ESAIM: COCV 19 (2013) 404–437
DOI: 10.1051/cocv/2012015

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

HAMILTON–JACOBI EQUATIONS AND TWO-PERSON ZERO-SUM DIFFERENTIAL GAMES WITH UNBOUNDED CONTROLS*

Hong Qiu[1,2] and Jiongmin Yong[2]

Abstract. A two-person zero-sum differential game with unbounded controls is considered. Under proper coercivity conditions, the upper and lower value functions are characterized as the unique viscosity solutions to the corresponding upper and lower Hamilton–Jacobi–Isaacs equations, respectively. Consequently, when the Isaacs' condition is satisfied, the upper and lower value functions coincide, leading to the existence of the value function of the differential game. Due to the unboundedness of the controls, the corresponding upper and lower Hamiltonians grow super linearly in the gradient of the upper and lower value functions, respectively. A uniqueness theorem of viscosity solution to Hamilton–Jacobi equations involving such kind of Hamiltonian is proved, without relying on the convexity/concavity of the Hamiltonian. Also, it is shown that the assumed coercivity conditions guaranteeing the finiteness of the upper and lower value functions are sharp in some sense.

Mathematics Subject Classification. 49L25, 49N70, 91A23.

Received May 1st, 2011. Revised September 22, 2011.
Published online January 23, 2013.

1. Introduction

Let us begin with the following control system:

$$\begin{cases} \dot{y}(s) = f(s, y(s), u_1(s), u_2(s)), & s \in [t, T], \\ y(t) = x. \end{cases} \tag{1.1}$$

where $f : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}^n$ is a given map. In the above, $y(\cdot)$ is the state trajectory taking values in $\mathbb{R}^n$, and $(u_1(\cdot), u_2(\cdot))$ is the control pair taken from the set $\mathcal{U}_1^{\sigma_1}[t, T] \times \mathcal{U}_2^{\sigma_2}[t, T]$ of *admissible controls*, defined by the following:

$$\mathcal{U}_i^{\sigma_i}[t, T] = \left\{ u_i : [t, T] \to U_i \;\middle|\; \|u_i(\cdot)\|_{L^{\sigma_i}(t,T)} \equiv \left[\int_t^T |u_i(s)|^{\sigma_i} \mathrm{d}s \right]^{\frac{1}{\sigma_i}} < \infty \right\}, \qquad i = 1, 2,$$

Keywords and phrases. Two-person zero-sum differential games, unbounded control, Hamilton–Jacobi equation, viscosity solution.

* *This work is supported in part by the NSF Grant DMS-1007514, the NSFC Grant 11171081, and the Postgraduate Scholarship Program of China.*

[1] Department of Mathematics, Harbin Institute of Technology, Weihai 264209, Shandong, P.R. China

[2] Department of Mathematics, University of Central Florida, Orlando, 32816 FL, USA. Jiongmin.Yong@ucf.edu

Article published by EDP Sciences

with U_i being a closed subset of $\mathbb{R}^{m_i}$ and with some $\sigma_i \geq 1$. We point out that U_1 and U_2 are allowed to be unbounded, and they could even be $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively. Hereafter, we suppress $\mathbb{R}^{m_i}$ in $\|u_i(\cdot)\|_{L^{\sigma_i}(t,T;\mathbb{R}^{m_i})}$ for notational simplicity and this will not cause confusion. The *performance functional* associated with (1.1) is the following:

$$J(t,x;u_1(\cdot),u_2(\cdot)) = \int_t^T g(s,y(s),u_1(s),u_2(s))\mathrm{d}s + h(y(T)), \tag{1.2}$$

with $g:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}$ and $h:\mathbb{R}^n\to\mathbb{R}$ being some given maps.

The above setting can be used to describe a two-person zero-sum differential game: Player 1 wants to select a control $u_1(\cdot)\in\mathcal{U}_1^{\sigma_1}[t,T]$ so that the functional (1.2) is minimized and Player 2 wants to select a control $u_2(\cdot)\in\mathcal{U}_2^{\sigma_2}[t,T]$ so that the functional (1.2) is maximized. Therefore, $J(t,x;u_1(\cdot),u_2(\cdot))$ is a *cost functional* for Player 1 and a *payoff functional* for Player 2, respectively. If U_2 is a singleton, the above is reduced to a standard optimal control problem.

Under some mild conditions, for any *initial pair* $(t,x)\in[0,T]\times\mathbb{R}^n$ and control pair $(u_1(\cdot),u_2(\cdot))\in\mathcal{U}_1^{\sigma_1}[t,T]\times\mathcal{U}_2^{\sigma_2}[t,T]$, the state equation (1.1) admits a unique solution $y(\cdot)\equiv y(\cdot\,;t,x,u_1(\cdot),u_2(\cdot))$, and the performance functional $J(t,x;u_1(\cdot),u_2(\cdot))$ is well-defined. By adopting the notion of *Elliott–Kalton strategies* [11], we can define the *upper* and *lower value functions* $V^\pm:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ (see Sect. 3 for details). Further, when $V^\pm(\cdot,\cdot)$ are differentiable, they should satisfy the following upper and lower Hamilton–Jacobi–Isaacs (HJI, for short) equations, respectively:

$$\begin{cases} V_t^\pm(t,x)+H^\pm(t,x,V_x^\pm(t,x))=0, & (t,x)\in[0,T]\times\mathbb{R}^n,\\ V^\pm(T,x)=h(x), & x\in\mathbb{R}^n,\end{cases} \tag{1.3}$$

where $H^\pm(t,x,p)$ are the so-called *upper* and *lower Hamiltonians* defined by the following, respectively:

$$\begin{cases} H^+(t,x,p)=\inf_{u_1\in U_1}\sup_{u_2\in U_2}\left[\langle\, p,f(t,x,u_1,u_2)\,\rangle+g(t,x,u_1,u_2)\right],\\ H^-(t,x,p)=\sup_{u_2\in U_2}\inf_{u_1\in U_1}\left[\langle\, p,f(t,x,u_1,u_2)\,\rangle+g(t,x,u_1,u_2)\right],\end{cases} \quad (t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n. \tag{1.4}$$

When the sets U_1 and U_2 are bounded, the above differential game is well-understood [12,17]: under reasonable conditions, the upper and lower value functions $V^\pm(\cdot,\cdot)$ are the unique viscosity solutions to the corresponding upper and lower HJI equations, respectively. Consequently, in the case that the following *Isaacs condition*:

$$H^+(t,x,p)=H^-(t,x,p), \qquad \forall(t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n, \tag{1.5}$$

holds, the upper and lower value functions coincide and the two-person zero-sum differential game admits the value function

$$V(t,x)=V^+(t,x)=V^-(t,x), \qquad (t,x)\in[0,T]\times\mathbb{R}^n. \tag{1.6}$$

For comparison purposes, let us now take a closer look at the properties that the upper and lower value functions $V^\pm(\cdot,\cdot)$ and the upper and lower Hamiltonians $H^\pm(\cdot,\cdot,\cdot)$ have, under classical assumptions. To this end, let us recall the following classical assumption:

(B) Functions $f:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}^n$, $g:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}$, and $h:\mathbb{R}^n\to\mathbb{R}$ are continuous. There exists a constant $L>0$ and a continuous function $\omega:[0,\infty)\times[0,\infty)\to[0,\infty)$, increasing in each of its arguments and $\omega(r,0)=0$ for all $r\geq0$, such that for all $t,s\in[0,T],\ x,y\in\mathbb{R}^n,\ (u_1,u_2)\in U_1\times U_2$,

$$\begin{cases} |f(t,x,u_1,u_2)-f(s,y,u_1,u_2)|\leq L|x-y|+\omega\big(|x|\vee|y|,|t-s|\big),\\ |g(t,x,u_1,u_2)-g(s,y,u_1,u_2)|\leq\omega\big(|x|\vee|y|,|x-y|+|t-s|\big),\\ |h(x)-h(y)|\leq\omega\big(|x|\vee|y|,|x-y|\big),\\ |f(t,0,u_1,u_2)|+|g(t,0,u_1,u_2)|+|h(0)|\leq L,\end{cases} \tag{1.7}$$

where $|x|\vee|y|=\max\{|x|,|y|\}$.

Condition (1.7) implies that the continuity and the growth of $(t,x) \mapsto (f(t,x,u_1,u_2),\ g(t,x,u_1,u_2))$ are uniform in $(u_1,u_2) \in U_1 \times U_2$. This essentially will be the case if U_1 and U_2 are bounded (or compact metric spaces). Let us state the following proposition.

Proposition 1.1. *Under assumption* (B)*, one has the following:*

(i) *The upper and lower value functions $V^{\pm}(\cdot,\cdot)$ are well-defined continuous functions. Moreover, they are the unique viscosity solutions to the upper and lower HJI equations* (1.3)*, respectively. In particular, if Isaacs' condition* (1.5) *holds, the upper and lower value functions coincide.*

(ii) *The upper and lower Hamiltonians $H^{\pm}(\cdot,\cdot,\cdot)$ satisfy the following: for all $t \in [0,T],\ x,y,p,q \in \mathbb{R}^n$,*

$$\begin{cases} |H^{\pm}(t,x,p) - H^{\pm}(t,y,q)| \le L(1+|x|)|p-q| + \omega\big(|x| \vee |y|, |x-y|\big), \\ |H^{\pm}(t,x,p)| \le L(1+|x|)|p| + L + \omega(|x|,|x|). \end{cases} \tag{1.8}$$

Condition (1.8) plays an important role in the proof of the uniqueness of viscosity solution to HJI equations (1.3) [5,16]. Note that, in particular, (1.8) implies that $p \mapsto H^{\pm}(t,x,p)$ is at most of linear growth.

Unfortunately, the above property (1.8) fails, in general, when the control domains U_1 and/or U_2 is unbounded. To make this more convincing, let us look at a one-dimensional linear-quadratic (LQ, for short) optimal control problem (which amounts to saying that $U_1 = \mathbb{R}$ and $U_2 = \{0\}$). Consider the state equation

$$\dot y(s) = y(s) + u(s), \qquad s \in [t,T],$$

with a quadratic cost functional

$$J(t,x;u(\cdot)) = \frac{1}{2}\left[\int_t^T \big(|y(s)|^2 + |u(s)|^2\big)\mathrm{d}s + |y(T)|^2\right].$$

Then the Hamiltonian is

$$H(t,x,p) = \inf_{u\in\mathbb{R}}\left[p(x+u) + \frac{|x|^2+|u|^2}{2}\right] = xp + \frac{x^2}{2} - \frac{p^2}{2}\cdot$$

Thus, $p \mapsto H(t,x,p)$ is of quadratic growth and (1.8) fails.

Optimal control problems with unbounded control domains were studied in [2,8]. Uniqueness of viscosity solution to the corresponding Hamilton–Jacobi–Bellman equation was proved by some arguments relying on the convexity/concavity of the corresponding Hamiltonian with respect to p. Recently, the above results were substantially extended to stochastic optimal control problems [10] for which the Hamiltonian is convex in the gradient of the value function. On the other hand, as an extension of [24], two-person zero-sum differential games with (only) one player having unbounded control were studied in [21]. Some nonlinear H_∞ problems can also be treated as such kind of differential games [20,22]. Further, stochastic two-person zero-sum differential games were studied in [9] with one player having unbounded control and with the two players' controls being separated both in the state equation and the performance functional.

The main purpose of this paper is to study two-person zero-sum differential games with *both* players having unbounded controls, and the controls of two players are not necessarily separated. One motivation comes from the problem of what we call the affine-quadratic (AQ, for short) two-person zero-sum differential games, by which we mean that the right hand side of the state equation is affine in the controls, and the integrand of the performance functional is quadratic in the controls (see Sect. 2). This is a natural generalization of the classical LQ problems. For general two-person zero-sum differential games with (both players having) unbounded controls, under some mild coercivity conditions, the upper and lower Hamiltonians $H^{\pm}(t,x,p)$ are proved to be well-defined, continuous, and locally Lipschitz in p. Therefore, the upper and lower HJI equations can be formulated. Then we will establish the uniqueness of viscosity solutions to a general first order Hamilton–Jacobi equation which includes our upper and lower HJI equations of the differential game. Comparing with a relevant result found in [6], the conditions we assumed here are a little different from theirs and we present a detailed proof for reader's convenience. By assuming a little stronger coercivity conditions, together with some additional conditions (guaranteeing the well-posedness of the state equation, etc.), we show that the upper and lower value functions can be well-defined and are continuous. Combining the above results, one obtains a characterization of the upper and lower value functions of the differential game as the unique viscosity solutions to the corresponding upper and lower HJI equations. Then if in addition, the Isaacs' condition holds, the upper and lower value functions coincide which yields the existence of the value function of the differential game.

We would like to mention here that due to the unboundedness of the controls, the continuity of the upper and lower value functions $V^{\pm}(t,x)$ in t is quite subtle. To prove that, we need to establish a modified principle of optimality and fully use the coercivity conditions. It is interesting to indicate that the assumed coercivity conditions that ensuring the finiteness of the upper and lower value functions are actually sharp in some sense, which was illustrated by a one-dimensional LQ situation.

For some other relevant works in the literature, we would like to mention [1, 13–15, 19, 25], and references cited therein.

The rest of the paper is organized as follows. In Section 2, we make some brief observations on an AQ two-person differential game, for which we have a situation that the Isaacs' condition holds and the upper and lower Hamiltonians $H^{\pm}(t,x,p)$ are quadratic in p but may be neither convex nor concave. Section 3 is devoted to a study of upper and lower Hamiltonians. The uniqueness of viscosity solutions to a class of HJ equations will be proved in Section 4. In Section 5, we will show that under certain conditions, the upper and lower value functions are well-defined and continuous. Finally, in Section 6, we show that the assumed coercivity conditions ensuring the upper and lower value functions to be well-defined are sharp in some sense.

