ESAIM: COCV 17 (2011) 1174–1197
DOI: 10.1051/cocv/2010042

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

MAXIMUM PRINCIPLE FOR FORWARD-BACKWARD DOUBLY STOCHASTIC CONTROL SYSTEMS AND APPLICATIONS *

Liangquan Zhang[1, 2] and Yufeng Shi[1]

Abstract. The maximum principle for optimal control problems of fully coupled forward-backward doubly stochastic differential equations (FBDSDEs in short) in the global form is obtained, under the assumptions that the diffusion coefficients do not contain the control variable, but the control domain need not to be convex. We apply our stochastic maximum principle (SMP in short) to investigate the optimal control problems of a class of stochastic partial differential equations (SPDEs in short). And as an example of the SMP, we solve a kind of forward-backward doubly stochastic linear quadratic optimal control problems as well. In the last section, we use the solution of FBDSDEs to get the explicit form of the optimal control for linear quadratic stochastic optimal control problem and open-loop Nash equilibrium point for nonzero sum stochastic differential games problem.

Mathematics Subject Classification. 93E20, 60H10.

Received June 30, 2009. Revised January 4, 2010 and August 14, 2010.
Published online November 8, 2010.

1. Introduction

It is well known that optimal control problem is one of the central themes of control science. The necessary conditions of optimal problem were established for deterministic control system by Pontryagin [24] in the 1950's and 1960's. Since then, a lot of work has been done on the forward stochastic system such as Kushner [13], Bismut [5], Bensoussan [2,3], Haussmann [10,11] and Peng [20], etc.

Peng [20] studied the following type of stochastic optimal control problem. Minimize a cost function

$$J\left(v_{(\cdot)}\right) = \mathbf{E}\int_0^T l\left(x_t, v_t\right) \mathrm{d}t + \mathbf{E}\left(h_T\right),$$

Keywords and phrases. Maximum principle, stochastic optimal control, forward-backward doubly stochastic differential equations, spike variations, variational equations, stochastic partial differential equations, nonzero sum stochastic differential game.

* *L. Zhang has been partially supported by Marie Curie Initial Training Network (ITN) project: "Deterministic and Stochastic Controlled System and Application", FP7-PEOPLE-2007-1-1-ITN, No. 213841-2. Y. Shi has been partially supported by National Natural Science Foundation of China Grants 10771122 and 11071145, Natural Science Foundation of Shandong Province of China Grant Y2006A08, Independent Innovation Foundation of Shandong University Grant 2010JQ010 and National Basic Research Program of China (973 Program, No. 2007CB814900).*

[1] School of Mathematics, Shandong University, Jinan 250100, P.R. China. yfshi@sdu.edu.cn

[2] Laboratoire de Mathématiques, Université de Bretagne Occidentale, 29285 Brest Cedex, France.

subject to

$$\begin{cases} \mathrm{d}x_t = g\left(t, x_t, v_t\right)\mathrm{d}t + \sigma\left(t, x_t, v_t\right)\mathrm{d}B_t, \\ x_0 = x, \end{cases} \tag{1.1}$$

over an admissible control domain which need not be convex, and the diffusion coefficients contain the control variable. In his paper, by spike variational method and the second order adjoint equations, Peng [20] obtained a general stochastic maximum principle for the above optimal control problem. It was just the adjoint equations in stochastic optimal control problems that motivated the famous theory of backward stochastic differential equations (BSDEs in short) (see [18]). Later Peng [21] studied a stochastic optimal control problem where state variables are described by the system of forward and backward SDEs, that is

$$\begin{cases} \mathrm{d}x_t = f\left(t, x_t, v_t\right)\mathrm{d}t + \sigma\left(t, x_t, v_t\right)\mathrm{d}W_t, \\ x_0 = x, \\ \mathrm{d}y_t = g\left(t, x_t, v_t\right)\mathrm{d}t + z_t \mathrm{d}W_t, \\ y_T = y, \end{cases} \tag{1.2}$$

where x and y are given deterministic constants. The optimal control problem is to minimize the cost function

$$J\left(v_{(\cdot)}\right) = \mathbf{E}\left[\int_0^T l\left(t, x_t, y_t, v_t\right)\mathrm{d}t + h\left(x_T\right) + \gamma\left(y_0\right)\right], \tag{1.1}$$

over an admissible control domain which is convex. Xu [28] studied the following non-fully coupled forward-backward stochastic control system

$$\begin{cases} \mathrm{d}x_t = f\left(t, x_t, v_t\right)\mathrm{d}t + \sigma\left(t, x_t\right)\mathrm{d}W_t, \\ x_0 = x, \\ \mathrm{d}y_t = g\left(t, x_t, y_t, z_t, v_t\right)\mathrm{d}t + z_t \mathrm{d}W_t, \\ y_T = h\left(x_T\right). \end{cases} \tag{1.3}$$

The optimal control problem is to minimize the cost function

$$J\left(v_{(\cdot)}\right) = \mathbf{E}\gamma\left(y_0\right),$$

over $\mathcal{U}_{\mathrm{ad}}$, but the control domain is non-convex. Wu [26] firstly gave the maximum principle for optimal control problem of fully coupled forward-backward stochastic system

$$\begin{cases} \mathrm{d}x_t = f\left(t, x_t, y_t, z_t, v_t\right)\mathrm{d}t + \sigma\left(t, x_t, y_t, z_t, v_t\right)\mathrm{d}B_t, \\ \mathrm{d}y_t = -g\left(t, x_t, y_t, z_t, v_t\right)\mathrm{d}t + z_t \mathrm{d}B_t, \\ x_0 = x, \qquad y_T = \xi, \end{cases} \tag{1.4}$$

where ξ is a random variable and the cost function

$$J\left(v_{(\cdot)}\right) = \mathbf{E}\left[\int_0^T L\left(t, x_t, y_t, z_t, v_t\right)\mathrm{d}t + \Phi\left(x_T\right) + h\left(y_0\right)\right].$$

The optimal control problem is to minimize the cost function $J\left(v_{(\cdot)}\right)$ over an admissible control domain which is convex. Ji and Zhou [12] obtained a maximum principle for stochastic optimal control of non-fully coupled forward-backward stochastic system with terminal state constraints. Shi and Wu [25] studied the maximum principle for fully coupled forward-backward stochastic system

$$\begin{cases} \mathrm{d}x_t = b\left(t, x_t, y_t, z_t, v_t\right)\mathrm{d}t + \sigma\left(t, x_t, y_t, z_t\right)\mathrm{d}B_t, \\ \mathrm{d}y_t = -f\left(t, x_t, y_t, z_t, v_t\right)\mathrm{d}t + z_t \mathrm{d}B_t, \\ x_0 = x, \qquad y_T = h\left(x_T\right), \end{cases} \tag{1.5}$$

and the cost function is

$$J\left(v_{(\cdot)}\right)=\mathbf{E}\left[\int_0^T l\left(t,x_t,y_t,z_t,v_t\right)\mathrm{d}t+\Phi\left(x_T\right)+\gamma\left(y_0\right)\right].$$

The control domain is non-convex but the forward diffusion does not contain the control variable. For more details in this field, see Yong and Zhou [29].

In order to provide a probabilistic interpretation for the solutions of a class of semilinear stochastic partial differential equations (SPDEs in short), Pardoux and Peng [19] introduced the following backward doubly stochastic differential equation (BDSDE in short):

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)\mathrm{d}s+\int_t^T g(s,Y_s,Z_s)\hat{\mathrm{d}}B_s-\int_t^T Z_s\mathrm{d}W_s,\quad 0\leq t\leq T. \tag{1.6}$$

Note that the integral with respect to $\{B_t\}$ is a "backward Itô integral" and the integral with respect to $\{W_t\}$ is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Skorohod integral (for details see [16]). Peng and Shi [22] introduced a type of time-symmetric forward-backward stochastic differential equations, *i.e.*, so-called fully coupled forward-backward doubly stochastic differential equations (FBDSDEs in short):

$$\begin{cases} y_t=x+\int_0^t f\left(s,y_s,Y_s,z_s,Z_s\right)\mathrm{d}s+\int_0^t g\left(s,y_s,Y_s,z_s,Z_s\right)\mathrm{d}W_s-\int_0^t z_s\hat{\mathrm{d}}B_s,\\ Y_t=h\left(y_T\right)+\int_t^T F\left(s,y_s,Y_s,z_s,Z_s\right)\mathrm{d}s+\int_t^T G\left(s,y_s,Y_s,z_s,Z_s\right)\hat{\mathrm{d}}B_s+\int_t^T Z_s\mathrm{d}W_s. \end{cases} \tag{1.7}$$

In FBDSDEs (1.7), the forward equation is "forward" with respect to a standard stochastic integral $\mathrm{d}W_t$, as well as "backward" with respect to a backward stochastic integral $\hat{\mathrm{d}}B_t$; the coupled "backward equation" is "forward" under the backward stochastic integral $\hat{\mathrm{d}}B_t$ and "backward" under the forward one. In other words, both the forward equation and the backward one are types of BDSDE (1.6) with different directions of stochastic integrals. So (1.7) provides a very general framework of fully coupled forward-backward stochastic systems, which is more extensive than the one in [23]. Peng and Shi [22] proved the existence and uniqueness of solutions to FBDSDEs (1.7) with arbitrarily fixed time duration under some monotone assumptions. FBDSDEs (1.7) can provide a probabilistic interpretation for the solutions of a class of quasilinear SPDEs.

As we have known, stochastic control problem of the SPDEs arising from partial observation control has been studied by Mortensen [14], using a dynamic programming approach, and subsequently by Bensoussan, using a maximum principle method. See [4,15] and the references therein for more information. Our approach differs from the one of Bensoussan. More precisely, we relate the FBDSDEs to one kind of SPDEs with control variables where the control systems of SPDEs can be transformed to the relevant control systems of FBDSDEs. To our knowledge, this is the first time to treat the optimal control problems of SPDEs from a new perspective of FBDSDEs. It is worth mentioning that the quasilinear SPDEs in [17] Øksendal considered can just be related to our partially coupled FBDSDEs.

Besides, in Section 6 we investigate the nonzero sum stochastic differential game problem. This problem have been considered by Friedman [8], Bensoussan [1] and Eisele [7]. For stochastic case Hamadène [9] and Wu [27] (for more information see references therein) showed existence result of Nash equilibrium point under some assumptions, respectively. Here, we extend their result to doubly stochastic case in which we can regard the backward filtration as the disturbed information come from outside the "control system".

In this paper, we consider the following fully coupled forward-backward doubly stochastic control system

$$\begin{cases} y_t=x+\int_0^t f\left(s,y_s,Y_s,z_s,Z_s,v_s\right)\mathrm{d}s+\int_0^t g\left(s,y_s,Y_s,z_s,Z_s\right)\mathrm{d}W_s-\int_0^t z_s\hat{\mathrm{d}}B_s,\\ Y_t=h\left(y_T\right)+\int_t^T F\left(s,y_s,Y_s,z_s,Z_s,v_s\right)\mathrm{d}s+\int_t^T G\left(s,y_s,Y_s,z_s,Z_s\right)\hat{\mathrm{d}}B_s-\int_t^T Z_s\mathrm{d}W_s. \end{cases} \tag{1.8}$$

Our optimal control problem is to minimize the cost function:

$$J\left(v_{(\cdot)}\right) = \mathbf{E}\left[\int_0^T l\left(t, y_t, Y_t, z_t, Z_t, v_t\right) \mathrm{d}t + \Phi\left(y_T\right) + \gamma\left(Y_0\right)\right]$$

over an admissible control domain which need not be convex. It is obvious that (1.8) covers (1.3) and (1.5), so (1.8) can describe more intricate control systems. As for the fully coupled forward-backward doubly stochastic control systems such as (1.8) whose diffusion coefficients contain the control variables, this issue will be carried out in our future publications.

The notable difficulties to obtain the maximum principles for the fully coupled forward-backward doubly stochastic control systems within non-convex control domains are how to use the spike variational method to get variational equations with enough high order estimates and how to use the duality technique to obtain the adjoint equations. On account of the quadruple of variables in the FBDSDEs, we can not directly apply the methods used in [25,26,28]. In this paper, by virtue of the results of FBDSDEs in [22], we can ensure the existence and uniqueness of the solutions for the adjoint FBDSDEs which are obtained by applying the duality technique to the variational equations. Besides, we apply the technique of FBDSDEs to get the enough high order estimates for the solutions of the variational equations.

From the maximum principle for optimal control problems of FBDSDEs obtained in this paper, we can find the equations satisfied by Nash equilibrium points for linear quadratic nonzero sum doubly stochastic differential games problems. As an application, we study a linear quadratic nonzero sum doubly stochastic differential games problem in this paper.

This paper is organized as follows. In Section 2, we state the problems and some assumptions. In Section 3, we study the variational equations and variational inequalities. In Section 4, a stochastic maximum principle in global form is obtained, subsequently, an example of this kind of control problems is given in this section. As an application, we study the optimal control problem of a kind of SPDEs with control variable by the approach of FBDSDEs in Section 5. Lastly, we give the explicit form of Nash equilibrium point for a kind of stochastic differential game problem.

