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SCALING LAWS FOR NON-EUCLIDEAN PLATES AND THE $W^{2,2}$ ISOMETRIC IMMERSIONS OF RIEMANNIAN METRICS

Marta Lewicka[1] and Mohammad Reza Pakzad[2]

Abstract. Recall that a smooth Riemannian metric on a simply connected domain can be realized as the pull-back metric of an orientation preserving deformation if and only if the associated Riemann curvature tensor vanishes identically. When this condition fails, one seeks a deformation yielding the closest metric realization. We set up a variational formulation of this problem by introducing the non-Euclidean version of the nonlinear elasticity functional, and establish its Γ-convergence under the proper scaling. As a corollary, we obtain new necessary and sufficient conditions for existence of a $W^{2,2}$ isometric immersion of a given 2d metric into $\mathbb{R}^3$.

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1. Introduction

Recently, there has been a growing interest in the study of flat thin sheets which assume non-trivial configuration in the absence of exterior forces or imposed boundary conditions. This phenomenon has been observed in different contexts: growing leaves, torn plastic sheets and specifically engineered polymer gels [12]. The study of wavy patterns along the edges of a torn plastic sheet or the ruffled edges of leaves suggest that the sheet endeavors to reach a non-attainable equilibrium and hence necessarily assumes a non-zero stress rest configuration.

In this paper, we study a possible mathematical foundation of these phenomena, in the context of the nonlinear theory of elasticity. The basic model, called "three dimensional incompatible elasticity" [3], follows the findings of an experiment described in [12]. The experiment (see Fig. 1) consists in fabricating programmed flat disks of gels having a non-constant monomer concentration which induces a "differential shrinking factor". The disk is then activated in a temperature raised above a critical threshold, whereas the gel shrinks with a factor proportional to its concentration and the distances between the points on the surface are changed. This defines a new target metric on the disk, inducing hence a 3d configuration in the initially planar plate. One of the remarkable features of this deformation is the onset of some "transversal" oscillations.

Keywords and phrases. Non-Euclidean plates, nonlinear elasticity, Gamma convergence, calculus of variations, isometric immersions.

[1] University of Minnesota, Department of Mathematics, 206 Church St. S.E., Minneapolis, MN 55455, USA.
lewicka@math.umn.edu
[2] University of Pittsburgh, Department of Mathematics, 139 University Place, Pittsburgh, PA 15260, USA. pakzad@pitt.edu

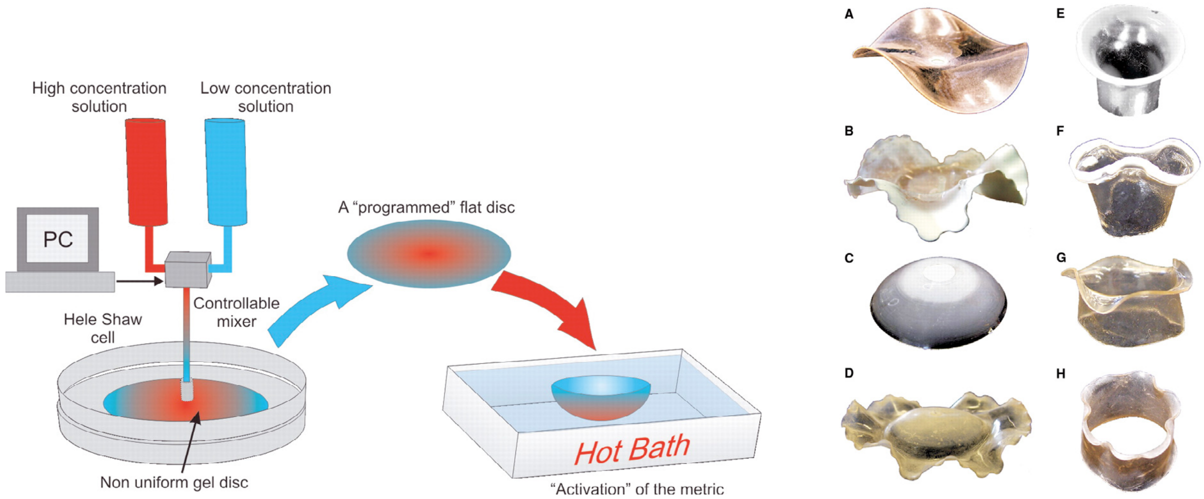

FIGURE 1. The experimental system and the obtained structures of sheets with radially symmetric target metrics. Reprinted from [12] with permission from AAAS. (Figure in colour available online at www.esaim-cocv.org.)

Trying to understand the above phenomena in the context of nonlinear elasticity theory, it has been postulated [3,12] that the 3d elastic body seeks to realize a configuration with a prescribed pull-back metric g. In this line, a 3d energy functional was introduced in [3], measuring the L^2 distance of the realized pull-back metric of the given deformation from g. Recall that, there always exist Lipschitz deformations with zero energy level [7]. They are, however, neither orientation preserving nor reversing in any neighborhood of a point where the Riemann curvature of the metric g does not vanish (*i.e.* when the metric is non-Euclidean).

In order to exclude such "infinitely oscillatory" deformations, here we study a modified energy $I(u)$ which measures, in an average pointwise manner, how far a given deformation u is from being an orientation preserving realization of the prescribed metric. An immediate consequence is that for non-Euclidean g, the infimum of I (in absence of any forces or boundary conditions) is strictly positive, which points to the existence of non-zero strain at free equilibria.

Several interesting questions arise in the study of the proposed functional. A first one is to determine the scaling of the infimum energy in terms of the vanishing thickness of a sheet. Another is to determine limiting zero-thickness theories under obtained scaling laws. The natural analytical tool in this regard is that of Γ-convergence, in the context of Calculus of Variations.

In this paper, we consider a first case where the prescribed metric is given by a tangential Riemannian metric $[g_{\alpha\beta}]$ on the 2d mid-plate, and is independent of the thickness variable. The 3d metric g is set-up such that no stretching may happen in the direction normal to the sheet in order to realize the metric. Consequently, if $[g_{\alpha\beta}]$ has non-zero Gaussian curvature, then such g is non-Euclidean. We further observe a correspondence between the scaling law for the infimum energy of the thin sheet in terms of the thickness, and the immersability of $[g_{\alpha\beta}]$ into $\mathbb{R}^3$ (Thm. 2.6). This result relates to a longstanding problem in differential geometry, depending heavily on the regularity of the immersion [9,13,14]. In our context, we deal with $W^{2,2}$ immersions not studied previously. We also derive the Γ-limit of the rescaled energies, expressed by a curvature functional on the space of all $W^{2,2}$ realizations of $[g_{\alpha\beta}]$ in $\mathbb{R}^3$ (Thms. 2.4 and 2.5).

To put these results in their proper context, recall the seminal work of Friesecke *et al.* [4] (see also [5,15,16]), where the nonlinear bending theory of plates (due to Kirchhoff) was derived as the Γ-limit of the classic theory of nonlinear elasticity, under the assumption that the latter energy per unit thickness h scales like h^2. The present paper recovers the non-Euclidean version of the same results under the same scaling law, and the 2d limit theory we obtain is hence the natural non-Euclidean generalization of the Kirchhoff model. Note that one should distinguish between our setting, where the reference configuration is generically not a physically relevant state, and the results in [5] on the Kirchhoff model for an arbitrary surface, where the surface is assumed to be embedded in the 3d space and be at rest. Contrary to the classic case, in our context the scaling law is the unique natural scaling of the energy for the free thin sheet with the associated prescribed metric.

The proofs follow the general framework set up in [4]. As a major ingredient of proofs, we also give a simple generalization of the geometric rigidity estimate [4] to the non-Euclidean setting (Thm. 2.3). We estimate the average L^2 oscillations of the deformation gradient from a fixed matrix in terms of the 3d non-Euclidean energy I and certain geometric parameters of the 3d domain. The main difference is an extra term of the bound, depending on the derivatives of the prescribed metric g and hence vanishing when g is Euclidean as in [4].

