STRONG UNIQUE CONTINUATION FOR THE LAMÉ SYSTEM WITH LIPSCHITZ COEFFICIENTS IN THREE DIMENSIONS∗

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Abstract. This paper studies the strong unique continuation property for the Lamé system of elasticity with variable Lamé coefficients λ, μ in three dimensions, div(μ(∇u + ∇u')) + ∇(λdiv u) + Vu = 0 where λ and μ are Lipschitz continuous and V ∈ L∞. The method is based on the Carleman estimate with polynomial weights for the Lamé operator.

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1. INTRODUCTION

The main purpose of this paper is to study the strong unique continuation for the Lamé system with C0,1 coefficients in three dimensions. Firstly, let us introduce the Lamé system of elasticity. Assume that Ω is a bounded domain in ℝ3, the Lamé moduli μ = μ(x), λ = λ(x) are C0,1(Ω) and satisfy the strong ellipticity conditions

\[ \mu \geq \alpha_0 > 0, \quad 2\mu + \lambda \geq \beta_0 > 0, \]

and the upper bound

\[ \|\mu\|_{C^{0,1}(\Omega)} + \|\lambda\|_{C^{0,1}(\Omega)} \leq C, \]

where α0, β0 and C are given positive constants. Without loss of generality, we may assume that Ω contains the origin and BR ⊂⊂ Ω for some R > 0 where BR is the open ball centered at the origin with radius R. The Lamé system with a perturbation V ∈ L∞(Ω) is given by

\[ \text{div}(\mu(\nabla u + (\nabla u)^\top)) + \nabla(\lambda \text{div} u) + Vu = 0, \]

where u = (u1, u2, u3)⊤ is the displacement vector and

\[ \nabla u = \begin{pmatrix} \partial_{x_1}u_1 & \partial_{x_2}u_1 & \partial_{x_3}u_1 \\ \partial_{x_1}u_2 & \partial_{x_2}u_2 & \partial_{x_3}u_2 \\ \partial_{x_1}u_3 & \partial_{x_2}u_3 & \partial_{x_3}u_3 \end{pmatrix} \]

is the gradient matrix of u.

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We recall that a function $u \in L^2_{\text{loc}}(\mathbb{R}^n)$ is said to vanish of infinite order at a point $x_0$ if for any $k > 0$,
\[ \int_{|x-x_0|<r} |u|^2 \, dx = O(r^k), \quad \text{as } r \to 0. \]

We say that (1.1) has the strong unique continuation property (SUCP) if any solution $u$ is identically zero whenever it vanishes of infinite order at a point of $\Omega$.

The previous results in the literature are the following. The result of the (weak) unique continuation for the Lamé system was first given by Dehman and Robbiano for Lamé operator with Lipschitz coefficients. Inspired by [6,11], one can find a way to obtain the Carleman estimates with polynomial weights for Lamé operator directly from a corresponding estimates for the Laplacian in three dimensions, unless $u$ is identically zero. This result can be seen in our previous work [14]. Next, we will prove if $x_0$ is a zero of “exponential” order, i.e.,
\[ \int_{|x-x_0|<r} |u|^2 \, dx = O \left( e^{-r^{-1}} \right), \quad \text{as } r \to 0, \]
(1.3)

unless $u$ is identically 0. This result can be seen in our previous work [14]. Next, we will prove if $x_0$ is a zero of infinite order for $u$, then $x_0$ is a zero of “exponential” order, i.e. (1.2) implies (1.3). For this step we need the Carleman estimates with polynomial weights.

2. Carleman estimates with polynomial weights

In [11], the authors give a result of the Carleman estimates for Dirac operators. The proof uses the fact that Dirac squared is the Laplacian. Inspired by [6,11], one can find a way to obtain the Carleman estimates with polynomial weights for Lamé operator directly from a corresponding estimates for the Laplacian in three dimensions, since the Lamé operator is a composition of two first order operators, one of which is a Dirac operator. For doing this, we need to use the semiclassical Sobolev spaces
\[ \|u\|_{H^s_{\text{sc}}} = \| (hD)^s u \|_{L^2} \]
where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. It is easy to prove that if $s$ is a positive integer, then
\[ \|u\|_{H^s_{\text{sc}}} \approx \sum_{0 \leq k \leq s} \| h^k D^k u \|_{L^2}. \]

