LIPSCHITZ REGULARITY FOR SOME ASYMPTOTICALLY
CONVEX PROBLEMS

LARS DIENING\(^1\), BIANCA STROFFOLINI\(^2\) AND ANNA VERDE\(^2\)

Abstract. We establish a local Lipschitz regularity result for local minimizers of asymptotically convex variational integrals.

Mathematics Subject Classification. 35B65, 35J70.

Received October 3rd, 2008.
Published online December 4, 2009.

1. Introduction

We consider local minimizers of variational integrals of the type
\[ F(u) = \int_\Omega f(\nabla u) \, dx, \tag{1.1} \]
where \( \Omega \) is a bounded, open subset of \( \mathbb{R}^n \), \( u : \Omega \to \mathbb{R}^N \) is a vector valued function and \( \nabla u \) stands for the total derivative of \( u \). A function \( u \in W^{1,p}(\Omega) \) is a local minimizer of \( F(u) \) if \( F(u) \leq F(u + \eta) \), for every test function \( \eta \in W^{1,p}_0(\Omega) \) with compact support in \( \Omega \).

In 1977 Uhlenbeck (see [26]) proved everywhere \( C^{1,\alpha} \) regularity for local minimizers of functional when the integrand \( f \in C^2 \) is assumed to behave like \( |\xi|^p \), with \( p \geq 2 \); Acerbi and Fusco considered the case \( 1 < p < 2 \). Later on a large number of generalizations have been made, see for example the survey [22].

For the \( (p, q) \) case and the general growth case, see the papers of Marcellini [18–21] and [6,7].

Another direction of research is the one arising in the model of electro-rheological fluids [2,3].

For the Lipschitz regularity, the results are available when \( f \in C^2 \) is asymptotically, in a \( C^2 \)-sense, quadratic or super-quadratic at infinity (see [4] for the case \( p = 2 \) and [15,24] for the case \( p > 2 \); for the subquadratic case see [17]). For related results, see [11–14,23].

Keywords and phrases. Local minimizers, decay estimates, asymptotic behaviour.

\(^*\) The work of B.S. was supported by PRIN 2007 Project: “Calcolo delle Variazioni e Teoria Geometrica della Misura”; the work of A.V. was supported in part by Prin 2007 Project “Calcolo delle Variazioni e Teoria Geometrica della Misura” and in part by the European Research Council under FP7 Advanced Grant n° 226234: “Analytic Techniques for Geometric and Functional Inequalities”.

\(^1\) Institute of Mathematics, Eckerstr. 1, 79104 Freiburg, Germany. diening@mathematik.uni-freiburg.de

\(^2\) Dipartimento di Matematica, Università di Napoli, Federico II, Via Cintia, 80126 Napoli, Italy.

bstroffo@unina.it; anverde@unina.it

Article published by EDP Sciences © EDP Sciences, SMAI 2009
The argument of such results is the following: if the gradients of minimizers are very large, the problem becomes “regular” and so good estimates are known.

Moreover, Dolzmann and Kristensen [10] have proved local higher integrability with large exponents of minimizers when \( f \in C^0 \) approaches at infinity, in a \( C^0 \)-sense, the \( p \)-Dirichlet integrand, for some arbitrary \( p > 1 \), see also [16].

In a recent paper Diening and Ettwein [8] considered fractional estimates for non-differentiable systems with \( \varphi \)-growth. Using some of their techniques, we were able to prove in [9] excess decay estimates for vectorial functionals with \( \varphi \)-growth. In this paper we extend the results found in [4,15,17,24] to the case of a convex function satisfying the \( \Delta_2 \)-condition with its conjugate (\( \Delta_2(\{\varphi, \varphi^*\}) < \infty \)), see Section 2 for the definitions.

More precisely we have the following theorem:

**Theorem 1.1.** Let \( \varphi \) be an \( N \)-function such that

1. \( \varphi \in C^2((0, \infty)) \cap C([0, \infty)) \) and \( \varphi \in \Delta_2(\{\varphi, \varphi^*\}) \).
2. \( \Delta_2(\{\varphi, \varphi^*\}) < \infty \).
3. \( \varphi'(t) \sim t \varphi''(t) \).
4. there exists \( \beta \in (0, 1] \) and \( c > 0 \) such that
   
   \[ |\varphi''(s + t) - \varphi''(t)| \leq c_1 \varphi''(t) \left( \frac{|s|}{t} \right)^\beta \]

for all \( t > 0 \) and \( s \in \mathbb{R} \) with \( |s| < \frac{1}{2} t \).

Moreover let \( f : \mathbb{R}^{n \times N} \rightarrow \mathbb{R} \) be such that

1. \( f \in C^2(\mathbb{R}^{n \times N}) \);
2. there exists \( L > 0 \) such that for all \( \xi \in \mathbb{R}^{n \times N} \setminus \{0\} \)
   
   \[ |\nabla^2 f(\xi)| \leq L \varphi''(|\xi|); \quad (1.2) \]
3. there holds\(^3\)
   
   \[ \lim_{|\xi| \rightarrow \infty} \frac{|\nabla^2 f(\xi) - \nabla^2 \varphi(\xi)|}{\varphi''(|\xi|)} = 0. \quad (1.3) \]

If \( u \in W^{1,\varphi}(\Omega) \) is a local minimizer of the functional \( \mathcal{F} \), see (1.1), then \( \nabla u \) is locally bounded in \( \Omega \). Moreover, for every ball \( B \subset \Omega \) we have

\[ \text{esssup}_{\in B} \varphi(|\nabla u|) \leq c \left( 1 + \int_{B} \varphi(|\nabla u|) \, dx \right), \quad (1.4) \]

where \( c \) depends only on \( n, N, L, \Delta_2(\{\varphi, \varphi^*\}), c_1, \beta \), and the convergence in (1.3).