2. Overview of the main results

Consider an open, bounded, connected domain $\mathcal{U} \subset \mathbb{R}^n$, with a given Riemannian metric $g = [g_{ij}]$ smooth up to the Lipschitz boundary $\partial\mathcal{U}$. The matrix field $g : \bar{\mathcal{U}} \longrightarrow \mathbb{R}^{n\times n}$ is therefore symmetric and strictly positive definite up to the boundary. Let $A = \sqrt{g}$ be the unique symmetric positive definite square root of g and define, for all $x \in \bar{\mathcal{U}}$:

$$\mathcal{F}(x) = \Big\{RA(x);\ R \in SO(n)\Big\}, \tag{2.1}$$

where $SO(n)$ stands for the special orthogonal group of rotations in $\mathbb{R}^n$. By polar decomposition theorem, it easily follows that u is an orientation preserving realization of g:

$$(\nabla u)^T \nabla u = g \quad \text{and} \quad \det \nabla u > 0 \qquad \text{a.e. in } \mathcal{U}$$

if and only if:

$$\nabla u(x) \in \mathcal{F}(x) \qquad \text{a.e. in } \mathcal{U}.$$

Motivated by this observation, we define:

$$I(u) = \int_{\mathcal{U}} \operatorname{dist}^2(\nabla u(x), \mathcal{F}(x))\ \mathrm{d}x \qquad \forall u \in W^{1,2}(\mathcal{U}, \mathbb{R}^n). \tag{2.2}$$

Notice that when $g = \mathrm{Id}$ then the above functional becomes $I(u) = \int \operatorname{dist}^2(\nabla u, SO(n))$ which is a standard quadratic nonlinear elasticity energy, obeying the frame invariance.

Remark 2.1. For a deformation $u : \mathcal{U} \longrightarrow \mathbb{R}^n$ one could define the energy as the difference between its pull-back metric on $\mathcal{U}$ and g:

$$I_{\mathrm{str}}(u) = \int_{\mathcal{U}} |(\nabla u)^T \nabla u - g|^2\ \mathrm{d}x.$$

However, such "stretching" functional is not appropriate from the variational point of view, for the following reason. It is known that there always exists $u \in W^{1,\infty}(\mathcal{U}, \mathbb{R}^n)$ such that $I_{\mathrm{str}}(u) = 0$. On the other hand [7], if the Riemann curvature tensor R associated to g does not vanish identically, say $R_{ijkl}(x) \neq 0$ for some $x \in \mathcal{U}$, then u must have a 'folding structure' around x; the realization u cannot be orientation preserving (or reversing) in any open neighborhood of x.

In view of the above remark, our first observation concerns the energy (2.2) in case of $R \not\equiv 0$.

Theorem 2.2. *If the Riemann curvature tensor $R_{ijkl} \not\equiv 0$, then* $\inf\Big\{I(u);\ u \in W^{1,2}(\mathcal{U}, \mathbb{R}^n)\Big\} > 0$.

In case $g = \mathrm{Id}$, the infimum as above is naturally 0 and is attained only by the rigid motions. In the celebrated paper [4], the authors proved an optimal estimate of the deviation in $W^{1,2}$ of a deformation u from rigid motions in terms of $I(u)$. In Section 4 we give a simple corollary of such quantitative rigidity estimate, applicable to our setting. Since there is no realization of $I(u) = 0$ if the Riemann curvature is non-zero, we choose to estimate the deviation of the deformation from a linear map at the expense of an extra term, proportional to the gradient of g.

Theorem 2.3. *For every $u \in W^{1,2}(\mathcal{U}, \mathbb{R}^n)$ there exists $Q \in \mathbb{R}^{n\times n}$ such that:*

$$\int_{\mathcal{U}} |\nabla u(x) - Q|^2 \, \mathrm{d}x \leq C \left(\int_{\mathcal{U}} \mathrm{dist}^2(\nabla u, \mathcal{F}(x)) \, \mathrm{d}x + \|\nabla g\|^2_{L^\infty} (\mathrm{diam}\, \mathcal{U})^2 |\mathcal{U}| \right),$$

where the constant C depends on $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$, and on the domain $\mathcal{U}$. The dependence on $\mathcal{U}$ is uniform for a family of domains which are bilipschitz equivalent with controlled Lipschitz constants.

We then consider a class of more general 3d non-Euclidean elasticity functionals:

$$I_W(u) = \int_{\mathcal{U}} W(x, \nabla u(x)) \, \mathrm{d}x,$$

where the inhomogeneous stored energy density $W : \mathcal{U} \times \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}_+$ satisfies the following assumptions of frame invariance, normalization, growth and regularity:

(i) $W(x, RF) = W(x, F)$ for all $R \in SO(n)$,
(ii) $W(x, A(x)) = 0$,
(iii) $W(x, F) \geq c\, \mathrm{dist}^2(F, \mathcal{F}(x))$, with some uniform constant $c > 0$,
(iv) W has regularity $\mathcal{C}^2$ in some neighborhood of the set $\{(x, F);\ x \in \mathcal{U},\ F \in \mathcal{F}(x)\}$.

The properties (i)–(iii) are assumed to hold for all $x \in \mathcal{U}$ and all $F \in \mathbb{R}^{n\times n}$. In case the Riemann curvature tensor of g does not vanish, by Theorem 2.2 the infimum of I_W is positive, in which case I_W is called a 3d incompatible elasticity functional.

We consider thin 3d plates of the form:

$$\Omega^h = \Omega \times (-h/2, h/2) \subset \mathbb{R}^3, \qquad 0 < h << 1,$$

with a given mid-plate Ω an open, bounded, connected and smooth domain of $\mathbb{R}^2$. Following [12], we assume that the metric g on Ω^h has the form:

$$g(x', x_3) = \begin{bmatrix} \begin{bmatrix} g_{\alpha\beta}(x') \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} \qquad \forall x' \in \Omega, \quad x_3 \in (-h/2, h/2), \tag{2.3}$$

where $[g_{\alpha\beta}]$ is a metric on Ω, smooth up to the boundary. In particular, g does not depend on the (small) thickness variable x_3. Accordingly, we assume that W does not depend on x_3:

(v) $W(x, F) = W(x', F)$, for all $x \in \mathcal{U}$ and all $F \in \mathbb{R}^{n\times n}$.

Define now the rescaled energy functionals:

$$I^h(u) = \frac{1}{h} \int_{\Omega^h} W(x, \nabla u(x)) \, \mathrm{d}x \qquad \forall u \in W^{1,2}(\Omega^h, \mathbb{R}^3),$$

where the energy well $\mathcal{F}(x) = \mathcal{F}(x') = SO(3)A(x)$ is given through the unique positive definite square root $A = \sqrt{g}$ of the form:

$$A(x', x_3) = \begin{bmatrix} \begin{bmatrix} A_{\alpha\beta}(x') \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} \qquad \forall x' \in \Omega, \quad x_3 \in (-h/2, h/2).$$

By an easy direct calculation, one notices that the Riemann curvature tensor $R_{ijkl} \equiv 0$, of g in Ω^h if and only if the Gaussian curvature of the 2d metric $\kappa_{[g_{\alpha\beta}]} \equiv 0$. Hence, by Theorem 2.2, $\inf I^h > 0$ for all h if this condition is violated. A natural question is now to investigate the behavior of the sequence $\inf I^h$ as $h \to 0$. We first prove (in Sect. 5) the following lower bound and compactness result:

Theorem 2.4. *Assume that a given sequence of deformations* $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ *satisfies:*

$$I^h(u^h) \leq Ch^2, \tag{2.4}$$

where $C > 0$ *is a uniform constant. Then, for some sequence of constants* $c^h \in \mathbb{R}^3$*, the following holds for the renormalized deformations* $y^h(x', x_3) = u^h(x', hx_3) - c^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$*:*

(i) y^h *converge, up to a subsequence, in* $W^{1,2}(\Omega^1, \mathbb{R}^3)$ *to* $y(x', x_3) = y(x')$ *and* $y \in W^{2,2}(\Omega, \mathbb{R}^3)$.

(ii) *The matrix field* $Q(x') = \Big[\partial_1 y(x'), \partial_2 y(x'), \vec{n}(x')\Big] \in \mathcal{F}(x')$*, for a.e.* $x' \in \Omega$*. Here:*

$$\vec{n} = \frac{\partial_1 y \times \partial_2 y}{|\partial_1 y \times \partial_2 y|} \tag{2.5}$$

is the (well defined) normal to the image surface $y(\Omega)$*. Consequently,* y *realizes the mid-plate metric:* $(\nabla y)^T \nabla y = [g_{\alpha\beta}]$.