Here and below $\| \cdot \| = \| \cdot \|_{L^2(\mathbb{R}^n)}$ and $C$ denoting a constant independent of $h$ and $u$, may vary from line to line.
Firstly, we give a generalized Carleman estimate with polynomial weights for the Laplacian. Below, we will use the conventions of semiclassical calculus. In particular we consider the usual semiclassical symbol classes $S^m$, and relate to symbols $a \in S^m$ operators $A = \text{Op}_h(a)$ via Weyl quantization. See [9] for more details.

**Lemma 2.1.** Let $0 \leq s \leq 1$. Then for any $u \in C_c^\infty(B_r \setminus \{0\})$ where $B_r \subset \mathbb{R}^n$ is a ball of radius $r$ around the origin $0$ and for any $\frac{1}{h} = k + \frac{n+1}{2}$, $k \in \mathbb{N}$, we have the estimate

$$\| |x|^{-\frac{2}{n}} u \|_{H^{s+1}_h} \leq Ch^r \| |x|^{-\frac{2}{n}} \Delta u \|_{H^{s+1}_h},$$

(2.1)

where $C$ depends only on $n$.

**Proof.** Consider the Carleman inequality for the Laplace operator proved by Regbaoui [10],

$$\int |x|^{-\frac{2}{n}-4}|u|^2 dx + h^2 \int |x|^{-\frac{2}{n}-2} |Du|^2 dx + h^4 \int |x|^{-\frac{2}{n}} |D^2 u|^2 dx \leq Ch^2 \int |x|^{-\frac{2}{n}} |\Delta u|^2 dx$$

(2.2)

for all $u \in C_c^\infty(B_r \setminus \{0\})$, where $\frac{1}{h} = k + \frac{n+1}{2}$, $k \in \mathbb{N}$.

Setting $t = hr$, we define

$$\| u \|_{H^{s+1}_h} = \| (tD)^{s} u \|_{L^2}$$

where $(\xi) = (1 + |\xi|^2)^{\frac{1}{2}}$. Let

$$P = |x|^{-\frac{2}{n}} \circ (-t^2 \Delta) \circ |x|^{\frac{2}{n}}$$

and

$$u = |x|^{\frac{2}{n}} v \in C_c^\infty(B_r \setminus \{0\}).$$

Putting $u$ into (2.2), one has

$$I_1 = \| |x|^{-2} v \| + h \| |x|^{-\frac{2}{n}-1} D(|x|^{\frac{2}{n}} v) \| + h^2 \| |x|^{-\frac{2}{n}} D^2(|x|^{\frac{2}{n}} v) \|$$

$$\leq C h \| |x|^{-\frac{2}{n}} \Delta(|x|^{\frac{2}{n}} v) \| = C t^{-1} r^{-1} \| P v \| = I_2.$$  

(2.3)

Let $0 < \eta \ll \epsilon \ll 1$.

$$I_1 \geq \| |x|^{-2} v \| + \epsilon h \| |x|^{-1} Dv \| + \frac{1}{h} \| |x|^{-2} (D|x|) v \| + \eta h^2 \| D^2 v \| + |x|^{-\frac{2}{n}} \| D^2 |x|^{\frac{2}{n}} v \|$$

$$\geq \| |x|^{-2} v \| + \epsilon h \| |x|^{-1} Dv \| + \frac{1}{h} \| |x|^{-2} (D|x|) v \| + \eta h^2 \| D^2 v \| + \frac{1}{h} \| |x|^{-1} D(|x|) Dv \| + \frac{1}{h} \left( \frac{1}{h} - 1 \right) \| |x|^{-2} v \|$$

$$\geq \| |x|^{-2} v \| + \epsilon \left( h \| |x|^{-1} Dv \| - C \| |x|^{-2} v \| \right) + \eta \left( h^2 \| D^2 v \| - h C \| |x|^{-1} Dv \| - C \| |x|^{-2} v \| \right)$$

$$\geq (1 - C \epsilon - C \eta) \| |x|^{-2} v \| + h(\epsilon - C \eta) \| |x|^{-1} Dv \| + h^2 \| D^2 v \|$$

$$\geq C \left( \| |x|^{-2} v \| + h \| |x|^{-1} Dv \| + h^2 \| D^2 v \| \right)$$

$$\geq C \left( r^{-2} \| v \| + hr^{-1} \| Dv \| + h^2 \| D^2 v \| \right).$$

(2.4)

