CONJUGATE AND CUT TIME IN THE SUB-RIEMANNNIAN PROBLEM ON THE GROUP OF MOTIONS OF A PLANE

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Abstract. The left-invariant sub-Riemannian problem on the group of motions (rototranslations) of a plane SE(2) is studied. Local and global optimality of extremal trajectories is characterized. Lower and upper bounds on the first conjugate time are proved. The cut time is shown to be equal to the first Maxwell time corresponding to the group of discrete symmetries of the exponential mapping. Optimal synthesis on an open dense subset of the state space is described.

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1. Introduction

This work is devoted to the study of the left-invariant sub-Riemannian problem on the group of motions of a plane. This problem can be stated as follows: given two unit vectors \( v_0 = (\cos \theta_0, \sin \theta_0) \), \( v_1 = (\cos \theta_1, \sin \theta_1) \) attached respectively at two given points \((x_0, y_0), (x_1, y_1)\) in the plane, one should find an optimal motion in the plane that transfers the vector \( v_0 \) to the vector \( v_1 \), see Figure 1. The vector can move forward or backward and rotate simultaneously. The required motion should be optimal in the sense of minimal length in the space \((x, y, \theta)\), where \( \theta \) is the slope of the moving vector.

The corresponding optimal control problem reads as follows:

\[
\dot{x} = u_1 \cos \theta, \quad \dot{y} = u_1 \sin \theta, \quad \dot{\theta} = u_2, \\
q = (x, y, \theta) \in M \cong \mathbb{R}^2_{x,y} \times S^1_\theta, \quad u = (u_1, u_2) \in \mathbb{R}^2, \\
q(0) = q_0 = (0, 0, 0), \quad q(t_1) = q_1 = (x_1, y_1, \theta_1), \\
l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} \, dt \to \min.
\]

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or, equivalently,

\[ J = \frac{1}{2} \int_0^{t_1} (u_1^2 + u_2^2) \, dt \to \min. \]  

\[ \text{(1.5)} \]

Problem (1.1)–(1.8) is a left-invariant sub-Riemannian problem on the group of motions of a plane \( \text{SE}(2) = \mathbb{R}^2 \ltimes \text{SO}(2) \). This problem is closely connected to models of vision \([5,10,11]\) and robotics \([8]\). On the other hand, this problem is important for understanding properties of the heat kernel of the diffusion equation on the group \( \text{SE}(2) \), see \([4]\). We expect that results of this work can be applied to these domains. On the other hand, problem (1.1)–(1.5) is the first completely studied sub-Riemannian problem without rotational symmetry; this problem has the local structure of generic contact sub-Riemannian problems as described in works \([1,6]\).

This is an immediate continuation of the previous work \([9]\). We use extensively the results obtained in that paper, and now we recall the most important of them.

The normal Hamiltonian system of Pontryagin Maximum Principle \([12]\) becomes triangular in appropriate coordinates on cotangent bundle \( T^*M \), and its vertical subsystem is the equation of mathematical pendulum:

\[ \dot{\gamma} = c, \quad \dot{c} = -\sin \gamma, \quad (\gamma, c) \in C \cong (2\mathbb{S}_1^1) \times \mathbb{R}_c, \]  

\[ \dot{x} = \sin \frac{\gamma}{2} \cos \theta, \quad \dot{y} = \sin \frac{\gamma}{2} \sin \theta, \quad \dot{\theta} = -\cos \frac{\gamma}{2}. \]  

\[ \text{(1.6)} \]

\[ \text{(1.7)} \]

The phase cylinder of pendulum (1.6) decomposes into invariant subsets according to values of the energy \( E = c^2/2 - \cos \gamma \):

\[ C = \bigcup_{i=1}^5 C_i, \]

\[ C_1 = \{ \lambda \in C \mid E \in (-1, 1) \}, \]  

\[ \text{(1.8)} \]

\[ C_2 = \{ \lambda \in C \mid E \in (1, +\infty) \}, \]  

\[ \text{(1.9)} \]

\[ C_3 = \{ \lambda \in C \mid E = 1, \ c \neq 0 \}, \]  

\[ \text{(1.10)} \]

\[ C_4 = \{ \lambda \in C \mid E = -1 \} = \{ (\gamma, c) \in C \mid \gamma = 2\pi n, \ c = 0 \}, \]  

\[ \text{(1.11)} \]

\[ C_5 = \{ \lambda \in C \mid E = 1, \ c = 0 \} = \{ (\gamma, c) \in C \mid \gamma = \pi + 2\pi n, \ c = 0 \}. \]  

\[ \text{(1.12)} \]
In the subsets $C_1$, $C_2$, $C_3$ were introduced elliptic coordinates $(\varphi, k)$ that rectify the flow of the pendulum: $\varphi$ is the phase, and $k$ a reparameterized energy of pendulum (1.6):

$$k = \sqrt{(E + 1)/2} \text{ in } C_1 \cup C_3, \quad k = \sqrt{2/(E + 1)} \text{ in } C_2.$$  

The Hamiltonian system (1.6), (1.7) was integrated in Jacobi’s functions [22]. The equation of pendulum (1.6) has a discrete group of symmetries $G = \{\text{Id}, \varepsilon^1, \ldots, \varepsilon^7\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by reflections in the axes of coordinates $\gamma, c$, and translations $(\gamma, c) \mapsto (\gamma + 2\pi, c)$. Action of the group $G$ is naturally extended to extremal trajectories $(x_t, y_t)$, this action modulo rotations is represented at Figures 2–4.

Reflections $\varepsilon^i$ are symmetries of the exponential mapping $\text{Exp} : N = C \times \mathbb{R}_+ \to M$, $\text{Exp}(\lambda, t) = q_t$.

The main result of work [9] is an upper bound on cut time

$$t_{\text{cut}} = \sup \{t_1 > 0 \mid q_s \text{ is optimal for } s \in [0, t_1]\}$$

along extremal trajectories $q_s$. It is based on the fact that a sub-Riemannian geodesic cannot be optimal after a Maxwell point, i.e., a point where two distinct geodesics of equal sub-Riemannian length meet one another. A natural idea is to look for Maxwell points corresponding to discrete symmetries of the exponential mapping. For each extremal trajectory $q_s = \text{Exp}(\lambda, s)$, we described Maxwell times $t_{\varepsilon^i}^n(\lambda)$, $i = 1, \ldots, 7$, $n = 1, 2, \ldots$, corresponding to discrete symmetries $\varepsilon^i$. The following upper bound is the main result of work [9]:

$$t_{\text{cut}}(\lambda) \leq t(\lambda), \quad \lambda \in C,$$  

(1.13)

where $t(\lambda) = \min(t_{\varepsilon^i}^n(\lambda))$ is the first Maxwell time corresponding to the group of symmetries $G$. We recall the explicit definition of the function $t(\lambda)$ below in equations (2.20)–(2.24).

In this work we continue the study of problem (1.1)–(1.5).

First we consider the local optimality of sub-Riemannian geodesics (Sect. 2). We show that extremal trajectories corresponding to oscillating pendulum (1.6) do not have conjugate points, thus they are locally optimal forever. In the case of rotating pendulum we prove that the first conjugate time is bounded from below and from above by the first Maxwell times $t_{\varepsilon^2}^1(\lambda)$ and $t_{\varepsilon^5}^1(\lambda)$ respectively. For critical values of energy of the pendulum, there are no conjugate points.

In Section 3 we study the global optimality of geodesics. We construct open dense subsets in the preimage and image of exponential mapping and prove that the exponential mapping transforms these domains diffeomorphically. As a consequence, we show that inequality (1.13) is in fact an equality. Moreover, we describe the optimal synthesis on the open dense subset of the state space.

In Section 4 we present plots of the sub-Riemannian caustic and spheres obtained in Mathematica [23]. In the subsequent work [19] we complete our study of problem (1.1)–(1.5). There we describe explicitly the Maxwell strata and cut locus, and characterize the optimal synthesis in this problem. For some special terminal points $q_1$, we provide explicit optimal solutions.
2. Conjugate points

In this section we obtain bounds on conjugate time in the sub-Riemannian problem on SE(2), see Theorem 2.5.

2.1. General facts

First we recall some known facts from the theory of conjugate points in optimal control problems. For details see, e.g., [2,3,20].