(iii) *Define the following quadratic forms respectively defined on* $\mathbb{R}^{3\times 3}$ *and* $\mathbb{R}^{2\times 2}$*:*

$$\mathcal{Q}_3(x')(F) = \nabla^2 W(x', \cdot)_{|A(x')}(F, F), \quad \mathcal{Q}_2(x')(F_{\tan}) = \min\{\mathcal{Q}_3(x')(\tilde{F});\ \ \tilde{F}_{\tan} = F_{\tan}\}.$$

Then we have the lower bound:

$$\liminf_{h\to 0} \frac{1}{h^2} I^h(u^h) \geq \frac{1}{24} \int_\Omega \mathcal{Q}_2(x')\Big(A^{-1}_{\alpha\beta} (\nabla y)^T \nabla \vec{n}\Big)\ \mathrm{d}x'.$$

We further prove that the lower bound in (iii) above is optimal, in the following sense. Let $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ be a Sobolev regular isometric immersion of the given mid-plate metric, that is $(\nabla y)^T \nabla y = [g_{\alpha\beta}]$. The normal vector $\vec{n} \in W^{1,2}(\Omega, \mathbb{R}^3)$ is then given by (2.5) and it is well defined because $|\partial_1 y \times \partial_2 y| = (\det g)^{1/2} > 0$.

Theorem 2.5. *For every isometric immersion* $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ *of* g*, there exists a sequence of "recovery" deformations* $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ *such that the assertion* (i) *of Theorem* 2.4 *hold, together with:*

$$\lim_{h\to 0} \frac{1}{h^2} I^h(u^h) = \frac{1}{24} \int_\Omega \mathcal{Q}_2(x')\Big(A^{-1}_{\alpha\beta} (\nabla y)^T \nabla \vec{n}\Big)\ \mathrm{d}x'. \tag{2.6}$$

A corollary of Theorems 2.4 and 2.5, proved in Section 8, provides a necessary and sufficient condition for the existence of $W^{2,2}$ isometric immersions of $(\Omega, [g_{\alpha\beta}])$:

Theorem 2.6. *Let* $[g_{\alpha\beta}]$ *be a smooth metric on the midplate* $\Omega \subset \mathbb{R}^2$*. Then:*

(i) $[g_{\alpha\beta}]$ *has an isometric immersion* $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ *if and only if for all* $h > 0$, $\frac{1}{h^2} \inf I^h \leq C$*, for a uniform constant* C.

(ii) $[g_{\alpha\beta}]$ *has an isometric immersion* $y \in W^{2,2}(\Omega, \mathbb{R}^2)$ *(or, equivalently, the Gaussian curvature* $\kappa_{[g_{\alpha\beta}]} \equiv 0$*) if and only if* $\lim_{h\to 0} \frac{1}{h^2} \inf I^h = 0$.

(iii) *If the Gaussian curvature* $\kappa_{[g_{\alpha\beta}]} \not\equiv 0$ *in* Ω *then* $\frac{1}{h^2} \inf I^h \geq c > 0$.

The existence of local or global isometric immersions of a given 2d Riemannian manifold into $\mathbb{R}^3$ is a long-standing problem in differential geometry, its main feature consisting of finding the optimal regularity. By a classical result of Kuiper [13,14], a $\mathcal{C}^1$ isometric embedding into $\mathbb{R}^3$ can be obtained by means of convex integration (see also [7]). This regularity is far from $W^{2,2}$, where information about the second derivatives is also available. On the other hand, a smooth isometry exists for some special cases, *e.g.* for smooth metrics with uniformly positive Gaussian curvatures on bounded domains in $\mathbb{R}^2$ (see [9], Thm. 9.0.1). Counterexamples to such theories are largely unexplored. By [11], there exists an analytic metric $[g_{\alpha\beta}]$ with nonnegative Gaussian curvature on 2d sphere, with no $\mathcal{C}^3$ isometric embedding. However such metric always admits a $\mathcal{C}^{1,1}$ embedding (see [8,10]). For a related example see also [18].

Finally, notice that Theorems 2.4 and 2.5 can be summarized using the language of Γ-convergence [2]. Recall that a sequence of functionals $\mathcal{F}^h : X \longrightarrow \overline{\mathbb{R}}$ defined on a metric space X, is said to Γ-converge, as $h \to 0$, to $\mathcal{F} : X \longrightarrow \overline{\mathbb{R}}$ provided that the following two conditions hold:

(i) For any converging sequence $\{x^h\}$ in X:

$$\mathcal{F}\left(\lim_{h\to 0} x^h\right) \leq \liminf_{h\to 0} \mathcal{F}^h(x^h).$$

(ii) For every $x \in X$, there exists a sequence $\{x^h\}$ converging to x and such that:

$$\mathcal{F}(x) = \lim_{h\to 0} \mathcal{F}^h(x^h).$$

Corollary 2.7. *The sequence of functionals* $\mathcal{F}^h : W^{1,2}(\Omega^1, \mathbb{R}^3) \longrightarrow \overline{\mathbb{R}}$, *given by:*

$$\mathcal{F}^h(y(x)) = \frac{1}{h^2} I^h(y(x', hx_3))$$

Γ-*converges, as* $h \to 0$, *to:*

$$\mathcal{F}(y) = \begin{cases} \frac{1}{24}\int_\Omega \mathcal{Q}_2(x')\Big(A_{\alpha\beta}^{-1}(\nabla y)^T \nabla\vec{n}\Big)\,\mathrm{d}x' & \text{if } y \text{ is a } W^{2,2} \text{ isometric immersion on } [g_{\alpha\beta}] \\ +\infty & \text{otherwise.} \end{cases}$$

Consequently, the (global) approximate minimizers of $\mathcal{F}^h$ *converge to a global minimizer of* $\mathcal{F}$.

3. The non-Euclidean elasticity functional – A proof of Theorem 2.2

In the sequel, we shall write $|g| = \det g$ and $g^{-1} = [g^{ij}]$. By ∇ we denote the covariant gradient of a scalar/vector field or a differential form, while by ∇_g we denote the contravariant gradient. The covariant divergence of a vector field u can be written as:

$$\mathrm{div}_g u = (\nabla_i u)^i = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}u^i)$$

and the scalar product of two vector fields (that is of two $(1,0)$ contravariant tensors) has the form: $\langle u, v\rangle_g = u^i g_{ij} v^j$. We shall often use the Laplace-Beltrami operator Δ_g of scalar fields f:

$$\Delta_g f = \mathrm{div}_g(\nabla_g f) = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j f).$$

By $R = [R_{ijkl}]$ we mean the ((0, 4) covariant) Riemann curvature tensor.

Towards the proof of Theorem 2.2 and for the completeness of presentation, we first derive an auxiliary result, which is somewhat standard in differential geometry. For a more general statement on the regularity of isometries between equi-dimensional Riemannian manifolds see [1].

Lemma 3.1. *Let* $u \in W^{1,1}(\mathcal{U}, \mathbb{R}^n)$ *satisfy* $\nabla u(x) \in \mathcal{F}(x)$ *for a.e.* $x \in \mathcal{U}$. *Then* u *is smooth and* $R \equiv 0$.

Proof. Write $u = (u^1, \ldots, u^n)$ and notice that in view of the assumption, each $u^i \in W^{1,\infty}$. Moreover:

$$\det \nabla u = \sqrt{|g|}, \qquad \mathrm{cof}\, \nabla u = \sqrt{|g|}(\nabla u)g^{-1}.$$

Recall that for a matrix $F \in \mathbb{R}^{n\times n}$, cof F denotes the matrix of cofactors of F, that is $(\mathrm{cof}\ F)_{ij} = (-1)^{i+j} \det \hat{F}_{ij}$, where $\hat{F}_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ is obtained from F by deleting its ith row and jth column. Since $\mathrm{div}(\mathrm{cof}\ \nabla u) = 0$ (the divergence of the cofactor matrix is always taken row-wise), the Laplace-Beltrami operator of each component u^m is zero:

$$\Delta_g u^m = 0,$$

and therefore we conclude that $u^m \in \mathcal{C}^\infty$. The second statement follows immediately since $u : \mathcal{U} \to \mathbb{R}^n$ is a smooth isometric immersion of $(\mathcal{U}, g)$ into the Euclidean space $\mathbb{R}^n$. □

Remark 3.2. The last conclusion is standard in differential geometry and its converse is also (locally) true. See for example [19], Volume II, Chapter 4.

We now make two further auxiliary observations.