Composing (2.3) and (2.4), we rewrite (2.2) as follows

$$\| v \| + rh \| Dv \| + r^2 h^2 \| D^2 v \| \leq C t r^{-1} \| P v \|.$$  

(2.5)
Then we have
\[ ||v||_{H^s_{\text{ext}}} \leq Crt^{-1}||Pv||, \text{ for all } v \in C_c^\infty(B_r \setminus \{0\}). \] (2.6)

Suppose \( v \in C_c^\infty(B_r \setminus \{0\}) \), then \( \langle tD \rangle^j v \in \mathcal{S}, \forall j \in \mathbb{R} \). In order to use the Carleman estimate we have to cut \( \langle tD \rangle^j v \) off.

Let \( \chi \in C_c^\infty(B_{2r} \setminus \{0\}) \) such that
\[ (1 - \chi)v = 0. \]

Note that \( \text{supp}(1 - \chi) \cap \text{supp} v = \emptyset \), and this yields the estimate
\[ \|(1 - \chi)\langle tD \rangle^j v\|_{H^s_{\text{ext}}} \leq C_M t^M \|v\|_{H^{s+1}_{\text{ext}}}, \text{ for any } t, k, j, M. \] (2.7)

This and (2.6) imply
\[ \|v\|_{H^{s+1}_{\text{ext}}} = \|\langle tD \rangle^{s-1}v\|_{H^s_{\text{ext}}} \]
\[ \leq \|\chi\langle tD \rangle^{s-1}v\|_{H^s_{\text{ext}}} + \|(1 - \chi)\langle tD \rangle^{s-1}v\|_{H^s_{\text{ext}}} \]
\[ \leq C \left( rt^{-1}\|P(\chi\langle tD \rangle^{s-1}v)\| + t\|v\|_{H^{s+1}_{\text{ext}}} \right) \]
\[ \leq Ct^{-1} \left( r\|\chi\langle tD \rangle^{s-1}Pv\| + r\|\chi[P, (\langle tD \rangle^{s-1})v]\| + r\|[P, \chi]\langle tD \rangle^{s-1}v\| + t^2\|v\|_{H^{s+1}_{\text{ext}}} \right). \] (2.8)

By the property (2.7) we have
\[ \|\chi[P, \chi]\langle tD \rangle^{s-1}v\| \leq C_M t^M \|v\|_{H^{s+1}_{\text{ext}}}, \text{ for any } M. \] (2.9)

Noting that \(|\xi|^2 \text{ and } \langle t|\xi| \rangle^{s-1} \) are main symbols of differential operator \( P \) and pseudo differential operator \( \langle tD \rangle^{s-1} \), respectively, we have \( \chi[P, \langle tD \rangle^{s-1}] \in \text{Op}_t(S^{s-1}) \), since \( \{|\xi|^2, \langle t|\xi| \rangle^{s-1}\} \) vanishes identically, where \( \{a, b\} \) denotes the Poisson bracket of \( a \) and \( b \), defined by
\[ \{a, b\} = \frac{\partial a}{\partial \xi} \frac{\partial b}{\partial x} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial \xi}. \]

Hence
\[ \|\chi[P, \langle tD \rangle^{s-1}]v\| \leq Ct\|\langle tD \rangle^{s-1}v\| \] (2.10)
where \( C \) depends only on the dimension. Then by (2.8), (2.9), (2.10) and choosing \( 0 < t \ll r \ll 1 \), we have
\[ t\|v\|_{H^{s+1}_{\text{ext}}} \leq Cr\|Pv\|_{H^{s-1}_{\text{ext}}}. \]

Noting that \( v = |x|^{-\frac{n}{2}}u \), we conclude (2.1).