2. An affine-quadratic two-person differential game

To better understand two-person zero-sum differential games with unbounded controls, in this section, we look at a nontrivial special case which is a main motivation of this paper. Consider the following state equation:

$$\begin{cases} \dot y(s) = A(s,y(s)) + B_1(s,y(s))u_1(s) + B_2(s,y(s))u_2(s), \qquad s\in[t,T],\\ y(t)=x, \end{cases} \tag{2.1}$$

for some suitable matrix valued functions $A(\cdot,\cdot)$, $B_1(\cdot,\cdot)$, and $B_2(\cdot,\cdot)$. The state $y(\cdot)$ takes values in $\mathbb{R}^n$ and the control $u_i(\cdot)$ takes values in $U_i=\mathbb{R}^{m_i}$ $(i=1,2)$. The performance functional is given by

$$\begin{aligned} J(t,x;u_1(\cdot),u_2(\cdot)) = \int_t^T \Big[& Q(s,y(s)) + \frac12 \langle R_1(s,y(s))u_1(s),u_1(s)\rangle \\ & + \langle S(s,y(s))u_1(s),u_2(s)\rangle - \frac12 \langle R_2(s,y(s))u_2(s),u_2(s)\rangle \\ & + \langle \theta_1(s,y(s)),u_1(s)\rangle + \langle \theta_2(s,y(s)),u_2(s)\rangle \Big]\,\mathrm{d}s + G(y(T)), \end{aligned} \tag{2.2}$$

for some scalar functions $Q(\cdot,\cdot)$ and $G(\cdot)$, some vector valued functions $\theta_1(\cdot,\cdot)$ and $\theta_2(\cdot,\cdot)$, and some matrix valued functions $R_1(\cdot,\cdot)$, $R_2(\cdot,\cdot)$, and $S(\cdot,\cdot)$. Note that the right hand side of the state equation is affine in the controls $u_1(\cdot)$ and $u_2(\cdot)$, and the integrand in the performance functional is up to quadratic in $u_1(\cdot)$ and $u_2(\cdot)$. Therefore, we refer to such a problem as an *affine-quadratic* (AQ, for short) *two-person zero-sum differential game*. We also note that due to the presence of the term $\langle S(s,y(s))u_1(s),u_2(s)\rangle$, controls $u_1(\cdot)$ and $u_2(\cdot)$ cannot be completely separated. Let us now introduce the following basic hypotheses concerning the above AQ two-person zero-sum differential game.

(AQ1) The maps

$$A:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n,\quad B_1:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{n\times m_1},\quad B_2:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{n\times m_2},$$

are continuous.

(AQ2) The maps

$$Q:[0,T]\times\mathbb{R}^n\to\mathbb{R},\quad G:\mathbb{R}^n\to\mathbb{R},\quad R_1:[0,T]\times\mathbb{R}^n\to\mathcal{S}^{m_1},\quad R_2:[0,T]\times\mathbb{R}^n\to\mathcal{S}^{m_2},$$
$$S:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{m_2\times m_1},\quad \theta_1:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{m_1},\quad \theta_2:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{m_2}$$

are continuous (where $\mathcal{S}^m$ stands for the set of all $(m\times m)$ symmetric matrices), and $R_1(t,x)$ and $R_2(t,x)$ are positive definite for all $(t,x)\in[0,T]\times\mathbb{R}^n$.

With the above hypotheses, we let

$$\begin{aligned}\mathbb{H}(t,x,p,u_1,u_2)=&\langle\, p, A(t,x)+B_1(t,x)u_1+B_2(t,x)u_2\,\rangle+Q(t,x)+\frac{1}{2}\langle\, R_1(t,x)u_1,u_1\,\rangle\\&+\langle\, S(t,x)u_1,u_2\,\rangle-\frac{1}{2}\langle\, R_2(t,x)u_2,u_2\,\rangle+\langle\,\theta_1(t,x),u_1\,\rangle+\langle\,\theta_2(t,x),u_2\,\rangle.\end{aligned}\tag{2.3}$$

Our result concerning the above-defined function is the following proposition.

Proposition 2.1. *Let* (AQ1)–(AQ2) *hold. Then the matrix* $\begin{matrix}R_1(t,x) & S(t,x)^T\\ S(t,x) & -R_2(t,x)\end{matrix}$ *is invertible, and*

$$\begin{aligned}\mathbb{H}(t,x,p,u_1,u_2)=&\frac{1}{2}\langle\, R_1(t,x)(u_1-\bar u_1),u_1-\bar u_1\,\rangle+\langle\, S(t,x)(u_1-\bar u_1),u_2-\bar u_2\,\rangle\\&-\frac{1}{2}\langle\, R_2(t,x)(u_2-\bar u_2),u_2-\bar u_2\,\rangle+Q_0(t,x,p),\end{aligned}\tag{2.4}$$

where

$$\begin{pmatrix}\bar u_1\\ \bar u_2\end{pmatrix}=-\begin{pmatrix}R_1(t,x) & S(t,x)^T\\ S(t,x) & -R_2(t,x)^{-1}\end{pmatrix}\begin{pmatrix}B_1(t,x)^Tp+\theta_1(t,x)\\ B_2(t,x)^Tp+\theta_2(t,x)\end{pmatrix},\tag{2.5}$$

and

$$\begin{aligned}Q_0(t,x,p)=&\,Q(t,x)+\langle\, p,A(t,x)\,\rangle\\&-\frac{1}{2}\begin{pmatrix}B_1(t,x)^Tp+\theta_1(t,x)\\ B_2(t,x)^Tp+\theta_2(t,x)\end{pmatrix}^T\begin{pmatrix}R_1(t,x) & S(t,x)^T\\ S(t,x) & -R_2(t,x)\end{pmatrix}^{-1}\begin{pmatrix}B_1(t,x)^Tp+\theta_1(t,x)\\ B_2(t,x)^Tp+\theta_2(t,x)\end{pmatrix}.\end{aligned}\tag{2.6}$$

Further, $(\bar u_1,\bar u_2)$ *given by* (2.5) *is the unique saddle point of* $(u_1,u_2)\mapsto\mathbb{H}(t,x,p,u_1,u_2)$, *namely,*

$$\mathbb{H}(t,x,p,\bar u_1,u_2)\le\mathbb{H}(t,x,p,\bar u_1,\bar u_2)\le\mathbb{H}(t,x,p,u_1,\bar u_2),\qquad\forall(u_1,u_2)\in U_1\times U_2,\tag{2.7}$$

and consequently, the Isaacs' condition is satisfied:

$$\begin{aligned}H^+(t,x,p)\equiv&\inf_{u_1\in U_1}\sup_{u_2\in U_2}\mathbb{H}(t,x,p,u_1,u_2)=\sup_{u_2\in U_2}\inf_{u_1\in U_1}\mathbb{H}(t,x,p,u_1,u_2)\\ \equiv&\,H^-(t,x,p)=Q_0(t,x,p),\qquad\forall(t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n.\end{aligned}\tag{2.8}$$

Proof. For simplicity of notation, let us suppress (t,x) below. We may write

$$\mathbb{H}(p,u_1,u_2)=\frac{1}{2}\langle\, R_1(u_1-\bar u_1),u_1-\bar u_1\,\rangle+\langle\, S(u_1-\bar u_1),u_2-\bar u_2\,\rangle-\frac{1}{2}\langle\, R_2(u_2-\bar u_2),u_2-\bar u_2\,\rangle+Q_0,$$

with $\bar u_1$, $\bar u_2$, and Q_0 undetermined. Then

$$\begin{aligned}&\langle\, p,A\,\rangle+Q+\langle\, B_1^Tp+\theta_1,u_1\,\rangle+\langle\, B_2^Tp+\theta_2,u_2\,\rangle+\frac{1}{2}\langle\, R_1u_1,u_1\,\rangle+\langle\, Su_1,u_2\,\rangle-\frac{1}{2}\langle\, R_2u_2,u_2\,\rangle\\&=\mathbb{H}(p,u_1,u_2)=\frac{1}{2}\langle\, R_1u_1,u_1\,\rangle+\langle\, Su_1,u_2\,\rangle-\frac{1}{2}\langle\, R_2u_2,u_2\,\rangle-\langle\, R_1\bar u_1,u_1\,\rangle\\&\quad-\langle\, S^T\bar u_2,u_1\,\rangle-\langle\, S\bar u_1,u_2\,\rangle+\langle\, R_2\bar u_2,u_2\,\rangle+\frac{1}{2}\langle\, R_1\bar u_1,\bar u_1\,\rangle+\langle\, S\bar u_1,\bar u_2\,\rangle-\frac{1}{2}\langle\, R_2\bar u_2,\bar u_2\,\rangle+Q_0.\end{aligned}$$

Hence, we must have

$$\begin{cases} B_1^T p + \theta_1 = -R_1\bar u_1 - S^T\bar u_2, \qquad B_2^T p + \theta_2 = -S\bar u_1 + R_2\bar u_2, \\ \langle\, p, A\,\rangle + Q = \frac{1}{2}\langle\, R_1\bar u_1, \bar u_1\,\rangle + \langle\, S\bar u_1, \bar u_2\,\rangle - \frac{1}{2}\langle\, R_2\bar u_2, \bar u_2\,\rangle + Q_0. \end{cases} \tag{2.9}$$

Consequently, from the first two equations in (2.9), we have

$$\begin{pmatrix} R_1 & S^T \\ S & -R_2 \end{pmatrix}\begin{pmatrix} \bar u_1 \\ \bar u_2 \end{pmatrix} = -\begin{pmatrix} B_1^T p + \theta_1 \\ B_2^T p + \theta_2 \end{pmatrix}.$$

Note that

$$\det\begin{pmatrix} R_1 & S^T \\ S & -R_2 \end{pmatrix} = \det\begin{pmatrix} R_1 & 0 \\ 0 & -(R_2 + SR_1^{-1}S^T) \end{pmatrix} = (-1)^{m_2}\det(R_1)\det(R_2 + SR_1^{-1}S^T) \neq 0.$$

Thus, $\begin{pmatrix} R_1 & S^T \\ S & -R_2 \end{pmatrix}$ is invertible, which yields

$$\begin{pmatrix} \bar u_1 \\ \bar u_2 \end{pmatrix} = -\begin{pmatrix} R_1 & S^T \\ S & -R_2 \end{pmatrix}^{-1}\begin{pmatrix} B_1^T p + \theta_1 \\ B_2^T p + \theta_2 \end{pmatrix}.$$

Then from the last equality in (2.9), one has

$$\begin{aligned} Q_0 &= \langle\, p, A\,\rangle + Q - \frac{1}{2}\begin{pmatrix} \bar u_1 \\ \bar u_2 \end{pmatrix}^T\begin{pmatrix} R_1 & S^T \\ S & -R_2 \end{pmatrix}\begin{pmatrix} \bar u_1 \\ \bar u_2 \end{pmatrix} \\ &= \langle\, p, A\,\rangle + Q - \frac{1}{2}\begin{pmatrix} B_1^T p + \theta_1 \\ B_2^T p + \theta_2 \end{pmatrix}^T\begin{pmatrix} R_1 & S^T \\ S & -R_2 \end{pmatrix}^{-1}\begin{pmatrix} B_1^T p + \theta_1 \\ B_2^T p + \theta_2 \end{pmatrix}, \end{aligned}$$

proving (3.5). Now, we see that

$$\begin{aligned} \mathbb{H}(p, \bar u_1, u_2) &= -\frac{1}{2}\langle\, R_2(u_2 - \bar u_2), u_2 - \bar u_2\,\rangle + Q_0 \le Q_0 = \mathbb{H}(p, \bar u_1, \bar u_2) \\ &\le \frac{1}{2}\langle\, R_1(u_1 - \bar u_1), u_1 - \bar u_1\,\rangle + Q_0(t, x, p) = \mathbb{H}(p, u_1, \bar u_2), \end{aligned}$$

which means that $(\bar u_1, \bar u_2)$ is a saddle point of $\mathbb{H}(t, x, p, u_1, u_2)$. Then the Isaacs condition (2.8) follows easily. Finally, since R_1 and R_2 are positive definite, the saddle point must be unique. □

We see that in the current case, $p \mapsto H^{\pm}(t, x, p)$ is quadratic, and is neither convex nor concave in general. As a matter of fact, the Hessian $H^{\pm}_{pp}(t, x, p)$ of $H^{\pm}(t, x, p)$ is given by the following:

$$H^{\pm}_{pp}(t, x, p) = -\frac{1}{2}\begin{pmatrix} B_1(t,x)^T \\ B_2(t,x)^T \end{pmatrix}^T\begin{pmatrix} R_1(t,x) & S(t,x)^T \\ S(t,x) & -R_2(t,x) \end{pmatrix}^{-1}\begin{pmatrix} B_1(t,x)^T \\ B_2(t,x)^T \end{pmatrix}.$$

which is indefinite in general.

We have seen from the above that in order the upper and lower Hamiltonians to be well-defined, the only crucial assumption that we made is the positive definiteness of the matrix-valued maps $R_1(\cdot,\cdot)$ and $R_2(\cdot,\cdot)$. Whereas, in order to study the AQ two-person zero-sum differential games, we need a little stronger hypotheses. For example, in order the state equation to be well-posed, we need the right hand side of the state equation is Lipschitz continuous in the state variable, for any given pair of controls, etc. We will look at the general situation a little later.

3. Upper and lower Hamiltonians

In this section, we will carefully look at the upper and lower Hamiltonians associated with general two-person zero-sum differential games with unbounded controls. First of all, we introduce the following standing assumptions.

(H0) For $i=1,2$, the set $U_i\subseteq\mathbb{R}^{m_i}$ is closed and

$$0\in U_i,\qquad i=1,2. \tag{3.1}$$

The time horizon $T>0$ is fixed.

Note that both U_1 and U_2 could be unbounded and may even be equal to $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively. Condition (3.1) is for convenience. We may make a translation of the control domains and make corresponding changes in the control systems and performance functional to achieve this.

Inspired by the AQ two-person zero-sum differential games, let us now introduce the following assumptions for the involved functions f and g in the state equation (1.1) and the performance functional (1.2). We denote $\langle\, x\,\rangle=\sqrt{1+|x|^2}$.

(H1) Map $f:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}^n$ is continuous and there are constants $\sigma_1,\sigma_2\ge 0$ such that

$$|f(t,x,u_1,u_2)|\le L\big(\langle\, x\,\rangle+|u_1|^{\sigma_1}+|u_2|^{\sigma_2}\big),\qquad \forall(t,x,u_1,u_2)\in[0,T]\times\mathbb{R}^n\times U_1\times U_2. \tag{3.2}$$

(H2) Map $g:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}$ is continuous and there exist constants $L,c,\rho_1,\rho_2>0$ and $\mu\ge 1$ such that

$$c|u_1|^{\rho_1}-L\big(\langle\, x\,\rangle^{\mu}+|u_2|^{\rho_2}\big)\le g(t,x,u_1,u_2)\le L\big(\langle\, x\,\rangle^{\mu}+|u_1|^{\rho_1}\big)-c|u_2|^{\rho_2},\forall(t,x,u_1,u_2)\in[0,T]\times\mathbb{R}^n\times U_1\times U_2. \tag{3.3}$$

Further, we introduce the following compatibility condition which will be crucial below.