Lemma 3.3. *There is a constant $M > 0$, depending only on $\|g\|_{L^\infty}$ and such that for every $u \in W^{1,2}(\mathcal{U}, \mathbb{R}^n)$ there exists a truncation $\bar{u} \in W^{1,2}(\mathcal{U}, \mathbb{R}^n)$ with the properties:*

$$\|\nabla \bar{u}\|_{L^\infty} \leq M, \qquad \|\nabla u - \nabla \bar{u}\|^2_{L^2(\mathcal{U})} \leq 4I(u) \qquad \text{and} \qquad I(\bar{u}) \leq 10 I(u).$$

Proof. Use the approximation result of Proposition A.1 in [4] to obtain the truncation $\bar{u} = u^\lambda$, for $\lambda > 0$ having the property that if a matrix $F \in \mathbb{R}^{n\times n}$ satisfies $|F| \geq \lambda$ then:

$$|F| \leq 2\mathrm{dist}^2(F, \mathcal{F}(x)) \qquad \forall x \in \mathcal{U}.$$

Then $\|\nabla u^\lambda\|_{L^\infty} \leq C\lambda := M$ and further:

$$\|\nabla u - \nabla u^\lambda\|^2_{L^2} \leq \int_{\{|\nabla u| > \lambda\}} |\nabla u|^2 \leq 4 \int_{\{|\nabla u| > \lambda\}} \mathrm{dist}^2(\nabla u, \mathcal{F}(x))\ \mathrm{d}x \leq 4I(u).$$

The last inequality of the lemma follows from the above by triangle inequality. □

Lemma 3.4. *Let $u \in W^{1,\infty}(\mathcal{U}, \mathbb{R}^n)$ and define vector field w whose each component w^m satisfies:*

$$\Delta_g w^m = 0 \quad \text{in } \mathcal{U}, \qquad w^m = u^m \quad \text{on } \partial\mathcal{U}.$$

Then $\|\nabla(u - w)\|^2_{L^2(\mathcal{U})} \leq CI(u)$, where the constant C depends only on the coercivity constant of g and (in a nondecreasing manner) on $\|\nabla u\|_{L^\infty}$.

Proof. The unique solvability of the elliptic problem in the statement follows by the usual Lax-Milgram and compactness arguments. Further, the correction $z = u - w \in W^{1,2}_0(\mathcal{U}, \mathbb{R}^n)$ satisfies:

$$\int_\mathcal{U} g^{ij}\sqrt{|g|}\partial_i z^m \partial_j \phi = \int_\mathcal{U} g^{ij}\sqrt{|g|}\partial_i u^m \partial_j \phi - \int_\mathcal{U} \nabla\phi (\mathrm{cof}\ \nabla u)_{m\text{-th row}}$$

for all $\phi \in W^{1,2}_0(\mathcal{U})$. Indeed, the last term above equals to 0, since the row-wise divergence of the cofactor matrix of ∇u is 0, in view of u being Lipschitz continuous. Use now $\phi = z^m$ to obtain:

$$\begin{aligned}\int_\mathcal{U} \sqrt{|g|}|\nabla z^m|^2_g = \int_\mathcal{U} \sqrt{|g|}|\nabla_g z^m|^2_g = \int_\mathcal{U} \partial_j z^m \left(g^{ij}\sqrt{|g|}\partial_i u^m - (\mathrm{cof}\ \nabla u)_{mj}\right)\\ \leq \|\nabla z^m\|_{L^2(\mathcal{U})} \left(\int_\mathcal{U} \left|\sqrt{|g|}(\nabla u)g^{-1} - \mathrm{cof}\ \nabla u\right|^2\right)^{1/2} \leq C_M \|\nabla z^m\|_{L^2(\mathcal{U})} I(u)^{1/2},\end{aligned} \tag{3.1}$$

which proves the lemma.

In order to deduce the last bound in (3.1), consider the function $f(x,F) = \sqrt{|g|}Fg^{-1} - \mathrm{cof}\, F$, which is locally Lipschitz continuous, uniformly in $x \in \mathcal{U}$. Clearly, when $F \in \mathcal{F}(x)$ then $F = RA$ for some $R \in SO(n)$, and so: $\mathrm{cof}\, F = \mathrm{cof}\,(RA) = (\det A)RA^{-1} = \sqrt{|g|}(RA)g^{-1}$, implying: $f(x,F) = 0$. Hence:

$$|f(x,\nabla u(x))|^2 \leq C_M^2 \mathrm{dist}^2(\nabla u(x), \mathcal{F}(x)),$$

where C_M stands for the Lipschitz constant of f on a sufficiently large ball, whose radius is equal to the bound $M = \|\nabla u\|_{L^\infty}$. □

Proof of Theorem 2.2. We argue by contradiction, assuming that for some sequence of deformations $u_n \in W^{1,2}(\mathcal{U},\mathbb{R}^n)$, there holds $\lim_{n\to\infty} I(u_n) = 0$. By Lemma 3.3, replacing u_n by $\bar{u}_n$, we may also and without loss of generality have $u_n \in W^{1,\infty}(\mathcal{U},\mathbb{R}^n)$ and $\|\nabla u_n\|_{L^\infty} \leq M$.

Clearly, the uniform boundedness of ∇u_n implies, *via* the Poincaré inequality, after a modification by a constant and after passing to a subsequence if necessary:

$$\lim u_n = u \qquad \text{weakly in } W^{1,2}(\mathcal{U}). \tag{3.2}$$

Consider the splitting $u_n = w_n + z_n$ as in Lemma 3.4. By the Poincaré inequality, Lemma 3.4 implies that the sequence $z_n \in W_0^{1,2}(\mathcal{U})$ converges to 0:

$$\lim z_n = 0 \qquad \text{strongly in } W^{1,2}(\mathcal{U}).$$

In view of the convergence in (3.2), the sequence w_n must be uniformly bounded in $W^{1,2}(\mathcal{U})$, and hence by the local elliptic estimates for the Laplace-Beltrami operator, each Δ_g-harmonic component w_n^m is locally uniformly bounded in a higher Sobolev norm:

$$\forall \mathcal{U}' \subset\subset \mathcal{U} \quad \exists C_{\mathcal{U}'} \qquad \|w_n^m\|_{W^{2,2}(\mathcal{U}')} \leq C_{\mathcal{U}'}\|w_n^m\|_{W^{1,2}(\mathcal{U})} \leq C.$$

Consequently, w_n converge to u strongly in $W^{1,2}_{\mathrm{loc}}(\mathcal{U})$ and recalling that $I(u_n)$ converge to 0, we finally obtain:

$$I(u) = 0.$$

Therefore $\nabla u \in \mathcal{F}(x)$ for a.e. $x \in \mathcal{U}$, which achieves the desired contradiction with the assumption $R \not\equiv 0$, by Lemma 3.1. □

4. A geometric rigidity estimate for Riemannian metrics – A proof of Theorem 2.3

Recall that according to the rigidity estimate [4], for every $v \in W^{1,2}(\mathcal{V},\mathbb{R}^n)$ defined on a connected, open, bounded set $\mathcal{V} \subset \mathbb{R}^n$, there exists $R \in SO(n)$ such that:

$$\int_{\mathcal{V}} |\nabla v - R|^2 \leq C_{\mathcal{V}} \int_{\mathcal{V}} \mathrm{dist}^2(\nabla v, SO(n)). \tag{4.1}$$

The constant $C_{\mathcal{V}}$ depends only on the domain $\mathcal{V}$ and it is uniform for a family of domains which are bilipschitz equivalent with controlled Lipschitz constants.

Proof of Theorem 2.3. For some $x_0 \in \mathcal{U}$ denote $A_0 = A(x_0)$ and apply (4.1) to the vector field $v(y) = u(A_0^{-1}y) \in W^{1,2}(A_0\mathcal{U},\mathbb{R}^n)$. After change of variables we obtain:

$$\exists R \in SO(n) \qquad \int_{\mathcal{U}} |(\nabla u)A_0^{-1} - R|^2 \leq C_{A_0\mathcal{U}} \int_{\mathcal{U}} \mathrm{dist}^2((\nabla u)A_0^{-1}, SO(n)).$$

Since the set $A_0\mathcal{U}$ is a bilipschitz image of $\mathcal{U}$, the constant $C_{A_0\mathcal{U}}$ has a uniform bound C depending on $\|A_0\|$, $\|A_0^{-1}\|$ and $\mathcal{U}$. Further:

$$\int_{\mathcal{U}} |\nabla u - RA_0|^2 \leq C\|A_0\|^4 \int_{\mathcal{U}} \mathrm{dist}^2(\nabla u, SO(n)A_0) \leq C\|A_0\|^4 \left(\int_{\mathcal{U}} \mathrm{dist}^2(\nabla u, \mathcal{F}(x)) + \int_{\mathcal{U}} |A(x) - A_0|^2\right)$$
$$\leq C\|g\|^2_{L^\infty} \left(\int_{\mathcal{U}} \mathrm{dist}^2(\nabla u, \mathcal{F}(x))\ \mathrm{d}x + C\|\nabla g\|^2_{L^\infty}(\mathrm{diam}\ \mathcal{U})^2|\mathcal{U}|\right),$$

which proves the claim. □

For completeness, we prove a version of the crucial approximation result in Theorem 6 of [6]:

Lemma 4.1. *There exist matrix fields $Q^h \in W^{1,2}(\Omega, \mathbb{R}^{3\times 3})$ such that:*

$$\frac{1}{h}\int_{\Omega^h} |\nabla u^h(x) - Q^h(x')|^2\ \mathrm{d}x \leq C(h^2 + I^h(u^h)), \tag{4.2}$$

$$\int_{\Omega} |\nabla Q^h|^2 \leq C(1 + h^{-2}I^h(u^h)), \tag{4.3}$$

with constant C independent of h.