Consider an optimal control problem of the form

\[ \dot{q} = f(q,u), \quad q \in M, \quad u \in U \subset \mathbb{R}^m, \quad q(0) = q_0, \quad q(t_1) = q_1, \quad t_1 \text{ fixed}, \]

\[ J = \int_0^{t_1} \varphi(q(t),u(t)) \, dt \to \min, \]

where \( M \) is a finite-dimensional analytic manifold, \( f(q,u) \) and \( \varphi(q,u) \) are respectively analytic in \( (q,u) \) families of vector fields and functions on \( M \) depending on the control parameter \( u \in U \), and \( U \) an open subset of \( \mathbb{R}^m \). Admissible controls are \( u(\cdot) \in L_\infty([0,t_1],U) \), and admissible trajectories \( q(\cdot) \) are Lipschitzian. Let

\[ h_u(\lambda) = \langle \lambda, f(q,u) \rangle - \varphi(q,u), \quad \lambda \in T^*_M, \quad q = \pi(\lambda) \in M, \quad u \in U, \]

be the normal Hamiltonian of PMP for the problem (2.1)–(2.3). Fix a triple \((\bar{u}(t), \lambda_\bar{t}, q(t))\) consisting of a normal extremal control \( \bar{u}(t) \), the corresponding extremal \( \lambda_\bar{t} \), and the extremal trajectory \( q(t) \) for the problem (2.1)–(2.3).

Let the following hypotheses hold:

(H1) For all \( \lambda \in T^*M \) and \( u \in U \), the quadratic form \( \frac{\partial^2 h_u}{\partial u^2}(\lambda) \) is negative definite.

(H2) For any \( \lambda \in T^*M \), the function \( u \mapsto h_u(\lambda), \quad u \in U \), has a maximum point \( \bar{u}(\lambda) \in U \):

\[ h_{\bar{u}(\lambda)}(\lambda) = \max_{u \in U} h_u(\lambda), \quad \lambda \in T^*M. \]

(H3) The extremal control \( \bar{u}(\cdot) \) is a corank one critical point of the endpoint mapping.

(H4) All trajectories of the Hamiltonian vector field \( \bar{H}(\lambda), \lambda \in T^*M \), are continued for \( t \in [0, +\infty) \).

An instant \( t_* > 0 \) is called a conjugate time (for the initial instant \( t = 0 \)) along the extremal \( \lambda_* \) if the restriction of the second variation of the endpoint mapping to the kernel of its first variation is degenerate, see [3] for details. In this case the point \( q(t_*) = \pi(\lambda_* \bar{t}) \) is called conjugate for the initial point \( q_0 \) along the extremal trajectory \( q(\cdot) \).

Under hypotheses (H1)–(H4), we have the following:

1. Normal extremal trajectories lose their local optimality (both strong and weak) at the first conjugate point, see [3].

2. An instant \( t > 0 \) is a conjugate time iff the exponential mapping \( \text{Exp}_t = \pi \circ e^{t\bar{H}} \) is degenerate, see [2].

3. Along each normal extremal trajectory, conjugate times are isolated one from another, see [20].

We will apply the following statement for the proof of absence of conjugate points via homotopy.

Proposition 2.1 (Cors. 2.2 and 2.3 [18]). Let \((u^s(t), \lambda_*^s), t \in [0, +\infty), s \in [0,1]\), be a continuous in parameter \( s \) family of normal extremal pairs in the optimal control problem (2.1)–(2.3) satisfying hypotheses (H1)–(H4).

1. Let \( s \mapsto t_*^s \) be a continuous function, \( s \in [0,1], t_*^s \in (0, +\infty) \). Assume that for any \( s \in [0,1] \) the instant \( t = t_*^s \) is not a conjugate time along the extremal \( \lambda_*^s \). If

   \[ \text{the extremal trajectory } q^s(t) = \pi(\lambda_*^s), \quad t \in [0,t_*^s], \]

   does not contain conjugate points, then the extremal trajectory \( q^s(t) = \pi(\lambda_*^s), \quad t \in [0,t_*^s] \), also does not contain conjugate points.
Lemma 2.3. Let for any \( s \in [0, 1] \) and \( T > 0 \) the extremal \( \lambda^s_t \) has no conjugate points for \( t \in (0, T] \). Then for any \( T > 0 \), the extremal \( \lambda^1_t \) also has no conjugate points for \( t \in (0, T] \).

One easily checks that the sub-Riemannian problem (1.1)–(1.5) satisfies all hypotheses (H1)–(H4), so the results cited in this subsection are applicable to this problem.

We denote the first conjugate time along an extremal trajectory \( q(t) = \text{Exp}(\lambda, t) \) as \( t_{\text{con}}^1(\lambda) \).

2.2. Conjugate points for the case of oscillating pendulum

In this section we assume that \( \lambda \in C_1 \) and prove that the corresponding extremal trajectories do not contain conjugate points, see Theorem 2.1.

Using the parameterization of extremal trajectories obtained in Section 3.3 [9], we compute explicitly Jacobian of the exponential mapping:

\[
J = \frac{\partial(x_1, y_1, \theta_1)}{\partial(t, \phi, k)} = \frac{4}{k^3(1 - k^2)(1 - k^2 \sin^2 p \sin^2 \tau)} J_1,
\]

where \( p = t/2, \quad \tau = \phi + t/2, \quad J_1(\tau, p, k) = v_1 \sin^2 \tau + v_2 \cos^2 \tau, \quad v_1 = (1 - k^2)(p - E(p))(E(p) - (1 - k^2)p), \)

\[
v_2 = (p - E(p))(E(p) - (1 - k^2)p) + k^2 \sin p \cos p \ln p (2E(p) + (k^2 - 2)p) \sin p
\]

so that \( \text{sgn} \ J = \text{sgn} \ J_1 \).

2.2.1. Preliminary lemmas

Lemma 2.1. For any \( k \in (0, 1) \) and \( p > 0 \) we have \( v_1(p, k) > 0 \).

Proof. The statement follows from the relations

\[
p - E(p) = k^2 \int_0^p \sin^2 t \, dt > 0, \quad E(p) - (1 - k^2)p = k^2 \int_0^p \cos^2 t \, dt > 0.
\]  (2.6)

Lemma 2.2. For any \( k \in (0, 1), n \in \mathbb{N}, \) and \( \tau \in \mathbb{R} \) we have \( J_1(\tau, 2Kn, k) > 0 \).

Proof. If \( p = 2Kn, n \in \mathbb{N} \), then \( v_2(p, k) = (p - E(p))(E(p) - (1 - k^2)p) > 0 \) by inequalities (2.6).

By virtue of Lemma 2.1 and decomposition (2.5), we obtain the inequality \( J_1(\tau, 2Kn, k) > 0 \).

Lemma 2.3. \( \forall p_1 > 0, \quad \exists \, \tilde{k} \in (0, 1), \quad \forall \tau \in \mathbb{R}, \quad J_1(\tau, p, k) > 0. \)

Proof. The statement of the lemma follows from the Taylor expansions:

\[
J_1 = \frac{k^4}{16} (4p^2 - \sin^2 2p) + o(k^4), \quad k \to 0, \quad k^2 + p^2 \to 0.
\]  (2.7)

By contradiction, if the statement is not verified, then there exists a converging sequence \( (\tau_n, p_n, k_n) \to (\tau_0, p_0, 0) \) such that \( J(\tau_n, p_n, k_n) \leq 0 \) for all \( n \in \mathbb{N} \). If \( p_0 > 0 \), then a standard calculus argument yields contradiction with (2.7). And if \( p_0 = 0 \), then similarly one obtains a contradiction with (2.8).
2.2.2. Absence of conjugate points in $C_1$

**Theorem 2.1.** If $\lambda \in C_1$, then the extremal trajectory $q(t) = \text{Exp}(\lambda, t)$, $t > 0$, does not contain conjugate points.

**Proof.** We choose any $\tilde{\lambda} \in C_1$, $\tilde{t} > 0$, and prove that the extremal trajectory $\tilde{q}(t) = \text{Exp}(\tilde{\lambda}, t)$ does not contain conjugate points for $t \in (0, \tilde{t})$.

Find the elliptic coordinates $$(k, \tilde{\varphi})$$ corresponding to the covector $\tilde{\lambda} \in C_1$ according to Section 3.2 [9], and let $\tilde{p} = \tilde{t}/2$, $\tilde{\varphi} = \tilde{\varphi} + \tilde{p}$. Find $n \in \mathbb{N}$ such that $p_1 = 2K(k)n > \tilde{p}$. Choose the following continuous curve in the plane $(k, p)$:

$$\{(k^s, p^s) \mid s \in [0, 1]\}, \quad k^s = sk, \quad p^s = 2K(k^s)n,$$

with the endpoints $(k^0, p^0) = (0, \pi n)$ and $(k^1, p^1) = (\hat{k}, 2K(\hat{k})n)$.

