

Γ -CONVERGENCE APPROACH TO VARIATIONAL PROBLEMS IN PERFORATED DOMAINS WITH FOURIER BOUNDARY CONDITIONS

VALERIA CHIADÒ PIAT¹ AND ANDREY PIATNITSKI²

Abstract. The work focuses on the Γ -convergence problem and the convergence of minimizers for a functional defined in a periodic perforated medium and combining the bulk (volume distributed) energy and the surface energy distributed on the perforation boundary. It is assumed that the mean value of surface energy at each level set of test function is equal to zero. Under natural coercivity and p -growth assumptions on the bulk energy, and the assumption that the surface energy satisfies p -growth upper bound, we show that the studied functional has a nontrivial Γ -limit and the corresponding variational problem admits homogenization.

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INTRODUCTION

This work is devoted to the asymptotic analysis of a variational problem for a functional defined in a perforated medium and combining the bulk (volume distributed) energy and the surface energy defined on the perforation boundary. In the studied model the perforation is obtained by a homothetic dilatation of a given periodic structure of holes, with a small scaling factor denoted by ε . Then the surface measure tends to infinity as ε goes to 0. To compensate this measure growth we assume that the mean value of surface energy at each level set of the unknown function is equal to zero. Then, under proper coercivity assumptions on the bulk energy, we show that the said functional has a nontrivial Γ -limit and the corresponding variational problem is well-posed and admits homogenization.

The behaviour of solutions to boundary value problems in perforated domains with Neumann boundary condition at the microstructure boundary is well understood now. There is an extensive literature on this subject. We refer here the works [9,13], where both Neumann and Dirichlet boundary conditions were considered. The paper [11] dealt with the Stokes and Navier-Stokes equations in perforated domains. In the work [6] the variational approach was used to study boundary value problems for Poisson equation in perforated domain.

The linear elliptic equations in perforated domain with Dirichlet and Fourier boundary conditions on the boundary of the perforation were considered in several mathematical works. The case of Dirichlet problem in a periodic perforated medium was investigated in [8,13]. It was shown that, if the volume fraction of the perforation does not vanish, then the solution vanishes at the rate ε^2 . If the volume fraction of the perforation

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¹ Dipartimento di Matematica, Politecnico di Torino, C.so Duca degli Abruzzi 24, 10129 Torino, Italy.

² Narvik University College, HiN, Postbox 385, 8505, Narvik, Norway and Lebedev Physical Institute RAS, Leninski prospect 53, Moscow 119991, Russia. andrey@sci.lebedev.ru

is asymptotically small, then, under a proper choice of the rate of its decay, the homogenized equation might receive an additional potential (the so called “strange term”). This phenomenon was observed in [8,13,14]. The problem with dissipative Fourier condition on the boundary of the perforation was considered in [7,10], and some other works. In the case of homothetic dilatation of a given periodic perforated structure, the solution vanishes, as the microstructure period tends to zero. However, if the coefficient of the Fourier boundary operator is small (of order ε), or the volume fraction of the holes vanishes at a certain critical rate, then the homogenization result holds, and the limit operator has an additional potential (see [3,15,16,18]).

The case of Fourier boundary condition with the coefficient having zero average over the perforation surface, has been considered in [4]; closely related spectral problems have also been studied in [17,19]. In this case the formally homogenized operator is well-defined and contains an additional potential. If this homogenized operator is coercive, then the original problem is well-posed for all sufficiently small ε and the studied family of problems admits homogenization. This is a linear version of the problem studied in the present paper, the corresponding Lagrangian in this case being purely quadratic.

Γ -convergence and homogenization of variational functionals with periodic and locally periodic Lagrangians have been widely studied in the existing literature, see for instance [5,12].

In the model studied in this work, the bulk energy density (denoted by $f(\frac{x}{\varepsilon}, Du)$) is periodic in the space variable and satisfies convexity, coercivity and p -growth conditions with respect to the gradient of the unknown function. The surface energy density (denoted by $g(\frac{x}{\varepsilon}, u(x))$) is a periodic function of the first argument, which admits a p -growth upper bound and satisfies a local Lipschitz condition with respect to u . We assume that the mean value of $g(\cdot, z)$ over the perforation surface is equal to zero for any $z \in \mathbb{R}$. This condition is crucial.

We show that under mentioned above conditions the studied functional Γ -converges to the limit functional defined in the solid domain. The limit Lagrangian is determined in terms of an auxiliary variational problem on the perforated torus. It is worth to note that, in contrast with the linear case mentioned above, the contributions of the bulk and surface energies to the limit Lagrangian are coupled.

We then prove that, if the coerciveness constant of the bulk energy is large enough, then the functional under consideration is coercive uniformly in ε . This allows us to study the asymptotic behaviour of the corresponding minimization problems and show that the minimal energies and minimizers of the ε -variational problems converge to those of the limit functional.

The paper is organized as follows:

Section 1 contains the problem setup and main definitions. Then, in Section 2 we introduce sufficient conditions for the coercivity of the studied energy functionals, and state our main results. In Section 3 we prove some auxiliary statements. Sections 4 and 5 deal with the proof of Γ -liminf and Γ -limsup inequalities respectively.

1. ASSUMPTIONS AND SETTING OF THE PROBLEM

Let $Y = [0, 1]^n$. Let also $E \subset \mathbb{R}^n$ be a Y -periodic, connected, open set, with Lipschitz boundary $\partial E = S$, and denote $B = \mathbb{R}^n \setminus E$. We also assume *for the presentation simplicity* that $E \cap Y$ is a connected set with Lipschitz boundary and that $B \cap Y \subset\subset Y$ so that $\mathbb{R}^n \setminus E$ is made of disconnected components. Denote $E_0 = E \cap Y$, $S_0 = S \cap Y$, $B_0 = B \cap Y$.

For any positive number ε and every set $A \subset \mathbb{R}^n$ we denote the corresponding ε -homothetic set by $\varepsilon A = \{x \in \mathbb{R}^n : x/\varepsilon \in A\}$. Now, for every $i \in \mathbb{Z}^n$ we set $Y_\varepsilon^i = \varepsilon(i + Y)$, $S_\varepsilon^i = \varepsilon S \cap Y_\varepsilon^i$, $B_\varepsilon^i = \varepsilon B \cap Y_\varepsilon^i$. Given a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, we consider the perforated domain Ω_ε defined by

$$\Omega_\varepsilon = \Omega \setminus \cup\{B_\varepsilon^i : i \in I_\varepsilon\}, \quad I_\varepsilon = \{i \in \mathbb{Z}^n : Y_\varepsilon^i \subset \Omega\}. \tag{1.1}$$

In this case Ω_ε remains connected and the perforation does not intersect the boundary of Ω , so that $\partial\Omega_\varepsilon$ is the union of the fixed surface $\partial\Omega$ and the varying surface S_ε

$$\partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon, \quad S_\varepsilon = \cup\{S_\varepsilon^i : i \in I_\varepsilon\}.$$

Let us denote by \mathcal{H}^{n-1} the Hausdorff $(n-1)$ -dimensional measure in \mathbb{R}^n , and let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. We consider the functional $F_\varepsilon : L^p(\Omega) \rightarrow \overline{\mathbb{R}}$ defined by

$$F_\varepsilon(u) = \begin{cases} \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, Du\right) dx + \int_{S_\varepsilon} g\left(\frac{x}{\varepsilon}, u\right) d\mathcal{H}^{n-1} & \text{if } u \in W^{1,p}(\Omega_\varepsilon) \\ +\infty & \text{otherwise.} \end{cases} \quad (1.2)$$

Here $f = f(y, \xi) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, $g = g(y, z) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ are given Borel functions, which satisfy the following conditions:

- $f(y, \xi)$ and $g(y, z)$ are Y -periodic in variable y .
- $f(y, \xi)$ is convex in ξ .
- p -growth: there is $p > 1$ such that

$$c_1 |\xi|^p \leq f(y, \xi) \leq c_2 (1 + |\xi|^p) \quad (1.3)$$

$$|g(y, z)| \leq c_3 (1 + |z|^p), \quad |g'_z(y, z)| \leq c_3 (1 + |z|^{p-1}) \quad (1.4)$$

for almost all $y \in \mathbb{R}^n$, all $\xi \in \mathbb{R}^n$ and $z \in \mathbb{R}$.

- Centering:

$$\int_S g(y, z) d\mathcal{H}^{n-1}(y) = 0 \quad \text{for all } z \in \mathbb{R}. \quad (1.5)$$

- Lipschitz continuity:

$$|g(y, z_1) - g(y, z_2)| \leq c_4 (1 + |z_1| + |z_2|)^{p-1} |z_1 - z_2|, \quad (1.6)$$

$$|g'_z(y, z_1) - g'_z(y, z_2)| \leq c_5 (1 + |z_1| + |z_2|)^{p-2} |z_1 - z_2|, \quad (1.7)$$

for all $z_1, z_2 \in \mathbb{R}$.

Actually, (1.6) is a consequence of (1.4). We formulate this condition explicitly for the sake of convenience. Also notice that, due to the convexity of $f(y, \cdot)$, (1.3) implies the estimate

$$|f(y, \xi_1) - f(y, \xi_2)| \leq c_6 (1 + |\xi_1| + |\xi_2|)^{p-1} |\xi_1 - \xi_2|. \quad (1.8)$$

Given $\Phi \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$, we consider a minimization problem with Dirichlet boundary conditions on the exterior boundary $\partial\Omega$, namely

$$m_\varepsilon = \min\{F_\varepsilon(u) : u = \Phi \text{ on } \partial\Omega\} \quad (1.9)$$

and study the asymptotic behaviour of m_ε and the corresponding minimizers as $\varepsilon \rightarrow 0$. We remark that the surface integral in the functional (1.2) plays the role of a boundary condition of Fourier-type on the varying part of $\partial\Omega_\varepsilon$.

Notice that the minimizers of F_ε are only defined in Ω_ε . It is convenient to extend them to the whole domain Ω .

Lemma 1.1. *Under our standing assumptions on the geometry of Ω_ε there exists a family of linear continuous extension operators*

$$T_\varepsilon : W^{1,p}(\Omega_\varepsilon) \rightarrow W^{1,p}(\Omega)$$

such that

$$T_\varepsilon u = u \text{ in } \Omega_\varepsilon$$

and

$$\int_\Omega |T_\varepsilon u|^p dx \leq C \int_{\Omega_\varepsilon} |u|^p dx, \quad \int_\Omega |D(T_\varepsilon u)|^p dx \leq C \int_{\Omega_\varepsilon} |Du|^p dx \quad (1.10)$$

for each $u \in W^{1,p}(\Omega_\varepsilon)$, the constant $C > 0$ here does not depend on ε .

Proof. In the case $p = 2$ the proof of the required statement can be found in [9] and in [10], Theorem 1.2.10. Following the line of this proof, one can easily show that this statement also holds for any $p \in (1, +\infty)$. Indeed, for an arbitrary function $u \in W^{1,p}(\Omega_\varepsilon)$, denote by u_ε^i , $i \in I_\varepsilon$, the restriction of u on Y_ε^i , and by U_ε^i the rescaled functions

$$U_\varepsilon^i(y) = u_\varepsilon^i(\varepsilon(y + i)).$$

By construction, $U_\varepsilon^i \in W^{1,p}(Y \setminus B_0)$. Since, under our assumptions, $E_0 = Y \cap E$ is a connected set with a Lipschitz boundary, then, according to Lemma 2.6 in [1], there exists an extension operator $T : W^{1,p}(Y \setminus B_0) \rightarrow W^{1,p}(Y)$ such that for any $U \in W^{1,p}(Y \setminus B_0)$ it holds $TU = U$ in $Y \setminus B_0$, and

$$\int_Y |TU|^p dy \leq C \int_{Y \setminus B_0} |U|^p dy, \quad \int_Y |D(TU)|^p dy \leq C \int_{Y \setminus B_0} |DU|^p dy.$$

The proof of quoted above Lemma 2.6 in [1] is based on subtracting from U its mean value and applying the Poincaré inequality to the obtained function (see [1] for the details). If we denote the extension TU_ε^i by \tilde{U}_ε^i and let $\tilde{u}_\varepsilon^i(x) = \tilde{U}_\varepsilon^i(\frac{x-i}{\varepsilon})$, then $\tilde{u}_\varepsilon^i = u$ in $Y_\varepsilon^i \setminus B_\varepsilon^i$, and

$$\int_{Y_\varepsilon^i} |\tilde{u}_\varepsilon^i|^p dx \leq C \int_{Y_\varepsilon^i \setminus B_\varepsilon^i} |u|^p dx, \quad \int_{Y_\varepsilon^i} |D\tilde{u}_\varepsilon^i|^p dx \leq C \int_{Y_\varepsilon^i \setminus B_\varepsilon^i} |Du|^p dx \tag{1.11}$$

for all $i \in I_\varepsilon$. Setting $T_\varepsilon u = u_\varepsilon^i$ for $x \in Y_\varepsilon^i$ and $T_\varepsilon u = u$ for $x \in \Omega \setminus \bigcup\{Y_\varepsilon^i : i \in I_\varepsilon\}$, we obtain the desired extension operator T_ε . The estimate (1.10) can be obtained by summing up the inequalities (1.11) over $i \in I_\varepsilon$. This completes the proof.

For the notation simplicity, in this paper we will keep the notation u also for the extended function $T_\varepsilon u$.

As a consequence of the existence of extension operators one can derive *Friedrichs inequality*: there exists a constant $k_f > 0$ depending only on p, n, Ω , such that

$$\int_{\Omega_\varepsilon} |u|^p dx \leq k_f \int_{\Omega_\varepsilon} |Du|^p dx \tag{1.12}$$

for all $\varepsilon > 0$ and all $u \in W^{1,p}(\Omega_\varepsilon)$, such that $u = 0$ on $\partial\Omega$.

Notice that the functional (1.2) need not be equi-coercive in $L^p(\Omega)$, and the infimum in (1.9) might be equal to $-\infty$. In this case the boundedness of the energies $F_\varepsilon(u)$ does not imply any *a priori* estimates for u . The corresponding example with quadratic functions $f(x, \cdot)$, $g(x, \cdot)$ can be constructed as follows.