For the simplicity of notations, we only consider the case where both y and Y are one-dimensional, and the control v is also one-dimensional. While in order to give the general results, we consider the multi-dimensional case in Section 6.

2. Statement of the problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $[0, T]$ be a given time duration throughout this paper. Let $\{W_t;\ 0 \le t \le T\}$ and $\{B_t;\ 0 \le t \le T\}$ be two mutually independent standard Brownian motions defined on $(\Omega, \mathcal{F}, P)$, with values respectively in $\mathbf{R}^d$ and in $\mathbf{R}^l$. Let $\mathcal{N}$ denote the class of P-null elements of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t \doteq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$$

where $\mathcal{F}_t^W = \mathcal{N} \vee \sigma\{W_r - W_0;\ 0 \le r \le t\}$, $\mathcal{F}_{t,T}^B = \mathcal{N} \vee \sigma\{B_r - B_t;\ t \le r \le T\}$. Note that the collection $\{\mathcal{F}_t,\ t \in [0, T]\}$ is neither increasing nor decreasing, and it does not constitute a classical filtration. We introduce the following:

Definition 2.1. A stochastic process $X = \{X_t;\ t \ge 0\}$ is called $\mathcal{F}_t$-progressively measurable, if for any $t \ge 0$, X on $\Omega \times [0, t]$ is measurable with respect to $\left(\mathcal{F}_t^W \times \mathcal{B}([0, t])\right) \vee \left(\mathcal{F}_{t,T}^B \times \mathcal{B}([t, T])\right)$.

Let $M^2\left(0, T; \mathbf{R}^n\right)$ denote the space of all (classes of $\mathrm{d}P \otimes \mathrm{d}t$ a.e. equal) $\mathbf{R}^n$-valued $\mathcal{F}_t$-progressively measurable stochastic processes $\{v_t;\ t \in [0, T]\}$ which satisfy

$$\mathbf{E} \int_0^T \left|v_t\right|^2 \mathrm{d}t < \infty.$$

Obviously $M^2(0,T;\mathbf{R}^n)$ is a Hilbert space. For a given $u \in M^2\left(0,T;\mathbf{R}^d\right)$ and $v \in M^2\left(0,T;\mathbf{R}^l\right)$, one can define the (standard) forward Itô's integral $\int_0^{\cdot} u_s \mathrm{d}W_s$ and the backward Itô's integral $\int_{\cdot}^T v_s \hat{\mathrm{d}}B_s$. They are both in $M^2(0,T;\mathbf{R})$ (see [19] for details). For the simplicity of notations, we only treat one dimension case. For multi-dimensional case, the results are the same.

Let $L^2(\Omega,\mathcal{F}_T,P;\mathbf{R})$ denote the space of all $\mathcal{F}_T$-measurable $\mathbf{R}$-valued random variable ξ satisfying $\mathbf{E}|\xi|^2<\infty$. Under this framework, we consider the following forward-backward doubly stochastic control system.

$$\begin{cases} \mathrm{d}y_t = f(t,y_t,Y_t,z_t,Z_t,v_t)\,\mathrm{d}t + g(t,y_t,Y_t,z_t,Z_t)\,\mathrm{d}W_t - z_t\hat{\mathrm{d}}B_t, \\ \mathrm{d}Y_t = -F(t,y_t,Y_t,z_t,Z_t,v_t)\,\mathrm{d}t - G(t,y_t,Y_t,z_t,Z_t)\,\hat{\mathrm{d}}B_t + Z_t\mathrm{d}W_t, \\ y_0 = x, \qquad Y_T = h(y_T), \qquad t\in[0,T], \end{cases} \tag{2.1}$$

where $\left(y_{(\cdot)},Y_{(\cdot)},z_{(\cdot)},Z_{(\cdot)},v_{(\cdot)}\right)\in\mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d\times\mathbf{R}$, $x\in\mathbf{R}$ is a given constant, $T>0$,

$$\begin{aligned} F &: [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d\times\mathbf{R}\to\mathbf{R}, \\ f &: [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d\times\mathbf{R}\to\mathbf{R}, \\ G &: [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d\times\mathbf{R}\to\mathbf{R}^l, \\ g &: [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d\times\mathbf{R}\to\mathbf{R}^d, \\ h &: \mathbf{R}\to\mathbf{R}. \end{aligned}$$

Let $\mathcal{U}$ be a nonempty subset of $\mathbf{R}$. We define the admissible control set

$$\mathcal{U}_{ad} \doteq \left\{v_{(\cdot)}\in M^2(0,T;\mathbf{R});\ v_t\in\mathcal{U},\ 0\le t\le T,\ \text{a.e., a.s.}\right\}.$$

Our optimal control problem is to minimize the cost function:

$$J\left(v_{(\cdot)}\right) \doteq \mathbf{E}\left[\int_0^T l(t,y_t,Y_t,z_t,Z_t,v_t)\,\mathrm{d}t + \Phi(y_T) + \gamma(Y_0)\right] \tag{2.2}$$

over $\mathcal{U}_{\mathrm{ad}}$, where

$$\begin{aligned} l &: [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d\times\mathbf{R}\to\mathbf{R}, \\ \Phi &: \mathbf{R}\to\mathbf{R}, \\ \gamma &: \mathbf{R}\to\mathbf{R}. \end{aligned}$$

An admissible control $u_{(\cdot)}$ is called an optimal control if it attains the minimum over $\mathcal{U}_{\mathrm{ad}}$. That is to say, we want to find a $u_{(\cdot)}$ such that

$$J\left(u_{(\cdot)}\right) \doteq \inf_{v_{(\cdot)}\in\mathcal{U}_{\mathrm{ad}}} J\left(v_{(\cdot)}\right).$$

(2.1) is called the state equation, the solution (y_t,Y_t,z_t,Z_t) corresponding to $u_{(\cdot)}$ is called the optimal trajectory.

Next we will give some notations:

$$\zeta = \begin{pmatrix} y \\ Y \\ z \\ Z \end{pmatrix}, \quad A(t,\zeta) = \begin{pmatrix} -F \\ f \\ -G \\ g \end{pmatrix}(t,\zeta).$$

We use the usual inner product $\langle\cdot,\cdot\rangle$ and Euclidean norm $|\cdot|$ in $\mathbf{R}$, $\mathbf{R}^l$ and $\mathbf{R}^d$. All the equalities and inequalities mentioned in this paper are in the sense of $\mathrm{d}t\otimes\mathrm{d}P$ almost surely on $[0,T]\times\Omega$. We assume that:

(H1) For each $\zeta\in\mathbf{R}^{1+1+l+d}$, $A(\cdot,\zeta)$ is an $\mathcal{F}_t$-measurable process defined on $[0,T]$ with $A(\cdot,0)\in M^2(0,T;\mathbf{R}^{1+1+l+d})$.

(H2) $A(t,\zeta)$ and $h(y)$ satisfy Lipschitz conditions: there exists a constant $k>0$, such that

$$\begin{cases} \left|A(t,\zeta)-A\left(t,\bar{\zeta}\right)\right|\le k\left|\zeta-\bar{\zeta}\right|, & \forall\zeta,\ \bar{\zeta}\in\mathbf{R}^{1+1+l+d},\ \forall t\in[0,T],\\ \left|h(y)-h(\bar{y})\right|\le k\left|y-\bar{y}\right|, & \forall y,\ \bar{y}\in\mathbf{R}. \end{cases}$$

The following monotonic conditions introduced in [22] are the main assumptions in this paper:

$$\text{(H3)}\quad\begin{cases} \left\langle A(t,\zeta)-A\left(t,\bar{\zeta}\right),\zeta-\bar{\zeta}\right\rangle\le-\mu\left|\zeta-\bar{\zeta}\right|^2,\\ \quad\forall\zeta=(y,Y,z,Z),\ \bar{\zeta}=\left(\bar{y},\bar{Y},\bar{z},\bar{Z}\right)\in\mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d,\ \forall t\in[0,T],\\ \left\langle h(y)-h(\bar{y}),y-\bar{y}\right\rangle\ge0,\ \forall y,\ \bar{y}\in\mathbf{R}, \end{cases}$$

or

$$\text{(H3)'}\quad\begin{cases} \left\langle A(t,\zeta)-A\left(t,\bar{\zeta}\right),\zeta-\bar{\zeta}\right\rangle\ge\mu\left|\zeta-\bar{\zeta}\right|^2,\\ \quad\forall\zeta=(y,Y,z,Z),\ \bar{\zeta}=\left(\bar{y},\bar{Y},\bar{z},\bar{Z}\right)\in\mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d,\ \forall t\in[0,T],\\ \left\langle h(y)-h(\bar{y}),y-\bar{y}\right\rangle\le0,\ \forall y,\ \bar{y}\in\mathbf{R}, \end{cases}$$

where μ is a positive constant.

Proposition 2.2. *For any given admissible control $v_{(\cdot)}$, we assume* (H1), (H2) *and* (H3) (*or* (H1), (H2) *and* (H3)') *hold. Then FBDSDE* (2.1) *has a unique solution* $(y_t,Y_t,z_t,Z_t)\in M^2\left(0,T;\mathbf{R}^{1+1+l+d}\right)$.

The proof is referred to [22]. We need a farther assumption as follows:

(H4) F, f, G, g, h, l, Φ, γ are continuously differentiable with respect to (y,Y,z,Z), y and Y. They and all their derivatives are bounded by a constant C.

Lastly, we need the following extension of Itô's formula (for details see [19]).

Proposition 2.3. *Let*

$$\alpha\in S^2\left(0,T;\mathbf{R}^k\right),\ \beta\in M^2\left(0,T;\mathbf{R}^k\right),\ \gamma\in M^2\left(0,T;\mathbf{R}^{k\times l}\right),\ \delta\in M^2\left(0,T;\mathbf{R}^{k\times d}\right)$$

satisfy:

$$\alpha_t=\alpha_0+\int_0^t\beta_s\mathrm{d}s+\int_0^t\gamma_s\hat{\mathrm{d}}B_s+\int_0^t\delta_s\mathrm{d}W_s,\qquad 0\le t\le T.$$

Then

$$|\alpha_t|^2=|\alpha_0|^2+2\int_0^t(\alpha_s,\beta_s)\,\mathrm{d}s+2\int_0^t\left(\alpha_s,\gamma_s\hat{\mathrm{d}}B_s\right)+2\int_0^t(\alpha_s,\delta_s\mathrm{d}W_s)-\int_0^t|\gamma_s|^2\,\mathrm{d}s+\int_0^t|\delta_s|^2\,\mathrm{d}s,$$

$$\mathbf{E}\,|\alpha_t|^2=\mathbf{E}\,|\alpha_0|^2+2\mathbf{E}\int_0^t(\alpha_s,\beta_s)\,\mathrm{d}s-\mathbf{E}\int_0^t|\gamma_s|^2\,\mathrm{d}s+\mathbf{E}\int_0^t|\delta_s|^2\,\mathrm{d}s.$$

Here $S^2\left(0,T;\mathbf{R}^k\right)$ denotes the space of (classes of $\mathrm{d}P\otimes\mathrm{d}t$ a.e. equal) all $\mathcal{F}_t$-progressively measurable k-dimensional processes v with

$$\mathbf{E}\left(\sup_{0\le t\le T}|v_t|^2\right)<\infty.$$

3. Variational equations and variational inequalities

Suppose $(y_t, Y_t, z_t, Z_t, u_t)$ is the solution to our optimal control problem. We introduce the following spike variational control:

$$u_t^\varepsilon = \begin{cases} v, & \tau \le t \le \tau+\varepsilon, \\ u_t, & \text{otherwise,} \end{cases}$$

where $\varepsilon > 0$ is sufficiently small, $\tau \in [0,T]$. v is an arbitrary $\mathcal{F}_\tau$-measurable random variable with values in $\mathcal{U}$, $0 \le t \le T$, and $\sup_{\omega\in\Omega} |v(\omega)| < \infty$. Let $(y_t^\varepsilon, Y_t^\varepsilon, z_t^\varepsilon, Z_t^\varepsilon)$ be the trajectory of the control system (2.1) corresponding to the control u_t^ε.

