

## PARTIAL REGULARITY OF MINIMIZERS OF HIGHER ORDER INTEGRALS WITH $(p, q)$ -GROWTH

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**Abstract.** We consider higher order functionals of the form

$$F[u] = \int_{\Omega} f(D^m u) \, dx \quad \text{for } u : \mathbb{R}^n \supset \Omega \rightarrow \mathbb{R}^N,$$

where the integrand  $f : \odot^m(\mathbb{R}^n, \mathbb{R}^N) \rightarrow \mathbb{R}$ ,  $m \geq 1$  is strictly quasiconvex and satisfies a non-standard growth condition. More precisely we assume that  $f$  fulfills the  $(p, q)$ -growth condition

$$\gamma|A|^p \leq f(A) \leq L(1 + |A|^q) \quad \text{for all } A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N),$$

with  $\gamma, L > 0$  and  $1 < p \leq q < \min\{p + \frac{1}{n}, \frac{2n-1}{2n-2}p\}$ . We study minimizers of the functional  $F[\cdot]$  and prove a partial  $C_{\text{loc}}^{m, \alpha}$ -regularity result.

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### INTRODUCTION

In this paper we are interested in the interior regularity theory for minimizers of higher order functionals with  $(p, q)$ -growth. For  $m \geq 1$  we consider variational integrals of the type

$$F[u] = \int_{\Omega} f(D^m u) \, dx, \tag{0.1}$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ ,  $u \in W^{m,1}(\Omega, \mathbb{R}^N)$ ,  $n \geq 2$ ,  $N \geq 1$  and  $f \in C^2(\odot^m(\mathbb{R}^n, \mathbb{R}^N))$  (here  $\odot^m(\mathbb{R}^n, \mathbb{R}^N)$  denotes the vectorspace of symmetric  $m$ -linear functions on  $\mathbb{R}^n$  with values in  $\mathbb{R}^N$ ). Furthermore, we assume that the integrand  $f$  is a strictly  $W^{m,p}$ -quasiconvex function (see (H3s), Sect. 1 for the definition) and satisfies the  $(p, q)$ -growth condition

$$\gamma|A|^p \leq f(A) \leq L(1 + |A|^q) \tag{0.2}$$

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for every  $A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$  with  $\gamma, L > 0$  and  $1 < p \leq q < \min\left\{p + \frac{1}{n}, \frac{2n-1}{2n-2}p\right\}$ . In the following we show existence and partial  $C_{\text{loc}}^{m,\alpha}$  regularity of  $W^{m,p}$ -minimizers of the functional (0.1) with respect to some fixed boundary values. Thereby  $W^{m,p}$ -minimizers of  $F[\cdot]$  are defined as follows:

**Definition 0.1.** Let  $p \in [1, \infty]$ . A map  $u \in W^{m,p}(\Omega, \mathbb{R})$  with  $F[u] < \infty$  is called a  $W^{m,p}$ -minimizer of  $F[\cdot]$  if

$$F[u] \leq F[u + \varphi] \quad \text{for every } \varphi \in W_0^{m,p}(\Omega, \mathbb{R}^N).$$

For second order functionals, *i.e.*  $m = 1$ , there are numerous papers dealing with existence and partial regularity of minimizers of quasiconvex functionals (see (1.7), Sect. 1 for the definition) which satisfy the standard  $p$ -growth condition

$$\gamma|A|^p \leq f(A) \leq L(1 + |A|^p)$$

for every  $A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$  and  $1 < p \leq \infty$ . Since under certain circumstances quasiconvexity is a necessary and sufficient condition for sequential weak lower semicontinuity of  $F[\cdot]$  it turned out to be a primary hypothesis in the existence theory of minimizers (see Sect. 1.1 for further details). Strengthening this condition slightly, Evans derived a first regularity result in [20]. More precisely, he showed for the superquadratic case ( $p \geq 2$ ) that minimizers of strictly quasiconvex functionals are partially  $C_{\text{loc}}^{1,\alpha}$ -regular. This result was extended to the subquadratic case ( $1 < p < 2$ ) in [11]. Further papers concerning this topic are for example [2,5,13,15,16,23,26] and also [31] investigating the Hausdorff-dimension of the singular set. For higher order functionals partial regularity results have been shown in [27,32] (see also [45]).

In the last years functionals with non-standard growth conditions attracted more attention. In this paper, we focus on the so called  $(p, q)$ -growth condition (0.2) which was introduced by Marcellini. Examples in the late 80's (see [25,29,35]) show that we cannot expect regularity in general if  $q$  is large enough with respect to  $p$ . As a start Marcellini studied in the scalar case variational functionals and elliptic equations with  $(p, q)$ -growth (see [36,37]) and deduced basic regularity results. Also various contributions in the vectorial case appeared, see for first examples [4,38,41]. In particular, in [41] it is proved under the assumptions that  $f$  is strictly convex and satisfies the  $(p, q)$ -growth condition with  $2 \leq p < q < \min\{p + 1, \frac{pn}{n-1}\}$  that minimizers of  $F[\cdot]$  are partially  $C_{\text{loc}}^{1,\alpha}$ -regular. Imposing an additional growth condition on the second derivatives of  $f$  in [7] this result was gained under the weaker condition  $q < p \frac{n+2}{n}$  and for integrands with  $x$  dependency in [8] (see also [12]). For a higher order version of [41] we refer to [10].

Furthermore, results concerning higher integrability of gradients of minimizers have been established in [18,19].

A first regularity result for quasiconvex functionals with  $m = 1$  was recently given in [43]. There it is proved that if  $f$  is strictly  $W^{1,p}$ -quasiconvex (see (H2), Sect. 1 for the definition) and satisfies the  $(p, q)$ -growth condition (0.2) with

$$1 < p \leq q < p + \frac{\min\{2, p\}}{2n}, \tag{0.3}$$

then  $u$  is partial  $C_{\text{loc}}^{1,\alpha}$ -regular. The same author also follows an alternative approach of treating regularity of minimizers [44]. This approach relies on a natural relaxation procedure in the spirit of the Lebesgue-Serrin extension which was introduced in [9,21,34].

Following the strategy in [43] this paper aims to give a higher order version of the result in [43]. Under the condition

$$1 < p \leq q < \min\left\{p + \frac{1}{n}, \frac{2n-1}{2n-2}p\right\} \tag{0.4}$$

we show that a minimizer  $u$  of (0.1) is partial  $C_{\text{loc}}^{m,\alpha}$ -regular. We remark that condition (0.4) is more general than (0.3). But this improvement has only technical reasons. It appears also in [42] where strong local minimizers of first order functionals are treated. We show the partial regularity of  $u$  via  $\mathcal{A}$ -harmonic approximation, a method developed in [14]. The main step to proof the regularity result is the derivation of a decay estimate

for the excess function

$$\Phi(\rho) = \left( \int_{B_\rho(x_0)} |V(D^m u) - V((D^m u)_{x_0, \rho})|^2 dx \right)^{\frac{1}{2}},$$

where  $(D^m u)_{x_0, \rho} := \int_{B_\rho(x_0)} D^m u dx$ . For the proof of this decay estimate we need as a key tool the trace preserving operator  $T$  which was introduced by Fonseca and Malý in [21] and extended by the same authors to higher orders in [22].

### 1. PRELIMINARIES AND STATEMENT OF THE RESULT

In this section we state our main result concerning partial regularity of minimizers. Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain,  $1 < p \leq q < \infty$  and  $u \in W^{m,1}(\Omega, \mathbb{R}^N)$ ,  $n \geq 2, N \geq 1$ . We consider  $W^{m,p}$ -minimizers of the functional

$$F[u] = \int_{\Omega} f(D^m u) dx,$$

where  $f$  is a  $C^2$ -function which satisfies some of the following conditions:

(H1)  **$q$ -Growth:** There exists a bound  $L > 0$  such that we have

$$0 \leq f(A) \leq L(1 + |A|^q) \quad \text{for every } A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N).$$

(H2)  **$p$ -Coercivity:** There is a coercivity constant  $\gamma > 0$  such that holds

$$f(A) \geq \gamma|A|^p \quad \text{for every } A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N).$$

(H3)  **$W^{m,p}$ -Quasiconvexity:**  $f$  is  $W^{m,p}$ -quasiconvex, iff

$$\int_{\Omega} f(A + D^m \varphi) dx \geq \int_{\Omega} f(A) dx$$

for all  $A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$  and for all  $\varphi \in W_0^{m,p}(\Omega, \mathbb{R}^N)$ .

**Remark.** The  $W^{m,p}$ -quasiconvexity condition has been introduced by Ball and Murat in [6]. There it is proved for  $m = 1$  that (H3) is a necessary condition for the sequential weak lower semicontinuity of the functional  $F[\cdot]$  on  $W^{m,p}(\Omega, \mathbb{R}^N)$  (an easy adaptation of the arguments gives this result also for  $m > 1$ ). We will later see that the  $W^{m,p}$ -quasiconvexity condition together with the  $q$ -growth condition (H1) and the  $p$ -coercivity condition (H2) implies the existence of at least one minimizer.

(H3s) **Strict  $W^{m,p}$ -quasiconvexity:**  $f$  is strictly non-degenerate  $W^{m,p}$ -quasiconvex, *i.e.* for each  $M > 0$  there is a convexity constant  $\lambda_M > 0$  such that

$$\int_{\Omega} (f(A + D^m \varphi) - f(A)) dx \geq \lambda_M \int_{\Omega} (1 + |D^m \varphi|^2)^{\frac{p-2}{2}} |D^m \varphi|^2 dx \tag{1.1}$$

for all  $A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$  with  $|A| \leq M + 1$  and for all  $\varphi \in W_0^{m,p}(\Omega, \mathbb{R}^N)$ .

**Remark.** This strong quasiconvexity assumption is necessary in particular for the proof of the partial regularity result, whereas for the existence result, condition (H3) is sufficient. Note that (H3s) is even stronger than the strict quasiconvexity of  $f$  that has to be assumed in the corresponding standard growth case: There, we require that (1.1) be satisfied for all smooth functions  $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$ .