Proof. Let $D_{x',h} = B(x',h) \cap \Omega$ be 2d curvilinear discs in Ω of radius h and centered at a given x'. Let $B_{x',h} = D_{x',h} \times (-h/2, h/2)$ be the corresponding 3d cylinders. On each $B_{x',h}$ use Theorem 2.3 to obtain:

$$\int_{B_{x',h}} |\nabla u^h - Q_{x',h}|^2 \leq C\left(\int_{B_{x',h}} \mathrm{dist}^2(\nabla u^h, \mathcal{F}(z))\ \mathrm{d}z + h^2|B_{x',h}|\right) \leq C\int_{B_{x',h}} h^2 + \mathrm{dist}^2(\nabla u^h, \mathcal{F}(z))\ \mathrm{d}z, \tag{4.4}$$

with a universal constant C in the right hand side above, depending only on the metric g and the Lipschitz constant of $\partial\Omega$.

Consider now the family of mollifiers $\eta_{x'} : \Omega \longrightarrow \mathbb{R}$, parameterized by $x' \in \Omega$ and given by:

$$\eta_{x'}(z') = \frac{\theta(|z'-x'|/h)}{h\int_\Omega \theta(|y'-x'|/h)\ \mathrm{d}y'},$$

where $\theta \in \mathcal{C}_c^\infty([0,1))$ is a nonnegative cut-off function, equal to a nonzero constant in a neighborhood of 0. Then $\mathrm{spt}(\eta_{x'}) \cap \Omega \subset D_{x,h}$ and:

$$\int_\Omega \eta_{x'} = h^{-1}, \quad \|\eta_{x'}\|_{L^\infty} \leq Ch^{-3}, \quad \|\nabla_{x'}\eta_{x'}\|_{L^\infty} \leq Ch^{-4}.$$

Define the approximation $Q^h \in W^{1,2}(\Omega, \mathbb{R}^{3\times 3})$:

$$Q^h(x') = \int_{\Omega^h} \eta_{x'}(z')\nabla u^h(z)\ \mathrm{d}z.$$

By (4.4), we obtain the following pointwise estimates, for every $x' \in \Omega$:

$$|Q^h(x') - Q_{x',h}|^2 \leq \left(\int_{\Omega^h} \eta_{x'}(z')\left(\nabla u^h(z) - Q_{x',h}\right)\ \mathrm{d}z\right)^2 \leq \int_{\Omega^h} |\eta_{x'}(z')|^2\ \mathrm{d}z \cdot \int_{B_{x',h}} |\nabla u^h - Q_{x',h}|^2$$
$$\leq Ch^{-3}\int_{B_{x',h}} h^2 + \mathrm{dist}^2(\nabla u^h, \mathcal{F}(z))\ \mathrm{d}z,$$

$$|\nabla Q^h(x')|^2 = \left(\int_{\Omega^h} (\nabla_{x'}\eta_{x'}(z'))\nabla u^h(z)\ \mathrm{d}z\right)^2 = \left(\int_{\Omega^h} (\nabla_{x'}\eta_{x'}(z'))\left(\nabla u^h(z) - Q_{x',h}\right)\ \mathrm{d}z\right)^2$$
$$\le \int_{\Omega^h} |\nabla_{x'}\eta_{x'}(z')|^2\ \mathrm{d}z \cdot \int_{\Omega^h} |\nabla u^h - Q_{x',h}|^2$$
$$\le Ch^{-5}\int_{B_{x',h}} h^2 + \mathrm{dist}^2(\nabla u^h, \mathcal{F}(z))\ \mathrm{d}z.$$

Applying the same estimate on doubled balls $B_{x',2h}$ we arrive at:

$$\int_{B_{x',h}} |\nabla u^h(x) - Q^h(x')|^2\ \mathrm{d}x \le C\left(\int_{B_{x',h}} |\nabla u^h(z) - Q_{x',h}|^2\ \mathrm{d}z + \int_{B_{x',h}} |Q_{x',h} - Q^h(z')|^2\ \mathrm{d}z\right)^2$$
$$\le C\int_{B_{x',2h}} h^2 + \mathrm{dist}^2(\nabla u^h, \mathcal{F}(z))\ \mathrm{d}z,$$
$$\int_{D_{x',h}} |\nabla Q^h|^2 \le Ch^{-3}\int_{B_{x',2h}} h^2 + \mathrm{dist}^2(\nabla u^h, \mathcal{F}(z))\ \mathrm{d}z.$$

Consider a finite covering $\Omega = \bigcup D_{x',h}$ whose intersection number is independent of h (as it depends only on the Lipschitz constant of $\partial\Omega$). Summing the above bounds and applying the uniform lower bound $W(x,F) \ge c\ \mathrm{dist}^2(F,\mathcal{F}(x))$ directly yields (4.2) and (4.3). □

5. Compactness and the lower bound on rescaled energies – A proof of Theorem 2.4

1. From (4.2), (4.3) in Lemma 4.1 and the assumption on the energy scaling, it follows directly that the sequence Q^h, obtained in Lemma 4.1, is bounded in $W^{1,2}(\Omega,\mathbb{R}^{3\times 3})$. Hence, Q^h converges weakly in this space, to some matrix field Q and:

$$\int_{\Omega^1} |\nabla u^h(x', hx_3) - Q(x')|^2\ \mathrm{d}x \le \int_\Omega |Q^h - Q|^2 + h^{-1}\int_{\Omega^h} |\nabla u^h(x) - Q^h(x')|\ \mathrm{d}x,$$

converges to 0 by (4.2). Therefore we obtain the following convergence of the matrix field with given columns:

$$\lim_{h\to 0}\left[\partial_1 y^h(x), \partial_2 y^h(x), h^{-1}\partial_3 y^h(x)\right] = Q(x) \qquad \text{in } L^2(\Omega^1,\mathbb{R}^{3\times 3}).$$

We immediately conclude that $\|\partial_3 y^h\|_{L^2(\Omega^1)}$ converges to 0.

Now, setting $c^h = ⨍_{\Omega^1} u^h(x', hx_3)\ \mathrm{d}x$, by means of the Poincaré inequality there follows the assertion (i) of the theorem. The higher regularity of y can be deduced from $\nabla y \in W^{1,2}$, in view of the established $W^{1,2}$ regularity of the limiting approximant Q.

To prove (ii), notice that by (4.2), (4.4) and the lower bound on W:

$$\int_\Omega \mathrm{dist}^2(Q^h, \mathcal{F}(x')) \le Ch^2. \tag{5.1}$$

Hence $Q(x') \in \mathcal{F}(x)$ a.e. in Ω and consequently $\partial_\alpha y \cdot \partial_\beta y = g_{\alpha\beta}$. To see that the last column of the matrix field Q coincides with the unit normal to the image surface: $Qe_3 = \vec{n}$, write $Q = RA$ for some $R \in SO(3)$ and notice that:

$$\partial_1 y \times \partial_2 y = (RAe_1)\times(RAe_2) = R((Ae_1)\times(Ae_2)) = cRAe_3 = cQe_3,$$

where $c = |\partial_1 y \times \partial_2 y| = |(Ae_1)\times(Ae_2)| = \det A > 0$, by the form of the matrix A. On the other hand $|Qe_3| = |Re_3| = 1$, so indeed there must be $Qe_3 = \vec{n} = (\partial_1 y \times \partial_2 y)/|\partial_1 y \times \partial_2 y|$.

