MULTI-BUMP SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH ELECTROMAGNETIC FIELDS

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Abstract. In this paper, we are concerned with the existence of multi-bump solutions for a nonlinear Schrödinger equations with electromagnetic fields. We prove under some suitable conditions that for any positive integer \( m \), there exists \( \epsilon(m) > 0 \) such that, for \( 0 < \epsilon < \epsilon(m) \), the problem has an \( m \)-bump complex-valued solution. As a result, when \( \epsilon \to 0 \), the equation has more and more multi-bump complex-valued solutions.

Mathematics Subject Classification. 35J10, 35B99, 35J60.

Received May 2, 2011. Revised November 30, 2011. Published online 1st March 2012.

1. Introduction

In this paper, we are interested in the existence of multi-bump solutions for the following nonlinear problem

\[
\left( \frac{\nabla}{i} - A_\epsilon(x) \right)^2 u + (1 + \epsilon a(x))u = |u|^{p-2} u, \quad x \in \mathbb{R}^N,
\]

where \( 2 < p < \frac{2N}{N-2} \) if \( N \geq 3 \) and \( 2 < p < +\infty \) if \( N = 1 \) or \( N = 2 \) and \( \epsilon > 0 \) is a parameter. \( a(x) \) is a positive continuous function on \( \mathbb{R}^N \), and \( A_\epsilon = (A_{\epsilon,1}(x), A_{\epsilon,2}(x), \ldots, A_{\epsilon,N}(x)) \) is such that \( A_{\epsilon,j}(x)(j = 1, 2, \ldots, N) \) is a real \( C^1 \) function on \( \mathbb{R}^N \). Throughout this paper we assume that \( a(x) \) and \( A_\epsilon(x) \) satisfy the following conditions respectively:

- \( (H_1) \) \( a(x) \in C(\mathbb{R}^N, \mathbb{R}^+) \), \( \lim_{|x| \to \infty} a(x) = 0 \), and \( \lim_{|x| \to \infty} \frac{\ln(a(x))}{|x|} = 0 \);

- \( (H_2) \) \( A_\epsilon(x) = \epsilon B(x) \), where \( B(x) \in C^1(\mathbb{R}^N, \mathbb{R}^N) \) is bounded.

Equation (1.1) arises in many fields of physics, in particular condensed matter physics and nonlinear optics (see [35])

\[
\frac{i}{\hbar} \frac{\partial \Psi}{\partial t} = \left( \frac{\hbar}{i} \nabla - A(x) \right)^2 \Psi + G(x)\Psi - f(x, \Psi), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N \quad (N \geq 2).
\]

Keywords and phrases. Contraction map, electromagnetic fields, multi-bump solutions, nonlinear Schrödinger equation, variational reduction method.

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The function $\Psi(x, t)$ takes on complex values, $\hbar$ is the Planck constant, $i$ is the imaginary unit. Here $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ denotes a magnetic potential and the Schrödinger operator is defined by

$$\left(\frac{\hbar}{i} \nabla - A(x)\right)^2 \Psi := -\hbar^2 \Delta \Psi - \frac{2\hbar}{i} A \cdot \nabla \Psi + |A|^2 \Psi - \frac{\hbar}{i} \Psi \text{div} A.$$ 

Actually, in general dimension $N \geq 4$, the magnetic field $D$ is a 2-form where $D_{k,j} = \partial_j A_k - \partial_k A_j$. In the case $N = 3$, $D = \text{curl} A$. The function $G : \mathbb{R}^N \rightarrow \mathbb{R}$ represents an electric potential.

We intend to find standing waves for (1.2), namely solutions of the form $\Psi(x, t) = e^{-i\omega t} u(x)$ for some function $u : \mathbb{R}^N \rightarrow \mathbb{C}$. Substituting this ansatz into (1.2), one is led to solve the complex equation

$$\left(\frac{\hbar}{i} \nabla - A(x)\right)^2 u + V(x) u = f(x, u), \ x \in \mathbb{R}^N,$$ 

(1.3)

where $V(x) = G(x) - E$. If $\hbar = 1, A(x) = A_e(x), V(x) = 1 + \epsilon a(x)$ and $f(x, u) = |u|^{p-2} u$, then (1.3) is reduced to (1.1). The transition from quantum mechanics to classical mechanics can be formally described by letting $\hbar \rightarrow 0$, and thus the existence of solutions for $\hbar$ small has physical interest. Standing waves for $\hbar$ small are usually referred as semi-classical bound states (see [22]).

When $A(x) \equiv 0$, (1.3) reads

$$-\hbar^2 \Delta u + V(x) u = f(x, u), \ x \in \mathbb{R}^N.$$ 

(1.4)

In recent years, much attention has been paid to the study of the existence and uniqueness for one- or multi-bump bound states of (1.4). In [21], using a Lyapunov-Schmidt reduction, Floer and Weinstein established the existence of a standing wave solution of (1.4) when $N = 1, f(x, u) = |u|u$ and $V(x)$ was a bounded function having a nondegenerate critical point for sufficiently small $\hbar$. Moreover, they showed that $u$ concentrated near the given non-degenerate critical point of $V$ when $\hbar$ tended to 0. Their methods and results were later generalized by Oh [32, 33] to the higher-dimensional case. For a potential $V$ without any nondegenerate critical point, Rabinowitz [34] obtained an existence result for (1.4) with $\hbar$ small, provided that $0 < \inf_{x \in \mathbb{R}^N} V(x) < \liminf_{|x| \rightarrow \infty} V(x)$, using a global variational argument. These solutions concentrate near the global minima of $V$ as $\hbar \rightarrow 0$, as shown by Wang [38]. For more general case, one can see [40]. del Pino and Felmer [16, 17] obtained multi-peak solutions having exactly $k$ maximum points provided that there were $k$ disjoint open bounded sets $\Lambda_i$ at its bottom. For more results concerning (1.4), see [7, 9, 10, 18, 19, 27, 39].

When $A(x) \neq 0$, we first mention a paper by Esteban and Lions [20], in which concentration-compactness principle of Lions was applied to solve some minimization problems under suitable assumptions on the magnetic field. Results concerning bounded vector potentials, when $V$ had a manifold of stationary points, were obtained by Cingolani and Secchi in [13] using a perturbation approach given by Ambrosetti et al. in [2]. Semiclassical multi-peak solutions for (1.3) for bounded vector potentials were constructed in [11] by Cao and Tang. In [14], using a penalization procedure (see [18]), Cingolani and Secchi extended the result in [13] to the case of a vector potential $A$, possibly unbounded. The penalization approach was also used by Bartsch et al. in [5], and later by Cingolani et al. in [15] to obtain multi-bump semiclassical bound for problem (1.2) with more general nonlinear term $f(x, \Psi)$. Concerning other papers on the topic, we mention that Kurata in [25] proved the existence of least energy solution of (1.3) for $\hbar > 0$ under a condition relating $V(x)$ and $A(x)$. In [22, 23], Helffer studied asymptotic behavior of the eigenfunctions of the Schrödinger operators with magnetic fields in the semiclassical limit. See also [6] for generalization of the results in [24] for potentials which were degenerate at infinity. For more related results, we can refer to [3, 12, 15, 36, 37] and the references therein.

We should point out that in almost all papers listed above, the solutions obtained will concentrate around some points when the parameter which is the Planck constant $\hbar(\epsilon)$ tends to 0$^+$. However, in this paper, we want to find solutions to (1.1) which do not concentrate near any point in the space. More precisely, we intend to look for solutions to (1.1) whose bumps are separated far apart and the distance between two bumps goes to infinity when $\epsilon \rightarrow 0$. Moreover, the size of each bump does not shrink and is fixed when $\epsilon \rightarrow 0$. This is greatly different...
from the concentration phenomenon described above. To this end, we use the idea introduced in the paper of Lin et al. [31], where \( A_\epsilon(x) \equiv 0 \), (1.1) was considered as a real-valued problem and multi-bump real-valued solutions were found.

When \( \epsilon \to 0 \), the limiting equation of (1.1) is

\[
-\Delta u + u = |u|^{p-2}u, \quad u(x) \in H^1(\mathbb{R}^N, \mathbb{C}).
\]

(1.5)

We will use the solutions of (1.5) to build up the approximate solutions for (1.1).

If we denote \( U_\epsilon : \mathbb{R}^N \to \mathbb{C} \) a least-energy solution to equation (1.5). By energy comparison (see [25]), one has that

\[
U_\epsilon(x) = e^{i\sigma}w(x - y_0),
\]

for some choice of \( \sigma \in [0, 2\pi] \) and \( y_0 \in \mathbb{R}^N \), where \( w \in C^\infty(\mathbb{R}^N, \mathbb{R}) \) is the unique solution of the problem

\[
\begin{aligned}
\left\{ \begin{array}{ll}
-\Delta u + u = w^{p-1}, & u > 0 \text{ in } \mathbb{R}^N, \\
u(x) \to 0, & \text{as } |x| \to +\infty.
\end{array} \right.
\end{aligned}
\]

(1.6)

It is well-known that every positive solution of equation (1.6) has the form \( w_y := w(\cdot - y) \) for some \( y \in \mathbb{R}^N \), \( w \) satisfies, for some \( c > 0 \),

\[
w(r)r^{\frac{N+1}{2}}e^r \to 0, \quad w'(r)r^{\frac{N+1}{2}}e^r \to -c, \quad \text{as } r = |x| \to \infty,
\]

(see [26]). This solution \( w \) will be employed as a building block to construct multi-bump solutions for (1.1). Let \( m \geq 1 \) be an integer. For sufficiently separated \( y_1, y_2, \ldots, y_m \) in \( \mathbb{R}^N \) and some choice of \( \sigma \in [0, 2\pi] \), a solution of (1.1) which is close to \( \sum_{j=1}^{m} e^{i\sigma}w(x - y_j) := \sum_{j=1}^{m} \eta w(x - y_j) \) in a sense which will be made clear later is called an \( m \)-bump solution.