Due to the  $q$ -growth of the functional, this amounts to allowing all  $\varphi \in W^{m,q}(\Omega, \mathbb{R}^N)$  as test functions. This condition does not suffice here, since we want to test with functions which are in general only in  $W^{m,p}(\Omega, \mathbb{R}^N)$  and not in  $W^{m,q}(\Omega, \mathbb{R}^N)$ . Therefore, we assume that  $f$  is  $W^{m,p}$ -quasiconvex.

**Remark 1.1.** The following properties of  $f$  are consequences of the preceding assumptions.

- (1) From the quasiconvexity condition (H3s) and the growth condition (H1) we can derive (see e.g. [33]) that

$$|Df(A)| \leq c(1 + |A|)^{q-1} \quad \text{for every } A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N), \tag{1.2}$$

with  $c = c(n, N, q, L)$ .

- (2) Let  $M > 0$  and consider  $A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$  with  $|A| \leq M + 1$ . Then (H3s) implies that  $D^2f(A)$  fulfills the strong Legendre-Hadamard condition, i.e. we have

$$D^2f(A)(\xi^m \otimes \eta, \xi^m \otimes \eta) \geq \frac{2\lambda_M}{m!} |\xi|^{2m} |\eta|^2 \tag{1.3}$$

for all  $\xi \in \mathbb{R}^n$  and  $\eta \in \mathbb{R}^N$ , see [32] and [39], Theorem 7.

- (3) We have no explicit growth condition on the second derivatives of  $f$ . But here it is sufficient that for every  $M > 0$  there exists a constant  $K_M < \infty$  such that there holds

$$\sup_{\substack{|A| \leq M \\ A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)}} |D^2f(A)| \leq K_M. \tag{1.4}$$

- (4) Since  $f$  is a  $C^2$  function for each  $M > 0$  exists a monotone non-decreasing, concave function  $\nu_M : [0, \infty) \rightarrow [0, \infty)$  continuous at 0 such that  $\nu_M(0) = 0$  and

$$|D^2f(A) - D^2f(B)| \leq \nu_M(|A - B|) \tag{1.5}$$

for any  $A, B \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$  with  $|A| \leq M + 1$  and  $|B| \leq M + 1$ .

We can now formulate our main result:

**Theorem 1.2.** *Let*

$$1 < p \leq q < \min \left\{ p + \frac{1}{n}, \frac{2n-1}{2n-2} p \right\} \tag{1.6}$$

and suppose that  $f \in C^2$  satisfies (H1) and (H3s). Then for each  $W^{m,p}(\Omega, \mathbb{R}^N)$ -minimizer (see Def. 0.1) of the functional  $F[u] = \int_{\Omega} f(D^m u) \, dx$  there exists an open subset  $\Omega_0 \subset \Omega$  with

$$\mathcal{L}^n(\Omega \setminus \Omega_0) = 0 \quad \text{and} \quad u \in C_{\text{loc}}^{m,\alpha}(\Omega_0, \mathbb{R}^N)$$

for every  $0 < \alpha < 1$ . Furthermore  $\Omega \setminus \Omega_0 = \Sigma_1 \cup \Sigma_2$  where

$$\begin{aligned} \Sigma_1 &= \left\{ x_0 \in \Omega : \liminf_{\rho \searrow 0} \int_{B_{\rho}(x_0)} |D^m u - (D^m u)_{x_0, \rho}|^p \, dx > 0 \right\}, \\ \Sigma_2 &= \left\{ x_0 \in \Omega : \limsup_{\rho \searrow 0} |(D^m u)_{x_0, \rho}| = \infty \right\}. \end{aligned}$$

### 1.1. Existence

Although our main result concerns the regularity of minimizers for the sake of completeness we also want to investigate whether minimizers of  $F$  exists.

First Morrey [40] has proved for  $m = 1$  that the quasiconvexity condition on  $f$  is a necessary and sufficient condition for weak lower semicontinuity of certain functionals on certain Sobolev spaces. Thereby  $f$  is called quasiconvex if it fulfills

$$\int_{\Omega} f(A) \, dx \leq \int_{\Omega} f(A + D^m \varphi) \, dx \tag{1.7}$$

for all  $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$  and all  $A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$ . Later Acerbi and Fusco [1] showed for  $m = 1$  that if  $f$  is a continuous function and satisfies the  $p$ -growth condition

$$0 \leq f(A) \leq L(1 + |A|^p) \quad (L > 0, 1 \leq p < \infty, A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)),$$

then the related functional  $F[u] = \int_{\Omega} f(D^m u) \, dx$  is sequentially weakly lower semicontinuous on  $W^{m,p}(\Omega, \mathbb{R}^N)$  iff  $f$  is quasiconvex. For a higher order version of this result we refer to Meyers [39] and Guidorzi and Poggiolini [28]. Assuming additionally the following coercivity condition

$$\frac{1}{L}(1 + |A|^p) \leq f(A) \quad \text{for all } A \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$$

it is known – by the direct method in the calculus of variations – that for given boundary data there exists at least one minimizer  $u \in W^{m,p}(\Omega, \mathbb{R}^N)$  of the functional  $F$ .

Now, if we have  $(p, q)$ -growth in [21,30] it is shown for  $m = 1$  and  $1 < p \leq q < \frac{np}{n-1}$  that the quasiconvexity condition (1.7) together with the  $q$ -growth condition (H1) implies that  $F[\cdot]$  is lower semicontinuous with respect to weak  $W^{1,p}$ -convergence of  $W^{1,q}$ -functions. Then, using the  $p$ -coercivity condition (H2) we only get that for given boundary data there exists a  $u \in W^{1,p}(\Omega, \mathbb{R}^N)$  such that

$$F[u] \leq F[v] \quad \text{for all } v \in W_{\text{loc}}^{1,q}(\Omega, \mathbb{R}^N) \cap [u + W_0^{1,p}(\Omega, \mathbb{R}^N)].$$

But if we use instead of the quasiconvexity condition the stronger  $W^{m,p}$ -quasiconvexity condition (H3) we can deduce that  $F[\cdot]$  is sequentially weakly lower semicontinuous on  $W^{m,p}$  (see [43] for the case  $m = 1$ ; note that  $W^{m,p}$ -quasiconvexity is also a necessary condition for sequential weak lower semicontinuity on  $W^{m,p}$ ). More precisely we can prove the following higher order version of [43], Theorem 4.4:

**Theorem 1.3** (semicontinuity). *Suppose that  $f$  fulfills the  $q$ -growth condition (H1) with  $1 < p \leq q < \frac{np}{n-1}$  and the  $W^{m,p}(\Omega, \mathbb{R}^N)$ -quasiconvexity condition (H3). Then  $F[\cdot]$  is sequentially weakly lower semicontinuous on  $W^{m,p}(\Omega, \mathbb{R}^N)$ .*

*Proof.* As mentioned in [43], for  $m = 1$  the proof is basically identical to the proof of [21], Theorem 4.1, except for one difference. In Theorem 4.1 of [21] the authors consider sequences of functions  $(u_k)_k$  with  $u_k \in W^{1,q}(\Omega, \mathbb{R}^N)$  and not general  $W^{1,p}$ -functions  $u_k$ . Therefore, on the other hand in [21] they do not have to assume  $W^{1,p}(\Omega, \mathbb{R}^N)$ -quasiconvexity for  $f$  but only quasiconvexity. However, this does not change the proof in a significant way. For an improvement of the result in [21] see also [30].

In [17], Theorem 4.1, Esposito and Mingione have generalized Theorem 4.1 of [21] to higher orders under the condition  $q < \frac{pmn}{mn-1}$ .

The proof of our theorem follows beside slight differences exactly the proofs of [17], Theorem 4.1, and [21], Theorem 4.1. For the readers comfort we carry out the first step of the proof and give a sketch of the second one.

**Step 1.** We suppose that  $\Omega = B := B_1(0)$ . Let  $u, u_k \in W^{m,p}(B, \mathbb{R}^N)$ ,  $u_k \rightharpoonup u$  weakly in  $W^{m,p}(B, \mathbb{R}^N)$  and let us further assume that  $u$  is a polynomial function of degree  $m$ ,  $u = \sum_{|\alpha| \leq m} c_\alpha x^\alpha$  with  $c_\alpha \in \mathbb{R}$ . Since  $u_k \rightharpoonup u$  in  $W^{m,p}(B, \mathbb{R}^N)$ , passing to a subsequence we may assume that

$$\|u - u_k\|_{W^{m-1,p}(B, \mathbb{R}^N)} \leq k^{-1}.$$

Let  $\frac{1}{4} < r < 1$  and  $\rho = \frac{r+1}{2}$ . Now instead of [17], Lemma 3.1, we apply here a variant of [10], Lemma 2.4, which in turn goes back on a theorem of Fonseca and Malý [22], Theorem 3.3. Due to [10], Lemma 2.4, there exist functions  $z_k \in W^{m,p}(B, \mathbb{R}^N)$  and  $r < s'_k < r'_k < \rho < 1$  with  $r'_k$  and  $s'_k$  depending on  $u$  and  $u_k$ , such that

$$\begin{aligned} z_k &= u_k \quad \text{on } B_{s'_k}, \quad z_k = u \quad \text{on } B \setminus B_{r'_k}, \\ (\rho - r)k^{-1} &\geq r'_k - s'_k \geq (\rho - r)(3k)^{-1} \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \|D^m z_k\|_{L^q(B_{r'_k} \setminus B_{s'_k}, \mathbb{R}^N)} &\leq C \frac{(\rho - r)^\tau}{k^\tau} \left( \|u\|_{W^{m,p}(B_\rho \setminus B_r, \mathbb{R}^N)} + \|u_k\|_{W^{m,p}(B_\rho \setminus B_r, \mathbb{R}^N)} \right. \\ &\quad \left. + \frac{k}{\rho - r} \|u - u_k\|_{W^{m-1,p}(B_\rho \setminus B_r, \mathbb{R}^N)} \right) \end{aligned}$$

where  $C = C(n, p, q)$  and  $\tau = \tau(n, p, q) > 0$ . Therefore, we have

$$\mathcal{L}^n(B_{r'_k} \setminus B_{s'_k}) \leq \frac{c(r)}{k} \quad \text{and} \quad \int_{B_{r'_k} \setminus B_{s'_k}} |D^m z_k|^q \, dx \leq c(r)k^{-q\tau}.$$

