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# AN *A PRIORI* CAMPANATO TYPE REGULARITY CONDITION FOR LOCAL MINIMISERS IN THE CALCULUS OF VARIATIONS

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**Abstract.** An *a priori* Campanato type regularity condition is established for a class of $\mathrm{W}^1\mathrm{X}$ local minimisers $\overline{u}$ of the general variational integral

$$\int_\Omega F(\nabla u(x))\,\mathrm{d}x$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain, $F$ is of class $\mathrm{C}^2$, $F$ is strongly quasi-convex and satisfies the growth condition

$$F(\xi) \le c(1+|\xi|^p)$$

for a $p > 1$ and where the corresponding Banach spaces X are the Morrey-Campanato space $\mathcal{L}^{p,\mu}(\Omega,\mathbb{R}^{N\times n})$, $\mu < n$, Campanato space $\mathcal{L}^{p,n}(\Omega,\mathbb{R}^{N\times n})$ and the space of bounded mean oscillation $\mathrm{BMO}(\Omega,\mathbb{R}^{N\times n})$. The admissible maps $u\colon \Omega \to \mathbb{R}^N$ are of Sobolev class $\mathrm{W}^{1,p}$, satisfying a Dirichlet boundary condition, and to help clarify the significance of the above result the sufficiency condition for $\mathrm{W}^1\mathrm{BMO}$ local minimisers is extended from Lipschitz maps to this admissible class.

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## 1. INTRODUCTION

We are concerned with showing the partial regularity of a special class of local minimisers $\overline{u} \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ of the multiple integral

$$I[u] = \int_\Omega F(\nabla u), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is strongly quasiconvex, $\mathrm{C}^2$, and for a $p > 1$ satisfies the polynomial growth condition

$$F(\xi) \le c(1+|\xi|^p)$$

for all $\xi \in \mathbb{R}^{N\times n}$.

*Keywords and phrases.* Calculus of variations, local minimiser, partial regularity, strong quasiconvexity, Campanato space, Morrey space, Morrey-Campanato space, space of bounded mean oscillation, extremals, positive second variation.

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Let $(\mathrm{X}, \|\cdot\|)$ denote a normed space continuously embedded in $\mathrm{L}^p_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$. By a $\mathrm{W}^1\mathrm{X}$-local minimiser we mean a map $\overline{u}$ for which there exists a $\delta > 0$ such that $I[\overline{u}] \le I[u]$ whenever

$$u \in \overline{u} + \mathrm{W}^{1,p}_0(\Omega, \mathbb{R}^N) \tag{1.2}$$

and

$$\|\nabla u - \nabla \overline{u}\| \le \delta. \tag{1.3}$$

In this paper we restrict our attention to a special class of $\mathrm{W}^1\mathrm{X}$-local minimisers $\overline{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ with $\mathrm{X} = \mathcal{L}^{p,\mu}(\Omega, \mathbb{R}^{N\times n})$, the *Campanato space with exponents $p$ and $\mu \ge 0$*, for which we prove partial regularity for $\mu \le n$ under a $\delta$-smallness condition of the $\mathcal{L}^{p,\mu}$-norm of $\nabla \overline{u}$ over all open balls $B \subset \Omega$ in the limit as radius of the balls approach zero. It is important to note that the $\delta$ here is not arbitrarily small as, for example, in [16]. It is fixed by the local minimiser condition (1.3) and we impose no additional condition on its size to prove the above result.

We will show that the equivalent regularising condition for *Bounded Mean Oscillation* type local minimisers, $\mathrm{X} = \mathrm{BMO}(\Omega, \mathbb{R}^{N\times n})$, is (1.4) and that in the context of partial regularity such minimisers are interchangeable with $\mathrm{W}^1\mathcal{L}^{p,n}$-local minimisers. Thus we clarify our partial regularity result for the case $\mu = n$ by extending a sufficiency condition for Lipschitz extremals to be local minimisers of $\mathrm{X} = \mathrm{BMO}(\Omega, \mathbb{R}^{N\times n})$ type to the non-Lipschitz case, with a view to showing that there exists a local minimiser of (1.1) that is not strong in the sense of [15]. *I.e.* not partially regular without the regularising condition

$$\limsup_{R\to 0^+} \left( \sup_{\substack{x\in\Omega' \\ r\in(0,R)}} \fint_{\Omega(x,r)} |\nabla \overline{u} - (\nabla \overline{u})_{x,r}| \mathrm{d}y \right) < \delta \tag{1.4}$$

for every open set $\Omega'$ compactly contained in $\Omega$, and where $\delta$ corresponds to (1.3).

## A regularity theorem for a new class of local minimisers

We will need the following definition for the statement of our result.

**Definition 1** (Campanato space)**.** Let $\Omega \subset \mathbb{R}^n$ be open and bounded define $\Omega(x_0, R) := \Omega \cap B(x_0, R)$. Then for $p > 1$ and $\mu \ge 0$ the Campanato space $\mathcal{L}^{p,\mu}(\Omega)$ [4,11], consists of all $f \in \mathrm{L}^p_{\mathrm{loc}}(\Omega)$ such that

$$[f]_{p,\mu,\Omega} := \sup_{\substack{x_0\in\Omega \\ 0<R<\mathrm{diam}(\Omega)}} \left( \frac{1}{R^\mu} \int_{\Omega(x_0,R)} |f - f_{x_0,R}|^p \,\mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

The $\mathcal{L}^{p,\mu}(\Omega)$-norm is given by

$$\|f\|_{p,\mu,\Omega} \equiv \|f\|_{\mathrm{L}^p} + [f]_{p,\mu,\Omega}.$$

We say that $f$ is locally $\mathcal{L}^{p,\mu}$ in $\Omega$, denoted $\mathcal{L}^{p,\mu}_{\mathrm{loc}}(\Omega)$, if for each open $\Omega'$ compactly contained in $\Omega$, $[f]_{p,\mu,\Omega'} < \infty$.

We will also need the definition of the space of functions of bounded mean oscillation with domain $\mathbb{R}^n$ and an open and bounded set $\Omega \subset \mathbb{R}^n$, respectively.

**Definition 2** (BMO space)**.** Let $\Omega$ be open and bounded or the entire space $\mathbb{R}^n$. Then the John-Nirenberg space $\mathrm{BMO}(\Omega)$ [11,14] consists of all $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ such that

$$[f]_{*,\Omega} := \sup_{B\subset\Omega} \left( \fint_B |f - f_B| \,\mathrm{d}x \right) < \infty,$$

where the supremum is taken over all open balls contained in $u$. If $\Omega = \mathbb{R}^n$ then the BMO-norm is given by

$$\|f\|_{BMO} \equiv [f]_{*,\mathbb{R}^n}.$$

Otherwise the BMO($\Omega$)-norm is given by

$$\|f\|_{*,\Omega} \equiv \|f\|_{\mathrm{L}^1} + [f]_{*,\Omega}.$$

We say that $f$ is locally BMO in $\Omega$ if for each open $\Omega'$ compactly contained in $\Omega$, $[f]_{*,\Omega'} < \infty$.

**Notation.** We have used $f_{x_0,R}$ to denote the average of $f$ over $\Omega(x_0, R)$

$$f_{x_0,R} = \fint_{\Omega(x_0,R)} f = \frac{1}{|\Omega(x_0,R)|} \int_{\Omega(x_0,R)} f(x)\,\mathrm{d}x.$$

Depending on the context we may write $f_{x,r}$ for the average over the ball $B = B(x, r)$. We may also write this as $f_B$ and we will denote the unit ball as $B_1 = B(0, 1)$ to avoid confusion with $B$. Finally for $\mu < n$ and a sufficiently regular boundary $\partial\Omega$, the Campanato space $\mathcal{L}^{p,\mu}(\Omega)$ is equivalent to the Morrey space $\mathrm{L}^{p,\mu}(\Omega)$ defined as the space of functions $f$ for which the norm

$$\|f\|_{p,\mu} = \sup_{\substack{x_0\in\Omega \\ 0<R<\mathrm{diam}(\Omega)}} \left( \frac{1}{R^\mu} \int_{\Omega(x_0,R)} |f|^p \,\mathrm{d}x \right) \tag{1.5}$$

is finite. In this case we refer to the space as *Morrey-Campanato space*. The inclusion $\mathrm{L}^{p,\mu}(\Omega) \hookrightarrow \mathcal{L}^{p,\mu}(\Omega)$ is a trivial result of

$$\fint_{\Omega(x_0,R)} |u - u_{x_0,R}|^p \le 2^p \inf_{\xi\in\mathbb{R}^n} \fint_{\Omega(x_0,R)} |u-\xi|^p$$

and holds for all open $\Omega$. For the opposite inclusion some work is required to derive the relevant inequality,

$$\|u\|_{p,\mu,\Omega} \le c(n,\Omega,p,\mu) \left( |\Omega|^{-\frac{\mu}{np}} \|u\|_{p,\Omega} + [u]_{p,\mu,\Omega} \right) \tag{1.6}$$

which only holds for exponents $0 < \mu < n$ and for domains without external cusps, *e.g.* domains with Lipschitz boundary (see [11], Sect. 2.3).

For any normed space Y we let $\mathrm{Y}(\Omega, \mathbb{R}^N)$ denote the space of vector valued maps $u\colon \Omega \to \mathbb{R}^N$ and $\mathrm{Y}(\Omega, \mathbb{R}^{N\times n})$ the space of matrix valued maps $u\colon \Omega \to \mathbb{R}^{N\times n}$. We use $|\cdot|$ to denote the usual euclidean norms, *e.g.* for matrices $\xi \in \mathbb{R}^{N\times n}$ we let

$$|\xi| := \sqrt{\mathrm{trace}(\xi^T \xi)}.$$

We combine the assumptions on the integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ of $I[\cdot]$ in [5] for $1 < p < 2$ with those in [15] for $p \ge 2$. The assumptions are as follows:

(H1) $F \in \mathrm{C}^2$;
(H2) $|F(\xi)| \le c(1 + |\xi|^p)$ for every $\xi \in \mathbb{R}^{N\times n}$, some constant $c$ and $p > 1$;
(H3) for some constant $\nu > 0$, every $\xi \in \mathbb{R}^{N\times n}$ and every $\varphi \in \mathrm{C}^1_C(\mathbb{R}^n, \mathbb{R}^N)$,

$$\nu \int_{\mathbb{R}^n} (|\nabla\varphi|^2 + |\nabla\varphi|^p) \le \int_{\mathbb{R}^n} (F(\xi + \nabla\varphi) - F(\xi)) \text{ when } p \ge 2 \tag{1.7}$$

$$\nu \int_{\mathbb{R}^n} (1 + |\xi|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}} |\nabla\varphi|^2 \le \int_{\mathbb{R}^n} (F(\xi + \nabla\varphi) - F(\xi)) \text{ when } 1 < p < 2. \tag{1.8}$$

The conditions (1.7) and (1.8) of (H3), known as strong quasiconvexity, were first introduced by Evans in his paper on partial regularity of absolute minimisers of $I[\cdot]$ ($p \ge 2$) [8]. Note that for $p \ge 2$ (1.7) is the weaker of the two conditions. (H2) replaces the original assumption of [8], namely a condition controlling the second derivative of $F$. For strong quasiconvexity this generalisation is due to Acerbi and Fusco [1] and later adapted to the $1 < p < 2$ case by Carozza *et al.* [6].