Consider the following family of extremal trajectories:

$$\gamma^s = \{q^s(t) = \text{Exp}(\varphi^s, k^s, t) \mid t \in [0, t^s]\}, \quad s \in [0, 1],$$

$$t^s = 2p^s, \quad \varphi^s = \tilde{\varphi} - p^s.$$

The endpoint $q^s(t^s)$ of each trajectory $\gamma^s$, $s \in [0, 1]$, corresponds to the values of parameters $(\tau, p, k) = (\tilde{\varphi}, 2K(k^s)n, k^s)$. Thus Lemma 2.2 implies that for any $s \in [0, 1]$ the endpoint $q^s(t^s)$ is not a conjugate point.

Further, Lemma 2.3 states that

$$\exists k_0 \in (0, \hat{k}) \quad \forall \tau \in \mathbb{R} \quad \forall p \in (0, p_1) \quad J(\tau, p, k) > 0. \quad (2.9)$$

Denote $s_0 = k_0/\hat{k} \in (0, 1)$, so that $k^{s_0} = k_0$. Condition (2.9) means that the extremal trajectory $\gamma^{s_0}$ does not contain conjugate points for all $t \in [0, t^{s_0}]$.

Then Proposition 2.1 yields that for any $s \in [s_0, 1]$, the extremal trajectory $q^s(t)$ does not contain conjugate points for all $t \in [0, t^s]$. In particular, the trajectory $\tilde{q}(t) = q^s(t)$, $t \in (0, \tilde{t})$, is free of conjugate points. \hfill \Box

So we proved that extremal trajectories $q(t) = \text{Exp}(\lambda, t)$ with $\lambda \in C_1$ (i.e., corresponding to oscillating pendulum) are locally optimal at any segment $[0, t_1]$, $t_1 > 0$.

2.3. Conjugate points for the case of rotating pendulum

In this section we obtain bounds on conjugate points in the case $\lambda \in C_2$. Using the formulas for extremal trajectories of Section 3.3 [9], we get:

$$J = \frac{\partial(x_1, y_1, \theta_1)}{\partial(t, \varphi, k)} = -\frac{4k}{(1 - k^2)(1 - k^2 \text{sn}^2 p \text{sn}^2 \tau)} J_2,$$

$$p = t/(2k), \quad \tau = \psi + t/(2k) = (2\varphi + t)/(2k), \quad (2.10)$$

$$J_2 = \alpha \text{sn}^2 \tau + \beta \text{cn}^2 \tau, \quad (2.11)$$

$$\alpha = (1 - k^2) \text{sn} p \alpha_1, \quad (2.12)$$

$$\alpha_1 = \text{cn} p \text{dn} p (p - 2E(p)) + \text{sn} p (\text{dn}^2 p + E(p)(p - E(p))), \quad (2.13)$$

$$\beta = f_1(p)/\beta_1, \quad \beta_1 = \text{cn} p E(p) - \text{dn} p \text{sn} p, \quad (2.14)$$

where $f_1(p, k) = \text{cn} p (E(p) - p) - \text{dn} p \text{sn} p$, see equation (5.12) [9].

2.3.1. Preliminary lemmas

Recall that we denoted the first positive root of the function $f_1(p)$ by $p_1^1(k)$, see Lemma 5.3 [9].

**Lemma 2.4.** If $k \in (0, 1)$ and $p = p_1^1(k)$, then $\alpha(p, k) > 0$.

If additionally $\text{sn} \tau \neq 0$, then $J_2 > 0$ and $J < 0$. 

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Proof. In terms of the auxiliary function
\[ \varphi(p, k) = \sin p \frac{d}{dp} p - (2E(p) - p) \csc p, \]  
we have a decomposition
\[ \alpha_1 = \sin p \varphi(p) + \sin p E(p) (p - E(p)). \] (2.16)

Let \( k \in (0, 1) \) and \( p = p_1(k) \). Then \( f_1(p) = 0, \) i.e., \( \sin p \frac{d}{dp} p = \sin p (E(p) - p) \). Thus \( \varphi(p) = \sin p (E(p) - p) - (2E(p) - p) \csc p = -E(p) \csc p \). By virtue of Corollary 5.1 \([9]\), we have \( \csc p < 0 \), so \( \varphi(p) > 0 \). Moreover, \( \csc p > 0 \). Then decomposition (2.16) yields \( \alpha_1(p) > 0 \), consequently, \( \alpha(p) > 0 \).

If additionally \( \sin \tau \neq 0 \), then it is obvious that \( J_2 > 0 \) and \( J < 0 \). □

Lemma 2.5. \( \exists \hat{k} \in (0,1), \quad \forall k \in (0, \hat{k}), \quad \forall p \in (0, p_1], \quad \alpha(p, k) > 0. \)

Proof. The statement of this lemma follows by the argument used in the proof of Lemma 2.3 from the Taylor expansions
\[ \alpha = \sin p (\sin p - p \cos p) + o(1), \quad k \to 0, \]
\[ \alpha = \frac{p^4}{3} + o(p^2 + k^2)^2, \quad p^2 + k^2 \to 0. \] □

Lemma 2.6. \( \forall k \in (0, 1), \quad \forall p \in (0, 2K], \quad \beta_1(p, k) < 0. \)

Proof. Since \((\beta_1(p)/\csc p)' = -(1-k^2)\sin^2 p / \csc^2 p \), the function \( \beta_1(p)/\csc p \) decreases at the segments \( p \in [0, K] \) and \( p \in (K, 2K] \).

We have \( \beta_1(0) = 0 \), thus \( \beta_1(p)/\csc p < 0 \), so \( \beta_1(p) < 0 \) for \( p \in (0, K) \).

Further, \( \beta_1(K) = -\sqrt{1-k^2} < 0. \)

Since \( \beta_1(p)/\csc p \to +\infty \) as \( p \to K + 0 \), and \( \beta_1(2K)/\csc(2K) = E(2K) > 0 \), we have \( \beta_1(p)/\csc p > 0 \), so \( \beta_1(p) < 0 \) for \( p \in (K, 2K] \). □

Lemma 2.7. Let \( k \in (0, 1) \).

(1) Let \( \sin \tau = 0. \) Then \( J_2(\tau, p, k) > 0 \) for \( p \in (0, p_1) \), and \( J_2(\tau, p, k) = 0 \) for \( p = p_1 \).

(2) Let \( \sin \tau \neq 0. \) Then \( J_2(\tau, p, k) > 0 \) for \( p \in (0, p_1] \).

Proof. If \( p \in (0, p_1] \), then \( f_1(p, k) < 0 \) (Cor. 5.1 \([9]\)), and \( \beta_1(p, k) < 0 \) (Lem. 2.6), thus \( \beta(p, k) = f_1(p, k) \beta_1(p, k) > 0. \)

(1) Let \( \sin \tau = 0. \) If \( p \in (0, p_1) \), then \( J_2(\tau, p, k) = \beta(p, k) > 0. \) And if \( p = p_1 \), then \( f_1(p, k) = 0 \), thus \( J_2(\tau, p, k) = \beta(p, k) = 0. \)

(2) Let \( \sin \tau \neq 0. \)

(2.a) We prove that the function \( \varphi(p) \) given by (2.15) satisfies the inequality
\[ \varphi(p) > 0 \quad \forall p \in (0, K]. \]

First, \( \varphi(p) = p^3/3 + o(p^3) > 0 \) as \( p \to +0. \) Second,
\[ (\varphi(p)/\csc p)' = \sin^2 p \frac{d}{dp} p / \csc^2 p > 0 \quad \forall p \in (0, K]. \]

Thus \( \varphi(p) > 0 \) for \( p \in (0, K) \). And if \( p = K \), then \( \varphi(p) = \sqrt{1-k^2} > 0. \)

(2.b) By virtue of the decomposition \( \varphi(p) = -f_1(p) - E(p) \csc p \), we get the inequality \( \varphi(p) > 0 \) for all \( p \in (K, p_1] \). We proved that
\[ \varphi(p) > 0 \quad \forall p \in (0, p_1]. \]

(2.c) In view of (2.16), we obtain that \( \alpha_1(p) > 0 \) for \( p \in (0, p_1]. \) Then equation (2.12) yields \( \alpha(p) > 0 \) for \( p \in (0, p_1]. \) Finally, equation (2.11) gives \( J_2 > 0 \) for \( p \in (0, p_1]. \) □
Lemma 2.8. \( \forall z \in (0, 1], \ \exists \tilde{k} \in (0, 1), \ \forall k \in (0, \tilde{k}), \ \forall p \in (0, p_1], \ \text{we have} \ J_2(z, p, k) > 0. \)

Proof. Fix any \( z \in (0, 1]. \) By Lemma 2.5,
\[
\exists \tilde{k} \in (0, 1), \ \forall k \in (0, \tilde{k}), \ \forall p \in (0, p_1], \ \alpha(p, k) > 0.
\]

But if \( p \in (0, p_1], \) then \( p \in (0, 2K), \) thus \( \beta_1(p, k) < 0 \) by Lemma 2.6, so \( \beta(p, k) > 0 \) by Corollary 5.1 [9].