Since  $z_k - u \in W_0^{m,p}(B, \mathbb{R}^N)$  and  $f$  is  $W^{m,p}$ -quasiconvex we can conclude that

$$\int_B f(D^m u) \, dx \leq \int_B f(D^m z_k) \, dx.$$

If we use the previous inequalities and the condition (H1) it follows that

$$\begin{aligned} \int_B f(D^m u) \, dx - \int_B f(D^m u_k) \, dx &\leq \int_B (f(D^m z_k) - f(D^m u_k)) \, dx \\ &\leq \int_{B \setminus B_{r'_k}} f(D^m u) \, dx + \int_{B_{r'_k} \setminus B_{s'_k}} f(D^m z_k) \, dx \\ &\leq c \left( \mathcal{L}^n(B \setminus B_{r'_k}) + \int_{B_{r'_k} \setminus B_{s'_k}} (1 + |D^m z_k|^q) \, dx \right) \\ &\leq c(1 - r) + c(r)k^{-q\tau} + c(r)k^{-1}. \end{aligned}$$

Letting first  $k \rightarrow \infty$  and then  $r \rightarrow 1$  we finally conclude

$$\liminf_{k \rightarrow \infty} \int_B f(D^m u_k) \, dx \geq \int_B f(D^m u) \, dx.$$

**Step 2.** Now we treat the general case. Let  $u, u_k \in W^{m,p}(\Omega, \mathbb{R}^N)$ ,  $u_k \rightharpoonup u$  weakly in  $W^{m,p}(\Omega, \mathbb{R}^N)$ . This Step 2 is identical to Step 2 of [17], Theorem 4.1, and therefore, we will only give a brief sketch of it. Without loss of generality we may assume that

$$\lim_{k \rightarrow \infty} \int_\Omega f(D^m u_k) \, dx$$

exists and is finite. There exist a non-negative Radon measure  $\mu$  such that passing, if necessary, to a subsequence

$$f(D^m u_k) \mathcal{L}^n \rightharpoonup \mu$$

weakly in the sense of measures. Now using Step 1, it can be shown that

$$\frac{d\mu}{d\mathcal{L}^n}(x_0) \geq f(D^m u(x_0)) \tag{1.9}$$

for almost every  $x_0 \in \Omega$ . Once (1.9) is verified the assertion of the theorem follows immediately. For further details we refer to [17].  $\square$

Finally for  $q \in [p, \frac{pn}{n-1}]$ , Theorem 1.3 together with the coercivity assumption (H2) implies – using the direct method – the existence of at least one minimizer  $u \in W^{m,p}(\Omega, \mathbb{R}^N)$  of  $F[\cdot]$  in every Dirichlet class.

## 2. PRELIMINARY LEMMAS

### 2.1. The functions $V_\beta$ and $W_\beta$

Let  $\beta > 0$ . Throughout the paper we use the functions  $V_\beta : \mathbb{R}^k \mapsto \mathbb{R}^k$  and  $W_\beta : \mathbb{R}^k \mapsto \mathbb{R}^k$  which we define as

$$V_\beta(\xi) = (1 + |\xi|^2)^{\frac{\beta-2}{4}} \xi, \quad W_\beta(\xi) = (1 + |\xi|^2)^{\frac{\beta}{2}-1} \xi \tag{2.1}$$

for  $\xi \in \mathbb{R}^k, k \in \mathbb{N}$ . For  $t \geq 0$  the functions  $t \mapsto V_\beta(t)$  and  $t \mapsto W_\beta(t)$  are both monotone non-decreasing. Using elementary inequalities we can deduce that

$$c^{-1}(\beta)|W_\beta(\xi)| \leq |V_\beta(\xi)| \leq c(\beta)|W_\beta(\xi)|, \tag{2.2}$$

see [16], Section 3. Further we can calculate that  $|W_p|^2$  is convex for  $1 \leq p < \infty$  and  $|W_p|^{\frac{2}{p}}$  is convex only for  $p \leq 2$  (in contrast to  $|V_p|^2$  and  $|V_p|^{\frac{2}{p}}$ ). In the following we present some useful properties of the functions  $V_\beta$  and  $W_\beta$ .

**Lemma 2.1.** *Let  $1 < p < \infty, \beta > 0$  and  $V_\beta, W_\beta : \mathbb{R}^k \rightarrow \mathbb{R}^k$  be the functions defined by (2.1). Then for any  $\xi, \eta \in \mathbb{R}^k$  and  $t > 0$  we have:*

- (i)  $|V_p(t\xi)| \leq \max\{t, t^{\frac{p}{2}}\}|V_p(\xi)|$ ;
- (ii)  $|V_\beta(\xi + \eta)| \leq c(\beta)(|V_\beta(\xi)| + |V_\beta(\eta)|)$ ;
- (iii)  $c^{-1}(k, p)|\xi - \eta| \leq \frac{|V_p(\xi) - V_p(\eta)|}{(1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{4}}} \leq c(k, p)|\xi - \eta|$ ;
- (iv)  $c^{-1}(p, M)|V_p(\xi - \eta)| \leq |V_p(\xi) - V_p(\eta)| \leq c(k, p, M)|V_p(\xi - \eta)| \quad \forall \eta$  with  $|\eta| \leq M$ ;
- (v)  $(1 + |\xi|^2 + |\eta|^2)^{\frac{p}{2}} \leq c(p)(1 + |V_p(\xi)|^2 + |V_p(\eta)|^2)$ .

*Inequalities (i) and (ii) also hold if we replace  $V$  by  $W$ .*

*Proof.* The assertions (i)–(iv) for  $V_p$  and  $W_p$  can be found in [3], Lemma 2.1, [11], Lemma 2.1, and [16], Lemma 1, (v) is obvious.  $\square$

The next lemma is taken from [2].

**Lemma 2.2.** *Consider  $f \in C^2$  satisfying (H1) and (H3s) and  $A, B \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$  with  $|A| \leq M + 1$ . Then the following estimates hold:*

$$\begin{aligned} |f(A + B) - f(A) - Df(A)B| &\leq c|V_q(B)|^2 \\ |Df(A + B) - Df(A)| &\leq c(1 + |B|^2)^{\frac{q-2}{2}}|B|, \end{aligned}$$

where  $c$  depends on  $n, N, q, L, M$  and  $K_M$ .

**2.2. A Poincaré-type inequality**

The following Poincaré-type inequality is for  $p \geq 2$  a direct consequence of the usual Poincaré inequality. For  $1 < p < 2$  it is a special case of the Sobolev-Poincaré-type inequality stated in [16] (see also [11]).

**Lemma 2.3.** *Let  $1 < p < \infty$  and  $V_p$  the function defined in (2.1). Then there exists a constant  $c_p$  depending on  $n, N$  and  $p$  such that*

$$\int_{B_\rho(x_0)} \left| V_p \left( \frac{u - u_{x_0, \rho}}{\rho} \right) \right|^2 dx \leq c_p \int_{B_\rho(x_0)} |V_p(Du)|^2 dx \tag{2.3}$$

for all  $B_\rho(x_0) \subset\subset \mathbb{R}^n$  and every function  $u \in W^{1,p}(B_\rho(x_0), \mathbb{R}^N)$ . □

**2.3. An iteration lemma**

The next lemma is a variant of [43], Lemma 6.6, [11], Lemma 2.7, and [16], Lemma 2, which in turn traces back to the well-known iteration lemma [24], Lemma 3.1, Chapter 5.

**Lemma 2.4** (iteration lemma). *Let  $0 \leq \vartheta < 1, A, B \geq 0, v \in W^{m,p}(B_\rho(x_0), \mathbb{R}^N), \alpha_1, \alpha_2 \in \mathbb{R}$  and  $g$  be a non-negative bounded function satisfying*

$$g(r) \leq \vartheta g(s) + A \int_{B_\rho(x_0)} \sum_{k=0}^{m-1} \left| V_p \left( \frac{D^k v}{(s-r)^{m-k}} \right) \right|^2 dx + B (s-r)^{\alpha_1} \left( \int_{B_\rho(x_0)} \sum_{k=0}^m \left| V_p \left( \frac{D^k v}{(s-r)^{m-k}} \right) \right|^2 dx \right)^{\alpha_2}$$

for all  $\rho/2 \leq r < s \leq \rho$ . Then there exists a constant  $c = c(\vartheta, p, \alpha_1, \alpha_2)$  such that

$$g(\rho/2) \leq c \left[ A \int_{B_\rho(x_0)} \sum_{k=0}^{m-1} \left| V_p \left( \frac{D^k v}{\rho^{m-k}} \right) \right|^2 dx + B \rho^{\alpha_1} \left( \int_{B_\rho(x_0)} \sum_{k=0}^m \left| V_p \left( \frac{D^k v}{\rho^{m-k}} \right) \right|^2 dx \right)^{\alpha_2} \right].$$

**2.4. Trace preserving operator**

The trace preserving operator  $T$  was first introduced in [21] and later used in [9]. A version for higher derivatives can be found in [22], Theorem 3.3. We state the extension result here in a slightly changed form.