The following result is a consequence of the various embeddings and isomorphisms linking Campanato, Morrey and BMO spaces on balls (see Sect. 2), Poincaré's inequality and standard compactness arguments, allowing the extension of the local minimiser version [5,15] of the "blow up method" for quasiconvex functionals $I[\cdot]$ [1,3,6,8], to a class of local minimisers characterised by the Morrey-Campanato metric.

**Theorem 1.** *Consider the functional* $I[\cdot]$ *of* (1.1) *satisfying the hypotheses* (H1)–(H3). *Suppose that* $\overline{u} \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ *for* $p \in (1,\infty)$ *is a* $\mathrm{W}^1\mathcal{L}^{p,\mu}$*-local minimiser of* $I[\cdot]$*: there exists a* $\delta > 0$ *such that* $I[\overline{u}] \leq I[u]$ *whenever* $u \in \overline{u} + \mathrm{W}_0^{1,p}(\Omega,\mathbb{R}^N)$ *and* $\|\nabla u - \nabla\overline{u}\|_{p,\mu;\Omega} \leq \delta$*, so that* $\nabla\overline{u}$ *satisfies the regularising condition*

$$\limsup_{R\to 0+}\left(\sup_{\substack{x_0\in\Omega'\\ r\in(0,R)}} \frac{1}{r^{\mu}}\int_{\Omega(x,r)} |\nabla\overline{u} - (\nabla\overline{u})_{\Omega(x,r)}|^p \,\mathrm{d}x\right) < \delta \tag{1.9}$$

*for every open set* $\Omega'$ *compactly contained in* $\Omega$*. Then for* $\mu \leq n$ *there exists an open set* $\Omega_0 \subset \Omega$ *of full* $n$*-dimensional measure, such that the minimiser* $\overline{u} \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_0,\mathbb{R}^N)$ *for every* $\alpha \in (0,1)$*.*

Partial regularity of non-Lipschitz $\mathrm{W}^1$BMO-local minimisers follows from Lemma 3.3 in the proof of Theorem 1 and the isomorphism $\mathcal{L}^{n,p}(B,\mathbb{R}^{N\times n}) \cong \mathrm{BMO}(B,\mathbb{R}^{N\times n})$ on balls $B \subset \mathbb{R}^n$ (see Sect. 2 and Lem. 3.3 for details):

**Corollary 1.1.** *Consider the functional* $I[\cdot]$ *of* (1.1) *satisfying the hypotheses* (H1)–(H3). *Suppose that* $\overline{u} \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ *for* $p \in (1,\infty)$ *is a* $\mathrm{W}^1$BMO*-local minimiser of* $I[\cdot]$*: There exists a* $\delta > 0$ *such that* $I[\overline{u}] \leq I[u]$ *whenever* $u \in \overline{u} + \mathrm{W}_0^{1,p}(\Omega,\mathbb{R}^N)$ *and* $\|\nabla u - \nabla\overline{u}\|_{*;\Omega} \leq \delta$*, so that* $\nabla\overline{u}$ *satisfies the regularising condition* (1.4). *Then there exists an open set* $\Omega_0 \subset \Omega$ *of full* $n$*-dimensional measure, such that the minimiser* $\overline{u} \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_0,\mathbb{R}^N)$ *for every* $\alpha \in (0,1)$*.*

**Remark 1.** By Lemma 2.1, Section 2, the embedding inequality for Morrey and Campanato spaces, condition (1.9) is satisfied if we assume $\nabla\overline{u} \in \mathcal{L}^{p,\nu}_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$ for $\nu > \mu$. In this case the condition reduces to

$$\limsup_{R\to 0+}\left(\sup_{\substack{x_0\in\Omega\\ r\in(0,R)}} \frac{1}{r^{\mu}}\int_{\Omega(x,r)} |\nabla\overline{u} - (\nabla\overline{u})_{\Omega(x,r)}|^p \,\mathrm{d}x\right) = 0 \tag{1.10}$$

for every open set $\Omega'$ compactly contained in $\Omega$.

As mentioned above, our proof of Theorem 1, will be based on the standard blow-up argument to show a decay estimate on the excess defined for every ball $B(x,r) \subset \Omega$ by

$$E(x,r) = \begin{cases} \displaystyle\fint_{B(x,r)} |V(\nabla\overline{u}) - V((\nabla\overline{u})_{x,r})|^2 & 1 < p < 2 \\[2ex] \displaystyle\fint_{B(x,r)} \left(|\nabla\overline{u} - (\nabla\overline{u})_{x,r}|^2 + |\nabla\overline{u} - (\nabla\overline{u})_{x,r}|^p\right) & p \geq 2. \end{cases} \tag{1.11}$$

Here

$$V(\xi) = (1+|\xi|^2)^{\frac{p-2}{4}}\xi, \qquad \xi \in \mathbb{R}^{N\times n}.$$

From this decay estimate it is well known that partial regularity follows.

## Significance of the regularity result

In [15] partial regularity for $\mathrm{W}^{1,q}$-local minimisers $\overline{u} \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ $(q > p)$ was proved by assuming $\nabla\overline{u} \in \mathrm{L}^q_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$. Given the Sobolev class $\mathrm{W}^{1,q}(\Omega,\mathbb{R}^N)$ for $q > p$, the inclusion $\mathrm{W}^{1,q}(\Omega,\mathbb{R}^N) \subset \mathrm{W}^1\mathcal{L}^{p,\mu}(\Omega,\mathbb{R}^N)$ follows directly from Hölders inequality for the exponents $\mu \leq n(1 - p/q)$. Thus for each $q > p$, $\mathrm{W}^1\mathcal{L}^{p,\mu}$

($\mu \leq n(1-p/q)$) possess a weaker topology than $W^{1,q}$ and thus in this case a $W^1\mathcal{L}^{p,\mu}$ local minimiser is a stronger notion of a local minimum than a $W^{1,q}$ local minimiser. The *a priori* $\delta$-smallness condition (1.9) is certainly a weaker requirement than condition $\overline{u} \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ when $\mu < n(1-p/q)$ as the later condition implies the arbitrary smallness condition (1.10). However it is not clear that the $W^{1,q}_{\mathrm{loc}}$ condition placed on the $W^{1,q}$-local minimisers of [5,15] is necessary for partial regularity. In any case our *a priori* condition for the general Morrey-Campanato class of minimisers fits in neatly with previous results for weaker notions of local-minimisers, namely the results for $W^1$BMO, $W^{1,\infty}$ local minimisers of Lipschitz class derived in [15]. In fact given the equivalence of Campanato and BMO spaces when Campanato exponent $\mu = n$ we will show that the results for $W^1$BMO local minimisers follow when the minimiser $\overline{u}$ is of class $W^{1,p}(\Omega)$, $1 < p < \infty$.

In particular results of [15] include a sufficiency theorem for Lipschitz extremals of $I$ to be $W^1$BMO-local minima, demonstrating that there are many potential examples of Lipschitz maps that are also $W^1$BMO-local minimisers (a similar result was also obtained by Firoozye [10] under more restrictive conditions). Further drawing on an example of [17] it was shown that for $N = n = 2$ there exists a Lipschitz $W^1$BMO-local minimiser of $I$ satisfying the hypotheses (H1)–(H3), but which is non-differentiable on any open set. From this it is clear that a regularising condition like (1.4) is necessary for partial regularity for Lipschitz $W^1$BMO-local minimisers. We note that these results are made possible by the sufficiency condition of [15] and the earlier result of [10] (for an alternative sufficiency condition for $W^{1,\infty}$ sequential weak-* local minimisers with the assumption that the minimiser is $C^1$-smooth see [13]). Therefore before we prove Theorem 1 we pause to justify the regularity result for $W^1$BMO-local minimisers in the non-Lipschitz case. Following the spirit of [15] we extend the sufficiency condition for $W^1$BMO-local minimisers, to the non-Lipschitz case.

## Positive second variation

It is shown in [15] that for $C^2$ integrands $F$ of the functional $I[\cdot]$ that positivity of the second variation of $I[\cdot]$ at a given Lipschitz extremal $\overline{u}$ implies that $\overline{u}$ is not only a weak local minimiser, which is well known, but is in fact a $W^1$BMO local minimiser. As mentioned previously a similar result was also proved in [10] but the proof requires stronger assumptions on the integrand $F$.

