\beta\right). \tag{2.17}$$

Since (u, A) is a solution of (1.2)–(1.3), by elliptic estimates (Lem. 2.3), we have $|\nabla u| \le \frac{C}{\varepsilon}$. Therefore the vortex structure of u is well defined and (2.17) implies u is vortexless. □

Proposition 2.8. *There exists $\alpha \in (0, 1/24)$ and ε_0 such that if $\varepsilon < \varepsilon_0$ and $h_{ex} \le C\varepsilon^{-\alpha}$, $\limsup_\varepsilon \mu_\varepsilon < \infty$, there exists a solution (u, A) of (1.2)–(1.3) satisfying $|u| \ge \frac{9}{10}$, that is a local minimizer of J in V. In addition,*

$$\inf_{\theta\in[0,2\pi]} \left\|(u, \xi) - \left(\mathrm{e}^{\mathrm{i}\theta}, h_{ex}\xi_0\right)\right\| \longrightarrow 0 \quad as \quad \varepsilon \longrightarrow 0, \tag{2.18}$$

where

$$\|(u, z)\|^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\nabla z\|_{L^2}^2 + \|\Delta z\|_{L^2}^2.$$

Proof. Let

$$G_k(u, A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{\mu_\varepsilon^2}{4}\frac{|\mathrm{curl}\, A - h_{ex}|}{|u|^2 + \frac{1}{k^2}} + \frac{1}{\varepsilon^2}|u|^2\left(1 - |u|^2\right)^2.$$

Consider the open domain

$$U = \left\{(u, A) \in V:\ F(u) + \frac{1}{2}\int_\Omega |\nabla\zeta|^2 + |\Delta\zeta|^2 < \varepsilon^{\frac{\beta}{2}}\right\},$$

where β is given by Lemma 2.6. There exists $(v_k, A_k) \in \overline{U}$ which achieves $\min_{\overline{U}} G_k$ and (v_k, A_k) satisfies

$$0 = -\frac{\mu_\varepsilon^2}{4}\,\mathrm{curl}\left(\frac{\mathrm{curl}\, A_k - h_{ex}}{|v_k|^2 + \frac{1}{k^2}}\right) + j_{A_k}(v_k). \tag{2.19}$$

This can be shown by the following argument. Given (u_k^n, A_k^n) minimizing sequence of G_k, since

$$F(u_k^n) + \frac{1}{2}\int_\Omega |\nabla\zeta_k^n|^2 + |\Delta\zeta_k^n|^2 \le \varepsilon^{\frac{\beta}{2}},$$

$$A_k^n = h_{ex}d^*\xi_0 + d^*\zeta,$$

we conclude (u_k^n, A_k^n) is a bounded sequence in $H^1(\Omega, \mathbb{C}) \times H^1\left(\Omega, \mathbb{R}^2\right)$. Subject to a subsequence, we can assume $(u_k^n, A_k^n) \rightharpoonup (v_k, A_k)$ in $H^1(\Omega, \mathbb{C}) \times H^1\left(\Omega, \mathbb{R}^2\right)$ as $n \longrightarrow \infty$ and

$$G_k(v_k, A_k) \le \lim\inf_{n\longrightarrow\infty} G_k(u_k^n, A_k^n)$$

$$F(v_k) + \frac{1}{2}\int_\Omega |\nabla\zeta_k|^2 + |\Delta\zeta_k|^2 \le \lim\inf_{n\longrightarrow\infty} F(u_k^n) + \frac{1}{2}\int_\Omega |\nabla\zeta_k^n|^2 + |\Delta\zeta_k^n|^2.$$

Therefore (v_k, A_k) is a minimizer of G_k in $\overline{U}$. Applying Lemma 2.4 and Proposition 2.5 to v_k, we obtain

$$\begin{aligned}\varepsilon^{\frac{\beta}{2}} &> F(v_k) \geq F(v_k^{\gamma}) \\ &\geq \pi \sum_{i\in L} \frac{|d_i|}{3} |\log\varepsilon| - C,\end{aligned}$$

where L is the collection of vortex balls for v_k^{γ}. This implies $L = \emptyset$, *i.e.* v_k^{γ} has no vortex (since $d_i \neq 0$). Moreover, when $\frac{1}{k^2} < \varepsilon$, we can prove a similar energy splitting formula for G_k as Lemma 2.6,

$$G_k(v_k, A_k) = G_0 + F(v_k) + \frac{1}{2}\int_{\Omega} |\nabla\zeta_k|^2 + |\Delta\zeta_k|^2 + o(\varepsilon^{\beta}). \tag{2.20}$$