**Lemma 2.5.** *Let  $\tilde{\Omega} \subset \mathbb{R}^n$  be a bounded domain. Let  $u \in W^{m,p}(\tilde{\Omega}, \mathbb{R}^N), 1 \leq p < \infty, 0 < r < s$  and  $B_s \subset \tilde{\Omega}$ . Then there exists a bounded linear operator*

$$T_{r,s} : W^{m,p}(\tilde{\Omega}) \rightarrow W^{m,p}(\tilde{\Omega})$$

with the following properties:

- (i)  $u = T_{r,s}u$  a.e. on  $(\tilde{\Omega} \setminus B_s) \cup B_r$  and  $T_{r,s}u - u \in W_0^{m,p}(B_s \setminus \overline{B_r}, \mathbb{R}^N)$ ;
- (ii)  $\|D^k T_{r,s}u\|_{L^p(B_s \setminus B_r, \mathbb{R}^N)} \leq c(n, m, p) \|D^k u\|_{L^p(B_s \setminus B_r, \mathbb{R}^N)}$  for  $k = 0, 1, \dots, m$ ;
- (iii)  $\|D^k T_{r,s}u\|_{L^q(B_s \setminus B_r, \mathbb{R}^N)}$

$$\leq c(n, m, p, q) (s-r)^{\frac{n}{q} - \frac{n-1}{p}} \cdot \left( \sup_{t \in ]r, s[} (t-r)^{-1} \int_{B_t \setminus B_r} |D^k u|^p dx + \sup_{t \in ]r, s[} (s-t)^{-1} \int_{B_s \setminus B_t} |D^k u|^p dx \right)^{\frac{1}{p}}$$

for  $k = 0, 1, \dots, m$  and for all  $1 \leq p \leq q < \frac{n}{n-1}p$ ;

- (iv)  $\| |V_p(D^k T_{r,s}u)|^2 \|_{L^1(B_s \setminus B_r, \mathbb{R}^N)} \leq c(n, m, p) \| |V_p(D^k u)|^2 \|_{L^1(B_s \setminus B_r, \mathbb{R}^N)}$  for  $k = 0, 1, \dots, m$ ;



$$(v) \quad \| |V_p(D^k T_{r,s}u)|^2 \|_{L^{\tilde{q}}(B_s \setminus B_r, \mathbb{R}^N)} \leq c(n, m, p, q)(s-r)^{\frac{n}{\tilde{q}} - (n-1)} \cdot \left( \sup_{t \in ]r, s[} (t-r)^{-1} \int_{B_t \setminus B_r} |V_p(D^k u)|^2 dx + \sup_{t \in ]r, s[} (s-t)^{-1} \int_{B_s \setminus B_t} |V_p(D^k u)|^2 dx \right)$$

for  $k = 0, 1, \dots, m$  and for all  $1 \leq \tilde{q} < \frac{n}{n-1}$ .

*Proof.* As mentioned above this lemma can be found in [22], Theorem 3.3, in a slightly different form and (i)–(iii) follow from it directly by setting  $\Omega = B_s \setminus \overline{B_r}$  and  $\varepsilon = \frac{r-s}{2}$ . Since  $c^{-1}|W_p| \leq |V_p| \leq c|W_p|$ , it suffices to show (iv) and (v) for  $W_p$  instead of  $V_p$  and therefore we can exploit the convexity of  $|W_p|^2$ . In [22], Theorem 3.3, the following estimate is stated:

$$(T1) \quad |D^k T_{r,s}u(x)| \leq C \int_{B_{3r_i}(x_i)} |D^k u| dy \quad \text{for } x \in B_{r_i}(x_i), \quad B_{r_i}(x_i) \in \mathcal{W} \quad k = 0, 1, \dots, m,$$

where  $\mathcal{W}$  denotes a family of Whitney balls for  $\Omega = B_s \setminus \overline{B_r}$ . Then estimate (T1) together with Jensen’s inequality and Lemma 2.1 gives us the following inequality:

$$\begin{aligned} \int_{B_{r_i}(x_i)} |W_p(D^k T_{r,s}u(x))|^{2\tilde{q}} dx &\leq c \int_{B_{r_i}(x_i)} \left| W_p \left( \int_{B_{3r_i}(x_i)} |D^k u| dy \right) \right|^{2\tilde{q}} dx \\ &\leq c \left( \int_{B_{3r_i}(x_i)} |W_p(D^k u)|^2 dy \right)^{\tilde{q}}. \end{aligned}$$

If we use the last inequality instead of (T2) in [22], Theorem 3.3, and proceed otherwise as for the derivation of (T3) and (T5) in [22], Theorem 3.3, we can deduce (iv) and (v). □

Later we want to estimate the terms on the right hand side of Lemma 2.5 (iii) and (v) using the following lemma. A proof of this lemma is given *e.g.* in [21].

**Lemma 2.6.** *Let  $-\infty < r < s < \infty$  and a continuous nondecreasing function  $\Xi : [r, s] \rightarrow \mathbb{R}$  be given. Then there are  $\tilde{r} \in [r, \frac{2r+s}{3}]$  and  $\tilde{s} \in [\frac{r+2s}{3}, s]$  for which hold:*

$$\begin{aligned} \frac{\Xi(t) - \Xi(\tilde{r})}{t - \tilde{r}} &\leq 3 \frac{\Xi(s) - \Xi(r)}{s - r} \\ \frac{\Xi(\tilde{s}) - \Xi(t)}{\tilde{s} - t} &\leq 3 \frac{\Xi(s) - \Xi(r)}{s - r}, \end{aligned} \tag{2.4}$$

for every  $t \in ]\tilde{r}, \tilde{s}[$ . In particular we have

$$\frac{s-r}{3} \leq \tilde{s} - \tilde{r} \leq s-r. \tag{2.5}$$

□

### 3. CACCIOPPOLI INEQUALITY

Similarly to the case  $m = 1$  treated in [43], we use the trace operator  $T$  defined in Lemma 2.5 to prove a Caccioppoli-type inequality, which differs from the usual Caccioppoli inequality under standard growth conditions (see *e.g.* [16,32] for higher orders) by an additional term on the right hand side. Using common smallness assumptions, we will see that this additional term causes no problems when deriving the decay estimate for the excess function  $\Phi(\rho)$  (see Sect. 6).

**Lemma 3.1** (Caccioppoli inequality). *Let  $1 < p \leq q < \min \left\{ p + \frac{1}{n}, \frac{2n-1}{2n-2} p \right\}$ , i.e.  $q$  satisfies (1.6), and  $M > 0$ . We consider a  $W^{m,p}$ -minimizer  $u$  of the functional  $F[u] = \int_{\Omega} f(D^m u)$ , where the integrand  $f \in C^2$  fulfills (H1) and (H3s). Then for all balls  $B_{\rho} \subset\subset \Omega$  and for all polynomials  $P : \Omega \rightarrow \mathbb{R}^N$  of degree  $\leq m$  with  $|D^m P| \leq M + 1$  we have*

$$\int_{B_{\rho/2}} |V_p(D^m v)|^2 dx \leq c_c \left[ \int_{B_{\rho}} \sum_{k=0}^{m-1} \left| V_p \left( \frac{D^k v}{\rho^{m-k}} \right) \right|^2 dx + \left( \int_{B_{\rho}} \sum_{k=0}^m \left| V_p \left( \frac{D^k v}{\rho^{m-k}} \right) \right|^2 dx \right)^{\frac{n}{n-1}} \right], \tag{3.1}$$

where we have set  $v = u - P$ . The constant  $c_c$  is of the form  $c_c(n, N, m, p, q, L, M, \lambda_M, K_M)$ .

*Proof.* Choose  $\frac{\rho}{2} \leq r < s \leq \rho$  and set

$$\Xi(t) := \int_{B_t} \sum_{k=0}^m \left| V_p \left( \frac{D^k v}{(s-r)^{m-k}} \right) \right|^2 dx.$$

For this function  $\Xi$  we choose  $r \leq \tilde{r} < \tilde{s} \leq s$  as in Lemma 2.6. Now we define a cut-off function  $\eta \in C_0^{\infty}(B_{\rho})$  which satisfies  $0 \leq \eta \leq 1$ ,  $\eta \equiv 1$  in a neighborhood of  $\overline{B_{\tilde{r}}}$ ,  $\eta \equiv 0$  on  $B_{\rho} \setminus B_{\tilde{s}}$  and  $|D^k \eta| \leq (c_{\eta}(\tilde{s} - \tilde{r}))^{-k}$  for all  $1 \leq k \leq m$ . Using the operator  $T$  from Lemma 2.5 we define

$$\phi := (1 - \eta)v, \quad \psi := T_{\tilde{r}, \tilde{s}} \phi \quad \text{and} \quad \varphi := v - \psi,$$

where

$$D^m u - D^m P = D^m v = D^m \varphi + D^m \psi \quad \text{on } B_{\rho}.$$

According to (i) of Lemma 2.5 we have  $\varphi \in W_0^{m,p}(B_{\tilde{s}}, \mathbb{R}^N)$  and  $\varphi = v$  on  $B_{\tilde{r}}$ . With the product formula and (2.5) we calculate

$$|D^m \phi| = \left| \sum_{k=0}^m \binom{m}{k} D^{m-k}(1 - \eta) \otimes D^k v \right| \leq c \sum_{k=0}^m \frac{|D^k v|}{(s-r)^{m-k}}. \tag{3.2}$$

As abbreviation we further set

$$X := \int_{B_s \setminus B_r} \sum_{k=0}^m \left| V_p \left( \frac{D^k v}{(s-r)^{m-k}} \right) \right|^2 dx.$$