Let $\Omega = (0, 1)^n$ be a unit cube in \mathbb{R}^n , and suppose that B is a $[0, 1]^n$ -periodic cubic structure in \mathbb{R}^n , generated by the set $B_0 = [r, 1 - r]^n$ with $r = 1/4$. Then εB is a disperse cubic structure. Denote $S_\varepsilon = \varepsilon\partial B \cap \Omega$. We consider the functional

$$F_\varepsilon(u) = \int_{\Omega_\varepsilon} |\nabla u|^2 dx + \lambda \int_{S_\varepsilon} g_0\left(\frac{x}{\varepsilon}\right) u^2 d\mathcal{H}^{n-1}, \quad \text{with } \lambda > 0,$$

and we assume that $g_0(y)$ is a smooth Y -periodic function whose trace on S is nontrivial (not equal to 0) and satisfies the condition

$$\int_{S_0} g_0(y) d\mathcal{H}^{n-1} = 0.$$

This is a particular case of (1.2) with $f(y, z, \xi) = |\xi|^2$ and $g(y, z) = g_0(y)z^2$. Clearly, all the conditions (1.3)–(1.8) are fulfilled with $p = 2$. Denote

$$u_0(x) = [x_1(1 - x_1)]^2 [x_2(1 - x_2)]^2 \times \dots \times [x_n(1 - x_n)]^2$$

and

$$u_\varepsilon(x) = u_0(x) - \varepsilon g_0\left(\frac{x}{\varepsilon}\right)u_0(x).$$

In this case $u_\varepsilon = u_0 = \Phi$ on $\partial\Omega$, with $\Phi \equiv 0$. Evaluating $F_\varepsilon(u_\varepsilon)$ we obtain after straightforward rearrangements

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \int_{E_0} dy \int_{\Omega} |\nabla u_0(x)|^2 dx + \int_{\Omega} u_0^2(x) \int_{E_0} |\nabla g_0(y)|^2 dy dx \\ &\quad + \lambda \int_{S_\varepsilon} u_0^2(x) g_0\left(\frac{x}{\varepsilon}\right) d\mathcal{H}^{n-1} - 2\varepsilon\lambda \int_{S_\varepsilon} |u_0(x)|^2 g_0^2\left(\frac{x}{\varepsilon}\right) d\mathcal{H}^{n-1} + O(\varepsilon) \\ &= \left(1 - (1 - 2r)^n\right) \int_{\Omega} |\nabla u_0(x)|^2 dx + \int_{\Omega} u_0^2(x) \int_{E_0} |\nabla g_0(y)|^2 dy dx \\ &\quad + 2\lambda \int_{\Omega} \nabla(u_0^2(x)) dx \cdot \int_{S_0} y g_0(y) d\mathcal{H}^{n-1}(y) - 2\lambda \int_{\Omega} \int_{S_0} u_0^2(x) g_0^2(y) d\mathcal{H}^{n-1}(y) dx + O(\varepsilon). \end{aligned}$$

If we now choose $g_0(y)$ in such a way that

- $g_0(y)$ only depends on y_1 , i.e. $g_0(y) = g_0(y_1)$;
- $g_0(y_1 - 1/2)$ is an even function;
- $\text{supp}(g_0(\cdot)) \cap [0, 1] \subset (-1/4, 1/4)$,
- $\int_0^1 g_0(y_1) dy_1 = 0$,

then the third integral on the r.h.s. of the last formula is equal to zero, and we get

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \left(1 - (1/4)^n\right) \int_{\Omega} |\nabla u_0(x)|^2 dx + \int_{\Omega} u_0^2(x) \int_{E_0} |\nabla g_0(y)|^2 dy dx \\ &\quad - 2\lambda \int_{\Omega} \int_{S_0} u_0^2(x) g_0^2(y) d\mathcal{H}^{n-1}(y) dx + O(\varepsilon). \end{aligned}$$

Since $\int_{\Omega} u_0^2(x) dx > 0$ and $\int_{S_0} g_0^2(y) d\mathcal{H}^{n-1}(y) > 0$, then for large enough λ and small enough ε we obtain $F_\varepsilon(u_\varepsilon) \leq c < 0$. Therefore, if we denote $v_\varepsilon(x) = \varepsilon^{-1}u_\varepsilon(x)$, then, for some $\lambda > 0$,

$$F_\varepsilon(v_\varepsilon) = \varepsilon^{-2}F_\varepsilon(u_\varepsilon) \leq c\varepsilon^{-2} < 0, \quad \text{and} \quad \|v_\varepsilon\|_{L^2(\Omega)} \rightarrow \infty, \quad \text{as } \varepsilon \rightarrow 0.$$

In this paper we will show that the functional F_ε does Γ -converge, as $\varepsilon \rightarrow 0$ (see, for instance [12] for the definition of Γ -convergence), and that the limit functional F takes the form

$$F(u) = \begin{cases} \int_{\Omega} L(u, Du) dx & \text{if } u \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise} \end{cases} \quad (1.13)$$

where

$$L(z, \xi) = \inf \left\{ \int_{E \cap Y} f(y, \xi + Dw) dy + \int_{S \cap Y} g'_z(y, z)(\xi \cdot y + w) d\mathcal{H}^{n-1} : w \in W_{\text{per}}^{1,p}(Y \cap E) \right\}. \quad (1.14)$$

Here $W_{\text{per}}^{1,p}(E \cap Y)$ denotes the space of functions $w : E \rightarrow \mathbb{R}$ being the restriction to E of Y -periodic functions in $W_{\text{loc}}^{1,p}(\mathbb{R}^n)$.

We also show that, under the assumption that the coercivity constant c_1 in (1.3) is sufficiently large, the minimizer u_ε in (1.9) converges in $L^p(\Omega)$, as $\varepsilon \rightarrow 0$, towards a minimizer u of the limit functional F . Moreover, the corresponding minimum values also converge, *i.e.*, $m_\varepsilon \rightarrow m$ where

$$m = \min\{F(u) : u = \Phi \text{ on } \partial\Omega\}. \tag{1.15}$$

Let us compute the effective Lagrangian in the quadratic case:

$$f(y, \xi) = a(y)\xi \cdot \xi, \quad g(y, z) = g_0(y)z^2.$$

According to the above formula (1.14), in this case we have

$$L(z, \xi) = \min_{w \in H_{\text{per}}^1(Y \cap E)} \left\{ \int_{E \cap Y} a(y)(\xi + Dw) \cdot (\xi + Dw) dy + 2 \int_{S \cap Y} g_0(y)z(\xi \cdot y + w) d\mathcal{H}^{n-1} \right\}. \tag{1.16}$$

The corresponding Euler equation reads

$$\begin{cases} \operatorname{div}(a(y)(\xi + \nabla w(y))) = 0 & \text{in } E_0, \\ \frac{\partial}{\partial n_a} w \Big|_{S_0} = -a(y)n \cdot \xi + z g_0(y), & w \in H_{\text{per}}^1(Y). \end{cases}$$

By linearity, a solution of this equation can be represented as the sum $w(x) = w_1(x) + w_2(x)$, where w_1 and w_2 are solutions to the problems

$$\begin{cases} \operatorname{div}(a(y)(\xi + \nabla w_1(y))) = 0 & \text{in } E_0, \\ \frac{\partial}{\partial n_a} w_1 \Big|_{S_0} = -a(y)n \cdot \xi, & w_1 \in H_{\text{per}}^1(Y) \end{cases}$$

and

$$\begin{cases} \operatorname{div}(a(y)\nabla w_2(y)) = 0 & \text{in } E_0, \\ \frac{\partial}{\partial n_a} w_2 \Big|_{S_0} = z g_0(y) & w_2 \in H_{\text{per}}^1(Y). \end{cases}$$

Substituting w_1 and w_2 in (1.16) and considering the above equations and the fact that w_1 and w_2 depend linearly on ξ and z respectively, we obtain after simple rearrangements

$$\begin{aligned} L(z, \xi) &= \int_{E_0} a(y)(\xi + \nabla w_1) \cdot (\xi + \nabla w_1) dy - \int_{E_0} a(y)\nabla w_2 \cdot \nabla w_2 dy \\ &\quad + 2 \int_{S_0} g_0(y)z(\xi \cdot y + w_1(y)) d\mathcal{H}^{n-1} = \hat{a}^N \xi \cdot \xi - \hat{g}z^2 + \hat{b} \cdot \xi z. \end{aligned}$$

It should be noted that the matrix \hat{a}^N here coincides with the effective matrix for the classical homogenization problem with homogeneous Neumann conditions on the perforation boundary.

Notice also that $\hat{g} > 0$ unless $g_0(y) = 0$. The contribution of the last term $\hat{b} \cdot \xi z$ can be computed explicitly. Indeed, if $u = \Phi$ on $\partial\Omega$, then

$$\frac{1}{2} \int_{\Omega} \hat{b} \cdot \nabla(u^2(x)) dx = \frac{1}{2} \int_{\partial\Omega} \hat{b} \cdot n \Phi^2(x) d\mathcal{H}^{n-1}.$$

Therefore, this term does not depend on u , and hence it does not have an influence on the limit minimizer.

2. MAIN RESULTS

First of all, we obtain the estimate for the surface term of $F_\varepsilon(u)$ in terms of its volume term and the L^p -norm of u on Ω_ε . This estimate relies crucially on the assumption (1.5) and plays important role in the further analysis.

Lemma 2.1. *For any $\gamma > 0$ there exists a positive constant $c(\gamma)$ such that for each $\varepsilon > 0$ and every set A_ε of the form*

$$A_\varepsilon = \cup\{Y_\varepsilon^i \cap (\varepsilon E) : i \in I\}, \quad \text{with } I \subset I_\varepsilon$$

the inequality holds

$$\left| \int_{S_\varepsilon \cap A_\varepsilon} g\left(\frac{x}{\varepsilon}, u\right) d\mathcal{H}^{n-1} \right| \leq (\gamma + \varepsilon^p c(\gamma)) \int_{A_\varepsilon} |Du|^p dx + c(\gamma) \int_{A_\varepsilon} (1 + |u|^p) dx \quad (2.1)$$

for all $u \in W^{1,p}(A_\varepsilon)$. Moreover,

$$\left| \int_{S_\varepsilon} g\left(\frac{x}{\varepsilon}, u\right) d\mathcal{H}^{n-1} \right| \leq (\gamma + \varepsilon^p c(\gamma)) \int_{\Omega_\varepsilon} |Du|^p dx + c(\gamma) \int_{\Omega_\varepsilon} (1 + |u|^p) dx \quad (2.2)$$

for all $u \in W^{1,p}(\Omega_\varepsilon)$. In particular,

$$\left| \int_{S_\varepsilon} g\left(\frac{x}{\varepsilon}, u\right) d\mathcal{H}^{n-1} \right| \leq k_0 \int_{\Omega_\varepsilon} (1 + |u|^p + |Du|^p) dx \quad (2.3)$$

for some $k_0 > 0$ which does not depend on ε .

The proof of Lemma 2.1 is given in Section 3.

For the reader's convenience, we recall now the definition of Γ -convergence that we will use in the rest of the paper.

Definition 2.2. Let $F_\varepsilon, F : L^p(\Omega) \rightarrow \overline{\mathbb{R}}$ for every $\varepsilon > 0$. The family F_ε is said to Γ -converge to F , as $\varepsilon \rightarrow 0$, if the following two properties hold

(a) (Γ -lim inf inequality) For any sequence $u_\varepsilon \in L^p(\Omega)$, such that u_ε converges to u in $L^p(\Omega)$, as $\varepsilon \rightarrow 0$, we have

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq F(u).$$

(b) (Γ -lim sup inequality) For any $u \in L^p(\Omega)$ there is a sequence $u_\varepsilon \in L^p(\Omega)$ such that u_ε converges to u in $L^p(\Omega)$ and

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \leq F(u).$$

We state now our main result.

Theorem 2.3. *Let $F_\varepsilon, F : L^p(\Omega) \rightarrow \overline{\mathbb{R}}$ be the functional given by (1.2), (1.13), (1.14), and suppose that all the conditions specified in Section 1 are fulfilled. Then F_ε does Γ -converge to F , as $\varepsilon \rightarrow 0$.*

Proposition 2.4. *Let $u \in W^{1,p}(\Omega)$, and suppose that $u|_{\partial\Omega} = \Phi$. Then there is a family $u_\varepsilon \in W^{1,p}(\Omega)$, $u_\varepsilon|_{\partial\Omega} = \Phi$, such that*

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) = F(u). \quad (2.4)$$

The proof of Theorem 2.3 and Proposition 2.4 is presented in the following sections. In the rest of this section we derive a number of consequences of these results. Consider minimization problems (1.9) and (1.15).

Corollary 2.5. *If all the assumptions of Theorem 2.3 hold true and if, furthermore,*

$$c_1 > k_0(1 + k_f), \tag{2.5}$$

where k_f is given by (1.12), then for every $\Phi \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ and for all sufficiently small $\varepsilon > 0$ problem (1.9) is well-posed and has a minimizer $u_\varepsilon \in W^{1,p}(\Omega)$. Moreover, the limit problem (1.15) is also well-posed, and

$$\int_{\Omega_\varepsilon} |u_\varepsilon - u|^p dx \rightarrow 0, \quad m_\varepsilon \rightarrow m, \tag{2.6}$$

as $\varepsilon \rightarrow 0$, where u is a solution to problem (1.15), and m is the corresponding minimum.

Proof. Let us first show that under condition (2.5) the functionals $\{F_\varepsilon(u) : u|_{\partial\Omega} = \Phi\}$ are equi-coercive. For any $u \in W^{1,p}(\Omega_\varepsilon)$ by (1.3) and (2.3) we have

$$F_\varepsilon(u) \geq -k_0 \int_{\Omega_\varepsilon} (1 + |u|^p + |Du|^p) dx + c_1 \int_{\Omega_\varepsilon} |Du|^p dx. \tag{2.7}$$

Since $u|_{\partial\Omega} = \Phi$, (1.12) we get

$$\int_{\Omega_\varepsilon} |u - \Phi|^p dx \leq k_f \int_{\Omega_\varepsilon} |Du - D\Phi|^p dx.$$

We transform this estimate using the following simple inequality: for any $\kappa > 0$ there is $c(p, \kappa) > 0$ such that $(a + b)^p \leq (1 + \kappa)a^p + c(p, \kappa)b^p$ for all positive a and b . After simple rearrangements this yields

$$\int_{\Omega_\varepsilon} |u|^p dx \leq (1 + \kappa)k_f \int_{\Omega_\varepsilon} |Du|^p dx + c(p, \kappa)(1 + k_f) \int_{\Omega_\varepsilon} (|\Phi|^p + |D\Phi|^p) dx. \tag{2.8}$$

Combining the last estimate with (2.7), we obtain

$$(c_1 - k_0(1 + (1 + \kappa)k_f)) \int_{\Omega_\varepsilon} |Du|^p dx \leq F_\varepsilon(u) + k_0|\Omega| + c_1(\kappa, p) \int_{\Omega_\varepsilon} (|\Phi|^p + |D\Phi|^p) dx.$$

According to (2.5) we can choose $\kappa > 0$ in such a way that $(c_1 - k_0(1 + (1 + \kappa)k_f)) > 0$. Considering (2.8) we conclude that for some $c_1(p) > 0$ and $c_2(p) > 0$ the inequality

$$c_1(p) \int_{\Omega_\varepsilon} (|u|^p + |Du|^p) dx \leq F_\varepsilon(u) + k_0|\Omega_\varepsilon| + c_2(p) \int_{\Omega_\varepsilon} (|\Phi|^p + |D\Phi|^p) dx \tag{2.9}$$

holds for all $u \in W^{1,p}(\Omega_\varepsilon)$ such that $u = \Phi$ on $\partial\Omega$. This completes the proof of equi-coercivity.