2. We now modify the sequence Q^h to retain its convergence properties and additionally get $\tilde{Q}^h(x') \in \mathcal{F}(x)$ for a.e. $x \in \Omega$. Define $\tilde{Q}^h \in L^2(\Omega, \mathbb{R}^3)$ with:

$$\tilde{Q}^h(x') = \begin{cases} \pi_{\mathcal{F}(x)}(Q^h(x')) & \text{if } Q^h(x') \in \text{small neighborhood of } \mathcal{F}(x) \\ A(x) & \text{otherwise} \end{cases}$$

where $\pi_{\mathcal{F}(x)}$ denotes the projection onto the compact set $\mathcal{F}(x)$ of its (sufficiently small) neighborhood. One can easily see that:

$$\int_\Omega |\tilde{Q}^h - Q^h|^2 \leq C \int_\Omega \mathrm{dist}^2(Q^h(x'), \mathcal{F}(x'))\ \mathrm{d}x' \leq Ch^2$$

by (5.1). In particular, $\tilde{Q}^h$ converge to Q in $L^2(\Omega)$.

Write $\tilde{Q}^h = R^h A$ for a matrix field $R^h \in SO(3)$ and consider the rescaled strain:

$$G^h(x', x_3) = \frac{1}{h}\Big((R^h)^T(x')\nabla u^h(x', hx_3) - A(x')\Big) \in L^2(\Omega^1, \mathbb{R}^{3\times 3}).$$

We obtain:

$$\int_{\Omega^1} |G^h|^2 \leq Ch^{-3}\int_{\Omega^h} |\nabla u^h - \tilde{Q}^h|^2 \leq Ch^{-2} I^h(u^h) + Ch^{-2}\int_\Omega |\tilde{Q}^h - Q^h|^2 \leq C.$$

Hence there exists a limit:

$$\lim_{h\to 0} G^h = G \qquad \text{weakly in } L^2(\Omega^1, \mathbb{R}^{3\times 3}). \tag{5.2}$$

3. Fix now a small $s > 0$ and consider the difference quotients:

$$f^{s,h}(x) = \frac{1}{h}\frac{1}{s}(y^h(x + se_3) - y^h(x)) \in W^{1,2}(\Omega^1, \mathbb{R}^3).$$

Since $h^{-1}\partial_3 y^h$ converges in $L^2(\Omega^1, \mathbb{R}^3)$ to $\vec{n}(x')$, then also:

$$\lim_{h\to 0} f^{s,h}(x) = \lim_{h\to 0} \frac{1}{h} \fint_0^s \partial_3 y^h(x + te_3)\ \mathrm{d}t = \vec{n}(x').$$

There also follows convergence of normal derivatives, strongly in $L^1(\Omega^1)$:

$$\lim_{h\to 0} \partial_3 f^{s,h}(x) = \lim_{h\to 0} (\partial_3 y^h(x + se_3) - \partial_3 y^h(x)) = 0,$$

and of tangential derivatives, weakly in $L^2(\Omega^1)$:

$$\lim_{h\to 0} \partial_\alpha f^{s,h}(x) = \lim_{h\to 0} \frac{1}{s} R^h(x')\Big(G^h(x + se_3) - G^h(x)\Big)e_\alpha = \frac{1}{s}(QA^{-1})(x')\Big(G(x + se_3) - G(x)\Big)e_\alpha,$$

where we have used that the L^∞ sequence $R^h = \tilde{Q}^h A^{-1}$ converges in $L^2(\Omega)$ to $QA^{-1} \in SO(3)$.

Consequently, the sequence $f^{s,h}$ converges, as $h \to 0$ weakly in $L^2(\Omega^1)$ to $\vec{n}(x')$. Equating the derivatives, we obtain:

$$\partial_\alpha \vec{n}(x') = \frac{1}{s}(QA^{-1})(x')\Big(G(x + se_3) - G(x)\Big)e_\alpha.$$

Therefore:

$$G(x', x_3)e_\alpha = G(x', 0)e_\alpha + x_3\Big((AQ^{-1})(x')\partial_\alpha \vec{n}(x')\Big), \qquad \alpha = 1, 2. \tag{5.3}$$

4. We now calculate the lower bound of the rescaled energies. To this end, define the sequence of characteristic functions:

$$\chi_h = \chi_{\{x\in\Omega^1;\ \ |G^h(x)| \leq h^{-1/2}\}},$$

which by (5.2) converge in $L^1(\Omega^1)$ to 1. Using frame invariance and noting that in the Taylor expansion of the function $F \mapsto W(x,F)$ at $A(x)$ the first two terms are 0, we obtain:

$$\begin{aligned}\frac{1}{h^2}I^h(u^h) &\geq \frac{1}{h^2}\int_{\Omega^1}\chi_h(x)W(x,\nabla u^h(x',hx_3))\,\mathrm{d}x = \frac{1}{h^2}\int_{\Omega^1}\chi_h(x)W(x,R^h(x')^T\nabla u^h(x',hx_3))\,\mathrm{d}x\\ &= \frac{1}{h^2}\int_{\Omega^1}\chi_h(x)W(x,A(x)+hG^h(x))\,\mathrm{d}x\\ &\geq \int_{\Omega^1}\chi_h(x)\left[\frac{1}{2}\nabla^2W(x,\cdot)_{|A(x)}(G^h(x),G^h(x)) - o(1)|G^h(x)|^2\right]\,\mathrm{d}x.\end{aligned}$$

Hence:

$$\begin{aligned}\liminf_{h\to 0}\frac{1}{h^2}I^h(u^h) &\geq \frac{1}{2}\liminf_{h\to 0}\int_{\Omega^1}\chi_h(x)\mathcal{Q}_3(x')\Big(G^h(x)\Big)\,\mathrm{d}x = \frac{1}{2}\liminf_{h\to 0}\int_{\Omega^1}\mathcal{Q}_3(x')\Big(\chi_h(x)G^h(x)\Big)\,\mathrm{d}x\\ &\geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_3(x')\Big(G(x)\Big)\,\mathrm{d}x \geq \frac{1}{2}\int_{\Omega^1}\mathcal{Q}_2(x')\Big(G(x)_{\tan}\Big)\,\mathrm{d}x.\end{aligned}$$

Above, we used the fact that $\chi_h G^h$ converges weakly in $L^2(\Omega^1,\mathbb{R}^{3\times 3})$ to G (compare with the convergence in (5.2)) and the nonnegative definiteness of the quadratic forms $\mathcal{Q}_3(x')$, following from $A(x')$ being the minimizer of the mapping W, as above.

By (5.3):

$$\begin{aligned}\mathcal{Q}_2(x')\Big(G(x',x_3)_{\tan}\Big) = \mathcal{Q}_2(x')\Big(G(x',0)_{\tan}\Big) + 2x_3\mathcal{L}_2(x')\Big(G(x',0)_{\tan},[AQ^{-1}\nabla\vec{n}]_{\tan}(x')\Big)\\ + x_3^2\mathcal{Q}_2(x')\Big([AQ^{-1}\nabla\vec{n}]_{\tan}(x')\Big).\end{aligned}$$

The second term above, expressed in terms of the bilinear operator $\mathcal{L}_2(x')$ representing the quadratic form $\mathcal{Q}_2(x')$, integrates to 0 on the domain Ω^1 symmetric in x_3. After dropping the first nonnegative term, we arrive at:

$$\liminf_{h\to 0}\frac{1}{h^2}I^h(u^h) \geq \frac{1}{24}\int_{\Omega}\mathcal{Q}_2(x')\Big([AQ^{-1}\nabla\vec{n}]_{\tan}(x')\Big)\,\mathrm{d}x'.$$

This yields the formula in (iii), as $AQ^{-1}\in SO(3)$ so $AQ^{-1} = (QA^{-1})^T = A^{-1}Q^T$. The proof of Theorem 2.4 is now complete.