First we derive some useful inequalities. If we apply (vi) and (v) of Lemma 2.5 to our operator  $T_{\tilde{r}, \tilde{s}}$  and use then Lemma 2.1 (i), (ii) and inequality (3.2) we get the following estimates:

$$\int_{B_{\tilde{s}} \setminus B_{\tilde{r}}} |V_p(D^m T_{\tilde{s}, \tilde{r}} \phi)|^2 \leq c \int_{B_{\tilde{s}} \setminus B_{\tilde{r}}} |V_p(D^m \phi)|^2 dx \leq cX \tag{3.3}$$

and

$$\begin{aligned} \int_{B_{\tilde{s}} \setminus B_{\tilde{r}}} |V_p(D^m T_{\tilde{s}, \tilde{r}} \phi)|^{2\tilde{q}} &\leq c(\tilde{s} - \tilde{r})^n \left( \sup_{t \in ]\tilde{r}, \tilde{s}[} \frac{(\tilde{s} - \tilde{r})^{1-n}}{(t - \tilde{r})} \int_{B_t \setminus B_{\tilde{r}}} |V_p(D^m \phi)|^2 dx \right. \\ &\quad \left. + \sup_{t \in ]\tilde{r}, \tilde{s}[} \frac{(\tilde{s} - \tilde{r})^{1-n}}{(\tilde{s} - t)} \int_{B_{\tilde{s}} \setminus B_t} |V_p(D^m \phi)|^2 dx \right)^{\tilde{q}} \\ &\leq c(\tilde{s} - \tilde{r})^n \left( \sup_{t \in ]\tilde{r}, \tilde{s}[} (\tilde{s} - \tilde{r})^{1-n} \frac{\Xi(t) - \Xi(\tilde{r})}{(t - \tilde{r})} + \sup_{t \in ]\tilde{r}, \tilde{s}[} (\tilde{s} - \tilde{r})^{1-n} \frac{\Xi(\tilde{s}) - \Xi(t)}{(\tilde{s} - t)} \right)^{\tilde{q}} \end{aligned}$$

for  $1 \leq \tilde{q} < \frac{n}{n-1}$ . Combining the last inequality with the estimates of Lemma 2.6 we obtain

$$\int_{B_{\tilde{s}} \setminus B_{\tilde{r}}} |V_p(D^m T_{\tilde{s}, \tilde{r}} \phi)|^{2\tilde{q}} \leq c(s-r)^n \left( \frac{X}{(s-r)^n} \right)^{\tilde{q}}, \quad \text{for } 1 \leq \tilde{q} < \frac{n}{n-1}. \tag{3.4}$$

We remark that for  $\tilde{q} = 1$  this formula is identical to equation (3.3).

Now, using the quasiconvexity condition (H3s) we get

$$\begin{aligned} \lambda_M \int_{B_{\tilde{r}}} |V_p(D^m v)|^2 dx &= \lambda_M \int_{B_{\tilde{r}}} |V_p(D^m \varphi)|^2 dx \\ &\leq \lambda_M \int_{B_{\tilde{s}}} (1 + |D^m \varphi|^2)^{\frac{p-2}{2}} |D^m \varphi|^2 dx \leq \int_{B_{\tilde{s}}} (f(D^m P + D^m \varphi) - f(D^m P)) dx \\ &= \int_{B_{\tilde{s}}} (f(D^m u - D^m \psi) - f(D^m u)) dx + \int_{B_{\tilde{s}}} (f(D^m u) - f(D^m u - D^m \varphi)) dx \\ &\quad + \int_{B_{\tilde{s}}} (f(D^m P + D^m \psi) - f(D^m P)) dx. \end{aligned}$$

The second term is smaller or equal to zero because of the minimality of  $u$ . Therefore we deduce with Lemmas 2.2 and 2.1 (i), (ii) (for  $\beta = 2q - 2$ )

$$\begin{aligned} \lambda_M \int_{B_{\tilde{r}}} |V_p(D^m v)|^2 dx &\leq \int_{B_{\tilde{s}}} \left[ \int_0^1 (Df(D^m P) - Df(D^m u - \tau D^m \psi)) d\tau D^m \psi \right. \\ &\quad \left. + f(D^m P + D^m \psi) - f(D^m P) - Df(D^m P) D^m \psi \right] dx \\ &\leq c \int_{B_{\tilde{s}}} \left[ \int_0^1 (1 + |D^m v - \tau D^m \psi|^2)^{\frac{q-2}{2}} |D^m v - \tau D^m \psi| d\tau |D^m \psi| + |V_q(D^m \psi)|^2 \right] dx \\ &\leq c \int_{B_{\tilde{s}} \setminus B_{\tilde{r}}} \left[ |V_q(D^m \psi)|^2 + (1 + |D^m v|^2)^{\frac{q-2}{2}} |D^m v| |D^m \psi| \right] dx \\ &= c[(I) + (II)], \end{aligned} \tag{3.5}$$

where  $c$  is a constant depending on  $n, N, p, q, M, K_M$  and  $L$ .

**Estimate for (I).** To estimate (I) we apply inequality (v) of Lemma 2.1 and deduce

$$\begin{aligned} (I) &= c \int_{B_{\tilde{s}} \setminus B_{\tilde{r}}} |V_p(D^m \psi)|^2 (1 + |D^m \psi|^2)^{\frac{p}{2} \frac{q-p}{p}} dx \\ &\leq c \int_{B_{\tilde{s}} \setminus B_{\tilde{r}}} |V_p(D^m \psi)|^2 (1 + |V_p(D^m \psi)|^2)^{\frac{q-p}{p}} dx \\ &\leq c \int_{B_{\tilde{s}} \setminus B_{\tilde{r}}} \left( |V_p(D^m \psi)|^2 + |V_p(D^m \psi)|^{\frac{2q}{p}} \right) dx. \end{aligned}$$

The first term on the right-hand side of the preceding inequality can be estimated using (3.3) (recall  $\psi := T_{\bar{r}, \bar{s}}\phi$ ), while the second one is treated using inequality (3.4) with  $\tilde{q} = \frac{q}{p}$  (recall that  $q < \frac{np}{n-1}$ ). Altogether we get

$$(I) \leq c \left[ X + (s-r)^n \left( \frac{X}{(s-r)^n} \right)^{\frac{q}{p}} \right] \leq c \left[ X + (s-r)^n \left( \frac{X}{(s-r)^n} \right)^{\frac{n}{n-1}} \right] \tag{3.6}$$

where  $c = c(n, N, m, p, q)$ .

**Estimate for (II).** For the estimation of (II) we distinguish the cases  $p > 2 \frac{n-1}{n}$  and  $1 < p \leq 2 \frac{n-1}{n}$ .

**Case 1:**  $1 < p \leq 2 \frac{n-1}{n}$ . In this case (1.6) reads  $q < p \frac{2n-1}{2n-2}$  and therefore we have also  $q < \frac{p}{2} + 1$ . We estimate the integrand of (II) pointwise. If  $|D^m v| \leq |D^m \psi|$  we use that  $(1+t^2)^{\frac{q-2}{2}}t$  is non-decreasing in  $t \geq 0$  to get

$$(1 + |D^m v|^2)^{\frac{q-2}{2}} |D^m v| |D^m \psi| \leq |V_q(D^m \psi)|^2$$

and if  $|D^m v| > |D^m \psi|$  we obtain (recall  $2q - p - 1 < 0$ )

$$\begin{aligned} (1 + |D^m v|^2)^{\frac{q-2}{2}} |D^m v| |D^m \psi| &= (1 + |D^m v|^2)^{\frac{p-2}{4}} (1 + |D^m v|^2)^{\frac{2q-p-2}{4}} |D^m v| |D^m \psi| \\ &\leq |V_p(D^m v)| |V_{2q-p}(D^m \psi)|. \end{aligned}$$

These pointwise estimates together with Hölder’s inequality imply the following estimate for (II):

$$\begin{aligned} (II) &\leq \int_{B_{\bar{s}} \setminus B_{\bar{r}}} \left( |V_q(D^m \psi)|^2 + |V_p(D^m v)| |V_{2q-p}(D^m \psi)| \right) dx \\ &\leq \int_{B_{\bar{s}} \setminus B_{\bar{r}}} |V_q(D^m \psi)|^2 dx + \left( \int_{B_{\bar{s}} \setminus B_{\bar{r}}} |V_{2q-p}(D^m \psi)|^2 dx \right)^{\frac{1}{2}} \left( \int_{B_{\bar{s}} \setminus B_{\bar{r}}} |V_p(D^m v)|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

We have already estimated the first term (see estimation of (I)). The second integral can be estimated analogously recalling that  $q < p \frac{2n-1}{2n-2}$ :

$$\int_{B_{\bar{s}} \setminus B_{\bar{r}}} |V_{2q-p}(D^m \psi)|^2 dx \leq cX + c(s-r)^n \left( \frac{X}{(s-r)^n} \right)^{\frac{2q}{p}-1} \leq c \left[ X + (s-r)^n \left( \frac{X}{(s-r)^n} \right)^{\frac{n}{n-1}} \right].$$

Collecting terms we deduce

$$(II) \leq c \left[ X + (s-r)^n \left( \frac{X}{(s-r)^n} \right)^{\frac{n}{n-1}} \right]$$

with  $c = c(n, N, m, p, q)$ .

**Case 2:**  $p > 2 \frac{n-1}{n}$ . In this case (1.6) reads  $q < p + \frac{1}{n}$  and we have  $\frac{p}{2} + 1 < p + \frac{1}{n}$ . If  $p \leq q \leq \frac{p}{2} + 1$  we can estimate (II) exactly as in the Case 1. Otherwise  $q$  satisfies  $2q - p - 2 > 0$  and applying Young’s inequality we estimate the integrand of (II) for  $|D^m v| \leq 1$  by

$$\begin{aligned} (1 + |D^m v|^2)^{\frac{q-2}{2}} |D^m v| |D^m \psi| &= (1 + |D^m v|^2)^{\frac{p-2}{4} + \frac{2q-p-2}{4}} |D^m v| |D^m \psi| \\ &\leq c(1 + |D^m v|^2)^{\frac{p-2}{4}} |D^m v| |D^m \psi| \\ &\leq c(p, q) (|V_p(D^m v)|^2 + |D^m \psi|^2), \end{aligned}$$

and for  $|D^m v| > 1$  by

$$\begin{aligned} (1 + |D^m v|^2)^{\frac{q-2}{2}} |D^m v| |D^m \psi| &= (1 + |D^m v|^2)^{\frac{p-2}{2} \frac{q-1}{p} + \frac{2q-p-2}{2p}} |D^m v| |D^m \psi| \\ &\leq c(1 + |D^m v|^2)^{\frac{p-2}{2} \frac{q-1}{p}} |D^m v|^2 \frac{q-1}{p} |D^m \psi| \\ &\leq c(p, q) (|V_p(D^m v)|^2 + |D^m \psi|^{\frac{p}{p+1-q}}). \end{aligned}$$

Thus, since  $\frac{p}{p+1-q} \geq 2$  this pointwise estimates lead us to

$$\begin{aligned} (II) &\leq c \int_{B_{\bar{s}} \setminus B_{\bar{r}}} (|V_p(D^m v)|^2 + |D^m \psi|^2 + |D^m \psi|^{\frac{p}{p+1-q}}) \, dx \\ &\leq c \left( X + \int_{B_{\bar{s}} \setminus B_{\bar{r}}} (|V_p(D^m \psi)|^2 + |V_p(D^m \psi)|^{\frac{2}{p+1-q}}) \, dx \right). \end{aligned}$$

We use the estimates (3.3) and (3.4) (remember that  $q < p + \frac{1}{n}$ ) to get

$$(II) \leq c \left[ X + (s-r)^n \left( \frac{X}{(s-r)^n} \right)^{\frac{1}{p+1-q}} \right] \leq c \left[ X + (s-r)^n \left( \frac{X}{(s-r)^n} \right)^{\frac{n}{n-1}} \right],$$

where  $c = c(n, N, m, p, q)$ .