Another important property of F_ε is its lower semi-continuity in the space $L^p(\Omega)$. To prove this property notice that, by Lemma 2.1, the surface integral $\int_{S_\varepsilon} g(\frac{x}{\varepsilon}, u) d\mathcal{H}^{n-1}$ is uniformly in ε continuous in the space $W^{1,p}$ equipped with the topology of weak convergence. This implies that the said surface integral is uniformly in ε continuous with respect to the strong L^p topology in any bounded subset of $W^{1,p}(\Omega)$. The lower semi-continuity of F_ε is then a consequence of the assumptions of Section 1 and of the coerciveness.

Notice also that, according to Proposition 2.4, for any $u \in W^{1,p}(\Omega)$ a recovery sequence $\{u_\varepsilon\}$ which satisfies Γ -limsup inequality can be chosen in such a way that $u_\varepsilon|_{\partial\Omega} = u|_{\partial\Omega}$ for all $\varepsilon > 0$.

The statement of Corollary 2.5 now follows from the standard properties of Γ -convergence (see, for instance [12]). □

Remark 2.6. It should be noted that the functional F_ε is coercive only in the presence of a dissipative boundary conditions on $\partial\Omega$. Indeed, in the case of the homogeneous Neumann boundary condition on $\partial\Omega$, letting $u_\varepsilon(x) = 1/\varepsilon$ in Ω we obtain the sequence of zero energy functions which tends to infinity as $\varepsilon \rightarrow 0$.

3. PRELIMINARY RESULTS

We begin this section by recalling some inequalities valid in Sobolev spaces. For their proof see, for instance [2,20]. Under our assumptions on E , and S , there exist positive constants k_p, k_t such that for each $u \in W^{1,p}(Y)$ the following inequalities hold:

Poincaré-Wirtinger inequality

$$\int_{S \cap Y} |u - \bar{u}|^p d\mathcal{H}^{n-1} \leq k_p \int_{E \cap Y} |Du|^p dy \quad (3.1)$$

where

$$\bar{u} = |E \cap Y|^{-1} \int_{E \cap Y} u dx \quad (3.2)$$

(the inequality remains valid if \bar{u} is replaced by the surface average of u on $S \cap Y$).

Trace inequality

$$\int_{S \cap Y} |u|^p d\mathcal{H}^{n-1} \leq k_t \int_{E \cap Y} (|u|^p + |Du|^p) dy. \quad (3.3)$$

By performing the change of variable $y = \frac{x}{\varepsilon}$ it is easy to obtain the corresponding re-scaled estimates. Given a function $u \in W^{1,p}(\Omega)$, denote by $\bar{u}_\varepsilon(\cdot)$ the piecewise-constant function obtained by taking the mean value of u over each cell Y_ε^i , i.e.,

$$\bar{u}_\varepsilon(x) = \frac{1}{|Y_\varepsilon^i \cap \varepsilon E|} \int_{Y_\varepsilon^i \cap \varepsilon E} u(y) dy, \quad \text{if } x \in Y_\varepsilon^i \cap \varepsilon E. \quad (3.4)$$

Then, it is easy to check that, for every $u \in W^{1,p}(\Omega)$ and every $\varepsilon > 0$

$$\int_{S_\varepsilon} |u - \bar{u}_\varepsilon|^p d\mathcal{H}^{n-1} \leq k_p \varepsilon^{p-1} \int_{\Omega_\varepsilon} |Du|^p dy \quad (3.5)$$

$$\int_{S_\varepsilon} |u|^p d\mathcal{H}^{n-1} \leq k_t \left(\varepsilon^{-1} \int_{\Omega_\varepsilon} |u|^p dy + \varepsilon^{p-1} \int_{\Omega_\varepsilon} |Du|^p dy \right). \quad (3.6)$$

Using the preceding inequalities we can prove the statement of Lemma 2.1.

Proof of Lemma 2.1. We first prove auxiliary inequalities for $W^{1,p}(Y)$ functions. Let $u \in W^{1,p}(Y)$ and \bar{u} be defined by (3.2); then, by (1.5) and (1.6) we have

$$\begin{aligned} \left| \int_{Y \cap S} g(y, u) d\mathcal{H}^{n-1} \right| &= \left| \int_{Y \cap S} \left(g(y, u) - g(y, \bar{u}) + g(y, \bar{u}) \right) d\mathcal{H}^{n-1} \right| \\ &\leq c_4 \int_{S \cap Y} |u - \bar{u}| (1 + |u| + |\bar{u}|)^{p-1}. \end{aligned}$$

By using Holder inequality, (3.7), (3.1) and (3.3) we obtain

$$\begin{aligned} \left| \int_{Y \cap S} g(y, u) d\mathcal{H}^{n-1} \right| &\leq c_4 \left(\int_{Y \cap S} |u - \bar{u}|^p d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \left(\int_{Y \cap S} (1 + |u| + |\bar{u}|)^p d\mathcal{H}^{n-1} \right)^{\frac{1}{p'}} \\ &\leq c_4 \left(k_p \int_{Y \cap E} |Du|^p dy \right)^{\frac{1}{p}} \left(k_t \int_{Y \cap E} \left((1 + |u| + |\bar{u}|)^p + |Du|^p \right) dy \right)^{\frac{1}{p'}}. \end{aligned}$$

Now, consider $u \in W^{1,p}(\Omega_\varepsilon)$. Making the change of variable $y = \frac{x}{\varepsilon}$, for each $Y_\varepsilon^i \subseteq \Omega$ we have

$$\begin{aligned} \varepsilon^{1-n} \left| \int_{Y_\varepsilon^i \cap S_\varepsilon} g\left(\frac{x}{\varepsilon}, u\right) d\mathcal{H}^{n-1} \right| &\leq c \left(\varepsilon^{-n} \int_{Y_\varepsilon^i \cap \varepsilon E} \varepsilon^p |Du|^p dx \right)^{\frac{1}{p}} \left(\varepsilon^{-n} \int_{Y_\varepsilon^i \cap \varepsilon E} (1 + |u|^p + |\bar{u}_\varepsilon|^p + \varepsilon^p |Du|^p) dx \right)^{\frac{1}{p'}} \\ &\leq c \varepsilon^{-\frac{n}{p}} \left(\int_{Y_\varepsilon^i \cap \varepsilon E} \varepsilon^p |Du|^p dx \right)^{1/p} \cdot \varepsilon^{-\frac{n}{p'}} \left(\int_{Y_\varepsilon^i \cap \varepsilon E} ((1 + |u| + |\bar{u}_\varepsilon|)^p + \varepsilon^p |Du|^p) dx \right)^{\frac{1}{p'}} \\ &= c \varepsilon^{1-n} \left(\int_{Y_\varepsilon^i \cap \varepsilon E} |Du|^p dx \right)^{1/p} \left(\int_{Y_\varepsilon^i \cap \varepsilon E} ((1 + |u| + |\bar{u}_\varepsilon|)^p + \varepsilon^p |Du|^p) dx \right)^{\frac{1}{p'}}. \end{aligned}$$

By the Young inequality, for any $\gamma > 0$ we get

$$\left| \int_{Y_\varepsilon^i \cap S_\varepsilon} g(y, u) d\mathcal{H}^{n-1} \right| \leq \gamma \int_{Y_\varepsilon^i \cap \varepsilon E} |Du|^p dx + c\gamma^{-p'/p} \int_{Y_\varepsilon^i \cap \varepsilon E} \left((1 + |u| + |\bar{u}_\varepsilon|)^p + \varepsilon^p |Du|^p \right) dx.$$

For \bar{u}_ε Jensen's inequality yields the bound

$$|\bar{u}_\varepsilon|^p \leq \frac{1}{|\varepsilon E \cap Y_\varepsilon^i|} \int_{\varepsilon E \cap Y_\varepsilon^i} |u|^p dx, \quad (3.7)$$

and we finally obtain

$$\left| \int_{Y_\varepsilon^i \cap S_\varepsilon} g\left(\frac{x}{\varepsilon}, u\right) d\mathcal{H}^{n-1} \right| \leq (\gamma + \varepsilon^p c(\gamma)) \int_{Y_\varepsilon^i \cap \varepsilon E} |Du|^p dx + c(\gamma) \int_{Y_\varepsilon^i \cap \varepsilon E} (1 + |u|)^p dx.$$

Taking the sum over $i \in I \subset I_\varepsilon$ or $i \in I_\varepsilon$ we obtain the estimates (2.1) and (2.2), respectively. The estimate (2.3) easily follows from (2.2). \square

We proceed to coercivity properties of functionals F_ε .

Lemma 3.1. *Assume that $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$ satisfies the bound*

$$\int_{\Omega_\varepsilon} |u_\varepsilon|^p dx \leq c$$

with a constant $c > 0$ independent of ε , and let $F_\varepsilon(u_\varepsilon) \leq c$. Then there exists $\varepsilon_0 > 0$ such that

$$\int_{\Omega_\varepsilon} (|u_\varepsilon|^p + |Du_\varepsilon|^p) dx \leq c' \quad \forall \varepsilon \in (0, \varepsilon_0)$$

for a suitable constant $c' > 0$ which does not depend on ε . In other words, there are constants $c_1, c_2 > 0$ and $\varepsilon_0 > 0$, such that

$$F_\varepsilon(u) + c_1 \|u\|_{L^p(\Omega_\varepsilon)}^p \geq c_2 \|u\|_{W^{1,p}(\Omega_\varepsilon)}^p$$

for all $\varepsilon < \varepsilon_0$.

Proof. The desired statement follows immediately from the estimate (2.2) and assumption (1.3). \square

Our next aim is to obtain some properties of the Lagrangian L , defined in (1.14).

Proposition 3.2. *The function L defined by (1.14) has the following properties:*

- (a) $L(z, \cdot)$ is convex for every $z \in \mathbb{R}$.
- (b) $L(\cdot, \xi)$ is Lipschitz-continuous for every $\xi \in \mathbb{R}^n$, i.e.,

$$|L(z_2, \xi) - L(z_1, \xi)| \leq c_5(1 + |z_1|^{p-1} + |z_2|^{p-1} + |\xi|^{p-1})|z_1 - z_2| \quad (3.8)$$

for every $z_1, z_2 \in \mathbb{R}$ and every $\xi \in \mathbb{R}^n$.

- (c) There are positive constants k_1, k_2, k_3 such that

$$L(z, \xi) \geq k_1|\xi|^p - k_2|z|^p - k_3 \quad (3.9)$$

for every $z \in \mathbb{R}$, and every $\xi \in \mathbb{R}^n$.

- (d) There is a positive constant k_4 such that

$$L(z, \xi) \leq k_4(|\xi|^p + |z|^p + 1) \quad (3.10)$$

for every $z \in \mathbb{R}$, and every $\xi \in \mathbb{R}^n$.

Proof.

- (a) Let us rewrite L as

$$L(z, \xi) = \int_{Y \cap S} g'_z(y, z)(\xi \cdot y) d\mathcal{H}^{n-1}(y) + \inf_w \left(\int_{Y \cap E} f(y, \xi + Dw) dy + \int_{Y \cap S} g'_z(y, z)w d\mathcal{H}^{n-1}(y) \right).$$

The first term is linear in ξ , while the second one is easily proved to be convex, since $f(y, \xi)$ is convex with respect to ξ . Hence the function $L(z, \cdot)$ is convex.

- (b) For brevity we denote by \mathcal{L} the function

$$\mathcal{L}(z, \xi, w) = \int_{Y \cap E} f(y, \xi + Dw) dy + \int_{S \cap Y} g'_z(y, z)(\xi \cdot y + w) d\mathcal{H}^{n-1}. \quad (3.11)$$

Let us fix $\xi \in \mathbb{R}^n$, and $z_1, z_2 \in \mathbb{R}$ such that $L(z_1, \xi) < L(z_2, \xi)$. For every $\eta > 0$ there exists $w_\eta \in W_{\text{per}}^{1,p}(Y \cap E)$ such that

$$L(z_1, \xi) + \eta > \mathcal{L}(z_1, \xi, w_\eta).$$

The function w_η is defined up to an additive constant. Choosing this additive constant in a proper way, one can assume without loss of generality, that the mean value of either w_η or $(\xi \cdot y + w_\eta)$ is equal to zero, so that the Poincaré inequality holds. It is not difficult to show that

$$\|Dw_\eta\|_{L^p(Y \cap E)}^p \leq k_6(1 + |\xi|^p + |z_1|^p), \quad \|\xi + Dw_\eta\|_{L^p(Y \cap E)}^p \leq k_6(1 + |\xi|^p + |z_1|^p) \quad (3.12)$$

with $k_6 > 0$ which does not depend on η . Indeed, by (1.3) and (3.10) we get

$$\begin{aligned}
 \int_{Y \cap E} |\xi + Dw_\eta|^p dy &\leq \frac{1}{c_1} \int_{Y \cap E} \leq f(y, \xi + Dw_\eta) dy \\
 &\leq \frac{1}{c_1} L(\xi, z_1) + \frac{1}{c_1} \left| \int_{S \cap Y} g'_z(y, z_1)(\xi \cdot y + w_\eta) d\mathcal{H}^{n-1} \right| + \frac{\eta}{c_1} \\
 &\leq C(1 + |\xi|^p + |z_1|^p) + C(1 + |z_1|^{p-1}) \int_{S \cap Y} |\xi \cdot y + w_\eta| d\mathcal{H}^{n-1} \\
 &\leq C(1 + |\xi|^p + |z_1|^p) + C(1 + |z_1|^{p-1}) \left(\int_{Y \cap E} |\xi + Dw_\eta|^p dy \right)^{1/p} \\
 &\leq C(1 + |\xi|^p + |z_1|^p) + \frac{1}{2} \int_{Y \cap E} |\xi + Dw_\eta|^p dy;
 \end{aligned}$$

here we have also used the Young and the trace inequalities. This yields the second upper bound in (3.12). The first upper bound easily follows from the second one. Similar arguments are used in the proof of Proposition 3.4. The reader can find more detail proof there.

Now, by the definition of L , we have

$$\begin{aligned}
 0 < L(z_2, \xi) - L(z_1, \xi) &\leq \mathcal{L}(z_2, \xi, w_\eta) - \mathcal{L}(z_1, \xi, w_\eta) + \eta \\
 &= \int_{Y \cap S} \left(g'_z(y, z_2) - g'_z(y, z_1) \right) (\xi \cdot y + w_\eta) d\mathcal{H}^{n-1} + \eta.
 \end{aligned}$$

By the Lipschitz-continuity of g'_z (see (1.7)) we conclude that

$$\begin{aligned}
 0 < L(z_2, \xi) - L(z_1, \xi) &\leq c_5(1 + |z_1|^{p-2} + |z_2|^{p-2})|z_1 - z_2| \|\xi + Dw_\eta\|_{L^p(Y \cap E)}^{1/p} + \eta \\
 &\leq c_6(1 + |z_1|^{p-1} + |z_2|^{p-1} + |\xi|^{p-1})|z_1 - z_2| + \eta.
 \end{aligned}$$

If $L(z_2, \xi) < L(z_1, \xi)$ the proof is analogous. Since η is an arbitrary positive number, then (3.8) follows.