6. Two lemmas on the quadratic forms $\mathcal{Q}_3$ and $\mathcal{Q}_2$

We now gather some facts regarding the quadratic forms $\mathcal{Q}_3$ and $\mathcal{Q}_2$. First, it is easy to notice that the tangent space to $\mathcal{F}(x)$ at $A(x)$ coincides with the products of the skew-symmetric matrices with $A(x)$, denoted by skew $\cdot\, A(x)$. Here 'skew' stands for the space of all skew matrices of appropriate dimension; in the present case 3×3. The orthogonal complement of this space equals to:

$$\Big(T_{A(x)}\mathcal{F}(x)\Big)^{\perp} = \mathrm{sym}\cdot A(x)^{-1},$$

where 'sym' denotes the space of all symmetric matrices (again, of appropriate dimension). The quadratic, nonnegative definite form $\mathcal{Q}_3(x')$ is strictly positive definite on the space above, and it depends only on the projection of its argument on this space:

$$\mathcal{Q}_3(x')(F) = \mathcal{Q}_3(x')\left(\mathbb{P}_{\{\mathrm{sym}\cdot A(x')^{-1}\}}F\right). \tag{6.1}$$

Lemma 6.1. *We have:*

$$\mathbb{P}_{\{\mathrm{sym}\cdot A^{-1}\}}\left[\begin{array}{cc} \left[F_{\alpha\beta}\right] & \begin{array}{c} f_{13} \\ f_{23}\end{array} \\ \begin{array}{cc} f_{31} & f_{32}\end{array} & f_{33}\end{array}\right] = \left[\begin{array}{cc} \left[\mathbb{P}_{\{\mathrm{sym}\cdot A_{\alpha\beta}^{-1}\}}F_{\alpha\beta}\right] & \begin{array}{c} b_1 \\ b_2\end{array} \\ \left[\begin{array}{cc} b_1 & b_2\end{array}\right] A_{\alpha\beta}^{-1} & b_3\end{array}\right]$$

with $b_3 = -f_{33}$ *and:* $\left[\begin{array}{cc} b_1 & b_2\end{array}\right]\left(\mathrm{Id} + A_{\alpha\beta}^{-2}\right) = -\left[\begin{array}{cc} f_{13} & f_{23}\end{array}\right] - \left[\begin{array}{cc} f_{31} & f_{32}\end{array}\right] A_{\alpha\beta}^{-1}$.

Proof. Since the projection $\mathbb{P}$ is a linear operator, we will separately prove the above formula in two cases: when $F_{\alpha\beta} = 0$ and when $f_{ij} = 0$. Notice first that $\mathbb{P}_{\{\mathrm{sym}\cdot A^{-1}\}}F = BA^{-1}$, for a symmetric matrix B, uniquely determined through the formula:

$$\forall S \in \mathrm{sym} \qquad 0 = (F - BA^{-1}) : (SA^{-1}).$$

Since the right hand side above equals to $(FA^{-1} - BA^{-2}) : S$, we obtain:

$$FA^{-1} - BA^{-2} \in \mathrm{skew}. \tag{6.2}$$

Also, we notice the form of the matrix:

$$B = \left[\begin{array}{cc} \left[B_{\alpha\beta}\right] & \begin{array}{c} b_1 \\ b_2\end{array} \\ \begin{array}{cc} b_1 & b_2\end{array} & b_3\end{array}\right], \qquad BA^{-1} = \left[\begin{array}{cc} \left[B_{\alpha\beta}A_{\alpha\beta}^{-1}\right] & \begin{array}{c} b_1 \\ b_2\end{array} \\ \left[\begin{array}{cc} b_1 & b_2\end{array}\right] A_{\alpha\beta}^{-1} & b_3\end{array}\right]. \tag{6.3}$$

In the first case when $f_{ij} = 0$, let $B_{\alpha\beta} = \left[\mathbb{P}_{\{\mathrm{sym}\cdot A^{-1}\}}F_{\alpha\beta}\right]A_{\alpha\beta}$. Then $F_{\alpha\beta}A_{\alpha\beta}^{-1} - B_{\alpha\beta}A_{\alpha\beta}^{-2} \in \mathrm{skew}$, and the same matrix provides the only non-zero, principal 2×2 minor of the 3×3 matrix $FA^{-1} - BA^{-2}$, where B is taken so that all $b_i = 0$ and $B_{\tan} = B_{\alpha\beta}$. By uniqueness of the symmetric matrix B satisfying (6.2), this proves the claim.

In the second case when $F_{\alpha\beta} = 0$, define B as in (6.3) with $B_{\alpha\beta} = 0$. The result follows, since:

$$FA^{-1} - BA^{-2} = \left[\begin{array}{cc} [0] & \begin{array}{c} f_{13} \\ f_{23}\end{array} \\ \left[\begin{array}{cc} f_{31} & f_{32}\end{array}\right] A_{\alpha\beta}^{-1} & f_{33}\end{array}\right] + \left[\begin{array}{cc} [0] & \begin{array}{c} b_1 \\ b_2\end{array} \\ \left[\begin{array}{cc} b_1 & b_2\end{array}\right] A_{\alpha\beta}^{-2} & b_3\end{array}\right],$$

and (6.2) is equivalent to the conditions on b_i given in the statement of the lemma. We remark that since $A_{\alpha\beta}^{-2} = [g_{\alpha\beta}]^{-1}$ is strictly positive definite, then the same is true for the matrix $\mathrm{Id} + A_{\alpha\beta}^{-2}$, which implies its invertibility. □

Now, the quadratic and nonnegative definite form $\mathcal{Q}_2(x')$ is likewise strictly positive definite on the space $\mathrm{sym}\cdot A_{\alpha\beta}^{-1}$ and:

$$\mathcal{Q}_2(x')(F_{\tan}) = \mathcal{Q}_2(x')\left(\mathbb{P}_{\{\mathrm{sym}\cdot A(x')^{-1}\}}F_{\tan}\right). \tag{6.4}$$

Lemma 6.2. *There exists linear maps* $b, c : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^3$ *related by:*

$$c\left(F_{\tan}\right) = -\left[\begin{array}{cc} \left[\mathrm{Id} + A_{\alpha\beta}^{-2}\right] & \begin{array}{c} 0 \\ 0\end{array} \\ \begin{array}{cc} 0 & 0\end{array} & 1\end{array}\right] \cdot b\left(F_{\tan}\right)$$

and such that:

$$\mathcal{Q}_2(x')(F_{\tan}) = \mathcal{Q}_3(x')\left[\begin{array}{cc} \left[\mathbb{P}_{\{\mathrm{sym}\cdot A_{\alpha\beta}^{-1}\}}F_{\tan}\right] & \begin{array}{c} b_1 \\ b_2\end{array} \\ \left[\begin{array}{cc} b_1 & b_2\end{array}\right] A_{\alpha\beta}^{-1} & b_3\end{array}\right] = \mathcal{Q}_3(x')\left[\begin{array}{cc} \left[F_{\tan}\right] & \begin{array}{c} c_1 \\ c_2\end{array} \\ \begin{array}{cc} 0 & 0\end{array} & c_3\end{array}\right]. \tag{6.5}$$

Proof. By (6.1), Lemma 6.1 and the definition of $\mathcal{Q}_2$ it follows that:

$$\mathcal{Q}_2(x')(F_{\tan}) = \min_{b\in\mathbb{R}^3} \mathcal{Q}_3(x') \left[\begin{array}{cc} \left[\mathbb{P}_{\{\mathrm{sym}\cdot A_{\alpha\beta}^{-1}\}} F_{\tan}\right] & \begin{array}{c} b_1 \\ b_2 \end{array} \\ \left[\begin{array}{cc} b_1 & b_2 \end{array}\right] A_{\alpha\beta}^{-1} & b_3 \end{array} \right].$$

Hence we obtain the first equality in the representation (6.5). The second one follows again from (6.1) and Lemma 6.1, provided that $c_3 = -b_3$ and $\left[\begin{array}{cc} c_1 & c_2 \end{array}\right] = -\left[\begin{array}{cc} b_1 & b_2 \end{array}\right]\Big(\mathrm{Id} + A_{\alpha\beta}^{-2}\Big)$, which is exactly the condition defining the vector c in the statement of the lemma. □

7. The recovery sequence – A proof of Theorem 2.5

Following the reasoning in step 1 of the proof of Theorem 2.4, we first notice that the matrix field Q whose columns are given by:

$$Q(x') = \Big[\partial_1 y(x'), \partial_2 y(x'), \vec{n}(x')\Big] \in \mathcal{F}(x').$$

Hence in particular: $QA^{-1} \in SO(3)$. With the help of the above definition and Lemma 6.2, we put:

$$d(x') = Q(x')A^{-1}(x') \cdot c\Big(A_{\alpha\beta}^{-1}(\nabla y)^T \nabla\vec{n}(x')\Big) \in L^2(\Omega, \mathbb{R}^3). \tag{7.1}$$

Let $d^h \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ be such that:

$$\lim_{h\to 0} d^h = d \quad \text{in } L^2(\Omega) \qquad \text{and} \qquad \lim_{h\to 0} h\|d^h\|_{W^{1,\infty}} = 0. \tag{7.2}$$

Note that a sequence d^h with properties (7.2) can always be derived by reparameterizing (slowing down) a sequence of smooth approximations of the given vector field $d \in L^2(\Omega)$.