From [4] we know that in three dimensions the Lamé operator \( L \) can be written as a composition of two first order operators. Indeed, consider the elliptic operator \( A : H^1(\Omega; \mathbb{R}^4) \rightarrow L^2(\Omega; \mathbb{R}^4) \) defined by
\[ A(\partial)(u, v) = (\nabla 	imes u + \nabla v, -\nabla \cdot u), \]
and its perturbation \( A_\alpha : H^1(\Omega; \mathbb{R}^4) \rightarrow L^2(\Omega; \mathbb{R}^4) \) defined by
\[ A_\alpha(x, \partial)(u, v) = (\nabla 	imes u + \alpha \nabla v, -\nabla \cdot u), \]
where $u$ is a vector-valued function with three components, $v$ and $\alpha$ are scalar-valued functions. $A(\partial)$ is a Dirac operator since $A^2 = -\Delta I_4$. We refer [11] for Carleman estimates for general Dirac operators.

Now we turn to the Lamé system (1.1). Assume $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ denoting the displacement vector. Let $L$ be the Lamé operator

$$Lu := \mu \Delta u + (\lambda + \mu) \nabla \text{div} u.$$ 

It is easy to check that

$$Lu := \mu \Delta u + (\lambda + \mu) \nabla \text{div} u = (2\mu + \lambda) \nabla \nabla \cdot u - \mu \nabla \times \nabla \times u.$$ 

Then we have

$$(Lu, 0) = (\mu \Delta u + (\lambda + \mu) \nabla \text{div} u, 0) = -\mu A_\alpha(x, \partial) A(\partial)(u, 0),$$

where we choose $\alpha = \frac{2(\mu + \lambda)}{\mu}$.

Firstly we give the Carleman estimates for $A$ and $A_\alpha$ by using our generalized Carleman estimate for Laplace operator.

**Lemma 2.2.** For any $v \in C^\infty_c(B_r \setminus \{0\}; \mathbb{R}^4)$ where $B_r \subset \mathbb{R}^3$ is a ball of radius $r$ around the origin $0$ and for any $\frac{1}{k} = k + \frac{1}{2}$, $k \in \mathbb{N}$, we have the estimates

$$\|v\|_{H^1_{scl}} \leq C_1 r \| |x|^{-\frac{3}{2}} A(|x|^\frac{3}{2} u)\|,$$

$$\|v\|_{H^1_{scl}} \leq C_2 r \| |x|^{-\frac{3}{2}} A_\alpha(|x|^\frac{3}{2} v)\|,$$

where $C_1$ and $C_2$ are two positive constants depending only on $\alpha$.

**Proof.** Let

$$\widehat{A} = |x|^{-\frac{3}{2}} \circ (tA) \circ |x|^\frac{3}{2}.$$ 

Then we have

$$\widehat{A}^2 = |x|^{-\frac{3}{2}} \circ (-t^2 \Delta I_4) \circ |x|^\frac{3}{2}.$$ 

We recall the composition formula for semiclassical symbols $p$ and $q$: the operator $PQ$ has symbol

$$\sigma(PQ) \sim \sum_{\alpha, \beta} \frac{h^{\alpha + \beta}(\alpha)!}{(2i)^{\alpha} \alpha! \beta!} (\partial_x^\alpha \partial_\xi^\beta p(x, \xi)) (\partial_x^\alpha \partial_\xi^\beta q(x, \xi)).$$

(2.14)

Denoting $b(x, \xi)$ the symbol of $\langle hD\rangle^{-1} \widehat{A}$, we have by (2.14)

$$b(x, \xi) \sim \sum_{\alpha, \beta} \frac{h^{\alpha}}{(2i)^{\alpha} \alpha!} (\partial_\xi^\beta (1 + |h\xi|^2)^{-\frac{1}{2}}) (\partial_\xi^\alpha q(x, \xi)) = 1 + O(h), \quad \text{in } S^0.$$ 