Let us point out that in the power case, with \( 1 < p < 2 \) [17], the authors considered the asymptotic behaviour like \( (\mu + t^2)^{\frac{2}{p}} \), \( \mu > 0 \). Here we are able to recover also the case \( \mu = 0 \).

\(^3\)We use that \( \varphi \) can also be interpreted as a function from \( \mathbb{R}^{n \times N} \) to \( \mathbb{R}^n \) by \( \varphi(\xi) := \varphi(|\xi|) \).
2. TECHNICAL LEMMAS

In the sequel Ω will denote a bounded, open set of \( \mathbb{R}^n \). To simplify the notation, the letter \( c \) will denote any positive constant, which may vary throughout the paper. For \( w \in L^1_{\text{loc}}(\mathbb{R}^n) \) and a ball \( B \subset \mathbb{R}^n \) we define

\[
\langle w \rangle_B := \frac{1}{|B|} \int_B w(x) \, dx,
\]

(2.1)

where \( |B| \) is the \( n \)-dimensional Lebesgue measure of \( B \). For \( \lambda > 0 \) we denote by \( \lambda B \) the ball with the center as \( B \) but \( \lambda \)-times the radius. We write \( B_r(x) \) for the ball with radius \( R \) and center \( x \). For \( U, \Omega \subset \mathbb{R}^n \) we write \( U \Subset \Omega \) if the closure of \( U \) is a compact subset of \( \Omega \). We define \( \delta_{i,j} := 0 \) for \( i \neq j \) and \( \delta_{i,i} = 1 \).

The following definitions and results are standard in the context of \( \nu \)-functions. A real function \( \varphi : [0,\infty) \to [0,\infty) \) is a Banach space. By \( W^{1,\varphi} \) where the constants only depend on \( \Delta \).

For \( \phi \in \mathcal{L}^1(\Omega) \) we define the \( \lambda \)-dimensional Hausdorff measure of \( \phi \).

The following definitions and results are standard in the context of \( \nu \)-functions. A real function \( \varphi : [0,\infty) \to [0,\infty) \) is again an \( \nu \)-function and \( (\varphi) = \text{sup} \{ \varphi((t)) \} \) is the inverse function of \( \varphi \). If \( \varphi \) is strictly increasing then \( (\varphi)^{-1} \) is the inverse function of \( \varphi^\prime \). Then \( \varphi^* : \mathbb{R}^n \to \mathbb{R}^n \) with

\[
\varphi^*(t) := \int_0^t (\varphi^\prime)^{-1}(s) \, ds
\]

is again an \( \nu \)-function and \( (\varphi^*)^\prime(t) = (\varphi^\prime)^{-1}(t) \) for \( t > 0 \). It is the complementary function of \( \varphi \). Note that \( \varphi^*(t) = \text{sup}_{s>0}(st - \varphi(s)) \) and \( (\varphi^*)^* = \varphi \). For all \( \delta > 0 \) there exists \( c_\delta \) (only depending on \( \Delta_2(\{\varphi, \varphi^*\}) \)) such that for all \( t, s \geq 0 \) holds

\[
t s \leq \delta \varphi(t) + c_\delta \varphi^*(s).
\]

(2.2)

For \( \delta = 1 \) we have \( c_1 = 1 \). This inequality is called Young’s inequality. For all \( t \geq 0 \)

\[
\frac{t}{2} \varphi\left(\frac{t}{2}\right) \leq \varphi(t) \leq t \varphi^\prime(t),
\]

(2.3)

\[
\varphi\left(\frac{\varphi^*(t)}{t}\right) \leq \varphi^*(t) \leq \varphi\left(\frac{2 \varphi^*(t)}{t}\right).
\]

Therefore, uniformly in \( t \geq 0 \)

\[
\varphi(t) \sim \varphi^\prime(t) t, \quad \varphi^*(\varphi^\prime(t)) \sim \varphi(t),
\]

(2.4)

where the constants only depend on \( \Delta_2(\{\varphi, \varphi^*\}) \).

We define the \textit{shifted \( \nu \)-function} \( \varphi_\alpha : \mathbb{R}^n \to \mathbb{R}^n \) by

\[
\varphi_\alpha(t) = \int_0^t \varphi^\prime_a(s) \, ds \quad \text{where} \quad \varphi^\prime_a(t) = \frac{\varphi^\prime(a + t) - \varphi^\prime(a)}{a + t} t.
\]

(2.5)
The shifted N-functions have been introduced in [8]. See [25] for a detailed study of the shifted N-functions. The function \( \varphi_a \) and its dual \( \varphi_a^* \) are again N-functions and satisfy the \( \Delta_2 \)-condition uniformly in \( a \geq 0 \). In particular, \( \Delta_2((\varphi_a, (\varphi_a^*)_a)_{a \geq 0}) < \infty \). For given \( \varphi \) we define the N-function \( \psi \) by

\[
\psi(t) := \left( \varphi'(t) \right)^{\gamma}.
\]

(2.6)

It is shown in [8] that \( \psi \) also satisfies (H2)–(H3) and uniformly in \( t > 0 \) holds \( \psi''(t) \sim \sqrt[\gamma]{\varphi''(t)} \). We define the function \( V(Q) \):

\[
V(Q) := \frac{\psi(|Q|)}{|Q|}. \]

The following lemma can be found in [1].