Then the inequalities \( \alpha(p, k) > 0, \beta(p, k) > 0 \) imply the required inequality \( J_2(\tau, p, k) > 0. \)

2.3.2. Conjugate points in \( C_2 \)

First we obtain a lower bound on the first conjugate time. It will play a crucial role in the subsequent analysis of the global structure of the exponential mapping in Section 3 and in the subsequent work [19].

Theorem 2.2. If \( \lambda \in C_2, \) then \( t_1^\text{conj} (\lambda) \geq 2k\tau^1(k). \)

Proof. Given any \( \lambda \in C_2, \) compute the corresponding elliptic coordinates \((\varphi, k), \) If additionally we have \( t > 0, \) find the corresponding parameters \( p = t/(2k), \) \( \tau = \varphi/k + t/(2k) \)

We should prove that for any \( \lambda \in C_2 \) the interval \( t \in (0, 2k\tau^1(k)) \) does not contain conjugate times for
the extremal trajectory \( q(t) = \text{Exp}(\lambda, t). \)

Take any \( \lambda_1 \in C_2 \) and denote the corresponding elliptic coordinates \((\varphi^1, k^1), \) For \( t^1 = 2k^1\tau^1(k^1) \) we denote the corresponding parameters \( p^1, \tau^1, z^1. \) In order to prove that the extremal trajectory \( q^1(t) = \text{Exp}(\lambda^1, t) \) does not have conjugate points at the interval \( t \in (0, t^1), \) we show that
\[
p \in (0, p^1) \quad \Rightarrow \quad J_2(z^1, p, k^1) > 0 \quad \Rightarrow \quad J(z^1, p, k^1) < 0.
\]

(1) Assume first that \( z^1 = \text{sn}^2(\tau^1, k^1) \neq 0, \) i.e., \( z^1 \in (0, 1]. \) We prove that in this case
\[
p \in (0, p^1) \quad \Rightarrow \quad J(z^1, p, k^1) < 0.
\]

Consider the following continuous curve in the space \((z, p, k), \)
\[
\{(z^1, p^1, k^1) \mid s \in (0, 1], \quad k^1 = sk^1, \quad p^1 = p_1^1(k^1).
\]
The corresponding curve in the space \((\tau, p, k), \)
\[
\{(\tau^1, p^1, k^1) \mid s \in (0, 1], \quad \tau^1 = F(\text{am}(\tau^1, k^1), k^1),
\]
and in the space \((t, \varphi, k), \)
\[
\{(t^*, \varphi^*, k^*) \mid s \in (0, 1], \quad t^* = 2k^*p^*, \quad \varphi^* = (t^* - p^*)k^*.
\]

Let \( \lambda^* = (\varphi^*, k^*), s \in (0, 1], \) be the corresponding curve in \( C_2, \) and consider the continuous one-parameter family of extremal trajectories
\[
q^*(t) = \text{Exp}(\lambda^*, t), \quad t \in [0, t^*], \quad s \in (0, 1].
\]

For any \( s \in (0, 1], \) if \( t = t^*, \) then by Lemma 2.4 we have \( J_2(z^1, p_1^1(k^*), k^*) < 0, \) i.e., the terminal instant \( t = t^* \) is not a conjugate time along the extremal trajectory \( q^*(t). \)

Further, by Lemma 2.8, for \( z^1 \in (0, 1] \)
\[
\exists \tilde{k} \in (0, 1), \ \forall k \in (0, \tilde{k}), \ \forall p \in (0, p_1^1(k)], \quad J_2(z^1, p, k) > 0.
\]
Consequently, there exists $s_0 \in (0, 1)$ such that the whole trajectory $q^{s_0}(t)$, $t \in (0,t^{s_0}]$, is free of conjugate points.

Then Proposition 2.1 implies that the trajectory $q^1(t)$, $t \in (0,t^1]$, also does not contain conjugate points.

We proved that if $z^1 \neq 0$, then the trajectory $q^1(t) = \text{Exp}(\lambda^1, t)$, $t \in (0,t^1]$, does not have conjugate points.

(2) Now consider the case $z^1 = \text{sn}^2(t^1, k^1) = 0$. Then Lemma 2.7 states that the terminal instant $t = 2k^1p^1_1(k^1)$ is a conjugate point. We prove that all the least instants are not conjugate.

Since conjugate points are isolated one from another at each extremal trajectory, there exists $p < p^1_1(k^1)$ arbitrarily close to $p^1_1(k^1)$ such that the corresponding time $t = 2k^1p$ is not conjugate.

Consider the continuous curve in the space $(z,p,k)$:

$$\{(z_s, p, k^1) \mid s \in [0, \varepsilon]\}, \quad z_s = sz^1.$$ 

By item (1) of this proof, there exists $\varepsilon > 0$ such that for any $s \in (0, \varepsilon)$ the extremal trajectory $q^s(t)$, $t \in (0,t^s]$, $t^s = 2k^1p$, does not have conjugate points. By Proposition 2.1, for $s = 0$ the initial extremal trajectory $q^0(t)$, $t \in (0,t^0]$, also does not contain conjugate points.

Then items (1), (2) of this proposition follow directly from the corresponding items of Lemma 2.7, and from Theorem 2.3.

Now we obtain the final result on the first conjugate time in the domain $C_2$ — the following two-side bound.

**Theorem 2.3.** If $\lambda \in C_2$, then

$$2kp_1^1(k) \leq t_1^\text{conj}(\lambda) \leq 4kK(k). \tag{2.17}$$

**Proof.** We proved in Theorem 2.2 that $2kp_1^1(k) \leq t_1^\text{conj}(\lambda)$; moreover, if $t \in (0,2kp_1^1)$, then $J < 0$.

Let $t = 4kK$, then $p = 2K$, thus $\alpha = 0$, $f_1 = p - E(p) > 0$, $\beta_1 = -E(p) < 0$, so

$$J = -\frac{4k}{1-k^2}J_2 = \frac{4k}{1-k^2} \text{cn}^2 \tau \text{ E}(p) (p - E(p)) \geq 0.$$ 

It follows that for any $\lambda \in C_2$ the function $t \mapsto J$ has a root at the segment $t \in [2kp_1^1, 4kK]$. Consequently, also the first root $t_1^\text{conj} \in [2kp_1^1, 4kK]$.

One can show that the bound (2.17) can be a little bit improved. The precise bound on the first conjugate time is

$$2kp_1^1(k) \leq t_1^\text{conj}(\lambda) \leq \gamma(k) = \min \{4kK, 2kp_1^0(k)\}, \tag{2.18}$$

where $p = p_1^0(k)$ is the first positive root of the equation $\alpha_1(p) = 0$, and the function $\alpha_1$ is given by equation (2.13). One can show that $\gamma(k) = 4kK$ for $k \in (0,k_0]$ and $\gamma(k) = 2kp_1^0(k)$ for $[k_0,1)$, where $k_0 \approx 0.909$ is the unique root of the equation $2E(k) - K(k) = 0$, see Proposition 11.5 [17]. Thus for $k \in (k_0,1)$ the bound (2.17) is not exact and can be replaced by the following exact one:

$$2kp_1^1(k) \leq t_1^\text{conj}(\lambda) \leq 2kp_1^0(k), \quad k \in (k_0,1). \tag{2.19}$$

The bound (2.17) is illustrated at Figures 5 and 6; and the bound (2.19) — at Figure 7. The exact bounds (2.18) are plotted at Figure 8.

**Proposition 2.2.** Let $\lambda \in C_2$ and $\tau = (2p_2 + 2kp_1^1)/(2k)$.

1. If $\text{sn} \tau = 0$, then $t_1^\text{conj}(\lambda) = 2kp_1^1$.
2. If $\text{sn} \tau \neq 0$, then $t_1^\text{conj}(\lambda) \in (2kp_1^1, 4kK]$.

**Proof.** Notice first that by Theorem 2.2, the interval $(0,2kp_1^1)$ does not contain conjugate times. Then items (1), (2) of this proposition follow directly from the corresponding items of Lemma 2.7, and from Theorem 2.3. \qed
2.4. Conjugate points for the cases of critical energy of pendulum

The subset $C_3 \cup C_4 \cup C_5$ of the cylinder $C$ is the boundary of the domain $C_1$, see equations (1.8)–(1.12) above, and Figure 2 in [9]. Thus absence of conjugate points for the corresponding extremal trajectories follows by limit passage from $C_1$.