Inserting the preceding estimates for (I) and (II) into (3.5) we finally arrive at

$$\int_{B_r} |V_p(D^m v)|^2 \, dx \leq C_1 X + C_1 (s-r)^n \left( \frac{X}{(s-r)^n} \right)^{\frac{n}{n-1}}$$

with  $C_1 = C_1(n, N, m, p, q, L, K_M, \lambda_M, M)$ . Adding  $C_1 \int_{B_r} |V_p(D^m v)|^2$  on both sides leads to

$$\begin{aligned} \int_{B_r} |V_p(D^m v)|^2 \, dx &\leq \frac{C_1}{1+C_1} \int_{B_s} |V_p(D^m v)|^2 \, dx + \int_{B_\rho} \sum_{k=0}^{m-1} \left| V_p \left( \frac{D^k v}{(s-r)^{m-k}} \right) \right|^2 \, dx \\ &\quad + (s-r)^n \left( \frac{1}{(s-r)^n} \int_{B_\rho} \sum_{k=0}^m \left| V_p \left( \frac{D^k v}{(s-r)^{m-k}} \right) \right|^2 \, dx \right)^{\frac{n}{n-1}}. \end{aligned}$$

Now we can apply the iteration Lemma 2.4 with  $\vartheta = \frac{C_1}{1+C_1}$  and  $g(t) = \int_{B_t} |V_p(D^m v)|^2 \, dx$  to infer the desired estimate:

$$\int_{B_{\rho/2}} |V_p(D^m v)|^2 \, dx \leq c \left[ \int_{B_\rho} \sum_{k=0}^{m-1} \left| V_p \left( \frac{D^k v}{\rho^{m-k}} \right) \right|^2 \, dx + \left( \int_{B_\rho} \sum_{k=0}^m \left| V_p \left( \frac{D^k v}{\rho^{m-k}} \right) \right|^2 \, dx \right)^{\frac{n}{n-1}} \right],$$

with  $c = c(n, N, m, p, q, L, K_M, \lambda_M, M)$ . □

#### 4. $\mathcal{A}$ -HARMONIC APPROXIMATION

In this section we consider a bilinear form  $\mathcal{A} \in \odot^2(\odot^m(\mathbb{R}^n, \mathbb{R}^N), \mathbb{R})$  which is elliptic in the sense of Legendre-Hadamard with ellipticity constant  $\kappa$  and upper bound  $K$ , that means

$$\mathcal{A}(\xi^m \otimes \eta, \xi^m \otimes \eta) \geq \kappa |\xi|^{2m} |\eta|^2 \tag{4.1}$$

for all  $\xi \in \mathbb{R}^n$  and  $\eta \in \mathbb{R}^N$  and

$$\mathcal{A}(A, B) \leq K|A||B| \quad \forall A, B \in \odot^m(\mathbb{R}^n, \mathbb{R}^N). \tag{4.2}$$

The next lemma provides a standard *a-priori* estimate for solutions of linear elliptic systems with constant coefficients.

**Lemma 4.1.** *Consider  $\mathcal{A}$  satisfying (4.1) and (4.2) and a function  $h \in W^{m,1}(B_\rho(x_0), \mathbb{R}^N)$  with*

$$\int_{B_\rho(x_0)} \mathcal{A}(D^m h, D^m \varphi) \, dx = 0$$

for all  $\varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N)$ . Then  $h \in C^\infty(B_\rho(x_0), \mathbb{R}^N)$  and there exists a constant  $c_a = c_a(n, N, m, \kappa, K)$  such that the following estimate holds:

$$\sup_{B_{\rho/2}(x_0)} |D^m h| + \rho \sup_{B_{\rho/2}(x_0)} |D^{m+1} h| \leq c_a \int_{B_\rho(x_0)} |D^m h| \, dx.$$

*Proof.* This lemma can be found in [32] for  $h \in W^{m,2}$ . A standard approximation argument (see [11], Prop. 2.1) shows the desired estimates for  $h \in W^{m,1}$ . □

The main tool for proving partial regularity *via*  $\mathcal{A}$ -harmonic approximation is the so called  $\mathcal{A}$ -harmonic approximation lemma. For  $m = 1$  a basic version of this lemma is given in [14]. In our case we need the following higher order version:

**Lemma 4.2** ( $(\mathcal{A}, m)$ -harmonic approximation lemma). *Let  $\kappa, K$  be positive constants and  $1 < \tilde{p} \leq 2$ . Then for any  $\varepsilon > 0$  there exists  $\delta = \delta(n, N, m, \kappa, K, \varepsilon) \in (0, 1]$  with the following property: for any given  $\mathcal{A}$  satisfying (4.1) and (4.2) and for any  $v \in W^{m,\tilde{p}}(B_\rho(x_0), \mathbb{R}^N)$  with*

$$\int_{B_\rho(x_0)} |W_{\tilde{p}}(D^m v)|^2 \, dx \leq \gamma^2 \leq 1 \quad \text{and}$$

$$\left| \int_{B_\rho(x_0)} \mathcal{A}(D^m v, D^m \varphi) \, dx \right| \leq \gamma \delta \sup_{B_\rho(x_0)} |D^m \varphi| \quad \forall \varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N),$$

there exists an  $(\mathcal{A}, m)$ -harmonic function  $h \in W^{m,\tilde{p}}(B_\rho(x_0), \mathbb{R}^N)$  satisfying

$$\int_{B_\rho(x_0)} |W_{\tilde{p}}(D^m h)|^2 \, dx \leq 1 \quad \text{and} \quad \int_{B_\rho(x_0)} \sum_{k=0}^{m-1} \left| W_{\tilde{p}} \left( \frac{D^k(v - \gamma h)}{\rho} \right) \right|^2 \, dx \leq \gamma^2 \varepsilon.$$

A function  $h$  is called  $(\mathcal{A}, m)$ -harmonic if  $\int_{B_\rho(x_0)} \mathcal{A}(D^m h, D^m \varphi) \, dx = 0$  for all  $\varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N)$ .

*Proof.* For  $\tilde{p} = 2$  a proof of this lemma is presented in [32]. In the case  $1 < \tilde{p} < 2$  the lemma is shown in [16], Lemma 6, for  $m = 1$ . The here stated higher order version is an easy adaption of [16], Lemma 6. □

Since we want to treat the cases  $p \geq 2$  and  $1 < p < 2$  simultaneously it is useful to combine the last two lemmas and state them here in the form of [43], Lemma 6.8.

**Corollary 4.3.** *Let  $\kappa, K$  be positive constants and  $1 < p < \infty$ . Then for any  $\varepsilon > 0$  there exists  $\delta = \delta(n, N, m, \kappa, K, \varepsilon) \in (0, 1]$  with the following property: for any given  $\mathcal{A}$  satisfying (4.1) and (4.2) and for any  $v \in W^{m,p}(B_\rho(x_0), \mathbb{R}^N)$  with*

$$\int_{B_\rho(x_0)} |V_p(D^m v)|^2 dx \leq \gamma^2 \leq 1 \quad \text{and}$$

$$\left| \int_{B_\rho(x_0)} \mathcal{A}(D^m v, D^m \varphi) dx \right| \leq \gamma \delta \sup_{B_\rho(x_0)} |D^m \varphi| \quad \forall \varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N)$$

exists an  $(\mathcal{A}, m)$ -harmonic function  $h \in C^\infty(B_\rho(x_0), \mathbb{R}^N)$  with

$$\sup_{B_{\rho/2}(x_0)} |D^m h| + \rho \sup_{B_{\rho/2}(x_0)} |D^{m+1} h| \leq c_a \quad \text{and} \quad \int_{B_{\rho/2}(x_0)} \sum_{k=0}^{m-1} \left| V_p \left( \frac{D^k(v - \gamma h)}{\rho} \right) \right|^2 dx \leq \gamma^2 \varepsilon,$$

where  $c_a = c_a(n, N, m, \kappa, K)$  denotes the constant of Lemma 4.1.

*Proof.* In the case  $1 < p \leq 2$  this follows immediately combining Lemmas 4.2 and 4.1. In the case  $p \geq 2$  we can apply Lemma 4.2 for  $\tilde{p} = 2$  to the function  $v$ . Assuming  $x_0 = 0$  and  $\rho = 1$  (scaling argument,  $B := B_1(0)$ ) we get then an  $(\mathcal{A}, m)$ -harmonic  $h \in W^{m,2}(B, \mathbb{R}^N)$  with

$$\int_B |D^m h|^2 dx \leq 1 \quad \text{and} \quad \int_B \sum_{k=0}^{m-1} |D^k(v - \gamma h)|^2 \leq \tilde{\varepsilon} \gamma^2,$$

$\tilde{\varepsilon}$  will be chosen later.