**Lemma 2.1.** Let \( \alpha > -1 \) then uniformly in \( \xi_0, \xi_1 \in \mathbb{R}^{n \times N} \) with \( |\xi_0| + |\xi_1| > 0 \) holds

\[
\left( |\xi_0| + |\xi_1| \right)^{\alpha} \sim \int_0^1 |\xi_0|^1 d\theta,
\]

(2.7)

where \( \xi_\theta := (1 - \theta)\xi_0 + \theta\xi_1 \).

Moreover, we need the following generalization of Lemma 2.1.

**Lemma 2.2** ([8], Lem. 20). Let \( \varphi \) be an N-function with \( \Delta_2((\varphi, \varphi^*)_\gamma) < \infty \). Then uniformly for all \( \xi_0, \xi_1 \in \mathbb{R}^{n \times N} \) with \( |\xi_0| + |\xi_1| > 0 \) holds

\[
\int_0^1 \frac{\varphi(|\xi_0|)}{|\xi_0|} d\theta \sim \frac{\varphi'(|\xi_0| + |\xi_1|)}{|\xi_0| + |\xi_1|},
\]

(2.8)

where \( \xi_\theta := (1 - \theta)\xi_0 + \theta\xi_1 \). The constants only depend on \( \Delta_2((\varphi, \varphi^*)_\gamma) \).

**Remark 2.3.** Let \( \varphi \) be an N-function with \( \Delta_2((\varphi, \varphi^*)_\gamma) < \infty \). Then it has been shown in [8], p. 546, and [25], Lemma 5.19, that there exists \( 0 < \gamma < 1 \) and and N-function \( \rho \) with \( \Delta_2((\rho, \rho^*)_\gamma) < \infty \) such that \( (\varphi(t))^{\gamma} \sim \rho(t) \) uniformly in \( t \geq 0 \). It is important to remark that \( \gamma \) and \( \Delta_2((\rho, \rho^*)_\gamma) \) only depend on \( \Delta_2((\varphi, \varphi^*)_\gamma) \). Note that \( \varphi(t) \sim t\varphi'(t), \varphi(t) \sim (\rho(t))^{1/\gamma}, \) and \( \rho(t) \sim t\rho'(t) \) imply \( \varphi(t) \sim (\rho(t))^{1/\gamma} \).

The next Lemma contains useful properties of the function \( V \) (see [8], Lem. 3, or [9,25]).

**Lemma 2.4.** For every \( \xi_0, \xi_1 \in \mathbb{R}^{n \times N} \) with \( |\xi_0| + |\xi_1| > 0 \) holds

\[
|V(\xi_0) - V(\xi_1)|^2 \sim |\xi_0 - \xi_1|^2 \varphi''(|\xi_0| + |\xi_1|)
\]

\[
|V(\xi_0)|^2 \sim \varphi(|\xi_0|).
\]

(2.9)

### 3. Proof of the Main Result

We need two lemmas that measures the differences of \( f \) and \( \varphi \) in a \( C^2 \) sense. The first lemma is a rough estimate using only the upper estimates for \( \nabla^2 f \) and \( \nabla^2 \varphi \). The second lemma is more subtle using that \( \nabla^2 f \) and \( \nabla^2 \varphi \) are close for large arguments. It is the analogue of Lemma 5.1 in [15] and Lemma 2.4 in [17].

**Lemma 3.1.** Let \( \varphi \) satisfy (H1)–(H4) and \( f \) satisfy (F1)–(F3). Then there exists \( c > 0 \) such that for all \( \xi_0, \xi_1 \in \mathbb{R}^{n \times N} \) holds

\[
\int_0^1 \|\nabla^2 f(\xi_0) - \nabla^2 \varphi(\xi_0)\|d\theta |\xi_1 - \xi_0|^2 \leq c |V(\xi_1) - V(\xi_0)|^2,
\]

(3.1)

where \( \xi_\theta := (1 - \theta)\xi_0 + \theta\xi_1 \). Note that \( c \) depends only on \( n, N, L, \) and \( \Delta_2((\varphi, \varphi^*)_\gamma) \).
Proof. Due to (1.2), Lemmas 2.2 and 2.4 we estimate

\[
\int_0^1 |[\nabla^2 f(\xi_\theta) - \nabla^2 \varphi(\xi_\theta)]| d\theta |\xi_1 - \xi_0|^2 \leq (L + 1) \int_0^1 \varphi''(\xi_\theta) d\theta |\xi_1 - \xi_0|^2 \\
\leq c(L + 1)\varphi''(|\xi_0| + |\xi_1|) |\xi_1 - \xi_0|^2 \\
\leq c(L + 1)|V(\xi_1) - V(\xi_0)|^2.
\]