In the following we extend the result of [15] for extremals $\overline{u}$ of $I[\cdot]$ that are in $W^{1,p}(\Omega, \mathbb{R}^N)$ for $1 \leq p < \infty$ by adding a uniform continuity condition to the second derivative of $F$. We assume that $F''$ is uniformly continuous with a modulus of continuity $\omega\colon [0,\infty) \to \mathbb{R}$, which is continuous, increasing, $\omega(0) = 0$ and

$$\sup_{t>0} \frac{\omega(2t)}{\omega(t)} < \infty. \tag{1.12}$$

The result is as follows:

**Theorem 2.** *Let the integrand of* (1.1)*,* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *be a* $C^2$ *function,* $\Omega \subset \mathbb{R}^n$ *be open and bounded and* $\overline{u} \in W^{1,p}(\Omega, \mathbb{R}^N)$ $1 \leq p < \infty$ *be an extremal of* (1.1) *with strong positive second variation: for some* $\delta_s > 0$ *and all* $\varphi \in W^1\mathrm{BMO}(\mathbb{R}^n, \mathbb{R}^N) \cap W^{1,1}_0(\Omega, \mathbb{R}^N)$*,*

$$\int_\Omega F'(\nabla\overline{u})[\nabla\varphi] = 0 \tag{1.13}$$

$$\int_\Omega F''(\nabla\overline{u})[\nabla\varphi, \nabla\varphi] \geq \delta_s \int_\Omega |\nabla\varphi|^2. \tag{1.14}$$

*Further assume*

$$|F''(\xi) - F''(\eta)| \leq \omega(|\xi - \eta|) \tag{1.15}$$

*for all* $\xi, \eta \in \mathbb{R}^{N\times n}$*. Then there exists a* $\delta_*(n, N, c, q) > 0$ *such that*

$$\int_\Omega F(\nabla\overline{u} + \nabla\varphi) \geq \int_\Omega F(\nabla\overline{u})$$

*holds for all* $\varphi \in W^1\mathrm{BMO}(\mathbb{R}^n, \mathbb{R}^N) \cap W^{1,1}_0(\Omega, \mathbb{R}^N)$*, with* $\|\nabla\varphi\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{R}^N)} \leq \delta_*$*.*

**Remark 2.**

(i) The space $\mathrm{W}^1\mathrm{BMO}(\mathbb{R}^n,\mathbb{R}^N)\cap \mathrm{W}_0^{1,1}(\Omega,\mathbb{R}^N)$ is exactly the space of $\mathrm{W}^1\mathrm{BMO}(\mathbb{R}^n,\mathbb{R}^N)$ functions $f$, for which $f$ and $\nabla f$ are extended by 0 outside of $\Omega$.

(ii) Beside excluding exponential growth of $\omega$ the doubling condition also excludes certain classes of piecewise polynomial growth. However we can accommodate the subclass of piecewise polynomials $\omega$ (not necessarily increasing) that do not satisfy (1.12) but instead satisfy

$$\tilde{\omega}(t) := \sup_{s\geq 1}\left(s^{-k}\sup_{r\leq st}\omega(r)\right) < \infty$$

for some $k>0$ and all $t>0$. In this case one may easily show that $\omega(t)\leq\tilde{\omega}(t)$ and $\tilde{\omega}(\alpha t)\leq \alpha^k\tilde{\omega}(t)$ for $\alpha\geq 0$. Thus we can replace $\omega$ with $\tilde{\omega}$ in the proof of the theorem.

Finally this straight forward corollary to the above theorem gives the sufficiency conditions for non-Lipschitz extremals of $I[\cdot]$ to be partially regular.

**Corollary 1.2.** *Let the integrand of $I[\cdot]$, $F\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ be $\mathrm{C}^2$, $\Omega\subset\mathbb{R}^n$ open and bounded. Let $\overline{u}\in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$, $1\leq p<\infty$ be an extremal of $I[\cdot]$ with strongly positive second variation such that for some $\delta_s>0$ and all $\varphi\in \mathrm{W}^1\mathrm{BMO}(\mathbb{R}^n,\mathbb{R}^N)\cap \mathrm{W}_0^{1,1}(\Omega,\mathbb{R}^N)$ we have (1.13) and (1.14). Suppose also that we have*

$$|F''(\xi)-F''(\eta)|\leq\omega(|\xi-\eta|) \tag{1.16}$$

*such that $F$ satisfies* (H1)–(H3). *Then $\overline{u}$ is partially regular in the sense of Theorem 1 provided $\nabla\overline{u}$ satisfies the regularity condition* (1.4) *with $\delta=\delta_*$ where $\delta_*$ is given in Theorem 2.*

**Remark 3.** In the case of $p=\infty$, $F''$ does not need to satisfy (1.16), see Theorem 6.1 of [15].

## 2. Preliminaries

We remind the reader of some well known relationships between Morrey, Campanato and BMO spaces important for the proof of our result (for further reading see [11], Sects. 2.3 and 2.4). The following lemma provides the inequality between Morrey space norms (Campanato space semi-norms) of different exponents and is easily derived with Hölders inequality:

**Lemma 2.1** (Morrey-Campanato embeddings)**.** *Let $\Omega\subset\mathbb{R}^n$ be open and bounded, $1\leq p\leq q<\infty$ and $\frac{n-\mu}{p}-\frac{n-\nu}{q}\geq 0$ then $\mathrm{L}^{q,\nu}(\Omega)$ is continuously embedded in $\mathrm{L}^{p,\mu}(\Omega)$ and $\mathcal{L}^{q,\nu}(\Omega)$ is continuously embedded in $\mathcal{L}^{p,\mu}(\Omega)$ with*

$$\|f\|_{p,\mu,\Omega}\leq \operatorname{diam}(\Omega)^{\frac{n-\mu}{p}-\frac{n-\nu}{q}}\|f\|_{q,\nu,\Omega},\quad f\in \mathrm{L}^{q,\nu}(\Omega)$$

*and*

$$[f]_{p,\mu,\Omega}\leq \operatorname{diam}(\Omega)^{\frac{n-\mu}{p}-\frac{n-\nu}{q}}[f]_{q,\nu,\Omega},\quad f\in \mathcal{L}^{q,\nu}(\Omega) \tag{2.1}$$

*respectively.*

The next lemma summarises the relationships between Campanato and BMO spaces:

**Lemma 2.2** (Campanato-BMO Isometry)**.** *Let $1\leq p<\infty$:*

(i) *For general $\Omega$ open and bounded in $\mathbb{R}^n$, $\mathcal{L}^{p,n}(\Omega)$ is continuously embedded in* $\mathrm{BMO}(\Omega)$.

(ii) *If $\Omega=B_0$ where $B_0$ is an arbitrary ball in $\mathbb{R}^n$, $\mathcal{L}^{p,n}(\Omega)$ is isomorphic to* $\mathrm{BMO}(\Omega)$.

*Proof.* Given the open bounded set $\Omega\subset\mathbb{R}^n$, it follows from Definitions 1 and 2 and Lemma 2.1 that

$$[f]_{*,\Omega}\leq\frac{1}{|B_1|}[f]_{1,n,\Omega}\leq\frac{1}{|B_1|}[f]_{p,n,\Omega}$$

for $f \in \mathcal{L}^{p,n}(\Omega)$ proving (i). Given

$$|B_1|\left(\frac{1}{2}r\right)^n \leq |B_0 \cap B|$$

for $B$ of radius $0 < r \leq \mathrm{diam}(B_0)$, centre $x_0 \in B_0$, it is easily shown that

$$\frac{1}{r^n}\int_{B_0\cap B} |f - f_{B_0\cap B}|^p \leq 2^{p+n}|B_1| \fint_B |f - f_B|^p, \quad r > 0.$$

Thus part (ii) follows from the inequality, bounding $\mathrm{L}^p(B, \mathbb{R}^{N\times n})$ by $\mathrm{BMO}(B_0, \mathbb{R}^{N\times n})$,

$$\fint_B |f - f_B|^p \leq c[f]^p_{*,B_0} \tag{2.2}$$

for all $B \subset B_0$. This inequality can be shown with a well known argument, reproduced here for the convenience of the reader, that uses the celebrated result of John and Nirenberg [14]. This result states that for every $f \in \mathrm{BMO}(B_0)$ and $\sigma > 0$ there exist positive constants $A$ and $\alpha$ that are independent of $f$ and $\sigma$ such that

$$|\lambda_{\sigma,B}| \leq A\exp\left(-\frac{\alpha\sigma}{[f]_{*,B_0}}\right)|B|,$$

where $\lambda_{\sigma,B} := \{x \in B : |f - f_B| > \sigma\}$. Given this we have by standard formula for integrals in terms of distribution functions

$$\begin{aligned}\int_B |f - f_B|^p &= p\int_0^\infty \sigma^{p-1}|\lambda_{\sigma,B}|\mathrm{d}\sigma \\ &\leq pA\int_0^\infty \sigma^{p-1}\exp\left(-\frac{\alpha\sigma}{[f]_{*,B_0}}\right)|B|\mathrm{d}\sigma \\ &= A\cdot\left(\frac{[f]_{*,B_0})}{\alpha}\right)^p|B|\cdot p\int_0^\infty t^{p-1}\mathrm{e}^{-t}\mathrm{d}t \\ &\leq c_*|B|[f]_{*,B_0},\end{aligned}$$

here the improper integral of the penultimate estimate is equal to the Gamma function of $p$. Thus $c_*$ is dependent on $p$, $\alpha$ and $A$ proving (2.2). □

As in [5] we will use the properties of $V$ highlighted in the following lemma. The lemma is proved in [6], for $1 < p < 2$.

**Lemma 2.3.** *Let $1 < p < 2$ and $V\colon \mathbb{R}^{K\times k} \to \mathbb{R}^{K\times k}$. Then, for any $\eta, \xi \in \mathbb{R}^{K\times k}$, $t > 0$:*

(i) $2^{\frac{p-2}{4}}\min\{|\xi|, |\xi|^{\frac{p}{2}}\} \leq |V(\xi)| \leq \min\{|\xi|, |\xi|^{\frac{p}{2}}\}$;
(ii) $|V(t\xi)| \leq \max\{t, t^{\frac{p}{2}}\}|V(\xi)|$;
(iii) $|V(\xi+\eta)| \leq 2^{\frac{p}{2}}\,[|V(\xi)| + |V(\eta)|]$;
(iv) $\frac{p}{2}(1+|\xi|^2+|\eta|^2)^{\frac{(p-2)}{4}}|\xi-\eta| \leq |V(\xi)-V(\eta)| \leq c(1+|\xi|^2+|\eta|^2)^{\frac{(p-2)}{4}}|\xi-\eta|$;
(v) $|V(\xi)-V(\eta)| \leq c|V(\xi-\eta)|$;
(vi) *For each $m > 0$ there exists a $c_m < \infty$ such that* $|V(\xi-\eta)| \leq c_M|V(\xi)-V(\eta)|$ *if* $|\eta| \leq M$

*where $c$ depends on $k$ and $p$ and $c_M$ on $M$ and $p$.*

Again following [5] we will use the extension of the theory of linear elliptic systems with weak solutions in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$ to weak solutions in $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$, observed in [6]. For the statement of the lemma we use the summation convention.