**Theorem 2.4.** If $\lambda \in C_3 \cup C_4 \cup C_5$, then the corresponding extremal trajectory $q(t) = \text{Exp}(\lambda, t)$ does not have conjugate points for $t > 0$.

**Proof.** For any $\lambda \in C_3 \cup C_4 \cup C_5$, there exists a continuous curve $\lambda^s$, $s \in [0, 1]$, such that $\lambda^s \in C_1$ for $s \in [0, 1)$ and $\lambda^1 = \lambda$. By Theorem 2.1, the trajectories $q^s(t) = \text{Exp}(\lambda^s, t)$, $t > 0$, are free of conjugate points. Then Proposition 2.1 implies the same for the trajectory $q^1(t) = q(t)$. $\square$

2.5. General bound of conjugate points

We collect the bounds on the first conjugate time obtained in the previous subsections. Moreover, we conclude that the first conjugate time $t_1^{\text{conj}}$ admits the lower bound by the function $t : C \to (0, +\infty]$ on the phase cylinder.
of pendulum $(2S_1^1) \times \mathbb{R}_c = C = \bigcup_{i=1}^{5} C_i$ introduced in [9]:

$$\lambda \in C_1 \Rightarrow t(\lambda) = 2K, \quad (2.20)$$
$$\lambda \in C_2 \Rightarrow t(\lambda) = 2k p_1 (k), \quad (2.21)$$
$$\lambda \in C_3 \Rightarrow t(\lambda) = +\infty, \quad (2.22)$$
$$\lambda \in C_4 \Rightarrow t(\lambda) = \pi, \quad (2.23)$$
$$\lambda \in C_5 \Rightarrow t(\lambda) = +\infty. \quad (2.24)$$

**Theorem 2.5.**

1. If $\lambda \in C_1 \cup C_3 \cup C_4 \cup C_5$, then $t_{\text{conj}}(\lambda) = +\infty$.
2. If $\lambda \in C_2$, then $t_{\text{conj}}(\lambda) \in [2kp_1, 4kK]$.
3. Consequently, $t_{\text{conj}}(\lambda) \geq t(\lambda)$ for all $\lambda \in C$.

### 3. Exponential mapping of open stratas and cut time

In this section we show that there exist open dense domains $\tilde{N} \subset N$, $\tilde{M} \subset M$ transformed diffeomorphically by the exponential mapping. As a consequence, we prove that $t_{\text{cut}}(\lambda) = t(\lambda)$ for any $\lambda \in C$, and describe the optimal synthesis on $\tilde{M}$.

#### 3.1. Decompositions in preimage and image of exponential mapping

Denote $\tilde{M} = M \setminus \{q_0\}$. For any point $q \in \tilde{M}$ there exists an optimal trajectory $q(s) = \text{Exp}(\lambda, s)$ such that $q(t) = q$, $(\lambda, t) \in N$. Thus the mapping $\text{Exp} : N \to \tilde{M}$ is surjective. By Theorem 5.4 [9], the optimal instant $t$ satisfies the inequality $t \leq t(\lambda)$. So the restriction

$$\text{Exp} : \tilde{N} \to \tilde{M},$$
$$\tilde{N} = \{ (\lambda, t) \in N \mid t \leq t(\lambda) \},$$

is surjective as well.

**3.1.1. Decomposition in $\tilde{N}$**

Now we select open dense subsets of $\tilde{N}$ such that restriction of Exp to these subsets will turn out to be a diffeomorphism. Let

$$N_i = C_i \times \mathbb{R}_+, \quad i = 1, \ldots, 5,$$
$$\tilde{N} = \{ (\lambda, t) \in \bigcup_{i=1}^{5} N_i \mid t < t(\lambda), \text{ sn } \tau \text{ cn } \tau \neq 0 \}, \quad (3.1)$$
$$N' = \{ (\lambda, t) \in \bigcup_{i=1}^{5} N_i \mid t = t(\lambda) \text{ or } \text{ sn } \tau \text{ cn } \tau = 0 \} \cup \tilde{N}_4 \cup N_5,$$
$$\tilde{N}_4 = \tilde{N} \cap N_4.$$

We have the obvious decomposition $\tilde{N} = \tilde{N} \cup N'$ (we denote by $\cup$ the union of mutually non-intersecting sets).

There hold the following implications, see [9]:

$$\lambda \in N_1 \Rightarrow t(\lambda) = 2K, \quad \tau \in \mathbb{R}/(4K\mathbb{Z}),$$
$$\lambda \in N_2 \Rightarrow t(\lambda) = 2kp_1, \quad \tau \in \mathbb{R}/(4K\mathbb{Z}),$$
$$\lambda \in N_3 \Rightarrow t(\lambda) = +\infty, \quad \tau \in \mathbb{R}.$$
Proof. We prove the statement only for the set $D_1$ by the same inequalities as $D_2$ by a shift of origin in elliptic coordinate $\varphi$. Taking into account Table 1

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
$\lambda$ & $C_{1}^{0}$ & $C_{1}^{0}$ & $C_{1}^{0}$ & $C_{1}^{0}$ & $C_{1}^{1}$ & $C_{1}^{1}$ & $C_{1}^{1}$ & $C_{1}^{1}$ \\
$\tau$ & $(3K,4K)$ & $(0,K)$ & $(2K,3K)$ & $(-K,0)$ & $(0,K)$ & $(0,K)$ & $(0,K)$ & $(2K,3K)$ \\
p & $(0,K)$ & $(0,K)$ & $(0,K)$ & $(0,K)$ & $(0,K)$ & $(0,K)$ & $(0,K)$ & $(0,K)$ \\
\hline
$\lambda$ & $C_{2}^{+}$ & $C_{2}^{+}$ & $C_{2}^{-}$ & $C_{2}^{-}$ & $C_{2}^{+}$ & $C_{2}^{+}$ & $C_{2}^{-}$ & $C_{2}^{-}$ \\
$\tau$ & $(-K,0)$ & $(0,K)$ & $(-K,0)$ & $(0,K)$ & $(K,2K)$ & $(K,2K)$ & $(-3K,-2K)$ & $(-2K,-K)$ \\
p & $(0,p_{1})$ & $(0,p_{1})$ & $(0,p_{1})$ & $(0,p_{1})$ & $(0,p_{1})$ & $(0,p_{1})$ & $(0,p_{1})$ & $(0,p_{1})$ \\
\hline
$\lambda$ & $C_{3}^{0+}$ & $C_{3}^{0+}$ & $C_{3}^{0-}$ & $C_{3}^{0-}$ & $C_{3}^{1+}$ & $C_{3}^{1+}$ & $C_{3}^{1-}$ & $C_{3}^{1-}$ \\
$\tau$ & $(-\infty,0)$ & $(0,\infty)$ & $(-\infty,0)$ & $(0,\infty)$ & $(-\infty,0)$ & $(0,\infty)$ & $(-\infty,0)$ & $(0,\infty)$ \\
p & $(0,\infty)$ & $(0,\infty)$ & $(0,\infty)$ & $(0,\infty)$ & $(0,\infty)$ & $(0,\infty)$ & $(0,\infty)$ & $(0,\infty)$ \\
\hline
\end{tabular}
\caption{Definition of domains $D_i$.}
\end{table}

Consequently, there holds the following decomposition:

$$\tilde{N} = \bigcup_{i=1}^{8} D_i,$$

where the sets $D_i$, $i = 1, \ldots, 8$, are defined by Table 1.

Table 1 should be read by columns. For example, the first column means that

\begin{align*}
D_1 &= (D_1 \cap N_1) \cup (D_1 \cap N_2) \cup (D_1 \cap N_3), \\
D_1 \cap N_1 &= \{(\tau, p, k) \in N_1 \mid \lambda \in C_{1}^{0}, \tau \in (3K,4K), p \in (0,K), k \in (0,1)\}, \\
D_1 \cap N_2 &= \{(\tau, p, k) \in N_2 \mid \lambda \in C_{1}^{0}, \tau \in (-K,0), p \in (0,p_{1}), k \in (0,1)\}, \\
D_1 \cap N_3 &= \{(\tau, p, k) \in N_3 \mid \lambda \in C_{1}^{0}, \tau \in (-\infty,0), p \in (0,\infty), k = 1\}.
\end{align*}

Projections of the sets $D_i$ to the phase cylinder of the pendulum $(\gamma, c)$ are shown in Figure 9.

Lemma 3.1. Each set $D_i$, $i = 1, \ldots, 8$, is homeomorphic to $\mathbb{R}^3$.