This proves the assertion. □

Lemma 3.2. Let \( \varphi \) satisfy (H1)--(H4) and \( f \) satisfy (F1)--(F3). Then for every \( \varepsilon > 0 \) there exist \( \sigma(\varepsilon) > 0 \) such that for all \( \xi_0, \xi_1 \in \mathbb{R}^{n \times N} \) with \( \max\{|\xi_0|, |\xi_1|\} \geq \sigma(\varepsilon) \) holds

\[
\int_0^1 |[\nabla^2 f(\xi_\theta) - \nabla^2 \varphi(\xi_\theta)]| d\theta |\xi_1 - \xi_0|^2 \leq \varepsilon |V(\xi_1) - V(\xi_0)|^2, \tag{3.2}
\]

where \( \xi_\theta = (1 - \theta)\xi_0 + \theta \xi_1. \) Note that \( \sigma \) depends only on \( \varepsilon, n, N, \Delta_2(\{\varphi, \varphi^*\}) \), and the limit in (1.3).

Proof. Fix \( \varepsilon > 0. \) In the following let \( \delta = \delta(\varepsilon) > 0. \) The precise value of \( \delta \) will be chosen later. Due to (1.3) there exists \( \Lambda(\delta) > 0 \) such that

\[
|\nabla^2 f(\xi) - \nabla^2 \varphi(\xi)| \leq \delta \varphi''(|\xi|) \tag{3.3}
\]

for all \( \xi \in \mathbb{R}^{n \times N} \) with \( |\xi| \geq \Lambda(\delta). \)

Let \( \sigma(\varepsilon) := K \Lambda(\delta) \) with \( K \geq 2, \) where the precise value of \( K \) will be chosen later. Let \( \xi_0, \xi_1 \in \mathbb{R}^{n \times N} \) with \( \max\{|\xi_0|, |\xi_1|\} \geq \sigma(\varepsilon). \) By symmetry we can assume without loss of generality \( |\xi_1| \geq \sigma(\varepsilon). \) For \( \theta \in (0, 1) \) define \( \xi_\theta := (1 - \theta)\xi_0 + \theta \xi_1. \) We split the domain of integration on the left hand side of (3.2) into \( I^\prec := \{\theta \in [0, 1] : |\xi_\theta| \leq \Lambda(\delta)\} \) and \( I^\succ = \{\theta \in [0, 1] : |\xi_\theta| > \Lambda(\delta)\}. \) Thanks to (3.3), Lemmas 2.2 and 2.4 we get

\[
(I) := \int_{I^\succ} |[\nabla^2 f(\xi_\theta) - \nabla^2 \varphi(\xi_\theta)]| d\theta |\xi_1 - \xi_0|^2 \\
\leq 2\varepsilon \varphi''(|\xi_0| + |\xi_1|) |\xi_1 - \xi_0|^2 \\
\leq 2c\delta |V(\xi_1) - V(\xi_0)|^2.
\]

If we choose \( \delta > 0 \) small enough, then

\[
(I) \leq \frac{\varepsilon}{2}|V(\xi_1) - V(\xi_0)|^2.
\]

Assumptions (F2) and (H3) yield

\[
(II) := \int_{I^\succ} |[\nabla^2 f(\xi_\theta) - \nabla^2 \varphi(\xi_\theta)]| d\theta |\xi_1 - \xi_0|^2 \\
\leq c(L + 1) \int_{I^\succ} \frac{\varphi''(|\xi_\theta|)}{|\xi_\theta|} d\theta |\xi_1 - \xi_0|^2.
\]

Due to Remark 2.3 there exists \( 0 < \gamma < 1 \) and an N-function \( \rho \) with \( \Delta_2(\{\rho, \rho^*\}) < \infty \) such that \( (\varphi(t))^{\gamma} \sim \rho(t) \) uniformly in \( t \geq 0. \) Since \( 1/\gamma - 2 > -1 \) we can find \( \alpha > 1 \) such that \( \alpha'(1/\gamma - 2) > -1, \) where \( 1 = \frac{1}{\alpha} + \frac{1}{\alpha'}. \)
With the previous estimate, Hölder’s inequality, and \( \varphi' (t) \sim (\rho'(t))^{1/\gamma} t^{1/\gamma - 1} \) we get
\[
(II) \leq c (L + 1)|I^\leq|^{\frac{1}{\gamma}} \left( \int_0^1 \frac{\varphi'(|\xi_0|)^{\alpha'}}{|\xi_0|^{\alpha'}} d\theta \right) |\xi_1 - \xi_0|^2
\]
\[
\leq c (L + 1)|I^\leq|^{\frac{1}{\gamma}} \left( \int_0^1 \frac{(\rho'(|\xi_0|))^{\alpha'/\gamma}}{|\xi_0|^{\alpha' (1/\gamma - 2)/\gamma}} d\theta \right) |\xi_1 - \xi_0|^2.
\]