(H3) The constants $\sigma_1,\sigma_2,\rho_1,\rho_2$ in (H1)–(H2) satisfy the following:

$$\sigma_i<\rho_i,\qquad i=1,2. \tag{3.4}$$

It is not hard to see that the above (H1)–(H3) includes the AQ two-person zero-sum differential game described in the previous section as a special case. Now, we let

$$\mathbb{H}(t,x,p,u_1,u_2)=\langle\, p,f(t,x,u_1,u_2)\,\rangle+g(t,x,u_1,u_2),\qquad (t,x,u_1,u_2)\in[0,T]\times\mathbb{R}^n\times U_1\times U_2. \tag{3.5}$$

Then the *upper* and *lower Hamiltonians* are defined as follows:

$$\begin{cases} H^+(t,x,p)=\inf_{u_1\in U_1}\sup_{u_2\in U_2}\mathbb{H}(t,x,p,u_1,u_2),\\ H^-(t,x,p)=\sup_{u_2\in U_2}\inf_{u_1\in U_1}\mathbb{H}(t,x,p,u_1,u_2),\end{cases}\qquad (t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n, \tag{3.6}$$

provided the involved infimum and supremum exist. Note that the upper and lower Hamiltonians are nothing to do with the function $h(\cdot)$ (appears as the terminal cost/payoff in (1.2)). The main result of this section is the following.

Proposition 3.1. *Under* (H1)–(H3)*, the upper and lower Hamitonians* $H^{\pm}(\cdot,\cdot,\cdot)$ *are well-defined and continuous. Moreover, there are constants* $C>0$, $\lambda_i,\nu_i\ge 0$, $(i=1,2,\dots,k)$ *such that*

$$-L\langle x\rangle^{\mu}-L\langle x\rangle|p|-C|p|^{\frac{\rho_1}{\rho_1-\sigma_1}}\le H^{\pm}(t,x,p)\le L\langle x\rangle^{\mu}+L\langle x\rangle|p|+C|p|^{\frac{\rho_2}{\rho_2-\sigma_2}},\forall(t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n, \quad (3.7)$$

and

$$|H^{\pm}(t,x,p)-H^{\pm}(t,x,q)|\le C\sum_{i=1}^{k}\langle x\rangle^{\lambda_i}\big(|p|\vee|q|\big)^{\nu_i}|p-q|,\forall(t,x)\in[0,T]\times\mathbb{R}^n,\ p,q\in\mathbb{R}^n. \quad (3.8)$$

To prove the above, we will use the following lemma.

Lemma 3.2. *Let* $0<\sigma<\rho$ *and* $c,N>0$*. Let*

$$\theta(r)=Nr^{\sigma}-cr^{\rho},\qquad r\in[0,\infty).$$

Then

$$\max_{r\in[0,\infty)}\theta(r)=\max_{r\in[0,\bar r]}\theta(r)=\theta(\bar r)=(\rho-\sigma)\left(\frac{\sigma^{\sigma}N^{\rho}}{\rho^{\rho}c^{\sigma}}\right)^{\frac{1}{\rho-\sigma}}, \quad (3.9)$$

with

$$\bar r=\left(\frac{\sigma N}{\rho c}\right)^{\frac{1}{\rho-\sigma}}. \quad (3.10)$$

Proof. From

$$\theta(0)=0,\quad \lim_{r\to\infty}\theta(r)=-\infty,$$

we see that the maximum of $\theta(\cdot)$ on $[0,\infty)$ is achieved at some point $\bar r\in(0,\infty)$. Set

$$0=\theta'(r)=N\sigma r^{\sigma-1}-c\rho r^{\rho-1}.$$

Then

$$r^{\rho-\sigma}=\frac{N\sigma}{c\rho}>0,$$

which implies that the maximum is achieved at $\bar r$ given by (3.10), and

$$\begin{aligned}\max_{r\in[0,\infty)}\theta(r)=\max_{r\in[0,\bar r]}\theta(r)=\theta(\bar r)&=N\left(\frac{N\sigma}{c\rho}\right)^{\frac{\sigma}{\rho-\sigma}}-c\left(\frac{N\sigma}{c\rho}\right)^{\frac{\rho}{\rho-\sigma}}\\&=\left[\left(\frac{\sigma}{c\rho}\right)^{\frac{\sigma}{\rho-\sigma}}-c\left(\frac{\sigma}{c\rho}\right)^{\frac{\rho}{\rho-\sigma}}\right]N^{\frac{\rho}{\rho-\sigma}}=\frac{(\rho-\sigma)\sigma^{\frac{\sigma}{\rho-\sigma}}}{c^{\frac{\sigma}{\rho-\sigma}}\rho^{\frac{\rho}{\rho-\sigma}}}N^{\frac{\rho}{\rho-\sigma}}.\end{aligned}$$

This proves our conclusion. □

Proof of Proposition 3.1. Let us look at $H^+(t,x,p)$ carefully ($H^-(t,x,p)$ can be treated similarly). First, by our assumption, we have

$$\begin{aligned}\mathbb{H}(t,x,p,u_1,u_2)&\le|p|\,|f(t,x,u_1,u_2)|+g(t,x,u_1,u_2)\le L\big(\langle x\rangle+|u_1|^{\sigma_1}+|u_2|^{\sigma_2}\big)|p|+L\big(\langle x\rangle^{\mu}+|u_1|^{\rho_1}\big)-c|u_2|^{\rho_2}\\&=L\big(\langle x\rangle^{\mu}+\langle x\rangle|p|+|p|\,|u_1|^{\sigma_1}+|u_1|^{\rho_1}\big)+L|p|\,|u_2|^{\sigma_2}-c|u_2|^{\rho_2},\end{aligned} \quad (3.11)$$

and

$$\begin{aligned}\mathbb{H}(t,x,p,u_1,u_2)&\ge-|p|\,|f(t,x,u_1,u_2)|+g(t,x,u_1,u_2)\\&\ge-L\big(\langle x\rangle+|u_1|^{\sigma_1}+|u_2|^{\sigma_2}\big)|p|-L\big(\langle x\rangle^{\mu}+|u_2|^{\rho_2}\big)+c|u_1|^{\rho_2}\\&=-L\big(\langle x\rangle^{\mu}+\langle x\rangle|p|+|p|\,|u_2|^{\sigma_2}+|u_2|^{\rho_2}\big)-L|p|\,|u_1|^{\sigma_1}+c|u_1|^{\rho_1}.\end{aligned} \quad (3.12)$$

Noting $\sigma_1 < \rho_1$, from (3.11), we see that for any fixed $(t,x,p,u_1)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times U_1$, the map $u_2\mapsto\mathbb{H}(t,x,p,u_1,u_2)$ is coercive from above. Consequently, since U_2 is closed, for any given $(t,x,p,u_1)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times U_1$, there exists a $\bar u_2\equiv\bar u_2(t,x,p,u_1)\in U_2$ such that

$$\begin{aligned}\mathcal{H}^+(t,x,p,u_1)&\equiv\sup_{u_2\in U_2}\mathbb{H}(t,x,p,u_1,u_2)=\sup_{u_2\in U_2,\,|u_2|\le|\bar u_2|}\mathbb{H}(t,x,p,u_1,u_2)=\mathbb{H}(t,x,p,u_1,\bar u_2)\\&\le L\big(\langle x\rangle^{\mu}+\langle x\rangle|p|+|p|\,|u_1|^{\sigma_1}+|u_1|^{\rho_1}\big)+L|p|\,|\bar u_2|^{\sigma_2}-c|\bar u_2|^{\rho_2}\\&\le L\big(\langle x\rangle^{\mu}+\langle x\rangle|p|+|p|\,|u_1|^{\sigma_1}+|u_1|^{\rho_1}\big)+(\rho_2-\sigma_2)\left(\frac{\sigma_2^{\sigma_2}\big(L|p|\big)^{\rho_2}}{\rho_2^{\rho_2}c^{\sigma_2}}\right)^{\frac{1}{\rho_2-\sigma_2}}\\&\le L\big(\langle x\rangle^{\mu}+\langle x\rangle|p|+|p|\,|u_1|^{\sigma_1}+|u_1|^{\rho_1}\big)+K_2|p|^{\frac{\rho_2}{\rho_2-\sigma_2}},\end{aligned}\tag{3.13}$$

where

$$K_2=(\rho_2-\sigma_2)\left(\frac{\sigma_2^{\sigma_2}L^{\rho_2}}{\rho_2^{\rho_2}c^{\sigma_2}}\right)^{\frac{1}{\rho_2-\sigma_2}}.$$

Here, we have used Lemma 3.2. On the other hand, from (3.12), for any $(t,x,p,u_1)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times U_1$, we have

$$\mathcal{H}^+(t,x,p,u_1)=\sup_{u_2\in U_2}\mathbb{H}(t,x,p,u_1,u_2)\ge\mathbb{H}(t,x,p,u_1,0)\ge-L\big(\langle x\rangle^{\mu}+\langle x\rangle|p|\big)-L|p|\,|u_1|^{\sigma_1}+c|u_1|^{\rho_1}.\tag{3.14}$$

By Young's inequality, we have

$$L|p|\,|u_i|^{\sigma_i}\le\frac{c}{2}|u_i|^{\rho_i}+\bar K_i|p|^{\frac{\rho_i}{\rho_i-\sigma_i}},\qquad i=1,2,$$

for some absolute constants $\bar K_i$ (depending on L,c,ρ_i,σ_i only), which leads to

$$\frac{c}{2}|u_i|^{\rho_i}\le c|u_i|^{\rho_i}-L|p|\,|u_i|^{\sigma_i}+\bar K_i|p|^{\frac{\rho_i}{\rho_i-\sigma_i}},\qquad i=1,2.\tag{3.15}$$

Hence, combining the first inequality in (3.13) and (3.14), we obtain

$$\begin{aligned}\frac{c}{2}|\bar u_2|^{\rho_2}&\le c|\bar u_2|^{\rho_2}-L|p|\,|\bar u_2|^{\sigma_2}+\bar K_2|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}\\&\le L\big(\langle x\rangle^{\mu}+\langle x\rangle|p|+|p|\,|u_1|^{\sigma_1}+|u_1|^{\rho_1}\big)-\mathcal{H}^+(t,x,u_1)+\bar K_2|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}\\&\le 2L\big(\langle x\rangle^{\mu}+\langle x\rangle|p|+|p|\,|u_1|^{\sigma_1}\big)+(L-c)|u_1|^{\rho_1}+\bar K_2|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}\equiv\widehat K_2\big(|x|,|p|,|u_1|\big).\end{aligned}\tag{3.16}$$

The above implies that for any compact set $G\subseteq[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times U_1$, there exists a compact set $\widehat U_2(G)\subseteq U_2$, depending on G, such that

$$\mathcal{H}^+(t,x,p,u_1)=\sup_{u_2\in\widehat U_2(G)}\mathbb{H}(t,x,p,u_1,u_2),\qquad\forall(t,x,p,u_1)\in G.$$

Hence, $\mathcal{H}^+(\cdot,\cdot,\cdot,\cdot)$ is continuous. Next, from (3.14), noting $\sigma_1<\rho_1$, we have that for any fixed $(t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$, the map $u_1\mapsto\mathcal{H}^+(t,x,p,u_1)$ is coercive from below. Therefore, using the continuity of $\mathcal{H}^+(\cdot,\cdot,\cdot,\cdot)$, one can find a $\bar u_1\equiv\bar u_1(t,x,p)$ such that (note (3.15))

$$\begin{aligned}H^+(t,x,p)&=\inf_{u_1\in U_1}\sup_{u_2\in U_2}\mathbb{H}(t,x,p,u_1,u_2)=\inf_{u_1\in U_1}\mathcal{H}^+(t,x,p,u_1)=\mathcal{H}^+(t,x,p,\bar u_1)\\&\ge\inf_{u_1\in U_1}\mathbb{H}(t,x,p,u_1,0)\ge\inf_{u_1\in U_1}\Big\{-L\big(\langle x\rangle^{\mu}+\langle x\rangle|p|\big)-L|p|\,|u_1|^{\sigma_1}+c|u_1|^{\rho_1}\Big\}\\&\ge-L\big(\langle x\rangle^{\mu}+\langle x\rangle|p|\big)-\bar K_1|p|^{\frac{\rho_1}{\rho_1-\sigma_1}}.\end{aligned}\tag{3.17}$$

This means that $H^+(t,x,p)$ is well-defined for all $(t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$, and it is locally bounded from below. Also, from (3.13), we obtain

$$\begin{aligned}H^+(t,x,p)&=\inf_{u_1\in U_1}\sup_{u_2\in U_2}\mathbb{H}(t,x,p,u_1,u_2)\le\sup_{u_2\in U_2}\mathbb{H}(t,x,p,0,u_2)\equiv\mathcal{H}^+(t,x,p,0)\\&\le L\big(\langle\,x\,\rangle^{\mu}+\langle\,x\,\rangle|p|\big)+K_2|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}.\end{aligned}\tag{3.18}$$

This proves (3.7) for $H^+(\cdot,\cdot,\cdot)$.