For convenience, we use the following notations in this paper:

$$\begin{aligned} \Xi_y &= \Xi_y(t, y_t, Y_t, z_t, Z_t, u_t), \\ \Xi_y(u_t^\varepsilon) &= \Xi_y(t, y_t, Y_t, z_t, Z_t, u_t^\varepsilon), \\ \Xi(u_t) &= \Xi(t, y_t, Y_t, z_t, Z_t, u_t), \\ \Xi(u_t^\varepsilon) &= \Xi(t, y_t, Y_t, z_t, Z_t, u_t^\varepsilon), \\ &\text{etc.,} \end{aligned}$$

where $\Xi = f,\ F,\ g,\ G$, respectively. We introduce the following variational equations:

$$\begin{cases} \mathrm{d}y_t^1 &= \left[f_y y_t^1 + f_Y Y_t^1 + f_z z_t^1 + f_Z Z_t^1 + f(u_t^\varepsilon) - f(u_t)\right]\mathrm{d}t \\ &\quad + \left[g_y y_t^1 + g_Y Y_t^1 + g_z z_t^1 + g_Z Z_t^1\right]\mathrm{d}W_t - z_t^1 \hat{\mathrm{d}} B_t, \\ y_0^1 &= 0, \\ \mathrm{d}Y_t^1 &= -\left[F_y y_t^1 + F_Y Y_t^1 + F_z z_t^1 + F_Z Z_t^1 + F(u_t^\varepsilon) - F(u_t)\right]\mathrm{d}t \\ &\quad - \left[G_y y_t^1 + G_Y Y_t^1 + G_z z_t^1 + G_Z Z_t^1\right]\hat{\mathrm{d}} B_t + Z_t^1 \mathrm{d}W_t, \\ Y_T^1 &= h_y(y_T)\, y_T^1. \end{cases} \tag{3.1}$$

Owing to (H4), it is easy to check that the variational equation (3.1) same as (2.1), also satisfies (H1), (H2) and (H3). Thus by Proposition 2.2, there exists a unique solution $\left(y_t^1, Y_t^1, z_t^1, Z_t^1\right) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d$, $0 \le t \le T$, satisfying (3.1). The variational inequalities can be derived from the fact $J\left(u_{(\cdot)}^\varepsilon\right) - J\left(u_{(\cdot)}\right) \ge 0$. The following lemmas play important roles to establish the inequalities.

Lemma 3.1. *We assume* (H1)–(H4) *hold. Then we have*

$$\mathbf{E}\int_0^T \left|y_t^1\right|^2 \mathrm{d}t \le C\varepsilon, \tag{3.2}$$

$$\mathbf{E}\int_0^T \left|Y_t^1\right|^2 \mathrm{d}t \le C\varepsilon, \tag{3.3}$$

$$\mathbf{E}\int_0^T \left|z_t^1\right|^2 \mathrm{d}t \le C\varepsilon, \tag{3.4}$$

$$\mathbf{E}\int_0^T \left|Z_t^1\right|^2 \mathrm{d}t \le C\varepsilon, \tag{3.5}$$

where $C > 0$ is some constant.

Proof. Using the Itô's formula to $\langle y_t^1, Y_t^1 \rangle$, it follows that

$$\begin{aligned}\mathbf{E}\left(\left|y_T^1\right|^2 h_y(y_T)\right) =& \mathbf{E}\int_0^T \left(f_y y_t^1 + f_Y Y_t^1 + f_z z_t^1 + f_Z Z_t^1\right) Y_t^1 \mathrm{d}t \\ &- \mathbf{E}\int_0^T \left(F_y y_t^1 + F_Y Y_t^1 + F_z z_t^1 + F_Z Z_t^1\right) y_t^1 \mathrm{d}t \\ &- \mathbf{E}\int_0^T \left(G_y y_t^1 + G_Y Y_t^1 + G_z z_t^1 + G_Z Z_t^1\right) z_t^1 \mathrm{d}t \\ &+ \mathbf{E}\int_0^T \left(g_y y_t^1 + g_Y Y_t^1 + g_z z_t^1 + g_Z Z_t^1\right) Z_t^1 \mathrm{d}t \\ &+ \mathbf{E}\int_0^T \left(f(u_t^\varepsilon) - f(u_t)\right) Y_t^1 \mathrm{d}t \\ &- \mathbf{E}\int_0^T \left(F(u_t^\varepsilon) - F(u_t)\right) y_t^1 \mathrm{d}t. \end{aligned} \tag{3.6}$$

Since (3.1) satisfies the monotonic condition (H3), it is easy to see that

$$\begin{aligned}&\mathbf{E}\left(\left|y_T^1\right|^2 h_y(y_T)\right) + \mu\mathbf{E}\int_0^T \left(\left|y_t^1\right|^2 + \left|Y_t^1\right|^2 + \left|z_t^1\right|^2 + \left|Z_t^1\right|^2\right) \mathrm{d}t \\ &\leq \mathbf{E}\int_0^T \left(f(u_t^\varepsilon) - f(u_t)\right) Y_t^1 \mathrm{d}t - \mathbf{E}\int_0^T \left(F(u_t^\varepsilon) - F(u_t)\right) y_t^1 \mathrm{d}t \\ &\leq \frac{1}{\mu}\mathbf{E}\int_0^T \left|f(u_t^\varepsilon) - f(u_t)\right|^2 \mathrm{d}t + \frac{\mu}{4}\mathbf{E}\int_0^T \left|Y_t^1\right|^2 \mathrm{d}t \\ &\quad + \frac{1}{\mu}\mathbf{E}\int_0^T \left|F(u_t^\varepsilon) - F(u_t)\right|^2 \mathrm{d}t + \frac{\mu}{4}\mathbf{E}\int_0^T \left|y_t^1\right|^2 \mathrm{d}t. \end{aligned} \tag{3.7}$$

From (H4) and (3.7), it is easy to know that (3.2)–(3.5) hold. The proof is complete. □

However, the order of the estimate for $\left(y_t^1, Y_t^1, z_t^1, Z_t^1\right)$ is too low to get the variational inequalities. We need to give some more elaborate estimates. For that, we firstly give the following lemma.

Lemma 3.2. *Assuming* (H1)–(H4) *hold, then we have*

$$\sup_{0\leq t\leq T}\left(\mathbf{E}\left|y_t^1\right|^2\right) \leq C\varepsilon, \tag{3.8}$$

$$\sup_{0\leq t\leq T}\left(\mathbf{E}\left|Y_t^1\right|^2\right) \leq C\varepsilon. \tag{3.9}$$

Proof. Squaring both sides of

$$\begin{aligned}y_t^1 + \int_0^t z_s^1 \hat{\mathrm{d}}B_s =& \int_0^t \left(f_y y_s^1 + f_Y Y_s^1 + f_z z_s^1 + f_Z Z_s^1 + f(u_s^\varepsilon) - f(u_s)\right) \mathrm{d}s \\ &+ \int_0^t \left(g_y y_s^1 + g_Y Y_s^1 + g_z z_s^1 + g_Z Z_s^1\right) \mathrm{d}W_s, \end{aligned}$$

noting that

$$\mathbf{E}\left[y_t^1 \int_0^t z_s^1 \hat{\mathrm{d}}B_s\right] = \mathbf{E}\left[\mathbf{E}^{\mathcal{F}_t}\left(y_t^1 \int_0^t z_s^1 \hat{\mathrm{d}}B_s\right)\right] = \mathbf{E}\left[y_t^1 \mathbf{E}^{\mathcal{F}_t}\left(\int_0^t z_s^1 \hat{\mathrm{d}}B_s\right)\right] = 0,$$

we have

$$\begin{aligned}\mathbf{E}\left|y_t^1\right|^2+\mathbf{E}\int_0^t\left|z_s^1\right|^2\mathrm{d}s=\mathbf{E}\bigg[&\int_0^t\left(f_yy_s^1+f_YY_s^1+f_zz_s^1+f_ZZ_s^1+f\left(u_s^\varepsilon\right)-f\left(u_s\right)\right)\mathrm{d}s\\&+\int_0^t\left(g_yy_s^1+g_YY_s^1+g_zz_s^1+g_ZZ_s^1\right)\mathrm{d}W_s\bigg]^2\\\le C\mathbf{E}&\int_0^t\left[\left|y_s^1\right|^2+\left|Y_s^1\right|^2+\left|z_s^1\right|^2+\left|Z_s^1\right|^2\right]\mathrm{d}s+C\mathbf{E}\left(\int_0^t\left(f\left(u_s^\varepsilon\right)-f\left(u_s\right)\right)\mathrm{d}s\right)^2.\end{aligned}$$

Thus

$$\sup_{0\le t\le T}\left(\mathbf{E}\left|y_t^1\right|^2\right)\le C\varepsilon.$$

By the similar argument, we can have

$$\sup_{0\le t\le T}\left(\mathbf{E}\left|Y_t^1\right|^2\right)\le C\varepsilon.$$

The proof is complete. □

Lemma 3.3. *Assuming* (H1)–(H4) *hold, then we have*

$$\mathbf{E}\left(\sup_{0\le t\le T}\left|y_t^1\right|^2\right)\le C\varepsilon, \tag{3.10}$$

$$\mathbf{E}\left(\sup_{0\le t\le T}\left|Y_t^1\right|^2\right)\le C\varepsilon. \tag{3.11}$$

Proof. Squaring both sides of

$$\begin{aligned}y_t^1=&\int_0^t\left(f_yy_s^1+f_YY_s^1+f_zz_s^1+f_ZZ_s^1+f\left(u_s^\varepsilon\right)-f\left(u_s\right)\right)\mathrm{d}s\\&+\int_0^t\left(g_yy_s^1+g_YY_s^1+g_zz_s^1+g_ZZ_s^1\right)\mathrm{d}W_s-\int_0^t z_s^1\hat{\mathrm{d}}B_s,\end{aligned}$$

we have

$$\begin{aligned}\left|y_t^1\right|^2\le\;&3\left(\int_0^t\left(f_yy_s^1+f_YY_s^1+f_zz_s^1+f_ZZ_s^1+f\left(u_s^\varepsilon\right)-f\left(u_s\right)\right)\mathrm{d}s\right)^2\\&+3\left(\int_0^t\left(g_yy_s^1+g_YY_s^1+g_zz_s^1+g_ZZ_s^1\right)\mathrm{d}W_s\right)^2+3\left(\int_0^t z_s^1\hat{\mathrm{d}}B_s\right)^2\\\le\;&3t\int_0^t\left(f_yy_s^1+f_YY_s^1+f_zz_s^1+f_ZZ_s^1+f\left(u_s^\varepsilon\right)-f\left(u_s\right)\right)^2\mathrm{d}s\\&+3\left(\int_0^t\left(g_yy_s^1+g_YY_s^1+g_zz_s^1+g_ZZ_s^1\right)\mathrm{d}W_s\right)^2+3\left(\int_0^T z_s^1\hat{\mathrm{d}}B_s-\int_t^T z_s^1\hat{\mathrm{d}}B_s\right)^2\\\le\;&C\int_0^t\left[\left|y_s^1\right|^2+\left|Y_s^1\right|^2+\left|z_s^1\right|^2+\left|Z_s^1\right|^2+\left|f\left(u_s^\varepsilon\right)-f\left(u_s\right)\right|^2\right]\mathrm{d}s\\&+3\left(\int_0^t\left(g_yy_s^1+g_YY_s^1+g_zz_s^1+g_ZZ_s^1\right)\mathrm{d}W_s\right)^2+6\left(\int_0^T z_s^1\hat{\mathrm{d}}B_s\right)^2+6\left(\int_t^T z_s^1\hat{\mathrm{d}}B_s\right)^2,\end{aligned}$$

then

$$\begin{aligned}\sup_{0\le t\le T}\left|y_t^1\right|^2 &\le C\int_0^T\left[\left|y_s^1\right|^2+\left|Y_s^1\right|^2+\left|z_s^1\right|^2+\left|Z_s^1\right|^2+\left|f\left(u_s^\varepsilon\right)-f\left(u_s\right)\right|^2\right]\mathrm{d}s+6\left(\int_0^T z_s^1\hat{\mathrm{d}}B_s\right)^2\\&\quad+3\sup_{0\le t\le T}\left(\int_0^t\left(g_y y_s^1+g_Y Y_s^1+g_z z_s^1+g_Z Z_s^1\right)\mathrm{d}W_s\right)^2+3\sup_{0\le t\le T}\left(\int_t^T z_s^1\hat{\mathrm{d}}B_s\right)^2\\&\le C\int_0^T\left[\left|y_s^1\right|^2+\left|Y_s^1\right|^2+\left|z_s^1\right|^2+\left|Z_s^1\right|^2+\left|f\left(u_s^\varepsilon\right)-f\left(u_s\right)\right|^2\right]\mathrm{d}s+6\left(\int_0^T z_s^1\hat{\mathrm{d}}B_s\right)^2\\&\quad+3\left(\sup_{0\le t\le T}\left|\int_0^t\left(g_y y_s^1+g_Y Y_s^1+g_z z_s^1+g_Z Z_s^1\right)\mathrm{d}W_s\right|\right)^2+3\left(\sup_{0\le t\le T}\left|\int_t^T z_s^1\hat{\mathrm{d}}B_s\right|\right)^2,\end{aligned}$$

where $C>0$ is some constant. Hereafter, C will be some generic constant, which can be different from line to line. Taking expectation, by B-D-G inequality and Hölder inequality, it follows that

$$\begin{aligned}\mathbf{E}\left(\sup_{0\le t\le T}\left|y_t^1\right|^2\right) &\le C\mathbf{E}\int_0^T\left[\left|y_s^1\right|^2+\left|Y_s^1\right|^2+\left|z_s^1\right|^2+\left|Z_s^1\right|^2+\left|f\left(u_s^\varepsilon\right)-f\left(u_s\right)\right|^2\right]\mathrm{d}s+6\mathbf{E}\int_0^T\left|z_s^1\right|^2\mathrm{d}s\\&\quad+C\mathbf{E}\int_0^T\left|g_y y_s^1+g_Y Y_s^1+g_z z_s^1+g_Z Z_s^1\right|^2\mathrm{d}s+C\mathbf{E}\int_0^T\left|z_s^1\right|^2\mathrm{d}s\\&\le C\mathbf{E}\int_0^T\left[\left|y_s^1\right|^2+\left|Y_s^1\right|^2+\left|z_s^1\right|^2+\left|Z_s^1\right|^2\right]\mathrm{d}s+C\mathbf{E}\int_0^T\left|f\left(u_s^\varepsilon\right)-f\left(u_s\right)\right|^2\mathrm{d}s.\end{aligned}$$