Now Lemma A.2.1 implies, for any $v \in C^\infty_c(B_r \setminus \{0\}; \mathbb{R}^4)$

$$t \|v\|_{H^1_{scl}} \leq Cr \| |x|^{-\frac{3}{2}} \circ (-t^2 \Delta) \circ |x|^\frac{3}{2} v\|_{H^1_{scl}}$$

$$= Cr \| \widehat{A} v\|_{H^1_{scl}}$$

$$= Cr \| (tD)^{-1} \widehat{A} v\|$$

$$\leq Cr \| \widehat{A} v\|$$

(2.15)

since $\langle hrD\rangle^{-1} \widehat{A}$ is of order 0.
Substituting $|x|^{-\frac{\lambda}{2}}v$ for $v$ in (2.15), we have
\[
\| |x|^{-\frac{\lambda}{2}}v \|_{H^1_{|x|}} \leq Cr \| |x|^{-\frac{\lambda}{2}}Av \|.
\] (2.16)

By regularizing, we know (2.16) is valid if we suppose $v \in H^1(\mathbb{R}^3)$ with compact support in $B_r \setminus \{0\}$ that is a three dimensional ball of radius $r$ centered at 0 but removing the center 0.

Next, with
\[
I_\alpha = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \alpha
\end{pmatrix}
\]
we have
\[
A_\alpha(x, \partial) = A(\partial)I_\alpha.
\]

Noting that $C^{0,1}(\Omega)$ is equivalent to $W^{1,\infty}(\Omega)$, we have
\[
\alpha = \frac{(2\mu + \lambda)}{\mu} \in W^{1,\infty}(\Omega).
\]

Thus $\tilde{v} \in H^1(\mathbb{R}^3)$ has compact support in $B_r \setminus \{0\}$. Hence by putting $\tilde{v}$ into (2.16), we have
\[
\| |x|^{-\frac{\lambda}{2}}I_\alpha v \|_{H^1_{|x|}} \leq C r \| |x|^{-\frac{\lambda}{2}}A(I_\alpha v) \|
\leq C r \| |x|^{-\frac{\lambda}{2}}A_\alpha v \| + C r \| |x|^{-\frac{\lambda}{2}}(A_\alpha) v \|
\leq C r \| |x|^{-\frac{\lambda}{2}}A_\alpha v \| + C r \| |x|^{-\frac{\lambda}{2}}\tilde{v} \|,
\]

where $C_\alpha$ is a constant depending on $\alpha$. Noting that the second term of the right hand side can be absorbed by the left hand side, we have (2.13). \qed

The Carleman estimate for the Lamé operator follows immediately by (2.11), (2.12) and (2.13). Indeed, noticing that $|x|^{-\frac{\lambda}{2}}A \circ |x|^{\frac{\lambda}{2}}$ is a differential operator, we have
\[
|x|^{-\frac{\lambda}{2}}A(|x|^{\frac{\lambda}{2}}v) \in C^\infty_c(\mathbb{R}^3 \setminus \{0\}; \mathbb{R}^4),
\]
for any $u \in C^\infty_c(\mathbb{R}^3 \setminus \{0\}; \mathbb{R}^3)$, let
\[
v = (u, 0) \in C^\infty_c(\mathbb{R}^3 \setminus \{0\}; \mathbb{R}^4),
\]
we have
\[
\| |x|^{-\frac{\lambda}{2}}A(|x|^{\frac{\lambda}{2}}v) \| \leq \| |x|^{-\frac{\lambda}{2}}A(|x|^{\frac{\lambda}{2}}v) \|_{H^1_{|x|}}
\leq C r \| |x|^{-\frac{\lambda}{2}}A_\alpha v \| + C r \| |x|^{-\frac{\lambda}{2}}\tilde{v} \|
= C r \| |x|^{-\frac{\lambda}{2}}L(|x|^{\frac{\lambda}{2}}u) \|.
\] (2.17)

With (2.12) and (2.17), we have a key theorem as follows.

**Theorem 2.1.** There exists $C$ depending only on $\nu$ and Lamé coefficients such that for any $u \in C^\infty_c(\mathbb{B}_r \setminus \{0\})$ where $B_r \subset \mathbb{R}^3$ is a ball of radius $r$ around the origin 0 and for any $\frac{1}{h} = k + \frac{n + 1}{2}$, $k \in \mathbb{N}$, we have the estimate
\[
\| u \| + rh \| Du \| \leq C r^2 \| |x|^{-\frac{\lambda}{2}}L(|x|^{\frac{\lambda}{2}}u) \|.
\] (2.18)
Proof. We need only to prove Lemma 3.1. Suppose $u \in H^{2}_0(\mathbb{R}^3)$ with compact support and satisfying for all $|\alpha| \leq 2$ and all $N > 0$, $\int_{B_r}|D^\alpha u|^2dx = O(r^N)$ as $r \to 0$. We can easily see this by cutting $u$ off for small $r$ and regularizing.