Using that \( \rho'(|\xi_0|) \leq \rho'(|\xi_0| + |\xi_1|) \), Lemma 2.1, \( \varphi'(t) \sim (\rho'(t))^{1/\gamma} t^{1/\gamma - 1} \), and Lemma 2.4 we get
\[
(II) \leq c (L + 1)|I^\leq|^{\frac{1}{\gamma}} \left( \frac{\rho'(|\xi_0| + |\xi_1|))^{1/\gamma}}{|\xi_0| + |\xi_1|^{1/\gamma - 2}} |\xi_1 - \xi_0|^2
\]
\[
\leq c (L + 1)|I^\leq|^{\frac{1}{\gamma}} \left( \frac{2|\xi_1 - \xi_0|^2}{\xi_1 - \xi_0} \right).
\]

Let us now estimate \( |I^\leq| \). Recall that \( |\xi_0| \geq \sigma (\varepsilon) = K \Lambda (\delta) \). If \( |\xi_1 - \xi_0| \geq (K - 1) \Lambda (\delta) \), then
\[
|I^\leq| \leq \frac{2 \Lambda (\delta)}{|\xi_1 - \xi_0|} \leq \frac{2}{K - 1} \quad (3.4)
\]
If on the other hand \( |\xi_1 - \xi_0| < (K - 1) \Lambda (\delta) \), then \( |I^\leq| = 0 \). Thus, (3.4) holds in both cases. It follows that
\[
(II) \leq c (L + 1) \left( \frac{2}{K - 1} \right)^{\frac{1}{\gamma}} |V(\xi_1) - V(\xi_0)|^2.
\]

If we choose \( K \geq 2 \) large enough, then
\[
(II) \leq \frac{\varepsilon}{2} |V(\xi_1) - V(\xi_0)|^2.
\]

Combining the estimates for (I) and (II) we get the claim. \( \square \)

We define the functional \( \mathcal{F}_\varphi : W^{1,\varphi}(\Omega) \to \mathbb{R} \) by
\[
\mathcal{F}_\varphi (u) := \int_\Omega \varphi (|V u|) dx. \quad (3.5)
\]

**Lemma 3.3** (comparison estimate). Let \( \varphi, f, \) and \( u \) be as in Theorem 1.1. Then for every \( \varepsilon > 0 \) there exists \( \kappa (\varepsilon) > 0 \) such that the following holds: If \( B \) be a ball with \( B \subset \Omega \) and \( v \) is the local minimizer of the functional \( \mathcal{F}_\varphi \), see (3.5), satisfying \( v - u \in W^{1,\varphi}_0 (B) \), then
\[
\frac{1}{2} |V(\xi_1)|^2 dx \leq \kappa (\varepsilon) \quad (3.6)
\]

or
\[
\frac{1}{2} |V(\xi_1) - V(\xi_0)|^2 dx \leq \varepsilon \int_B |V(\xi_1) - V(\xi_0)|^2 dx. \quad (3.7)
\]

Note that \( \kappa (\varepsilon) \) and \( \gamma (\varepsilon) \) depend only on \( \varepsilon, n, N, L, \Delta_2 (\{ \varphi, \varphi^\gamma \}) \), and the convergence in (1.3).
Proof. In the following let $B$ always be a ball and let $v$ be the local minimizer of the functional $F_\varphi$, see (3.5), satisfying $v - u \in W_0^{1,\varphi}(B)$. Since $V$ is surjective we can choose $\xi_0 \in \mathbb{R}^{n \times N}$ such that $V(\xi_0) = \langle V(\nabla u) \rangle_B$. Let $\sigma$ be as in Lemma 3.2.

We start the proof with an auxiliary result.

Claim. There holds
\[
\int_B |V(\nabla u) - V(\nabla v)|^2 \, dx \leq \frac{\varepsilon}{2} \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 \, dx + \Gamma_B,
\]
(3.8)
where $\Gamma_B := 0$ if $|\xi_0| \geq \sigma(\varepsilon/16)$ and $\Gamma_B := 2\varepsilon \varphi(c \sigma(\varepsilon/16))$ if $|\xi_0| < \sigma(\varepsilon/16)$. The constant $c$ depends only on $n, N, L$, and $\Delta_2(\varphi, \varphi^*)$.