- (c) By the definition (3.11) of L and by the coercivity of f (see (1.3)), for every $w \in W_{\text{per}}^{1,p}(Y \cap E)$ with zero average on $Y \cap E$ we have the following estimate

$$\begin{aligned}
 \mathcal{L}(z, \xi, w) &\geq \int_{Y \cap E} f(y, z, \xi + Dw) dy + \int_{S \cap Y} g'_z(y, z)(\xi \cdot y + w) d\mathcal{H}^{n-1} \\
 &\geq c_1 \int_{Y \cap E} |\xi + Dw|^p dy + \int_{S \cap Y} g'_z(y, z)(\xi \cdot y + w) d\mathcal{H}^{n-1}.
 \end{aligned}$$

By assumption (1.4) we can estimate the second integral above as follows

$$\begin{aligned}
 \left| \int_{S \cap Y} g'_z(y, z)(\xi \cdot y + w) d\mathcal{H}^{n-1} \right| &\leq c_3 \int_{S \cap Y} (1 + |z|^{p-1}) |\xi \cdot y + w| d\mathcal{H}^{n-1} \\
 &\leq c\mathcal{H}^{n-1}(Y \cap S)^{1-1/p} (1 + |z|^{p-1}) \left(\int_{S \cap Y} |\xi \cdot y + w|^p d\mathcal{H}^{n-1} \right)^{1/p},
 \end{aligned}$$

where c is a suitable positive constant, and we have applied Holder's inequality. Now, by the trace inequality (3.3)

$$\begin{aligned} c\mathcal{H}^{n-1}(Y \cap S)^{1-1/p}(1 + |z|^{p-1}) \left(\int_{S \cap Y} |\xi \cdot y + w|^p d\mathcal{H}^{n-1} \right)^{1/p} \leq \\ c'(1 + |z|^{p-1}) \left(\int_{E \cap Y} (|\xi \cdot y + w|^p + |\xi + Dw|^p) dy \right)^{1/p}. \end{aligned}$$

By means of Poincaré inequality (3.1), we obtain also that

$$c'(1 + |z|^{p-1}) \left(\int_{E \cap Y} (|\xi \cdot y + w|^p + |\xi + Dw|^p) dy \right)^{1/p} \leq c'k_p(1 + |z|^{p-1}) \left(\int_{E \cap Y} |\xi + Dw|^p dy \right)^{1/p}.$$

Now, by Young's inequality, for every $\eta > 0$ there exists $c_\eta > 0$ such that

$$c'k_p(1 + |z|^{p-1}) \left(\int_{E \cap Y} |\xi + Dw|^p dy \right)^{1/p} \leq c_\eta(1 + |z|^p) + \eta \int_{E \cap Y} |\xi + Dw|^p dy.$$

Hence, we have obtained that

$$\left| \int_{S \cap Y} g'_z(y, z)(\xi \cdot y + w) d\mathcal{H}^{n-1} \right| \leq c_\eta(1 + |z|^p) + \eta \int_{E \cap Y} |\xi + Dw|^p dy. \quad (3.13)$$

According to [1] there is an extension operator from $W^{1,p}(Y \cap E)$ to $W^{1,p}(Y)$ such that the extended function (still denoted w) satisfies the inequality

$$\int_Y |\xi + Dw|^p dy \leq C \int_{Y \cap E} |\xi + Dw|^p dy$$

with a constant C which does not depend on w . Hence, using Jensen's inequality we can estimate L as follows

$$\begin{aligned} L(z, \xi) = \inf_w \mathcal{L}(z, \xi, w) &\geq \inf \left(c_1 \int_{Y \cap E} |\xi + Dw|^p dy - \eta \int_{E \cap Y} |\xi + Dw|^p dy \right) \\ &\quad - c(1 + |z|^p) \geq \inf \left(\frac{c_1}{C} \int_Y |\xi + Dw|^p dy - \eta \int_Y |\xi + Dw|^p dy \right) \\ &\quad - c(1 + |z|^p) \geq \left(\frac{c_1}{C} - \eta \right) \int_Y |\xi|^p dy - c(1 + |z|^p). \end{aligned}$$

From the arbitrariness of η inequality (3.9) follows immediately.

- (d) This upper bound is straightforward. Indeed, it suffices to substitute $w = 0$ in the definition of $L(\xi, z)$, and the required bound follows from (1.3)–(1.4).

Remark 3.3. In the case of disjoint inclusions studied here, one can improve the upper bound (3.10) for $L(\xi, z)$ by choosing the test function $w(y)$ equal to $-\xi \cdot y$ on S and zero on ∂Y , in such a way that $|\nabla w| \leq C|\xi|$. This yields an upper bound

$$L(\xi, z) \leq k_5(|\xi|^p + 1)$$

with a positive constant k_5 . However, having in mind more general case of connected perforation, we prefer not to use this estimate.

Proposition 3.4. *For every $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, there exists at least one solution $w(\cdot, z, \xi) \in W^{1,p}(Y \cap E)$ to the minimum problem (1.14). Moreover, for every $k \in \mathbb{R}$ the function $w + k$ is also a solution. Finally, there exists a positive constant c_0 such that*

$$\int_{Y \cap E} |w(\cdot, z, \xi)|^p dy \leq c_0(1 + |z|^p + |\xi|^p), \tag{3.14}$$

$$\int_{Y \cap E} |Dw(\cdot, z, \xi)|^p dy \leq c_0(1 + |z|^p + |\xi|^p) \tag{3.15}$$

for every $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, and for every solution w that has zero average on the set $Y \cap E$.

Proof. Let $\mathcal{L}(z, \xi, w)$ be defined by (3.11) for every $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, $w \in W^{1,p}(Y \cap E)$. It is easy to see that $\mathcal{L}(z, \xi, \cdot)$ is lower-semicontinuous on $W^{1,p}(Y \cap E)$ with respect to $W^{1,p}$ -weak convergence. Moreover, $\mathcal{L}(z, \xi, \cdot)$ is also coercive; in fact, using (1.3) and (3.13), one can show that, for every $\eta > 0$ there exist c_η such that

$$\mathcal{L}(z, \xi, w) \geq (c_1 - \eta) \int_{Y \cap E} |\xi + Dw|^p dy - 2c_\eta(1 + |z|^p), \tag{3.16}$$

for any w with zero average. Therefore, problem (1.14) has at least one solution. If w is a solution and $k \in \mathbb{R}$, then also $w + k$ is a solution to problem (1.14) since, by (1.5)

$$\int_{Y \cap S} g'_z(y, z) d\mathcal{H}^{n-1} = 0 \quad \text{for all } z \in \mathbb{R}.$$

In order to prove estimate (3.14), let $w \in W^{1,p}(Y \cap E)$ be a solution of (1.14) with zero mean value

$$\int_{Y \cap E} w dy = 0.$$

By definition, $\mathcal{L}(z, \xi, w) \leq \mathcal{L}(z, \xi, 0)$ and hence

$$\int_{Y \cap E} f(y, z, \xi + Dw) dy + \int_{Y \cap S} g'_z(y, z) w d\mathcal{H}^{n-1} \leq \int_{Y \cap E} f(y, z, \xi) dy. \tag{3.17}$$

By (3.16) we obtain that for every $\eta > 0$ there exists $c_\eta > 0$, such that

$$(c_1 - \eta) \int_{Y \cap E} |\xi + Dw|^p dy - c_\eta(1 + |z|^p) \leq c_2(1 + |z|^p + |\xi|^p) |Y \cap E|.$$

From this inequality (3.15) follows easily. The bound (3.14) follows, thanks to the Poincaré inequality.

We now state a lemma, that will be used both in the proof of Γ -lim inf and Γ -lim sup inequality.

Lemma 3.5. *There exists a constant $c > 0$ such that if $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$ and \bar{u}_ε is the piece-wise constant function defined by (3.4) as the integral average of u_ε in each cell Y_ε^i , then*

$$\left| \int_{S_\varepsilon} g\left(\frac{x}{\varepsilon}, u_\varepsilon\right) d\mathcal{H}^{n-1} - \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \bar{u}_\varepsilon\right)(u_\varepsilon - \bar{u}_\varepsilon) d\mathcal{H}^{n-1} \right| \leq c \max\{\varepsilon, \varepsilon^{p-1}\} \left(1 + \int_{\Omega_\varepsilon} (|u_\varepsilon|^p + |Du_\varepsilon|^p) dx\right) \tag{3.18}$$

for all $\varepsilon > 0$.

Proof. We make use of the representation

$$\int_{S_\varepsilon} g\left(\frac{x}{\varepsilon}, u_\varepsilon\right) d\mathcal{H}^{n-1} = \sum_{i \in I_\varepsilon} \int_{S_\varepsilon \cap Y_\varepsilon^i} g\left(\frac{x}{\varepsilon}, u_\varepsilon\right) d\mathcal{H}^{n-1}$$

and estimate separately the integral over different cells $S_\varepsilon \cap Y_\varepsilon^i$. Since \bar{u}_ε is constant in Y_ε^i , by (1.5) and the regularity of $g(y, \cdot)$ we have

$$\begin{aligned} \int_{S_\varepsilon \cap Y_\varepsilon^i} g\left(\frac{x}{\varepsilon}, u_\varepsilon\right) d\mathcal{H}^{n-1} &= \int_{S_\varepsilon \cap Y_\varepsilon^i} g\left(\frac{x}{\varepsilon}, \bar{u}_\varepsilon + (u_\varepsilon - \bar{u}_\varepsilon)\right) d\mathcal{H}^{n-1} \\ &= \int_0^1 \int_{S_\varepsilon \cap Y_\varepsilon^i} g'_z\left(\frac{x}{\varepsilon}, \bar{u}_\varepsilon + t(u_\varepsilon - \bar{u}_\varepsilon)\right) (u_\varepsilon - \bar{u}_\varepsilon) d\mathcal{H}^{n-1} dt. \end{aligned}$$

From (1.7) we have

$$\begin{aligned} \left| \int_0^1 \int_{S_\varepsilon \cap Y_\varepsilon^i} \left[g'_z\left(\frac{x}{\varepsilon}, \bar{u}_\varepsilon + t(u_\varepsilon - \bar{u}_\varepsilon)\right) - g'_z\left(\frac{x}{\varepsilon}, \bar{u}_\varepsilon\right) \right] (u_\varepsilon - \bar{u}_\varepsilon) d\mathcal{H}^{n-1} dt \right| \leq \\ c_5 \int_{S_\varepsilon \cap Y_\varepsilon^i} (1 + |u_\varepsilon| + |\bar{u}_\varepsilon|)^{p-2} |u_\varepsilon - \bar{u}_\varepsilon|^2 d\mathcal{H}^{n-1}. \end{aligned}$$

If $p \geq 2$ then by applying Holder's inequality we get

$$\begin{aligned} \int_{S_\varepsilon \cap Y_\varepsilon^i} (1 + |u_\varepsilon| + |\bar{u}_\varepsilon|)^{p-2} |u_\varepsilon - \bar{u}_\varepsilon|^2 d\mathcal{H}^{n-1} \leq \\ c \left(\int_{S_\varepsilon \cap Y_\varepsilon^i} |u_\varepsilon - \bar{u}_\varepsilon|^p d\mathcal{H}^{n-1} \right)^{2/p} \cdot \left(\int_{S_\varepsilon \cap Y_\varepsilon^i} (1 + |u_\varepsilon|^p + |\bar{u}_\varepsilon|^p) d\mathcal{H}^{n-1} \right)^{\frac{p-2}{p}}. \end{aligned}$$

Since we can estimate

$$\int_{S_\varepsilon \cap Y_\varepsilon^i} |\bar{u}_\varepsilon|^p d\mathcal{H}^{n-1} \leq c\varepsilon^{-1} \int_{\Omega_\varepsilon \cap Y_\varepsilon^i} |\bar{u}_\varepsilon|^p dx \leq c\varepsilon^{-1} \int_{\Omega_\varepsilon \cap Y_\varepsilon^i} |u_\varepsilon|^p dx,$$

then by (3.5), (3.6) we have

$$\begin{aligned} \left(\int_{S_\varepsilon \cap Y_\varepsilon^i} |u_\varepsilon - \bar{u}_\varepsilon|^p d\mathcal{H}^{n-1} \right)^{2/p} \cdot \left(\int_{S_\varepsilon \cap Y_\varepsilon^i} (1 + |u_\varepsilon|^p + |\bar{u}_\varepsilon|^p) d\mathcal{H}^{n-1} \right)^{\frac{p-2}{p}} \leq \\ c \left(\varepsilon^{p-1} \int_{\Omega_\varepsilon \cap Y_\varepsilon^i} |Du_\varepsilon|^p dx \right)^{2/p} \cdot \left(\varepsilon^{-1} \int_{\Omega_\varepsilon \cap Y_\varepsilon^i} (1 + |u_\varepsilon|^p) dx + \varepsilon^{p-1} \int_{\Omega_\varepsilon \cap Y_\varepsilon^i} |Du_\varepsilon|^p dx \right)^{\frac{p-2}{p}} \\ = c\varepsilon^{p-1} \int_{\Omega_\varepsilon \cap Y_\varepsilon^i} |Du_\varepsilon|^p dx + c\varepsilon \left(\int_{\Omega_\varepsilon \cap Y_\varepsilon^i} |Du_\varepsilon|^p dx \right)^{2/p} \cdot \left(\int_{\Omega_\varepsilon \cap Y_\varepsilon^i} (1 + |u_\varepsilon|^p) dx \right)^{\frac{p-2}{p}}. \end{aligned}$$

By Young's inequality with powers $p/2$ and $(p-2)/p$ for any $t \in [0, 1]$ we finally get

$$\begin{aligned} \left| \int_{S_\varepsilon \cap Y_\varepsilon^i} \left[g'_z \left(\frac{x}{\varepsilon}, \bar{u}_\varepsilon + t(u_\varepsilon - \bar{u}_\varepsilon) \right) - g'_z \left(\frac{x}{\varepsilon}, \bar{u}_\varepsilon \right) \right] (u_\varepsilon - \bar{u}_\varepsilon) d\mathcal{H}^{n-1} \right| \leq \\ c \max\{\varepsilon^{p-1}, \varepsilon\} \int_{\Omega_\varepsilon \cap Y_\varepsilon^i} |Du_\varepsilon|^p dx + c\varepsilon \int_{\Omega_\varepsilon \cap Y_\varepsilon^i} (1 + |u_\varepsilon|^p) dx. \end{aligned}$$

By summing up over i we complete the proof in the case $p \geq 2$.