For convenience, we denote

\[ V_\epsilon(x) = 1 + \epsilon a(x). \]

Let \( E \) be a Hilbert space defined as the closure of \( C_0^\infty(\mathbb{R}^N, \mathbb{C}) \) under the scalar product

\[
(u,v)_\epsilon = \text{Re} \int \left( \frac{\nabla u}{i} - A_\epsilon(x)u \right) \left( \frac{\nabla v}{i} - A_\epsilon(x)v \right) + V_\epsilon(x)u\overline{v}.
\]

The norm induced by the product \((\cdot , \cdot)_\epsilon\) is

\[
\|u\|_\epsilon = \left( \int \left| \frac{\nabla u}{i} - A_\epsilon(x)u \right|^2 + V_\epsilon(x)|u|^2 \right)^{\frac{1}{2}}
= \left( \int |\nabla u|^2 + |A_\epsilon(x)|^2|u|^2 + V_\epsilon(x)|u|^2 - 2 \text{Re} \int \frac{1}{i} \nabla u \cdot A_\epsilon(x) \overline{u} \right)^{\frac{1}{2}}.
\]

We use \( \| \cdot \| \) and \((\cdot , \cdot)\) to denote the usual norm and inner product of \( H^1(\mathbb{R}^N, \mathbb{C}) \). By the assumptions of \( A_\epsilon(x) \) and \( a(x) \) and Lemma A.2 we know that \( \| \cdot \|_\epsilon \) in \( E \) is equivalent to \( \| \cdot \| \) in \( H^1(\mathbb{R}^N, \mathbb{C}) \). The energy functional associated with (1.1) is defined by

\[
I_\epsilon(u) = \frac{1}{2} \int \left| \frac{\nabla u}{i} - A_\epsilon(x)u \right|^2 + V_\epsilon(x)|u|^2 - \frac{1}{p} \int |u|^p, \quad \forall u \in E.
\]

(1.7)

Denote the functional related to (1.5) by \( I_0(u) \), that is

\[
I_0(u) = \frac{1}{2} \int \left( |\nabla u|^2 + |u|^2 \right) - \frac{1}{p} \int |u|^p, \quad \forall u \in H^1(\mathbb{R}^N, \mathbb{C}).
\]
Let 
\[ Z = \{ e^{i\sigma} w(x); (x, \sigma) \in \mathbb{R}^N \times [0, 2\pi] \simeq \mathbb{R}^N \times S^1 \}. \]

From [13, 14], we know that \( Z \) is non-degenerate, that is
\[ \ker I''_0(qw) = \text{span}_\mathbb{R} \left\{ \frac{\partial(qw)}{\partial x_1}, \ldots, \frac{\partial(qw)}{\partial x_N}, \frac{\partial(qw)}{\partial \sigma} = iqw \right\}. \]

Our main result is as follows:

**Theorem 1.1.** Let \((H_1)\) and \((H_2)\) hold. Then for any positive integer \( m \) there exists \( \epsilon(m) > 0 \) such that for \( 0 < \epsilon < \epsilon(m) \), problem \((1.1)\) has an \( m \)-bump complex-valued solution \( u \) with the following form:

\[ u = \sum_{j=1}^{m} e^{i\sigma} w(x - y_j^\epsilon) + v_{\epsilon, \sigma, y^\epsilon}, \]

where \( \sigma \) is any constant in \([0, 2\pi]\), \( y^\epsilon = (y_1^\epsilon, y_2^\epsilon, \ldots, y_m^\epsilon) \in (\mathbb{R}^N)^m \) with \( |y_i^\epsilon - y_j^\epsilon| \to +\infty \) as \( \epsilon \to 0 \) for any \( i \neq j \), \( \|v_{\epsilon, \sigma, y^\epsilon}\|_e = O(\epsilon) \).

**Remark:** By the very similar argument, we can obtain the following result (see also [30]):
Suppose that \( A_e(x) \) satisfies \((H_2)\) and \( a(x) \) satisfies
\[(H_1') \ a(x) \in C(\mathbb{R}^N, \mathbb{R}^+), \lim_{|x|\to\infty} a(x) = 0, \ a(x) \geq ce^{-\delta|x|} \text{ for some } c > 0, \ \delta > 0.\]

If \( m \in \mathbb{N} \) satisfies \( m < 1 + \frac{p-2}{2(p-1)} \), then there exists \( \epsilon(m) > 0 \) such that for \( 0 < \epsilon < \epsilon(m) \), the following equation
\[(\nabla_i - A_e(x))^2 u + u = (1 - \epsilon a(x))|u|^{p-2} u, \ x \in \mathbb{R}^N, \ (1.8)\]
has an \( m \)-bump complex-valued solution.

Moreover, if \( A_e(x) \) satisfies \((H_2)\) and \( a(x) \in C(\mathbb{R}^N, \mathbb{R}^+) \) satisfies
\[(H_1'') \ \lim_{|x|\to\infty} a(x) = 0, \text{ and there exists } c > 0 \text{ such that } a(x) \geq ce^{-\delta|x|}, \ \forall \delta > 0.\]

Then for any \( m \in \mathbb{N} \), there exists \( \epsilon(m) > 0 \) such that for \( 0 < \epsilon < \epsilon(m) \), \((1.8)\) has an \( m \)-bump complex-valued solution. As a consequence, when \( \epsilon \to 0 \), \((1.8)\) has more and more multi-bump complex-valued solutions.

We mainly use the variational reduction method to prove Theorem 1.1. Our argument is partially inspired by [28–31]. We first reduce the problem to look for solutions of \((1.1)\) to the problem to find the critical points of a function defined on a open subset of a finite dimensional Euclidian space. Then we prove that the function achieves its maximum at the interior of that open subset. We remark that differently from [28,30,31], we need to overcome many additional difficulties which arise because of the appearance of the magnetic field \( A_e(x) \). Problem \((1.1)\) cannot be changed into a pure real-valued problem, hence we should deal with a complex-valued problem directly, which causes more new difficulties in employing the methods in dealing with singularly perturbed problems (see [1]).

This paper is organized as follows. In Section 2, we will carry out the reduction. Then, we will study the reduced finite dimensional problem and prove Theorem 1.1 in Section 3. In Appendix A, we give some elementary inequalities which are useful in our estimates.

**Notation:**
1. We simply write \( \int f \) to mean the Lebesgue integral of \( f(x) \) in \( \mathbb{R}^N \);
2. the complex conjugate of any number \( z \in \mathbb{C} \) will be denoted by \( \bar{z} \);
3. the real part of a number \( z \in \mathbb{C} \) will be denoted by \( \text{Re} z \);
4. the ordinary inner product between two vectors \( a, b \in \mathbb{R}^N \) will be denoted by \( a \cdot b \);
5. \( C, c_i, C_i, C_i' \ (i = 1, 2, \ldots) \) denote generic constants, which may vary inside a chain of inequalities.
2. Variational reduction

Fix $m \in \mathbb{N}$. For $\lambda > 0$ and $m \geq 2$, define
\[
\Omega_\lambda = \{(y_1, y_2, \ldots, y_m) \in (\mathbb{R}^N)^m : |y_k - y_j| > \lambda, \text{ for any } k \neq j\}.
\]

For simplicity, we make the convention
\[
\Omega_\lambda = \mathbb{R}^N \ (\forall \lambda > 0),
\]
if $m = 1$. For $y \in \Omega_\lambda$, let
\[
z_y = \sum_{j=1}^{m} \epsilon w(x - y_j) = \sum_{j=1}^{m} \eta w_y = \eta u_y,
\]
where $w_y = w(\cdot - y_j), u_y = \sum_{j=1}^{m} w_{y_j}$.

Let $y \in \Omega_y$. Define
\[
W_y = \left\{ v \in E \bigg| \Re \int \eta w_y^{p-2} \frac{\partial w_y}{\partial x_i} \bar{v} = 0 \text{ and } \Re \int \eta w_y^{p-1} \bar{v} = 0 \right\},
\]
where $\alpha = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, m$.

It is easy to check that
\[
\Re \int \left[ \left( \frac{\nabla}{i} - A_\epsilon(x) \right) v_1 \right] \left( \frac{\nabla}{i} - A_\epsilon(x) \right) v_2 + \Re \int V_\theta(x) v_1 \bar{v}_2 - (p-2) \Re \int |z_y|^{p-4} \Re(z_y \bar{v}_2) z_y \bar{v}_1 + \int |z_y|^{p-2} \Re(v_1 \bar{v}_2), \forall v_1, v_2 \in W_y,
\]
is a bounded bi-linear functional in $W_y$. Hence there is a bounded linear operator $L_y$ from $W_y$ to $W_y$, such that
\[
\langle L_y v_1, v_2 \rangle = \Re \int \left[ \left( \frac{\nabla}{i} - A_\epsilon(x) \right) v_1 \right] \left( \frac{\nabla}{i} - A_\epsilon(x) \right) v_2 + \Re \int V_\theta(x) v_1 \bar{v}_2 - (p-2) \Re \int |z_y|^{p-4} \Re(z_y \bar{v}_2) z_y \bar{v}_1 + \int |z_y|^{p-2} \Re(v_1 \bar{v}_2), \forall v_1, v_2 \in W_y.
\]

The following lemma shows that $L_y$ is invertible in $W_y$.