From Lemma 3.1, (3.10) holds. By the similar argument, we can prove (3.11). Squaring both sides of

$$\begin{aligned}Y_t^1 &= h_y\left(y_T\right)y_T^1+\int_t^T\left(F_y y_s^1+F_Y Y_s^1+F_z z_s^1+F_Z Z_s^1+F\left(u_s^\varepsilon\right)-F\left(u_s\right)\right)\mathrm{d}s\\&\quad+\int_t^T\left(G_y y_s^1+G_Y Y_s^1+G_z z_s^1+G_Z Z_s^1\right)\hat{\mathrm{d}}B_s-\int_t^T Z_s^1\mathrm{d}W_s,\end{aligned}$$

it follows that

$$\begin{aligned}\left|Y_t^1\right|^2 &\le 5\left|h_y\left(y_T\right)y_T^1\right|^2+5\left(\int_t^T\left(F_y y_s^1+F_Y Y_s^1+F_z z_s^1+F_Z Z_s^1+F\left(u_s^\varepsilon\right)-F\left(u_s\right)\right)\mathrm{d}s\right)^2\\&\quad+5\left(\int_t^T\left(G_y y_s^1+G_Y Y_s^1+G_z z_s^1+G_Z Z_s^1\right)\hat{\mathrm{d}}B_s\right)^2+5\left(\int_0^T Z_s^1\mathrm{d}W_s\right)^2+5\left(\int_0^t Z_s^1\mathrm{d}W_s\right)^2.\end{aligned}$$

Thus

$$\begin{aligned}\sup_{0\le t\le T}\left|Y_t^1\right|^2 &\le 5\left|h_y\left(y_T\right)y_T^1\right|^2+5\left(\int_0^T Z_s^1\mathrm{d}W_s\right)^2+5\sup_{0\le t\le T}\left(\int_0^t Z_s^1\mathrm{d}W_s\right)^2\\&\quad+5\left(T-t\right)\int_t^T\left|F_y y_s^1+F_Y Y_s^1+F_z z_s^1+F_Z Z_s^1+F\left(u_s^\varepsilon\right)-F\left(u_s\right)\right|^2\mathrm{d}s\\&\quad+5\sup_{0\le t\le T}\left(\int_t^T\left(G_y y_s^1+G_Y Y_s^1+G_z z_s^1+G_Z Z_s^1\right)\hat{\mathrm{d}}B_s\right)^2.\end{aligned}$$

Taking expectation and by B-D-G inequality, it follows that

$$\begin{aligned}\mathbf{E}\left(\sup_{0\le t\le T}\left|Y_t^1\right|^2\right) \le\ & 5\mathbf{E}\left|h_y\left(y_T\right)y_T^1\right|^2 + 5\mathbf{E}\int_0^T\left|Z_s^1\right|^2\mathrm{d}s + C\mathbf{E}\int_0^T\left|Z_s^1\right|^2\mathrm{d}s\\ &+ C\mathbf{E}\int_0^T\left(\left|y_s^1\right|^2+\left|Y_s^1\right|^2+\left|z_s^1\right|^2+\left|Z_s^1\right|^2+\left|F\left(u_s^\varepsilon\right)-F\left(u_s\right)\right|^2\right)\mathrm{d}s\\ &+ C\mathbf{E}\int_0^T\left|G_yy_s^1+G_YY_s^1+G_zz_s^1+G_ZZ_s^1\right|^2\mathrm{d}s.\end{aligned}$$

Noting (3.10), from Lemmas 3.1 and 3.2, it is easy to see that (3.11) holds. The proof is complete. □

Next, we will give some elaborate estimates for $\left(y_t^1, Y_t^1, z_t^1, Z_t^1\right)$ by virtue of the techniques of FBDSDEs.

Lemma 3.4. *Assuming* (H1)–(H4) *hold, then we have*

$$\mathbf{E}\int_0^T\left|y_t^1\right|^2\mathrm{d}t\le C\varepsilon^{\frac{3}{2}}, \tag{3.12}$$

$$\mathbf{E}\int_0^T\left|Y_t^1\right|^2\mathrm{d}t\le C\varepsilon^{\frac{3}{2}}, \tag{3.13}$$

$$\mathbf{E}\int_0^T\left|z_t^1\right|^2\mathrm{d}t\le C\varepsilon^{\frac{3}{2}}, \tag{3.14}$$

$$\mathbf{E}\int_0^T\left|Z_t^1\right|^2\mathrm{d}t\le C\varepsilon^{\frac{3}{2}}. \tag{3.15}$$

Proof. By (3.7), we have

$$\begin{aligned}&\mathbf{E}\left[\left|y_T^1\right|^2h_y\left(y_T\right)\right]+\mu\mathbf{E}\int_0^T\left(\left|y_t^1\right|^2+\left|Y_t^1\right|^2+\left|z_t^1\right|^2+\left|Z_t^1\right|^2\right)\mathrm{d}t\\ \le\ &\mathbf{E}\int_0^T\left(f\left(u_t^\varepsilon\right)-f\left(u_t\right)\right)Y_t^1\mathrm{d}t-\mathbf{E}\int_0^T\left(F\left(u_t^\varepsilon\right)-F\left(u_t\right)\right)y_t^1\mathrm{d}t\\ \le\ &\mathbf{E}\left[\sup_{0\le t\le T}\left|Y_t^1\right|\int_0^T\left|f\left(u_t^\varepsilon\right)-f\left(u_t\right)\right|\mathrm{d}t\right]\\ &+\mathbf{E}\left[\sup_{0\le t\le T}\left|y_t^1\right|\mathrm{d}t\int_0^T\left|F\left(u_t^\varepsilon\right)-F\left(u_t\right)\right|\mathrm{d}t\right]\\ \le\ &\left[\mathbf{E}\left(\sup_{0\le t\le T}\left|y_t^1\right|^2\right)\right]^{\frac{1}{2}}\left[\mathbf{E}\left(\int_0^T\left|F\left(u_t^\varepsilon\right)-F\left(u_t\right)\right|\mathrm{d}t\right)^2\right]^{\frac{1}{2}}\\ &+\left[\mathbf{E}\left(\sup_{0\le t\le T}\left|Y_t^1\right|^2\right)\right]^{\frac{1}{2}}\left[\mathbf{E}\left(\int_0^T\left|f\left(u_t^\varepsilon\right)-f\left(u_t\right)\right|\mathrm{d}t\right)^2\right]^{\frac{1}{2}}\\ \le\ &C\varepsilon^{\frac{3}{2}},\end{aligned}$$

where C is a sufficiently large positive constant. From (H3), the desired results are obtained. □

In order to obtain variational inequality, we need the following lemma.

Lemma 3.5. *Assuming* (H1)–(H4) *hold, then we have*

$$\mathbf{E}\int_0^T \left|y_t^\varepsilon - y_t - y_t^1\right|^2 \mathrm{d}t \le C\varepsilon^{\frac{3}{2}}, \tag{3.16}$$

$$\mathbf{E}\int_0^T \left|Y_t^\varepsilon - Y_t - Y_t^1\right|^2 \mathrm{d}t \le C\varepsilon^{\frac{3}{2}}, \tag{3.17}$$

$$\mathbf{E}\int_0^T \left|z_t^\varepsilon - z_t - z_t^1\right|^2 \mathrm{d}t \le C\varepsilon^{\frac{3}{2}}, \tag{3.18}$$

$$\mathbf{E}\int_0^T \left|Z_t^\varepsilon - Z_t - Z_t^1\right|^2 \mathrm{d}t \le C\varepsilon^{\frac{3}{2}}, \tag{3.19}$$

$$\sup_{0\le t\le T}\left[\mathbf{E}\left|y_t^\varepsilon - y_t - y_t^1\right|^2\right] \le C\varepsilon^{\frac{3}{2}}, \tag{3.20}$$

$$\sup_{0\le t\le T}\left[\mathbf{E}\left|Y_t^\varepsilon - Y_t - Y_t^1\right|^2\right] \le C\varepsilon^{\frac{3}{2}}. \tag{3.21}$$

Proof. For notational convenience, we denote

$$\begin{aligned}
\tilde{y}_t &= y_t^\varepsilon - y_t - y_t^1,\\
\tilde{Y}_t &= Y_t^\varepsilon - Y_t - Y_t^1,\\
\tilde{z}_t &= z_t^\varepsilon - z_t - z_t^1,\\
\tilde{Z}_t &= Z_t^\varepsilon - Z_t - Z_t^1.
\end{aligned}$$

We have the following FBDSDEs

$$\begin{aligned}
\tilde{y}_t &= \int_0^t \left[\tilde{f}_y\tilde{y}_s + \tilde{f}_Y\tilde{Y}_s + \tilde{f}_z\tilde{z}_s + \tilde{f}_Z\tilde{Z}_s\right]\mathrm{d}s + \int_0^t V_s^\varepsilon \mathrm{d}s + \int_0^t H_s \mathrm{d}s\\
&\quad + \int_0^t \left[\tilde{g}_y\tilde{y}_s + \tilde{g}_Y\tilde{Y}_s + \tilde{g}_z\tilde{z}_s + \tilde{g}_Z\tilde{Z}_s\right]\mathrm{d}W_s - \int_0^t \tilde{z}_s \hat{\mathrm{d}}B_s,\\
\tilde{Y}_t &= h\left(y_T^\varepsilon\right) - h\left(y_T + y_T^1\right) + \int_t^T \left[\tilde{F}_y\tilde{y}_s + \tilde{F}_Y\tilde{Y}_s + \tilde{F}_z\tilde{z}_s + \tilde{F}_Z\tilde{Z}_s\right]\mathrm{d}s\\
&\quad + \int_t^T \left[\tilde{G}_y\tilde{y}_s + \tilde{G}_Y\tilde{Y}_s + \tilde{G}_z\tilde{z}_s + \tilde{G}_Z\tilde{Z}_s\right]\hat{\mathrm{d}}B_s + \int_t^T \tilde{V}_s^\varepsilon \mathrm{d}s + \int_t^T \tilde{H}_s \mathrm{d}s\\
&\quad + \int_0^1 \left(h_y\left(y_T + y_T^1\lambda\right) - h_y\left(y_T\right)\right) y_T^1 \mathrm{d}\lambda - \int_t^T \tilde{Z}_s \mathrm{d}W_s,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{f}_y &= \int_0^1 f_y\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s, u_s^\varepsilon\right)\mathrm{d}\lambda,\\
\tilde{f}_Y &= \int_0^1 f_Y\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s, u_s^\varepsilon\right)\mathrm{d}\lambda,\\
\tilde{f}_z &= \int_0^1 f_z\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s, u_s^\varepsilon\right)\mathrm{d}\lambda,\\
\tilde{f}_Z &= \int_0^1 f_Z\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s, u_s^\varepsilon\right)\mathrm{d}\lambda,
\end{aligned}$$