On the other hand, $\left(1, h_{ex}\nabla^{\perp}\xi_0\right) \in U$ is a comparison map, by minimality of (v_k, A_k), we obtain $G_{csh}(v_k, A_k) \leq G_0$. This together with (2.20) implies

$$F(v_k) + \frac{1}{2}\int_{\Omega} |\nabla\zeta_k|^2 + |\Delta\zeta_k|^2 \leq o(\varepsilon^{\beta}).$$

This guarantees $(v_k, A_k) \in \overset{\circ}{U}$, *i.e.* (v_k, A_k) is a local minimizer of G_k and satisfies

$$-\frac{\mu_{\varepsilon}^2}{4}\frac{|\mathrm{curl}\, A_k - h_{ex}|^2}{\left(|v_k|^2 + \frac{1}{k^2}\right)^2} u = \nabla_A^2 v_k + \frac{1}{\varepsilon^2} v_k\left(1 - |v_k|^2\right)\left(3|v_k|^2 - 1\right) \tag{2.21}$$

$$0 = -\frac{\mu^2}{4}\,\mathrm{curl}\left(\frac{\mathrm{curl}\, A_k - h_{ex}}{|v_k|^2 + \frac{1}{k^2}}\right) + j_{A_k}(v_k). \tag{2.22}$$

By elliptic estimates (similar to Lem. 2.1), (v_k, A_k) is bounded in $H^1 \times H^1$. Up to a subsequence, we assume $(v_k, A_k) \rightharpoonup (u, A)$ in $H^1 \times H^1$ where (u, A) satisfies (1.2)–(1.3) and

$$G_{csh}(u, A) \leq \lim\inf_{k\longrightarrow\infty} G_k(v_k, A_k). \tag{2.23}$$

Given a minimizing sequence (u_k, B_k) of G_{csh} in U, we have

$$G_{csh}(u_k, B_k) \geq G_k(u_k, B_k) \geq G_k(v_k, A_k).$$

(2.23) implies (u, A) is a minimizer of G_{csh} in U and $(u, A) \in \overline{U}$. We repeat the regularization argument for u and conclude u^{γ} is vortexless. By Lemma 2.7, u is vortexless. Finally, since $|u| = 1$ on $\partial\Omega$, energy estimates imply $\left\|1 - |u|^2\right\|_{L^2} \leq o(1)$, from here (2.18) can be proved following exact same argument of step 2 in the proof of Proposition 3.1 in [9]. □

3. Proof of uniqueness

We assume that $h_{ex} \leq C\varepsilon^{-\alpha}$ and $\mu_{\varepsilon} \geq \varepsilon^{\frac{1}{9}}$. We prove that if a Meissner solution (u, A) exists and stable under perturbation among vortexless mappings, then it is unique among the solutions satisfying $\|\nabla u\|_{L^2}^2 \leq o(\varepsilon^{\beta})$. (Here β is given by Lem. 2.6.) In particular, a solution (u, A) that is minimizing among all vortexless solutions is unique.

We prove uniqueness by contradiction. If there are two distinct stable solutions (u_1, A_1) and (u_2, A_2) of (1.2) and (1.3) with $\mathrm{div}\, A_j = 0$, $A_j \cdot \nu = 0$ on $\partial\Omega$ and $\|\nabla u_j\|_{L^2}^2 \leq o(\varepsilon^{\beta})$. We assume $G_{csh}(u_1, A_1) \leq G_{csh}(u_2, A_2)$. Denote $\eta_j = |u_j|$.

Lemma 3.1. *For $j = 1, 2$, $(u_j,\ A_j)$ is gauge equivalent to (η_j, B_j) with*

$$\operatorname{div}\left(\eta_j^2 B_j\right) = 0 \tag{3.1}$$

$$G_{csh}\left(u_j, A_j\right) = \frac{1}{2}\int_\Omega \eta_j^2 B_j^2 + \left|\nabla \eta_j\right|^2 + \frac{1}{\varepsilon^2}\eta_j^2\left(1-\eta_j^2\right)^2 + \frac{\mu_\varepsilon^2}{4}\frac{\left|\operatorname{curl} B_j - h_{ex}\right|^2}{\eta_j^2}. \tag{3.2}$$

Proof. Since $\eta_j \geq \frac{9}{10}$, we can write $u_j = \eta_j e^{i\phi_j}$ globally on Ω. We write $B_j = A_j - \nabla\phi_j$, then (u_j, A_j) is gauge equivalent to

$$\left(u_j e^{-i\phi_j}, A_j - \nabla\phi_j\right) = \left(\eta_j, B_j\right)$$

and $\operatorname{curl} A_j = \operatorname{curl} B_j$. Since $\int_\Omega |\nabla_A u|^2$ is invariant under gauge-transformations,

$$\int_\Omega \left|\nabla_{A_j} u_j\right|^2 = \int_\Omega \left|\nabla_{B_j}\eta_j\right|^2 = \int_\Omega \left|\nabla\eta_j - iB_j\eta_j\right|^2 = \int_\Omega \eta_j^2 B_j^2 + \left|\nabla\eta_j\right|^2.$$