3. PROOF OF THE MAIN RESULT

We need the following auxiliary lemma.

Lemma 3.1. Suppose $u \in H^2_0(\mathbb{R}^3)$ and $u$ vanishes of infinite order at $x_0$. Then $|x|^{-\frac{3}{2}}u$ also vanishes of infinite order at $x_0$.

Proof. $u$ vanishes of infinite order at $x_0$, then

$$\int_{B_r}|u|^2dx = O(r^k), \quad \forall k > 0 \quad r \to 0.$$ 

We have

$$\int_{B_r}|x|^{-\frac{3}{2}}|u|^2dx = \sum_{n>0} \int_{B_{\frac{r}{2n+1}}} B_{\frac{r}{2n+1}} - \frac{r}{2n+1} |x|^{-\frac{3}{2}}|u|^2dx$$

$$\leq \sum_{n>0} \left( \frac{r}{2n+1} \right)^{-\frac{3}{2}} \left( \frac{r}{2n} \right)^k$$

$$= r^{-\frac{3}{2}+k} \sum_{n>0} \frac{1}{2^{nk-(n+1)\frac{3}{2}}}$$

$$\leq r^{-\frac{3}{2}+k}.$$ (3.1)

That is

$$\int_{B_r}|x|^{-\frac{3}{2}}|u|^2dx = O(r^{k-\frac{3}{2}}), \quad \forall k > 0 \text{ as } r \to 0.$$ 

This is the conclusion of the lemma. □

By Lemma 3.1, we have another version of Theorem 2.1.

Corollary 3.1. Assume $u \in C_0^\infty(B_\varepsilon \setminus \{0\})$ and $u$ vanishes of infinite order at $x_0$. There exists $C$ depending on the Lamé coefficients, such that for all $\frac{1}{h} = k + \frac{n+1}{2}$, $k \in \mathbb{N}$, it holds

$$\| |x|^{-\frac{3}{2}}u \| + rh\| |x|^{-\frac{3}{2}}Du \| \leq C r^2 \| |x|^{-\frac{3}{2}}Lu \|.$$ (3.2)

Proof. We need only to prove

$$\| |x|^{-\frac{3}{2}}u \| + rh\| D(|x|^{-\frac{3}{2}}u) \| \geq C \left( \| |x|^{-\frac{3}{2}}u \| + rh\| |x|^{-\frac{3}{2}}Du \| \right).$$ (3.3)

Indeed, there exists $c > 1$, such that

$$\| |x|^{-\frac{3}{2}}u \|^2 + r^2h^2\| D(|x|^{-\frac{3}{2}}u) \|^2 \geq \| |x|^{-\frac{3}{2}}u \|^2 + \left(1 - \frac{1}{c}\right)r^2h^2\| |x|^{-\frac{3}{2}}Du \|^2 + (1-c)r^2\| |x|^{-\frac{3}{2}+1}u \|^2$$

$$= \| |x|^{-\frac{3}{2}}u \|^2 + (1-c)r^2\| |x|^{-\frac{3}{2}+1}u \|^2 + \left(1 - \frac{1}{c}\right)r^2h^2\| |x|^{-\frac{3}{2}}Du \|^2.$$ (3.4)
Noting that \( u \) vanishes of infinite order at \( x_0 \), we set \( w = |x|^{-1} h u \). Then by the proof of Lemma 3.1, we know
\[
\int_{B_r} |w|^2 dx = O(r^{k - \frac{1}{2}}), \quad \forall k > 0 \text{ as } r \to 0.
\]

Similarly to (3.1),
\[
r^2 \int_{B_r} |x|^{-2} |w|^2 dx = r^2 O(r^{k - \frac{1}{2} - 2}) = O(r^{k - \frac{3}{2}}), \quad \forall k > 0 \text{ as } r \to 0.
\]