Define $g : \mathbb{R}^{n \times N} \to \mathbb{R}$ by
\[
g(\xi) = \varphi(\xi) + f(\xi_0) - \varphi(\xi_0) + [\nabla f(\xi_0) - \nabla \varphi(\xi_0)](\xi - \xi_0).
\]
It is easy to see that $v$ is also a local minimizer of
\[
\int_B g(\nabla v) \, dx.
\]
(3.9)
such that $v - u \in W_0^{1,\varphi}(B)$. The Euler equation for (3.9) and the ellipticity of $g$ yield
\[
\int_B [g(\nabla u) - g(\nabla v)] \, dx = \int_B \left( \int_0^1 (1 - \theta) \nabla^2 g((1 - \theta)\nabla v + \theta \nabla u) \, d\theta \right) \nabla(\nabla u - \nabla v) : (\nabla u - \nabla v) \, dx
\]
\[
\geq c \int_B \int_0^1 (1 - \theta) \varphi''(|(1 - \theta)\nabla v + \theta \nabla u|) \, d\theta \, |\nabla u - \nabla v|^2 \, dx.
\]
Now with $\varphi''(t) \sim \varphi'(t)$, Lemmas 2.2 and 2.4 it follows
\[
\int_B [g(\nabla u) - g(\nabla v)] \, dx \geq c \int_B \varphi''(|\nabla u| + |\nabla v|) |\nabla u - \nabla v|^2 \, dx
\]
\[
\geq c \int_B |V(\nabla u) - V(\nabla v)|^2 \, dx.
\]
(3.10)
Now, since $u$ is a local minimizer for $F$, $u - v \in W_0^{1,\varphi}(B)$, and $B \subset \Omega$ it follows that
\[
\int_B [g(\nabla u) - g(\nabla v)] \, dx = \int_B [g(\nabla u) - f(\nabla u)] \, dx + \int_B [f(\nabla u) - f(\nabla v)] \, dx
\]
\[
+ \int_B [f(\nabla v) - g(\nabla v)] \, dx
\]
\[
\leq \int_B [g(\nabla u) - f(\nabla u)] \, dx + \int_B [f(\nabla v) - g(\nabla v)] \, dx =: (I).
\]
Observe that for every $\xi_1 \in \mathbb{R}^{n \times N}$ holds
\[
f(\xi_1) - g(\xi_1) = \left\langle \int_0^1 (1 - \theta)[\nabla^2 f(\xi_0) - \nabla^2 g(\xi_0)] \, d\theta \, (\xi_1 - \xi_0), (\xi_1 - \xi_0) \right\rangle,
\]
(3.11)
where $\xi_\theta := (1 - \theta)\xi_0 + \theta \xi_1$. 

If \(|\xi_0| \geq \sigma(\varepsilon/16)\), then it follows from (3.11) and Lemma 3.2 that

\[
(I) \leq \frac{\varepsilon}{16} \left( \int_B |V(\nabla u) - V(\xi_0)|^2 \, dx + \int_B |V(\nabla v) - V(\xi_0)|^2 \, dx \right).
\]

If on the other hand \(|\xi_0| \leq \sigma(\varepsilon/16)\), then it follows from (3.11), Lemmas 3.2 and 3.1, and (2.9) that

\[
(I) \leq \frac{\varepsilon}{16} \left( \int_B \chi_{\{|V(\nabla u)| \geq \sigma(\varepsilon/16)\}} |V(\nabla u) - V(\xi_0)|^2 \, dx + \int_B \chi_{\{|V(\nabla v)| \geq \sigma(\varepsilon/16)\}} |V(\nabla v) - V(\xi_0)|^2 \, dx \right)
+ c \left( \int_B \chi_{\{|V(\nabla u)| < \sigma(\varepsilon/16)\}} |V(\nabla u) - V(\xi_0)|^2 \, dx + \int_B \chi_{\{|V(\nabla v)| < \sigma(\varepsilon/16)\}} |V(\nabla v) - V(\xi_0)|^2 \, dx \right)
\leq \frac{\varepsilon}{16} \left( \int_B |V(\nabla u) - V(\xi_0)|^2 \, dx + \int_B |V(\nabla v) - V(\xi_0)|^2 \, dx \right) + c\varphi\left(c\sigma(\varepsilon/16)\right).
\]

This and the previous estimate prove

\[
\int_B |V(\nabla u) - V(\nabla v)|^2 \, dx \leq \frac{\varepsilon}{16} \left( \int_B |V(\nabla u) - V(\xi_0)|^2 \, dx + \int_B |V(\nabla v) - V(\xi_0)|^2 \, dx \right) + \frac{1}{2} \Gamma_B,
\]

where \(\Gamma_B := 0\) if \(|\xi_0| \geq \sigma(\varepsilon/16)\) and \(\Gamma_B := 2c\varphi(c\sigma(\varepsilon/16))\) if \(|\xi_0| < \sigma(\varepsilon/16)\). We estimate by adding and subtracting \(V(\nabla v)\) in the second integrand

\[
\frac{\varepsilon}{16} \left( \int_B |V(\nabla u) - V(\nabla u)_B|^2 \, dx + \int_B |V(\nabla v) - V(\nabla u)_B|^2 \, dx \right)
\leq \frac{\varepsilon}{4} \left( \int_B |V(\nabla u) - V(\nabla u)_B|^2 \, dx + \int_B |V(\nabla v) - V(\nabla u)|^2 \, dx \right).
\]

This and the previous estimate shows

\[
\int_B |V(\nabla u) - V(\nabla v)|^2 \, dx \leq \frac{\varepsilon}{2} \left( \int_B |V(\nabla u) - V(\xi_0)|^2 \, dx \right) + \Gamma_B,
\]

which proves the auxiliary result (3.8).

Let us now prove the claim of the lemma. If \(|\xi_0| \geq \sigma(\varepsilon/16)\), then the claim follows from (3.8), since in this case \(\Gamma_B = 0\). So let us assume in the following that \(|\xi_0| < \sigma(\varepsilon/16)\), which implies \(\Gamma_B = 2c\varphi(c\sigma(\varepsilon/16))\). If

\[
\Gamma_B \leq \frac{\varepsilon}{2} \int_B |V(\nabla u) - V(\nabla u)_B|^2 \, dx,
\]

then the claim follows again from (3.8). So we can assume in the following that

\[
\int_B |V(\nabla u) - V(\nabla u)_B|^2 \, dx \leq 2\Gamma_B / \varepsilon.
\]
This, \(|\xi| \leq \sigma(\varepsilon/16)|\), and (2.9) imply

\[
\int_B |\nabla u|^2 \, dx \leq 2 \int_B |\nabla u - (\nabla u)|_B|^2 \, dx + 2 |\nabla \zeta_0|^2 \leq 4 \Gamma_B + c \sigma(\varepsilon/16) =: \kappa(\varepsilon).
\]

This proves the lemma. \(\square\)

The following result on the decay of the excess functional for local minimizers can be found in [9], Theorem 6.4.