The expression (3.2) follows. For (3.1), notice that equation (1.3) gives

$$-\frac{\mu_\varepsilon^2}{4}\operatorname{curl}\left(\frac{\operatorname{curl} A_j - h_{ex}}{\left|u_j\right|^2}\right) = \left(iu_j, \nabla_{A_j} u_j\right) = \left(i\eta_j, \nabla_{B_j}\eta_j\right) = -\eta_j^2 B_j,$$

take divergence on both sides, we get $\operatorname{div}\left(\eta_j^2 B_j\right) = 0$. □

A direct corollary of Lemmas 2.1 and 2.3 is the following

Lemma 3.2. *If (u, A) is weak solution of (1.2)–(1.3) satisfying $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, the following holds for any $1 < q \leq 4$, $\frac{3}{4} \leq \delta < 1$,*

$$\left\|j_A(u)\right\|_{L^q(\Omega)} \leq \frac{C_q}{\mu_\varepsilon^2} M_\varepsilon^{\frac{3}{2}} + Ch_{ex} + \frac{C(\Omega,\delta)}{\varepsilon^\delta}\left(\sqrt{M_\varepsilon}\right)^{1-\delta}, \tag{3.3}$$

$$\left\|\frac{h - h_{ex}}{\eta^2}\right\|_{W^{1,q}(\Omega)} \leq \left(\frac{C_q}{\mu_\varepsilon^2} M_\varepsilon^{\frac{3}{2}} + Ch_{ex} + \frac{C(\Omega,\delta)}{\varepsilon^\delta}\left(\sqrt{M_\varepsilon}\right)^{1-\delta}\right)\frac{1}{\mu_\varepsilon^2}. \tag{3.4}$$

In particular, this implies

$$\|\operatorname{curl} A\|_{L^\infty(\Omega)} \leq \left(\frac{C_q}{\mu_\varepsilon^2} M_\varepsilon^{\frac{3}{2}} + Ch_{ex} + \frac{C(\Omega,\delta)}{\varepsilon^\delta}\left(\sqrt{M_\varepsilon}\right)^{1-\delta}\right)\frac{1}{\mu_\varepsilon^2}. \tag{3.5}$$

Proof. Since $j_A(u) = (iu, \nabla_A u) = (iu, \nabla u - iAu)$, it follows from (2.5), (2.12) and Lemma 2.3 that for $1 < q \leq 4$,

$$\begin{aligned}\left\|j_A(u)\right\|_{L^q(\Omega)} &\leq \|\nabla u\|_{L^\infty(\Omega)}^\delta \left\||u|\,|\nabla u|^{1-\delta}\right\|_{L^q(\Omega)} + \|A\|_{L^\infty(\Omega)}\left\||u|^2\right\|_{L^q} \\ &\leq \frac{C_q}{\mu_\varepsilon^2} M_\varepsilon^{\frac{3}{2}} + Ch_{ex} + \frac{C(\Omega,\delta)}{\varepsilon^\delta}\left(\sqrt{M_\varepsilon}\right)^{1-\delta}.\end{aligned}$$

(3.4) follows from elliptic estimates for equations (1.3) and (3.3). Finally (3.5) follows from (2.10), (3.4) and Sobolev embedding. □

Lemma 3.3. *Given (u_j, A_j) stable Meissner state solution and satisfying $\|\nabla u_j\|_{L^2}^2 \leq o\left(\varepsilon^\beta\right)$, β is given by Lemma 2.6. If $G_{csh}\left(u_j, A_j\right) \leq C\mu_\varepsilon^2 h_{ex}^2$, $h_{ex} \leq C\varepsilon^{-\alpha}$, $0 < \alpha < \frac{1}{24}$ and $\mu_\varepsilon \geq \varepsilon^{\frac{1}{9}}$, $\limsup \mu_\varepsilon < \infty$, then as $\varepsilon \longrightarrow 0$,*

$$\left\|B_j\right\|_{L^\infty(\Omega)} \leq o\left(\frac{1}{\varepsilon}\right) \tag{3.6}$$

$$\left\|\operatorname{curl} B_j\right\|_{L^\infty(\Omega)} \leq o\left(\frac{1}{\varepsilon}\right). \tag{3.7}$$