**Lemma 2.4.** *Suppose* $u \in \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$ *is a solution to the variational system*

$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \varphi^i \,\mathrm{d}x = 0, \quad \forall \varphi \in \mathrm{C}_0^1(\Omega,\mathbb{R}^N)$$

*with constants* $A_{ij}^{\alpha\beta}$ *satisfying the strong Legendre-Hadamard condition*

$$A_{ij}^{\alpha\beta}\xi^i\xi^j\eta_\alpha\eta_\beta \geq \nu|\xi|^2|\eta|^2 \quad (\xi \in \mathbb{R}^N, \eta \in \mathbb{R}^n).$$

*Then* $u \in \mathrm{C}^\infty(\Omega,\mathbb{R}^N)$ *and for any* $B(x_0,R) \subset \Omega$ *and* $0 < \rho \leq R \leq \mathrm{dist}(x_0,\partial\Omega)$ *the following estimates hold*

$$\int_{B(x_0,\rho)} |\nabla u - (\nabla u)_{x_0,\rho}|^2 \,\mathrm{d}x \leq c\left(\frac{\rho}{R}\right)^{n+2} \int_{B(x_0,R)} |\nabla u - (\nabla u)_{x_0,R}|^2 \,\mathrm{d}x \tag{2.3}$$

$$\sup_{B(x_0,R/2)} |\nabla u| \leq c ⨍_{B(x_0,R)} |\nabla u| \,\mathrm{d}x, \tag{2.4}$$

*where* $c$ *is dependent only on* $n$, $N$, $p$, $\nu$ *and* $\max\{|A_{ij}^{\alpha\beta}|\}$.

*Proof.* From [6], with (2.3) following from [11], Section 10.2, Theorem 10.7. □

## 3. Proof of Theorem 1

The proof is based on a blow-up technique originally developed by De Giorgi and Almgren in the context of geometric measure theory, see [11], Section 9.6, and the references therein, and adapted to the setting of partial regularity for elliptic systems by Giusti and Miranda [12]. Specifically once the following proposition is proved partial regularity follows.

**Proposition 3.1.** *For every* $L > 0$, *there exists* $C = C(L) > 0$ *with the property that for each* $\tau \in (0, \frac{1}{2})$, *there exists* $\epsilon = \epsilon(L,\tau) > 0$ *such that for all* $B(x,r) \subset \Omega$ *with* $|(\nabla\overline{u})_{x,r}| \leq L$ *and* $E(x,r) < \epsilon$, *we have*

$$E(x,\tau r) \leq C(L)\tau^2 E(x,r).$$

The proof is indirect and was originally adapted for minimisers of the quasiconvex integral $I[\cdot]$ by Evans [8]. The basic idea is to assume blow up of the solution for a sequence of small balls around $x$ and study the convergence in the unit ball of the sequence of solutions for suitably re-scaled functionals so to obtain a contradiction. This argument involves three main steps. In Step 1 we show that the limit of the blow up sequence of solutions converges weakly in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ for $1 < p < 2$ and $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$ for $p \geq 2$. In Step 2 we show that the weak limit of these solutions satisfies a linear uniformly elliptic system with constant coefficients. Finally in Step 3, we show the strong convergence of the sequence of solutions to obtain the contradiction. To show this we use the standard construction of comparison maps from a suitably rescaled version of the minimiser $\overline{u} \in \mathrm{W}^{1,p}(\Omega)$, and thus must prove that these maps satisfy the Morrey-Campanato local minimiser condition (1.3). It is in showing that the local minimiser condition is satisfied, Lemma 3.3, that it is necessary to introduce the condition (1.9), a generalisation of the condition for Lipschitz maps introduced in [15]. Having verified this we can proceed with the methods of [5,15] without modification, deriving a pre-Caccioppoli inequality and using the measure theoretic argument therein to obtain our result.

It is well known (see [6] and the reference therein) that given (H2) and (H3) control on $F'$ follows and which by simple manipulation implies

$$|F'(\xi)| \leq c_0(1+|\xi|^2)^{\frac{p-1}{2}} \tag{3.1}$$

for $p > 1$. In the sequel we will use the following lemma, a consolidation of Lemma 3.3 [6] and Lemma 2.3 [2] for functions satisfying the above estimate. Note that Lemma 3.3 of [6] is proved in the same way as Lemma 2.3 of [2].

**Lemma 3.2.** *Let $F\colon \mathbb{R}^{K\times k}\to\mathbb{R}$ be a function of class $\mathrm{C}^2$ with*

$$|F'(\xi)|\le c_0(1+|\xi|^2)^{\frac{p-1}{2}},\quad p\ge 1.$$

*Then for any $\lambda>0$ and $\xi_0\in\mathbb{R}^{K\times k}$ with $|\xi_0|\le L$, setting*

$$F_{\xi_0,\lambda}(\xi)=\lambda^{-2}\left[F(\xi_0+\lambda\xi)-F(\xi_0)-\lambda F'(\xi_0)\xi\right] \tag{3.2}$$

*there exist constants $c_1$ and $c_2$ dependent only on $c_0, L, p$ such that for $p\ge 1$,*

$$|F_{\xi_0,\lambda}(\xi)|\le \min\left\{c_1(1+|\lambda\xi|^2)^{\frac{p-2}{2}}|\xi|^2, c_2(|\xi|^2+\lambda^{p-2}|\xi|^p)\right\}. \tag{3.3}$$

*Proof of Proposition* 3.1. Suppose the proposition is false. Then there exists an $L>0$ and a sequence of balls $\{B(x_j,r_j)\}$ with the properties that

$$|(\nabla\overline{u})_{x_j,r_j}|\le L \text{ for all } j,$$

and

$$E(x_j,r_j)\to 0 \text{ as } j\to\infty$$

such that for every $C>0$ there exists a $\tau\in(0,\frac{1}{2})$ with

$$E(x_j,r_j\tau)>C\tau^2E(x_j,r_j) \text{ for all } j. \tag{3.4}$$

We look for a $C$ that contradicts this.

**Step 1**: We suppose the sequence of balls satisfies the above with vanishing radii, $r_j\to 0$ as $j\to\infty$. We rescale the minimiser on each ball to a sequence of maps, $u_j$, on the unit ball in the usual way

$$u_j(y):=\frac{\overline{u}(x_j+r_jy)-\overline{u}(x_j)-\xi_jr_jy}{\lambda_jr_j},\qquad y\in B_1$$

where the scaling is given by $\lambda_j^2:=E(x_j,r_j)$, and $\xi_j:=(\nabla\overline{u})_{x_j,r_j}$.

By assumption $|\xi_j|\le L$, so for a subsequence (for convenience not relabeled)

$$\xi_j\to\xi_\infty \text{ as } j\to\infty.$$

From the definition of $u_j$, $(u_j)_{0,1}=0$, $(\nabla u_j)_{0,1}=0$, so for $p\ge 2$

$$\fint_{B_1}\left(|\nabla u_j|^2+\lambda_j^{p-2}|\nabla u_j|^p\right)\le 1 \tag{3.5}$$

and for $1<p<2$, utilising part (vi) of Lemma 2.3,

$$\begin{aligned}\fint_{B_1}|V(\nabla u_j)|^2 &\le c_0(p,L)\frac{1}{\lambda_j^2}\fint_{B_j}|V(\nabla\overline{u})-V((\nabla\overline{u})_{x_j,r_j})|^2\\ &= c(p,L).\end{aligned} \tag{3.6}$$

This implies

$$\|\nabla u_j\|_{\mathrm{L}^{s(p)}(B_1,\mathbb{R}^{N\times n})}<c_B(p,L),\quad p>1 \tag{3.7}$$

where $s(p):=\min\{2,p\}$. Note that part (i) of Lemma 2.3 is used in the derivation for $1<p<2$. Thus by weak compactness (3.7) implies for a further subsequence (again not relabeled)

$$\nabla u_j\rightharpoonup\nabla u \text{ in } \mathrm{L}^{s(p)}(B_1,\mathbb{R}^{N\times n}). \tag{3.8}$$

Now setting $F_j := F_{\xi_j,\lambda_j}$ in (3.2) of Lemma 3.2, so that $F_j$ satisfies the associated growth estimates, we replace the integral (1.1) with the sequence of integrals

$$I_j[u] = \int_{B_1} F_j(\nabla u). \tag{3.9}$$

It follows using strong quasiconvexity of $F$ that each $F_j$ satisfies a quasi-convexity condition

$$\nu \int_{B_1} (|\nabla\varphi|^2 + \lambda_j^{p-2}|\nabla\varphi|^p) \leq \int_{B_1} (F_j(\xi + \nabla\varphi) - F_j(\xi))$$

for all $\varphi \in \mathrm{W}_0^{1,p}(B_1, \mathbb{R}^N)$ when $p \geq 2$ and

$$\nu \int_{B_1} (1 + |\xi_j + \lambda_j\xi|^2 + |\lambda_j\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2 \leq \int_{B_1} (F_j(\xi + \nabla\varphi) - F_j(\xi)) \tag{3.10}$$

for all $\varphi \in \mathrm{W}_0^{1,p}(B_1, \mathbb{R}^N)$ when $1 < p < 2$. Finally using the local minimality of $\overline{u}$ it follows that $u_j$ is a $\mathrm{W}^1\mathrm{X}$-local minimiser of $I_j$ defined at (3.9). Precisely, $I_j[u_j] \leq I_j[u]$ whenever

$$\|\nabla u - \nabla u_j\| \leq \delta_j := \begin{cases} \frac{\delta}{\lambda_j r_j^{\frac{n-\mu}{p}}}, & \mathrm{X} = \mathcal{L}^{p,\mu}(B_1), \quad \mu \leq n \\ \frac{\delta}{\lambda_j}, & \mathrm{X} = \mathrm{BMO}(B_1), \end{cases} \tag{3.11}$$

with

$$u \in u_j + \mathrm{W}_0^{1,p}(B_1, \mathbb{R}^N). \tag{3.12}$$

**Step 2** ($u$ solves linear elliptic system): We wish to show that the limit $u$ satisfies

$$\int_{B_1} F''(\xi_\infty)\,[\nabla u, \nabla\varphi] = 0, \quad \forall\varphi \in \mathrm{C}_0^1(\Omega, \mathbb{R}^N) \tag{3.13}$$

since it then follows (given (H1) and (H3)) by Lemma 2.4 of the preliminaries that $u$ is $C^\infty$ and

$$\fint_{B(0,\tau)} |\nabla u - (\nabla u)_{0,\tau}|^2 \mathrm{d}y \leq C^*\tau^2 \qquad (p > 1). \tag{3.14}$$

From this we may use part (i) of Lemma 2.3 to attain

$$\fint_{B(0,\tau)} |V(\nabla u - (\nabla u)_{0,\tau})|^2 \mathrm{d}y \leq C^*\tau^2 \tag{3.15}$$

for the case $1 < p < 2$. The proof of (3.13) is given in [5] for $1 < p < 2$ and [15] for $p \geq 2$ and remains unchanged in this case. It only uses the following properties: that $\overline{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ is an extremal of $I$; $F \in \mathrm{C}^2$, (H1), and satisfies growth condition (3.1). We do not include the proof for brevity, but remark that the result follows from these properties by application of Lemma 3.2 for the growth estimate of $F_j$ and the bound (3.7) of Step 1.