Proof. We prove the statement only for the set $D_2$ since all other sets $D_i$ can be defined in the coordinates $(\tau, p, k)$ by the same inequalities as $D_2$ by a shift of origin in elliptic coordinate $\varphi$. Taking into account Table 1

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
& $D_1$ & $D_2$ & $D_5$ & $D_6$ \\
& $\gamma$ & $\gamma$ & $\gamma$ & $\gamma$ \\
$\pi$ & $D_4$ & $D_3$ & $D_8$ & $D_7$ \\
& $\pi$ & $\pi$ & $\pi$ & $\pi$ \\
\hline
$\gamma$ & $\gamma$ & $\gamma$ & $\gamma$ & $\gamma$ \\
$\gamma$ & $\gamma$ & $\gamma$ & $\gamma$ & $\gamma$ \\
\hline
\end{tabular}
\caption{Projections of domains $D_i$ to the phase cylinder of pendulum $(2S_{1}^{1}) \times \mathbb{R}_{c}^{1}$.}
\end{table}
and equations (2.4), (2.10), we get:

\[ D_2 = \{ (\lambda, t) \in N_1 \mid \lambda \in C_1^0, \ k \in (0, 1), \ t \in (0, 2K), \ \varphi \in (-t, -t + 2K) \}, \]

\[ D_2 \cap N_1 = \{ (\lambda, t) \in N_1 \mid \lambda \in C_0^1, \ k \in (0, 1), \ t \in (0, 2K), \ \varphi \in (-t, -t + 2kK) \}, \]

\[ D_2 \cap N_2 = \{ (\lambda, t) \in N_1 \mid \lambda \in C_1^0, \ k \in (0, 1), \ t \in (0, 2K), \ \varphi \in (-t, -t + 2kK) \}, \]

\[ D_2 \cap N_3 = \{ (\lambda, t) \in N_1 \mid \lambda \in C_0^1, \ k \in (0, 1), \ t \in (0, 2K), \ \varphi \in (-t, -t + 2kK) \}. \]

As shown in [17], one can choose regular system of coordinates \((k_1, \varphi, t)\) on the set \(D_2\), where \(k_1 = k\) for \(\lambda \in C_1; \ k_1 = 1/k\) for \(\lambda \in C_2; \ k_1 = 1\) for \(\lambda \in C_3\).

In this system of coordinates

\[ D_2 = \{ \nu = (k_1, \varphi, t) \mid k_1 \in (0, +\infty), \ t \in (0, t_1(k_1), \ \varphi \in (-t, -t + t_2(k_2))) \}, \]

(3.2)

where \(t_1(k_1) = 2K(k_1)\) for \(k_1 \in (0, 1), \ t_1(k_1) = +\infty\) for \(k_1 = 1, \ t_1(k_1) = (2/k_1)p_1^1(1/k_1)\) for \(k_1 \in (1, +\infty)\);

and \(t_2(k_1) = 2K(k_1)\) for \(k_1 \in (0, 1), \ t_2(k_1) = +\infty\) for \(k_1 = 1, \ t_2(k_1) = (2/k_1)K(1/k_1)\) for \(k_1 \in (1, +\infty)\). The both functions \(t_i : (0, +\infty) \to (0, +\infty), \ i = 1, 2, \) are continuous. Thus representation (3.2) implies that the domain \(D_2\) is homeomorphic to \(\mathbb{R}^3\).

Consequently, all domains \(D_i\) are open, connected, and simply connected. These domains are schematically represented in the left-hand side of Figure 10.

3.1.2. **Decomposition in \(\widehat{M}\)**

The state space of the problem admits a decomposition of the form

\[ M = \widehat{M} \cup M', \]

\[ \widehat{M} = \{ q \in M \mid R_1(q)R_2(q)\sin\theta \neq 0 \}, \]

\[ M' = \{ q \in M \mid R_1(q)R_2(q)\sin\theta = 0 \}, \]

where

\[ R_1 = y \cos\frac{\theta}{2} - x \sin\frac{\theta}{2}, \quad R_2 = x \cos\frac{\theta}{2} + y \sin\frac{\theta}{2}. \]

Further,

\[ \widehat{M} = \bigcup_{i=1}^{2} M_i, \]

where each of the sets \(M_i\) is characterized by constant signs of the functions \(\sin\theta, \ R_1, \ R_2\) described in Table 2.

For example, \(M_1 = \{ q \in M \mid \sin\theta < 0, \ R_1 > 0, \ R_2 > 0 \}\). The numeration of the sets \(M_i\) is chosen so that it correspond to numeration of the sets \(N_i\) (we prove below in Thm. 3.1 that each mapping \(\text{Exp} : N_i \to M_i\) is a diffeomorphism). It is obvious that all the sets \(M_i\) are diffeomorphic to \(\mathbb{R}^3\).

All the domains \(M_i\) are contained in the set \(\{ q \in M \mid \theta \neq 0 \}\). At this set \(\theta\) is a single-valued function, and we choose the branch \(\theta \in (0, 2\pi)\). Thus in the sequel we assume that \(\theta \in (0, 2\pi)\) on the sets \(M_i\).

Table 2. Definition of domains \(M_i\).

<table>
<thead>
<tr>
<th>(M_i)</th>
<th>(M_1)</th>
<th>(M_2)</th>
<th>(M_3)</th>
<th>(M_4)</th>
<th>(M_5)</th>
<th>(M_6)</th>
<th>(M_7)</th>
<th>(M_8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{sgn}(\sin\theta))</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>(\text{sgn}(R_1))</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>(\text{sgn}(R_2))</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>
Then $R_1$, $R_2$ become single-valued functions, and the last two rows of Table 2 reflect the signs of these single-valued functions $R_i$ on the sets $M_i$.

The boundary $M'$ of the domain $\tilde{M}$ decomposes into four mutually orthogonal surfaces: two planes $\{\theta = 0\}$, $\{\theta = \pi\}$ and two Moebius strips $\{R_1 = 0\}$, $\{R_2 = 0\}$, see the right-hand side of Figure 10, and Figure 7 [9].

3.2. Diffeomorphic properties of exponential mapping

Lemma 3.2. The restriction $\text{Exp}|_{\tilde{N}}$ is nondegenerate.

Proof. If $\nu = (\lambda, t) \in \tilde{N}$, then $t < t(\lambda)$. By Theorem 2.5, $t^{\text{conj}}_1(\lambda) \geq t(\lambda)$, thus $t < t^{\text{conj}}_1(\lambda) = \inf\{s > 0 \mid \text{Exp}(\lambda, s) \text{ is degenerate}\}$. Consequently, the exponential mapping is nondegenerate at the point $$(\lambda, t)$$. □

Lemma 3.3. For any $i = 1, \ldots, 8$, we have $\text{Exp}(D_i) \subset M_i$.

Proof. We prove only the inclusion $\text{Exp}(D_1) \subset M_1$ since the rest inclusions are proved similarly.

Let $(\lambda, t) \in D_1 \cap N_1 = \{(\lambda, t) \in N_1 \mid \lambda \in C^0_1, \tau \in (3K, 4K), p \in (0, K), k \in (0, 1)\}$, see Table 1. Since $\lambda \in C^0_1$, then $s_1 = \text{sgn}(\gamma_t/2) = 1$. Moreover, we have $cn p sn p dn \tau > 0$, thus $\sin \theta_t < 0$ by virtue of equation (5.2) [9]. So $\theta_t/2 \in (\pi/2, \pi)$. Consequently, $\cos(\theta_t/2) < 0$, $\sin \theta_t/2 > 0$ on $D_1 \cap N_1$, thus $s_2 = -1, s_4 = 1$ in equations (5.3)–(5.6) [9]. Then we get $R_1 > 0$ from equation (5.5) [9], and $R_2 > 0$ from equation (5.6) [9] and Lemma 5.2 [9].

We proved that if $\nu \in D_1 \cap N_1$, then $\sin \theta_t < 0, R_1 > 0, R_2 > 0$, i.e., $\text{Exp}(\nu) \in M_1$, see Table 2. That is, $\text{Exp}(D_1 \cap N_1) \subset M_1$.