**Lemma 2.1.** There exist positive constants $\lambda_0, \epsilon_0$ and $\zeta_0$ such that for any $\lambda > \lambda_0, 0 < \epsilon < \epsilon_0, \sigma \in [0, 2\pi], y \in \Omega_\lambda$ and $v \in W_y$,
\[
\| L_y v \|_e \geq \zeta_0 \| v \|_e.
\]

**Proof.** We argue by contradiction argument. Suppose that there exist \( \{y_{k,n}\}_{n=1}^{\infty} \subset \mathbb{R}^N, k = 1, 2, \ldots, m \), with $|y_{k,n} - y_{j,n}| \to \infty$ for $k \neq j$ and $v_n \in W_{y_n}$ with $\| v_n \|_e = 1$ such that
\[
\| L_{y_n} v_n \|_e = o(1) \| v_n \|_e = o(1),
\]
where $y_n = (y_{1,n}, y_{2,n}, \ldots, y_{m,n})$. Without loss of generality, we may assume that $|y_{k,n}| \to \infty, k = 1, 2, \ldots, m$ as $n \to \infty$. Assume that
\[
v_n(\cdot + y_{k,n}) \rightarrow v_k^*, \text{ in } E, \ k = 1, 2, \ldots, m, \ \text{ as } n \rightarrow \infty
\]
and
\[
v_n(\cdot + y_{k,n}) \rightarrow v_k^*, \text{ strongly in } L_{loc}^2(\mathbb{R}^N), \ k = 1, 2, \ldots, m, \ \text{ as } n \rightarrow \infty.
\]
From
\[ \text{Re} \int \eta w^{p-2} \frac{\partial w}{\partial x_\alpha} \varphi = 0 \quad \text{and} \quad \text{Re} \int \eta w^{p-1} \bar{\varphi} = 0 \]
for \( \alpha = 1, 2, \ldots, N \) and \( k = 1, 2, \ldots, m \), we obtain
\[ \text{Re} \int \eta w^{p-2} \frac{\partial w}{\partial x_\alpha} v_n(\cdot + y_{k,n}) = 0 \quad \text{and} \quad \text{Re} \int \eta w^{p-1} v_n(\cdot + y_{k,n}) = 0 \]
for \( \alpha = 1, 2, \ldots, N \) and \( k = 1, 2, \ldots, m \). So \( v_k^* \) satisfies
\[ \text{Re} \int \eta w^{p-2} \frac{\partial w}{\partial x_\alpha} v_k^* = 0 \quad \text{and} \quad \text{Re} \int \eta w^{p-1} v_k^* = 0 \quad (2.3) \]
for \( \alpha = 1, 2, \ldots, N \) and \( k = 1, 2, \ldots, m \).

Now we prove that \( v_k^* \in \ker I^\prime_0(\eta w) \), that is
\[ \text{Re} \int \nabla v_k^* \nabla \varphi + \text{Re} \int v_k^* \bar{\varphi} - \left[(p - 2) \text{Re} \int \eta w^{p-3} \text{Re}(\eta w \varphi) v_k^* + \text{Re} \int w^{p-2} \text{Re}(v_k^* \varphi) \right] = 0, \quad \forall \varphi \in E. \]

Define
\[ \tilde{W}_y = \left\{ \varphi : \varphi \in E, \text{Re} \int \eta w^{p-2} \frac{\partial w}{\partial x_\alpha} \varphi = 0 \quad \text{and} \quad \text{Re} \int \eta w^{p-1} \varphi = 0, \alpha = 1, 2, \ldots, N \right\}. \]

Note that
\[ o(1) \| \varphi \| = \langle L_y, v_n, \varphi \rangle = \text{Re} \int \left[ \left( \frac{\nabla}{\bar{\lambda}} - A_\varepsilon(x) \right) v_n \right] \varphi + \text{Re} \int V_\varepsilon(x) v_n \varphi - \left[(p - 2) \text{Re} \int |z_n|^{p-4} \text{Re}(z_n \varphi) z_n \bar{\varphi} + \int |z_n|^{p-2} \text{Re}(v_n \varphi) \right], \quad \forall \varphi \in \tilde{W}_y. \quad (2.4) \]

Let \( \varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{C}) \cap \tilde{W}_y \) and take \( \varphi_n(x) =: \varphi(x + y_{k,n}) \in C_0^\infty(\mathbb{R}^N, \mathbb{C}) \). Inserting \( \varphi_n \) into (2.4) and choosing \( \epsilon > 0 \) small enough and \( \lambda > 0 \) big enough, we find
\[ \text{Re} \int \nabla v_k^* \nabla \varphi + \text{Re} \int v_k^* \bar{\varphi} - \left[(p - 2) \text{Re} \int \eta w^{p-3} \text{Re}(\eta w \varphi) v_k^* + \text{Re} \int w^{p-2} \text{Re}(v_k^* \varphi) \right] = 0. \]

Since \( C_0^\infty(\mathbb{R}^N, \mathbb{C}) \) is dense in \( H^1(\mathbb{R}^N, \mathbb{C}) \) and the norm \( \| \cdot \|_\epsilon \) in \( E \) is equivalent to \( \| \cdot \| \) in \( H^1(\mathbb{R}^N, \mathbb{C}), C_0^\infty(\mathbb{R}^N, \mathbb{C}) \) is dense in \( E \). It is easy to show that
\[ \text{Re} \int \nabla v_k^* \nabla \varphi + \text{Re} \int v_k^* \bar{\varphi} - \left[(p - 2) \text{Re} \int \eta w^{p-3} \text{Re}(\eta w \varphi) v_k^* + \text{Re} \int w^{p-2} \text{Re}(v_k^* \varphi) \right] = 0, \quad \forall \varphi \in \tilde{W}_y. \quad (2.5) \]
But (2.5) holds for \( \varphi = c_0 \eta w + \sum_{\alpha=1}^{N} c_\alpha \frac{\partial (w^2)}{\partial x_\alpha} \). Hence (2.5) is true for any \( \varphi \in E \), which means that \( v_k^* \in \ker I''_0(\eta w) \) and hence \( v_k^* = c_0 \eta w + \sum_{\alpha=1}^{N} c_\alpha \frac{\partial (w^2)}{\partial x_\alpha} \). From (2.3), we find

\[
\begin{aligned}
&\left\{ 
\begin{array}{l}
0 < c_\alpha < \infty \\
0 < \alpha \ll 1
\end{array}
\right. \quad \forall \alpha \in \mathbb{N}, \quad 0 < \kappa \ll 1
\end{aligned}
\]

(2.6)

Consequently, \( c_0 = c_1 = \ldots = c_N = 0 \) and \( v_k^* = 0 \). As a result,

\[
\int_{B_R(0)} |v_n|^2 = o(1), \text{ for any } R > 0.
\]

Thus, choosing \( \epsilon > 0 \) small enough, we have

\[
\langle L_y v_n, v_n \rangle = \int \left\{ \left| \left( \frac{\nabla}{i} - A_{\epsilon}(x) \right) v_n \right|^2 + V_{\epsilon}(x)|v_n|^2 - \left[ (p - 2)|z_{y_n}|^{p-4}(\text{Re}(z_{y_n} \tilde{v}_n))^2 + |z_{y_n}|^{p-2}|v_n|^2 \right] \right\}
\]

\[
\geq \int \left[ \left| \left( \frac{\nabla}{i} - A_{\epsilon}(x) \right) v_n \right|^2 + V_{\epsilon}(x)|v_n|^2 \right] - (p - 1) \int |z_{y_n}|^{p-2}|v_n|^2
\]

\[
= \|v_n\|^2 - (p - 1) \int |z_{y_n}|^{p-2}|v_n|^2
\]

\[
= 1 + O(e^{-(p-2)R}) \int_{B_R(0)} |v_n|^2 \geq \frac{1}{2} + o(1).
\]

This is a contradiction to (2.2). \( \square \)

Let

\[
J(\varphi) = I_y(z_{y} + \varphi), \quad \varphi \in W_y.
\]

We have the following result:

**Lemma 2.2.** There exist positive constants \( \epsilon_0 \) and \( \lambda_0 \) such that for \( 0 < \epsilon < \epsilon_0 \) and \( \lambda \geq \lambda_0 \), there exists a \( C^1 \)

\[
v_{\lambda, \epsilon} : \Omega_\lambda \times [0, 2\pi] \to E
\]

depending on \( \lambda \) and \( \epsilon \), satisfying

(i) for any \( (y, \sigma) \in \Omega_\lambda \times [0, 2\pi] \), \( v_{\lambda, \epsilon, \sigma, y} \in W_y \) and

\[
\langle \frac{\partial J(v_{\lambda, \epsilon, \sigma, y})}{\partial v_{\lambda, \epsilon, \sigma, y}}, \varphi \rangle = 0, \quad \forall \varphi \in W_y;
\]

(ii)

\[
\|v_{\lambda, \epsilon, \sigma, y}\| \leq \epsilon^{1-\tau} + \sum_{k \neq j} e^{-\frac{\kappa}{\beta} |y_k - y_j|},
\]

where \( \tau > 0 \) is an arbitrary small constant. Moreover, \( v_{\lambda, \epsilon, \sigma, y} = e^{i\sigma} V_{\lambda, \epsilon, y} \) with \( V_{\lambda, \epsilon, y} \in E \) independent of \( \sigma \).
Proof. Noting that
\[
\text{Re} \int \nabla z_y \nabla \bar{v}_{\lambda, \epsilon, \sigma, y} + \text{Re} \int z_y \bar{v}_{\lambda, \epsilon, \sigma, y} = \sum_{j=1}^{m} \text{Re} \int \eta u_{y_j}^{p-1} \bar{v}_{\lambda, \epsilon, \sigma, y},
\]
we see
\[
J(v_{\lambda, \epsilon, \sigma, y}) =: J(0) + L_y (v_{\lambda, \epsilon, \sigma, y}) + \frac{1}{2} \left( L_y v_{\lambda, \epsilon, \sigma, y}, v_{\lambda, \epsilon, \sigma, y} \right) - R_y (v_{\lambda, \epsilon, \sigma, y}),
\]
where
\[
l_y (v_{\lambda, \epsilon, \sigma, y}) = \sum_{j=1}^{m} \text{Re} \int \eta u_{y_j}^{p-1} \bar{v}_{\lambda, \epsilon, \sigma, y} - \text{Re} \int |z_y|^{p-2} z_y \bar{v}_{\lambda, \epsilon, \sigma, y}
\]
\[
- \epsilon \text{Re} \int \frac{\nabla}{i} z_y B(x) \bar{v}_{\lambda, \epsilon, \sigma, y} + \epsilon^2 \text{Re} \int |B(x)|^2 z_y \bar{v}_{\lambda, \epsilon, \sigma, y}.
\]

Direct calculation yields
\[
\text{Re} \int \left( \frac{\nabla}{i} - A_{\epsilon} (x) \right) v_{\lambda, \epsilon, \sigma, y} \bar{v}_{\lambda, \epsilon, \sigma, y} = \text{Re} \int \nabla z_y \nabla \bar{v}_{\lambda, \epsilon, \sigma, y} - \epsilon \text{Re} \int \frac{\nabla}{i} z_y B(x) \bar{v}_{\lambda, \epsilon, \sigma, y}
\]
\[
- \epsilon \text{Re} \int B(x) z_y \frac{\nabla}{i} v_{\lambda, \epsilon, \sigma, y} + \epsilon^2 \text{Re} \int |B(x)|^2 z_y \bar{v}_{\lambda, \epsilon, \sigma, y}.
\]

Hence,
\[
J(v_{\lambda, \epsilon, \sigma, y}) =: J(0) + L_y (v_{\lambda, \epsilon, \sigma, y}) + \frac{1}{2} \left( L_y v_{\lambda, \epsilon, \sigma, y}, v_{\lambda, \epsilon, \sigma, y} \right) - R_y (v_{\lambda, \epsilon, \sigma, y}),
\]
Thus, to find a critical point for $J(v_{\lambda, \epsilon, \sigma, y})$, we only need to solve

$$l_y(v_{\lambda, \epsilon, \sigma, y}, v_{\lambda, \epsilon, \sigma, y}) = 0.$$  

Lemma 2.3 below implies that $l_y(v_{\lambda, \epsilon, \sigma, y}, v_{\lambda, \epsilon, \sigma, y})$.