**Proposition 3.4** (decay estimate for \(v\)). Let \(\varphi\) satisfy (H1)–(H4), let \(B \subset \Omega\) be a ball, and let \(v\) be the local minimizer of the functional \(F_\varphi\), see (3.5), satisfying \(v - u \in W_0^{1,q}(B)\). Then there exists \(\beta > 0\) and \(c > 0\) such that for every ball \(B \subset \Omega\) and every \(\lambda \in (0, 1)\) holds

\[
\int_{\lambda B} |\nabla v - (\nabla v)|_B|^2 \, dx \leq c \lambda^\beta \int_B |\nabla v - (\nabla v)|_B|^2 \, dx.
\]

Note that \(c\) and \(\beta\) depend only on \(n, N, \Delta_2(\{\varphi, \varphi^*\}), \) and \(c_1\).

We will now combine Proposition 3.4 and Lemma 3.3 to derive a decay estimate for the excess functional of \(u\).

**Lemma 3.5** (decay estimate for \(u\)). Let \(\varphi, f,\) and \(u\) be as in Theorem 1.1. Then exists \(\kappa_0 > 0\) and \(\lambda_0\) such that the following holds: if \(B\) is a ball with \(B \subset \Omega\), then

\[
\int_B |\nabla u|^2 \leq \kappa_0
\]

or

\[
\int_{\lambda_0 B} |\nabla u - (\nabla u)|_{\lambda_0 B}|^2 \, dx \leq \frac{1}{2} \int_B |\nabla u - (\nabla u)|_B|^2 \, dx.
\]

Note that \(\kappa_0\) and \(\lambda_0\) depend only on \(n, N, \Delta_2(\{\varphi, \varphi^*\}), \) \(c_1\) \(\beta\), and the limit in (1.3).

**Proof.** Let \(B\) be a ball with \(B \subset \Omega\). With Proposition 3.4 we estimate for any \(\lambda \in (0, 1)\)

\[
\int_{\lambda B} |\nabla u - (\nabla u)|_{\lambda B}|^2 \, dx \leq 2 \int_{\lambda B} |\nabla u - (\nabla v)|^2 \, dx + \int_{\lambda B} |\nabla v - (\nabla v)|_{\lambda B}|^2 \, dx
\]

\[
\leq 2 \lambda^{-n} \int_B |\nabla u - \nabla v|^2 \, dx + c \lambda^\sigma \int_B |\nabla v - (\nabla v)|_B|^2 \, dx
\]

\[
\leq c \lambda^{-n} \int_B |\nabla u - \nabla v|^2 \, dx + c \lambda^\sigma \int_B |\nabla u - (\nabla u)|_B|^2 \, dx.
\]

In the following we fix \(\lambda \in (0, 1)\) such that \(c \lambda^\sigma \leq \frac{1}{4}\), which implies

\[
\int_{\lambda B} |\nabla u - (\nabla u)|_{\lambda B}|^2 \, dx \leq c \lambda^{-n} \int_B |\nabla u - \nabla v|^2 \, dx + \frac{1}{4} \int_B |\nabla u - (\nabla u)|_B|^2 \, dx.
\]
Due to Lemma 3.3 there exists \( \kappa_0 > 0 \) such that
\[
\int_B |V(\nabla u)|^2 \, dx \leq \kappa_0 \tag{3.15}
\]
or
\[
c\lambda^{-n} \int_B |V(\nabla u) - V(\nabla v)|^2 \, dx \leq \frac{1}{4} \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 \, dx. \tag{3.16}
\]
In combination with (3.14) we get that (3.15) holds or
\[
\int_B |V(\nabla u) - \langle V(\nabla u) \rangle_{\lambda B}|^2 \, dx \leq \frac{1}{2} \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 \, dx.
\]
This proves the claim. \( \square \)

We are now in position to prove our main result.

**Proof of Theorem 1.1.** Let \( B \subset \Omega \) and let \( R \) denote the radius of \( B \). Due to (2.9) it suffices to show that
\[
|V(\nabla u(z))|^2 \leq c \left( 1 + \int_B |V(\nabla u)|^2 \, dx \right)
\]
for almost all \( z \in \frac{1}{2}B \). Since \( u \in W^{1,\infty}(\Omega) \), it follows by Lemma 2.4 that \( V(\nabla u) \in L^2(\Omega) \). Thus for almost every \( z \in \frac{1}{2}B \) holds
\[
\lim_{r \to 0} \int_{B_r(z)} |V(\nabla u(z)) - \langle V(\nabla u) \rangle_{B_r(z)}|^2 \, dx = 0. \tag{3.18}
\]
Let \( E \) denote the set of \( z \in \frac{1}{2}B \) such (3.18) holds. To prove the theorem it suffices to show that
\[
|V(\nabla u(z))|^2 \leq c \left( 1 + \int_{B_n(z)} |V(\nabla u)|^2 \, dx \right)
\]
for every \( z \in E \).