$$
\begin{aligned}
\tilde{F}_y &= \int_0^1 F_y\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s, u_s^\varepsilon\right)\mathrm{d}\lambda,\\
\tilde{F}_Y &= \int_0^1 F_Y\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s, u_s^\varepsilon\right)\mathrm{d}\lambda,\\
\tilde{F}_z &= \int_0^1 F_z\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s, u_s^\varepsilon\right)\mathrm{d}\lambda,\\
\tilde{F}_Z &= \int_0^1 F_Z\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s, u_s^\varepsilon\right)\mathrm{d}\lambda,\\
\tilde{g}_y &= \int_0^1 g_y\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s\right)\mathrm{d}\lambda,\\
\tilde{g}_Y &= \int_0^1 g_Y\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s\right)\mathrm{d}\lambda,\\
\tilde{g}_z &= \int_0^1 g_z\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s\right)\mathrm{d}\lambda,\\
\tilde{g}_Z &= \int_0^1 g_Z\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s\right)\mathrm{d}\lambda,\\
\tilde{G}_y &= \int_0^1 G_y\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s\right)\mathrm{d}\lambda,\\
\tilde{G}_Y &= \int_0^1 G_Y\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s\right)\mathrm{d}\lambda,\\
\tilde{G}_z &= \int_0^1 G_z\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s\right)\mathrm{d}\lambda,\\
\tilde{G}_Z &= \int_0^1 G_Z\left(y_s + y_s^1 + \lambda\tilde{y}_s, Y_s + Y_s^1 + \lambda\tilde{Y}_s, z_s + z_s^1 + \lambda\tilde{z}_s, Z_s + Z_s^1 + \lambda\tilde{Z}_s\right)\mathrm{d}\lambda,\\
V_s^\varepsilon &= \int_0^1 \left[f_y\left(y_s + \lambda y_s^1, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1, u_s^\varepsilon\right) - f_y\right] y_s^1 \mathrm{d}\lambda\\
&\quad + \int_0^1 \left[f_Y\left(y_s + \lambda y_s^1, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1, u_s^\varepsilon\right) - f_Y\right] Y_s^1 \mathrm{d}\lambda\\
&\quad + \int_0^1 \left[f_z\left(y_s + \lambda y_s^1, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1, u_s^\varepsilon\right) - f_z\right] z_s^1 \mathrm{d}\lambda\\
&\quad + \int_0^1 \left[f_Z\left(y_s + \lambda y_s^1, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1, u_s^\varepsilon\right) - f_Z\right] Z_s^1 \mathrm{d}\lambda,\\
H_s &= \int_0^1 \left[g_y\left(y_s + y_s^1\lambda, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1\right) - g_y\right] y_s^1 \mathrm{d}\lambda\\
&\quad + \int_0^1 \left[g_Y\left(y_s + y_s^1\lambda, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1\right) - g_Y\right] Y_s^1 \mathrm{d}\lambda\\
&\quad + \int_0^1 \left[g_z\left(y_s + y_s^1\lambda, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1\right) - g_z\right] z_s^1 \mathrm{d}\lambda\\
&\quad + \int_0^1 \left[g_Z\left(y_s + y_s^1\lambda, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1\right) - g_Z\right] Z_s^1 \mathrm{d}\lambda,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{V}_s^\varepsilon = & \int_0^1 \left[F_y\left(y_s + y_s^1\lambda, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1, u_s^\varepsilon\right) - F_y\right] y_s^1 \mathrm{d}\lambda \\
& + \int_0^1 \left[F_Y\left(y_s + y_s^1\lambda, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1, u_s^\varepsilon\right) - F_Y\right] Y_s^1 \mathrm{d}\lambda \\
& + \int_0^1 \left[F_z\left(y_s + y_s^1\lambda, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1, u_s^\varepsilon\right) - F_z\right] z_s^1 \mathrm{d}\lambda \\
& + \int_0^1 \left[F_Z\left(y_s + y_s^1\lambda, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1, u_s^\varepsilon\right) - F_Z\right] Z_s^1 \mathrm{d}\lambda, \\
\tilde{H}_s = & \int_0^1 \left[G_y\left(y_s + y_s^1\lambda, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1\right) - G_y\right] y_s^1 \mathrm{d}\lambda \\
& + \int_0^1 \left[G_Y\left(y_s + y_s^1\lambda, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1\right) - G_Y\right] Y_s^1 \mathrm{d}\lambda \\
& + \int_0^1 \left[G_z\left(y_s + y_s^1\lambda, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1\right) - G_z\right] z_s^1 \mathrm{d}\lambda \\
& + \int_0^1 \left[G_Z\left(y_s + y_s^1\lambda, Y_s + \lambda Y_s^1, z_s + \lambda z_s^1, Z_s + \lambda Z_s^1\right) - G_Z\right] Z_s^1 \mathrm{d}\lambda.
\end{aligned}
$$

It is easy to check that

$$
\begin{aligned}
\mathbf{E}\int_0^T \left|\tilde{V}_s^\varepsilon\right|^2 \mathrm{d}s \le C\varepsilon^{\frac{3}{2}}, \\
\mathbf{E}\int_0^T \left|\tilde{H}_s\right|^2 \mathrm{d}s \le C\varepsilon^{\frac{3}{2}}, \\
\mathbf{E}\int_0^T \left|V_s^\varepsilon\right|^2 \mathrm{d}s \le C\varepsilon^{\frac{3}{2}}, \\
\mathbf{E}\int_0^T \left|H_s\right|^2 \mathrm{d}s \le C\varepsilon^{\frac{3}{2}}.
\end{aligned}
$$

By Lemma 3.4, applying Itô's formula to $\left\langle \tilde{y}_t, \tilde{Y}_t \right\rangle$ on $[0,T]$, we get

$$
\begin{aligned}
& \mathbf{E}\left\langle h\left(y_T^\varepsilon\right) - h\left(y_T\right) - h_y\left(y_T\right) y_T^1, y_T^\varepsilon - y_T - y_T^1 \right\rangle + \mu \mathbf{E}\int_0^T \left[\left|\tilde{y}_s\right|^2 + \left|\tilde{Y}_s\right|^2 + \left|\tilde{z}_s\right|^2 + \left|\tilde{Z}_s\right|^2 \right] \mathrm{d}s \\
\le & -\mathbf{E}\int_0^T \left\langle \tilde{y}_s, \tilde{V}_s^\varepsilon + \tilde{H}_s \right\rangle \mathrm{d}s + \mathbf{E}\int_0^T \left\langle \tilde{Y}_s, V_s^\varepsilon + H_s \right\rangle \mathrm{d}s \\
\le & \mathbf{E}\frac{\mu}{2}\int_0^T \left|\tilde{y}_s\right|^2 \mathrm{d}s + \mathbf{E}\frac{1}{\mu}\int_0^T \left|\tilde{V}_s^\varepsilon\right|^2 \mathrm{d}s + \frac{1}{\mu}\mathbf{E}\int_0^T \left|\tilde{H}_s\right|^2 \mathrm{d}s \\
& + \mathbf{E}\frac{\mu}{2}\int_0^T \left|\tilde{Y}_s\right|^2 \mathrm{d}s + \mathbf{E}\frac{1}{\mu}\int_0^T \left|V_s^\varepsilon\right|^2 \mathrm{d}s + \frac{1}{\mu}\mathbf{E}\int_0^T \left|H_s\right|^2 \mathrm{d}s.
\end{aligned}
$$

Noting that by means of the same arguments in Lemma 3.2, from Lemma 3.4, we easily have

$$
\begin{aligned}
\sup_{0\le t\le T}\left(\mathbf{E}\left|y_t^1\right|^2\right) \le C\varepsilon^{\frac{3}{2}}, \\
\sup_{0\le t\le T}\left(\mathbf{E}\left|Y_t^1\right|^2\right) \le C\varepsilon^{\frac{3}{2}}.
\end{aligned}
$$

Thus it is obvious that

$$\mathbf{E}h\left(y_T+y_T^1\right)=\mathbf{E}h\left(y_T\right)+\mathbf{E}h_y\left(y_T\right)y_T^1+C\varepsilon^{\frac{3}{2}},$$

so by (H3) it follows that

$$\begin{aligned}\mathbf{E}\left\langle h\left(y_T^\varepsilon\right)-h\left(y_T\right)-h_y\left(y_T\right)y_T^1,y_T^\varepsilon-y_T-y_T^1\right\rangle&=\mathbf{E}\left\langle h\left(y_T^\varepsilon\right)-h\left(y_T+y_T^1\right)+C\varepsilon^{\frac{3}{2}},y_T^\varepsilon-y_T-y_T^1\right\rangle\\&\geq\mathbf{E}\left(y_T^\varepsilon-y_T-y_T^1\right)\cdot C\varepsilon^{\frac{3}{2}},\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\mu\int_0^T\left[\frac{1}{2}\left|\tilde{y}_s\right|^2+\frac{1}{2}\left|\tilde{Y}_s\right|^2+\left|\tilde{z}_s\right|^2+\left|\tilde{Z}_s\right|^2\right]\mathrm{d}s&\leq\mathbf{E}\frac{1}{\mu}\int_0^T\left|\tilde{V}_s^\varepsilon\right|^2\mathrm{d}s+\frac{1}{\mu}\mathbf{E}\int_0^T\left|\tilde{H}_s\right|^2\mathrm{d}s\\&\quad+\mathbf{E}\frac{1}{\mu}\int_0^T\left|V_s^\varepsilon\right|^2\mathrm{d}s+\frac{1}{\mu}\mathbf{E}\int_0^T\left|H_s\right|^2\mathrm{d}s-\mathbf{E}\left(\tilde{y}_T\right)\cdot C\varepsilon^{\frac{3}{2}}.\end{aligned}$$

It is not difficult to see that $\mathbf{E}\left(\tilde{y}_T\right)$ is bounded. Consequently, from that, (3.16)–(3.19) hold. Further using the similar arguments in Lemma 3.2, we can obtain (3.20) and (3.21). The proof is complete. □

Lemma 3.6 (variational inequality)**.** *Under the assumptions* (H1)–(H4)*, it holds that*

$$\mathbf{E}\int_0^T\left[l_y y_t^1+l_Y Y_t^1+l_z z_t^1+l_Z Z_t^1+l\left(u_t^\varepsilon\right)-l\left(u_t\right)\right]\mathrm{d}t+\mathbf{E}\left[\Phi_y\left(y_T\right)y_T^1\right]+\mathbf{E}\left[\gamma_Y\left(Y_0\right)Y_0^1\right]\geq o\left(\varepsilon\right).\tag{3.22}$$

Proof. According to the definition of u_t^ε, we have

$$J\left(u_{(\cdot)}^\varepsilon\right)\geq J\left(u_{(\cdot)}\right),$$

moreover

$$\mathbf{E}\int_0^T\left[l\left(t,y_t^\varepsilon,Y_t^\varepsilon,z_t^\varepsilon,Z_t^\varepsilon,u_t^\varepsilon\right)-l\left(t,y_t,Y_t,z_t,Z_t,u_t\right)\right]\mathrm{d}t+\mathbf{E}\left[\Phi\left(y_T^\varepsilon\right)-\Phi\left(y_T\right)\right]+\mathbf{E}\left[\gamma\left(Y_0^\varepsilon\right)-\gamma\left(Y_0\right)\right]\geq0,$$

or

$$\begin{aligned}&\mathbf{E}\int_0^T\left[l\left(t,y_t^\varepsilon,Y_t^\varepsilon,z_t^\varepsilon,Z_t^\varepsilon,u_t^\varepsilon\right)-l\left(t,y_t+y_t^1,Y_t+Y_t^1,z_t+z_t^1,Z_t+Z_t^1,u_t^\varepsilon\right)\right]\mathrm{d}t\\&+\mathbf{E}\int_0^T\left[l\left(t,y_t+y_t^1,Y_t+Y_t^1,z_t+z_t^1,Z_t+Z_t^1,u_t^\varepsilon\right)-l\left(t,y_t,Y_t,z_t,Z_t,u_t\right)\right]\mathrm{d}t\\&+\mathbf{E}\left[\Phi\left(y_T^\varepsilon\right)-\Phi\left(y_T+y_T^1\right)\right]+\mathbf{E}\left[\Phi\left(y_T+y_T^1\right)-\Phi\left(y_T\right)\right]\\&+\mathbf{E}\left[\gamma\left(Y_0^\varepsilon\right)-\gamma\left(Y_0+Y_T^1\right)\right]+\mathbf{E}\left[\gamma\left(Y_0+Y_T^1\right)-\gamma\left(Y_0\right)\right]\geq0.\end{aligned}$$

By Lemma 3.5, it follows that

$$\begin{aligned}&\mathbf{E}\int_0^T\left[l\left(t,y_t^\varepsilon,Y_t^\varepsilon,z_t^\varepsilon,Z_t^\varepsilon,u_t^\varepsilon\right)-l\left(t,y_t+y_t^1,Y_t+Y_t^1,z_t+z_t^1,Z_t+Z_t^1,u_t^\varepsilon\right)\right]\mathrm{d}t\\&+\mathbf{E}\left[\Phi\left(y_T^\varepsilon\right)-\Phi\left(y_T+y_T^1\right)\right]+\mathbf{E}\left[\gamma\left(Y_0^\varepsilon\right)-\gamma\left(Y_0+Y_T^1\right)\right]\leq C\varepsilon^{\frac{3}{2}},\end{aligned}$$