Thus, $S_y = \left\{ v_{\lambda, \epsilon, \sigma, y} : v_{\lambda, \epsilon, \sigma, y} \in W_y, \|v_{\lambda, \epsilon, \sigma, y}\|_e \leq \epsilon^{1-\tau} + \sum_{k \neq j} e^{-\frac{p-1-\tau}{p}|y_k-y_j|} \right\}$.

If $2 < p \leq 3$, we can check that

$$\|R'_y(v_{\lambda, \epsilon, \sigma, y})\|_e \leq C\|v_{\lambda, \epsilon, \sigma, y}\|_e^{p-1} \text{ and } \|R''_y(v_{\lambda, \epsilon, \sigma, y})\|_e \leq C\|v_{\lambda, \epsilon, \sigma, y}\|_e^{p-2}.$$  

Thus,

$$\|A_y(v_{\lambda, \epsilon, \sigma, y}) - A_y^2(v_{\lambda, \epsilon, \sigma, y})\|_e = \|L^{-1}yR'_y(v_{\lambda, \epsilon, \sigma, y}) - L^{-1}yR'_y(v_{\lambda, \epsilon, \sigma, y})\|_e$$

$$\leq C\|R'_y(v_{\lambda, \epsilon, \sigma, y}) - R'_y(v_{\lambda, \epsilon, \sigma, y})\|_e$$

$$\leq C\|R'_y(\theta v_{\lambda, \epsilon, \sigma, y} + (1-\theta)v_{\lambda, \epsilon, \sigma, y})\|_e\|v_{\lambda, \epsilon, \sigma, y} - v_{\lambda, \epsilon, \sigma, y}\|_e$$

$$\leq C(\|v_{\lambda, \epsilon, \sigma, y}\|_e^{p-2} + \|v_{\lambda, \epsilon, \sigma, y}\|_e^{p-2})\|v_{\lambda, \epsilon, \sigma, y} - v_{\lambda, \epsilon, \sigma, y}\|_e$$

$$\leq \frac{1}{2}\|v_{\lambda, \epsilon, \sigma, y} - v_{\lambda, \epsilon, \sigma, y}\|_e,$$

where $\theta \in (0, 1)$.

Thus, we have proved that if $2 < p \leq 3$, $A_y$ is a contraction map.

When $\epsilon \to 0$ and $\lambda \to \infty$, Lemma 2.3 below implies that

$$\|l_{y,k}\|_e \leq C \left( \epsilon + \sum_{k \neq j} e^{-\frac{p-1}{p}|y_k-y_j|} \right).$$

Hence,

$$\|A_y(v_{\lambda, \epsilon, \sigma, y})\|_e = \|A_y(v_{\lambda, \epsilon, \sigma, y}) - A_y(0)\|_e + \|A_y(0)\|_e$$

$$\leq Ce^{p-2}\|v_{\lambda, \epsilon, \sigma, y}\|_e + C\|l_{y,k}\|_e$$

$$\leq Ce^{p-2}\|v_{\lambda, \epsilon, \sigma, y}\|_e + C \left( \epsilon + \sum_{k \neq j} e^{-\frac{p-1}{p}|y_k-y_j|} \right)$$

$$\leq \epsilon^{1-\tau} + \sum_{k \neq j} e^{-\frac{p-1-\tau}{p}|y_k-y_j|}.$$  

Thus, $A_y$ maps $S_y$ into $S_y$ if $2 < p \leq 3$. 

$L_y$ is the bounded linear map from $W_y$ to $W_y$ in Lemma 2.1, and

$$R_y(v_{\lambda, \epsilon, \sigma, y}) = \int \left\{ \left| z_y + v_{\lambda, \epsilon, \sigma, y} \right|^p - \left| z_y \right|^p - pRe \left( \left| z_y \right|^{p-2} z_y \bar{v}_{\lambda, \epsilon, \sigma, y} \right) \right\} - \frac{p}{2} \left( (p-2)\left| z_y \right|^p - 2Re(\left| z_y \bar{v}_{\lambda, \epsilon, \sigma, y} \right|) + \left| z_y \right|^{p-2} \left| v_{\lambda, \epsilon, \sigma, y} \right|^2 \right).$$

It is easy to check that $l_y(v_{\lambda, \epsilon, \sigma, y})$ is a bounded linear functional in $W_y$, so there exists an $l_{y,k} \in W_y$ such that

$$l_y(v_{\lambda, \epsilon, \sigma, y}) = \langle l_{y,k}, v_{\lambda, \epsilon, \sigma, y} \rangle.$$
Suppose that $p > 3$. Note that for any $a \in \mathbb{C}$, $|\text{Re}a| \leq |a|$. Then by Lemma A.4, Hölder inequality and the Sobolev inequality, we get

$$
\left| \langle R'_y(v_{\lambda,\epsilon,\sigma,y}), \xi \rangle \right| = \left| \text{Re} \int \left| z_y + v_{\lambda,\epsilon,\sigma,y} \right|^{p-2}(z_y + v_{\lambda,\epsilon,\sigma,y})\xi - \text{Re} \int |z_y|^{p-2}z_y\xi \right. \\
- \text{Re} \int \left[ (p-2)|z_y|^{p-4}\text{Re}(\overline{z_y}v_{\lambda,\epsilon,\sigma,y})z_y\xi + |z_y|^{p-2}\overline{v_{\lambda,\epsilon,\sigma,y}}\xi \right] \\
\leq \int \left| z_y + v_{\lambda,\epsilon,\sigma,y} \right|^{p-2}(z_y + v_{\lambda,\epsilon,\sigma,y}) - |z_y|^{p-2}z_y \\
- \left[ (p-2)|z_y|^{p-4}\text{Re}(\overline{z_y}v_{\lambda,\epsilon,\sigma,y})z_y + |z_y|^{p-2}\overline{v_{\lambda,\epsilon,\sigma,y}} \right] \left| \xi \right| \\
\leq C \int \left| z_y \right|^{p-3}\left| v_{\lambda,\epsilon,\sigma,y} \right|^2 \left| \xi \right| \\
\leq C \left[ \int \left( \left| z_y \right|^{p-3}\left| v_{\lambda,\epsilon,\sigma,y} \right|^2 \right) \right]^{\frac{p-1}{p}} \left\| \xi \right\|_{\epsilon}.
$$

Hence, we get

$$
\left\| R'_y(v_{\lambda,\epsilon,\sigma,y}) \right\|_{\epsilon} \leq C \left[ \int \left( \left| z_y \right|^{p-3}\left| v_{\lambda,\epsilon,\sigma,y} \right|^2 \right) \right]^{\frac{p-1}{p}} \left\| \xi \right\|_{\epsilon}.
$$

Since $z_y$ is bounded, we have

$$
\left\| R'_y(v_{\lambda,\epsilon,\sigma,y}) \right\|_{\epsilon} \leq C \left( \int \left| v_{\lambda,\epsilon,\sigma,y} \right|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{p}} \leq C \left\| v_{\lambda,\epsilon,\sigma,y} \right\|_{\epsilon}^2.
$$

For the estimate of $\left\| R''_y(v_{\lambda,\epsilon,\sigma,y}) \right\|_{\epsilon}$, by Hölder inequality and the Sobolev inequality, we have

$$
\left| R''_y(v_{\lambda,\epsilon,\sigma,y})(\xi, \vartheta) \right| = \left| \text{Re} \int \left[ (p-2)|z_y|^{p-4}\text{Re}(\overline{z_y}v_{\lambda,\epsilon,\sigma,y})z_y\vartheta + |z_y|^{p-2}\overline{z_y}\vartheta \right] \\
- \text{Re} \int \left[ (p-2)|z_y|^{p-4}\text{Re}(\overline{z_y}v_{\lambda,\epsilon,\sigma,y})z_y\vartheta + |z_y|^{p-2}\overline{z_y}\vartheta \right] \\
= \left| \text{Re} \int (p-2)|z_y|^{p-4}\text{Re}(\overline{v_{\lambda,\epsilon,\sigma,y}}z_y)\vartheta \right| \\
\leq \int (p-2)|z_y|^{p-3}\left\| v_{\lambda,\epsilon,\sigma,y} \right\|_{\epsilon} \left\| \xi \right\|_{\epsilon} \left\| \vartheta \right\|_{\epsilon} \\
\leq C \left( \int \left| v_{\lambda,\epsilon,\sigma,y} \right|^3 \right)^{\frac{1}{3}} \left( \int \left| \xi \right|^3 \right)^{\frac{1}{3}} \left( \int \left| \vartheta \right|^3 \right)^{\frac{1}{3}} \\
\leq C \left\| v_{\lambda,\epsilon,\sigma,y} \right\|_{\epsilon} \left\| \xi \right\|_{\epsilon} \left\| \vartheta \right\|_{\epsilon},
$$