If $1 < p < 2$, then

$$\begin{aligned} \int_{S_\varepsilon \cap Y_\varepsilon^i} (1 + |u_\varepsilon| + |\bar{u}_\varepsilon|)^{p-2} |u_\varepsilon - \bar{u}_\varepsilon|^2 d\mathcal{H}^{n-1} = \\ \int_{S_\varepsilon \cap Y_\varepsilon^i} \frac{|u_\varepsilon - \bar{u}_\varepsilon|^{2-p}}{(1 + |u_\varepsilon| + |\bar{u}_\varepsilon|)^{2-p}} |u_\varepsilon - \bar{u}_\varepsilon|^p d\mathcal{H}^{n-1} \leq c \int_{S_\varepsilon \cap Y_\varepsilon^i} |u_\varepsilon - \bar{u}_\varepsilon|^p d\mathcal{H}^{n-1} \\ \leq c\varepsilon^{p-1} \int_{\Omega_\varepsilon \cap Y_\varepsilon^i} |Du_\varepsilon|^p dx. \end{aligned}$$

Summing up over i we obtain the desired bound.

Lemma 3.6. *The family of functionals F_ε is continuous with respect to the strong topology of $W^{1,p}(\Omega)$, uniformly with respect to ε .*

Proof. We can prove separately that volume integral

$$F_\varepsilon^v(u) = \int_{\Omega_\varepsilon} f \left(\frac{x}{\varepsilon}, u, Du \right) dx$$

and surface integral

$$F_\varepsilon^s(u) = \int_{S_\varepsilon} g \left(\frac{x}{\varepsilon}, u \right) d\mathcal{H}^{n-1}$$

are continuous, uniformly with respect to ε . We first proceed with F_ε^v . From the growth and convexity conditions stated in Section 1 it follows that there exists a constant $c > 0$ such that

$$|f(y, \xi) - f(y, \eta)| \leq c(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|$$

for a.e. $y \in \mathbb{R}^n$, and every $\xi, \eta \in \mathbb{R}^n$. Therefore,

$$\begin{aligned} |F_\varepsilon^v(u) - F_\varepsilon^v(w)| &\leq c \int_{\Omega_\varepsilon} (1 + |Du|^{p-1} + |Dw|^{p-1}) |Du - Dw| dx \\ &\leq c \left(\int_{\Omega_\varepsilon} (1 + |Du|^p + |Dw|^p) dx \right)^{\frac{p-1}{p}} \cdot \left(\int_{\Omega_\varepsilon} |Du - Dw|^p dx \right)^{\frac{1}{p}} \\ &\leq c(1 + \|u\|_{W^{1,p}(\Omega_\varepsilon)}^{p-1} + \|w\|_{W^{1,p}(\Omega_\varepsilon)}^{p-1}) \|u - w\|_{W^{1,p}(\Omega_\varepsilon)}. \end{aligned}$$

Now we consider the surface integral F_ε^s . By Lemma 3.5, we have

$$\begin{aligned} |F_\varepsilon^s(u) - F_\varepsilon^s(w)| &\leq \left| \int_{S_\varepsilon} g'_z \left(\frac{x}{\varepsilon}, \bar{u} \right) (u - \bar{u} - (w - \bar{w})) d\mathcal{H}^{n-1} \right| \\ &\quad + \left| \int_{S_\varepsilon} \left(g'_z \left(\frac{x}{\varepsilon}, \bar{u} \right) - g'_z \left(\frac{x}{\varepsilon}, \bar{w} \right) \right) (w - \bar{w}) d\mathcal{H}^{n-1} \right| \\ &\quad + c \max\{\varepsilon, \varepsilon^{p-1}\} (1 + \|u\|_{W^{1,p}(\Omega_\varepsilon)}^p + \|w\|_{W^{1,p}(\Omega_\varepsilon)}^p). \end{aligned} \tag{3.19}$$

For the first integral on the right hand side, by Poincaré inequality we obtain

$$\left| \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \bar{u}\right)(u - \bar{u} - (w - \bar{w})) d\mathcal{H}^{n-1} \right| c \left(\int_{S_\varepsilon} (1 + |u|^{p-1}) d\mathcal{H}^{n-1} \right)^{\frac{p-1}{p}} \left(\int_{S_\varepsilon} |u - \bar{u} - (w - \bar{w})|^p d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \leq c(1 + \|u\|_{W^{1,p}(\Omega_\varepsilon)}^{p-1}) \|u - w\|_{W^{1,p}(\Omega_\varepsilon)}.$$

The second term on the right hand side of (3.19) can be estimated in a similar way, in view of the Lipschitz continuity of $g'_z(y, \cdot)$. Combining the above bounds and taking into account the fact that for $\varepsilon \in (\varepsilon_0, 1)$, $\varepsilon_0 > 0$, the uniform continuity trivially follows from the trace inequality, we obtain the desired uniform continuity for all $\varepsilon \in (0, 1)$.

4. PROOF OF THEOREM 2.3: THE Γ -LIM INF INEQUALITY

By Definition 2.2, the family $F_\varepsilon(\cdot)$ Γ -converges to a functional $F = F(u)$ if the following two properties hold:

(1) (Γ -lim inf inequality). For any sequence $\{u_\varepsilon\}$, $u_\varepsilon \in L^p(\Omega)$, such that u_ε converges to u in $L^p(\Omega)$, we have

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq F(u)$$

(by (1.2)), if $u_\varepsilon \notin W^{1,p}(\Omega_\varepsilon)$, we set $F_\varepsilon(u_\varepsilon) = +\infty$.

(2) (Γ -lim sup inequality). For any $u \in L^p(\Omega)$ there is a sequence $u_\varepsilon \in L^p(\Omega)$ such that u_ε converges to u in $L^p(\Omega)$ and

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \leq F(u).$$

In this section we are going to prove the Γ -lim inf inequality. To this end, given any $u, u_\varepsilon \in W^{1,p}(\Omega)$ such that $u_\varepsilon \rightarrow u$ strongly in $L^p(\Omega)$, we have to show that

$$F(u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon). \quad (4.1)$$

Since F is continuous on $W^{1,p}(\Omega)$ and F_ε is continuous on $W^{1,p}(\Omega)$ uniformly with respect to ε (see Lem. 3.6), then it is sufficient to prove the above inequality for piecewise affine function u . Moreover, without loss of generality we can assume that $F_\varepsilon(u_\varepsilon)$ has a finite limit, as $\varepsilon \rightarrow 0$. Then, by Lemma 3.1,

$$\sup\{\|u_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)} : \varepsilon > 0\} < +\infty.$$

As we said in Section 1, by the extension theorem in [1] we can assume that u_ε are also bounded in $W^{1,p}(\Omega)$. For any open subset $A \subset \Omega$ we denote

$$F_\varepsilon(u, A) = \int_{\Omega_\varepsilon \cap A} f\left(\frac{x}{\varepsilon}, Du\right) dx + \int_{S_\varepsilon \cap A} g\left(\frac{x}{\varepsilon}, u\right) d\mathcal{H}^{n-1}.$$

Step 1. Let \bar{u}_ε be defined by (3.4) as the integral average of u_ε in each cell Y_ε^i . Then, by Lemma 3.5,

$$\left| \int_{S_\varepsilon} g\left(\frac{x}{\varepsilon}, u_\varepsilon\right) d\mathcal{H}^{n-1} - \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \bar{u}_\varepsilon\right)(u_\varepsilon - \bar{u}_\varepsilon) d\mathcal{H}^{n-1} \right| \leq c \max\{\varepsilon, \varepsilon^{p-1}\} \left(1 + \int_{\Omega_\varepsilon} (|u_\varepsilon|^p + |Du_\varepsilon|^p) dx \right) \leq c' \max\{\varepsilon, \varepsilon^{p-1}\} \quad (4.2)$$

for all $\varepsilon > 0$.

Step 2. Let A^δ be the periodic grid with period δ and thickness $\tilde{\delta}$, $\tilde{\delta} = \delta^2 + o(\delta^2)$, defined by $A^\delta = \mathbb{R}^n \setminus \cup\{\delta\alpha + [-\frac{\delta+\tilde{\delta}}{2}, \frac{\delta-\tilde{\delta}}{2}]^n : \alpha \in \mathbb{Z}^n\}$. We assume in what follows that both δ and $\tilde{\delta}$ are integer multipliers of ε . Covering the domain Ω by $\tilde{\delta}$ shifts of A^δ in each coordinate directions so that

$$\Omega = \bigcup_{\alpha \in \mathbb{Z}^n \cap [0, \delta/\tilde{\delta}]^n} \left((A^\delta + \alpha\tilde{\delta}) \cap \Omega \right),$$

one can conclude in the standard way that for every $\varepsilon > 0$ there exists $x_\varepsilon^\delta \in \mathbb{R}^n$ such that

$$\begin{aligned} F_\varepsilon(u_\varepsilon, (A^\delta + x_\varepsilon^\delta) \cap \Omega) &\leq c_n \delta F_\varepsilon(u_\varepsilon, \Omega), \\ \int_{(A^\delta + x_\varepsilon^\delta) \cap \Omega} (|Du_\varepsilon|^p + |u_\varepsilon|^p) dx &\leq c_n \delta \int_{\Omega} (|Du_\varepsilon|^p + |u_\varepsilon|^p) dx \end{aligned} \quad (4.3)$$

where the constant $c_n > 0$ only depend on the dimension n . Moreover, the shift x_ε^δ can be chosen in such a way that $(A^\delta + x_\varepsilon^\delta)$ is compatible with ε -grid, *i.e.* it consists of integer number of solid periods $\{Y_\varepsilon^i\}$. From now on, we set $A_\varepsilon^\delta = A^\delta + x_\varepsilon^\delta$. In view of Lemmas 2.1 and 3.1 the estimate (4.3) yields

$$\int_{(A^\delta + x_\varepsilon^\delta) \cap \Omega} (|u_\varepsilon|^p + |Du_\varepsilon|^p) dx \leq c_n \delta (F_\varepsilon(u_\varepsilon, \Omega) + C) \quad (4.4)$$

for every $\delta > 0$.

Step 3. Let $\varphi_\varepsilon^\delta$ be a δ -periodic cut-off function associated to the set A_ε^δ , *i.e.*, $\varphi_\varepsilon^\delta \in \mathcal{C}_{\text{per}}^\infty(\mathbb{R}^n)$, $0 \leq \varphi_\varepsilon^\delta \leq 1$, $\varphi_\varepsilon^\delta = 1$ in $\mathbb{R}^n \setminus A_\varepsilon^\delta$, $\varphi_\varepsilon^\delta = 0$ on $\cup(\{\delta\alpha + \partial[-\frac{\delta}{2}, \frac{\delta}{2}]^n : \alpha \in \mathbb{Z}^n\} + x_\varepsilon^\delta)$, and $|D\varphi_\varepsilon^\delta| \leq c/\delta^2$. Let us define

$$u_\varepsilon^\delta = u + \varphi_\varepsilon^\delta(u_\varepsilon - u). \quad (4.5)$$

Then $u_\varepsilon^\delta = u$ on the boundary of each δ -cell $\delta\alpha + [-\frac{\delta}{2}, \frac{\delta}{2}]^n + x_\varepsilon^\delta$. For the volume integral we claim that

$$\begin{aligned} \left| \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) dx - \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, Du_\varepsilon^\delta\right) dx \right| &= \int_{A_\varepsilon^\delta} \left| f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) - f\left(\frac{x}{\varepsilon}, Du_\varepsilon^\delta\right) \right| dx \\ &\leq c \int_{A_\varepsilon^\delta} \left(1 + |Du|^p + |Du_\varepsilon|^p + \frac{1}{\delta^2} |u_\varepsilon - u|^p \right) dx \\ &\leq c\delta \int_{\Omega_\varepsilon} (|Du|^p + |Du_\varepsilon|^p + 1) dx + \frac{1}{\delta^2} \int_{\Omega_\varepsilon} |u_\varepsilon - u|^p dx \leq c\delta + \frac{1}{\delta^2} \kappa_0(\varepsilon) \end{aligned} \quad (4.6)$$

with $\kappa_0(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, uniformly in δ .

Step 4. Let x_δ^α be the center of the cell $Y_\delta^\alpha = \delta\alpha + x_\varepsilon^\delta + [-\frac{\delta}{2}, \frac{\delta}{2}]$, and let

$$u_\delta(x) = \sum_i u(x_\delta^i) \chi_{Y_\delta^i}(x), \quad \forall x \in \mathbb{R}^n.$$

We claim that

$$\left| \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \bar{u}_\varepsilon\right) (u_\varepsilon - \bar{u}_\varepsilon) d\mathcal{H}^{n-1} - \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, u_\delta\right) u_\varepsilon^\delta d\mathcal{H}^{n-1} \right| \leq c \left(\delta + \delta^{1/p} + \kappa(\varepsilon) \left(1 + \frac{1}{\delta^2} \right) \right) \quad (4.7)$$

with $\kappa(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

First we estimate

$$\left| \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \bar{u}_\varepsilon\right)(u_\varepsilon - \bar{u}_\varepsilon) d\mathcal{H}^{n-1} - \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, u_\delta\right)(u_\varepsilon - \bar{u}_\varepsilon) d\mathcal{H}^{n-1} \right| \leq \int_{S_\varepsilon} (1 + |\bar{u}_\varepsilon| + |u_\delta|)^{p-2} |\bar{u}_\varepsilon - u_\delta| |u_\varepsilon - \bar{u}_\varepsilon| d\mathcal{H}^{n-1}.$$