Having shown (3.13) we note as a consequence of (H3) and the continuity of $F$, (H1), that $F$ is strongly rank-1-convex, *i.e.* $F$ satisfies the strong Legendre-Hadamard condition, $F''(\xi_\infty)[\eta, \eta] \geq 2\nu|\eta|^2$ with $\mathrm{rank}(\eta) \leq 1$. Further by the continuity of $F''$, (H1), we have $|F''(\xi_\infty)| \leq M(L)$ where $M(L) := \sup_{|\xi|\leq L} |F''(\xi)|$. Thus the coefficients of the Legendre-Hadamard condition are finite (and constant) and we may apply Lemma 2.4 to the system (3.13), obtaining immediately that $u \in \mathrm{C}^\infty(B_1, \mathbb{R}^N)$, and by (2.3) and (2.4) of the same lemma,

$$\begin{aligned}
\fint_{B(0,\tau)} |\nabla u - (\nabla u)_\tau|^2 \mathrm{d}y &\le c\tau^2 \fint_{B(0,\frac{1}{2})} |\nabla u - (\nabla u)_{\frac{1}{2}}|^2 \\
&\le c\tau^2 \fint_{B(0,\frac{1}{2})} |\nabla u|^2 \\
&\le c\tau^2 \left( \sup_{B(0,\frac{1}{2})} |\nabla u| \right)^2 \\
&\le c_1\tau^2 \left( \fint_{B(0,1)} |\nabla u|^{s(p)} \right)^{\frac{2}{s(p)}} .
\end{aligned}$$

Finally, by $\|\nabla u_j\|_{\mathrm{L}^{s(p)}(B_1,\mathbb{R}^{N\times n})} < c_B$ for all $j$, inequality (3.14) follows. Hence we have the estimate (3.15) for a constant $C^*$ that only depends on $\nu$ and $L$ (and $n$, $N$, $F''$).

As we mentioned earlier we are looking for a constant $C$ that contradicts (3.4). By part (v) of Lemma 2.3 and the definition of $u_j$ we find,

$$\limsup_{j\to\infty} \frac{E(x_j, \tau r_j)}{\lambda_j^2} \le \mathrm{RHS} \tag{3.16}$$

where

$$\mathrm{RHS} \le \begin{cases} \limsup\limits_{j\to\infty} \dfrac{c}{\lambda_j^2} \fint_{B(0,\tau)} |V(\lambda_j(\nabla u_j - (\nabla u_j)_{0,\tau}))|^2 & 1 < p < 2 \\[2ex] \limsup\limits_{j\to\infty} c \fint_{B(0,\tau)} \Big( |\nabla u_j - (\nabla u_j)_{0,\tau}|^2 & p \ge 2 \\ \qquad\qquad + \lambda_j^{p-2} |\nabla u_j - (\nabla u_j)_{0,\tau}|^p \Big). & \end{cases}$$

We will show at the end of Step 3, with a simple argument, that if $\nabla u_j$ converges strongly in $\mathrm{L}^{s(p)}(B_1, \mathbb{R}^{N\times n})$, (3.15) together with (3.16) gives the desired contradiction (recall $\lambda_j^2 := E(x_j, r_j)$). Therefore our third and final step in proving Proposition 3.1 is to show suitable strong convergence of $\nabla u_j$ in $\mathrm{L}^{s(p)}(B_1, \mathbb{R}^{N\times n})$ as defined below.

**Step 3** (Strong convergence of $u_j$): In this step we will show that, for every $\sigma < 1$:

$$\lim_{j\to\infty} \int_{B(0,\sigma)} \frac{1}{\lambda_j^2} |V(\lambda_j(\nabla u_j - \nabla u))|^2 = 0 \tag{3.17}$$

for $1 < p < 2$ and similarly

$$\lim_{j\to\infty} \int_{B(0,\sigma)} \Big( |\nabla u_j - \nabla u|^2 + \lambda_j^{p-2} |\nabla u_j - \nabla u|^p \Big) = 0 \tag{3.18}$$

for $p \ge 2$. The standard way to obtain (3.17)-(3.18) for global minimisers is by use of a Caccioppoli inequality. In the local minimiser case we can not use the standard method to obtain an inequality of full Caccioppoli type (see [15]). Instead we stop short of deriving the full inequality and use direct techniques introduced in [15] and modified for $1 < p < 2$ in [5] to complete our proof. This 'pre-Caccioppoli' inequality is proved as in the global minimiser case with the construction of suitable comparison maps.

Fix $\alpha \in (0,1)$, $B(x_0,r) \subset B(0,1)$ and let $a_j\colon \mathbb{R}^n \to \mathbb{R}^N$ be the affine map such that $\nabla a_j = (\nabla u_j)_{x_0,r}$ and $(u_j - a_j)_{x_0,r} = 0$. It follows from (3.7) that there exists a constant $M$ such that

$$|\nabla a_j| \le M, \quad \text{for all } j. \tag{3.19}$$

Now let $\rho\colon \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz cut off function satisfying $1_{B(x_0,\alpha r)} \le \rho \le 1_{B(x_0,r)}$ and $|\nabla\rho| \le \frac{2}{(1-\alpha)r}$. The standard comparison maps $\varphi_j$ and $\psi_j$ are defined by

$$\varphi_j := \rho(u_j - a_j) \quad \text{and} \quad \psi_j := (1-\rho)(u_j - a_j).$$

We prove that $u := a_j + \psi_j$ satisfies the local minimiser condition (3.11) according to the following lemma:

**Lemma 3.3.** *Define $\psi_j$ as above and $I_j$ as in (3.9). Let $B_j = B(x_j, r_j)$ and assume that*

$$\limsup_{j\to\infty} [\nabla\overline{u}]_{p,\mu,B_j} < \delta \tag{3.20}$$

*where $\delta > 0$ is given by (1.9). Then if $\mu \le n$, $u := a_j + \psi_j$ satisfies the $\mathrm{W}^1\mathcal{L}^{p,\mu}$-local minimiser condition* i.e. *condition (3.11) with $\mathrm{X} = \mathrm{W}^1\mathcal{L}^{p,\mu}(B_1)$, so that $I_j[u_j] \le I_j[a_j + \psi_j]$.*

**Corollary 3.4.** *Let*

$$\limsup_{j\to\infty} [\nabla\overline{u}]_{*,B_j} < \delta.$$

*Then $u := a_j + \psi_j$ satisfies the $\mathrm{W}^1\mathrm{BMO}$-local minimiser condition* i.e. *condition (3.11) with $\mathrm{X} = \mathrm{W}^1\mathrm{BMO}(B_1)$, so that $I_j[u_j] \le I_j[a_j + \psi_j]$.*

*Proof.* First note $u_j - a_j = \varphi_j + \psi_j$, thus

$$\begin{aligned}[\nabla u - \nabla u_j]_{p,\mu,B_1} &= [\nabla\varphi_j]_{p,\mu,B_1}\\ &= [\rho(\nabla u_j - \nabla a_j) + \nabla\rho \otimes (u_j - a_j)]_{p,\mu,B_1}.\end{aligned}$$

For $\mu \le n$,

$$[\nabla u_j - \nabla a_j]_{p,\mu,B_1} = \sup_{\substack{x\in B_1\\ R\in(0,2)}} \frac{1}{\lambda_j}\left(\frac{r_j^\mu}{r_j^n R^\mu}\int_{B(x,R)} |\nabla\overline{u} - (\nabla\overline{u})_{B(x,R)}|^p\right)^{\frac{1}{p}}. \tag{3.21}$$

Therefore it follows that

$$[\nabla u - \nabla u_j]_{p,\mu,B_1} \le \frac{1}{\lambda_j r_j^{\frac{n-\mu}{p}}}\left([\nabla\overline{u}]_{p,\mu,B_j} + \mathcal{R}_j[\overline{u},\alpha,r]\right), \tag{3.22}$$

where

$$\mathcal{R}_j[\overline{u},\alpha,r] := \frac{\lambda_j r_j^{\frac{n-\mu}{p}}}{(1-\alpha)r}[1_{B(x_0,r)}(u_j - a_j)]_{p,\mu,B_1}. \tag{3.23}$$

Clearly the first term in (3.22) is bounded by $\delta/(\lambda_j r_j^{\frac{n-\mu}{p}})$ for sufficiently large $j \ge J$ as a result of (3.20). To show that $u$ satisfies (3.11) we must show that $\mathcal{R}_j[\overline{u},\alpha,r] \to 0$ as $j \to \infty$ for arbitrarily fixed $\alpha, r \in (0,1)$. Although it is only necessary in the proof of Theorem 1 for a subsequence of $\{\mathcal{R}_j\}$ to converge to zero, we prove that the full sequence converges to zero in the case $\mu < n$.

**Case $\mu < n$:** For convenience we rewrite the sequence of functionals $\mathcal{R}_j$ as the functional $\overline{\mathcal{R}}_{\alpha,r}$ of the sequence of functions $f_j^r$ *i.e.* we set $\overline{\mathcal{R}}_{\alpha,r}[f_j^r] := \mathcal{R}_j[\overline{u},\alpha,r]$ where $f_j^r$ is given by

$$f_j^r := \lambda_j^p r_j^{n-\mu} 1_{B(x_0,r)}(u_j - a_j). \tag{3.24}$$

Our strategy is to show first that $\{f_j^r\}$ is bounded in $\mathrm{W}^{1,p}(B_1)$ as are all subsequences (it is actually uniformly bounded in $r$ but this is not important here). Then show the full sequence $\{f_j^r\}$ converges strongly to zero in $\mathrm{L}^p(B_1)$, $1 < p < \infty$. We do this by using Rellich-Kondrackov to show that given any subsequence of $\{f_j^r\}$ a further subsequence converges strongly to zero in $\mathrm{L}^p(B_1)$, $1 < p < \infty$. Following from the boundedness of $\{f_j^r\}$ in $\mathrm{W}^{1,p}(B_1)$ we then show that $\{f_j^r\}$ is also bounded in $\mathrm{W}^1\mathcal{L}^{p,\mu}(B_1)$. This allows the use of strong convergence to zero in $\mathrm{L}^p(B_1)$ to prove $[f_j^r]_{1,p,\mu} \to 0$ for the full sequence $\{f_j^r\}$.