Proof. We follow idea of [9] to prove (3.6). If we assume (u_j, A_j) is energy minimizing among vortexless solutions, then

$$G_{csh}(u_j, A_j) \le G_{csh}\left(1, h_{ex}\nabla^{\perp}\xi_0\right) = G_0 \le C\mu_\varepsilon^2 h_{ex}^2.$$

Decomposing $\xi = h_{ex}\xi_0 + \zeta$ and dropping the subscript j, we obtain

$$\begin{aligned} G_0 &\ge G_{csh}(u, A) \\ &\ge \frac{1}{2}\int_\Omega |\nabla u|^2 + |\nabla \xi|^2 \\ &\quad + \frac{\mu_\varepsilon^2}{4}\frac{|\Delta\xi - h_{ex}|^2}{|u|^2} + \frac{1}{\varepsilon^2}|u|^2\left(1 - |u|^2\right)^2 \\ &\quad + o\left(\varepsilon^\beta\right) \\ &= G_0 + F(u) + \frac{1}{2}\int_\Omega |\Delta\zeta|^2 + |\nabla\zeta|^2 + o\left(\varepsilon^\beta\right). \end{aligned}$$

Therefore

$$\int_\Omega |\nabla u|^2 = \int_\Omega |\nabla\eta|^2 + \eta^2|\nabla\phi|^2 \le o\left(\varepsilon^\beta\right)$$

for some $\beta > 0$. We now assume this condition is satisfied. From Lemma 2.1, we have $\|A_j\|_{L^\infty} \le \frac{C_q}{\mu^2}M_\varepsilon^{\frac{3}{2}} + Ch_{ex}$. Therefore

$$\begin{aligned} \|B_j\|_{L^\infty} &\le \|A_j\|_{L^\infty} + \|\nabla\phi\|_{L^\infty} \\ &\le \frac{C}{\varepsilon}. \end{aligned}$$

For any $p > 1$, by interpolation, we have

$$\begin{aligned} \|\nabla\eta\|_{L^p} &\le C\|\nabla\eta\|_{L^\infty}^{1-\frac{2}{p}}\|\nabla\eta\|_{L^2}^{\frac{2}{p}} \\ &\le C\varepsilon^{-1+\frac{2}{p}}\varepsilon^{\frac{\beta}{p}} \\ &\le C\varepsilon^\gamma \end{aligned} \tag{3.8}$$

for some $\gamma > 0$, provided $p < \beta + 2$. On the other hand, from (3.1), we have

$$\eta^2 \operatorname{div} B_j = -2\eta\nabla\eta\cdot B_j,$$

which implies

$$-\Delta\phi = -\frac{2}{\eta}\nabla\eta\cdot B_j.$$

We deduce that

$$\|\Delta\phi\|_{L^p} \le C\|B_j\|_{L^\infty}\|\nabla\eta\|_{L^p}.$$

Choosing $2 < p < \beta + 2$, we have

$$\|\Delta\phi\|_{L^p} \le C\frac{\varepsilon^\gamma}{\varepsilon} \le o\left(\frac{1}{\varepsilon}\right).$$

Since $\frac{\partial u}{\partial\nu} = 0$ implies $\frac{\partial\phi}{\partial\nu} = 0$ on $\partial\Omega$. From elliptic estimates and Sobolev embedding we deduce that

$$\|\nabla\phi\|_{L^\infty} \le o\left(\frac{1}{\varepsilon}\right),$$

from which follows

$$\|B_j\|_{L^\infty} \le o\left(\frac{1}{\varepsilon}\right).$$

Finally since $\operatorname{curl} B_j = \operatorname{curl} A_j$, if $\mu_\varepsilon \ge \varepsilon^{\frac{1}{9}}$, taking $\delta = \frac{3}{4}$ in (3.5), (3.7) follows directly. □

We are going to prove that

$$G_{csh}\left(\frac{\eta_1+\eta_2}{2}, \frac{B_1+B_2}{2}\right) < \frac{G_{csh}(\eta_1, B_1) + G_{csh}(\eta_2, B_2)}{2} \le G_{csh}(\eta_2, B_2),$$

thus getting a contradiction to the assumption that (u_2, A_2) is stable.