while

$$
\begin{aligned}
0 \leq\ & \mathbf{E} \int_0^T \left[l\left(t, y_t + y_t^1, Y_t + Y_t^1, z_t + z_t^1, Z_t + Z_t^1, u_t^\varepsilon\right) - l\left(t, y_t, Y_t, z_t, Z_t, u_t\right) \right] \mathrm{d}t \\
& + \mathbf{E}\left[\Phi\left(y_T + y_T^1\right) - \Phi\left(y_T\right) \right] + \mathbf{E}\left[\gamma\left(Y_0 + Y_T^1\right) - \gamma\left(Y_0\right) \right] + C\varepsilon^{\frac{3}{2}} \\
=\ & \mathbf{E} \int_0^T \left[l\left(t, y_t + y_t^1, Y_t + Y_t^1, z_t + z_t^1, Z_t + Z_t^1, u_t\right) - l\left(t, y_t, Y_t, z_t, Z_t, u_t\right) \right] \mathrm{d}t \\
& + \mathbf{E} \int_0^T l\left(t, y_t + y_t^1, Y_t + Y_t^1, z_t + z_t^1, Z_t + Z_t^1, u_t^\varepsilon\right) \mathrm{d}t \\
& - \mathbf{E} \int_0^T l\left(t, y_t + y_t^1, Y_t + Y_t^1, z_t + z_t^1, Z_t + Z_t^1, u_t\right) \mathrm{d}t \\
& + \mathbf{E}\left[\Phi\left(y_T + y_T^1\right) - \Phi\left(y_T\right) \right] + \mathbf{E}\left[\gamma\left(Y_0 + Y_T^1\right) - \gamma\left(Y_0\right) \right] + C\varepsilon^{\frac{3}{2}} \\
=\ & \mathbf{E} \int_0^T \left[l_y y_t^1 + l_Y Y_t^1 + l_z z_t^1 + l_Z Z_t^1 \right] \mathrm{d}t + \mathbf{E} \int_0^T \left[l\left(u_t^\varepsilon\right) - l\left(u_t\right) \right] \mathrm{d}t \\
& + \mathbf{E} \int_0^T \left\{ \left[l_y\left(u_t^\varepsilon\right) - l_y\left(u_t\right) \right] y_t^1 + \left[l_Y\left(u_t^\varepsilon\right) - l_Y\left(u_t\right) \right] Y_t^1 \right\} \mathrm{d}t \\
& + \mathbf{E} \int_0^T \left\{ \left[l_z\left(u_t^\varepsilon\right) - l_z\left(u_t\right) \right] z_t^1 + \left[l_Z\left(u_t^\varepsilon\right) - l_Z\left(u_t\right) \right] Z_t^1 \right\} \mathrm{d}t + \mathbf{E}\left[\Phi_y\left(y_T\right) y_T^1 \right] + \mathbf{E}\left[\gamma_Y\left(Y_0\right) Y_0^1 \right] + C\varepsilon^{\frac{3}{2}} \\
\leq\ & \mathbf{E} \int_0^T \left[l_y y_t^1 + l_Y Y_t^1 + l_z z_t^1 + l_Z Z_t^1 \right] \mathrm{d}t + \mathbf{E} \int_0^T \left[l\left(u_t^\varepsilon\right) - l\left(u_t\right) \right] \mathrm{d}t \\
& + \mathbf{E}\left[\sup_{0 \leq t \leq T} \left| y_t^1 \right| \int_0^T \left| l_y\left(u_t^\varepsilon\right) - l_y\left(u_t\right) \right| \mathrm{d}t \right] + \mathbf{E}\left[\sup_{0 \leq t \leq T} \left| Y_t^1 \right| \mathrm{d}t \int_0^T \left| l_Y\left(u_t^\varepsilon\right) - l_Y\left(u_t\right) \right| \mathrm{d}t \right] \\
& + \left[\mathbf{E} \int_0^T \left| l_z\left(u_t^\varepsilon\right) - l_z\left(u_t\right) \right|^2 \mathrm{d}t \right]^{\frac{1}{2}} \left[\mathbf{E} \int_0^T \left| z_t^1 \right|^2 \mathrm{d}t \right]^{\frac{1}{2}} \\
& + \left[\mathbf{E} \int_0^T \left| l_Z\left(u_t^\varepsilon\right) - l_Z\left(u_t\right) \right|^2 \mathrm{d}t \right]^{\frac{1}{2}} \left[\mathbf{E} \int_0^T \left| Z_t^1 \right|^2 \mathrm{d}t \right]^{\frac{1}{2}} + \mathbf{E}\left[\Phi_y\left(y_T\right) y_T^1 \right] + \mathbf{E}\left[\gamma_Y\left(Y_0\right) Y_0^1 \right] + C\varepsilon^{\frac{3}{2}} \\
\leq\ & \mathbf{E} \int_0^T \left[l_y y_t^1 + l_Y Y_t^1 + l_z z_t^1 + l_Z Z_t^1 \right] \mathrm{d}t + \mathbf{E} \int_0^T \left[l\left(u_t^\varepsilon\right) - l\left(u_t\right) \right] \mathrm{d}t \\
& + \left[\mathbf{E}\left(\sup_{0 \leq t \leq T} \left| y_t^1 \right|^2 \right) \right]^{\frac{1}{2}} \left[\mathbf{E}\left(\int_0^T \left| l_y\left(u_t^\varepsilon\right) - l_y\left(u_t\right) \right| \mathrm{d}t \right)^2 \right]^{\frac{1}{2}} \\
& + \left[\mathbf{E} \sup_{0 \leq t \leq T} \left(\left| Y_t^1 \right|^2 \right) \right]^{\frac{1}{2}} \left[\mathbf{E}\left(\int_0^T \left| l_Y\left(u_t^\varepsilon\right) - l_Y\left(u_t\right) \right| \mathrm{d}t \right)^2 \right]^{\frac{1}{2}} \\
& + C\varepsilon^{\frac{1}{2}} \cdot C\varepsilon^{\frac{3}{4}} + C\varepsilon^{\frac{1}{2}} \cdot C\varepsilon^{\frac{3}{4}} + \mathbf{E}\left[\Phi_y\left(y_T\right) y_T^1 \right] + \mathbf{E}\left[\gamma_Y\left(Y_0\right) Y_0^1 \right] + C\varepsilon^{\frac{3}{2}} \\
=\ & \mathbf{E} \int_0^T \left[l_y y_t^1 + l_Y Y_t^1 + l_z z_t^1 + l_Z Z_t^1 + l\left(u_t^\varepsilon\right) - l\left(u_t\right) \right] \mathrm{d}t \\
& + \mathbf{E}\left[\Phi_y\left(y_T\right) y_T^1 \right] + \mathbf{E}\left[\gamma_Y\left(Y_0\right) Y_0^1 \right] + o\left(\varepsilon\right).
\end{aligned}
$$

From that, the desired result is obtained. □

4. The maximum principle in global form

We introduce the adjoint equations by virtue of dual technique and Hamilton function for our control problem. From the variational inequality obtained in Lemma 3.6, the maximum principle can be proved by means of Itô's formula. The adjoint equations are as follows:

$$\begin{cases} \mathrm{d}p_t = (F_Y p_t - f_Y q_t + G_Y k_t - g_Y h_t - l_Y)\mathrm{d}t \\ \qquad\quad + (F_Z p_t - f_Z q_t + G_Z k_t - g_Z h_t - l_Z)\mathrm{d}W_t - k_t \hat{\mathrm{d}} B_t, \\ \mathrm{d}q_t = (F_y p_t - f_y q_t + G_y k_t - g_y h_t - l_y)\mathrm{d}t \\ \qquad\quad + (F_z p_t - f_z q_t + G_z k_t - g_z h_t - l_z)\hat{\mathrm{d}} B_t + h_t \mathrm{d}W_t, \\ p_0 = -\gamma_Y(Y_0), \qquad q_T = -h_y(y_T) P_T + \Phi_y(y_T), \qquad 0 \le t \le T, \end{cases} \tag{4.1}$$

where $\left(p_{(\cdot)}, q_{(\cdot)}, k_{(\cdot)}, h_{(\cdot)}\right) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d$. It is easy to verify that FBDSDE (4.1) satisfies (H1), (H2) and (H3)'. From Proposition 2.2, we know that (4.1) has a unique solution $\left(p_{(\cdot)}, q_{(\cdot)}, k_{(\cdot)}, h_{(\cdot)}\right) \in M^2\left(0,T; \mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d\right)$. Now we define the Hamilton function as follows:

$$\begin{aligned} H(t,y,Y,z,Z,v,p,q,k,h) \doteq{}& \langle q, f(t,y,Y,z,Z,v)\rangle - \langle p, F(t,y,Y,z,Z,v)\rangle \\ &- \langle k, G(t,y,Y,z,Z)\rangle + \langle h, g(t,y,Y,z,Z)\rangle + l(t,y,Y,z,Z,v), \end{aligned} \tag{4.2}$$

where $H: [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}^l\times\mathbf{R}^d \to \mathbf{R}$. (4.1) can be rewritten as

$$\begin{cases} \mathrm{d}p_t = -H_Y \mathrm{d}t - H_Z \mathrm{d}W_t - k_t \hat{\mathrm{d}} B_t, \\ \mathrm{d}q_t = -H_y \mathrm{d}t - H_z \hat{\mathrm{d}} B_t + h_t \mathrm{d}W_t, \\ p_0 = -\gamma_Y(Y_0), \\ q_T = -h_y(y_T) P_T + \Phi_y(y_T), \quad 0 \le t \le T, \end{cases} \tag{4.3}$$

where $H_\beta = H_\beta(t, y(t), Y(t), z(t), Z(t), u(t), p(t), q(t), k(t), h(t))$, $\beta = y,\ Y,\ z,\ Z$, respectively.

From Lemma 3.6 and (4.2), we can obtain the main result in this paper.

Theorem 4.1. *Suppose* (H1)–(H4) *hold. Let* $\left(y_{(\cdot)}, Y_{(\cdot)}, z_{(\cdot)}, Z_{(\cdot)}, u_{(\cdot)}\right)$ *be an optimal control and its corresponding trajectory of* (2.1), $\left(p_{(\cdot)}, q_{(\cdot)}, k_{(\cdot)}, h_{(\cdot)}\right)$ *be the corresponding solution of* (4.1). *Then the maximum principle holds, that is*

$$H(t, y_t, Y_t, z_t, Z_t, v, p_t, q_t, k_t, h_t) \ge H(t, y_t, Y_t, z_t, Z_t, u_t, p_t, q_t, k_t, h_t), \ \forall v \in \mathcal{U}, \ a.e, \ a.s. \tag{4.4}$$

Proof. By applying Itô's formula to $\langle p_t, Y_t^1\rangle + \langle q_t, y_t^1\rangle$, and noting the variational equation (3.1), the adjoint equation (4.1) and the variational inequality (3.22), we get

$$\begin{aligned} &\mathbf{E}\left[\Phi_y(y_T) y_T^1\right] + \mathbf{E}\left[\gamma_Y(Y_0) Y_0^1\right] + \mathbf{E}\int_0^T \left[l_y y_t^1 + l_Y Y_t^1 + l_z z_t^1 + l_Z Z_t^1 + l(u_t^\varepsilon) - l(u_t)\right]\mathrm{d}t \\ &\qquad = \mathbf{E}\int_0^T \left[H(t, y_t, Y_t, z_t, Z_t, u_t^\varepsilon, p_t, q_t, k_t, h_t) - H(t, y_t, Y_t, z_t, Z_t, u_t, p_t, q_t, k_t, h_t)\right]\mathrm{d}t \ge o(\varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ can be arbitrarily small, from the above inequality, (4.4) can be easily obtained. The proof is complete. □

In the last part of this section, we provide a concrete example of forward-backward doubly stochastic LQ control problems. We give the explicit optimal control and validate our major theoretical result in Theorem 4.1.

Example 4.2. Let the control domain be $\mathcal{U} = [-1, 1]$. Consider the following linear forward-backward doubly stochastic control system which is a simple case of (2.1). We assume that $l = d = 1$:

$$\begin{cases} \mathrm{d}y_t = (z_t - Z_t + v_t)\,\mathrm{d}W_t - z_t \hat{\mathrm{d}}B_t, \\ \mathrm{d}Y_t = -(z_t + Z_t + v_t)\,\hat{\mathrm{d}}B_t + Z_t \mathrm{d}W_t, \\ y_0 = 0, \quad Y_T = 0, \quad t \in [0, T], \end{cases} \tag{4.5}$$

where $T > 0$ is a given constant and the cost function is

$$J\left(v_{(\cdot)}\right) = \frac{1}{2}\mathbf{E}\int_0^T \left(y_t^2 + Y_t^2 + z_t^2 + Z_t^2 + v_t^2\right)\mathrm{d}t + \frac{1}{2}\mathbf{E}y_T^2 + \frac{1}{2}\mathbf{E}Y_0^2. \tag{4.6}$$

Note that (4.5) are a linear control system. According to the existence and uniqueness for (4.5), it is straightforward to know the optimal control is $u_{(\cdot)} \equiv 0$, with the corresponding optimal state trajectory $(y_t, Y_t, z_t, Z_t) \equiv 0$, $t \in [0, T]$. Notice that the adjoint equation associated with the optimal quadruple $(y_t, Y_t, z_t, Z_t) \equiv 0$ are

$$\begin{cases} \mathrm{d}p_t = -Y_t \mathrm{d}t + (-k_t - h_t - Z_t)\,\mathrm{d}W_t - k_t \hat{\mathrm{d}}B_t, \\ \mathrm{d}q_t = -y_t \mathrm{d}t + (-k_t - h_t - z_t)\,\hat{\mathrm{d}}B_t + h_t \mathrm{d}W_t, \\ p_0 = 0, \quad q_T = 0, \quad t \in [0, T]. \end{cases} \tag{4.7}$$

Obviously, $(p_t, q_t, k_t, h_t) \equiv 0$ is the unique solution of (4.7). Instantly, we give the Hamilton function is

$$\begin{aligned} H\left(t, y_t, Y_t, z_t, Z_t, v, p_t, q_t, k_t, h_t\right) &= \frac{1}{2}\left(y_t^2 + Y_t^2 + z_t^2 + Z_t^2 + v^2\right) - k_t\left(z_t + Z_t + v\right) + h_t\left(z_t - Z_t + v\right) \\ &= \frac{1}{2}v^2. \end{aligned}$$

It is clear that, for any $v \in \mathcal{U}$, we always have

$$H\left(t, y_t, Y_t, z_t, Z_t, v, p_t, q_t, k_t, h_t\right) \geq H\left(t, y_t, Y_t, z_t, Z_t, u_t, p_t, q_t, k_t, h_t\right) = 0, \quad \text{a.e, a.s.}$$