A similar argument works in the case $(\lambda, t) \in D_2 \cap N_2 = \{(\lambda, t) \in N_2 \mid \lambda \in C^0_2, \tau \in (-K, 0), p \in (0, p), k \in (0, 1)\}$. We have $s_2 = \text{sgn} c_2 = 1; \sin \theta_t < 0$ by equation (5.7) [9]; $R_1 > 0$ and $R_2 > 0$ by equation (5.10) and (5.11) [9]. Thus $\text{Exp}(D_1 \cap N_2) \subset M_1$.\n
**Figure 10.** Global structure of exponential mapping.
It follows from the definition of \( N_3 \) and \( N_2 \) (see Tab. 1) that \( D_1 \cap N_3 \subset \text{cl}(D_2 \cap N_3) \), thus \( \text{Exp}(D_1 \cap N_3) \subset \text{cl}(M_1) \). So

\[
\text{Exp}(D_1) \subset \text{cl}(M_1). \tag{3.3}
\]

Let \( \nu = (\lambda, t) \in D_1 \cap N_3 \), then \( q_\nu = \text{Exp}(\nu) \in \text{cl}(M_1) \). In order to prove that \( q_\nu \in M_1 \), assume by contradiction that \( q_\nu \in \partial M_1 \). By Lemma 3.2, the exponential mapping is a local diffeomorphism near \( \nu \). Thus there exist a neighborhood \( U \subset D_1 \) of the point \( \nu \) and a neighborhood \( V \subset M \) of the point \( q_\nu \) such that \( \text{Exp} : U \rightarrow V \) is a diffeomorphism. Since \( q_\nu \in \partial M_1 \), there exists a point \( \tilde{q} \in V \setminus \text{cl}(M_1) \). Then \( \tilde{\nu} = \text{Exp}^{-1}(\tilde{q}) \in U \subset D_1 \), but \( \tilde{\nu} = \text{Exp}(\tilde{\nu}) \notin \text{cl}(M_1) \), which contradicts to (3.3). Thus \( q_\nu \in M_1 \). So \( \text{Exp}(D_1 \cap N_3) \subset M_1 \), and the required inclusion \( \text{Exp}(D_1) \subset M_1 \) follows.

\[\square\]

**Lemma 3.4.** For any \( i = 1, \ldots, 8 \), the mapping \( \text{Exp} : D_i \rightarrow M_i \) is proper.

**Proof.** Similarly to Lemma 3.1, we can consider only the case \( i = 2 \). Let \( K \subset M_2 \) be a compact, we show that \( S = \text{Exp}^{-1}(K) \subset D_2 \) is a compact as well, i.e., \( S \) is bounded and closed.

There exists \( \varepsilon > 0 \) such that

\[
|\sin \theta| \geq \varepsilon, \quad \varepsilon \leq |R_1|, |R_2| \leq 1/\varepsilon \quad \text{for all} \ q \in K.
\]

(1) We show that \( S \) is bounded. By contradiction, let \( \nu_n = (k_n, \varphi_n, t_n) \rightarrow \infty \) for some sequence \( \{\nu_n\} \subset S \). Then there exists a sequence \( \{\nu_n\} \subset S \cap N_i \) for some \( i = 1, 2, 3 \) with \( \nu_n \rightarrow \infty \).

Let \( S \cap N_1 \ni \nu_n = (k_n, \varphi_n, t_n) \rightarrow \infty \). Then \( t_n = 2p_n \in \{0, K(k_n)\} \), \( \tau_n = (\varphi_n + t_n)/2 \in \{0, K(k_n)\} \). If \( k_n \) is separated from 1, then \( p_n, \tau_n \) are bounded, thus \( t_n, \varphi_n \) are bounded, a contradiction. Thus \( k_n \rightarrow 1 \) for a subsequence (we will assume that this holds for the initial sequence).

If \( (\gamma_n, c_n) \rightarrow (\pm \pi, 0) \), then \((\theta, y) \rightarrow 0\), thus \( R_1 \rightarrow 0 \), a contradiction. Thus the sequence \( (\gamma_n, c_n) \) is separated from the point \( (\pm \pi, 0) \).

Then there exists a sequence such that \( k_n \rightarrow 1 \) and \( \varphi_n \rightarrow \varphi \subset (-\infty, +\infty) \), thus \( t_n \rightarrow +\infty, p_n \rightarrow +\infty, \tau_n \rightarrow +\infty \). Then \( (p_n - E(p_n))/(k_n \sqrt{\Delta}) \rightarrow \infty \), \( f_2(p_n, k_n)/(k_n \sqrt{\Delta}) \rightarrow \infty \).

If \( cn(\tau_n) \) is separated from zero, then \( R_1 \rightarrow \infty \) (see equation (5.5) [9]). And if \( cn(\tau_n) \) is not separated from zero, then there exists a sequence such that \( cn(\tau_n) \rightarrow 0 \), thus \( sn(\tau_n) \) is separated from zero, then \( R_2 \rightarrow \infty \) (see equation (5.6) [9]).

So the hypothesis \( S \cap C_1 \ni \nu_n = (k_n, \varphi_n, t_n) \rightarrow \infty \) leads to a contradiction.

Similarly the hypotheses \( C \cap C_1 \ni \nu_n \rightarrow \infty \), \( i = 2, 3 \), lead to a contradiction.

Thus the set \( S = \text{Exp}^{-1}(K) \) is bounded.

(2) We show that \( S \) is closed. Let \( \{\nu_n\} \subset S \), we have to prove that there exists a subsequence \( \nu_{n_k} \) converging in \( D_2 \). By contradiction, let \( \nu_n \rightarrow \infty \) or \( \nu_n \rightarrow \nu \in \partial D_2 \).

Consider the case \( \nu_n = (\tau_n, p_n, k_n) \in S \cap N_i \). If \( k_n \rightarrow 0 \), then \((x, y) \rightarrow 0\), thus \( R_1, R_2 \rightarrow 0 \), a contradiction.

Let \( k_n \rightarrow 1 \). If \( (\gamma_n, c_n) \rightarrow (\pm \pi, 0) \), then \((\theta, y) \rightarrow (0, 0)\), thus \( R_1 \rightarrow 0 \), a contradiction.

If \( (\gamma_n, c_n) \rightarrow (\gamma, c) \neq (\pm \pi, 0) \), then \( \nu \in N_3 \), a contradiction.

Thus \( k_n \rightarrow k \in (0, 1) \). Then \( \tau_n \rightarrow \tau \in [3K(k), 4K(k)] \). If \( \tau = 3K \), then \( R_1 \rightarrow 0 \), and if \( \tau = 4K \), then \( R_2 \rightarrow 0 \), a contradiction. Thus \( \tau_n \rightarrow \tau \in (3K(k), 4K(k)) \).

Further, \( p_n \rightarrow p \in [0, K(k)] \). If \( p = 0 \), then \( t = 0 \) and \( R_1, R_2 \rightarrow 0 \). If \( p = K \), then \( R_1 \rightarrow 0 \). Thus \( p_n \rightarrow p \in (0, K) \).

So \( (\tau_n, p_n, k_n) \rightarrow (\tau, p, K) \in N_1 \), a contradiction.

We proved that any sequence \( \nu_n \in S \cap N_i \) contains a subsequence converging in \( D_2 \). Similarly one proves the same for a sequence \( \nu_n \in S \cap N_i, i = 2, 3 \).

Thus any sequence \( \nu_n \in S \) contains a subsequence converging in \( D_2 \), thus converging in \( S \). So the set \( S = \text{Exp}^{-1}(K) \) is closed. \[\square\]
Theorem 3.1. For any $i = 1, \ldots, 8$, we have $\text{Exp}(D_i) \subset M_i$, and the mapping $\text{Exp} : D_i \to M_i$ is a diffeomorphism.

Proof. The inclusion $\text{Exp}(D_i) \subset M_i$ was proved in Lemma 3.3. The mapping $\text{Exp} : D_i \to M_i$ is smooth, nondegenerate (Lem. 3.2), and proper (Lem. 3.4), thus it is a covering. Since $M_i$ is simply connected, the mapping $\text{Exp} : D_i \to M_i$ is a diffeomorphism. □

Lemma 3.5. $\text{Exp}(N_4) = \{q \in M \mid R_1 = R_2 = 0\} = \{q \in M \mid x = y = 0\}$, $\text{Exp}(N_5) = \{q \in M \mid R_1 = 0, R_2 \neq 0, \theta = 0\} = \{q \in M \mid x \neq 0, y = 0, \theta = 0\}$.

Proof. Follows immediately from the corresponding formulas for extremal trajectories of Section 3.3 [9]. □

Lemma 3.6. $\text{Exp}(N') \subset M'$.

Proof. Follows from formulas (5.2)–(5.11) [9]. □

Theorem 3.1 implies the following statement.

Corollary 3.1. The mapping $\text{Exp} : \tilde{N} \to \tilde{M}$ is a diffeomorphism.

In view of Lemma 3.6, for any $q \in \tilde{M}$ there exists a unique $\nu = (\lambda, t) = \text{Exp}^{-1}(q) \in \tilde{N}$, $\lambda = \lambda(q), t = t(q)$.