Lemma 3.4. *If $(\eta_1, B_1) \ne (\eta_2, B_2)$, then*

$$\int_\Omega \left(\frac{\eta_1+\eta_2}{2}\right)^2 \left|\frac{B_1+B_2}{2}\right|^2 + \frac{\mu_\varepsilon^2}{4}\left|\frac{\operatorname{curl}\frac{B_1+B_2}{2} - h_{ex}}{\frac{\eta_1+\eta_2}{2}}\right|^2 + \int_\Omega \frac{1}{\varepsilon^2}\left(\frac{\eta_1+\eta_2}{2}\right)^2\left(1-\left(\frac{\eta_1+\eta_2}{2}\right)^2\right)^2 \le$$
$$\frac{1}{2}\int_\Omega \eta_1^2|B_1|^2 + \frac{\mu_\varepsilon^2}{4}\left|\frac{\operatorname{curl} B_1 - h_{ex}}{\eta_1}\right|^2 + \frac{1}{\varepsilon^2}\eta_1^2\left(1-\eta_1^2\right)^2 + \frac{1}{2}\int_\Omega \eta_2^2|B_2|^2 + \frac{\mu_\varepsilon^2}{4}\left|\frac{\operatorname{curl} B_2 - h_{ex}}{\eta_2}\right|^2 + \frac{1}{\varepsilon^2}\eta_2^2\left(1-\eta_2^2\right)^2.$$

Proof. We compute $X = X_1 + X_2 + X_3,$ where

$$X_1 = \frac{1}{2}\int_\Omega \eta_1^2|B_1|^2 + \eta_2^2|B_2|^2 - \int_\Omega \left(\frac{\eta_1+\eta_2}{2}\right)^2\left|\frac{B_1+B_2}{2}\right|^2, \tag{3.9}$$

$$X_2 = \frac{1}{2}\int_\Omega \frac{1}{\varepsilon^2}\eta_1^2\left(1-\eta_1^2\right)^2 + \frac{1}{\varepsilon^2}\eta_2^2\left(1-\eta_2^2\right)^2 - \int_\Omega \frac{1}{\varepsilon^2}\left(\frac{\eta_1+\eta_2}{2}\right)^2\left(1-\left(\frac{\eta_1+\eta_2}{2}\right)^2\right)^2 \tag{3.10}$$

$$X_3 = \frac{1}{2}\int_\Omega \frac{\mu_\varepsilon^2}{4}\left|\frac{\operatorname{curl} B_1 - h_{ex}}{\eta_1}\right|^2 + \frac{\mu_\varepsilon^2}{4}\left|\frac{\operatorname{curl} B_2 - h_{ex}}{\eta_2}\right|^2 - \int_\Omega \frac{\mu_\varepsilon^2}{4}\left|\frac{\operatorname{curl}\frac{B_1+B_2}{2} - h_{ex}}{\frac{\eta_1+\eta_2}{2}}\right|^2. \tag{3.11}$$

Following [9], we have

$$X_1 = \frac{1}{16}\int_\Omega (\eta_1-\eta_2)^2|B_1+B_2|^2 + 4\eta_1^2|B_1-B_2|^2 + (\eta_2-\eta_1)(B_2-B_1)\cdot\left(B_1(-2\eta_1-4\eta_2) + B_2(-6\eta_1-8\eta_2)\right). \tag{3.12}$$

Since u_1, u_2 are vortexless solutions, we know that $\frac{9}{10} \le \eta_j$ for $j = 1, 2$. This guarantees η_1, η_2 lie in the domain of convexity of function $f(x) = x^2\left(1-x^2\right)^2$. In particular, when $x_1, x_2 \ge \frac{9}{10}$, through Taylor expansion, we have (assuming $x_1 \leq x_2$)

$$\begin{aligned}\frac{1}{2}\left(f\left(x_1\right)+f\left(x_2\right)\right)-f\left(\frac{x_1+x_2}{2}\right) &= \frac{1}{2}\left(f\left(x_1\right)-f\left(\frac{x_1+x_2}{2}\right)\right)+\frac{1}{2}\left(f\left(x_2\right)-f\left(\frac{x_1+x_2}{2}\right)\right)\\ &= \frac{1}{2}\left[f'\left(\frac{x_1+x_2}{2}\right)\left(x_1-\frac{x_1+x_2}{2}\right)+f''\left(\widetilde{x_1}\right)\left(\frac{x_1-x_2}{2}\right)^2\right]\\ &\quad+\frac{1}{2}\left[f'\left(\frac{x_1+x_2}{2}\right)\left(x_2-\frac{x_1+x_2}{2}\right)+f''\left(\widetilde{x_2}\right)\left(\frac{x_1-x_2}{2}\right)^2\right]\\ &= \frac{1}{2}\left(f''\left(\widetilde{x_1}\right)+f''\left(\widetilde{x_2}\right)\right)\left(\frac{x_1-x_2}{2}\right)^2\\ &\geq 2\cdot\left(\frac{x_1-x_2}{2}\right)^2. \end{aligned} \tag{3.13}$$