5. Applications to optimal control problems of stochastic partial differential equations

Let us first give some notations from [19]. For convenience, all the variables in this section are one-dimensional. From now on $C^k(\mathbf{R};\mathbf{R})$, $C^k_{l,b}(\mathbf{R};\mathbf{R})$, $C^k_p(\mathbf{R};\mathbf{R})$ will denote respectively the set of functions of class C^k from $\mathbf{R}$ into $\mathbf{R}$, the set of those functions of class C^k whose partial derivatives of order less than or equal to k are bounded (and hence the function itself grows at most linearly at infinity), and the set of those functions of class C^k which, together with all their partial derivatives of order less than or equal to k, grow at most like a polynomial function of the variable x at infinity. We consider the following quasilinear SPDEs with control variable:

$$\begin{cases} u(t,x) = \tilde{h}(x) + \int_t^T \left[\mathcal{L}u(s,x) + f\left(s, x, u(s,x), (\nabla u\sigma)(s,x), v_s\right)\right]\mathrm{d}s \\ \qquad\qquad + \int_t^T g\left(s, x, u(s,x), (\nabla u\sigma)(s,x)\right)\hat{\mathrm{d}}B_s, \quad 0 \leq t \leq T, \end{cases} \tag{5.1}$$

where $u : [0, T] \times \mathbf{R} \to \mathbf{R}$ and $\nabla u(s, x)$ denotes the first order derivative of $u(s, x)$ with respect to x, and

$$\mathcal{L}u = \begin{pmatrix} Lu_1 \\ \vdots \\ Lu_k \end{pmatrix},$$

with $L\phi(x) = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}(x)\frac{\partial^2\phi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x,v)\frac{\partial\phi(x)}{\partial x_i}$. In the present paper, we set $d = k = 1$, and

$$\begin{aligned}
b &: \mathbf{R}\times\mathbf{R}\to\mathbf{R},\\
\sigma &: \mathbf{R}\to\mathbf{R},\\
f &: [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\to\mathbf{R},\\
g &: [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\to\mathbf{R},\\
\tilde{h} &: \mathbf{R}\to\mathbf{R}.
\end{aligned}$$

In order to assure the existence and uniqueness of solutions for (5.1) and (5.3) below, we give the following assumptions for sake of completeness (see [19] for more details).

(A1)

$$\begin{cases}
b\in C_{l,b}^3(\mathbf{R}\times\mathbf{R};\mathbf{R}), \quad \sigma\in C_{l,b}^3(\mathbf{R};\mathbf{R}), \quad \tilde{h}\in C_p^3(\mathbf{R};\mathbf{R}),\\
f(t,\cdot,\cdot,\cdot,v)\in C_{l,b}^3(\mathbf{R}\times\mathbf{R}\times\mathbf{R};\mathbf{R}), \quad f(\cdot,x,y,z,v)\in M^2(0,T;\mathbf{R}),\\
g(t,\cdot,\cdot,\cdot)\in C_{l,b}^3(\mathbf{R}\times\mathbf{R}\times\mathbf{R};\mathbf{R}), \quad g(\cdot,x,y,z)\in M^2(0,T;\mathbf{R})\\
\forall t\in[0,T],\ x\in\mathbf{R},\ y\in\mathbf{R},\ z\in\mathbf{R},\ v\in\mathbf{R}.
\end{cases}$$

(A2) There exist some constants $c > 0$ and $0 < \alpha < 1$ such that for all $(t,x,y_i,z_i,v)\in[0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}$ $(i = 1,2)$,

$$\begin{cases}
|f(t,x,y_1,z_1,v) - f(t,x,y_2,z_2,v)|^2 \le c\left(|y_1-y_2|^2 + |z_1-z_2|^2\right),\\
|g(t,x,y_1,z_1) - g(t,x,y_2,z_2)|^2 \le c|y_1-y_2|^2 + \alpha|z_1-z_2|^2.
\end{cases}$$

Let $\mathcal{U}_{\mathrm{ad}}$ be an admissible control set. The optimal control problem of SPDE (5.1) is to find an optimal control, such that

$$J\left(v_{(\cdot)}^*\right) \doteq \inf_{v_{(\cdot)}\in\mathcal{U}_{\mathrm{ad}}} J\left(v_{(\cdot)}\right),$$

where $J\left(v_{(\cdot)}\right)$ is its cost function as follows:

$$J\left(v_{(\cdot)}\right) = \mathbf{E}\left[\int_0^T l\left(s,x,u(s,x),(\nabla u\sigma)(s,x),v_s\right)\mathrm{d}s + \gamma\left(u(0,x)\right)\right]. \tag{5.2}$$

Here we assume l and γ satisfy (H4). We can transform the optimal control problem of SPDE (5.1) into one of the following FBDSDE with control variable:

$$\begin{cases}
X_s^{t,x} = x + \int_t^s b\left(X_r^{t,x},v_r\right)\mathrm{d}r + \int_t^s \sigma\left(X_r^{t,x}\right)\mathrm{d}W_r,\\
Y_s^{t,x} = \tilde{h}\left(X_T^{t,x}\right) + \int_s^T f\left(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x},v_r\right)\mathrm{d}r + \int_s^T g\left(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\right)\hat{\mathrm{d}}B_r\\
\qquad - \int_s^T Z_r^{t,x}\mathrm{d}W_r, \qquad 0\le t\le s\le T,
\end{cases} \tag{5.3}$$

where $\left(X_{(\cdot)}^{t,x}, Y_{(\cdot)}^{t,x}, Z_{(\cdot)}^{t,x}, v_{(\cdot)}\right) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}$, $x\in\mathbf{R}$. The corresponding optimal control problem of FBDSDE (5.3) is to find an optimal control $v_{(\cdot)}^*\in\mathcal{U}_{\mathrm{ad}}$, such that

$$J\left(v_{(\cdot)}^*\right) \doteq \inf_{v_{(\cdot)}\in\mathcal{U}_{\mathrm{ad}}} J\left(v_{(\cdot)}\right),$$

where $J\left(v_{(\cdot)}\right)$ is the cost function same as (5.2):

$$J\left(v_{(\cdot)}\right) = \mathbf{E}\left[\int_0^T l\left(s,X_s,Y_s,Z_s,v_s\right)\mathrm{d}s + \gamma\left(Y_0\right)\right].$$

Now we consider the following adjoint FBDSDEs involving the four unknown processes (p_t, q_t, k_t, h_t):

$$\begin{cases} \mathrm{d}p_t = (f_Y p_t + g_Y k_t - l_Y)\,\mathrm{d}t + (f_Z p_t - g_Z k_t - l_Z)\,\mathrm{d}W_t - k_t \hat{\mathrm{d}} B_t, \\ \mathrm{d}q_t = (f_X p_t - b_X q_t + g_X k_t - \sigma_X h_t - l_X)\,\mathrm{d}t + h_t \mathrm{d}W_t, \\ p_0 = -\gamma_Y (Y_0), \quad q_T = -\tilde{h}_X (X_T)\, p_T, \qquad 0 \le t \le T. \end{cases} \tag{5.4}$$

It is easy to see that the first equation of (5.4) is a "forward" BDSDE, so it is uniquely solvable by virtue of the result in [19]. The second equation of (5.4) is a standard BSDE, so it is uniquely solvable by virtue of the result in [18]. Therefore we know that (5.4) has a unique solution $\left(p_{(\cdot)}, q_{(\cdot)}, k_{(\cdot)}, h_{(\cdot)}\right) \in M^2 (0, T; \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R})$. Define the Hamilton function as follows:

$$\begin{aligned} \bar{H} (t, X, Y, Z, v, p, q, k, h) &= H (t, X, Y, 0, Z, v, p, q, k, h) \\ &= l (t, X, Y, Z, v) - k \cdot g (t, X, Y, Z) \\ &\quad + q \cdot b (X, v) - p \cdot f (t, X, Y, Z, v) + h \cdot \sigma (X) . \end{aligned} \tag{5.5}$$

We now formulate a maximum principle for the optimal control system of (5.3).

Theorem 5.1. *Suppose* (A1)–(A2) *hold. Let* $\left(X_{(\cdot)}, Y_{(\cdot)},\ Z_{(\cdot)}, u_{(\cdot)}\right)$ *be an optimal control and its corresponding trajectory of* (5.3), $\left(p_{(\cdot)}, q_{(\cdot)}, k_{(\cdot)}, h_{(\cdot)}\right)$ *be the solution of* (5.4). *Then the maximum principle holds, that is, for* $t \in [0, T]$, $\forall v \in \mathcal{U}$,

$$\bar{H} (t, X_t, Y_t, Z_t, v, p_t, q_t, k_t, h_t) \ge \bar{H} (t, X_t, Y_t, Z_t, v_t^*, p_t, q_t, k_t, h_t) , \ \text{a.e., a.s.}$$

Proof. Noting that the forward equation of (5.3) is independent of the backward one, we easily know that it is uniquely solvable. It is straightforward to use the same arguments in Section 3 to obtain the desired results. We omit the detailed proof. □

From the results in [19], we easily have the following propositions.

Proposition 5.2. *For any given admissible control* $v_{(\cdot)}$, *we assume* (A1) *and* (A2) *hold. Then* (5.3) *has a unique solution* $\left(X_{(\cdot)}^{t,x}, Y_{(\cdot)}^{t,x}, Z_{(\cdot)}^{t,x}\right) \in M^2 (0, T; \mathbf{R} \times \mathbf{R} \times \mathbf{R})$.

Proposition 5.3. *For any given admissible control* $v_{(\cdot)}$, *we assume* (A1) *and* (A2) *hold. Let* $\{u (t, x) ;\ 0 \le t \le T,\ x \in \mathbf{R}\}$ *be a random field such that* $u (t, x)$ *is* $\mathcal{F}_{t,T}^B$*-measurable for each* (t, x), $u \in C^{0,2} ([0, T] \times \mathbf{R}; \mathbf{R})$ *a.s., and* u *satisfies SPDE* (5.1). *Then* $u (t, x) = Y_t^{t,x}$.

Proposition 5.4. *For any given admissible control* $v_{(\cdot)}$, *we assume* (A1) *and* (A2) *hold. Then* $\{u (t, x) = Y_t^{t,x};\ 0 \le t \le T,\ x \in \mathbf{R}\}$ *is a unique classical solution of SPDE* (5.1).

Set the Hamilton function

$$\bar{H} (t, x, u, \nabla u\sigma, v, p, q, k, h) = l (t, x, u, \nabla u\sigma, v) - k \cdot g (t, x, u, \nabla u\sigma) + q \cdot b (x, v) - p \cdot f (t, x, u, \nabla u\sigma, v) + h \cdot \sigma (x) .$$

Now we can state the maximum principle for the optimal control problem of SPDE (5.1).

Theorem 5.5. *Suppose* $u (t, x)$ *is the optimal solution of SPDE* (5.1) *corresponding to the optimal control* $v_{(\cdot)}^*$ *of* (5.1). *Then we have, for any* $v \in \mathcal{U}$ *and* $t \in [0, T]$, $x \in \mathbf{R}$,

$$\bar{H} (t, x, u (t, x) , (\nabla u\sigma) (t, x) , v, p_t, q_t, k_t, h_t) \ge \bar{H} (t, x, u (t, x) , (\nabla u\sigma) (t, x) , v_t^*, p_t, q_t, k_t, h_t) , \ \text{a.e., a.s.}$$

Proof. By virtue of Propositions 5.2, 5.3 and 5.4, the optimal control problem of SPDE (5.1) can be transformed into the one of FBDSDE (5.3). Hence, from Theorem 5.1, the desired result is easily obtained. □

Remark 5.1. In Section 5, we study the optimal control problem of a kind of quasilinear SPDE which was similar to the SPDE considered by Øksendal in [17]. It is worth mentioning that the quasilinear SPDEs in [19] can also be related to a class of partially coupled FBDSDEs. Consequently the results in [17] can be obtained by the approach of FBDSDEs.