If $p \geq 2$ then by the Holder inequality, (3.3), and the fact that \bar{u}_ε is constant in each ε -cell, we obtain

$$\begin{aligned} \int_{S_\varepsilon} (1 + |\bar{u}_\varepsilon| + |u_\delta|)^{p-2} |\bar{u}_\varepsilon - u_\delta| |u_\varepsilon - \bar{u}_\varepsilon| d\mathcal{H}^{n-1} &\leq \\ &\left(\int_{S_\varepsilon} (1 + |\bar{u}_\varepsilon| + |u_\delta|)^p d\mathcal{H}^{n-1} \right)^{\frac{p-2}{p}} \left(\int_{S_\varepsilon} |\bar{u}_\varepsilon - u_\delta|^p d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \left(\int_{S_\varepsilon} |u_\varepsilon - \bar{u}_\varepsilon|^p d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \\ &\leq \left(c\varepsilon^{-1} \int_{\Omega_\varepsilon} (1 + |\bar{u}_\varepsilon| + |u_\delta|)^p dx \right)^{\frac{p-2}{p}} \left(\varepsilon^{-1} \int_{\Omega_\varepsilon} |\bar{u}_\varepsilon - u_\delta|^p dx \right)^{\frac{1}{p}} \left(\varepsilon^{p-1} \int_{\Omega_\varepsilon} |Du_\varepsilon|^p dx \right)^{\frac{1}{p}} \\ &\leq c \left(\int_{\Omega_\varepsilon} |\bar{u}_\varepsilon - u_\delta|^p dx \right)^{\frac{1}{p}} \leq c \left(\int_{\Omega_\varepsilon} |\bar{u}_\varepsilon - u|^p dx \right)^{\frac{1}{p}} + c \left(\int_{\Omega_\varepsilon} |u - u_\delta|^p dx \right)^{\frac{1}{p}} \leq \kappa_1(\varepsilon) + c_1\delta, \end{aligned}$$

where $\kappa_1(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

If $1 < p < 2$, for the same reason we have

$$\begin{aligned} \int_{S_\varepsilon} (1 + |\bar{u}_\varepsilon| + |u_\delta|)^{p-2} |\bar{u}_\varepsilon - u_\delta| |u_\varepsilon - \bar{u}_\varepsilon| d\mathcal{H}^{n-1} &\leq \\ &\int_{S_\varepsilon} \frac{|\bar{u}_\varepsilon - u_\delta|^{2-p}}{(1 + |\bar{u}_\varepsilon| + |u_\delta|)^{2-p}} |\bar{u}_\varepsilon - u_\delta|^{p-1} |u_\varepsilon - \bar{u}_\varepsilon| d\mathcal{H}^{n-1} \leq \int_{S_\varepsilon} |\bar{u}_\varepsilon - u_\delta|^{p-1} |u_\varepsilon - \bar{u}_\varepsilon| d\mathcal{H}^{n-1} \\ &\leq \left(\int_{S_\varepsilon} |\bar{u}_\varepsilon - u_\delta|^p d\mathcal{H}^{n-1} \right)^{\frac{p-1}{p}} \left(\int_{S_\varepsilon} |u_\varepsilon - \bar{u}_\varepsilon|^p d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \\ &\leq \left(\varepsilon^{-1} \int_{\Omega_\varepsilon} |\bar{u}_\varepsilon - u_\delta|^p dx \right)^{\frac{p-1}{p}} \left(\varepsilon^{p-1} \int_{\Omega_\varepsilon} |Du_\varepsilon|^p dx \right)^{\frac{1}{p}} \leq \kappa_2(\varepsilon) + c\delta^{1-\frac{1}{p}}, \end{aligned}$$

where again $\kappa_2(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Next, we should estimate

$$\begin{aligned} \left| \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, u_\delta\right) ((u_\varepsilon - \bar{u}_\varepsilon) - u_\varepsilon^\delta) d\mathcal{H}^{n-1} \right| &= \left| \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, u_\delta\right) ((u_\varepsilon - u_\varepsilon^\delta) - \overline{(u_\varepsilon - u_\varepsilon^\delta)}) d\mathcal{H}^{n-1} \right| \leq \\ &c \int_{S_\varepsilon} |(u_\varepsilon - u_\varepsilon^\delta) - \overline{(u_\varepsilon - u_\varepsilon^\delta)}| d\mathcal{H}^{n-1} \leq c \left(\int_{\Omega_\varepsilon} |D(u_\varepsilon - u_\varepsilon^\delta)|^p dx \right)^{1/p} \\ &= c \left(\int_{\Omega_\varepsilon} |D((1 - \varphi_\varepsilon^\delta)(u_\varepsilon - u))|^p dx \right)^{1/p} \leq \left(\int_{A_\varepsilon^\delta} |D(u_\varepsilon - u)|^p dx \right)^{1/p} + \frac{1}{\delta^2} \left(\int_{A_\varepsilon^\delta} |u_\varepsilon - u|^p dx \right)^{1/p} \\ &\leq c\delta^{1/p} + \frac{c}{\delta^2} \kappa_3(\varepsilon) \end{aligned}$$

with $\kappa_3(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$; the bound (4.4) and the estimate $|D\varphi_\varepsilon^\delta| \leq c/\delta^2$ have also been used here.

Combining the above estimates we obtain (4.7).

Step 5. Denote by \mathcal{C} the discontinuity set of Du . Since u is a piecewise affine function, then \mathcal{C} consists of finite number of hyperplanes. For every $\delta > 0$, let us consider the set $B_\delta = \cup\{Y_\delta^\alpha : Y_\delta^\alpha \cap \mathcal{C} \neq \emptyset \text{ or } Y_\delta^\alpha \cap \partial\Omega \neq \emptyset\}$. It is easy to see that $|B_\delta| \leq c\delta$. Under our standing assumptions,

$$F_\varepsilon(u_\varepsilon, B_\delta) \geq -c\delta^{\frac{np-n+p}{np} \wedge 1}, \quad |F(u, B_\delta)| \leq c_1\delta. \quad (4.8)$$

Indeed, by Lemma 2.1

$$F_\varepsilon(u_\varepsilon, B_\delta) \geq -c \int_{B_\delta} (1 + |u_\varepsilon|^p) \geq -c(|B_\delta|^{\frac{np-n+p}{np} \wedge 1} + |B_\delta|);$$

the last inequality here follows from the bound $\|u_\varepsilon\|_{W^{1,p}(\Omega)} \leq C$ and the Sobolev imbedding theorem. The second inequality in (4.8) is trivial.

Taking into account the definitions of $L(z, \xi)$ and of u_ε^δ (see (4.5)), we obtain

$$\int_{\Omega_\varepsilon \setminus B_\delta} f\left(\frac{x}{\varepsilon}, Du_\varepsilon^\delta\right) dx + \int_{S_\varepsilon \setminus B_\delta} g'_z\left(\frac{x}{\varepsilon}, u_\delta\right)(u_\varepsilon^\delta - u_\delta) d\mathcal{H}^{n-1} \geq \int_{\Omega \setminus B_\delta} L(u_\delta, Du) dx. \quad (4.9)$$

In order to prove this inequality we denote $\mathcal{I}_\delta = \{\alpha \in \mathbb{Z}^n : Y_\delta^\alpha \subset \Omega \setminus B_\delta\}$, and $\mathcal{J}_\varepsilon = \{i : Y_\varepsilon^i \subset \Omega \setminus B_\delta\}$. Then

$$\begin{aligned} \int_{\Omega_\varepsilon \setminus B_\delta} f\left(\frac{x}{\varepsilon}, Du_\varepsilon^\delta\right) dx + \int_{S_\varepsilon \setminus B_\delta} g'_z\left(\frac{x}{\varepsilon}, u_\delta\right)(u_\varepsilon^\delta - u_\delta) d\mathcal{H}^{n-1} &\geq \\ &\sum_{\alpha \in \mathcal{I}_\delta} \inf_{w \in W_{\text{per}}^{1,p}(Y_\delta^\alpha)} \int_{Y_\delta^\alpha \cap \Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, Du + Dw\right) dx + \int_{Y_\delta^\alpha \cap S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, u_\delta\right)(u + w) d\mathcal{H}^{n-1} \\ &= \sum_{i \in \mathcal{J}_\varepsilon} \inf_{w \in W_{\text{per}}^{1,p}(Y_\varepsilon^i)} \int_{Y_\varepsilon^i \cap \Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, Du + Dw\right) dx + \int_{Y_\varepsilon^i \cap S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, u_\delta\right)(u + w) d\mathcal{H}^{n-1} \\ &= \int_{\Omega \setminus B_\delta} L(u_\delta, Du) dx, \end{aligned}$$

the first equality here follows from the convexity of the Lagrangian with respect to the function w and its gradient.

Step 6. Considering the properties of $L(z, \xi)$ it is easy to see that

$$\left| \int_{\Omega \setminus B_\delta} L(u_\delta, Du) dx - \int_{\Omega \setminus B_\delta} L(u, Du) dx \right| \leq c\delta \quad (4.10)$$

as $\delta \rightarrow 0$.

The last two inequalities together with the estimates (4.6), (4.7) and (4.8) imply

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \int_{\Omega} L(u, Du) dx - c(\delta + \delta^{1/p} + \delta^{\frac{np-n+p}{np} \wedge 1}).$$

Since δ is an arbitrary positive number, the desired Γ -lim inf inequality (4.1) follows.

5. PROOF OF THEOREM 2.3: THE Γ -LIM SUP INEQUALITY

According to the definition of Γ -convergence, we have to prove that for every $u \in W^{1,p}(\Omega)$ there exists a sequence $u_\varepsilon \in W^{1,p}(\Omega) \rightarrow u$ strongly in $L^p(\Omega)$, such that

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \leq F(u). \quad (5.1)$$

Since the functional F is continuous with respect to strong convergence in $W^{1,p}(\Omega)$, it is enough to show that (5.1) holds for every piecewise affine function u . Moreover, since

$$F(u) = \int_{\Omega} L(u, Du) dx,$$

then, by localization, we can reduce to the case where u is affine, *i.e.*, $u(x) = \hat{u} + \xi \cdot x$, where $\hat{u} \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$.

First of all, given $z \in \mathbb{R}$, and $\xi \in \mathbb{R}^n$, we fix a solution $w(\cdot, z, \xi) \in W_{\text{per}}^{1,p}(Y)$ of the minimum problem (1.14) such that

$$\bar{w} = \mathcal{H}^{n-1}(S \cap Y)^{-1} \int_{S \cap Y} w \, d\mathcal{H}^{n-1} = 0.$$

For every $\delta > 0$, with $0 < \varepsilon < \delta < 1$, let us define by Q^δ the open cube $Q^\delta =]-\frac{\delta}{2}, \frac{\delta}{2}[^n$, and by $Q^{\delta+}$ and $Q^{\delta-}$ the slightly bigger and smaller ones $Q^{\delta+} =]-\frac{\delta+\delta^2}{2}, \frac{\delta+\delta^2}{2}[^n$, $Q^{\delta-} =]-\frac{\delta-\delta^2}{2}, \frac{\delta-\delta^2}{2}[^n$. For every $x_j^\delta \in \delta\mathbb{Z}^n$, we denote their translated images by $Q_j^\delta, Q_j^{\delta+}, Q_j^{\delta-}$, *i.e.* $Q_j^\delta + x_j^\delta, Q_j^{\delta+} + x_j^\delta, Q_j^{\delta-} + x_j^\delta$. Let $J_\delta = \{j \in \mathbb{Z}^n : Q_j^{\delta+} \cap \bar{\Omega} \neq \emptyset\}$. To the family of cubes $(Q_j^{\delta+})$, with $j \in J_\delta$, we can associate a finite partition of unity on the set $\bar{\Omega}$, by choosing $\Phi_j^\delta \in \mathcal{C}_0^\infty(Q_j^{\delta+})$, such that $0 \leq \Phi_j^\delta \leq 1$, $\Phi_j^\delta \equiv 1$ on $Q_j^{\delta-}$, $|D\Phi_j^\delta| \leq \frac{c}{\delta^2}$ with fixed $c > 0$, and $\sum_{j \in J_\delta} \Phi_j^\delta = 1$ on $\bar{\Omega}$.

For every $\varepsilon, \delta > 0$ and every $j \in J$ let us define the ε -periodic function

$$w_{\varepsilon j}^\delta(x) = w\left(\frac{x}{\varepsilon}, \hat{u} + \xi \cdot x_j^\delta, \xi\right), \quad \forall x \in \mathbb{R}^n.$$

Set also

$$v_\varepsilon^\delta(x) = \sum_{j \in J} \Phi_j^\delta(x) w_{\varepsilon j}^\delta(x) \quad \forall x \in \Omega \tag{5.2}$$

and

$$u_\varepsilon^\delta = \hat{u} + \xi \cdot x + \varepsilon v_\varepsilon^\delta. \tag{5.3}$$

By construction and by (3.14)

$$\|v_\varepsilon^\delta\|_{L^p(\Omega)} \leq c \quad \forall \varepsilon, \delta, \tag{5.4}$$

and hence

$$\|u_\varepsilon^\delta - (\hat{u} + \xi \cdot x)\|_{L^p(\Omega)} \leq c\varepsilon \quad \forall \varepsilon, \delta. \tag{5.5}$$

Moreover, from (3.15) we also have

$$\|Du_\varepsilon^\delta\|_{L^p(\Omega)} \leq c \left(1 + \frac{\varepsilon}{\delta^2}\right) \quad \forall \varepsilon, \delta. \tag{5.6}$$

Now we claim that, provided δ is an integer multiple of ε , we have

$$F_\varepsilon(u_\varepsilon^\delta) \leq \sum_{j \in J_\delta} L(\hat{u} + \xi \cdot x_j^\delta, \xi) |Q_j^\delta| + k(\varepsilon, \delta), \tag{5.7}$$

where

$$k(\varepsilon, \delta) \leq c \left(\varepsilon + \delta + \frac{\varepsilon}{\delta^2}\right) + c \max\{\varepsilon, \varepsilon^{p-1}\} \left(1 + \left(\frac{\varepsilon}{\delta^2}\right)^p\right).$$

Note that, for a proper choice of $\delta = \delta(\varepsilon)$, $\lim_{\varepsilon \rightarrow 0} k(\varepsilon, \delta(\varepsilon)) = 0$. Hence, from (5.7), by choosing $\delta = \delta(\varepsilon) \rightarrow 0$ and $u_\varepsilon = u_\varepsilon^\delta \rightarrow u$, we will obtain finally that

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \leq \limsup_{\delta \rightarrow 0} \sum_{j \in J_\delta} L(\hat{u} + \xi \cdot x_j^\delta, \xi) |Q_j^\delta| = \int_{\Omega} L(u, Du) dx = F(u).$$

The the inequality (5.7) will be derived through several steps. We recall that

$$F_\varepsilon(u_\varepsilon^\delta) = \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, Du_\varepsilon^\delta\right) dx + \int_{S_\varepsilon} g\left(\frac{x}{\varepsilon}, u_\varepsilon^\delta\right) d\mathcal{H}^{n-1}, \quad (5.8)$$

and $Du_\varepsilon^\delta = \xi + \varepsilon Dv_\varepsilon^\delta$.