Here $\widetilde{x_1} \in \left(x_1, \frac{x_1+x_2}{2}\right)$ and $\widetilde{x_2} \in \left(\frac{x_1+x_2}{2}, x_2\right)$ satisfying $\widetilde{x_1}, \widetilde{x_2} \geq \frac{9}{10}$, in the last step, we used this and the fact that $f''(\widetilde{x}_i) \geq f''(\frac{9}{10}) \geq 2$. From (3.13), we obtain estimates for X_2:

$$\begin{aligned}X_2 &= \frac{1}{2\varepsilon^2}\int_\Omega\left[\eta_1^2\left(1-\eta_1^2\right)^2-\left(\frac{\eta_1+\eta_2}{2}\right)^2\left(1-\left(\frac{\eta_1+\eta_2}{2}\right)^2\right)^2\right]\\ &\quad+\frac{1}{2\varepsilon^2}\int_\Omega\left[\eta_2^2\left(1-\eta_2^2\right)^2-\left(\frac{\eta_1+\eta_2}{2}\right)^2\left(1-\left(\frac{\eta_1+\eta_2}{2}\right)^2\right)^2\right]\\ &\geq \frac{1}{\varepsilon^2}\int_\Omega 2\cdot\left(\frac{\eta_1-\eta_2}{2}\right)^2=\frac{1}{2\varepsilon^2}\int_\Omega\left(\eta_1-\eta_2\right)^2. \end{aligned} \tag{3.14}$$

For X_3, we denote $y_j = \frac{\mu_\varepsilon}{2}\left(\operatorname{curl} B_j - h_{ex}\right)$, $j = 1, 2$. Then

$$\begin{aligned}X_3 &= \frac{1}{2}\int_\Omega\left(\frac{y_1}{\eta_1}\right)^2+\left(\frac{y_2}{\eta_2}\right)^2-2\left(\frac{y_1+y_2}{\eta_1+\eta_2}\right)^2\\ &= \frac{1}{2}\int_\Omega\left(\frac{y_1}{\eta_1}+\frac{y_1+y_2}{\eta_1+\eta_2}\right)\left(\frac{y_1}{\eta_1}-\frac{y_1+y_2}{\eta_1+\eta_2}\right)\\ &\quad+\frac{1}{2}\int_\Omega\left(\frac{y_2}{\eta_2}+\frac{y_1+y_2}{\eta_1+\eta_2}\right)\left(\frac{y_2}{\eta_2}-\frac{y_1+y_2}{\eta_1+\eta_2}\right)\\ &= \frac{1}{2}\int_\Omega\frac{y_1\eta_2-y_2\eta_1}{\eta_1+\eta_2}\left(\frac{y_1}{\eta_1^2}-\frac{y_2}{\eta_2^2}+\frac{y_1+y_2}{\eta_1+\eta_2}\left(\frac{1}{\eta_1}-\frac{1}{\eta_2}\right)\right)\\ &= \frac{1}{2}\int_\Omega\frac{y_1\left(\eta_2-\eta_1\right)+\left(y_1-y_2\right)\eta_1}{\eta_1+\eta_2}\cdot\frac{y_1\left(\eta_2^2-\eta_1^2\right)+\left(y_1-y_2\right)\eta_1^2}{\eta_1^2\eta_2^2}\\ &\quad+\frac{1}{2}\int_\Omega\frac{y_1\left(\eta_2-\eta_1\right)+\left(y_1-y_2\right)\eta_1}{\left(\eta_1+\eta_2\right)^2}\cdot\left(y_1+y_2\right)\cdot\frac{\left(\eta_2-\eta_1\right)}{\eta_1\eta_2}\\ &= \frac{1}{2}\int_\Omega\frac{y_1^2\left(\eta_2-\eta_1\right)^2}{\eta_1^2\eta_2^2}+\frac{y_1\cdot\left(\eta_2-\eta_1\right)\left(y_1-y_2\right)}{\eta_2^2\left(\eta_1+\eta_2\right)}+\frac{y_1\cdot\left(\eta_2-\eta_1\right)\left(y_1-y_2\right)}{\eta_1\eta_2^2}\\ &\quad+\frac{1}{2}\int_\Omega\frac{y_1\left(y_1+y_2\right)\left(\eta_2-\eta_1\right)^2}{\left(\eta_1+\eta_2\right)^2\eta_1\eta_2}+\frac{\left(y_1+y_2\right)\left(\eta_2-\eta_1\right)\left(y_1-y_2\right)}{\eta_2\left(\eta_1+\eta_2\right)^2}+\frac{\left(y_1-y_2\right)^2\eta_1}{\eta_2^2\left(\eta_1+\eta_2\right)}. \end{aligned}$$