Fix \( z \in E \). Then due to Lemma 3.5 there exists \( \kappa_0 > 0 \) and \( \lambda_0 \in (0, 1) \) such that for every \( r \in (0, R/2) \) holds
\[
\int_{B_r(z)} |V(\nabla u)|^2 \, dx \leq \kappa_0 \tag{3.20}
\]
or
\[
\int_{B_{\lambda_0 r}(z)} |V(\nabla u) - \langle V(\nabla u) \rangle_{B_{\lambda_0 r}(z)}|^2 \, dx \leq \frac{1}{2} \int_{B_r(z)} |V(\nabla u) - \langle V(\nabla u) \rangle_{B_r(z)}|^2 \, dx. \tag{3.21}
\]
This allows us to distinguish two cases:

(i) There exists a sequence of radii \( r_j \to 0 \) such that (3.20) holds for every \( r_j \).

(ii) There exists \( R_0 > 0 \) such that (3.20) holds for all \( r \leq R_0 \).

In the case (i) it follows with (3.18) that

\[ |V(\nabla u)(z)|^2 \leq \kappa_0. \]

Let us now consider the case (i). Let \( r_0 := \sup \{ s \in (0, R/2) : (3.21) \text{ holds for all } r \leq s \} \), then \( r_0 \geq R_0 > 0 \).

By continuity of the expressions in (3.21) with respect to \( r \in (0, R) \), it follows that also \( r_0 \) satisfies (3.21). Let \( r_k := \lambda_0^{-k}r_0 \) for \( k \in \mathbb{N}_0 \). Repeated use of (3.21) shows

\[
\int_{B_{r_k}(z)} |V(\nabla u) - \langle V(\nabla u) \rangle_{B_{r_k}(z)}|^2 \, dx \leq 2^{-k} \int_{B_{r_0}(z)} |V(\nabla u) - \langle V(\nabla u) \rangle_{B_{r_0}(z)}|^2 \, dx
\]

for every \( k \in \mathbb{N} \). But then, since

\[
\left| \langle V(\nabla u) \rangle_{B_{r_k}(z)} - \langle V(\nabla u) \rangle_{B_{r_{k+1}}(z)} \right| \leq \lambda_0^{-n} \left( \int_{B_{r_k}(z)} |V(\nabla u) - \langle V(\nabla u) \rangle_{B_{r_k}(z)}|^2 \, dx \right)^{\frac{1}{2}},
\]

we get using (3.18)

\[
|V(\nabla u)(z)| \leq \sum_{k=0}^{\infty} \left| \langle V(\nabla u) \rangle_{B_{r_{k+1}}(z)} - \langle V(\nabla u) \rangle_{B_{r_k}(z)} \right| + \left| \langle V(\nabla u) \rangle_{B_{r_0}(z)} \right|
\]

\[
\leq \lambda_0^{-n} \sum_{k=0}^{\infty} 2^{-k} \left( \int_{B_{r_0}(z)} |V(\nabla u) - \langle V(\nabla u) \rangle_{B_{r_0}(z)}|^2 \, dx \right)^{\frac{1}{2}} + \left| \langle V(\nabla u) \rangle_{B_{r_0}(z)} \right|
\]

\[
\leq (2 \lambda_0^{-n}) \left( \int_{B_{r_0}(z)} |V(\nabla u) - \langle V(\nabla u) \rangle_{B_{r_0}(z)}|^2 \, dx \right)^{\frac{1}{2}} + \left| \langle V(\nabla u) \rangle_{B_{r_0}(z)} \right|
\]

\[
\leq (4 \lambda_0^{-n} + 1) \left( \int_{B_{r_0}(z)} |V(\nabla u)|^2 \, dx \right)^{\frac{1}{2}}.
\]

If \( r_0 = R/2 \), then we estimate with (3.22), \( B_{R/2}(z) \subset B \), and \( |B_{R/2}(z)| = 2^{-n}|B|\)

\[
|V(\nabla u)(z)|^2 \leq 2^n (4 \lambda_0^{-n} + 1)^2 \int_B |V(\nabla u)|^2 \, dx,
\]

which proves (3.19). So we can continue under the assumption that \( 0 < r_0 < R/2 \). We will show in the following that in this case \( r_0 \) satisfies (3.20). The definition of \( r_0 \) and \( r_0 < R/2 \) imply that for every \( j \in \mathbb{N} \) there exists \( r_j \in [r_0, \min \{ r_0 + \frac{1}{j}, R/2 \} ) \) such that (3.20) holds. Since \( r_j \to r_0 \) for \( j \to \infty \), we conclude by continuity of \( r \mapsto \int_{B_r(z)} |V(\nabla u)|^2 \, dx \) on \( (0, R) \) that also \( r_0 \) satisfies (3.20). This and (3.22) imply

\[
|V(\nabla u)(z)|^2 \leq (4 \lambda_0^{-n} + 1)^2 \kappa_0.
\]

This concludes the proof of Theorem 1.1. \( \square \)
LIPSCHITZ REGULARITY FOR SOME ASYMPTOTICALLY CONVEX PROBLEMS

REFERENCES