Since \( 1 - c < 0 \), then we have
\[
\int_{B_r} |w|^2 dx + (1 - c)r^2 \int_{B_r} |x|^{-2} |w|^2 dx \geq \int_{B_r} |w|^2 dx + (1 - c) \int_{B_r} |w|^2 dx
\]
\[
= (2 - c) \int_{B_r} |w|^2 dx.
\]

Choosing \( 1 < c < 2 \), we have (3.3) by (3.4) and (3.6). □

Now we show that if \( x_0 \) is a zero of infinite order for \( u \) which is a solution of system (1.1), then \( x_0 \) is a zero of “exponential” order. Without loss of generality, we suppose \( x_0 = 0 \).

By Remark 2.1 and Lemma 3.1, we have:

**Theorem 3.1.** Suppose that \( u \in H^2_{\text{loc}}(\mathbb{R}^3) \), and satisfies (1.2). Then
\[
\int_{B_r} |u|^2 dx = O(e^{-r^{-1}}), \quad \text{as } r \to 0,
\]

**Proof.** Suppose \( u \in H^2_{\text{loc}}(\mathbb{R}^3) \) is a solution (1.1), and satisfies (1.2). We can obtain that for all \( |\alpha| \leq 2 \)
\[
\int_{B_r} |D^\alpha u|^2 dx = O(r^k), \quad \forall k > 0 \text{ as } r \to 0.
\]

Assume that \( h \) satisfies the conditions of Lemma 2.1. Choose \( k \) sufficiently larger than \( \frac{k}{2} \) and a cut-off function \( \tilde{\chi} \in C^\infty_c(\mathbb{R}^3) \) such that
\[
\begin{align*}
\tilde{\chi}(x) &= 1 \quad \text{as } |x| < r, \\
\tilde{\chi}(x) &= 0 \quad \text{as } |x| > 2r.
\end{align*}
\]

It is easy to check that \( \tilde{\chi} \) satisfies
\[
|D^\alpha \tilde{\chi}(x)| \leq Cr^{-|\alpha|},
\]