Note the integrand in X_3 is symmetric in indices $1, 2$, we deduce

$$X_3 = \frac{1}{2}\int_\Omega \frac{y_2^2(\eta_2-\eta_1)^2}{\eta_1^2\eta_2^2} + \frac{y_2\cdot(\eta_2-\eta_1)(y_1-y_2)}{\eta_1^2(\eta_1+\eta_2)} + \frac{y_2\cdot(\eta_2-\eta_1)(y_1-y_2)}{\eta_2\eta_1^2}$$
$$+\frac{1}{2}\int_\Omega \frac{y_2(y_1+y_2)(\eta_2-\eta_1)^2}{(\eta_1+\eta_2)^2\eta_1\eta_2} + \frac{(y_1+y_2)(\eta_2-\eta_1)(y_1-y_2)}{\eta_1(\eta_1+\eta_2)^2} + \frac{(y_1-y_2)^2\eta_2}{\eta_1^2(\eta_1+\eta_2)}.$$

Therefore

$$X_3 = \frac{1}{4}\int_\Omega \frac{(y_1^2+y_2^2)(\eta_2-\eta_1)^2}{\eta_1^2\eta_2^2} + \frac{(y_1+y_2)^2(\eta_2-\eta_1)^2}{(\eta_1+\eta_2)^2\eta_1\eta_2} + \frac{1}{4}\int_\Omega \frac{(y_1-y_2)^2}{(\eta_1+\eta_2)}\left(\frac{\eta_1}{\eta_2^2}+\frac{\eta_2}{\eta_1^2}\right)$$
$$+\frac{1}{4}\int_\Omega (\eta_2-\eta_1)(y_1-y_2)\left[\frac{y_2}{\eta_1^2(\eta_1+\eta_2)} + \frac{y_1}{\eta_2^2(\eta_1+\eta_2)}\right.$$
$$\left.+\frac{y_2}{\eta_2\eta_1^2} + \frac{y_1}{\eta_1\eta_2^2} + \frac{(y_1+y_2)}{\eta_1(\eta_1+\eta_2)^2} + \frac{(y_1+y_2)}{\eta_2(\eta_1+\eta_2)^2}\right]. \tag{3.15}$$

By (3.7),

$$\|y_j\|_{L^\infty} \le \|\text{curl}\, B_j\|_{L^\infty} + h_{ex} \le o\left(\frac{1}{\varepsilon}\right).$$

If we assume for contradiction that $X \le 0$, combining (3.12), (3.14) and (3.15) we obtain

$$\frac{1}{4}\int_\Omega \frac{(y_1^2+y_2^2)(\eta_2-\eta_1)^2}{\eta_1^2\eta_2^2} + \frac{(y_1+y_2)^2(\eta_2-\eta_1)^2}{(\eta_1+\eta_2)^2\eta_1\eta_2} + \frac{1}{4}\int_\Omega \frac{(y_1-y_2)^2}{(\eta_1+\eta_2)}\left(\frac{\eta_1}{\eta_2^2}+\frac{\eta_2}{\eta_1^2}\right)$$
$$+\frac{1}{2\varepsilon^2}\int_\Omega (\eta_2-\eta_1)^2 + \frac{1}{16}\int_\Omega (\eta_1-\eta_2)^2|B_1+B_2|^2 + 4\eta_1^2|B_1-B_2|^2 \le$$
$$C\|\eta_1-\eta_2\|_{L^2}\|B_1-B_2\|_{L^2}(\|B_1\|_{L^\infty}+\|B_2\|_{L^\infty}) + C\|\eta_1-\eta_2\|_{L^2}\|y_1-y_2\|_{L^2}(\|y_1\|_{L^\infty}+\|y_2\|_{L^\infty}).$$

We remark here that in the first term of the last inequality, we used the boundedness of η_i. In fact, taking $p=4$ and β close to $\frac{1}{2}$ in (2.4), we conclude

$$\|1-\eta_i\|_{L^4} \le C.$$

From here and (3.8), boundedness of η_i follows from Sobolev embedding. On the other hand,

$$\frac{1}{4}\int_\Omega \frac{(y_1^2+y_2^2)(\eta_2-\eta_1)^2}{\eta_1^2\eta_2^2} + \frac{(y_1+y_2)^2(\eta_2-\eta_1)^2}{(\eta_1+\eta_2)^2\eta_1\eta_2} + \frac{1}{4}\int_\Omega \frac{(y_1-y_2)^2}{(\eta_1+\eta_2)}\left(\frac{\eta_1}{\eta_2^2}+\frac{\eta_2}{\eta_1^2}\right) + \frac{1}{2\varepsilon^2}\int_\Omega (\eta_2-\eta_1)^2$$
$$+\frac{1}{16}\int_\Omega (\eta_1-\eta_2)^2|B_1+B_2|^2 + 4\eta_1^2|B_1-B_2|^2 \ge$$
$$\frac{C}{\varepsilon}(\|\eta_1-\eta_2\|_{L^2}\|B_1-B_2\|_{L^2} + \|\eta_1-\eta_2\|_{L^2}\|y_1-y_2\|_{L^2}).$$