In particular for the first step using $(u_j - a_j)_{B(x_0,r)} = 0$ and (3.20), it follows by Poincarés inequality on balls that $\{f_j^r\}$ and any subsequence is bounded in $\mathrm{W}^{1,p}(B_1)$ for $1 < p < \infty$. Thus for any subsequence $\{f_{j_k}^r\}$, using $\lambda_j \nabla u_j \to 0$ $\mathcal{L}^n$ a.e. and once again $(u_j - a_j)_{B(x_0,r)} = 0$, we have by Rellich-Kondrachov

$$f_{j_k}^r \to 0 \text{ in } \mathrm{L}^p(B_1), \quad 1 < p < \infty$$

for a further (suitably relabeled) subsequence. Therefore the full sequence $\{f_j^r\}$ converges strongly to zero in $\mathrm{L}^p(B_1)$, $1 < p < \infty$. Next, given boundedness of the full sequence $\{f_j^r\}$ in $\mathrm{W}^{1,p}(B_1)$ we use the following estimate derived from Poincarés inequality and the Morrey-Campanato inclusion (1.6) applicable to bounded domains $\Omega$ without external cusps and valid for Morrey-Campanato exponent $0 < \mu < n$,

$$[f_j^r]_{p,\mu,\Omega} \le \begin{cases} c(p,\mu,\Omega)\|\nabla f_j^r\|_{p,\Omega}, & \mu \le p \\ c(n,p,\mu,\Omega)\left(|\Omega|^{\frac{-\mu}{np}}\|\nabla f_j^r\|_{p,\Omega} + [\nabla f_j^r]_{p,\mu-p,\Omega}\right), & \mu > p. \end{cases} \tag{3.25}$$

This gives us boundedness of $\{f_j^r\}$ in $\mathrm{W}^1\mathcal{L}^{p,\mu}(B_1)$ since

$$[\nabla f_j^r]_{p,\mu,B_1} \le [\nabla \overline{u}]_{p,\mu,B_j}$$

and

$$[\nabla f_j^r]_{p,\mu-p,B_1} \le c[\nabla f_j^r]_{p,\mu,B_1}, \quad \mu > p$$

by the Campanato embedding (2.1). Finally to prove $[f_j^r]_{p,\mu,B_1} \to 0$ we split the family of intersections of balls with $B_1$ over which we take the supremum in the semi-norm $[\cdot]_{p,\mu,B_1}$ into the family of balls with radius $s \in (S, \operatorname{diam}(B_1))$ and $s \in (0,S)$. We deal with these two cases separately. In the first case $\operatorname{diam}(B_1) > s > S$, by strong convergence of $\{f_j^r\}$ to zero in $\mathrm{L}^p(B_1)$,

$$\begin{aligned} s^{-\mu}\int_{B_1(x,s)} |f_j^r - (f_j^r)_{x,r}|^p &< c(S)\int_{B_1(x,s)} |f_j^r - (f_j^r)_{x,r}|^p \\ &< 2^{p-1}c(S)\left(\int_{B_1}|f_j^r|^p + \int_{B_1}|f_j^r|^p\right) \to 0 \end{aligned}$$

as $j \to \infty$. For the second case the boundedness of $\{f_j^r\}$ in $\mathrm{W}^1\mathcal{L}^{p,\mu}(B_1)$ allows us to write the following. Given $\epsilon > 0$, take $S$ such that $cS < \epsilon$ where $c$ is a constant defined according to the inequality

$$\int_{B_1(x,s)} |f_j^r - (f_j^r)_{x,s}|^p \le cs^{p+\mu}.$$

Using Poincarés inequality for balls the above inequality follows from the Morrey-Campanato isomorphism (on balls and their intersections) and the boundedness of $\{f_j^r\}$ in $\mathrm{W}^1\mathcal{L}^{p,\mu}(B_1)$. Hence given any $\epsilon > 0$ there exists a $J$ such that for $j \geq J$

$$[f_j^r]_{p,\mu,B_1} < \epsilon$$

for the full sequence defined in (3.24). We remark that $J$ is independent of $r$ since convergence is uniform in $r$. However this is not the case for $\overline{\mathcal{R}}_{\alpha,r}[f_j^r]$ which converges to zero for each pair $(\alpha, r)$ as required, but not uniformly in either $\alpha$ or $r$.

**Case $\mu = n$:** By the Campanato-BMO isometry, Lemma 2.1, there exists a $c \in [|B_1|, 2^{p+n}|B_1|c_*]$ such that

$$[u_j - a_j]_{p,n,B_1} = c[u_j - a_j]_{*,B_1}. \tag{3.26}$$

We estimate the above semi-norm using the $\mathrm{L}^\infty$ norm,

$$[u_j - a_j]_{*,B_1} \leq \sup_{B \subset B_1} \left( \operatorname{ess.sup}_{x \in B} |(u_j - a_j)(x) - (u_j - a_j)_B| \right). \tag{3.27}$$

To make sense of this estimate we use the fact that $\mathrm{W}^1\mathrm{BMO}(\Omega) \hookrightarrow \mathrm{W}^{1,q}(\Omega)$ for all $1 \leq q < \infty$ and general open and bounded $\Omega$. We set $q > n$, then make use of Morrey's inequality. Our aim is to show that the sequence

$$\left\{ \operatorname{ess.sup}_{x \in B_1} |\lambda_j (u_j - a_j)(x)| \right\} \tag{3.28}$$

converges to zero as $j \to \infty$ (note that direct estimation of (3.27) results in $\sup_j [u_j - a_j]_{*,B_1} \leq \infty$, not sufficient to show $\lambda_j [u_j - a_j]_{*,B_1} \to 0$). We start by showing that the sequence is bounded. By Morrey's inequality

$$|(u_j - a_j)(x) - (u_j - a_j)(y)| \leq cR_{x,y} \left( \fint_{B(0,R_{x,y})} |\nabla u_j - \nabla a_j|^q \right)^{\frac{1}{q}} \tag{3.29}$$

for $\mathcal{L}^n$-a.e. $x, y \in B_1$ and every $R_{x,y} \geq 1$. The integral on the right may be estimated as follows

$$\left( \fint_{B_1} |\nabla u_j - \nabla a_j|^q \right)^{\frac{1}{q}} \leq |(\nabla u_j - \nabla a_j)_{B_1}| + \left( \fint_{B_1} |\nabla u_j - \nabla a_j - (\nabla u_j - \nabla a_j)_{B_1}|^q \right)^{\frac{1}{q}}.$$

By noting that $(\nabla u_j)_{B_1} = 0$ and $|\nabla a_j| < M$ we see immediately that the first term on the right is uniformly bounded. For the remainder we apply the equality of (3.21) for change of variables. Thus

$$\begin{aligned} \left( \fint_{B_1} |\nabla u_j - \nabla a_j|^q \right)^{\frac{1}{q}} &\leq M + [\nabla u_j - \nabla a_j]_{p,n,B_j} \\ &= M + \frac{1}{\lambda_j} [\nabla \overline{u}]_{p,n,B_j}. \end{aligned} \tag{3.30}$$

Therefore, given that we can extend $\nabla u_j - \nabla a_j = 0$ off $B_j$, choosing $R_{x,y} = 2|x - y|$ (so that $B(0, R_{x,y}) \subset B(0, 4)$), we find that $\lambda_j (u_j^* - a_j)$ where $u_j^*$ denotes the precise representative of $u_j$, has a uniformly bounded $(1 - \frac{n}{q})$th-Hölder semi-norm over $B_1$. Thus by the implied continuity of $u_j^*$ there exists for each component $(u_j - a_j)^{(k)}$, $k = 1, \ldots, N$, a point $y_k \in B_1$ such that $(u_j^* - a_j)^{(k)}(y_k) = (u_j^* - a_j)^{(k)}_{x_0,r} = (u_j - a_j)^{(k)}_{x_0,r} = 0$ and so

$$|(u_j^* - a_j)^{(k)}(x)| \leq |(u_j^* - a_j)(x) - (u_j^* - a_j)(y_k)|. \tag{3.31}$$

Therefore by taking $R_{x,y} = 1$ and substituting $u_j^*$ for $u_j$ in (3.29) it follows from (3.31) that the sequence $\{\lambda_j (u_j^* - a_j)^{(k)}\}$ is bounded uniformly on $B_1$ for each $k = 1, \ldots, N$. Thus the whole sequence (3.28) is bounded as required. It now follows that $\lambda_j(u_j^* - a_j)$ has a uniformly bounded $(1-\frac{n}{q})$th-Hölder norm over $B_1$ and thus the sequence $\{\lambda_j(u_j^* - a_j)\}$ is Hölder equicontinuous on $B_1$. Therefore $\{\lambda_j(u_j^* - a_j)\} \subset \mathrm{C}(B_1)$ and by its boundedness can uniquely be extended to $\mathrm{C}(\overline{B_1})$ as can any subsequence $\{\lambda_{j_k}(u_{j_k}^* - a_{j_k})\}$. Hence, after extracting a further subsequence if required, by Arzel-Ascoli combined with the properties $\lambda_j \nabla u_j \to 0$ $\mathcal{L}^n$-a.e. and $(u_j - a_j)_{x_0,r} = 0$,

$$\lambda_{j_k}(u_{j_k}^* - a_{j_k}) \to 0$$

uniformly on $B_1$. This means, after extracting to a subsequence where necessary, that (3.28) tends to zero as required and $\mathcal{R}_{j_k}[\overline{u}, r, \alpha] \to 0$ then follows from (3.27). □

Now it is straight forward to prove the Corollary to Lemma 3.3:

*Proof of Corollary* 3.4. From the proof of Lemma 3.3 it is clear, as a result of equivalence of $\mathcal{L}^{p,n}$ and BMO on $B_1$ and in particular equivalence relation (3.26), that we may replace $[\cdot]_{p,n,B_j}$ and $[\cdot]_{p,n,B_1}$ semi-norms with $[\cdot]_{*,B_j}$ and $[\cdot]_{*,B_1}$ semi-norms in the proof of the lemma. □