Step 1. We start by estimating the bulk energy. We want to show that there is a constant $c > 0$ such that

$$\int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, \xi + \varepsilon Dv_\varepsilon^\delta\right) dx \leq \sum_{j \in J} \int_{Q_j^\delta \cap \Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, \xi + D_y w\left(\frac{x}{\varepsilon}, \hat{u} + \xi \cdot x_j^\delta, \xi\right)\right) dx + c\varepsilon \left(1 + \frac{1}{\delta^2}\right) + c\delta \quad (5.9)$$

for all ε and δ , such that $0 < \varepsilon < \delta < 1$. Since

$$Du_\varepsilon^\delta = \xi + \varepsilon Dv_\varepsilon^\delta = \xi + \varepsilon \sum_{j \in J} w_{\varepsilon j}^\delta D\Phi_j^\delta + \sum_{j \in J} \Phi_j^\delta D_y w_{\varepsilon j}^\delta,$$

we can rewrite the volume integral as follows

$$\begin{aligned} \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, Du_\varepsilon^\delta\right) dx &= \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, \xi + \sum_{j \in J} \Phi_j^\delta D_y w_{\varepsilon j}^\delta\right) dx \\ &+ \left[\int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, Du_\varepsilon^\delta\right) dx - \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, \xi + \sum_{j \in J} \Phi_j^\delta D_y w_{\varepsilon j}^\delta\right) dx \right]. \end{aligned} \quad (5.10)$$

Due to the convexity of $f(y, \xi)$ with respect to ξ , we have

$$\int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, \xi + \sum_{j \in J} \Phi_j^\delta D_y w_{\varepsilon j}^\delta\right) dx \leq \sum_{j \in J_\delta} \int_{\Omega_\varepsilon \cap Q_j^{\delta+}} f\left(\frac{x}{\varepsilon}, \xi + D_y w_{\varepsilon j}^\delta\right) dx. \quad (5.11)$$

Moreover, since

$$|f(y, \xi) - f(y, \eta)| \leq c(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|, \quad \forall \xi, \eta \in \mathbb{R}^n,$$

we can estimate the difference in square brackets in (5.10) as follows

$$\begin{aligned} &\left| \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, Du_\varepsilon^\delta\right) dx - \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, \xi + \sum_{j \in J_\delta} \Phi_j^\delta D_y w_{\varepsilon j}^\delta\right) dx \right| \leq \\ &c\varepsilon \int_{\Omega_\varepsilon} \left(1 + |Du_\varepsilon^\delta|^{p-1} + \left|\xi + \sum_{j \in J_\delta} \Phi_j^\delta D_y w_{\varepsilon j}^\delta\right|^{p-1}\right) \left| \sum_{j \in J_\delta} w_{\varepsilon j}^\delta D\Phi_j^\delta \right| dx \stackrel{\text{def}}{=} J_1^{\varepsilon, \delta}. \end{aligned}$$

Due to Holder's and Young's inequalities, the last integral admits the estimate

$$\begin{aligned} J_1^{\varepsilon, \delta} &\leq c\varepsilon \left(\int_{\Omega_\varepsilon} (1 + |Du_\varepsilon^\delta|^p + \left|\xi + \sum_{j \in J_\delta} \Phi_j^\delta D_y w_{\varepsilon j}^\delta\right|^p) dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega_\varepsilon} \left| \sum_{j \in J_\delta} w_{\varepsilon j}^\delta D\Phi_j^\delta \right|^p dx \right)^{\frac{1}{p}} \\ &\leq c\varepsilon \int_{\Omega_\varepsilon} (1 + |Du_\varepsilon^\delta|^p + \left|\xi + \sum_{j \in J_\delta} \Phi_j^\delta D_y w_{\varepsilon j}^\delta\right|^p) dx + c\varepsilon \int_{\Omega_\varepsilon} \left| \sum_{j \in J_\delta} w_{\varepsilon j}^\delta D\Phi_j^\delta \right|^p dx. \end{aligned}$$

Notice that, by construction, $|D\Phi_j^\delta| \leq \frac{1}{\delta^2}$, and that by (3.14)

$$\|w_{\varepsilon_j}^\delta\|_{L^p(\Omega_\varepsilon)} \leq c(1 + |\hat{u}|^p + \xi^p), \quad \|Dw_{\varepsilon_j}^\delta\|_{L^p(\Omega_\varepsilon)} \leq c(1 + |\hat{u}|^p + \xi^p)$$

for all ε, δ and j . Therefore,

$$c\varepsilon \int_{\Omega_\varepsilon} \left(1 + |Du_\varepsilon^\delta|^p + \left| \xi + \sum_{j \in J_\delta} \Phi_j^\delta D_y w_{\varepsilon_j}^\delta \right|^p \right) dx + c\varepsilon \int_{\Omega_\varepsilon} \left| \sum_{j \in J_\delta} w_{\varepsilon_j}^\delta D\Phi_j^\delta \right|^p dx \leq c\varepsilon \left(1 + \frac{1}{\delta^2} \right)$$

for all ε, δ . We have obtained the estimate

$$\int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, \xi + \varepsilon Dv_\varepsilon^\delta\right) dx \leq \sum_{j \in J} \int_{Q_j^{\delta+} \cap \Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, \xi + D_y w\left(\frac{x}{\varepsilon}, \hat{u} + \xi \cdot x_j^\delta, \xi\right)\right) dx + c\varepsilon \left(1 + \frac{1}{\delta^2} \right). \quad (5.12)$$

To complete the proof of (5.9) it is sufficient to show that

$$\left| \sum_{j \in J} \int_{(Q_j^{\delta+} \setminus Q_j^\delta) \cap \Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, \xi + D_y w\left(\frac{x}{\varepsilon}, \hat{u} + \xi \cdot x_j^\delta, \xi\right)\right) dx \right| \leq c\delta.$$

Provided we choose δ^2 to be an integer multiple of ε , the above inequality follows easily from the ε periodicity of the integrand, and the fact that Lebesgue measure of the set

$$\cup_j (Q_j^{\delta+} \setminus Q_j^\delta)$$

can be estimated by $c\delta$.

Step 2. Now we estimate the surface integral on the right hand side of (5.8). We claim that

$$\begin{aligned} \int_{S_\varepsilon} g\left(\frac{x}{\varepsilon}, u_\varepsilon^\delta\right) d\mathcal{H}^{n-1} &\leq \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \hat{u} + \langle \xi \cdot x \rangle\right) (u_\varepsilon^\delta - \overline{u_\varepsilon^\delta}) d\mathcal{H}^{n-1} \\ &\quad + c \max\{\varepsilon, \varepsilon^{p-1}\} \left(1 + \left(\frac{\varepsilon}{\delta^2}\right)^p \right) + c(\varepsilon + \delta) \end{aligned} \quad (5.13)$$

where, for every function v , the symbol \overline{v} denotes the piecewise-constant function which coincides with the integral average of v on each ε -cell Y_ε^i , while $\langle \xi \cdot x \rangle$ is just defined by

$$\langle \xi \cdot x \rangle = \sum_{j \in J_\delta} \chi_{Q_j^\delta} \xi \cdot x_j^\delta.$$

The bound (5.13) will be obtained through the following intermediate steps.

Step 2a. By Lemma 3.5, considering the fact that u_ε^δ satisfies estimate (5.5) for all ε and δ , we deduce that

$$\int_{S_\varepsilon} g\left(\frac{x}{\varepsilon}, u_\varepsilon^\delta\right) d\mathcal{H}^{n-1} \leq \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \overline{u_\varepsilon^\delta}\right) (u_\varepsilon^\delta - \overline{u_\varepsilon^\delta}) d\mathcal{H}^{n-1} + c \max\{\varepsilon, \varepsilon^{p-1}\} \left(1 + \frac{\varepsilon}{\delta^2} \right) \quad (5.14)$$

for all ε, δ .

Step 2b. Let us recall that $u_\varepsilon^\delta = \hat{u} + \xi \cdot x + \varepsilon v_\varepsilon^\delta$. Our next aim is to prove the inequality

$$\int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \overline{u_\varepsilon^\delta}\right) (u_\varepsilon^\delta - \overline{u_\varepsilon^\delta}) d\mathcal{H}^{n-1} \leq \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \hat{u} + \langle \xi \cdot x \rangle\right) (u_\varepsilon^\delta - \overline{u_\varepsilon^\delta}) d\mathcal{H}^{n-1} + c(\varepsilon + \delta). \quad (5.15)$$

The proof relies on the Lipschitz-continuity of $g'_z(y, \cdot)$ (see (1.7)), and on the estimates of L^p -norms of the functions v_ε^δ over the surfaces S_ε . It is convenient to rearrange the left-hand side of (5.15) as follows

$$\begin{aligned} \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \overline{u_\varepsilon^\delta}\right)(u_\varepsilon^\delta - \overline{u_\varepsilon^\delta}) \, d\mathcal{H}^{n-1} &= \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \hat{u} + \langle \xi \cdot x \rangle\right)(u_\varepsilon^\delta - \overline{u_\varepsilon^\delta}) \, d\mathcal{H}^{n-1} \\ &+ \left[\int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \overline{u_\varepsilon^\delta}\right)(u_\varepsilon^\delta - \overline{u_\varepsilon^\delta}) \, d\mathcal{H}^{n-1} - \int_{S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \hat{u} + \langle \xi \cdot x \rangle\right)(u_\varepsilon^\delta - \overline{u_\varepsilon^\delta}) \, d\mathcal{H}^{n-1} \right]. \end{aligned}$$

Denote the term in the square brackets here by $J_2^{\varepsilon, \delta}$. Then by (1.7)

$$J_2^{\varepsilon, \delta} \leq c \int_{S_\varepsilon} \left(1 + |\overline{u_\varepsilon^\delta}| + |\hat{u} + \langle \xi \cdot x \rangle|\right)^{p-2} |u_\varepsilon^\delta - \hat{u} - \langle \xi \cdot x \rangle| |u_\varepsilon^\delta - \overline{u_\varepsilon^\delta}| \, d\mathcal{H}^{n-1}.$$

Note that

$$u_\varepsilon^\delta - \overline{u_\varepsilon^\delta} = \xi(x - \bar{x}) + \varepsilon(v_\varepsilon^\delta - \overline{v_\varepsilon^\delta}),$$

and that, by construction, $|x - \bar{x}| \leq \varepsilon$. Hence,

$$|u_\varepsilon^\delta - \overline{u_\varepsilon^\delta}| \leq \varepsilon(|\xi| + |v_\varepsilon^\delta - \overline{v_\varepsilon^\delta}|).$$

Taking this estimate into account and substituting

$$\overline{u_\varepsilon^\delta} = \hat{u} + \xi \cdot \bar{x} + \varepsilon \overline{v_\varepsilon^\delta},$$

we obtain that

$$J_2^{\varepsilon, \delta} \leq \varepsilon \int_{S_\varepsilon} \left(1 + |\hat{u}| + |\xi| + \varepsilon |\overline{v_\varepsilon^\delta}|\right)^{p-2} (|\xi \cdot \bar{x} - \langle \xi \cdot x \rangle + \varepsilon \overline{v_\varepsilon^\delta}|)(|\xi| + |v_\varepsilon^\delta - \overline{v_\varepsilon^\delta}|) \, d\mathcal{H}^{n-1}.$$

Applying Holder's inequality to the three terms of the integrand with the exponents $\frac{p}{p-2}$, p and p , respectively, and distributing the factor ε among these terms, we get

$$\begin{aligned} J_2^{\varepsilon, \delta} &\leq \left(\int_{S_\varepsilon} \varepsilon \left(1 + |\hat{u}| + |\xi| + \varepsilon |\overline{v_\varepsilon^\delta}|\right)^p \, d\mathcal{H}^{n-1} \right)^{\frac{p-2}{p}} \\ &\times \left(\int_{S_\varepsilon} \varepsilon |\xi \cdot \bar{x} - \langle \xi \cdot x \rangle + \varepsilon \overline{v_\varepsilon^\delta}|^p \, d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \left(\int_{S_\varepsilon} \varepsilon (|\xi| + |v_\varepsilon^\delta - \overline{v_\varepsilon^\delta}|)^p \, d\mathcal{H}^{n-1} \right)^{\frac{1}{p}}. \end{aligned} \quad (5.16)$$

The first and third terms of this product are bounded, while the second one is not greater than $c(\varepsilon + \delta)$. In order to prove this we should estimate the L^p -norm of v_ε^δ on S_ε . To this end we recall that, by definition (5.2), at each point $x \in \Omega$ the function $v_\varepsilon^\delta(x)$ is a finite combination of the local minimizers $w_{\varepsilon_j}^\delta$ with coefficients Φ_j^δ , and the number of terms involved is at most $2n$. Moreover, by (3.6),

$$\int_{S_\varepsilon} |w_{\varepsilon_j}^\delta|^p \, d\mathcal{H}^{n-1} \leq c \left(\varepsilon^{-1} \int_{\Omega_\varepsilon} |w_{\varepsilon_j}^\delta|^p \, dx + \varepsilon^{p-1} \int_{\Omega_\varepsilon} |Dw_{\varepsilon_j}^\delta|^p \, dx \right)$$

for every $j \in J$ and $\varepsilon, \delta > 0$. Since $w_{\varepsilon_j}^\delta$ is an ε -periodic function, it is easy to see that

$$\int_{\Omega_\varepsilon} |w_{\varepsilon_j}^\delta|^p \, dx \leq c, \quad \int_{\Omega_\varepsilon} |Dw_{\varepsilon_j}^\delta|^p \, dx \leq c\varepsilon^{-p}.$$

The last two estimates imply the bound

$$\int_{S_\varepsilon} |w_{\varepsilon j}^\delta|^p d\mathcal{H}^{n-1} \leq c\varepsilon^{-1}$$

for every $j \in J$ and $\varepsilon, \delta > 0$. By Jensen's inequality we then get

$$\int_{S_\varepsilon} |\overline{v_\varepsilon^\delta}|^p d\mathcal{H}^{n-1} \leq \int_{S_\varepsilon} |v_\varepsilon^\delta|^p d\mathcal{H}^{n-1} \leq c\varepsilon^{-1} \quad \forall \varepsilon, \delta > 0. \quad (5.17)$$

The first integral on the right-hand side of (5.16) is bounded for by (3.6), (5.5) and (5.6) we have

$$\begin{aligned} \int_{S_\varepsilon} \varepsilon (1 + |u_\varepsilon^\delta| + |\hat{u} + \langle \xi \cdot x \rangle|)^p d\mathcal{H}^{n-1} &\leq c \left(1 + \varepsilon \int_{S_\varepsilon} |u_\varepsilon^\delta|^p d\mathcal{H}^{n-1} \right) \leq \\ &\left(1 + \int_{\Omega_\varepsilon} |u_\varepsilon^\delta|^p dx + \varepsilon^p \int_{\Omega_\varepsilon} |Du_\varepsilon^\delta|^p dx \right) \leq c \left(1 + \varepsilon^p + \varepsilon^p \left(\frac{\varepsilon}{\delta^2} \right)^p \right). \end{aligned}$$

The second integral vanishes, as $\varepsilon, \delta \rightarrow 0$, since $|\xi \cdot \bar{x} - \langle \xi \cdot x \rangle| \leq c(\varepsilon + \delta)^p$ and, by (5.17), we have

$$\int_{S_\varepsilon} \varepsilon |\xi \cdot \bar{x} - \langle \xi \cdot x \rangle + \varepsilon \overline{v_\varepsilon^\delta}|^p d\mathcal{H}^{n-1} \leq c(\varepsilon + \delta) + c\varepsilon^{p+1} \int_{S_\varepsilon} |v_\varepsilon^\delta|^p d\mathcal{H}^{n-1} \leq c(\delta^p + \varepsilon^p).$$

Finally, the third integral is bounded for all $\varepsilon > 0$, since

$$\varepsilon \int_{S_\varepsilon} \left(|\xi| + |v_\varepsilon^\delta - \overline{v_\varepsilon^\delta}| \right)^p d\mathcal{H}^{n-1} \leq c(\xi) \left(1 + \varepsilon^{p+1} \int_{S_\varepsilon} |v_\varepsilon^\delta|^p \right) d\mathcal{H}^{n-1} \leq c(\xi)(1 + \varepsilon^p).$$

Combining the above bounds, we obtain (5.15).