Next, we want to get the local Lipschitz continuity of the map $p\mapsto H^+(t,x,p)$. To this end, we first let

$$U_1(|x|,|p|)=\Big\{u_1\in U_1\;\Big|\;\frac{c}{2}|u_1|^{\rho_1}\le 2L\big(\langle\,x\,\rangle^{\mu}+\langle\,x\,\rangle|p|\big)+\bar K_1|p|^{\frac{\rho_1}{\rho_1-\sigma_1}}+K_2|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}+1\Big\},\quad\forall x,p\in\mathbb{R}^n,$$

which, for any given $x,p\in\mathbb{R}^n$, is a compact set. Clearly, for any $u_1\in U_1\setminus U_1(|x|,|p|)$, one has (note (3.15))

$$c|u_1|^{\rho_1}-L|p|\,|u_1|^{\sigma_1}\ge\frac{c}{2}|u_1|^{\rho_1}-\bar K_1|p|^{\frac{\rho_1}{\rho_1-\sigma_1}}>2L\big(\langle\,x\,\rangle^{\mu}+\langle\,x\,\rangle|p|\big)+K_2|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}+1.$$

Thus, for such a u_1, by (3.14) and (3.18),

$$\begin{aligned}\mathcal{H}^+(t,x,p,u_1)&\ge -L\big(\langle\,x\,\rangle^{\mu}+\langle\,x\,\rangle|p|\big)-L|p|\,|u_1|^{\sigma_1}+c|u_1|^{\rho_1}\\&>L\big(\langle\,x\,\rangle^{\mu}+\langle\,x\,\rangle|p|\big)+K_2|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}+1\ge H^+(t,x,p)+1=\inf_{u_1\in U_1}\mathcal{H}^+(t,x,p,u_1)+1.\end{aligned}\tag{3.19}$$

Hence,

$$\inf_{u_1\in U_1}\mathcal{H}^+(t,x,p,u_1)=\inf_{u_1\in U_1(|x|,|p|)}\mathcal{H}^+(t,x,p,u_1).\tag{3.20}$$

Now, for any $u_1\in U_1(|x|,|p|)$, by (3.16), we have

$$\tfrac{c}{2}|\bar u_2|^{\rho_2}\le\widehat K_2\big(|x|,|p|,|u_1|\big)\le\widetilde K_2(|x|,|p|),\tag{3.21}$$

for some $\widetilde K_2(|x|,|p|)$. Hence, if we let

$$U_2(|x|,|p|)=\Big\{u_2\in U_2\,\Big|\,\frac{c}{2}\Big|\,u_2|^{\rho_2}\le\widetilde K_2(|x|,|p|)\Big\},$$

which is a compact set (for any given $x,p\in\mathbb{R}^n$), then for any $(t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$,

$$H^+(t,x,p)=\inf_{u_1\in U_1(|x|,|p|)}\sup_{u_2\in U_2(|x|,|p|)}\mathbb{H}(x,p,u_1,u_2).\tag{3.22}$$

This implies that $H^+(\cdot,\cdot,\cdot)$ is continuous. Next, we look at some estimates. By definition, for any $u_1\in U_1(|x|,|p|)$, we have

$$|u_1|^{\rho_1}\le C\Big\{\langle\,x\,\rangle^{\mu}+\langle\,x\,\rangle|p|+|p|^{\frac{\rho_1}{\rho_1-\sigma_1}}+|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}\Big\}.$$

Therefore,

$$\begin{aligned}|u_1|^{\sigma_1}&\le C\Big\{\langle\,x\,\rangle^{\mu}+\langle\,x\,\rangle|p|+|p|^{\frac{\rho_1}{\rho_1-\sigma_1}}+|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}\Big\}^{\frac{\sigma_1}{\rho_1}}\\&\le C\Big\{\langle\,x\,\rangle^{\frac{\sigma_1\mu}{\rho_1}}+\langle\,x\,\rangle^{\frac{\sigma_1}{\rho_1}}|p|^{\frac{\sigma_1}{\rho_1}}+|p|^{\frac{\sigma_1}{\rho_1-\sigma_1}}+|p|^{\frac{\sigma_1\rho_2}{\rho_1(\rho_2-\sigma_2)}}\Big\}.\end{aligned}$$

Also, by (3.16), one has

$$\begin{aligned}|\bar u_2|^{\rho_2}&\le C\Big\{\langle\,x\,\rangle^{\mu}+\langle\,x\,\rangle|p|+|p|\,|u_1|^{\sigma_1}+|u_1|^{\rho_1}+|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}\Big\}\\&\le C\Big\{\langle\,x\,\rangle^{\mu}+\langle\,x\,\rangle|p|+\Big[\langle\,x\,\rangle^{\frac{\sigma_1\mu}{\rho_1}}+\langle\,x\,\rangle^{\frac{\sigma_1}{\rho_1}}|p|^{\frac{\sigma_1}{\rho_1}}+|p|^{\frac{\sigma_1}{\rho_1-\sigma_1}}+|p|^{\frac{\sigma_1\rho_2}{\rho_1(\rho_2-\sigma_2)}}\Big]|p|\\&\quad+\langle\,x\,\rangle^{\mu}+\langle\,x\,\rangle|p|+|p|^{\frac{\rho_1}{\rho_1-\sigma_1}}+|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}\Big\}\\&\le C\Big\{\langle\,x\,\rangle^{\mu}+\langle\,x\,\rangle|p|+\langle\,x\,\rangle^{\frac{\sigma_1\mu}{\rho_1}}|p|+\langle\,x\,\rangle^{\frac{\sigma_1}{\rho_1}}|p|^{\frac{\sigma_1+\rho_1}{\rho_1}}+|p|^{\frac{\rho_1}{\rho_1-\sigma_1}}+|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}+|p|^{\frac{\sigma_1\rho_2}{\rho_1(\rho_2-\sigma_2)}+1}\Big\}.\end{aligned}\tag{3.23}$$

Hence,

$$
\begin{aligned}
|\bar u_2|^{\sigma_2} &\le C\Big\{\langle\,x\,\rangle^{\mu}+\langle\,x\,\rangle|p|+\langle\,x\,\rangle^{\frac{\sigma_1\mu}{\rho_1}}|p|+\langle\,x\,\rangle^{\frac{\sigma_1}{\rho_1}}|p|^{\frac{\sigma_1+\rho_1}{\rho_1}}+|p|^{\frac{\rho_1}{\rho_1-\sigma_1}}+|p|^{\frac{\rho_2}{\rho_2-\sigma_2}}+|p|^{\frac{\sigma_1\rho_2}{\rho_1(\rho_2-\sigma_2)}+1}\Big\}^{\frac{\sigma_2}{\rho_2}}\\
&\le C\Big\{\langle\,x\,\rangle^{\frac{\sigma_2\mu}{\rho_2}}+\langle\,x\,\rangle^{\frac{\sigma_2}{\rho_2}}|p|^{\frac{\sigma_2}{\rho_2}}+\langle\,x\,\rangle^{\frac{\sigma_1\sigma_2\mu}{\rho_1\rho_2}}|p|^{\frac{\sigma_2}{\rho_2}}+\langle\,x\,\rangle^{\frac{\sigma_1\sigma_2}{\rho_1\rho_2}}|p|^{\frac{\sigma_2(\sigma_1+\rho_1)}{\rho_1\rho_2}}\\
&\quad+|p|^{\frac{\sigma_2\rho_1}{\rho_2(\rho_1-\sigma_1)}}+|p|^{\frac{\sigma_2}{\rho_2-\sigma_2}}+|p|^{\frac{\sigma_1\sigma_2}{\rho_1(\rho_2-\sigma_2)}+\frac{\sigma_2}{\rho_2}}\Big\}.
\end{aligned}
$$

Consequently, for any $(t,x)\in[0,T]\times\mathbb{R}^n$, $p,q\in\mathbb{R}^n$ and $u_i\in U_i(|x|,|p|\vee|q|)$ $(i=1,2)$, we have (without loss of generality, let $|q|\le|p|$)

$$
\begin{aligned}
|\mathbb{H}(t,x,p,u_1,u_2)-\mathbb{H}(t,x,q,u_1,u_2)| &\le |p-q|\,|f(t,x,u_1,u_2)|\le L\big(\langle\,x\,\rangle+|u_1|^{\sigma_1}+|u_2|^{\sigma_2}\big)|p-q|\\
&\le C\Big\{\langle\,x\,\rangle+\langle\,x\,\rangle^{\frac{\sigma_1\mu}{\rho_1}}+\langle\,x\,\rangle^{\frac{\sigma_2\mu}{\rho_2}}+\langle\,x\,\rangle^{\frac{\sigma_1}{\rho_1}}|p|^{\frac{\sigma_1}{\rho_1}}+\langle\,x\,\rangle^{\frac{\sigma_2}{\rho_2}}|p|^{\frac{\sigma_2}{\rho_2}}\\
&\quad+|p|^{\frac{\sigma_1}{\rho_1-\sigma_1}}+|p|^{\frac{\sigma_2}{\rho_2-\sigma_2}}+|p|^{\frac{\sigma_1\rho_2}{\rho_1(\rho_2-\sigma_2)}}+|p|^{\frac{\sigma_2\rho_1}{\rho_2(\rho_1-\sigma_1)}}+\langle\,x\,\rangle^{\frac{\sigma_1\sigma_2\mu}{\rho_1\rho_2}}|p|^{\frac{\sigma_2}{\rho_2}}\\
&\quad+\langle\,x\,\rangle^{\frac{\sigma_1\sigma_2}{\rho_1\rho_2}}|p|^{\frac{\sigma_2(\sigma_1+\rho_1)}{\rho_1\rho_2}}+|p|^{\frac{\sigma_1\sigma_2}{\rho_1(\rho_2-\sigma_2)}+\frac{\sigma_2}{\rho_2}}\Big\}|p-q|\\
&\equiv C\sum_{i=1}^{12}\langle\,x\,\rangle^{\lambda_i}\big(|p|\vee|q|\big)^{\nu_i}|p-q|.
\end{aligned}
\tag{3.24}
$$

Due to the fact that the infimum and supremum in (3.22) can be taken on compact sets, we can prove the continuity of $(t,x)\mapsto H^+(t,x,p)$. □

A similar result as above can be proved under some much weaker conditions. In fact, we can relax (H1)–(H2) to the following.

(H1)* Map $f:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}^n$ is continuous and there are constants $\sigma_1,\sigma_2\ge0$ and $\mu_0,\mu_1,\mu_2\in\mathbb{R}$ such that

$$
|f(t,x,u_1,u_2)|\le L\big(\langle\,x\,\rangle^{\mu_0}+\langle\,x\,\rangle^{\mu_1}|u_1|^{\sigma_1}+\langle\,x\,\rangle^{\mu_2}|u_2|^{\sigma_2}\big),\quad\forall(t,x,u_1,u_2)\in[0,T]\times\mathbb{R}^n\times U_1\times U_2.\tag{3.25}
$$

(H2)* Map $g:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}$ is continuous and there exist constants $L,c,\rho_1,\rho_2>0$ and $\bar\mu_0,\bar\mu_1,\bar\mu_2\in\mathbb{R}$ such that

$$
\begin{aligned}
c\,\langle\,x\,\rangle^{\bar\mu_1}|u_1|^{\rho_1}-L\big(\langle\,x\,\rangle^{\bar\mu_0}+\langle\,x\,\rangle^{\bar\mu_2}|u_2|^{\rho_2}\big)\le g(t,x,u_1,u_2)\le L\big(\langle\,x\,\rangle^{\bar\mu_0}+\langle\,x\,\rangle^{\bar\mu_1}|u_1|^{\rho_1}\big)-c\,\langle\,x\,\rangle^{\bar\mu_2}|u_2|^{\rho_2},\\
\forall(t,x,u_1,u_2)\in[0,T]\times\mathbb{R}^n\times U_1\times U_2.
\end{aligned}
\tag{3.26}
$$

The following result can be proved in the same way as Proposition 3.1.

Proposition 3.3 (*)**.** *Under* (H1)*–(H2)* *and* (H3)*, the upper and lower Hamitonians* $H^{\pm}(\cdot,\cdot,\cdot)$ *are well-defined and continuous. Moreover, there are constants* $C>0$*,* $\nu_i\ge0$*, and* $\lambda_i\in\mathbb{R}$ $(i=1,2,\dots,k)$ *such that*

$$
\begin{aligned}
-L\,\langle\,x\,\rangle^{\bar\mu_0}-L\,\langle\,x\,\rangle^{\mu_0}|p|-C\,\langle\,x\,\rangle^{\frac{\mu_1\rho_1-\bar\mu_1\sigma_1}{\rho_1-\sigma_1}}|p|^{\frac{\rho_1}{\rho_1-\sigma_1}}&\le H^{\pm}(t,x,p)\\
&\le L\,\langle\,x\,\rangle^{\bar\mu_0}+L\,\langle\,x\,\rangle^{\mu_0}|p|+C\,\langle\,x\,\rangle^{\frac{\mu_2\rho_2-\bar\mu_2\sigma_2}{\rho_2-\sigma_2}}|p|^{\frac{\rho_2}{\rho_2-\sigma_2}},\\
&\qquad\forall(t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n,
\end{aligned}
\tag{3.27}
$$

and

$$
|H^{\pm}(t,x,p)-H^{\pm}(t,x,q)|\le C\sum_{i=1}^{k}\langle\,x\,\rangle^{\lambda_i}\big(|p|\vee|q|\big)^{\nu_i}|p-q|,\qquad\forall(t,x)\in[0,T]\times\mathbb{R}^n,\ p,q\in\mathbb{R}^n.\tag{3.28}
$$

We point out that different from Proposition 3.1, there are more terms in (3.28) than in (3.8), and the expressions of λ_i and ν_i are a little more complicated. In fact, instead of (3.24) we can prove the following: (for notational simplicity, we let $|q| \le |p|$))

$$
\begin{aligned}
|\mathbb{H}(t,x,p,u_1,u_2) - \mathbb{H}(t,x,q,u_1,u_2)| &\le |p-q|\,|f(t,x,u_1,u_2)| \le L\big(\langle x\rangle^{\mu_0} + \langle x\rangle^{\mu_1}|u_1|^{\sigma_1} + \langle x\rangle^{\mu_2}|u_2|^{\sigma_2}\big)|p-q| \\
&\le C\Big\{\langle x\rangle^{\mu_0} + \langle x\rangle^{\mu_1 - \frac{\sigma_1\bar\mu_1}{\rho_1}} + \langle x\rangle^{\mu_1 + \frac{\sigma_1(\bar\mu_0-\bar\mu_1)}{\rho_1}} + \langle x\rangle^{\mu_1 + \frac{\sigma_1(\mu_0-\bar\mu_1)}{\rho_1}}|p|^{\frac{\sigma_1}{\rho_1}} \\
&\quad + \langle x\rangle^{\mu_1 + \frac{\sigma_1(\mu_1-\bar\mu_1)}{\rho_1-\sigma_1}}|p|^{\frac{\sigma_1}{\rho_1-\sigma_1}} + \langle x\rangle^{\mu_1 + \frac{\sigma_1[\rho_2(\mu_2-\bar\mu_1)-\sigma_2(\bar\mu_2-\bar\mu_1)]}{\rho_1(\rho_2-\sigma_2)}}|p|^{\frac{\sigma_1\rho_2}{\rho_1(\rho_2-\sigma_2)}} \\
&\quad + \langle x\rangle^{\mu_2 - \frac{\sigma_2\bar\mu_2}{\rho_2}} + \langle x\rangle^{\mu_2 + \frac{\sigma_2(\bar\mu_0-\bar\mu_2)}{\rho_2}} + \langle x\rangle^{\mu_2 + \frac{\sigma_2(\mu_0-\bar\mu_2)}{\rho_2}}|p|^{\frac{\sigma_2}{\rho_2}} \\
&\quad + \langle x\rangle^{\mu_2 + \frac{\sigma_2(\mu_1-\bar\mu_2)}{\rho_2} - \frac{\sigma_1\sigma_2\bar\mu_1}{\rho_1\rho_2}}|p|^{\frac{\sigma_2}{\rho_2}} + \langle x\rangle^{\mu_2 + \frac{\sigma_2(\mu_1-\bar\mu_2)}{\rho_2} + \frac{\sigma_1\sigma_2(\bar\mu_0-\bar\mu_1)}{\rho_1\rho_2}}|p|^{\frac{\sigma_2}{\rho_2}} \\
&\quad + \langle x\rangle^{\mu_2 + \frac{\sigma_2(\mu_1-\bar\mu_2)}{\rho_2} + \frac{\sigma_1\sigma_2(\mu_0-\bar\mu_1)}{\rho_1\rho_2}}|p|^{\frac{\sigma_2(\sigma_1+\rho_1)}{\rho_1\rho_2}} \\
&\quad + \langle x\rangle^{\mu_2 + \frac{\sigma_2(\mu_1-\bar\mu_2)}{\rho_2} + \frac{\sigma_1\sigma_2(\mu_1-\bar\mu_1)}{\rho_2(\rho_1-\sigma_1)}}|p|^{\frac{\sigma_2\rho_1}{\rho_2(\rho_1-\sigma_1)}} \\
&\quad + \langle x\rangle^{\mu_2 + \frac{\sigma_2(\mu_1-\bar\mu_2)}{\rho_2} + \frac{\sigma_1\sigma_2[\rho_2(\mu_2-\bar\mu_1)-\sigma_2(\bar\mu_2-\bar\mu_1)]}{\rho_1\rho_2(\rho_2-\sigma_2)}}|p|^{\frac{\sigma_1\sigma_2}{\rho_1(\rho_2-\sigma_2)} + \frac{\sigma_2}{\rho_2}} \\
&\quad + \langle x\rangle^{\mu_2 + \frac{\sigma_2(\mu_1\rho_1-\bar\mu_1\sigma_1)}{\rho_2(\rho_1-\sigma_1)} - \frac{\sigma_2\bar\mu_2}{\rho_2}}|p|^{\frac{\sigma_2\rho_1}{\rho_2(\rho_1-\sigma_1)}} \\
&\quad + \langle x\rangle^{\mu_2 + \frac{\sigma_2(\mu_2\rho_2-\bar\mu_2\sigma_2)}{\rho_2(\rho_2-\sigma_2)} - \frac{\sigma_2\bar\mu_2}{\rho_2}}|p|^{\frac{\sigma_2}{\rho_2-\sigma_2}}\Big\}|p-q| \\
&\equiv C\sum_{i=1}^{16}\langle x\rangle^{\lambda_i}\big(|p|\vee|q|\big)^{\nu_i}|p-q|.
\end{aligned}
$$