Without loss of generality we may assume  $\int_{B_{1/2}} D^k(v - \gamma h) = 0$  for  $k = 0, \dots, m - 1$ . Due to Lemma 4.1 we know that  $h \in C^\infty(B, \mathbb{R}^N)$  and

$$\sup_{B_{1/2}} |D^m h| + \sup_{B_{1/2}} |D^{m+1} h| \leq c_a \int_B |D^m h| dx \leq c_a.$$

For  $p^* = \begin{cases} \frac{np}{n-p} & \text{if } 1 \leq p < n \\ p^* \in ]p, \infty[ \text{ fixed} & \text{if } p \geq n \end{cases}$  we define  $t \in [0, 1[$  by  $\frac{1}{p} = (1-t)\frac{1}{2} + t\frac{1}{p^*}$ . Applying the Sobolev-Poincaré inequality we conclude for  $k = 0, \dots, m - 1$

$$\begin{aligned} \int_{B_{1/2}} |D^k(v - \gamma h)|^p dx &\leq \left( \int_{B_{1/2}} |D^k(v - \gamma h)|^2 dx \right)^{(1-t)\frac{p}{2}} \left( \int_{B_{1/2}} |D^k(v - \gamma h)|^{p^*} dx \right)^{\frac{tp}{p^*}} \\ &\leq c (\gamma^2 \tilde{\varepsilon})^{(1-t)\frac{p}{2}} \left( \int_{B_{1/2}} |D^{k+1}(v - \gamma h)|^p dx \right)^t \\ &\leq c_k (\gamma^2 \tilde{\varepsilon})^{(1-t)\frac{p}{2}} \left( \int_{B_{1/2}} (|D^m v|^p + \gamma^p |D^m h|^p) dx \right)^t \\ &\leq c_k (\gamma^2 \tilde{\varepsilon})^{(1-t)\frac{p}{2}} (\gamma^2 + \gamma^p)^t \leq c_k \gamma^2 \tilde{\varepsilon}^{(1-t)\frac{p}{2}}. \end{aligned}$$

Choosing  $\tilde{\varepsilon}$  such that  $2^p \sum_{k=0}^{m-1} c_k \tilde{\varepsilon}^{(1-t)\frac{p}{2}} + 2^p \tilde{\varepsilon} = \varepsilon$  we get the claim. □

5. APPROXIMATE  $\mathcal{A}$ -HARMONICITY

If we want to apply the  $(\mathcal{A}, m)$ -harmonic approximation lemma of the last section to minimizers of the functional  $F$  the following lemma is required. Before stating this lemma we define the excess functional for the function  $u \in W^{m,p}(B_\rho(x_0), \mathbb{R}^N)$  as

$$\Phi(x_0, \rho, D^m P) = \left( \int_{B_\rho(x_0)} |V_p(D^m u) - V_p(D^m P)|^2 dx \right)^{\frac{1}{2}}, \tag{5.1}$$

where  $P$  is a polynomial of degree  $\leq m$  and  $B_\rho(x_0) \subset\subset \Omega$ .

**Lemma 5.1** (approximate  $(\mathcal{A}, m)$ -harmonicity). *Let  $u \in W^{m,p}(\Omega, \mathbb{R}^N)$  be a  $W^{m,p}$ -minimizer of the functional  $F[u] = \int_\Omega f(D^m u) dx$  where  $f$  satisfies the assumptions (H1) and (H3s) with  $q \leq p + 1$ . Then there exists for every  $M > 0$  a constant  $c_e = c_e(n, N, p, q, L, M, K_{M+1})$  such that for every ball  $B_\rho(x_0) \subset\subset \Omega$  and for every polynomial  $P$  of degree  $\leq m$  with  $|D^m P| \leq M$  we have*

$$\left| \int_{B_\rho(x_0)} D^2 f(D^m P)(D^m(u - P), D^m \varphi) dx \right| \leq c_e (\Phi \sqrt{\nu_M(\Phi)} + \Phi^2) \sup_{B_\rho(x_0)} |D^m \varphi| \tag{5.2}$$

for all  $\varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N)$ . Here we have abbreviated  $\Phi(x_0, \rho, D^m P)$  by  $\Phi$  and  $\nu_M$  denotes the modulus of continuity from Remark 1.1 (4).

*Proof.* Since  $q \leq p + 1$  holds and since  $f$  satisfies (1.2), the following Euler equation holds for all  $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$ :

$$\int_\Omega Df(D^m u) D^m \varphi dx = 0. \tag{5.3}$$

Setting  $v = u - P$  we can derive from (5.3)

$$\begin{aligned} \left| \int_{B_\rho(x_0)} D^2 f(D^m P)(D^m v, D^m \varphi) dx \right| &\leq \int_{B_\rho(x_0)} \left| \int_0^1 (D^2 f(D^m P) - D^2 f(D^m P + tD^m v)) dt (D^m v, D^m \varphi) \right| dx \\ &=: \int_{B_\rho(x_0)} I dx. \end{aligned}$$

To estimate  $I$  we distinguish the cases  $|D^m v| \leq 1$  and  $|D^m v| > 1$  on the ball  $B_\rho(x_0)$ .

**Case 1:**  $x \in B_\rho(x_0) \cap \{|D^m v| \leq 1\}$ . Since  $|D^m P + tD^m v| \leq M + 1$  we can use that  $D^2 f(B)$  is bounded on  $\{B \in \odot^m(\mathbb{R}^n, \mathbb{R}^N) : |B| \leq M + 1\}$ . We obtain with (1.4), (1.5), Lemma 2.1 and the fact that  $\nu_M$  is a monotone non-decreasing function

$$\begin{aligned} I &\leq \sup_{B_\rho(x_0)} |D^m \varphi| \sqrt{2K_{M+1}} \int_0^1 \sqrt{\nu_M(t|D^m v|)} |D^m v| dt \\ &\leq c \sup_{B_\rho(x_0)} |D^m \varphi| \sqrt{\nu_M(|V_p(D^m u) - V_p(D^m P)|) |V_p(D^m u) - V_p(D^m P)|}, \end{aligned}$$

where  $c = c(n, N, p, M, K_{M+1})$ .



**Case 2:**  $x \in B_\rho(x_0) \cap \{|D^m v| > 1\}$ . Using the fact  $|D^2 f(D^m P)| \leq K_M$  and inequality (1.2) (recall that  $q \leq p + 1$ ) we see that

$$\begin{aligned} I &= \left| D^2 f(D^m P)(D^m v, D^m \varphi) + Df(D^m P)D^m \varphi - Df(D^m u)D^m \varphi \right| \\ &\leq \sup_{B_\rho(x_0)} |D^m \varphi| (K_M |D^m v| + c |D^m v|^{q-1}) \leq c \sup_{B_\rho(x_0)} |D^m \varphi| |D^m v|^p \\ &\leq c \sup_{B_\rho(x_0)} |D^m \varphi| |V_p(D^m u) - V_p(D^m P)|^2, \end{aligned}$$

where the constant  $c$  depends on  $n, N, p, q, L, M$  and  $K_M$ .

Returning to the general case we apply the inequalities of Hölder and Jensen to conclude ( $\nu_M$  concave and non-decreasing)

$$\left| \int_{B_\rho(x_0)} D^2 f(D^m P)(D^m(u - P), D^m \varphi) dx \right| \leq c_e (\Phi \sqrt{\nu_M(\Phi)} + \Phi^2) \sup_{B_\rho(x_0)} |D^m \varphi|. \quad \square$$

### 6. PROOF OF THEOREM 1.2

In this section we prove the main theorem. For  $B_\rho(x_0) \subset\subset \Omega$  we set

$$\Phi(\rho) := \Phi(x_0, \rho, (D^m u)_{x_0, \rho}) = \left( \int_{B_\rho(x_0)} |V_p(D^m u) - V_p((D^m u)_{x_0, \rho})|^2 dx \right)^{\frac{1}{2}}.$$

We next derive an excess-improvement estimate assuming that the excess  $\Phi(\rho)$  is initially sufficient small.

**Lemma 6.1** (excess-improvement). *Let  $M > 0$  be fixed. For every  $\vartheta \in ]0, \frac{1}{4}]$  there exists  $\delta = \delta(n, N, m, \lambda_M, K_M, \vartheta, p)$  such that if the following smallness conditions are satisfied*

- (i)  $|(D^m u)_{x_0, \rho}| \leq M; \quad |(D^m u)_{x_0, \vartheta \rho}| \leq M;$
- (ii)  $\sqrt{\nu_M(\Phi(\rho))} + \Phi(\rho) \leq \delta/2;$
- (iii)  $c_1 c_e c_a \Phi(\rho) \leq 1;$
- (iv)  $\vartheta^{-\frac{n^2}{n-1}} \Phi^{2\frac{1}{n-1}}(\rho) \leq \vartheta^2$

then the following excess improvement holds:

$$\Phi^2(x_0, \vartheta \rho, (D^m u)_{x_0, \vartheta \rho}) \leq c_{\text{dec}} \vartheta^2 \Phi^2(x_0, \rho, (D^m u)_{x_0, \rho}), \tag{6.1}$$

where  $c_{\text{dec}} = c_{\text{dec}}(n, N, m, p, q, L, \lambda_M, M, K_{M+1})$ . The constants  $c_1, c_a$  and  $c_e$  denote the constants from Lemma 2.1 (iv), Corollary 4.3 and Lemma 5.1.