which implies

$$
\left\| R''_y(v_{\lambda,\epsilon,\sigma,y}) \right\|_{\epsilon} \leq C \left\| v_{\lambda,\epsilon,\sigma,y} \right\|_{\epsilon}.
$$
Consequently, we have
\[
\|A_y(v_{\lambda,z,s}, y) - A(v)\|_\epsilon = \|L_y^{-1}R_y'(v_{\lambda,z,s}, y) - L_y^{-1}R_y'(v)\|_\epsilon \\
\leq C\|R_y'(v_{\lambda,z,s}, y) - R_y'(v)\|_\epsilon \\
\leq C\|R_y'(v_{\lambda,z,s}, y) - R_y'(v_{\lambda,z,s}, y)\|_\epsilon \leq \epsilon \|v_{\lambda,z,s} - v_{\lambda,z,s}\|_\epsilon \\
\leq \frac{1}{2}\|v_{\lambda,z,s} - v_{\lambda,z,s}\|_\epsilon,
\]
where \(0 < \theta < 1\) and
\[
\|A_y(v_{\lambda,z,s}, y)\|_\epsilon = \|A_y(v_{\lambda,z,s}, y) - A_y(0)\|_\epsilon + \|A_y(0)\|_\epsilon \\
\leq C\|v_{\lambda,z,s}\|_\epsilon + C\|l_{y,k}\|_\epsilon \\
\leq C\|v_{\lambda,z,s}\|_\epsilon + C\left(\epsilon + \sum_{k \neq j} e^{-\frac{m-1}{p}|y_k-y_j|}\right) \\
\leq \epsilon^{1-\tau} + \sum_{k \neq j} e^{-\frac{m-1}{p}|y_k-y_j|},
\]
Hence, \(A_y\) is also a contraction map from \(S_y\) to \(S_y\). Now applying the contraction mapping theorem, for any \((y, \sigma) \in \Omega \times [0, 2\pi]\), we can find a unique \(v_{\lambda,z,s}, y\) such that (2.7) holds. By (2.8) and (2.9), we obtain
\[
\|v_{\lambda,z,s}\|_\epsilon \leq \epsilon^{1-\tau} + \sum_{k \neq j} e^{-\frac{m-1}{p}|y_k-y_j|}.
\]
To prove the \(C^1\)-continuity of \(v_{\lambda,z,s}, y\) with respect to \((y, \sigma)\), we can use the implicit function theorem to find a unique \(C^1\)-map: \((y, \sigma) \rightarrow v_{\lambda,z,s}, y\), which solves (2.7) (see [11]). By the uniqueness, we see \(v_{\lambda,z,s}, y = v_{\lambda,z,s}, y, \sigma\) and hence is a \(C^1\) map with respect to \((y, \sigma)\).

Finally, we prove \(v_{\lambda,z,s}, y = e^{i\sigma}V_{\lambda,z,s}, y\) with \(V_{\lambda,z,s}, y \in E\) independent of \(\sigma\). Since \(v_{\lambda,z,s}, y\) solves (2.7), from Lagrange multiplier theorem, there exist constants \(X_0 \in \mathbb{R}\) and \(Y_0 \in \mathbb{R}\) \((j = 1, 2, \ldots, m)\) such that
\[
I'(z + v_{\lambda,z,s}, y) = \sum_{j=1}^{m} iX_j e^{i\sigma} w_j + \sum_{j=1}^{m} \sum_{k=1}^{N} Y_j e^{i\sigma} \frac{\partial w}{\partial x_k}.
\]
Let \(v_{\lambda,z,s}, y = e^{i\sigma}V_{\lambda,z,s}, y\) with \(V_{\lambda,z,s}, y \in E\). Noting that for any \(\varphi \in E\)
\[
\langle I'(u), \varphi \rangle = \text{Re} \int \left(\frac{\nabla}{i} - A_\epsilon(x)\right) u \left(\frac{\nabla}{i} - A_\epsilon(x)\right) \varphi + \text{Re} \int V_\epsilon(x) w \varphi - \text{Re} \int |u|^{-2} u \varphi,
\]
we test (2.10) by \(e^{i\sigma} v(x)\) with \(v(x) \in E\) and derive that \(\sum_{j=1}^{N} w_j + V_{\lambda,z,s}, y\) is a solution of an equation independently of \(\sigma\). Thus, \(V_{\lambda,z,s}, y\) is independent of \(\sigma\) and we complete the proof.

Lemma 2.3. If \(\lambda \to \infty\) and for any \(y \in \Omega\), then
\[
\|l_{y,k}(v_{\lambda,z,s}, y)\|_\epsilon \leq C \left(\epsilon + \sum_{k \neq j} e^{-\frac{m-1}{p}|y_k-y_j|}\right) \|v_{\lambda,z,s}, y\|_\epsilon.
\]
Lemma A.6 and A.8, we have

\[ \text{Re} \int |z_y|^{p-2} z_y \bar{v}_{\lambda,\epsilon,\sigma,y} \leq \sum_{j=1}^m \text{Re} \int |\eta w_{y_j}|^{p-2} \eta w_{y_j} \bar{v}_{\lambda,\epsilon,\sigma,y} \leq \int |z_y|^{p-2} z_y \sum_{j=1}^m |\eta w_{y_j}|^{p-2} \eta w_{y_j} \bar{v}_{\lambda,\epsilon,\sigma,y} \]

\[ \leq \left( \int |z_y|^{p-2} z_y - \sum_{j=1}^m |\eta w_{y_j}|^{p-2} \eta w_{y_j} \right)^{\frac{p-1}{p}} \left( \int |\bar{v}_{\lambda,\epsilon,\sigma,y}|^p \right)^{\frac{1}{p}} \]

\[ \leq C \left( \sum_{k \neq j} w_{y_k}^{p-1} w_{y_j} \right)^{\frac{p-1}{p}} \|v_{\lambda,\epsilon,\sigma,y}\|_\epsilon \]

\[ \leq C e^{-\frac{p-1}{p} \|y_k - y_j\|_\epsilon} \|v_{\lambda,\epsilon,\sigma,y}\|_\epsilon, \quad (2.11) \]

as \( \lambda \to \infty. \)

On the other hand, we have

\[ |e \text{Re} \int \frac{\nabla}{i} z_y B(x) \bar{v}_{\lambda,\epsilon,\sigma,y}| \leq C e \int \left( \frac{\nabla}{i} z_y \right)^2 \|B(x)\|^2 \left( \int |v_{\lambda,\epsilon,\sigma,y}|^2 \right)^{\frac{1}{2}} \]

\[ \leq C e \int \left( \frac{\nabla}{i} z_y \right)^2 \|B(x)\|^2 \|v_{\lambda,\epsilon,\sigma,y}\|_\epsilon \]

\[ \leq C e \|v_{\lambda,\epsilon,\sigma,y}\|_\epsilon. \quad (2.12) \]

Similarly, we can get

\[ |e^2 \text{Re} \int \|B(x)\|^2 z_y \bar{v}_{\lambda,\epsilon,\sigma,y}| \leq C e^2 \int \left( \frac{\nabla}{i} z_y \right)^2 \|B(x)\|^2 \|v_{\lambda,\epsilon,\sigma,y}\|_\epsilon \]

\[ \leq C e^2 \|v_{\lambda,\epsilon,\sigma,y}\|_\epsilon \leq C e \|v_{\lambda,\epsilon,\sigma,y}\|_\epsilon \]

\[ \quad (2.14) \]

and

\[ |e \text{Re} \int \alpha(x) z_y \bar{v}_{\lambda,\epsilon,\sigma,y}| \leq C e \left( \int |A(x)|^2 \right)^{\frac{1}{2}} \|v_{\lambda,\epsilon,\sigma,y}\|_\epsilon \leq C e \|v_{\lambda,\epsilon,\sigma,y}\|_\epsilon. \quad (2.15) \]

From (2.11) to (2.15), we get

\[ \|l_{g,k}\|_\epsilon \leq C \left( \epsilon + \sum_{k \neq j} e^{-\frac{p-1}{p} \|y_k - y_j\|} \right) \|v_{\lambda,\epsilon,\sigma,y}\|_\epsilon, \]

as \( \lambda \to \infty. \)

For any \( y = (y_1, \ldots, y_m) \in \Omega_\lambda, \)

\[ f_{m,\epsilon}(y_1, y_2, \ldots, y_m) = I_\epsilon(z_y + v_{\lambda,\epsilon,\sigma,y}). \quad (2.16) \]

Since \( v_{\lambda,\epsilon,\sigma,y} = e^{i\sigma} V_{\lambda,\epsilon,\sigma,y} \) with \( V_{\lambda,\epsilon,\sigma,y} \in E \) independent of \( \sigma, \)

we know that \( I_\epsilon(z_y + v_{\lambda,\epsilon,\sigma,y}) \) does not depend on \( \sigma \) either.

From Lemma 2.2, we derive the following result, whose proof is standard and thus is omitted (see for example, [8, 31]).

\[ \]
Lemma 2.4. For large enough $\lambda > 0$ and small enough $\epsilon > 0$, if $y^0 = (y^0_1, \ldots, y^0_m) \in \Omega$ is a critical point of $f_{m, \epsilon}$, then $z_{y^0} + v_{\lambda, \epsilon, \sigma, y^0}$ is a critical point of $I_{\epsilon}$.

3. PROOF OF OUR MAIN RESULT

In this section, we will prove Theorem 1.1. In order to prove it, first we prove that for $\epsilon > 0$ small enough, we can choose $\mu = \mu(\epsilon)$ large enough such that the function $f_{m, \epsilon}(y_1, \ldots, y_m)$ defined in (2.16) attains its maximum in $\Omega_{\mu}$ at some point $y^{(0)} = (y^{(0)}_1, \ldots, y^{(0)}_m)$. We know that $z_{y^{(0)}} + v_{\lambda, \epsilon, \sigma, y^{(0)}}$ is a solution of (1.1) by Lemma 2.4. Here we mainly apply the technique in [28,31], but we make some minor modifications.