Note that (3.24) is a special case of the above with:

$$\mu_0 = 1,\ \bar\mu_0 = \mu,\ \mu_1 = \mu_2 = \bar\mu_1 = \bar\mu_2 = 0.$$

4. Uniqueness of viscosity solution

Consider the following HJ inequalities:

$$
\begin{cases} V_t(t,x) + H(t,x,V_x(t,x)) \ge 0, & (t,x)\in[0,T]\times\mathbb{R}^n, \\ V(T,x) \le h(x), & x\in\mathbb{R}^n, \end{cases} \tag{4.1}
$$

and

$$
\begin{cases} V_t(t,x) + H(t,x,V_x(t,x)) \le 0, & (t,x)\in[0,T]\times\mathbb{R}^n, \\ V(T,x) \ge h(x), & x\in\mathbb{R}^n, \end{cases} \tag{4.2}
$$

as well as the following HJ equation:

$$
\begin{cases} V_t(t,x) + H(t,x,V_x(t,x)) = 0, & (t,x)\in[0,T]\times\mathbb{R}^n, \\ V(T,x) = h(x), & x\in\mathbb{R}^n. \end{cases} \tag{4.3}
$$

We recall the following definition.

Definition 4.1.

(i) A continuous function $V(\cdot,\cdot)$ is called a *viscosity sub-solution* of (4.1) if

$$V(T,x) \le h(x), \qquad \forall x\in\mathbb{R}^n,$$

and for any continuous differentiable function $\varphi(\cdot,\cdot)$, if $(t_0,x_0)\in[0,T)\times\mathbb{R}^n$ is a local maximum of $(t,x)\mapsto V(t,x)-\varphi(t,x)$, then

$$\varphi_t(t_0,x_0)+H(t_0,x_0,\varphi_x(t_0,x_0))\ge 0.$$

(ii) A continuous function $V(\cdot,\cdot)$ is called a *viscosity super-solution* of (4.2) if

$$V(T,x)\ge h(x),\qquad \forall x\in\mathbb{R}^n,$$

and for any continuous differentiable function $\varphi(\cdot,\cdot)$, if $(t_0,x_0)\in[0,T)\times\mathbb{R}^n$ is a local minimum of $(t,x)\mapsto V(t,x)-\varphi(t,x)$, then

$$\varphi_t(t_0,x_0)+H(t_0,x_0,\varphi_x(t_0,x_0))\le 0.$$

(iii) A continuous function $V(\cdot,\cdot)$ is called a *viscosity solution* of (4.3) if it is a viscosity sub-solution of (4.1) and a viscosity super-solution of (4.2).

The following lemma is taken from [6].

Lemma 4.2. *Suppose $H:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is continuous. Let $V(\cdot,\cdot)$ and $\widehat V(\cdot,\cdot)$ be a viscosity sub- and super-solutions of* (4.1) *and* (4.2)*, respectively. Then*

$$W(t,x,y)=V(t,x)-\widehat V(t,y),\qquad (t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$$

is a viscosity sub-solution of the following:

$$\begin{cases} W_t(t,x,y)+H(t,x,W_x(t,x,y))-H(t,y,-W_x(t,x,y))\ge 0, & (t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n,\\ W(T,x,y)\le 0, & (x,y)\in\mathbb{R}^n\times\mathbb{R}^n.\end{cases}$$

Now for HJ equation (4.3), we assume the following.

(HJ) The maps $H:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ and $h:\mathbb{R}^n\to\mathbb{R}$ are continuous and there are constants $K_0>0$, $\mu\ge 1$, and $\lambda_i,\nu_i\ge 0$ $(i=1,2,\dots,k)$ with

$$\lambda_i+(\mu-1)\nu_i\le 1,\qquad 1\le i\le k, \tag{4.4}$$

and a continuous function $\omega:[0,\infty)^3\to[0,\infty)$ with property $\omega(r,s,0)=0$, such that

$$\begin{array}{ll} |H(t,x,p)-H(t,y,p)|\le\omega\big(|x|+|y|,|p|,|x-y|\big), & \forall t\in[0,T],\ x,y,p\in\mathbb{R}^n,\\ |H(t,x,p)-H(t,x,q)|\le K_0\sum_{i=1}^k\langle\, x\,\rangle^{\lambda_i}\big(|p|\vee|q|\big)^{\nu_i}|p-q|, & \forall t\in[0,T],\ x,p,q\in\mathbb{R}^n,\end{array} \tag{4.5}$$

and

$$|h(x)-h(y)|\le K_0\big(\langle\, x\,\rangle\vee\langle\, y\,\rangle\big)^{\mu-1}|x-y|,\qquad \forall x,y\in\mathbb{R}^n. \tag{4.6}$$

Our main result of this section is the following.

Theorem 4.3. *Let* (HJ) *hold. Suppose $V(\cdot,\cdot)$ and $\widehat V(\cdot,\cdot)$ are the viscosity sub- and super-solution of* (4.1) *and* (4.2)*, respectively. Moreover, let*

$$|V(t,x)-V(t,y)|,|\widehat V(t,x)-\widehat V(t,y)|\le K\big(\langle\, x\,\rangle\vee\langle\, y\,\rangle\big)^{\mu-1}|x-y|,\quad \forall t\in[0,T],\ x,y\in\mathbb{R}^n, \tag{4.7}$$

for some $K>0$. Then

$$V(t,x)\le\widehat V(t,x),\qquad \forall (t,x)\in[0,T]\times\mathbb{R}^n. \tag{4.8}$$

A similar result as above was proved in [7], with most technical details omitted. Our conditions are a little different from those assumed in [7]. For readers' convenience, we provide a detailed proof here.

Proof. Suppose $(\bar t,\bar x)\in[0,T)\times\mathbb{R}^n$ such that

$$V(\bar t,\bar x)-\widehat V(\bar t,\bar x)>0.$$

Let $C_0,\beta>0$ be undetermined. Define

$$Q\equiv Q(C_0,\beta)=\Big\{(t,x)\in[0,T]\times\mathbb{R}^n\ \Big|\ \langle\, x\,\rangle\le\langle\,\bar x\,\rangle\,\mathrm{e}^{C_0(t-\bar t)+\beta}\Big\},$$

and

$$G\equiv G(C_0,\beta)=\Big\{(t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\ \Big|\ (t,x),(t,y)\in Q\Big\}.$$

Now, for $\delta>0$ small, define

$$\psi(t,x)\equiv\psi^{C_0,\delta}(t,x)=\left[\frac{\langle\, x\,\rangle}{\langle\,\bar x\,\rangle}\mathrm{e}^{C_0(\bar t-t)}\right]^{\frac{1}{\delta}}\equiv\mathrm{e}^{\frac{1}{\delta}\left[\,\log\frac{\langle\, x\,\rangle}{\langle\,\bar x\,\rangle}+C_0(\bar t-t)\right]}.$$

Then

$$\psi(\bar t,\bar x)=1,\qquad\psi(T,x)=\left[\frac{\langle\, x\,\rangle}{\langle\,\bar x\,\rangle}\mathrm{e}^{C_0(\bar t-T)}\right]^{\frac{1}{\delta}},$$

and

$$\psi_t(t,x)=-\frac{C_0\psi(t,x)}{\delta},\qquad\psi_x(t,x)=\frac{x\psi(t,x)}{\delta\,\langle\, x\,\rangle^{\,2}}.$$

For any $(t,x)\in\bar Q$, we have

$$\langle\, x\,\rangle\le\langle\,\bar x\,\rangle\,\mathrm{e}^{\beta+C_0(t-\bar t)}\le\langle\,\bar x\,\rangle\,\mathrm{e}^{\beta+C_0(T-\bar t)}.$$

Thus, Q is bounded and $\bar G$ is compact. We introduce

$$\Psi(t,x,y)=V(t,x)-\widehat V(t,y)-\frac{|x-y|^2}{\varepsilon}-\sigma\psi(t,x)-\frac{\sigma(T-t)}{T-\bar t},\qquad(t,x,y)\in\bar G,$$

where $\varepsilon>0$ small and

$$0<\sigma\le\frac{V(\bar t,\bar x)-\widehat V(\bar t,\bar x)}{3}.$$

Clearly,

$$\Psi(\bar t,\bar x,\bar x)=V(\bar t,\bar x)-\widehat V(\bar t,\bar x)-\sigma-\sigma\ge3\sigma-2\sigma=\sigma>0.\tag{4.9}$$

Since $\Psi(\cdot,\cdot,\cdot)$ is continuous on the compact set $\bar G$, we may let $(t_0,x_0,y_0)\in\bar G$ be a maximum of $\Psi(\cdot,\cdot,\cdot)$ over $\bar G$. By the optimality of (t_0,x_0,y_0), we have

$$\begin{aligned}V(t_0,x_0)-\widehat V(t_0,x_0)-\sigma\psi(t_0,x_0)-\frac{\sigma(T-t_0)}{T-\bar t}&=\Psi(t_0,x_0,x_0)\le\Psi(t_0,x_0,y_0)\\&=V(t_0,x_0)-\widehat V(t_0,y_0)-\frac{|x_0-y_0|^2}{\varepsilon}-\sigma\psi(t_0,x_0)-\frac{\sigma(T-t_0)}{T-\bar t},\end{aligned}$$

which implies

$$\frac{|x_0-y_0|^2}{\varepsilon}\le\widehat V(t_0,x_0)-\widehat V(t_0,y_0)\le K\big(\langle\, x_0\,\rangle\vee\langle\, y_0\,\rangle\big)^{\mu-1}|x_0-y_0|.$$

Thus,

$$|x_0-y_0|\le K\big(\langle\, x_0\,\rangle\vee\langle\, y_0\,\rangle\big)^{\mu-1}\varepsilon.$$

Now, if $t_0=T$, then

$$\langle\, x_0\,\rangle,\ \langle\, y_0\,\rangle\le\langle\,\bar x\,\rangle\,\mathrm{e}^{\beta+C_0(T-\bar t)}.$$

Hence,

$$\begin{aligned}\Psi(T,x_0,y_0)&=h(x_0)-h(y_0)-\frac{|x_0-y_0|^2}{\varepsilon}-\sigma\psi(T,x_0)\le K_0\big(\langle\, x_0\,\rangle\vee\langle\, y_0\,\rangle\big)^{\mu-1}|x_0-y_0|\\&\le K_0K\big(\langle\, x_0\,\rangle\vee\langle\, y_0\,\rangle\big)^{2(\mu-1)}\varepsilon\le K_0K\,\langle\,\bar x\,\rangle^{\,2(\mu-1)}\mathrm{e}^{2(\mu-1)[\beta+C_0(T-\bar t)]}\varepsilon.\end{aligned}$$

Thus, for $\varepsilon>0$ small enough, the following holds:

$$\Psi(T,x_0,y_0)<\sigma\le\Psi(\bar t,\bar x,\bar x)\le\Psi(t_0,x_0,y_0),$$

which means that $t_0\in[0,T)$. Next, we note that for $(t,x)\in\big(\partial Q\big)\cap\big[(0,T)\times\mathbb{R}^n\big]$, one has

$$\log\frac{\langle\, x\,\rangle}{\langle\,\bar x\,\rangle}+C_0(\bar t-t)=\beta,\qquad\text{and}\qquad 0<t<T,$$

which implies

$$\psi(t,x)=\mathrm{e}^{\frac{\beta}{\delta}}\to\infty,\qquad\delta\to0,\quad\text{uniformly in }(t,x)\in\big(\partial Q\big)\cap\big[(0,T)\times\mathbb{R}^n\big].\tag{4.10}$$

This implies that for $\delta>0$ small (only depending on β),

$$(t_0,x_0,y_0)\in G\cup[\{0\}\times\mathbb{R}^n\times\mathbb{R}^n].$$

By Lemma 4.2, we have

$$\begin{aligned}0&\le\sigma\psi_t(t_0,x_0)-\frac{\sigma}{T-\bar t}+H\left(t_0,x_0,\frac{2(x_0-y_0)}{\varepsilon}+\sigma\psi_x(t_0,x_0)\right)-H\left(t_0,y_0,-\frac{2(y_0-x_0)}{\varepsilon}\right)\\&=\sigma\psi_t(t_0,x_0)-\frac{\sigma}{T-\bar t}+H\left(t_0,x_0,\frac{2(x_0-y_0)}{\varepsilon}+\sigma\psi_x(t_0,x_0)\right)-H\left(t_0,x_0,\frac{2(x_0-y_0)}{\varepsilon}\right)\\&\quad+H\left(t_0,x_0,\frac{2(x_0-y_0)}{\varepsilon}\right)-H\left(t_0,y_0,\frac{2(x_0-y_0)}{\varepsilon}\right)\\&\le\sigma\psi_t(t_0,x_0)-\frac{\sigma}{T-\bar t}+K_0\sum_{i=1}^k\langle\, x_0\,\rangle^{\,\lambda_i}\left(\frac{2|x_0-y_0|}{\varepsilon}+\sigma|\psi_x(t_0,x_0)|\right)^{\nu_i}\sigma|\psi_x(t_0,x_0)|\\&\quad+\omega\left(|x_0|+|y_0|,\frac{2|x_0-y_0|}{\varepsilon},|x_0-y_0|\right)\\&\le-\sigma\frac{C_0}{\delta}\psi(t_0,x_0)-\frac{\sigma}{T-\bar t}+\sigma K_0\sum_{i=1}^k\langle\, x_0\,\rangle^{\,\lambda_i}\left(2K\big(\langle\, x_0\,\rangle\vee\langle\, y_0\,\rangle\big)^{\mu-1}+\frac{\sigma\psi(t_0,x_0)}{\delta\,\langle\, x_0\,\rangle}\right)^{\nu_i}\Bigg]\frac{\psi(t_0,x_0)}{\delta\,\langle\, x_0\,\rangle}\\&\quad+\omega\left(|x_0|+|y_0|,\frac{2|x_0-y_0|}{\varepsilon},|x_0-y_0|\right).\end{aligned}$$