Using Lemma 3.3/Corollary 3.4 we can now follow the method of [5] and derive an inequality of pre-Caccioppoli type presented here for $1 < p < 2$:

$$\begin{aligned}\int_{B(x_0,\alpha r)} |V(\lambda_j(\nabla u_j - \nabla u))|^2 \leq \theta \int_{B(x_0,r)} |V(\lambda_j(\nabla u_j - \nabla u))|^2 + c\int_{B(x_0,r)} |V(\lambda_j(\nabla u - \nabla a_j))|^2 \\ + c\int_{B(x_0,r)} \frac{|V(\lambda_j(u_j - a_j))|^2}{(1-\alpha)^2 r^2} + c\int_{B(x_0,r)\setminus B(x_0,\alpha r)} |V(\lambda_j(\nabla a_j))|^2 \end{aligned} \tag{3.32}$$

with $\theta < 1$. In the case $p \geq 2$ one simply replaces the function $V(\xi)$ with $|\xi|^2 + |\xi|^p$. We summarise the proof of (3.32) given in [5,6]. To start we estimate

$$\frac{1}{\lambda_j^2}\int_{B(x_0,\alpha r)} |V(\lambda_j(\nabla u_j - \nabla a_j))|^2 = \int_{B(x_0,\alpha r)} (1 + |\lambda_j \nabla \varphi_j|^2)^{\frac{p-2}{2}} |\nabla \varphi_j|^2$$

in terms of $F_j$ using quasiconvexity of $F_j$, (3.10). Given $|\xi_j| \leq L$ and (3.19) for all $j$, there exists a constant $c_J > 0$ dependent only on $p$, $L$ and $\nu$ of (3.10) such that for $j \geq J$ ($J$ sufficiently large), $1 \leq c_J \nu (1 + |\xi_j + \lambda_j \nabla a_j|^2)^{\frac{p-2}{2}}$. Thus

$$\begin{aligned}\frac{1}{\lambda_j^2}\int_{B(x_0,\alpha r)} |V(\lambda_j(\nabla u_j - \nabla a_j))|^2 &\leq c_J \nu \int_{B(x_0,r)} (1 + |\xi_j + \lambda_j \nabla a_j|^2 + |\lambda_j \nabla \varphi_j|^2)^{\frac{p-2}{2}} |\nabla \varphi_j|^2 \\ &\leq c_J \int_{B(x_0,r)} (F_j(\nabla a_j + \nabla \varphi_j) - F_j(\nabla a_j)). \end{aligned} \tag{3.33}$$

To guarantee $\theta < 1$ in (3.32) we estimate the right hand integral in such a way that we may remove $B(x_0, \alpha r)$ from the domain of integration $B(x_0, r)$. By construction, $\nabla a_j + \nabla \varphi_j = \nabla u_j$ on $B(x_0, \alpha r)$, thus

$$\begin{aligned}\int_{B(x_0,r)} (F_j(\nabla a_j + \nabla \varphi_j) - F_j(\nabla a_j)) \leq \int_{B(x_0,r)\setminus B(x_0,\alpha r)} (F_j(\nabla a_j + \nabla \varphi_j) - F_j(\nabla u_j)) \\ + \int_{B(x_0,r)} (F_j(\nabla u_j) - F_j(\nabla a_j)).\end{aligned}$$

Now given Lemma 3.3/Corollary 3.4 (implying that for sufficiently large $j$, $I_j[u_j] \le I_j[u]$ where $u := a_j + \psi_j$) and using $\nabla\psi_j = 0$ on $B(x_0, \alpha r)$ we obtain

$$\int_{B(x_0,r)} (F_j(\nabla a_j + \nabla\varphi_j) - F_j(\nabla a_j)) \le \int_{B(x_0,r)\setminus B(x_0,\alpha r)} (F_j(\nabla a_j + \nabla\varphi_j) - F_j(\nabla u_j)) + \int_{B(x_0,r)\setminus B(x_0,\alpha r)} (F_j(\nabla a_j + \nabla\psi_j) - F_j(\nabla a_j)).$$

Next by (3.3) of Lemma 3.2 and properties of $V$, Lemma 2.3 (and $|\nabla\rho| \le 2/(1-\alpha)r$)

$$\int_{B(x_0,r)} (F_j(\nabla a_j + \nabla\varphi_j) - F_j(\nabla a_j)) \le \frac{c(c_1,p)}{\lambda_j^2} \int_{B(x_0,r)\setminus B(x_0,\alpha r)} \left( |V(\lambda_j(\nabla u_j - \nabla a_j))|^2 + \left| \frac{V(\lambda_j(u_j - a_j))}{(1-\alpha)r} \right|^2 + |V(\lambda_j \nabla a_j)|^2 \right). \tag{3.34}$$

Finally to obtain (3.32) with $\theta < 1$ we first add and subtract $\nabla u$ within the first instance of $V$ on the right hand side of (3.34). Thus using Lemma 2.3, combining the result with (3.33) and then adding

$$\frac{1}{\lambda_j^2} \int_{B(x_0,\alpha r)} |V(\lambda_j(\nabla u - \nabla a_j))|^2$$

to both sides, we obtain

$$\frac{1}{\lambda_j^2} \int_{B(x_0,\alpha r)} \left( |V(\lambda_j(\nabla u_j - \nabla a_j))|^2 + |V(\lambda_j(\nabla u - \nabla a_j))|^2 \right) \le \frac{c}{\lambda_j^2} \int_{B(x_0,r)\setminus B(x_0,\alpha r)} \left( |V(\lambda_j(\nabla u - \nabla a_j))|^2 + |V(\lambda_j(\nabla u_j - \nabla u))|^2 \right) + \frac{c}{\lambda_j^2} \int_{B(x_0,r)\setminus B(x_0,\alpha r)} \left( \left| \frac{V(\lambda_j(u_j - a_j))}{(1-\alpha)r} \right|^2 + |V(\lambda_j \nabla a_j)|^2 \right) \tag{3.35}$$

where the constant $c$ depends only on $p$, $c_1$ and $c_J$. Now using Lemma 2.3

$$\frac{1}{\lambda_j^2} \int_{B(x_0,\alpha r)} |V(\lambda_j(\nabla u_j - \nabla u))|^2 \le \frac{2^{p+1}}{\lambda_j^2} \int_{B(x_0,\alpha r)} \left( |V(\lambda_j(\nabla u_j - \nabla a_j))|^2 + |V(\lambda_j(\nabla u - \nabla a_j))|^2 \right).$$

Thus by multiplying (3.35) through by $2^{p+1}$ and combining with the above we finalise the calculation by filling the hole. *I.e.* by adding

$$\frac{\tilde{c}}{\lambda_j^2} \int_{B(x_0,\alpha r)} |V(\lambda_j(\nabla u_j - \nabla u))|^2$$

to both sides (where $\tilde{c} := 2^{p+1} \cdot c$). Hence obtaining (3.32) with $\theta = \frac{\tilde{c}}{\tilde{c}+1}$.

**Weak Convergence of measures:** We follow precisely the argument of [5] for $1 < p < 2$ and [15] for the case $p \ge 2$. Once again we reproduce it here for the convenience of the reader. In the case $1 < p < 2$ [5] required a Sobolev-Poincaré type inequality for the auxiliary function $V$ as introduced in [6]. We present a refined version of this inequality proved in [7]:

**Lemma 3.5.** *Let $p \in (1,2)$, $B(x_0,r) \subset \mathbb{R}^n$ with $n \geq 2$ and set $p^\# := \frac{2n}{n-p}$. Then*

$$\left(\fint_{B(x_0,r)} \left|V\left(\frac{u-u_{x_0,r}}{r}\right)\right|^{p^\#} \mathrm{d}x\right)^{\frac{1}{p^\#}} \leq c \left(\fint_{B(x_0,r)} |V(\nabla u)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \tag{3.36}$$

*for any $u \in \mathrm{W}^{1,p}(B(x_0,r), \mathbb{R}^N)$ and where $c$ depends only on $n$, $N$, and $p$.*

Unlike the inequality of [6], the radius of the ball is not increased on the right hand side but is kept the same. Note that this refinement marginally simplifies, but is not critical for, the proceeding proof.

First we claim that

$$\frac{1}{\lambda_j^2}|V(\lambda_j(\nabla u_j - \nabla u))|^2 \mathcal{L}^n \rightharpoonup \mu \text{ in } \mathrm{C}_0(\overline{B})^* \tag{3.37}$$

for $1 < p < 2$ and

$$\left(|\nabla u_j - \nabla u|^2 + \lambda_j^{p-2}|\nabla u_j - \nabla u|^p\right) \mathcal{L}^n \rightharpoonup \mu \text{ in } \mathrm{C}_0(\overline{B})^* \tag{3.38}$$

for $p \geq 2$ where $\mu$ is a Radon measure.

As in [5], this claim follows from the bound imposed on the sequence of measures in (3.37) by

$$\int_B \frac{1}{\lambda_j^2}|V(\lambda_j(\nabla u_j - \nabla u))|^2 \leq 2^{p+1} c_0(p,L)|B| \fint_{B_j} \frac{1}{\lambda_j^2}|V(\nabla \overline{u}) - V((\nabla \overline{u})_{x_j,r_j})|^2 + 2^{p+1}\int_B |\nabla u|^2$$

and estimate (3.6). Similarly the bound for the sequence in (3.38) follows from (3.5).

It is now straightforward to show that limit form of the pre-Caccioppoli inequality matches that of [15]. For $1 < p < 2$ using properties of $V$ as in [5]

$$\begin{aligned}
\lim_{j\to\infty} \frac{1}{\lambda_j^2} \int_{B(x_0,r)} |V(\lambda_j(\nabla u - \nabla a_j))|^2 &\leq \int_{B(x_0,r)} |\nabla u - \nabla a|^2 \\
&= \epsilon_1(r) r^n \\
\lim_{j\to\infty} \frac{1}{\lambda_j^2} \int_{B(x_0,r)\setminus B(x_0,\alpha r)} |V(\lambda_j \nabla a_j)|^2 &\leq c|\nabla a|^2 r^n (1-\alpha)^n \\
&= \epsilon_2(r) r^n (1-\alpha)^n.
\end{aligned}$$

The final estimate follows from the Sobolev Poincaré inequality (3.36) of Lemma 3.5, Rellich-Kondrachov compactness theorem and Vitali's lemma.