Considering that the case $m = 1$ is much easier, we will discuss the case $m \geq 2$. Define

$$d = \sup_{y \in (\mathbb{R}^N)^m} \left\{ \int a(x)|z_y|^2 \right\}.$$

We choose a number $l$ such that $l > \max\{1, 3dC^{-1}_4\}$. Then for any $\epsilon$ satisfying

$$0 < \epsilon < \min \left\{ \left( \frac{1}{2} \frac{C_4}{C_4^2} \right)^{\frac{1}{p-2}}; \frac{1}{l} |w|^p \right\}$$

there exists $\mu^* = \mu^*(\epsilon) > \mu = \mu(\epsilon) > 0$ such that, for $z \in \mathbb{R}^N$ with $|z| \in [\mu(\epsilon), \mu^*(\epsilon)]$,

$$l \epsilon \leq \int w^{p-1}(x)w(x - z)dx \leq 2l \epsilon. \quad (3.1)$$

Define

$$M_{\epsilon} := \sup \left\{ f_{m, \epsilon}(y) | y \in \Omega_{\mu(\epsilon)} \right\}.$$

Denote

$$C_0 = \frac{1}{2} \int [|
abla (\eta w)|^2 + |\eta w|^2] - \frac{1}{p} \int |\eta w|^p.$$

In order to get an $m$-bump solution of (1.1), it is sufficient to prove that $M_{\epsilon}$ is achieved in the interior of $\Omega_{\mu(\epsilon)}$.

Lemma 3.1. Let $m \geq 2$. Then for $\epsilon > 0$ sufficiently small,

$$M_{\epsilon} > \sup \left\{ f_{m, \epsilon}(y) | y \in \Omega_{\mu(\epsilon)} \text{ and } |y_k - y_j| \in [\mu(\epsilon), \mu^*(\epsilon)] \text{ for some } k \neq j \right\}.$$

Proof. Observe that $\mu(\epsilon) = O(\ln \frac{1}{\epsilon}) \rightarrow \infty$ as $\epsilon \rightarrow 0$.

By Lemma 2.2 (ii) and (3.1) we know that if $y \in \Omega_{\mu(\epsilon)}$, then

$$\|v_{\lambda, \epsilon, \sigma, y}\|_\epsilon \leq \epsilon^{1-\tau} + \sum_{k \neq j} e^{-\frac{\tau}{p-2} |y_k - y_j|} \leq C_1 \epsilon^{\frac{p-1-\tau}{p-2}}. \quad (3.2)$$

Note that

$$\frac{1}{2} \langle L_y v_{\lambda, \epsilon, \sigma, y}, v_{\lambda, \epsilon, \sigma, y} \rangle = \frac{1}{2} \int \left\{ \left| \left( \sum_{i} - A_{\epsilon}(x) \right) v_{\lambda, \epsilon, \sigma, y} \right|^2 + V_{\epsilon}(x) |v_{\lambda, \epsilon, \sigma, y}|^2 \right\}$$

$$- \left[ (p - 2)|z_y|^{p-2}(\text{Re}(z_y v_{\lambda, \epsilon, \sigma, y}))^2 + |z_y|^{p-2}|v_{\lambda, \epsilon, \sigma, y}|^2 \right]$$

$$\leq \frac{1}{2} \|v_{\lambda, \epsilon, \sigma, y}\|_\epsilon^2 + (p - 1) \int |z_y|^{p-2}|v_{\lambda, \epsilon, \sigma, y}|^2$$

$$\leq \frac{1}{2} \|v_{\lambda, \epsilon, \sigma, y}\|_\epsilon^2 + C \|v_{\lambda, \epsilon, \sigma, y}\|_\epsilon^2 = C_2 \|v_{\lambda, \epsilon, \sigma, y}\|_\epsilon^2$$
and by Lemma A.5
\[
|R_y(v_{\lambda,\epsilon,\sigma,y})| = \frac{1}{p} \int \left\{ |z_y + v_{\lambda,\epsilon,\sigma,y}|^p - |z_y|^p - p \left[ |z_y|^{p-2} \text{Re}(z_y \bar{v}_{\lambda,\epsilon,\sigma,y}) \right] \right. \\
\left. - \frac{p}{2} [(p-2) |z_y|^{p-4} (\text{Re}(z_y \bar{v}_{\lambda,\epsilon,\sigma,y}))^2 + |z_y|^{p-2} |v_{\lambda,\epsilon,\sigma,y}|^2] \right\} \\
\leq C \int |v_{\lambda,\epsilon,\sigma,y}|^{p^*} \leq C \|v_{\lambda,\epsilon,\sigma,y}\|_{\epsilon}^{p^*} \leq C_3 \|v_{\lambda,\epsilon,\sigma,y}\|_{\epsilon}^2,
\]
where \(p^* = \min\{3, p\} > 2\).

Hence, by Lemma 2.3, we have
\[
f_{m,\epsilon}(y_1, y_2, \ldots, y_m) = I_\epsilon(z_y + v_{\lambda,\epsilon,\sigma,y}) \\
= I_\epsilon(z_y) + I_y(v_{\lambda,\epsilon,\sigma,y}) + \frac{1}{2} \langle L_y v_{\lambda,\epsilon,\sigma,y}, v_{\lambda,\epsilon,\sigma,y} \rangle - R_y(v_{\lambda,\epsilon,\sigma,y}) \\
= I_\epsilon(z_y) + O \left( \|L_y\|_\epsilon \|v_{\lambda,\epsilon,\sigma,y}\|_\epsilon + \|v_{\lambda,\epsilon,\sigma,y}\|_\epsilon^2 \right) \\
= I_\epsilon(z_y) + O \left( \epsilon^{2(p-1)/p} \right) \\
= \frac{1}{2} \int \left( \left| \nabla \frac{x}{i} - A_\epsilon(x) \right| z_y \right|^2 + V_\epsilon(x) |z_y|^2 - \frac{1}{p} \int |z_y|^p + O \left( \epsilon^{2(p-1)/p} \right) \\
= \frac{1}{2} \int \left| \nabla \frac{x}{i} \right|^2 + |z_y|^2 + \frac{\epsilon}{2} \int a(x) |z_y|^2 - \frac{\epsilon}{2} \int |B(x)|^2 |z_y|^2 - \epsilon \text{Re} \int \nabla \frac{x}{i} z_y B(x) \bar{z}_y \\
- \frac{1}{p} \int |z_y|^p + O \left( \epsilon^{2(p-1)/p} \right) \\
= \left\{ \sum_{j=1}^m \frac{1}{2} \int \left| \nabla \frac{x}{i} (\eta w_{y_j}) \right|^2 + |\eta w_{y_j}|^2 - \sum_{j=1}^m \int \frac{1}{p} |\eta w_{y_j}|^p \right\} + \frac{\epsilon}{2} \int a(x) |z_y|^2 \\
+ \left\{ \sum_{k<j} \text{Re} \int \nabla \frac{x}{i} (\eta w_{y_k}) \nabla \frac{x}{i} (\eta w_{y_j}) + \text{Re} \sum_{k<j} \int \eta w_{y_k} \eta w_{y_j} \right\} \\
+ \frac{1}{p} \sum_{j=1}^m \int |\eta w_{y_j}|^p - \frac{1}{p} \int |z_y|^p + O \left( \epsilon^{2(p-1)/p} \right) \\
= mC_0 + \left\{ \sum_{k<j} \int w_{y_j}^{p-1} w_{y_k} - \frac{1}{p} \int u_y + \frac{m}{p} \sum_{j=1}^m \int w_{y_j}^p \right\} + \frac{\epsilon}{2} \int a(x) |z_y|^2 + O \left( \epsilon^{2(p-1)/p} \right) \\
=: mC_0 + \frac{\epsilon}{2} \int a(x) |z_y|^2 - \mathcal{L}_y,
\]
where
\[
\mathcal{L}_y = - \sum_{k<j} \int w_{y_j}^{p-1} w_{y_k} + \frac{1}{p} \int u_y - \frac{m}{p} \sum_{j=1}^m \int w_{y_j}^p + O \left( \epsilon^{2(p-1)/p} \right). \tag{3.3}
\]

Assume that \(y = (y_1, \ldots, y_m) \in O_{\mu(\epsilon)}\) and \(|y_k - y_j| \in [\mu(\epsilon), \mu^*(\epsilon)]\) for some \(k \neq j\).

On one hand, by Lemma A.7, we have
\[
\int u_y = \int \left( \sum_{j=1}^m w_{y_j} \right)^p \geq \sum_{j=1}^m \int w_{y_j}^p + 2(p-1) \sum_{1 \leq k<j \leq m} \int w_{y_k}^{p-1} w_{y_j},
\]
where we use the fact that
\[ \int u_{y_k}^{p-1} w_{y_j} = \int w_{y_j}^{p-1} w_{y_k}. \]

Therefore, by (3.1),
\[ \mathcal{L}_y \geq C_4 \sum_{1 \leq k < j \leq m} \int w_{y_k}^{p-1} w_{y_j} - C'_4 \epsilon^{2(p-1-r)/p} \geq C_4 \epsilon - C'_4 \epsilon^{2(p-1-r)/p} \geq \frac{1}{2} C_4 \epsilon > \frac{3}{2} d \epsilon. \]

Hence,
\[ f_{m,\epsilon}(y_1, y_2, \ldots, y_m) \leq m C_0 + \frac{\epsilon}{2} \int a(x) |z_y|^2 - \mathcal{L}_y \]
\[ \leq m C_0 + \frac{d}{2} \epsilon - \frac{3}{2} d \epsilon = m C_0 - d \epsilon. \]

On the other hand, by Lemma A.3, we get
\[ \int u_y^p = \int \left( \sum_{j=1}^{m} w_{y_j} \right)^p \leq \sum_{j=1}^{m} \int w_{y_j}^p + C \sum_{1 \leq k < j \leq m} \int w_{y_k}^{p-1} w_{y_j}, \]
where we also use the fact that
\[ \int u_{y_k}^{p-1} w_{y_j} = \int w_{y_j}^{p-1} w_{y_k}. \]

Then we have
\[ \mathcal{L}_y \leq C_5 \left( \sum_{k<j} \int w_{y_k}^{p-1} w_{y_j} \right) + O \left( \epsilon^{2(p-1-r)/p} \right). \]

Hence
\[ f_{m,\epsilon}(y_1, y_2, \ldots, y_m) = m C_0 + \frac{\epsilon}{2} \int a(x) |z_y|^2 - \mathcal{L}_y \]
\[ \geq m C_0 + \frac{\epsilon}{2} \int a(x) |z_y|^2 - C_6 \epsilon^{2(p-1-r)/p} + o(1), \]
where \( o(1) \) denotes some quantities which depend only on \( y \) and converge to 0 as \( |y_k - y_j| \to \infty \) for all \( k \neq j \).