Note that $(t_0,x_0,y_0)\equiv(t_{0,\varepsilon},x_{0,\varepsilon},y_{0,\varepsilon})\in\bar G(C_0,\beta)$ (a fixed compact set). Let $\varepsilon\to0$ along a suitable sequence, we have $|x_{0,\varepsilon}-y_{0,\varepsilon}|\to0$. For notational simplicity, we denote $(t_{0,\varepsilon},x_{0,\varepsilon},y_{0,\varepsilon})\to(t_0,x_0,x_0)$. In the above, by canceling σ, and then send $\varepsilon\to0$ and $\sigma\to0$, one obtains (canceling σ)

$$\begin{aligned}\frac{1}{T-\bar t}&\le-\frac{C_0\psi(t_0,x_0)}{\delta}+K_0\sum_{i=1}^k\langle\, x_0\,\rangle^{\,\lambda_i}\left(2K\langle\, x_0\,\rangle^{\,\mu-1}\right)^{\nu_i}\frac{\psi(t_0,x_0)}{\delta\,\langle\, x_0\,\rangle}\\&=-\Big\{C_0-K_0\sum_{i=1}^k(2K)^{\nu_i}\langle\, x_0\,\rangle^{\,\lambda_i+(\mu-1)\nu_i-1}\Big\}\frac{\psi(t_0,x_0)}{\delta}\\&\le-\Big\{C_0-K_0\sum_{i=1}^k(2K)^{\nu_i}\Big\}\frac{\psi(t_0,x_0)}{\delta}\equiv-\big(C_0-\widetilde K_0\big)\frac{\psi(t_0,x_0)}{\delta}.\end{aligned}$$

Thus, by taking $C_0>\widetilde K_0$, we obtain a contradiction, proving our conclusion. $\square$

We now make some comments on the uniqueness/non-uniqueness of viscosity solutions. First of all, let us look at the following example which is adopted from [4, 5],

Example 4.4. It is known that there are two different bounded strictly increasing continuous differentiable functions $f_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ such that

$$b(x) \equiv f_1'(f_1^{-1}(x)) = f_2'(f_2^{-1}(x)), \qquad x \in \mathbb{R}.$$

Further, if we define

$$X^i(t; x_0) = f_i(t + f_i^{-1}(x_0)), \qquad t \in \mathbb{R}$$

then $X^1(\cdot\,; x_0)$ and $X^2(\cdot\,; x_0)$ are two different solutions to the following initial value problem:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X(t; x_0) = b\big(X(t; x_0)\big), & t \in \mathbb{R}, \\ X(0; x_0) = x_0. \end{cases}$$

By defining

$$V^i(t, x) = h(X^i(T - t; x)), \qquad (t, x) \in [0, T] \times \mathbb{R},$$

we obtain two different viscosity solutions to the following HJ equation:

$$\begin{cases} V_t(t, x) + b(x)V_x(t, x) = 0, & (t, x) \in [0, T] \times \mathbb{R}, \\ V(T, x) = h(x), & x \in \mathbb{R}. \end{cases} \tag{4.11}$$

Therefore, the viscosity solution to the above HJ equation is not unique in the set of continuous functions. However, we note that in the current case,

$$H(t, x, p) = b(x)p, \qquad (t, x, p) \in [0, T] \times \mathbb{R} \times \mathbb{R}.$$

Thus,

$$|H(t, x, p) - H(t, x, q)| \le C|p - q|, \qquad \forall t \in [0, T],\ x, p, q \in \mathbb{R},$$

which means that (4.5) holds with $k = 1$, $\lambda_1 = \nu_1 = 0$. Hence, for any $\mu \ge 1$, as long as (4.6) holds, viscosity solution to (4.11) is unique in the class of continuous functions satisfying (4.7).

Example 4.5. Consider

$$-x^2 - axV_x(x) + |V_x(x)|^2 = 0, \qquad x \in \mathbb{R},$$

with $a \ge 0$. Thus,

$$H(x, p) = -x^2 - axp + p^2, \qquad (x, p) \in \mathbb{R}^2.$$

Then let

$$V(x) = \lambda x^2, \qquad x \in \mathbb{R}.$$

We should have

$$0 = -1 - 2a\lambda + 4\lambda^2.$$

Hence,

$$\lambda = \frac{2a \pm \sqrt{4a^2 + 16}}{8} = \frac{a \pm \sqrt{a^2 + 4}}{4}.$$

Therefore, there are two solutions to the HJ equation:

$$V^{\pm}(x) = \frac{a \pm \sqrt{a^2 + 4}}{4}\, x^2, \qquad x \in \mathbb{R}.$$

Both of these solutions are analytic. Note that

$$|H(x,p)-H(x,q)|\le \big(a|x|+2\big(|p|\vee|q|\big)\big)\,|p-q|,\qquad x,p,q\in\mathbb{R},$$

and

$$|V^{\pm}(x)-V^{\pm}(y)|\le \frac{|a\pm\sqrt{a^2+4}\,|}{4}\big(\langle\, x\,\rangle\vee\langle\, y\,\rangle\big)|x-y|.$$

Thus, in our terminology, $\mu=2$, $k=2$ with

$$\lambda_1=0,\quad \nu_1=1,\quad \lambda_2=0,\quad \nu_2=1.$$

Consequently,

$$\lambda_i+(\mu-1)\nu_i=1,\qquad i=1,2.$$

This means that although (4.4) is satisfied, the corresponding HJ equation has more than one viscosity solution. This example shows that stationary problems are different from evolution problems, as far as the uniqueness of viscosity solution is concerned.

5. Upper and lower value functions

In this section, we are going to define the upper and lower value functions *via* the so-called Elliott–Kalton strategies. Some basic properties of upper and lower value functions will be established carefully.

5.1. State trajectories and Elliott–Kalton strategies

Let us introduce the following hypotheses which are strengthened versions of (H1)–(H3).

(H1)′ Map $f:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}^n$ satisfies (H1). Moreover, for some μ_0,μ_1,μ_2,

$$\begin{aligned}|f(t,x,u_1,u_2)-f(t,y,u_1,u_2)|\le \big[\big(\langle\, x\,\rangle\vee\langle\, y\,\rangle\big)^{\mu_0}+\big(\langle\, x\,\rangle\vee\langle\, y\,\rangle\big)^{\mu_1}|u_1|^{\sigma_1}+\big(\langle\, x\,\rangle\vee\langle\, y\,\rangle\big)^{\mu_2}|u_2|^{\sigma_2}\big]|x-y|,\\ \forall(t,u_1,u_2)\in[0,T]\times U_1\times U_2,\ x,y\in\mathbb{R}^n,\end{aligned}\tag{5.1}$$

and

$$\langle\, f(t,x,u_1,u_2)-f(t,y,u_1,u_2),x-y\,\rangle\le L|x-y|^2,\forall(t,u_1,u_2)\in[0,T]\times U_1\times U_2,\ x,y\in\mathbb{R}^n.\tag{5.2}$$

We note that condition (5.1) implies the local Lipschitz continuity of the map $x\mapsto f(t,x,u_1,u_2)$, with the Lipschtiz constant possibly depending on $|u_1|^{\sigma_1}$ and $|u_2|^{\sigma_2}$. This is the case if we are considering AQ two-person zero-sum differential games (see Sect. 2). On the other hand, condition (5.2) will be used to establish the local Lipschitz continuity of the upper and lower value functions, with the Lipschitz constant being of polynomial order of $\langle\, x\,\rangle\vee\langle\, y\,\rangle$. It is important that the right hand side of (5.2) is independent of (u_1,u_2); Otherwise, the Lipschitz constant of the upper and lower value functions will be some exponential function of $\langle\, x\,\rangle\vee\langle\, y\,\rangle$, for which we do not know if the uniqueness of viscosity solution to the corresponding HJI equation holds. By the way, we point out that (5.2) does not imply the local Lipschitz continuity of the map $x\mapsto f(t,x,u_1,u_2)$. For example, $f(x)=x^{\frac13}$, with $x\in\mathbb{R}$.

(H2)′ Map $g:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}$ satisfies (H2). Moreover,

$$\begin{aligned}|g(t,x,u_1,u_2)-g(t,y,u_1,u_2)|\le\Big[\big(\langle\, x\,\rangle\vee\langle\, y\,\rangle\big)^{\mu-1}+|u_1|^{\frac{\rho_1(\mu-1)}{\mu}}+|u_2|^{\frac{\rho_2(\mu-1)}{\mu}}\Big]\,|x-y|,\\ \forall(t,u_1,u_2)\in[0,T]\times U_1\times U_2,\ x,y\in\mathbb{R}^n.\end{aligned}\tag{5.3}$$

Also, map $h:\mathbb{R}^n\to\mathbb{R}$ is continuous and

$$\begin{cases}|h(x)-h(y)|\le L\big(\langle\, x\,\rangle\vee\langle\, y\,\rangle\big)^{\mu-1}|x-y|, & \forall x,y\in\mathbb{R}^n,\\ |h(0)|\le L.\end{cases}\tag{5.4}$$

Further, the compatibility hypothesis (H3) is now replaced by the following:
(H3)′ The constants $\sigma_1, \sigma_2, \rho_1, \rho_2, \mu$ appear in (H1)′–(H2)′ satisfy the following:

$$\sigma_i \mu < \rho_i, \qquad i = 1, 2. \tag{5.5}$$

Let us first present the following Gronwall type inequality.

Lemma 5.1. *Let* $\theta, \alpha, \beta : [t, T] \to \mathbb{R}_+$ *and* $\theta_0 \geq 0$ *satisfy*

$$\theta(s)^2 \leq \theta_0^2 + \int_t^s \big[\alpha(r)\theta(r)^2 + \beta(r)\theta(r)\big]\mathrm{d}r, \qquad s \in [t, T]. \tag{5.6}$$

Then

$$\theta(s) \leq \mathrm{e}^{\frac{1}{2}\int_t^T \alpha(\tau)d\tau}\theta_0 + \frac{1}{2}\mathrm{e}^{\int_t^T \alpha(\tau)d\tau} \int_t^s \beta(r)\mathrm{d}r, \qquad s \in [t, T]. \tag{5.7}$$

Proof. First, by the usual Gronwall's inequality, we have

$$\theta(s)^2 \leq \mathrm{e}^{\int_t^s \alpha(\tau)d\tau}\theta_0^2 + \int_t^s \mathrm{e}^{\int_r^s \alpha(\tau)d\tau}\beta(r)\theta(r)\mathrm{d}r \leq \mathrm{e}^{\int_t^T \alpha(\tau)d\tau}\theta_0^2 + \mathrm{e}^{\int_t^T \alpha(\tau)d\tau}\int_t^s \beta(r)\theta(r)\mathrm{d}r \equiv \Theta(s).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\sqrt{\Theta(s)} = \frac{1}{2}\Theta(s)^{-\frac{1}{2}}\dot{\Theta}(s) = \frac{1}{2}\Theta(s)^{-\frac{1}{2}}\mathrm{e}^{\int_t^T \alpha(\tau)d\tau}\beta(s)\theta(s) \leq \frac{1}{2}\mathrm{e}^{\int_t^T \alpha(\tau)d\tau}\beta(s).$$

Consequently,

$$\theta(s) \leq \sqrt{\Theta(s)} \leq \mathrm{e}^{\frac{1}{2}\int_t^T \alpha(\tau)d\tau}\theta_0 + \frac{1}{2}\mathrm{e}^{\int_t^T \alpha(\tau)d\tau}\int_t^s \beta(r)\mathrm{d}r, \qquad s \in [t, T],$$

proving our conclusion. □

We now prove the following result concerning the state trajectories.