*Proof.* We define

$$w = u - \frac{1}{m!} (D^m u)_{x_0, \rho} (x - x_0)^m \quad \text{and} \quad \gamma = c_1 c_e \Phi(\rho),$$

where we have  $\gamma \leq 1$  from the smallness condition (iii). Now we want to show that the conditions of Corollary 4.3 are fulfilled for  $\mathcal{A} = D^2 f((D^m u)_{x_0, \rho})$  and  $v = w$ . We know from (1.3) and (1.4) that  $D^2 f((D^m u)_{x_0, \rho})$  is elliptic in the sense of Legendre-Hadamard with ellipticity constant  $\frac{2\lambda_M}{m!}$  and upper bound  $K = K_M = \sup_{|A| \leq M} |D^2 f(A)|$ . For  $\vartheta \in ]0, \frac{1}{4}]$  we set  $\varepsilon = \vartheta^{n+2+\max\{2,p\}}$  and denote with  $\delta = \delta(n, N, m, \lambda_M, K_M, \varepsilon) = \delta(n, N, m, \lambda_M, K_M, \vartheta, p)$  the constant from Corollary 4.3. Lemma 2.1 (vi) (recall  $|(D^m u)_{x_0, \rho}| \leq M$ ) and  $\gamma = c_1 c_e \Phi(\rho)$  imply:

$$\int_{B_\rho(x_0)} |V_p(D^m w)|^2 dx \leq c_1(p, M) \int_{B_\rho(x_0)} |V_p(D^m u) - V_p((D^m u)_{x_0, \rho})|^2 dx \leq \gamma^2.$$

Furthermore, from Lemma 5.1 with  $\mathcal{A} = D^2 f((D^m u)_{x_0, \rho})$ , the definition  $\gamma = c_1 c_e \Phi(\rho)$ , and the smallness condition  $\sqrt{\nu_M(\Phi(\rho))} + \Phi(\rho) \leq \delta/2$  we infer:

$$\left| \int_{B_\rho(x_0)} D^2 f((D^m u)_{x_0, \rho})(D^m w, D^m \varphi) \, dx \right| \leq c_e (\sqrt{\nu_M(\Phi(\rho))} \Phi(\rho) + \Phi^2(\rho)) \sup_{B_\rho(x_0)} |D^m \varphi| \leq \gamma \delta \sup_{B_\rho(x_0)} |D^m \varphi|.$$

Hence the assumptions of Corollary 4.3 are fulfilled and it follows that there exists a  $D^2 f((D^m u)_{x_0, \rho})$ -harmonic function  $h \in C^\infty(B_\rho(x_0), \mathbb{R}^N)$  such that

$$\sup_{B_{\rho/2}(x_0)} |D^m h| + \rho \sup_{B_{\rho/2}(x_0)} |D^{m+1} h| \leq c_a \quad \text{and} \quad \int_{B_{\rho/2}(x_0)} \sum_{k=0}^{m-1} \left| V_p \left( \frac{D^k(w - \gamma h)}{\rho} \right) \right|^2 \, dx \leq \gamma^2 \varepsilon. \tag{6.2}$$

With Taylor’s theorem applied to  $D^{m-1}h$  on  $B_{2\vartheta\rho}$  we obtain the estimate

$$\sup_{B_{2\vartheta\rho}(x_0)} \left| D^{m-1}h(x) - D^{m-1}h(x_0) - D^m h(x_0)(x - x_0) \right| \leq \frac{1}{2} (2\vartheta\rho)^2 \sup_{B_{\rho/2}(x_0)} |D^{m+1}h| \leq c \vartheta^2 \rho.$$

Next we define  $P$  as the unique polynomial of degree  $m$  satisfying

$$\begin{aligned} \int_{B_{2\vartheta\rho}(x_0)} D^k(u - P) \, dx &= 0 \quad \text{for } k = 0, \dots, m-2 \quad \text{and} \\ D^{m-1}P &= \gamma D^{m-1}h(x_0) + ((D^m u)_{x_0, \rho} + \gamma D^m h(x_0))(x - x_0) \end{aligned}$$

and set  $v = u - P$ . Using Lemma 2.1 (ii) and (iii) together with the Poincaré-type inequality (2.3) and estimate (6.2) we get:

$$\begin{aligned} I &:= \sum_{k=0}^{m-1} \int_{B_{2\vartheta\rho}(x_0)} \left| V_p \left( \frac{D^k v}{(2\vartheta\rho)^{(m-k)}} \right) \right|^2 \, dx \leq c \int_{B_{2\vartheta\rho}(x_0)} \left| V_p \left( \frac{D^{m-1}(u - P)}{2\vartheta\rho} \right) \right|^2 \, dx \\ &\leq c \left[ \vartheta^{-n - \max\{2, p\}} \int_{B_{\rho/2}(x_0)} \left| V_p \left( \frac{D^{m-1}(w - \gamma h)}{\rho} \right) \right|^2 \, dx \right. \\ &\quad \left. + \int_{B_{2\vartheta\rho}(x_0)} \left| V_p \left( \gamma \frac{D^{m-1}h(x) - D^{m-1}h(x_0) - D^m h(x_0)(x - x_0)}{2\vartheta\rho} \right) \right|^2 \, dx \right] \\ &\leq c \left[ \vartheta^{-n - \max\{2, p\}} \gamma^2 \varepsilon + |V_p(\vartheta\gamma)|^2 \right] \leq c \left[ \vartheta^{-n - \max\{2, p\}} \gamma^2 \varepsilon + (\vartheta\gamma)^2 \right]. \end{aligned}$$

Moreover, Lemma 2.1 (iv) ( $|(D^m u)_{x_0, \rho}| \leq M$ ) and (6.2) gives us

$$II := \int_{B_{2\vartheta\rho}(x_0)} |V_p(D^m(u - P))|^2 \leq c(\vartheta^{-n}\Phi^2(\rho) + |V_p(\gamma D^m h(x_0))|^2) \leq c(\vartheta^{-n}\Phi^2(\rho) + \gamma^2).$$

Recalling the choice  $\varepsilon = \vartheta^{n+2+\max\{2, p\}}$  and the definition of  $\gamma$  we finally derive

$$I \leq c \vartheta^2 \Phi^2(\rho) \quad \text{and} \quad II \leq c \vartheta^{-n} \Phi^2(\rho),$$

where  $c = c(n, N, m, p, \lambda_M, K_M, M)$ . With the help of Lemma 2.1 (ii) and (iv) ( $|(D^m u)_{x_0, \vartheta\rho}| \leq M$ ), inequality (2.2), the convexity of  $|W_p|^2$  and Jensen’s inequality we deduce:

$$\begin{aligned}
 \Phi^2(\vartheta\rho) &= \int_{B_{\vartheta\rho}(x_0)} |V_p(D^m u) - V_p((D^m u)_{x_0, \vartheta\rho})|^2 dx \\
 &\leq c(p, n, N, M) \int_{B_{\vartheta\rho}(x_0)} |V_p(D^m u - (D^m u)_{x_0, \vartheta\rho})|^2 dx \\
 &\leq c \int_{B_{\vartheta\rho}(x_0)} |V_p(D^m u - (D^m u)_{x_0, \rho} - \gamma D^m h(x_0))|^2 dx \\
 &\quad + c |W_p((D^m u)_{x_0, \vartheta\rho} - (D^m u)_{x_0, \rho} - \gamma D^m h(x_0))|^2 \\
 &\leq c \int_{B_{\vartheta\rho}(x_0)} |V_p(D^m u - (D^m u)_{x_0, \rho} - \gamma D^m h(x_0))|^2 dx = \int_{B_{\vartheta\rho}(x_0)} |V_p(D^m v)|^2 dx, \tag{6.3}
 \end{aligned}$$

where  $c$  depends on  $n, N, M$  and  $p$ . Using  $|(D^m u)_{x_0, \rho}| \leq M$ , (6.2), the definition of  $\gamma$  and the smallness assumption (iii) we conclude

$$|D^m P| \leq |(D^m u)_{x_0, \rho}| + |\gamma D^m h(x_0)| \leq M + \gamma c_a \leq M + c_1 c_e c_a \Phi(\rho) \leq M + 1.$$

Now we apply the Caccioppoli inequality (Lem. 3.1) on  $B_{\vartheta\rho}(x_0)$  and obtain with Lemma 2.1 (ii) and the estimates for  $I$  and  $II$ :

$$\begin{aligned}
 \Phi^2(\vartheta\rho) &\leq c \int_{B_{\vartheta\rho}(x_0)} |V_p(D^m v)|^2 dx \\
 &\leq c \left[ \int_{B_{2\vartheta\rho}(x_0)} \sum_{k=0}^{m-1} \left| V_p \left( \frac{D^k v}{(2\vartheta\rho)^{m-k}} \right) \right|^2 dx + \left( \int_{B_{2\vartheta\rho}(x_0)} \sum_{k=0}^m \left| V_p \left( \frac{D^k v}{(2\vartheta\rho)^{m-k}} \right) \right|^2 dx \right)^{\frac{n}{n-1}} \right] \\
 &\leq c \left[ I + I^{\frac{n}{n-1}} + II^{\frac{n}{n-1}} \right] \leq c \left[ \vartheta^2 \Phi^2(\rho) + \vartheta^{2\frac{n}{n-1}} \Phi^{2\frac{n}{n-1}}(\rho) + \vartheta^{-\frac{n^2}{n-1}} \Phi^{2\frac{n}{n-1}}(\rho) \right].
 \end{aligned}$$

The smallness assumption (iv) and  $\Phi(\rho) \leq 1$  imply

$$\Phi^2(x_0, \vartheta\rho, (D^m u)_{x_0, \vartheta\rho}) = \Phi^2(\vartheta\rho) \leq c_{\text{dec}} \vartheta^2 \Phi^2(\rho) = c_{\text{dec}} \vartheta^2 \Phi^2(x_0, \rho, (D^m u)_{x_0, \rho}),$$

where  $c_{\text{dec}}$  depends on  $n, N, m, p, q, L, \lambda_M, M$  and  $K_{M+1}$ . □

The smallness conditions of Lemma 6.1 are fulfilled in all points  $x_0 \in \Omega_0$ . We can now iterate inequality (6.1) of Lemma 6.1 to derive the following excess-decay estimate for  $\Phi(\rho)$  (for further details see [11,20])

$$\Phi(x_0, r, (D^m u)_{x_0, r}) \leq \left(\frac{r}{\rho}\right)^\alpha \Phi(x_0, \rho, (D^m u)_{x_0, \rho})$$

for all  $0 < r \leq \rho$  and  $0 < \alpha < 1$ . Using Campanato’s integral characterization of Hölder continuous functions (see e.g. [24]) and the property Lemma 2.1 (iii) of the function  $V_p$  we complete the proof of Theorem 1.2.

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