\( \tilde{\chi} u \in H^2 \) has a compact support, then by Remark 2.1, (2.18) also holds for \( \tilde{\chi} u \). Thus by Corollary 3.1, we have
\[
\int \tilde{\chi}|x|^{-\frac{1}{2}} |u|^2 dx + h^2 r^2 \int |x|^{-\frac{1}{2}} |D\tilde{\chi} u|^2 dx \leq Cr^4 \int |x|^{-\frac{3}{2}} |L(\tilde{\chi} u)|^2 dx.
\]
Noting that
\[
0 = \text{div}(\mu(\nabla u + (\nabla u)\top)) + \nabla(\lambda \text{div } u) + Vu = Lu + (\nabla u + (\nabla u)\top)\nabla \mu + (\text{div } u)\nabla \lambda + Vu.
\]
It means
\[
\int \tilde{\chi}|x|^{-\frac{n}{p}}|u|^2 \, dx + h^2r^2 \int \left| |x|^{-\frac{n}{p}} D\tilde{\chi}u \right|^2 \, dx \leq Cr^4 \int |x|^{-\frac{n}{p}} |L(\tilde{\chi}u)|^2 \, dx
\]
\[
\leq Cr^4 \int \tilde{\chi}|x|^{-\frac{n}{p}}|Lu|^2 + |x|^{-\frac{n}{p}} \left| L, \tilde{\chi} \right|^2 \, dx
\]
\[
\leq Cr^4 \left( \int_{|x|<r} |x|^{-\frac{n}{p}} |Du|^2 \, dx + \int_{|x|>r} |x|^{-\frac{n}{p}} |Vu|^2 \, dx \right.
\]
\[
\left. + \int_{r<|x|<2r} |x|^{-\frac{n}{p}} |u|^2 \, dx + \int_{r<|x|<2r} |x|^{-\frac{n}{p}} |Du|^2 \, dx \right). \quad (3.10)
\]
Recall that \( V \in L^\infty(\Omega) \), we have
\[
\int \tilde{\chi}|x|^{-\frac{n}{p}}|u|^2 \, dx + h^2r^2 \int \left| |x|^{-\frac{n}{p}} D\tilde{\chi}u \right|^2 \, dx \leq C_0r^4 \left( \int_{|x|<r} |x|^{-\frac{n}{p}} |Du|^2 \, dx + \int_{|x|<r} |x|^{-\frac{n}{p}} |u|^2 \, dx \right.
\]
\[
\left. + \int_{r<|x|<2r} |x|^{-\frac{n}{p}} |u|^2 \, dx + \int_{r<|x|<2r} |x|^{-\frac{n}{p}} |Du|^2 \, dx \right). \quad (3.11)
\]
After some calculations, we have
\[
(1 - C_0r^4) \int_{|x|<r} |x|^{-\frac{n}{p}} |u|^2 \, dx + (h^2r^2 - C_0r^4) \int |x|^{-\frac{n}{p}} |Du|^2 \, dx \leq C_0r^4 \left( \int_{r<|x|<2r} |x|^{-\frac{n}{p}} |u|^2 \, dx + \int_{r<|x|<2r} |x|^{-\frac{n}{p}} |Du|^2 \, dx \right). \quad (3.12)
\]
Noting \( r \) is sufficiently small, as long as \( 0 \ll r \ll h \ll 1 \), we have
\[
(h^2r^2 - C_0r^4) \int |x|^{-\frac{n}{p}} |Du|^2 \, dx \geq r^4 \int |x|^{-\frac{n}{p}} |D\tilde{\chi}u|^2 \, dx
\]
\[
\geq r^4 \int_{|x|<r} |x|^{-\frac{n}{p}} |Du|^2 \, dx - r^4 \int_{r<|x|<2r} |x|^{-\frac{n}{p}} |u|^2 \, dx. \quad (3.13)
\]
Summing up, we have
\[
\int_{|x|<r} |x|^{-\frac{n}{p}} |u|^2 \, dx \leq Cr^4 \int_{|x|>r} |x|^{-\frac{n}{p}} |u|^2 \, dx + Cr^4 \int_{|x|>r} |x|^{-\frac{n}{p}} |Du|^2 \, dx. \quad (3.14)
\]
That is
\[
\left( \frac{r}{2} \right)^{-\frac{n}{p}} \int_{|x|<r} |u|^2 \, dx \leq Cr^{-\frac{n}{p}+4} \int_{|x|>r} |u|^2 \, dx + Cr^{-\frac{n}{p}+4} \int_{|x|>r} |Du|^2 \, dx.
\]
It implies
\[ \int_{B_r} |u|^2 \, dx \leq C r^{-2} 2^{-\frac{2}{n}} \|u\|_{H^1}^2. \quad (3.15) \]
Choosing \( h = (1 + C_0) r \), we note that \( h \) has to satisfy \( h^{-1} \in A = \{ m + 2, m \in \mathbb{N} \} \). Then \( [(1 + C_0) r]^{-1} \) must be in \( A \). But but one can check without difficulty that (3.15) is available for all \( r > 0 \). Thus
\[ \int_{B_r} |u|^2 \, dx = O \left( e^{-r^{-1}} \right), \quad r \to 0. \]

Then we complete the proof the theorem. □

In our previous work [14], we gave a quantitative estimate of unique continuation for a three-dimensional Lamé system with \( C^1 \) coefficients in the form of three spheres inequalities. The property of the non faster than exponential vanishing of nonzero local solutions is also given as an application of the three spheres inequality (1.1).

**Theorem 3.2.** Assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \), the Lamé moduli \( \mu, \lambda \in C^{0,1}(\Omega) \) satisfy the strong elliptic conditions. Let \( u \in H^2(\Omega) \) be a solution to (1.1). If there is a point \( x_0 \in \Omega \) and \( k > 0 \) such that,
\[ \int_{|x-x_0|<r} |u|^2 \, dx = O \left( e^{-r^{-k}} \right), \quad as \ r \to 0^+. \]

Then \( u \equiv 0, \ in \ \Omega. \)

Then the SUCP follows immediately by Theorems 3.1 and 3.2.

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**References**