Proposition 5.2. *Let* (H1)′ *hold. Then, for any* $(t, x) \in [0, T) \times \mathbb{R}^n$, $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1^{\sigma_1}[t, T] \times \mathcal{U}_2^{\sigma_2}[t, T]$, *state equation* (1.1) *admits a unique solution* $y(\cdot) \equiv y(\cdot\,; t, x, u_1(\cdot), u_2(\cdot)) \equiv y_{t,x}(\cdot)$. *Moreover, there exists a constant* $C_0 > 0$ *only depends on* L, T, t *such that*

$$\langle\, y_{t,x}(s)\,\rangle \leq C_0\left\{\langle\, x\,\rangle + \int_t^s \big(|u_1(r)|^{\sigma_1} + |u_2(r)|^{\sigma_2}\big)\mathrm{d}r\right\}, \qquad s \in [t, T], \tag{5.8}$$

$$|y_{t,x}(s) - x| \leq C_0\left\{\langle\, x\,\rangle(s - t) + \int_t^s \big(|u_1(r)|^{\sigma_1} + |u_2(r)|^{\sigma_2}\big)\mathrm{d}r\right\}, \quad s \in [t, T]. \tag{5.9}$$

and for $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^n$ *with* $\bar{t} \in [t, T]$, *and* $y_{\bar{t},\bar{x}}(\cdot) \equiv y(\cdot\,; \bar{t}, \bar{x}, u_1(\cdot), u_2(\cdot))$

$$|y_{t,x}(s) - y_{\bar{t},\bar{x}}(s)| \leq C_0\left\{|x - \bar{x}| + \langle\, x\,\rangle(\bar{t} - t) + \textstyle\int_t^{\bar{t}} \big(|u_1(r)|^{\sigma_1} + |u_2(r)|^{\sigma_2}\big)\mathrm{d}r\right\}, \qquad s \in [t, T]. \tag{5.10}$$

Proof. First, under (H1)′, for any $(t, x) \in [0, T) \times \mathbb{R}^n$, and any $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1^{\sigma_1}[t, T] \times \mathcal{U}_2^{\sigma_2}[t, T]$, the map $y \mapsto f(s, y, u_1(s), u_2(s))$ is locally Lipschitz continuous. Thus, state equation (1.1) admits a unique local solution $y(\cdot) = y(\cdot\,; t, x, u_1(\cdot), u_2(\cdot))$. Next, by (5.2), we have

$$\begin{aligned}\langle\, x, f(t, x, u_1, u_2)\,\rangle &= \langle\, x, f(t, x, u_1, u_2) - f(t, 0, u_1, u_2)\,\rangle + \langle\, x, f(t, 0, u_1, u_2)\,\rangle\\ &\leq L|x|^2 + L|x|\big(1 + |u_1|^{\sigma_1} + |u_2|^{\sigma_2}\big), \qquad \forall (t, x, u_1, u_2) \in [0, T] \times \mathbb{R}^n \times U_1 \times U_2.\end{aligned}$$

Thus,

$$\begin{aligned}\langle\, y(s)\,\rangle^{2} &= \langle\, x\,\rangle^{2} + 2\int_t^s \langle\, y(r), f(r,y(r),u_1(r),u_2(r))\,\rangle\,\mathrm{d}r\\ &\le \langle\, x\,\rangle^{2} + 2\int_t^s L\left(\langle\, y(r)\,\rangle^{2} + \langle\, y(r)\,\rangle\left(1+|u_1(r)|^{\sigma_1}+|u_2(r)|^{\sigma_2}\right)\right)\mathrm{d}r.\end{aligned}$$

Then, it follows from Lemma 5.1 that

$$\langle\, y(s)\,\rangle \le \mathrm{e}^{L(T-t)}\langle\, x\,\rangle + L\mathrm{e}^{2L(T-t)}\int_t^s \left(1+|u_1(r)|^{\sigma_1}+|u_2(r)|^{\sigma_2}\right)\mathrm{d}r.$$

This implies that the solution $y(\cdot)$ of the state equation (1.1) globally exists on $[t,T]$ and (5.8) holds. Also, we have

$$\begin{aligned}|y(s)-x|^2 &= 2\int_t^s \langle\, y(r)-x, f(r,y(r),u_1(r),u_2(r))\,\rangle\,\mathrm{d}r\\ &\le 2\int_t^s \left(L|y(r)-x|^2 + \langle\, y(r)-x, f(r,x,u_1(r),u_2(r))\,\rangle\right)\mathrm{d}r\\ &\le 2L\int_t^s \left(|y(r)-x|^2 + |y(r)-x|\left(\langle\, x\,\rangle + |u_1(r)|^{\sigma_1}+|u_2(r)|^{\sigma_2}\right)\right)\mathrm{d}r.\end{aligned}$$

Thus, by Lemma 5.2 again, we obtain (5.9).

Now, for any $(t,x),(\bar t,\bar x)\in[0,T]\times\mathbb{R}^n$, with $0\le t\le \bar t<T$, denote $y_{t,x}(\cdot)=y(\cdot\,;t,x,u_1(\cdot),u_2(\cdot))$, and $y_{\bar t,\bar x}(\cdot)=y(\cdot\,;\bar t,\bar x,u_1(\cdot),u_2(\cdot))$. Then for $s\in[\bar t,T]$, we have

$$\begin{aligned}|y_{t,x}(s)-y_{\bar t,\bar x}(s)|^2 &= |y_{t,x}(\bar t)-\bar x|^2 + 2\int_{\bar t}^s \langle\, y_{t,x}(r)-y_{\bar t,\bar x}(r), f(r,y_{t,x}(r),u_1(r),u_2(r)) - f(r,y_{\bar t,\bar x}(r),u_1(r),u_2(r))\,\rangle\,\mathrm{d}r\\ &\le |y_{t,x}(\bar t)-x|^2 + 2L\int_{\bar t}^s |y_{t,x}(r)-y_{\bar t,\bar x}(r)|^2\mathrm{d}r.\end{aligned}$$

Thus, it follows from the Gronwall's inequality that

$$\begin{aligned}|y_{t,x}(s)-y_{\bar t,\bar x}(s)| &\le \mathrm{e}^{L(s-\bar t)}|y_{t,x}(\bar t)-x| \le \mathrm{e}^{L(s-\bar t)}\left(|x-\bar x| + |y_{t,x}(\bar t)-x|\right)\\ &\le \mathrm{e}^{L(s-\bar t)}\left\{|x-\bar x| + L\mathrm{e}^{2L(T-t)}\left(\langle\, x\,\rangle(\bar t-t) + \int_t^{\bar t}|u_1(r)|^{\sigma_1}\mathrm{d}r + \int_t^{\bar t}|u_2(r)|^{\sigma_2}\mathrm{d}r\right)\right\}\\ &\le C\left\{|x-\bar x| + \langle\, x\,\rangle(\bar t-t) + \int_t^{\bar t}\left(|u_1(r)|^{\sigma_1}+|u_2(r)|^{\sigma_2}\right)\mathrm{d}r\right\}.\end{aligned}$$

This completes the proof. □

From the above proposition, together with (H2)′, we see that for any $u_i(\cdot)\in\mathcal{U}_i^{\rho_i}[t,T]$ (which is smaller than $\mathcal{U}_i^{\sigma_i}[t,T]$), $i=1,2$, the performance functional $J(t,x;u_1(\cdot),u_2(\cdot))$ is well-defined. Let us now introduce the following definition which is a modification of the notion introduced in [11].

Definition 5.3. A map $\alpha_1:\mathcal{U}_2^1[t,T]\to\mathcal{U}_1^\infty[t,T]$ is called an Elliott–Kalton (E-K, for short) strategy for Player 1 if it is *non-anticipating*, namely, for any $u_2(\cdot),\bar u_2(\cdot)\in\mathcal{U}_2^1[t,T]$, and any $\hat t\in[t,T]$,

$$\alpha_1[u_2(\cdot)](s)=\alpha_1[\bar u_2(\cdot)](s),\qquad \text{a.e. } s\in[t,\hat t],$$

provided

$$u_1(s)=\bar u_1(s),\qquad \text{a.e. } s\in[t,\hat t].$$

The set of all E-K strategies for Player 1 is denoted by $\mathcal{A}_1[t,T]$. An E-K strategy $\alpha_2 : \mathcal{U}_2^1[t,T] \to \mathcal{U}_1^\infty[t,T]$ for Player 2 can be defined similarly. The set of all E-K strategies for Player 2 is denoted by $\mathcal{A}_2[t,T]$.

Note that as far as the state equation is concerned, one could define an E-K strategy α_1 for Player I as a map $\alpha_1 : \mathcal{U}_2^{\sigma_2}[t,T] \to \mathcal{U}_1^{\sigma_1}[t,T]$. Whereas, as far as the performance functional is concerned, one might have to restrictively define $\alpha_1 : \mathcal{U}_2^{\rho_2}[t,T] \to \mathcal{U}_1^{\rho_1}[t,T]$. We note that the numbers $\sigma_1, \sigma_2, \rho_1, \rho_2$ appeared in (H1)$'$–(H2)$'$ might not be the "optimal" ones, in some sense (for example, σ_1 and σ_2 might be larger than necessary, and ρ_1 and ρ_2 could be smaller than they should be, and so on). Our above definition is somehow "universal". The domain $\mathcal{U}_2^1[t,T]$ of α_1 is large enough to cover possible $u_2(\cdot)$ in some larger space than $\mathcal{U}_2^{\sigma_2}[t,T]$, and the co-domain $\mathcal{U}_1^\infty[t,T]$ is large enough so that the integrability of $\alpha_1[u_2(\cdot)]$ is ensured and the supremum will remain the same due to the density of $\mathcal{U}_1^\infty[t,T]$ in $\mathcal{U}_1^{\rho_1}[t,T]$. In what follows, we simply denote

$$\mathcal{U}_i[t,T] = \mathcal{U}_i^\infty[t,T], \qquad i = 1,2.$$

Recall that $0 \in U_i$ $(i = 1,2)$. For later convenience, we hereafter let $u_1^0(\cdot) \in \mathcal{U}_1[t,T]$ and $u_2^0(\cdot) \in \mathcal{U}_2[t,T]$ be defined by

$$u_1^0(s) = 0, \quad u_2^0(s) = 0, \qquad \forall s \in [t,T],$$

and let $\alpha_1^0 \in \mathcal{A}_1[t,T]$ be the E-K strategy that

$$\alpha_1^0[u_2(\cdot)](s) = 0, \qquad \forall s \in [t,T], \quad u_2(\cdot) \in \mathcal{U}_2^1[t,T].$$

We call such an α_1^0 the *zero E-K strategy* for Player 1. Similarly, we define zero E-K strategy $\alpha_2^0 \in \mathcal{A}_2[t,T]$ for Player 2.

Now, we define

$$\begin{cases} V^+(t,x) = \sup\limits_{\alpha_2 \in \mathcal{A}_2[t,T]} \inf\limits_{u_1(\cdot)\in\mathcal{U}_1[t,T]} J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)]), \\ V^-(t,x) = \inf\limits_{\alpha_1 \in \mathcal{A}_1[t,T]} \sup\limits_{u_2(\cdot)\in\mathcal{U}_2[t,T]} J(t,x;\alpha_1[u_2(\cdot)],u_2(\cdot)). \end{cases} \quad (t,x) \in [0,T]\times\mathbb{R}^n, \tag{5.11}$$

which are called *upper* and *lower value functions* of our two-person zero-sum differential game.

5.2. Upper and lower value functions, and principle of optimality

We now introduce the following notations: for $r > 0$,

$$\mathcal{U}_i[t,T;r] = \left\{ u_i \in \mathcal{U}_i[t,T] \;\Big|\; \int_t^T |u_i(s)|^{\rho_i} \mathrm{d}s \le r \right\}, \qquad i = 1,2,$$

and

$$\begin{cases} \mathcal{A}_1[t,T;r] = \Big\{ \alpha_1 : \mathcal{U}_2^1[t,T] \to \mathcal{U}_1[t,T;r] \;\big|\; \alpha_1 \in \mathcal{A}_1[t,T] \Big\}, \\ \mathcal{A}_2[t,T;r] = \Big\{ \alpha_2 : \mathcal{U}_1^1[t,T] \to \mathcal{U}_2[t,T;r] \;\big|\; \alpha_2 \in \mathcal{A}_2[t,T] \Big\}. \end{cases}$$

We point out that although the upper and lower value functions are formally defined in (5.11), there seems to be no guarantee that they are well-defined. The following result states that under suitable conditions, $V^\pm(\cdot,\cdot)$ are indeed well-defined.

Theorem 5.4. *Let* (H1)$'$–(H3)$'$ *hold. Then the upper and lower value functions* $V^\pm(\cdot,\cdot)$ *are well-defined and there exists a constant* $C > 0$ *such that*

$$|V^\pm(t,x)| \le C\langle x\rangle^\mu, \qquad (t,x) \in [0,T]\times\mathbb{R}^n. \tag{5.12}$$

Moreover,

$$\begin{cases} V^+(t,x) = \sup\limits_{\alpha_2 \in \mathcal{A}_2[t,T;N(|x|)]} \inf\limits_{u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|)]} J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)]), \\ V^-(t,x) = \inf\limits_{\alpha_1 \in \mathcal{A}_1[t,T;N(|x|)]} \sup\limits_{u_2(\cdot)\in\mathcal{U}_2[t,T;N(|x|)]} J(t,x;\alpha_1[u_2(\cdot)],u_2(\cdot)), \end{cases} \tag{5.13}$$

where $N(|x|) = C\langle x\rangle^\mu$, *for some constant* $C > 0$.

Proof. First of all, for any $(t,x)\in[0,T]\times\mathbb{R}^n$ and $u_1(\cdot)\in\mathcal{U}_1[t,T]$, by Proposition 5.2, we have

$$\langle\, y(s)\,\rangle \le C_0\left\{\langle\, x\,\rangle+\int_t^s |u_1(r)|^{\sigma_1}\mathrm{d}r\right\}\le C_0\left\{\langle\, x\,\rangle+\|u_1(\cdot)\|^{\sigma_1}_{L^{\sigma_1}(t,T)}\right\}.$$

Then

$$\begin{aligned}J(t,x;u_1(\cdot),0)&=\int_t^T g(s,y(s),u_1(s),0)\mathrm{d}s+h(y(T))\\&\ge\int_t^T\left[c|u_1(s)|^{\rho_1}-L\,\langle\, y(s)\,\rangle^{\mu}\right]\mathrm{d}s-L\,\langle\, y(T)\,\rangle^{\mu}\\&\ge\int_t^T\left[c|u_1(s)|^{\rho_1}-LC_0^{\mu}\Big(\langle\, x\,\rangle+\int_t^s|u_1(r)|^{\sigma_1}\mathrm{d}r\Big)^{\mu}\right]\mathrm{d}s-LC_0^{\mu}\left(\langle\, x\,\rangle+\|u_1(\cdot)\|^{\sigma_1}_{L^{\sigma_1}(t,T)}\right)^{\mu}\\&\ge -C\,\langle\, x\,\rangle^{\mu}-C\|u_1(\cdot)\|^{\sigma_1\mu}_{L^{\sigma_1}(t,T)}+\int_t^T c|u_1(s)|^{\rho_1}\mathrm{d}s.\end{aligned}$$

Since (note $\mu\ge 1$)

$$\|u_1(\cdot)\|^{\sigma_1\mu}_{L^{\sigma_1}(t,T)}=\left(\int_t^T|u_1(r)|^{\sigma_1}\mathrm{d}r\right)^{\mu}\le(T-t)^{\mu-1}\int_t^T|u_1(r)|^{\sigma_1\mu}\mathrm{d}r,$$

we obtain (taking into account $\sigma_1\mu<\rho_1$)

$$J(t,x;u_1(\cdot),0)\ge -C\,\langle\, x\,\rangle^{\mu}+\int_t^T\left[c|u_1(s)|^{\rho_1}-C|u_1(s)|^{\sigma_1\mu}\right]\mathrm{d}s\ge -C\,\langle\, x\,\rangle^{\mu}+\frac{c}{2}\int_t^T|u_1(s)|^{\rho_1}\mathrm{d}s\ge -C\,\langle\, x\,\rangle^{\mu}. \tag{5.14}$$

Consequently,

$$V^+(t,x)=\sup_{\alpha_2\in\mathcal{A}_2[t,T]}\inf_{u_1(\cdot)\in\mathcal{U}_1[t,T]}J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)])\ge\inf_{u_1(\cdot)\in\mathcal{U}_1[t,T]}J(t,x;u_1(\cdot),\alpha_2^0[u_1(\cdot)])\ge -C\,\langle\, x\,\rangle^{\mu}.$$

Likewise, for any $u_2(\cdot)\in\mathcal{U}_2[t,T]$, we have

$$J(t,x;0,u_2(\cdot))=\int_t^T g(s,y(s),0,u_2(s))\mathrm{d}s+h(y(T))\le C\,\langle\, x\,\rangle^{\mu}. \tag{5.15}$$

Thus,

$$V^+(t,x)=\sup_{\alpha_2\in\mathcal{A}_2[0,T]}\inf_{u_1(\cdot)\in\mathcal{U}_1[t,T]}J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)])\le\sup_{\alpha_2\in\mathcal{A}_2[t,T]}J(t,x;u_1^0(\cdot),\alpha_2[u_1^0(\cdot)])\le C\,\langle\, x\,\rangle^{\mu}.$$

Similar results also hold for the lower value function $V^-(\cdot,\cdot)$. Therefore, we obtain that $V^{\pm}(t,x)$ are well-defined for all $(t,x)\in[0,T]\times\mathbb{R}^n$ and (5.12) holds.