The diffeomorphism $\text{Exp} : \tilde{N} = \cup_{i=1}^{5} D_i \to \tilde{M} = \cup_{i=1}^{5} M_i$ is schematically shown at Figure 10. At this figure in the left, each set $N_i$, $i = 1, \ldots, 5$, projects down to the corresponding subset $C_i$ in the phase cylinder of the pendulum, see (1.8)–(1.12) and Figure 2 [9]. Each domain $D_i$, $i = 1, \ldots, 8$, is mapped to one of the domains $M_i$, $i = 1, \ldots, 8$, cut out in the solid torus by the discs $\{\theta = 0\}$, $\{\theta = \pi\}$, and the Moebius strips $\{R_1 = 0\}$, $\{R_2 = 0\}$. Boundaries of the domains $D_i$ are mapped to these discs and Moebius strips in a more complicated way to be described in detail in the forthcoming work [19].

3.3. Cut time

Theorem 3.2. For any $q_1 \in \tilde{M}$, let $(\lambda_1, t_1) = \text{Exp}^{-1}(q_1) \in \tilde{N}$. Then the extremal trajectory $q(s) = \text{Exp}(\lambda_1, s)$ is optimal with $q(t_1) = q_1$.

Thus optimal synthesis on the domain $\tilde{M}$ is given by

$$u_i(q) = h_i(\lambda), \quad i = 1, 2, \quad (\lambda, t) = \text{Exp}^{-1}(q) \in \tilde{N}, \quad q \in \tilde{M}.$$ 

Proof. Let $q_1 \in \tilde{M}$. There exists $\nu_1 = (\lambda_1, t_1) \in \tilde{N} = \tilde{N} \cup N'$ such that the trajectory $q(s) = \text{Exp}(\lambda_1, s)$ is optimal and $q(t_1) = \text{Exp}(\nu_1) = q_1$. By Lemmas 3.5 and 3.6, we have $\nu_1 \in \tilde{N}$. By Corollary 3.1, there exists a unique $\nu_1 \in \tilde{N}$ such that $\text{Exp}(\nu_1) = q_1$. So $q(s) = \text{Exp}(\lambda_1, s)$ is a unique optimal trajectory coming to $q_1$. □

In work [9] we proved the inequality $t_{\text{cut}}(\lambda) \leq t(\lambda)$. Now we prove the corresponding equality.

Theorem 3.3. For any $\lambda \in C$ we have $t_{\text{cut}}(\lambda) = t(\lambda)$.

Proof. We proved the inequality $t_{\text{cut}}(\lambda) \leq t(\lambda)$ in Theorem 5.4 [9].

(1) Consider first the generic case $\lambda_1 \in \cup_{i=1}^{5} C_i$. There exists $t_1 \in (0, t(\lambda_1))$ and arbitrarily close to $t(\lambda_1)$ such that $\text{sn} \tau_1 \cap \tau_1 \neq 0$. Then $\nu_1 = (\lambda_1, t_1) \in \tilde{N}$, thus $q_1 = \text{Exp}(\nu_1) \in \tilde{M}$. By Theorem 3.2, the trajectory $q(s) = \text{Exp}(\lambda_1, s), s \in [0, t_1]$, is optimal, thus $t_1 \leq t_{\text{cut}}(\lambda_1)$. So there exists $t_1 \in (0, t(\lambda_1))$ arbitrarily close to $t(\lambda_1)$ such that $t_1 \leq t_{\text{cut}}(\lambda_1)$. Consequently, $t(\lambda_1) \leq t_{\text{cut}}(\lambda_1)$. We proved that $t_{\text{cut}}(\lambda_1) = t(\lambda_1)$ for any $\lambda_1 \in \cup_{i=1}^{5} C_i$.

(2) If $\lambda \in C_4$, then the extremal trajectory $(x, y, \theta) = (0, 0, \pm t)$ is a Riemannian geodesic for the restriction of the sub-Riemannian problem on SE(2) to the circle $\{(0, 0, \theta) \mid \theta \in S^1\}$. It is optimal up to the antipodal point, thus $t_{\text{cut}}(\lambda) = \pi = t(\lambda)$.

(3) In the case $\lambda \in C_5$, the extremal trajectory is a line $(x, y, \theta) = (\pm t, 0, 0)$, thus it is optimal forever: $t_{\text{cut}}(\lambda) = +\infty = t(\lambda)$. □
4. PLOTS OF SUB-RIEMANNIAN CAUSTIC AND SPHERES

Here we collect 3-dimensional plots of some essential objects in the sub-Riemannian problem on SE(2). Figure 11 shows the sub-Riemannian caustic

$$\{\text{Exp}(\lambda, t) \mid \lambda \in C, \ t = t_{1}^{\text{con}}(\lambda)\}$$

in the rectifying coordinates \((R_1, R_2, \theta)\).

Figures 12–20 present sub-Riemannian spheres

$$S_R = \{q \in M \mid d(q_0, q) = R\} = \{\text{Exp}(\lambda, R) \mid \lambda \in C, \ t_{\text{cut}}(\lambda) \geq R\}$$

of different radii \(R\), where

$$d(q_0, q_1) = \inf \{l(q(\cdot)) \mid q(\cdot) \text{ trajectory of (1.1), } q(0) = q_0, \ q(t_1) = q_1\}$$

is the sub-Riemannian distance – the cost function in the sub-Riemannian problem (1.1)–(1.5).

In this problem sub-Riemannian spheres can be of three different topological classes. If \(R \in (0, \pi)\), then \(S_R\) is homeomorphic to the standard 2-dimensional Euclidean sphere \(S^2\), see Figure 14. For \(R = \pi\) the sphere \(S_R\) is homeomorphic to the sphere \(S^2\) with its north pole \(N\) and south pole \(S\) identified: \(S_\pi \cong S^2/\{N = S\}\), see Figure 17. And if \(R > \pi\), then \(S_R\) is homeomorphic to the 2-dimensional torus, see Figure 20.

Figure 12 shows the sphere \(S_{\pi/2}\) in rectifying coordinates \((R_1, R_2, \theta)\). Figure 13 represents the same sphere with a cut-out opening the singularities of the sphere: the sphere intersects itself at the local components of the cut locus \(\text{Cut}_{loc}\) described in [19]. Figure 14 shows embedding of the same sphere to the solid torus. Sub-Riemannian spheres of small radii resemble the well-known apple-shaped sub-Riemannian sphere in the Heisenberg group [21]. Although, there is a major difference: the sphere in the Heisenberg group has a one-parameter family of symmetries (rotations), but the sphere in SE(2) has only a discrete group of symmetries \(G = \{\text{Id}, \varepsilon^1, \ldots, \varepsilon^7\}\) (reflections).
Figures 12–17 represent similarly the sub-Riemannian sphere of the critical radius $\pi$. In addition to self-intersections at the local component $\text{Cut}_{\text{loc}}$, the sphere $S_{\pi}$ has one self-intersection point at the global component $\text{Cut}_{\text{glob}}$.

Figures 18–20 show similar images of the sphere of radius $3\pi/2$. For $R > \pi$ the sphere $S_{R}$ has two topological segments of self-intersection points at $\text{Cut}_{\text{loc}}^{\pm}$ respectively, and a topological circle of self-intersection points at $\text{Cut}_{\text{glob}}$.

Figures 21 and 22 show intersections of the spheres $S_{\pi/2}$, $S_{\pi}$, $S_{3\pi/2}$ with the half-spaces $\theta < 0$ and $R_{2} > 0$ respectively.
Figure 15. Sub-Riemannian sphere $S_\pi$.

Figure 16. Sub-Riemannian sphere $S_\pi$ with cut-out.

Figure 17. Sub-Riemannian sphere $S_\pi$, global view.

Figure 23 shows self-intersections of the wavefront

$$W_R = \{\text{Exp}(\lambda, R) \mid \lambda \in C\}$$

for $R = \pi$.

5. Conclusion

The solution to the sub-Riemannian problem on SE(2) obtained in the previous paper [9], this one, and the subsequent paper [19] is based on a detailed study of the action of the discrete group of symmetries of the exponential mapping. This techniques was already partially developed in the study of related optimal
control problems (nilpotent sub-Riemannian problem with the growth vector (2, 3, 5) [13–16] and Euler’s elastic problem [17,18]). The sub-Riemannian problem on SE(2) is the first problem in this series, where a complete solution was obtained (local and global optimality, cut time and cut locus, optimal synthesis). We believe that our approach based on the study of symmetries will provide such complete results in other symmetric invariant problems, such as the nilpotent sub-Riemannian problem with the growth vector (2,3,5), the ball-plate problem [7], and others.
Figure 21. Sub-Riemannian hemi-spheres $S_{\pi/2}$, $S_{\pi}$, $S_{3\pi/2}$ for $\theta < 0$.

Figure 22. Sub-Riemannian hemi-spheres $S_{\pi/2}$, $S_{\pi}$, $S_{3\pi/2}$ for $R_2 > 0$.

Figure 23. Wavefront $W_r$ with cut-out.
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REFERENCES