Recalling (2.5), we now approximate y and $\vec{n}$ respectively by sequences $y^h \in W^{2,\infty}(\Omega, \mathbb{R}^3)$ and $\vec{n}^h \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ such that:

$$\begin{aligned} &\lim_{h\to 0} \|y^h - y\|_{W^{2,2}(\Omega)} = 0, \qquad \lim_{h\to 0} \|\vec{n}^h - \vec{n}\|_{W^{1,2}(\Omega)} = 0, \\ &h\Big(\|y^h\|_{W^{2,\infty}(\Omega)} + \|\vec{n}^h\|_{W^{1,\infty}(\Omega)}\Big) \le \varepsilon_0, \\ &\lim_{h\to 0} \frac{1}{h^2}\Big|\Big\{x' \in \Omega;\ y^h(x') \ne y(x')\Big\} \cup \Big\{x' \in \Omega;\ \vec{n}^h(x') \ne \vec{n}(x')\Big\}\Big| = 0, \end{aligned} \tag{7.3}$$

for a sufficiently small, fixed number $\varepsilon_0 > 0$, to be chosen later. The existence of such approximation follows by partition of unity and a truncation argument, as a special case of the Lusin-type result for Sobolev functions in [17] (see also Prop. 2 in [6]).

Define:

$$u^h(x', x_3) = y^h(x') + x_3\vec{n}^h(x') + \frac{x_3^2}{2} d^h(x'). \tag{7.4}$$

Note that each map: $\Omega \ni x' \mapsto \mathrm{dist}(\nabla u^h(x'), \mathcal{F}(x'))$ vanishes on Ω_h and is Lipschitz in Ω, with Lipschitz constant of order $O(1/h)$. Here, we let:

$$\Omega_h = \Big\{x' \in \Omega;\ y^h(x') = y(x') \text{ and } \vec{n}^h(x') = \vec{n}(x')\Big\}.$$

For any point $x' \in \Omega \setminus \Omega_h$, we also have $\mathrm{dist}^2(x', \Omega_h) \le C|\Omega \setminus \Omega_h|$. The proof of the latter statement is standard, see for example [16], Lemma 6.1, for a similar argument. As a consequence, by (7.3) we obtain $1/h^2 \mathrm{dist}^2(x', \Omega_h) \to 0$ and hence:

$$\mathrm{dist}(\nabla u^h(x'), \mathcal{F}(x')) \le O(1/h)\mathrm{dist}(x', \Omega_h) = o(1). \tag{7.5}$$

The gradient of the deformation u^h is given by:

$$\nabla u^h(x',x_3) = Q^h(x') + x_3 A_2^h(x') + \frac{x_3^2}{2} D^h(x'),$$

where:

$$Q^h(x') = Q(x') \quad \text{in } \Omega_h, \qquad A_2^h(x') = \Big[\partial_1 \vec{n}^h(x'), \partial_2 \vec{n}^h(x'), d^h(x')\Big],$$
$$\lim_{h\to 0} A_2^h = A_2 = \Big[\partial_1 \vec{n}, \partial_2 \vec{n}, d\Big] \quad \text{in } L^2(\Omega),$$
$$D^h = \Big[\partial_1 d^h, \partial_2 d^h, 0\Big].$$

Note that by (7.5) and the local $\mathcal{C}^2$ regularity of W, the quantity $W(x, \nabla u^h(x))$ remains bounded upon choosing h and ε_0 in (7.3) small enough. The convergence in Theorem 2.4(i) follows immediately.

We now prove (2.6). Using Taylor's expansion of W in a neighborhood of $Q(x')$, we obtain:

$$\begin{aligned}\frac{1}{h^2} I^h(u^h) &= \frac{1}{h^2}\int_{\Omega_h^1} W\left(x, Q(x') + hx_3 A_2^h(x') + h^2 \frac{x_3^2}{2} D^h(x')\right)\,\mathrm{d}x + \frac{1}{h^2}\int_{\Omega^1\setminus\Omega_h^1} W(x, \nabla u^h(x))\,\mathrm{d}x\\ &= \int_{\Omega_h^1} \left(\frac{1}{2}\nabla^2 W(x',\cdot)_{|Q(x')}(x_3 A_2^h(x'), x_3 A_2^h(x')) + \mathcal{R}^h(x)\right)\,\mathrm{d}x + \frac{O(1)}{h^2}|\Omega\setminus\Omega_h|.\end{aligned}$$

Here the reminder $\mathcal{R}^h$ converges, by (7.2), to 0 pointwise almost everywhere, as $h \to 0$. Therefore, recalling the boundedness of $W(x, \nabla u^h(x))$ we deduce by dominated convergence and (7.3) that the above integral converges, as $h \to 0$, to:

$$\begin{aligned}\frac{1}{2}\int_{\Omega^1} x_3^2 \nabla^2 W(x',\cdot)_{|Q(x')}(A_2(x'), A_2(x'))\,\mathrm{d}x &= \frac{1}{2}\int_{\Omega^1} x_3^2 \mathcal{Q}_3(x')\Big(AQ^{-1}A_2\Big)\,\mathrm{d}x\\ &= \frac{1}{24}\int_\Omega \mathcal{Q}_3(x')\Big(A^{-1}Q^T A_2\Big)\,\mathrm{d}x' = \frac{1}{24}\int_\Omega \mathcal{Q}_2\Big(A_{\alpha\beta}^{-1}(\nabla y)^T \nabla\vec{n}\Big)\,\mathrm{d}x'\end{aligned}$$

where we applied frame invariance, (7.1) and (6.5).

8. Conditions for existence of $W^{2,2}$ isometric immersions of Riemannian metrics – A proof of Theorem 2.6

The assertions in (i) follow directly from Theorem 2.4 and Theorem 2.5. It remains to prove (ii), which clearly implies (iii).

Assume that $\lim_{h\to 0} \frac{1}{h^2} I^h(u^h) = 0$ for some sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$. Then, by Theorem 2.4 there exists a metric realization $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ such that:

$$\int_\Omega \mathcal{Q}_2(x')\Big(A_{\alpha\beta}^{-1}\Pi(x')\Big)\,\mathrm{d}x' = 0,$$

where $\Pi = (\nabla y)^T \nabla \vec{n}$ is the second fundamental form of the image surface $y(\Omega)$. Recalling (6.4) we obtain:

$$0 = \mathcal{Q}_2(x')\Big(A_{\alpha\beta}^{-1}\Pi\Big) = \mathcal{Q}_2(x')\Big(\mathbb{P}_{\mathrm{sym}\cdot A_{\alpha\beta}^{-1}}(A_{\alpha\beta}^{-1}\Pi)\Big) \qquad \forall x' \in \Omega.$$

Since the quadratic form $\mathcal{Q}_2(x')$ is nondegenerate on $\mathrm{sym}\cdot A_{\alpha\beta}^{-1}$, it follows that:

$$BA_{\alpha\beta}^{-1} = \mathbb{P}_{\mathrm{sym}\cdot A_{\alpha\beta}^{-1}}(A_{\alpha\beta}^{-1}\Pi) = 0, \tag{8.1}$$

for the symmetric matrix $B \in \mathbb{R}^{2\times 2}$ satisfying:

$$(A_{\alpha\beta}^{-1}\Pi - BA_{\alpha\beta}^{-1}) : (SA_{\alpha\beta}^{-1}) = 0 \qquad \forall S \in \mathrm{sym}.$$

The above condition is equivalent to $A_{\alpha\beta}^{-1}\Pi A_{\alpha\beta}^{-1} - BA_{\alpha\beta}^{-2} \in \mathrm{skew}$, but $B = 0$ in view of (8.1), so:

$$A_{\alpha\beta}^{-1}\Pi A_{\alpha\beta}^{-1} \in \mathrm{skew}.$$

Since $\Pi \in \mathrm{sym}$, there must be $\Pi = 0$ and therefore indeed effectively $y : \Omega \longrightarrow \mathbb{R}^2$.

On the other hand, if $y \in W^{2,2}(\Omega, \mathbb{R}^2)$ is a 2d realization of $[g_{\alpha\beta}]$ then clearly $\Pi = (\nabla y)^T \nabla \vec{n} = 0$, so for the recovery sequence corresponding to y and constructed in Theorem 2.5, there holds $\lim_{h\to 0} \frac{1}{h^2} I^h(u^h) = I(y) = 0$.

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