Since $\|y_j\|_{L^\infty} \le o\left(\frac{1}{\varepsilon}\right)$, $\|B_j\|_{L^\infty} \le o\left(\frac{1}{\varepsilon}\right)$, we must have $\eta_1 = \eta_2$ or $B_1 = B_2$. If $\eta_1 = \eta_2$, simple convexity argument gives

$$\int_\Omega (\eta_1)^2\left|\frac{B_1+B_2}{2}\right|^2 + \frac{\mu_\varepsilon^2}{4}\left|\frac{\text{curl}\,\frac{B_1+B_2}{2} - h_{ex}}{\eta_1}\right|^2 < \frac{1}{2}\int_\Omega \eta_1^2|B_1|^2 + \frac{\mu_\varepsilon^2}{4}\left|\frac{\text{curl}\, B_1 - h_{ex}}{\eta_1}\right|^2$$
$$+\frac{1}{2}\int_\Omega \eta_2^2|B_2|^2 + \frac{\mu_\varepsilon^2}{4}\left|\frac{\text{curl}\, B_2 - h_{ex}}{\eta_2}\right|^2,$$

thus $X > 0$ (since $B_1 \neq B_2$). If $B_1 = B_2$, again by convexity (since $\eta_i \geq \frac{9}{10}$)

$$\int_\Omega \left(\frac{\eta_1+\eta_2}{2}\right)^2 |B_1|^2 + \frac{\mu_\varepsilon^2}{4}\left|\frac{\operatorname{curl} B_1 - h_{ex}}{\frac{\eta_1+\eta_2}{2}}\right|^2 + \frac{1}{\varepsilon^2}\left(\frac{\eta_1+\eta_2}{2}\right)^2\left(1-\left(\frac{\eta_1+\eta_2}{2}\right)^2\right)^2 \leq$$

$$\frac{1}{2}\int_\Omega \eta_1^2 |B_1|^2 + \frac{\mu_\varepsilon^2}{4}\left|\frac{\operatorname{curl} B_1 - h_{ex}}{\eta_1}\right|^2 + \frac{1}{\varepsilon^2}\eta_1^2\left(1-\eta_1^2\right)^2 + \frac{1}{2}\int_\Omega \eta_2^2 |B_2|^2 + \frac{\mu_\varepsilon^2}{4}\left|\frac{\operatorname{curl} B_2 - h_{ex}}{\eta_2}\right|^2 + \frac{1}{\varepsilon^2}\eta_2^2\left(1-\eta_2^2\right)^2$$

and $X > 0$ (since $\eta_1 \neq \eta_2$). We are led to contradiction in all cases therefore $X > 0$ and lemma is proved. □

Lemma 3.5. *If $\mu_\varepsilon \geq \varepsilon^{\frac{1}{9}}$ and $\limsup \mu_\varepsilon < \infty$, there exists $\alpha \in (0, 1/24)$ and ε_0 such that, if $\varepsilon < \varepsilon_0$, a stable vortexless solution of (1.2)–(1.3) for $h_{ex} \leq C\varepsilon^{-\alpha}$ with $\int_\Omega |\nabla u|^2 \leq o\left(\varepsilon^\beta\right)$ for some $\beta > 0$ is unique. Let $E_0 = \left\{(u, A) \in D : |u| \geq \frac{9}{10}\right\}$. For $\varepsilon < \varepsilon_0$, if there exists a solution of (1.2)–(1.3) that minimizes G_{csh} over E_0, then it is unique.*

Proof. Lemma 3.4 implies

$$G_{csh}\left(\frac{\eta_1+\eta_2}{2}, \frac{B_1+B_2}{2}\right) < \frac{G_{csh}(\eta_1, B_1) + G_{csh}(\eta_2, B_2)}{2} \leq G_{csh}(\eta_2, B_2).$$

A standard argument gives

$$G_{csh}\left((1-t)\eta_1 + t\eta_2, (1-t)B_1 + tB_2\right) < G_{csh}(\eta_2, B_2)$$

for all $t \in (0, 1)$, this contradicts the stability of (η_2, B_2). Hence $\eta_1 = \eta_2$, $B_1 = B_2$. □

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