Hence, for \( \epsilon > 0 \) sufficiently small,
\[ \liminf_{|y_k - y_j| \to \infty, \forall k \neq j} f_{m,\epsilon}(y_1, y_2, \ldots, y_m) \geq m C_0. \]

(3.5)

Combining (3.4) and (3.5), we complete the proof of this lemma. \( \square \)

Choose \( y^{(h)}(\epsilon) = (y_1^{(h)}(\epsilon), \ldots, y_m^{(h)}(\epsilon)) \in \Omega_{\mu(\epsilon)} \) such that
\[ \lim_{h \to \infty} f_{m,\epsilon} \left( y_1^{(h)}(\epsilon), \ldots, y_m^{(h)}(\epsilon) \right) = M_\epsilon. \]

By Lemma 3.1, we may assume that
\[ \inf_{\epsilon} \min_{k \neq j} \left| y_k^{(h)}(\epsilon) - y_j^{(h)}(\epsilon) \right| \geq \mu^*. \]
Thus, for any $1 \leq k \leq m$, passing to a subsequence if necessary, we may assume either $\lim_{h \to \infty} y_k^{(h)}(\epsilon) = y_k^{(0)}(\epsilon) \in \mathbb{R}^N$ with
\[
|y_k^{(0)}(\epsilon) - y_j^{(0)}(\epsilon)| \geq \mu^* \text{ for } k \neq j \text{ or } \lim_{h \to \infty} |y_k^{(h)}(\epsilon)| = \infty.
\]
Define
\[
\Pi(\epsilon) = \left\{ 1 \leq k \leq m : |y_k^{(h)}(\epsilon)| \to \infty, \text{ as } h \to \infty \right\}.
\]
We will prove that $\Pi(\epsilon) = \emptyset$ for $\epsilon > 0$ small enough and hence $f_{m, \epsilon}$ attain its maximum at $(y_1^{(0)}(\epsilon), \ldots, y_m^{(0)}(\epsilon))$ in $\Omega_{\mu(\epsilon)}$.

**Lemma 3.2.** Let $m \geq 2$. Then there exists $\epsilon(m) > 0$ such that for any $\epsilon \in (0, \epsilon(m))$,
\[
\Pi(\epsilon) = \emptyset.
\]

**Proof.** We make a contradiction argument and assume that $\Pi(\epsilon) \neq \emptyset$ along a sequence $\epsilon_n \to 0$. Without loss of generality, we may assume $\Pi(\epsilon_n) = \{1, 2, \ldots, j_m\}$ for all $n \in \mathbb{N}$ and for some $1 \leq j_m < m$. When $j_m = m$, we can hand by the same argument. For notation of simplicity, we will denote $\epsilon = \epsilon_n$ and $(y_1^{(h)}, y_2^{(h)}, \ldots, y_m^{(h)}) = (y_1^{(\epsilon_n)}, y_2^{(\epsilon_n)}, \ldots, y_m^{(\epsilon_n)})$ for $h = 0, 1, 2, \ldots$. Then, when $h \to \infty$,
\[
|y_1^{(h)}| \to \infty, \ldots, |y_{j_m}^{(h)}| \to \infty \text{ and } y_{j_m+1}^{(h)} \to y_{j_m}^{(0)}, \ldots, y_m^{(h)} \to y_m^{(0)}.
\]
Set
\[
y^{(h)} = (y_1^{(h)}, y_2^{(h)}, \ldots, y_m^{(h)}) \text{ and } y^*_h = (y_{j_m+1}^{(h)}, y_{j_m+2}^{(h)}, \ldots, y_m^{(h)}).
\]
Let
\[
z_h = \sum_{k=1}^m \eta^k y_k^{(h)} := \eta u_h, z_{h,1} = \sum_{k=1}^{j_m} \eta^k y_k^{(h)} := \eta u_{h,1}, z_{h,2} = \sum_{k=j_m+1}^m \eta^k y_k^{(h)} := \eta u_{h,2}.
\]
Similar to (3.2), we get
\[
\|v_{\mu, \epsilon, \sigma, y^{(h)}}\|_e \leq C_7 \epsilon^{p-1-\frac{\tau}{p}}, \|v_{\mu, \epsilon, \sigma, y^*_h}\|_e \leq C_7 \epsilon^{p-1-\frac{\tau}{p}}. \tag{3.6}
\]
Now we rewrite $f_{m, \epsilon}(y_1^{(h)}, \ldots, y_m^{(h)})$
\[
f_{m, \epsilon}(y_1^{(h)}, \ldots, y_m^{(h)}) = I_e(z_h + v_{\mu, \epsilon, \sigma, y^{(h)}}) = mC_0 + \frac{\epsilon}{2} \int a(x)|z_h|^2 - \mathcal{L}_{y^{(h)}}
\]
\[
= j_mC_0 + \frac{\epsilon}{2} \int a(x)|z_{h,1}|^2 + \left[(m - j_m)C_0 + \frac{\epsilon}{2} \int a(x)|z_{h,2}|^2 - \mathcal{L}_{y^*_h} \right] + \epsilon \Re \int a(x) z_{h,1} \overline{z}_{h,2} + \mathcal{L}_{y^*_h} - \mathcal{L}_{y^{(h)}}. \tag{3.7}
\]
Since $|y_k^{(h)}| \to \infty, k = 1, \ldots, j_m$, we get, as $h \to \infty$,
\[
\left| \frac{\epsilon}{2} \int a(x)|z_{h,1}|^2 + \epsilon \Re \int a(x) z_{h,1} \overline{z}_{h,2} \right| \to 0. \tag{3.8}
\]
From (3.7) and (3.8), we obtain
\[
f_{m, \epsilon}(y^{(h)}) \leq j_mC_0 + f_{m-j_m, \epsilon}(y^*_h) + \mathcal{L}_{y^*_h} - \mathcal{L}_{y^{(h)}} + o(1). \tag{3.9}
\]
By (3.3), we infer that
\[
\mathcal{L}_{y^{(h)}} = -\sum_{k<j} \int w_{y_k}^{p-1} w_{y_{(k)}} + \frac{1}{p} \int w_{y_{(k)}} - \sum_{j=1}^{m} \frac{1}{p} \int w_{y_{(j)}} + O\left(\epsilon^{2(\frac{p-1}{p})}\right) \quad (3.10)
\]
and
\[
\mathcal{L}_{y^{(h)}} = -\sum_{j_m<k<j} \int w_{y_k}^{p-1} w_{y_{(k)}} + \frac{1}{p} \int u_{h,2} - \sum_{j=1}^{m} \frac{1}{p} \int w_{y_{(j)}} + O\left(\epsilon^{2(\frac{p-1}{p})}\right). \quad (3.11)
\]
Then, by (3.10) and (3.11),
\[
\mathcal{L}_{y^{(h)}} - \mathcal{L}_{y^{(h)}} = \sum_{k<j \leq j_m} \int w_{y_k}^{p-1} w_{y_{(k)}} + \sum_{k=1}^{j_m} \int w_{y_k}^{p-1} u_{h,2} + \frac{1}{p} \int \sum_{k=1}^{j_m} \int w_{y_{(j)}} + \frac{1}{p} \int u_{h,2} - \frac{1}{p} \int u_{h} + O\left(\epsilon^{2(\frac{p-1}{p})}\right) \\
< O\left(\epsilon^{2(\frac{p-1}{p})}\right). \quad (3.12)
\]
Letting \( h \to \infty \) in (3.9), we get
\[
M_{\epsilon} \leq j_m C_0 + f_{m-j_m,\epsilon} (y_{j_m+1}^{(0)}, \ldots, y_m^{(0)}) + C_9 \epsilon^{\frac{2(\frac{p-1}{p})}{\mu}}. \quad (3.13)
\]
Furthermore, by Lemma A.8 and (3.1), we see that
\[
C_9 \epsilon \leq \mu - \frac{N-1}{2} e^{-\mu} \leq C_{10} \epsilon, \quad (3.14)
\]
which means that
\[
\frac{2}{3} \ln \frac{1}{\epsilon} < \mu = \mu(\epsilon) < 2 \ln \frac{1}{\epsilon}, \quad (3.15)
\]
for \( \epsilon > 0 \) sufficiently small. Choose \( \delta \) such that \( 0 < \delta < \frac{e^{-2-2m}}{14mp} \). By \((H_1)\), there exists \( T > 0 \) such that
\[
a(x) \geq e^{-\delta|x|}, \quad |x| \geq T. \quad (3.16)
\]
For \( \epsilon > 0 \) sufficiently small, define
\[
\bar{y}_k^{\epsilon} = (14m \ln \epsilon^{-1} - 6s \mu(\epsilon) - 1, 0, \ldots, 0) \in \mathbb{R}^N, \quad s = 1, 2, \ldots, m. \quad (3.17)
\]
The open balls \( B(\bar{y}_k^{\epsilon}, 3\mu(\epsilon)) \) are mutually disjoint. Therefore there are \( j_m \) integers from \( \{1, 2, \ldots, m\} \), denoted by \( s_1 < s_2 < \cdots < s_{j_m} \), such that
\[
|\bar{y}_k^{\epsilon} - y_j^{(0)}| \geq 3\mu(\epsilon), \quad k = 1, 2, \ldots, j_m, \quad j = j_m + 1, \ldots, m. \quad (3.18)
\]
Denote \( \bar{y}_k^{\epsilon} \) by \( y_k^{\epsilon} \) for \( k = 1, 2, \ldots, j_m \). It follows that from (3.16)–(3.18), for \( \epsilon > 0 \) small enough,
\[
T + 1 \leq |y_k^{\epsilon}| \leq 14m \ln \epsilon^{-1} - 1, \quad k = 1, 2, \ldots, j_m, \quad (3.19)
\]
\[
|y_k^{\epsilon} - y_j^{(0)}| \geq 3\mu(\epsilon), \quad 1 \leq k < j \leq j_m, \quad (3.20)
\]
\[
|y_k^{\epsilon} - y_j^{(0)}| \geq 3\mu(\epsilon), \quad k = 1, 2, \ldots, j_m, \quad j = j_m + 1, \ldots, m. \quad (3.21)
\]
Thus,
\[
(y_1^{\epsilon}, \ldots, y_{j_m}^{\epsilon}, y_{j_m+1}^{(0)}, \ldots, y_m^{(0)}) \in \Omega_{\mu(\epsilon)}. 
\]
Denote \( y^{(c)} = (y_1^c, \ldots, y_j^c, y_{j_m+1}^0, \ldots, y_m^0) \) and \( y^{(o)} = (y_{j_m+1}^0, \ldots, y_m^0). \) Let