6. Linear quadratic nonzero sum doubly stochastic differential games

In this section, we investigate linear quadratic nonzero sum doubly stochastic differential games problems. Under the framework of uniqueness and existence result introduced above, we improve similar results in Hamadène [9] and Wu [27]. For notational simplification, we only consider two players, which is similar for n players. Now the control system is the following forward doubly stochastic differential equation involving two different directions of stochastic integrals.

$$\begin{cases} \mathrm{d}x_t^v = \left[Ax_t^v + B^1 v_t^1 + B_t^2 v_t^2 + Ck_t^v + \alpha_t\right]\mathrm{d}t + \left[Dx_t^v + Ek_t^v + \beta_t\right]\mathrm{d}W_t - k_t^v \hat{\mathrm{d}}B_t, \\ x_0^v = a \in \mathbf{R}^n, \qquad t \in [0,T], \end{cases} \tag{6.1}$$

where A, C, D and E are $n \times n$ bounded matrices, further, E satisfies $0 < |E| < 1$, v_t^1 and v_t^2, $t \in [0,T]$, are two admissible control processes, that is $\mathcal{F}_t$-progressively measurable square integrable processes taking values in R^k. B^1 and B^2 are $n \times k$ bounded matrices. α_t and β_t are two adapted square-integrable processes. We denote by

$$\begin{cases} J^1(v(\cdot)) = \frac{1}{2}\mathbf{E}\left[\int_0^T \left(\left\langle R^1 x_t^v, x_t^v\right\rangle + \left\langle N^1 v_t^1, v_t^1\right\rangle + \left\langle P^1 k_t^v, k_t^v\right\rangle\right)\mathrm{d}t + \left\langle Q^1 x_T^v, x_T^v\right\rangle\right], \\ J^2(v(\cdot)) = \frac{1}{2}\mathbf{E}\left[\int_0^T \left(\left\langle R^2 x_t^v, x_t^v\right\rangle + \left\langle N^2 v_t^2, v_t^2\right\rangle + \left\langle P^2 k_t^v, k_t^v\right\rangle\right)\mathrm{d}t + \left\langle Q^2 x_T^v, x_T^v\right\rangle\right]. \end{cases} \tag{6.2}$$

We denote $v(\cdot) = \left(v^1(\cdot), v^2(\cdot)\right)$. Here Q^i, R^i and P^i $(i = 1,2)$ are $n \times n$ nonnegative symmetric bounded matrices, N^1 and N^2 are $k \times k$ positive symmetric bounded matrices and inverses $\left(N^1\right)^{-1}$, $\left(N^2\right)^{-1}$ are also bounded. The problem is to find $\left(u^1(\cdot), u^2(\cdot)\right) \in R^k \times R^k$ which is called Nash equilibrium point for the game, such that

$$\begin{cases} J^1\left(u^1(\cdot), u^2(\cdot)\right) \le J^1\left(v^1(\cdot), u^2(\cdot)\right), \quad \forall v^1(\cdot) \in R^k; \\ J^2\left(u^1(\cdot), u^2(\cdot)\right) \le J^2\left(u^1(\cdot), v^2(\cdot)\right), \quad \forall v^2(\cdot) \in R^k. \end{cases} \tag{6.3}$$

Note that the actions of the two players are described by a "forward" BDSDE in which we indicates that the players should make some strategies to overcome the disturbed information. In order to introduce the main result, we need the following assumptions:

$$\begin{cases} B^i\left(N^i\right)^{-1}\left(B^i\right)^T A^T = A^T B^i\left(N^i\right)^{-1}\left(B^i\right)^T, \\ B^i\left(N^i\right)^{-1}\left(B^i\right)^T C^T = C^T B^i\left(N^i\right)^{-1}\left(B^i\right)^T, \\ B^i\left(N^i\right)^{-1}\left(B^i\right)^T D^T = D^T B^i\left(N^i\right)^{-1}\left(B^i\right)^T, \\ B^i\left(N^i\right)^{-1}\left(B^i\right)^T E^T = E^T B^i\left(N^i\right)^{-1}\left(B^i\right)^T, \\ B^i\left(N^i\right)^{-1}\left(B^i\right)^T P^1 = P^1 B^i\left(N^i\right)^{-1}\left(B^i\right)^T, \\ B^i\left(N^i\right)^{-1}\left(B^i\right)^T P^2 = P^2 B^i\left(N^i\right)^{-1}\left(B^i\right)^T, \end{cases} \quad (i = 1,2). \tag{6.4}$$

From the maximum principle for optimal control problems of FBDSDEs obtained above, we can find the equations satisfied by Nash equilibrium points for the linear quadratic nonzero sum doubly stochastic differential games problems. Now we can give an explicit form of Nash equilibrium point by virtue of solutions of linear FBDSDEs. That is, we have the following theorem.

Theorem 6.1. *The pair of functions*

$$\begin{cases} u_t^1 = -\left(N^1\right)^{-1}\left(B^1\right)^T y_t^1, \\ u_t^2 = -\left(N^1\right)^{-1}\left(B^1\right)^T y_t^2, \quad t \in [0,T], \end{cases}$$

is one Nash equilibrium point for the above game problem, where $\left(x_t, y_t^1, y_t^2, k_t, h_t^1, h_t^2\right)$ *is the solution of the following FBDSDE:*

$$\begin{cases} \mathrm{d}x_t = \left[Ax_t - B^1\left(N^1\right)^{-1}\left(B^1\right)^T y_t^1 - B^2\left(N^2\right)^{-1}\left(B^2\right)^T y_t^2 + Ck_t + \alpha_t\right]\mathrm{d}t \\ \qquad\quad + \left[Dx_t + Ek_t + \beta_t\right]\mathrm{d}W_t - k_t \hat{\mathrm{d}}B_t, \\ \mathrm{d}y_t^1 = -\left[Ay_t^1 + D^T h_t^1 + R^1 x_t\right]\mathrm{d}t - \left(C^T y_t^1 + E^T h_t^1 + P^1 k_t\right)\hat{\mathrm{d}}B_t + h_t^1 \mathrm{d}W_t, \\ \mathrm{d}y_t^2 = -\left[Ay_t^2 + D^T h_t^2 + R^2 x_t\right]\mathrm{d}t - \left(C^T y_t^2 + E^T h_t^2 + P^2 k_t\right)\hat{\mathrm{d}}B_t + h_t^2 \mathrm{d}W_t, \\ x_0 = a, \qquad y_T^1 = Q^1 x_T, \qquad y_T^2 = Q^2 x_T. \end{cases} \tag{6.5}$$

Proof. At the beginning, we prove the existence of the solutions of (6.5). Consider the following FBDSDE:

$$\begin{cases} \mathrm{d}X_t = \left(AX_t - Y_t + \alpha_t\right)\mathrm{d}t + \left[CX_t + \beta_t\right]\mathrm{d}W_t - K_t \hat{\mathrm{d}}B_t, \\ \mathrm{d}Y_t = -\left(A^T Y_t + \left(\left(B^1\left(N^1\right)^{-1}\right)\left(B^1\right)^T R^1 + \left(B^2\left(N^2\right)^{-1}\right)\left(B^2\right)^T R^2\right)X_t + D^T H_t\right)\mathrm{d}t \\ \qquad\quad - \left[C^T Y_t + E^T H_t + PK_t\right]\hat{\mathrm{d}}B_t + H_t \mathrm{d}W_t, \\ X_0 = a, \qquad Y_T = \left[\left(B^1\left(N^1\right)^{-1}\right)\left(B^1\right)^T R^1 + \left(B^2\left(N^2\right)^{-1}\right)\left(B^2\right)^T R^2\right]X_T. \end{cases} \tag{6.6}$$

Apparently, if $\left(x_t, y_t^1, y_t^2, k_t, h_t^1, h_t^2\right)$ is the solution of (6.5), then FBDSDE (6.6) with (6.4) can be satisfied by (X_t, Y_t, K_t, H_t):

$$\begin{cases} X_t = x_t, \\ K_t = k_t, \\ Y_t = B^1\left(N^1\right)^{-1}\left(B^1\right)^T y_t^1 + B^2\left(N^2\right)^{-1}\left(B^2\right)^T y_t^2, \\ H_t = B^1\left(N^1\right)^{-1}\left(B^1\right)^T h_t^1 + B^2\left(N^2\right)^{-1}\left(B^2\right)^T h_t^2, \\ P = P^1 B^1\left(N^1\right)^{-1}\left(B^1\right)^T + P^2 B^2\left(N^2\right)^{-1}\left(B^2\right)^T. \end{cases}$$

In fact, it is easy to check that there exists a unique solution (X_t, Y_t, K_t, H_t) for (6.6) according to Proposition 2.2. Hence we can first solve FBDSDE (6.6) to get solution (X_t, K_t) which is obviously the forward part (x_t, k_t) of the solution of (6.5). Thanks to $0 < |E| < 1$, by the existence and uniqueness of solutions of the classical backward doubly stochastic differential equations (BDSDEs in short) (see [19]), it immediately follows that $\left(y_t^1, h_t^1\right)$ and $\left(y_t^2, h_t^2\right)$ are obtained by solving the backward equations of (6.5). Therefore we get a solution $\left(x_t, y_t^1, y_t^2, k_t, h_t^1, h_t^2\right)$ of FBDSDE (6.5).

From now on we prove $\left(u^1(t), u^2(t)\right)$ is one Nash equilibrium point for our nonzero sum game problem. For that it suffices that

$$J^1\left(u^1(\cdot), u^2(\cdot)\right) \leq J^1\left(v^1(\cdot), u^2(\cdot)\right), \qquad \forall v^1(\cdot) \in R^k.$$

It is similar to give the other inequality by the same argument. Now we give the control system by $x_t^{v^1}$:

$$\begin{cases} \mathrm{d}x_t^{v^1} = \left[Ax_t^{v^1} + B^1 v_t^1 + B^2 u_t^2 + Ck_t^{v^1} + \alpha_t\right]\mathrm{d}t + \left[Dx_t^{v^1} + Ek_t^{v^1} + \beta_t\right]\mathrm{d}W_t - k_t^{v^1}\hat{\mathrm{d}}B_t, \\ x_0 = a, \qquad t \in [0,T]. \end{cases}$$

$$\begin{aligned} &J^1\left(v^1(\cdot), u^2(\cdot)\right) - J^1\left(u^1(\cdot), u^2(\cdot)\right) = \frac{1}{2}\mathbf{E}\Bigg[\int_0^T \left(\left\langle R^1 x_t^{v^1}, x_t^{v^1}\right\rangle - \left\langle R^1 x_t, x_t\right\rangle + \left\langle N^1 v_t^1, v_t^1\right\rangle\right. \\ &\quad \left. - \left\langle N^1 u_t^1, u_t^1\right\rangle + \left\langle P^1 k_t^{v^1}, k_t^{v^1}\right\rangle - \left\langle P^1 k_t, k_t\right\rangle\right)\mathrm{d}t + \left\langle Q^1 x_T^{v^1}, x_T^{v^1}\right\rangle - \left\langle Q^1 x_T, x_T\right\rangle\Bigg] \\ &= \frac{1}{2}\mathbf{E}\Bigg[\int_0^T \left(\left\langle R^1\left(x_t^{v^1} - x_t\right), x_t^{v^1} - x_t\right\rangle + \left\langle N^1\left(v_t^1 - u_t^1\right), v_t^1 - u_t^1\right\rangle\right. \\ &\quad \left. + \left\langle P^1\left(k_t^{v^1} - k_t\right), k_t^{v^1} - k_t\right\rangle + 2\left\langle R^1 x_t, x_t^{v^1} - x_t\right\rangle + 2\left\langle N^1 u_t^1, v_t^1 - u_t^1\right\rangle + 2\left\langle P^1 k_t, k_t^{v^1} - k_t\right\rangle\right)\mathrm{d}t \\ &\quad + \left\langle Q^1\left(x_T^{v^1} - x_T\right), x_T^{v^1} - x_T\right\rangle + 2\left(Q^1 x_T, x_T^{v^1} - x_T\right)\Bigg]. \end{aligned}$$

Note that

$$Q^1 x_T = y_T^1.$$

We apply Itô's formula to $\left\langle x_t^{v^1} - x_t, y_t^1\right\rangle$ on the $[0,T]$ and get

$$\mathbf{E}\left\langle x_T^{v^1} - x_T, y_T^1\right\rangle = \mathbf{E}\int_0^T \left(-\left\langle R^1 x_t, \left(x_t^{v^1} - x_t\right)\right\rangle + \left\langle B^1\left(v_t^1 - u_t^1\right), y_t^1\right\rangle - \left\langle P^1 k_t, k_t^{v^1} - k_t\right\rangle\right)\mathrm{d}t.$$

Under the assumption R^1, Q^1 and P^1 being nonnegative, N^1 being positive, and symmetry of B^1, we have

$$\begin{aligned} J^1\left(v^1(\cdot), u^2(\cdot)\right) - J^1\left(u^1(\cdot), u^2(\cdot)\right) &\geq \mathbf{E}\int_0^T \left(\left\langle N^1 u_t^1, v_t^1 - u_t^1\right\rangle + \left\langle B^1\left(v_t^1 - u_t^1\right), y_t^1\right\rangle\right)\mathrm{d}t \\ &= \mathbf{E}\int_0^T \left(\left\langle -N^1\left(N^1\right)^{-1}\left(B^1\right)^T y_t^1, v_t^1 - u_t^1\right\rangle + \left\langle \left(B^1\right)^T y_t^1, v_t^1 - u_t^1\right\rangle\right)\mathrm{d}t = 0. \end{aligned}$$

Lastly, we claim that

$$\begin{cases} u_t^1 = -\left(N^1\right)^{-1}\left(B^1\right)^T y_t^1, \\ u_t^2 = -\left(N^1\right)^{-1}\left(B^1\right)^T y_t^1, \quad t \in [0,T], \end{cases}$$

that is, $\left(u_t^1, u_t^2\right)$ is one Nash equilibrium point for our nonzero sum doubly stochastic game problem. □

Remark 6.1. As matter of fact, in Theorem 6.1, we use the adjoint equation, the idea is as in Theorem 4.1. Besides, the results of this section are clear and easy to understand. They can be applied in practice directly.

Acknowledgements. The authors would like to thank the referees for their helpful comments and suggestions.

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