Step 2c. To complete the proof of (5.13) it remains to show that

$$\int_{S_\varepsilon} g'_z \left(\frac{x}{\varepsilon}, \hat{u} + \langle \xi \cdot x \rangle \right) (u_\varepsilon^\delta - \overline{u_\varepsilon^\delta}) d\mathcal{H}^{n-1} = \int_{S_\varepsilon} g'_z \left(\frac{x}{\varepsilon}, \hat{u} + \langle \xi \cdot x \rangle \right) (\xi \cdot x - \langle \xi \cdot x \rangle + \varepsilon v_\varepsilon^\delta) d\mathcal{H}^{n-1}. \quad (5.18)$$

To this end, notice that the difference between $u_\varepsilon^\delta - \overline{u_\varepsilon^\delta}$ and $(\xi \cdot (x - \bar{x}) + \varepsilon(v_\varepsilon^\delta - \overline{v_\varepsilon^\delta}))$ is equal to a constant on each ε -cell provided that δ is an integer multiplier of ε . Then the desired relation (5.18) follows from (1.5).

Step 3. With the help of (5.13) one can prove that

$$\begin{aligned} \int_{S_\varepsilon} g'_z \left(\frac{x}{\varepsilon}, \hat{u} + \langle \xi \cdot x \rangle \right) (\xi \cdot x - \langle \xi \cdot x \rangle + \varepsilon v_\varepsilon^\delta) d\mathcal{H}^{n-1} &\leq \\ \sum_{j \in J_\delta} \int_{Q_j^\delta \cap S_\varepsilon} g'_z \left(\frac{x}{\varepsilon}, \hat{u} + \langle \xi \cdot x \rangle \right) (\xi \cdot x - \xi \cdot x_j^\delta + \varepsilon w_{\varepsilon j}^\delta) d\mathcal{H}^{n-1} &+ c \left(\delta + \frac{\varepsilon}{\delta^2} \right). \end{aligned} \quad (5.19)$$

Indeed, by the definition of v_ε^δ we have

$$\begin{aligned} \int_{S_\varepsilon} g'_z \left(\frac{x}{\varepsilon}, \hat{u} + \langle \xi \cdot x \rangle \right) (\xi \cdot (x - \langle x \rangle) + \varepsilon v_\varepsilon^\delta) d\mathcal{H}^{n-1} &= \\ \sum_{j \in J_\delta} \int_{Q_j^\delta \cap S_\varepsilon} g'_z \left(\frac{x}{\varepsilon}, \hat{u} + \xi \cdot x_j^\delta \right) (\xi \cdot x - \xi \cdot x_j^\delta + \varepsilon \Phi_j^\delta w_{\varepsilon j}^\delta) d\mathcal{H}^{n-1}. \end{aligned}$$

Therefore, it suffices to estimate the difference

$$J_4^{\varepsilon, \delta} = \left| \sum_{j \in J_\delta} \int_{Q_j^\delta \cap S_\varepsilon} g'_z \left(\frac{x}{\varepsilon}, \hat{u} + \xi \cdot x_j^\delta \right) \varepsilon (\Phi_j^\delta - 1) w_{\varepsilon j}^\delta \, d\mathcal{H}^{n-1} \right|.$$

Recall that $\Phi_j^\delta = 1$ in $Q_j^{\delta-}$ for any $j \in \mathbb{Z}^n$. If we denote $G_j^\delta = Q_j^\delta \setminus Q_j^{\delta-}$ and $G_\delta = \bigcup_j G_j^\delta$, then

$$J_4^{\varepsilon, \delta} \leq \left| \sum_{j \in J_\delta} \int_{G_j^\delta \cap S_\varepsilon} g'_z \left(\frac{x}{\varepsilon}, \hat{u} + \xi \cdot x_j^\delta \right) \varepsilon (\Phi_j^\delta - 1) w_{\varepsilon j}^\delta \, d\mathcal{H}^{n-1} \right|.$$

Note that $|\Phi_j^\delta - 1| \leq 1$ and, by (1.4)–(1.7),

$$\left| g'_z \left(\frac{x}{\varepsilon}, \hat{u} + \langle \xi \cdot x \rangle \right) \right| \leq c \quad \text{for a.e. } x \in \Omega,$$

so that

$$J_4^{\varepsilon, \delta} \leq c \sum_{j \in J_\delta} \int_{G_j^\delta \cap S_\varepsilon} \varepsilon |w_{\varepsilon j}^\delta| \, d\mathcal{H}^{n-1}.$$

Now, by the ε -periodicity of $w_{\varepsilon j}^\delta$ and (3.14), for each j we have

$$\int_{G_j^\delta \cap S_\varepsilon} \varepsilon |w_{\varepsilon j}^\delta| \, d\mathcal{H}^{n-1} = \varepsilon^{n-1} \int_{S \cap Y} \varepsilon |w(y, \hat{u} + \xi \cdot x_j^\delta, \xi)| \, d\mathcal{H}^{n-1} \cdot |G_j^\delta| \varepsilon^{-n} \leq c \delta^{n+1};$$

a simple bound $|G_j^\delta| \leq c \delta^{n+1}$ has also been used here. Since the cardinality of J_δ is of the order δ^{-n} , then, summing up over j and considering the fact that the cardinality of J_δ is not greater than bound $c \delta^{-n}$, we conclude that

$$\sum_{j \in J_\delta} \int_{G_j^\delta \cap S_\varepsilon} \varepsilon |w_{\varepsilon j}^\delta| \, d\mathcal{H}^{n-1} \leq c \delta,$$

which completes the proof of (5.19).

Step 4. Taking into account the estimates of Steps 1–3, we arrive at the inequality

$$\begin{aligned} F_\varepsilon(u_\varepsilon^\delta) &\leq \sum_{j \in J} \int_{Q_j^\delta \cap \Omega_\varepsilon} f \left(\frac{x}{\varepsilon}, \xi + D_y w \left(\frac{x}{\varepsilon}, \hat{u} + \xi \cdot x_j^\delta, \xi \right) \right) \, dx \\ &\quad + \sum_{j \in J} \int_{Q_j^\delta \cap S_\varepsilon} g'_z \left(\frac{x}{\varepsilon}, \hat{u} + \xi \cdot x_j^\delta \right) \left(\xi \cdot x - \xi \cdot x_j^\delta + \varepsilon w_{\varepsilon j}^\delta \right) + k(\varepsilon, \delta) \end{aligned}$$

with

$$k(\varepsilon, \delta) = c \left(\varepsilon + \delta + \frac{\varepsilon}{\delta^2} \right) + c \max\{\varepsilon, \varepsilon^{p-1}\} \left(1 + \left(\frac{\varepsilon}{\delta^2} \right)^p \right).$$

In order to prove (5.7) it is sufficient to observe that

$$\sum_{j \in J_\delta} \int_{Q_j^\delta \cap \Omega_\varepsilon} f\left(\frac{x}{\varepsilon}, \xi + D_y w\left(\frac{x}{\varepsilon}, \hat{u} + \xi \cdot x_j^\delta, \xi\right)\right) dx + \sum_{j \in J_\delta} \int_{Q_j^\delta \cap S_\varepsilon} g'_z\left(\frac{x}{\varepsilon}, \hat{u} + \xi \cdot x_j^\delta\right) (\xi \cdot x - \xi \cdot x_j^\delta + \varepsilon w_{\varepsilon j}^\delta) = \sum_{j \in J_\delta} L(\hat{u} + \xi \cdot x_j^\delta, \xi) |Q_j^\delta|.$$

The Γ -limsup inequality is proved. □

Proof of Proposition 2.4. For an arbitrary $u \in W^{1,p}(\Omega)$, $u|_{\partial\Omega} = \Phi$, we are going to modify the family u_ε so that for the modified functions the relation (2.4) still holds true and they satisfy the boundary condition $u_\varepsilon|_{\partial\Omega} = \Phi$. Clearly, it suffices to show that for any $\delta > 0$ there is a sequence $v_\varepsilon^\delta \in W^{1,p}(\Omega)$ such that $v_\varepsilon^\delta \rightarrow u$ in $W^{1,p}(\Omega)$ as $\varepsilon \rightarrow 0$, $v_\varepsilon^\delta|_{\partial\Omega} = \Phi$ and

$$\limsup_{\varepsilon \rightarrow 0} F^\varepsilon(v_\varepsilon^\delta) \leq F(u) + \kappa_5(\delta), \tag{5.20}$$

where $\kappa_5(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Let u^δ be a piece-wise affine function in \mathbb{R}^n such that $\|u - u^\delta\|_{W^{1,p}(\Omega)} \leq \delta$. As was shown in the beginning of this section, there is a family $\{u_\varepsilon^\delta\}$ (see (5.2), (5.3)) such that

$$\limsup_{\varepsilon \rightarrow 0} F^\varepsilon(u_\varepsilon^\delta) = F(u^\delta).$$

Denote by L_Ω^ν the ν -neighbourhood of $\partial\Omega$ intersected with Ω . By construction,

$$\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon^\delta\|_{W^{1,p}(L_\Omega^\nu)} \rightarrow 0,$$

as $\nu \rightarrow 0$. Let $\phi_\nu(x)$ be a cut-off function such that $\phi_\nu \in C_0^\infty(\Omega)$, $0 \leq \phi_\nu \leq 1$, $\phi_\nu(x) = 1$ if $\text{dist}(x, \partial\Omega) > \nu$, and $|\nabla \phi_\nu| \leq 2/\nu$. If we set $u_\varepsilon^{\delta,\nu} = u^\delta + (u_\varepsilon^\delta - u^\delta)\phi_\nu$, then $u_\varepsilon^{\delta,\nu} = u^\delta$ on $\partial\Omega$, and

$$\|u_\varepsilon^{\delta,\nu} - u_\varepsilon^\delta\|_{L^p(\Omega)} \leq C \left(\|u^\delta\|_{L^p(L_\Omega^\nu)} + \|u_\varepsilon^\delta\|_{L^p(L_\Omega^\nu)} \right),$$

$$\|Du_\varepsilon^{\delta,\nu} - Du_\varepsilon^\delta\|_{L^p(\Omega)} \leq C \left(\|Du^\delta\|_{L^p(L_\Omega^\nu)} + \|Du_\varepsilon^\delta\|_{L^p(L_\Omega^\nu)} + \frac{1}{\nu} \|u_\varepsilon^\delta - u^\delta\|_{L^p(\Omega)} \right).$$

The last inequalities yield

$$\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon^{\delta,\nu} - u_\varepsilon^\delta\|_{W^{1,p}(\Omega)} \rightarrow 0,$$

as $\nu \rightarrow 0$. Choosing now $\nu = \nu(\delta)$ in such a way that

$$\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon^{\delta,\nu} - u_\varepsilon^\delta\|_{W^{1,p}(\Omega)} \leq \delta$$

and letting $v_\varepsilon^\delta = u_\varepsilon^{\delta,\nu} + (u - u^\delta)$, by Lemma 3.6 we obtain the desired inequality (5.20).

REFERENCES

[1] E. Acerbi, V. Chiadò Piat, G. Dal Maso and D. Percivale, An extension theorem from connected sets, and homogenization in general periodic domains. *Nonlinear Anal.* **18** (1992) 481–496.
 [2] R.A. Adams, *Sobolev spaces*. Academic Press, New York (1975).
 [3] A.G. Belyaev, G.A. Chechkin and A.L. Piatnitski, Asymptotic behavior of a solution to a boundary value problem in a perforated domain with oscillating boundary. *Sib. Math. J.* **39** (1998) 621–644.
 [4] A.G. Belyaev, G.A. Chechkin and A.L. Piatnitski, Homogenization of second-order elliptic operators in a perforated domain with oscillating Fourier boundary conditions. *Sb. Math.* **192** (2001) 933–949.

- [5] A. Braides and A. Defranceschi, *Homogenization of multiple integrals*, *Oxford Lecture Series in Mathematics and its Applications* **12**. The Clarendon Press, Oxford University Press, New York (1998).
- [6] A. Brillard, Asymptotic analysis of two elliptic equations with oscillating terms. *RAIRO Modél. Math. Anal. Numér.* **22** (1988) 187–216.
- [7] D. Cioranescu and P. Donato, On a Robin problem in perforated domains, in *Homogenization and applications to material sciences*, D. Cioranescu *et al.* Eds., GAKUTO International Series, *Mathematical Sciences and Applications* **9**, Tokyo, Gakkotosho (1997) 123–135.
- [8] D. Cioranescu and F. Murat, A strange term coming from nowhere, in *Topics in the Mathematical Modelling of Composite Materials*, *Progr. Nonlinear Differential Equations Appl.* **31**, Birkhauser, Boston (1997) 45–93.
- [9] D. Cioranescu and J. Saint Jean Paulin, Homogenization in open sets with holes. *J. Math. Anal. Appl.* **71** (1979) 590–607.
- [10] D. Cioranescu and J. Saint Jean Paulin, Truss structures: Fourier conditions and eigenvalue problems, in *Boundary control and boundary variation*, J.P. Zolezio Ed., *Lecture Notes Control Inf. Sci.* **178**, Springer-Verlag (1992) 125–141.
- [11] C. Conca, On the application of the homogenization theory to a class of problems arising in fluid mechanics. *J. Math. Pures Appl.* **64** (1985) 31–75.
- [12] G. Dal Maso, *An introduction to Γ -convergence*. Birkhauser, Boston (1993).
- [13] V.A. Marchenko and E.Y. Khruslov, *Boundary value problems in domains with fine-grained boundaries*. Naukova Dumka, Kiev (1974).
- [14] V.A. Marchenko and E.Y. Khruslov, *Homogenization of partial differential equations*. Birkhauser (2006).
- [15] O.A. Oleinik and T.A. Shaposhnikova, On an averaging problem in a partially punctured domain with a boundary condition of mixed type on the boundary of the holes, containing a small parameter. *Differ. Uravn.* **31** (1995) 1150–1160, 1268. Translation in *Differ. Equ.* **31** (1995) 1086–1098.
- [16] O.A. Oleinik and T.A. Shaposhnikova, On the homogenization of the Poisson equation in partially perforated domains with arbitrary density of cavities and mixed type conditions on their boundary. *Rend. Mat. Acc. Linceis. IX* **7** (1996) 129–146.
- [17] S.E. Pastukhova, Tartar’s compensated compactness method in the averaging of the spectrum of a mixed problem for an elliptic equation in a punctured domain with a third boundary condition. *Sb. Math.* **186** (1995) 753–770.
- [18] S.E. Pastukhova, On the character of the distribution of the temperature field in a perforated body with a given value on the outer boundary under heat exchange conditions on the boundary of the cavities that are in accord with Newton’s law. *Sb. Math.* **187** (1996) 869–880.
- [19] S.E. Pastukhova, Spectral asymptotics for a stationary heat conduction problem in a perforated domain. *Mat. Zametki* **69** (2001) 600–612 [in Russian]. Translation in *Math. Notes* **69** (2001) 546–558.
- [20] W.P. Ziemer, *Weakly differentiable functions*. Springer-Verlag, New York (1989).