\[
z_{c,1} = \sum_{k=1}^{j_m} \eta w_{y_k^c} := \eta w_{1,c} \quad \text{and} \quad z_{c,2} = \sum_{k=j_m+1}^{m} \eta w_{y_k^{(o)}} = \eta w_{2,c}.
\]

Similar to (3.7), we get

\[
f_{m,c}(y_1^c, \ldots, y_j^c, y_{j_m+1}^0, \ldots, y_m^0) = j_m C_0 + f_{m-j_m,c}(y_{j_m+1}^0, \ldots, y_m^0) + \frac{\epsilon}{2} \int a(x) |z_{c,1}|^2 + \epsilon \text{Re} \int a(x) z_{c,1} \overline{z}_{c,2} + \mathcal{L}_{y^{(c)}} - \mathcal{L}_{y^{(o)}}.
\]

From (3.21), a similar argument shows that

\[
\sum_{1 \leq k < j \leq j_m} \int w_{y_k^{(c)}} w_{y_j} = o(\epsilon^{3\mu}) = o(\epsilon^2), \quad \text{as} \quad \epsilon \to 0.
\]

According to (3.21), a similar argument shows that

\[
\sum_{k=1}^{j_m} \int w_{y_k^{(c)}} u_{c,2} + \sum_{k=1}^{j_m} \int u_{c,2}^{-1} w_{y_k^{(c)}} = o(\epsilon^2).
\]

From (3.24) to (3.25), we get

\[
\mathcal{L}_{y^{(c)}} - \mathcal{L}_{y^{(o)}} \geq O \left( \epsilon^{\frac{2(\mu-1)}{p}} \right),
\]

which with (3.22) yields

\[
f_{m,c}(y_1^c, \ldots, y_j^c, y_{j_m+1}^0, \ldots, y_m^0) \geq j_m C_0 + f_{m-j_m,c}(y_{j_m+1}^0, \ldots, y_m^0) + \frac{\epsilon}{2} \int a(x) |z_{c,1}|^2 + \epsilon \text{Re} \int a(x) z_{c,1} \overline{z}_{c,2} - C_{11} \epsilon^{\frac{2(\mu-1)}{p}}.
\]

By (3.16) and (3.19), we have, for \( k = 1, 2, \ldots, j_m, \)

\[
\int a(x) |z_{c,1}|^2 = \int a(x) u_{c,1}^2 \geq \int_{|x-y_k^c| \leq 1} a(x) u_{c,1}^2 \geq \int_{|x-y_k^c| \leq 1} e^{-\delta|x|} u_{c,1}^2 \geq C_{12} e^{-\delta(|y_k^c|+1)} \geq C_{12} e^{-14m\delta \ln \epsilon^{-1}} = C_{12} \epsilon^{14m\delta}.
\]
Hence, we get for $\epsilon$ small enough,
\[
    f_{m,\epsilon}(y_1^{(0)}, \ldots, y_m^{(0)}) \geq j_m C_0 + f_{m-j_m,\epsilon}(y_{j_m+1}^{(0)}, \ldots, y_m^{(0)}) + C_{12} \epsilon^{14m\delta + 1} - C_{11} \epsilon^{2(\frac{1}{p} - \sigma) + 1},
\]
since $14m\delta + 1 < 2(\frac{1}{p} - \sigma)$. It contradicts to (3.13). Thus there exists $\epsilon(m) > 0$ such that if $0 < \epsilon < \epsilon(m)$, then $H(\epsilon) = \emptyset$ and $f_{m,\epsilon}$ achieves its maximum at some point $(y_1^{(0)}, \ldots, y_m^{(0)}) \in \Omega_{\mu(\epsilon)}$.

We are now in position to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 3.2, if $m \geq 2$ and $0 < \epsilon < \epsilon(m)$, then $f_{m,\epsilon}$ achieves its maximum at some point $(y_1^{(0)}, \ldots, y_m^{(0)}) \in \Omega_{\mu(\epsilon)}$. Then $z_y^{(0)} + v_{\lambda,\epsilon,\sigma,y^{(0)}}$ is an $m$-bump complex-valued solution of (1.1). By Lemma 2.2 (ii), if $m = 1$ and $\epsilon \in (0, \epsilon_0]$, then
\[
    \lim_{|y| \to \infty} f_{m,\epsilon}(y) = \lim_{|y| \to \infty} I_{\epsilon}(z_y + v_{\lambda,\epsilon,\sigma,y}) = I_0(\eta \nu) = C_0.
\]

For $m = 1$, since $f_{m,\epsilon}$ is defined on all $\mathbb{R}^N$, $f_{m,\epsilon}$ has a critical point $y^{(0)} \in \mathbb{R}^N$ and $z_y^{(0)} + v_{\lambda,\epsilon,\sigma,y^{(0)}}$ is a complex-valued solution of (1.1).

Set $\epsilon(1) = \epsilon_0$ and $\epsilon_1(m) = \min\{\epsilon(1), \epsilon(2), \ldots, \epsilon(m)\}$. If $0 < \epsilon < \epsilon_1(m)$, then (1.1) has at least $m$ nontrivial complex-valued solutions. $\Box$

**APPENDIX A**

Since the following two lemmas are very basic, we omit their proofs and one can refer [13].

**Lemma A.1.** For any $u \in E$, there exists $C > 0$ such that
\[
    \int \left| \left( \frac{\nabla}{i} - A_{\epsilon}(x) \right) u \right|^2 + V_{\epsilon}(x)|u|^2 \geq C \int (|\nabla u|^2 + |u|^2).
\]

**Lemma A.2.** The two norms $\| \cdot \|$ in $H^1(\mathbb{R}^N, \mathbb{C})$ and $\| \cdot \|_\epsilon$ in $E$ are equivalent.

Now we give some elementary inequalities which are applied in the previous subsections. For these inequalities, one can refer [29–31].

**Lemma A.3.** For $q > 1$, there exists $C > 0$ such that for any $a, b \in \mathbb{C}$,
\[
    |a + b|^q - |a|^q - |b|^q \leq C|a|^{q-1}|b| + C|a||b|^{q-1}.
\]

**Lemma A.4.** For $q > 1$, there exists $C > 0$ such that for any $a, b \in \mathbb{C}$ and $|a| > |b|$,
\[
    |a + b|^q(a + \bar{b}) - |a|^q a - |a|^{q-2}\text{Re}(a\bar{b})\bar{a} + |a|^{q}\bar{b} \leq C|a|^{q-1}|b|^2.
\]

**Lemma A.5.** For $q > 2$, there exists $C > 0$ such that for any $a, b \in \mathbb{C}$ and $|a| > |b|$,
\[
    |a + b|^q - |a|^q - |a|^{q-2}\text{Re}(a\bar{b}) - \frac{q}{2}(q - 2)|a|^{q-4}(\text{Re}(a\bar{b}))^2 + |a|^{q-2}|b|^2 \leq \begin{cases} C|a|^{q-3}|b|^3, & \text{if } q > 3, \\ C|a|^{3-q}|b|^q, & \text{if } 2 < q \leq 3. \end{cases}
\]
Lemma A.6. For \( q \geq 2 \) and \( m \in \mathbb{N} \), there exists \( C > 0 \) such that for any \( a_j \in \mathbb{C}, j = 1, 2, \ldots, m, \)
\[
\left\| \sum_{j=1}^{m} a_j \right\|^{q-2} \left\| \sum_{j=1}^{m} a_j \right\| - \sum_{j=1}^{m} |a_j|^{q-2} a_j \| \leq C \sum_{k \neq j} |a_k|^{q-1} |a_j|.
\]

Lemma A.7. For \( q \geq 2 \) and \( m \in \mathbb{N} \), there exists \( C > 0 \) such that for any \( a_j \geq 0, j = 1, 2, \ldots, m, \)
\[
\left( \sum_{j=1}^{m} a_j \right)^q \geq \sum_{j=1}^{m} a_j^q + (q-1) \sum_{1 \leq k \neq j \leq m} a_k^{q-1} a_j.
\]

Lemma A.8 (Lem. II.2, [4]). There exists a positive constant \( c > 0 \) such that for any \( |y_k - y_j| \to \infty, \)
\[
\int w_k^{-1} w_j \sim c |y_k - y_j|^{-\frac{n+1}{2}} e^{-|y_k-y_j|}.
\]

Acknowledgements. The authors thank sincerely Professors Shuangjie Peng and Gongbao Li for helpful discussions and suggestions. This paper was partially supported by NSFC (Nos. 11071092, 11071095) and the Ph.D. specialized grant of the Ministry of Education of China (20110144110001). C.H. Wang was also supported by the State Scholarship Fund of CSC (2011677017).

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MULTI-BUMP SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH ELECTROMAGNETIC FIELDS