From Sobolev Poincaré inequality (3.36)

$$\int_{B(x_0,r)} \left|\frac{1}{\lambda_j} V(\lambda_j(u_j - a_j))\right|^{p^\#} \leq c_1$$

and since $p^\# > 2$,

$$\int_{B(x_0,r)} \left|\frac{1}{\lambda_j} V(\lambda_j(u_j - a_j))\right|^2 \leq c_2.$$

Thus given $\frac{2n}{n-p} > 1$, the sequence $\{v_j\}$ defined by

$$v_j(x) := \frac{1}{\lambda_j} V(\lambda_j(u_j - a_j))$$

is equi-integrable. Now by Rellich-Kondrachov compactness theorem $u_j \to u$ in $\mathrm{L}^1(B_1)$. Thus for a suitably relabeled subsequence it follows from the definition of $V$ that

$$v_j(x) \to (u-a)(x) \text{ for } \mathcal{L}^n\text{-a.e. } x \in B_1.$$

Hence by Vitali's lemma

$$\begin{aligned}\lim_{j\to\infty} \frac{1}{\lambda_j^2} \int_{B(x_0,r)} \frac{|V(\lambda_j(u_j-a_j))|^2}{(1-\alpha)^2 r^2} &= \frac{1}{(1-\alpha)^2 r^2} \int_{B(x_0,r)} |u-a|^2 \\ &= \epsilon_3(r) \frac{1}{(1-\alpha)^2} r^n\end{aligned}$$

for a suitably relabeled subsequence, where

$$\begin{aligned}\epsilon_1 &:= \frac{1}{r^n} \int_{B(x_0,r)} |\nabla u - \nabla a|^2, \\ \epsilon_2 &:= c|\nabla a|^2, \\ \epsilon_3 &:= \frac{1}{r^{n+2}} \int_{B(x_0,r)} |\nabla u - \nabla a|^2.\end{aligned}$$

If we make the transformation $V(\xi) \mapsto |\xi|^2 + |\xi|^p$ it is easily verified that these limits hold for $p \geq 2$. Thus by the pre-Caccioppoli inequality (3.32)

$$\mu(B[x_0,\alpha r]) \leq \theta \mu(B[x_0,r]) + \left( \frac{\epsilon_3(r)}{(1-\alpha)^2} + \epsilon_2(r)(1-\alpha^n) + \epsilon_1(r) \right) r^n$$

for $p > 1$, and following the direct methods of [5,15] we obtain

$$\lim_{r\to 0^+} \inf \frac{\mu(B[x_0,r])}{r^n} = 0.$$

Hence by Vitali's covering theorem

$$\mu(B[0,\sigma]) = 0$$

for each fixed $\sigma \in (0,1)$ implying (3.17) and (3.18), completing Step 3.

We finish by recalling the estimate (3.16) from which

$$\lim_{j\to\infty} \frac{E(x_j,\tau r_j)}{\lambda_j^2} \leq \lim_{j\to\infty} \frac{c}{\lambda_j^2} \fint_{B(0,\tau)} \left[ |V(\lambda_j(\nabla u_j - \nabla u))|^2 + |V(\lambda_j(\nabla u - (\nabla u)_{0,\tau}))|^2 + |V(\lambda_j((\nabla u)_{0,\tau} - (\nabla u_j)_{0,\tau}))|^2 \right]$$

by (iii) of Lemma 2.3, and $(a+b)^p \leq 2^{p-1}(a^p + b^p)$. Thus by (3.15), (3.17) and (i) of the same lemma

$$\lim_{j\to\infty} \frac{E(x_j,\tau r_j)}{\lambda_j^2} \leq C^*(p,L)\tau^2 + \lim_{j\to\infty} |(\nabla u)_{0,\tau} - (\nabla u_j)_{0,\tau}|^2. \tag{3.39}$$

Similarly we show for $p \geq 2$ that

$$\lim_{j\to\infty} \frac{E(x_j,\tau r_j)}{\lambda_j^2} \leq C^*(p,L)\tau^2 + \lim_{j\to\infty} \left( |(\nabla u)_{0,\tau} - (\nabla u_j)_{0,\tau}|^2 + \lambda_j^{p-2} |(\nabla u)_{0,\tau} - (\nabla u_j)_{0,\tau}|^p \right). \tag{3.40}$$

Now since $\nabla u_j \rightharpoonup \nabla u$ weakly in $\mathrm{L}^{s(p)}(B(0,1), \mathbb{R}^{N\times n})$ ($s(p) = \min\{2,p\}$) the right hand limits in (3.39) and (3.40) are zero.

Thus

$$\lim_{j\to\infty} \frac{E(x_j, \tau r_j)}{\lambda_j^2} \le C^*(p, L)\tau^2$$

which contradicts (3.4) with $C_L = 2C^*(p, L)$. □

Having proved the proposition, Theorem 1 follows by well known arguments (see [11] and references therein).

## 4. Proof of Theorem 2

Following [15], we use Taylors formula together with (1.13) to obtain

$$\begin{aligned}\int_\Omega \big(F(\nabla\overline{u}+\nabla\varphi)-F(\nabla\overline{u})\big) &= \int_\Omega\int_0^1 (1-t)\big(F''(\nabla\overline{u}+t\nabla\varphi)-F''(\nabla\overline{u})\big)[\nabla\varphi,\nabla\varphi]\mathrm{d}t \\ &\quad + \frac{1}{2}\int_\Omega F''(\nabla\overline{u})[\nabla\varphi,\nabla\varphi]. \end{aligned} \tag{4.1}$$

Thus by the uniform continuity condition (1.15) and positive second variation at $\overline{u}$, (1.14), we have

$$\int_\Omega (F(\nabla\overline{u}+\nabla\varphi)-F(\nabla\overline{u})) \ge \frac{1}{2}\int_{\mathbb{R}^n} \big(\delta|\nabla\varphi|^2 - \omega(|\nabla\varphi|)|\nabla\varphi|^2\big). \tag{4.2}$$

Note that we have used the fact that $\nabla\varphi = 0$ off $\Omega$.

We next we use the Orlicz version of the inequality of Fefferman and Stein [9] derived in [15]. Noting that the derivation does not require $f$ to be bounded or have compact support in $\mathbb{R}^n$ we reproduce the relevant lemma for the convenience of the reader, omitting those conditions that are not relevant here. First we introduce the required notation.

The Hardy Littlewood and Fefferman-Stein maximal functions of $f\colon \mathbb{R}^n \to \mathbb{R}^{N\times n}$ are respectively

$$f^*(x) = \sup_{\{B: x\in B\}} \fint_B |f(y)|\mathrm{d}y$$

and

$$f^\#(x) = \sup_{\{B: x\in B\}} \fint_B |f(y)-f_B|\mathrm{d}y$$

where we have taken suprema over all open balls $B \subset \mathbb{R}^n$ containing $x$.

**Lemma 4.1.** *Let $\Phi\colon [0,\infty) \to [0,\infty)$ be an increasing and continuous function with $\Phi(0) = 0$ and consider the Borel map $f\colon \mathbb{R}^n \to \mathbb{R}^{N\times n}$ then*

$$\int_{\mathbb{R}^n} \Phi(|f^*|) \le \frac{5^n}{\epsilon}\int_{\mathbb{R}^n} \Phi\left(\frac{|f^\#|}{\epsilon}\right) + 2\cdot 5^{3n}\epsilon \int_{\mathbb{R}^n} \Phi(5^n\cdot 2^{n+1}|f^*|). \tag{4.3}$$

Now returning to (4.2), by applying Lemma 4.1 to $\Phi(t) = \omega(t)t^2$ with sufficiently small $\epsilon$ together with condition (1.12), we have the following for some positive finite constant $c_*$

$$\int_\Omega (F(\nabla\overline{u}+\nabla\varphi)-F(\nabla\overline{u})) \ge \frac{1}{2}\int_{\mathbb{R}^n} \big(\delta|\nabla\varphi|^2 - c_*\omega(|\nabla\varphi^\#|)|(\nabla\varphi)^\#|^2\big). \tag{4.4}$$

Now as in [15] we remark that by the Hardy Littlewood-Wiener maximal inequality there exists a constant $c_0(n,N) > 0$ such that

$$\int_{\mathbb{R}^n} |\nabla\varphi|^2 \geq c_0 \int_{\mathbb{R}^n} |(\nabla\varphi)^*|^2$$

and since $(\nabla\varphi)^\# \leq 2(\nabla\varphi)^*$ we have

$$\int_\Omega (F(\nabla\overline{u} + \nabla\varphi) - F(\nabla\overline{u})) \geq \frac{1}{2}\int_{\mathbb{R}^n} \left(\frac{\delta c_0}{4} - c_*\omega(|\nabla\varphi^\#|)\right) |(\nabla\varphi)^\#|^2. \tag{4.5}$$

The final integral is positive when

$$c_*\omega(|\nabla\varphi^\#|) \leq \frac{\delta c_0}{4}\cdot \tag{4.6}$$

It follows that integral is finite when

$$\sup_{\mathbb{R}^n} |(\nabla\varphi)^\#| \leq \omega^{-1}\left(\frac{\delta c_0}{4c_*}\right) =: \delta_*. \tag{4.7}$$

□

Finally we prove Corollary 1.2 of Theorem 2. The proof is straight forward and requires one to take note of the distinction between $\|\cdot\|_{\mathrm{BMO}}$ and $[\cdot]_{*,\Omega}$. For $f \in \mathrm{BMO}(\Omega, \mathbb{R}^{N\times n})$ we clearly have the inequality

$$[f]_{*,\Omega} \leq \|f\|_{\mathrm{BMO}}.$$

Obtaining a reverse inequality for functions of the type $\mathrm{BMO}(\mathbb{R}^n, \mathbb{R}^{N\times n})$ restricted to zero off $\Omega$, is not so easy and depends on the boundary of $\Omega$. Luckily the latter inequality is not required here.

*Proof.* By Theorem 2 we have $\nabla\overline{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ is a $\mathrm{W}^1\mathrm{BMO}$-local minimiser of $I[\cdot]$ for all $\varphi \in \mathrm{W}^1\mathrm{BMO}(\mathbb{R}^n, \mathbb{R}^N) \cap \mathrm{W}_0^{1,1}(\Omega, \mathbb{R}^N)$ (for any $1 \leq p < \infty$) with $\|\nabla\varphi\|_{\mathrm{BMO}} \leq \delta_*$. This implies $[\nabla\varphi]_{*,\Omega} \leq \delta_*$ and therefore is true for all $\varphi \in \mathrm{W}^1\mathrm{BMO}(\Omega, \mathbb{R}^N) \cap \mathrm{W}_0^{1,1}(\Omega, \mathbb{R}^N)$ with $[\nabla\varphi]_{*,\Omega} \leq \delta_*$. Hence all conditions of Theorem 1 are satisfied. □

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