Next, for the constant $C>0$ appearing in (5.12), we set

$$N(r)=\frac{4C}{c}\,\langle\, r\,\rangle^{\mu}.$$

Then for any $u_1(\cdot)\in\mathcal{U}_1[t,T]\setminus\mathcal{U}_1[t,T;N(|x|)]$, from (5.14), we see that

$$\begin{aligned}J(t,x;u_1(\cdot),\alpha_2^0[u_1(\cdot)])&\ge -C\,\langle\, x\,\rangle^{\mu}+\frac{c}{2}\int_t^T|u_1(s)|^{\rho_1}\mathrm{d}s> C\,\langle\, x\,\rangle^{\mu}\\&\ge V^+(t,x)=\sup_{\alpha_2\in\mathcal{A}_2[t,T]}\inf_{u_1(\cdot)\in\mathcal{U}_1[t,T]}J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)]).\end{aligned}$$

Thus,

$$V^+(t,x) = \sup_{\alpha_2\in\mathcal{A}_2[t,T]} \inf_{u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|)]} J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)]). \tag{5.16}$$

Consequently, from (5.15), for any $u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|)]$, we have

$$\begin{aligned} -C\langle x\rangle^{\mu} \le V^+(t,x) &\le \sup_{\alpha_2\in\mathcal{A}_2[t,T]} J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)]) \\ &\le C\langle x\rangle^{\mu} + C\int_t^T |u_1(s)|^{\rho_1}\mathrm{d}s - \frac{c}{2}\int_t^T |\alpha_2[u_1(\cdot)](s)|^{\rho_2}\mathrm{d}s \\ &\le C\langle x\rangle^{\mu} + 2C^2\langle x\rangle^{\mu} - \frac{c}{2}\int_t^T |\alpha_2[u_1(\cdot)](s)|^{\rho_2}\mathrm{d}s. \end{aligned}$$

This implies that

$$\frac{c}{2}\int_t^T |\alpha_2[u_1(\cdot)](s)|^{\rho_2}\mathrm{d}s \le \widetilde{C}\langle x\rangle^{\mu}, \qquad \forall u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|)], \tag{5.17}$$

with $\widetilde{C}=2C(C+1)>0$ being another absolute constant. Hence, if we replace the original $N(r)$ by the following:

$$N(r) = \frac{4\widetilde{C}}{c}\langle r\rangle^{\mu},$$

and let

$$\mathcal{A}_2[t,T;r] = \left\{\alpha_2\in\mathcal{A}_2[t,T] \;\Big|\; \int_t^T |\alpha_2[u_1(\cdot)](s)|^{\rho_2}\mathrm{d}s \le N(|x|)\right\},$$

then the first relation in (5.13) holds.

The second relation in (5.13) can be proved similarly. □

Next, we want to establish a modified Bellman's principle of optimality. To this end, for any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $\bar{t}\in(t,T]$, let

$$\mathcal{U}_i[t,\bar{t};r] = \left\{u_i(\cdot)\in\mathcal{U}_i[t,T] \;\Big|\; \int_t^{\bar{t}} |u_i(s)|^{\rho_i}\mathrm{d}s \le r\right\}, \qquad i=1,2,$$

and

$$\begin{cases} \mathcal{A}_1[t,\bar{t};r] = \left\{\alpha_1 : \mathcal{U}_2^1[t,T]\to\mathcal{U}_1[t,\bar{t};r] \;\big|\; \alpha_1\in\mathcal{A}_1[t,T]\right\}, \\ \mathcal{A}_2[t,\bar{t};r] = \left\{\alpha_2 : \mathcal{U}_1^1[t,T]\to\mathcal{U}_2[t,\bar{t};r] \;\big|\; \alpha_2\in\mathcal{A}_2[t,T]\right\}. \end{cases}$$

It is clear that

$$\begin{cases} \mathcal{U}_i[t,T;r]\subseteq\mathcal{U}_i[t,\bar{t};r]\subseteq\mathcal{U}_i[t,T], \\ \mathcal{A}_i[t,T;r]\subseteq\mathcal{A}_i[t,\bar{t};r]\subseteq\mathcal{A}_i[t,T], \end{cases} \qquad i=1,2.$$

Thus, from the proof of Theorem 5.4, we see that for a suitable choice of $N(\cdot)$, say, $N(r)=C(1+r^{\mu})$ for some large $C>0$, the following holds:

$$\begin{cases} V^+(t,x) = \displaystyle\sup_{\alpha_2\in\mathcal{A}_2[t,\bar{t};N(|x|)]} \inf_{u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|)]} J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)]), \\ V^-(t,x) = \displaystyle\inf_{\alpha_1\in\mathcal{A}_1[t,\bar{t};N(|x|)]} \sup_{u_2(\cdot)\in\mathcal{U}_2[t,\bar{t};N(|x|)]} J(t,x;\alpha_1[u_2(\cdot)],u_2(\cdot)). \end{cases} \tag{5.18}$$

We now state the following modified Bellman's principle of optimality.

Theorem 5.5. *Let* (H1)′–(H3)′ *hold. Let* $(t,x)\in[0,T)\times\mathbb{R}^n$ *and* $\bar t\in(t,T]$. *Let* $N:[0,\infty)\to[0,\infty)$ *be a nondecreasing continuous function such that* (5.18) *holds. Then*

$$V^+(t,x)=\sup_{\alpha_2\in\mathcal{A}_2[t,\bar t;N(|x|)]}\ \inf_{u_1(\cdot)\in\mathcal{U}_1[t,\bar t;N(|x|)]}\left\{\int_t^{\bar t}g(s,y(s),u_1(s),\alpha_2[u_1(\cdot)](s))\mathrm{d}s+V^+(\bar t,y(\bar t))\right\},\tag{5.19}$$

and

$$V^-(t,x)=\inf_{\alpha_1\in\mathcal{A}_1[t,\bar t;N(|x|)]}\ \sup_{u_2(\cdot)\in\mathcal{U}_2[t,\bar t;N(|x|)]}\left\{\int_t^{\bar t}g(s,y(s),\alpha_1[u_2(\cdot)](s),u_2(s))\mathrm{d}s+V^-(\bar t,y(\bar t))\right\}.\tag{5.20}$$

We note that if in (5.19) and (5.20), $\mathcal{A}_i[t,\bar t;N(|x|)]$ and $\mathcal{U}_i[t,\bar t;N(|x|)]$ are replaced by $\mathcal{A}_i[t,T]$ and $\mathcal{U}_i[t,T]$, respectively, the result is standard and the proof is routine. However, in the above case, some careful modification is necessary. For readers' convenience, we provide a proof in Appendix A.

We point out that our modified principle of optimality will play an essential role in the next subsection.

5.3. Continuity of upper and lower value functions

In this subsection, we are going to establish the continuity of the upper and lower value functions. Let us state the main results now.

Theorem 5.6. *Let* (H1)′–(H3)′ *hold. Then* $V^\pm(\cdot,\cdot)$ *are continuous. Moreover, there exists a constant* $C>0$ *and a nondecreasing continuous function* $N:[0,\infty)\to[0,\infty)$ *such that the following estimates hold:*

$$|V^\pm(t,x)-V^\pm(t,\bar x)|\le C\big(\langle x\rangle\vee\langle x\rangle\big)^{\mu-1}|x-\bar x|,\qquad t\in[0,T],\ x,\bar x\in\mathbb{R}^n,\tag{5.21}$$

and

$$|V^\pm(t,x)-V^\pm(\bar t,x)|\le N(|x|)|t-\bar t|^{\frac{\rho_1-\sigma_1}{\rho_1}\wedge\frac{\rho_2-\sigma_2}{\rho_2}},\qquad \forall t,\bar t\in[0,T],\ x\in\mathbb{R}^n.\tag{5.22}$$

Proof. We will only prove the conclusions for $V^+(\cdot,\cdot)$. The conclusions for $V^-(\cdot,\cdot)$ can be proved similarly.

First, let $0\le t\le T$, $x,\bar x\in\mathbb{R}^n$, and let $N(r)=C\langle r\rangle^\mu$ for some $C>0$, such that (5.13) holds. Take

$$u_1(\cdot)\in\mathcal{U}_1^{\rho_1}[t,T;N(|x|\vee|\bar x|)],\quad \alpha_2\in\widetilde{\mathcal{A}}_2^{\rho_2}[t,T;N(|x|\vee|\bar x|)].\tag{5.23}$$

Denote $u_2(\cdot)=\alpha_2[u_1(\cdot)]$. Then

$$\int_t^T|u_i(r)|^{\sigma_i}\mathrm{d}r\le C\left(\int_t^T|u_i(r)|^{\rho_i}\mathrm{d}r\right)^{\frac{\sigma_i}{\rho_i}}\le C\big(\langle x\rangle\vee\langle x\rangle\big)^{\frac{\sigma_i\mu}{\rho_i}}\le C\langle x\rangle\vee\langle\bar x\rangle,\qquad i=1,2.$$

Making use of Proposition 5.1, we have

$$|y_{t,x}(s)|,|y_{t,\bar x}(s)|\le C_0\left[\langle x\rangle\vee\langle\bar x\rangle+\int_t^T\big(|u_1(r)|^{\sigma_1}+|u_2(r)|^{\sigma_2}\big)\mathrm{d}r\right]\le C\big(\langle x\rangle\vee\langle\bar x\rangle\big),\qquad s\in[t,T],$$

and

$$|y_{t,x}(s)-y_{t,\bar x}(s)|\le C_0|x-\bar x|,\qquad s\in[t,T].$$

Consequently,

$$\begin{aligned}
&|J(t,x;u_1(\cdot),u_2(\cdot))-J(t,\bar{x};u_1(\cdot),u_2(\cdot))|\\
&\le \int_t^T |g(s,y_{t,x}(s),u_1(s),u_2(s))-g(s,y_{t,\bar{x}}(s),u(s))|\mathrm{d}s+|h(y_{t,x}(T))-h(y_{t,\bar{x}}(T))|\\
&\le \int_t^T L\left(\big(\langle\, y_{t,x}(s)\,\rangle\vee\langle\, y_{t,\bar{x}}(s)\,\rangle\big)^{\mu-1}+|u_1(s)|^{\frac{\rho_1(\mu-1)}{\mu}}+|u_2(s)|^{\frac{\rho_2(\mu-1)}{\mu}}\right)|y_{t,x}(s)-y_{t,\bar{x}}(s)|\mathrm{d}s\\
&\quad + L\left(\langle\, y_{t,x}(T)\,\rangle\vee\langle\, y_{t,\bar{x}}(T)\,\rangle\right)^{\mu-1}|y_{t,x}(T)-y_{t,\bar{x}}(T)|\\
&\le C\left\{\big(\langle\, x\,\rangle\vee\langle\,\bar{x}\,\rangle\big)^{\mu-1}+\left(\int_t^T|u_1(s)|^{\rho_1}\mathrm{d}s\right)^{\frac{\mu-1}{\mu}}+\left(\int_t^T|u_2(s)|^{\rho_2}\mathrm{d}s\right)^{\frac{\mu-1}{\mu}}\right\}|x-\bar{x}|\\
&\le C\big(\langle\, x\,\rangle\vee\langle\,\bar{x}\,\rangle\big)^{\mu-1}|x-\bar{x}|.
\end{aligned}$$

Since the above estimate is uniform in $(u_1(\cdot),\alpha_2)$ satisfying (5.23), we obtain (5.21) for $V^+(\cdot,\cdot)$.

We now prove the continuity in t. From the modified principle of optimality, we see that for any $\varepsilon>0$, there exists an $\alpha_2^\varepsilon\in\mathcal{A}_2[t,\bar{t};N(|x|)]$ such that

$$\begin{aligned}
V^+(t,x)-\varepsilon &\le \inf_{u_1(\cdot)\in\mathcal{U}_1[t,\bar{t};N(|x|)]}\left\{\int_t^{\bar{t}} g(s,y(s),u_1(\cdot),\alpha_2^\varepsilon[u_1(\cdot)](s))\mathrm{d}s+V^+(\bar{t},y(\bar{t}))\right\}\\
&\le \int_t^{\bar{t}} g(s,y(s),0,\alpha_2^\varepsilon[u_1^0(\cdot)](s))\mathrm{d}s+V^+(\bar{t},y(\bar{t}))\\
&\le \int_t^{\bar{t}} L\left(\langle\, y(s)\,\rangle^{\mu}-c|\alpha_2[u_1^0(\cdot)](s))|^{\rho_2}\right)\mathrm{d}s+V^+(\bar{t},x)+|V^+(\bar{t},y(\bar{t}))-V^+(\bar{t},x)|\\
&\le \int_t^{\bar{t}} L\,\langle\, y(s)\,\rangle^{\mu}\mathrm{d}s+V^+(\bar{t},x)+|V^+(\bar{t},y(\bar{t}))-V^+(\bar{t},x)|.
\end{aligned}$$

By Proposition 5.2, we have (denote $u_2^\varepsilon(\cdot)=\alpha_2^\varepsilon[u_1^0(\cdot)]$)

$$\begin{aligned}
|y(\bar{t})-x| &\le C\left[\langle\, x\,\rangle(\bar{t}-t)+\int_t^{\bar{t}}|u_2^\varepsilon(s)|^{\sigma_2}\mathrm{d}s\right]\le C\left[\langle\, x\,\rangle(\bar{t}-t)+\left(\int_t^{\bar{t}}|u_2^\varepsilon(s)|^{\rho_2}\mathrm{d}s\right)^{\frac{\sigma_2}{\rho_2}}(\bar{t}-t)^{\frac{\rho_2-\sigma_2}{\rho_2}}\right]\\
&\le C\left[\langle\, x\,\rangle(\bar{t}-t)+N(|x|)(\bar{t}-t)^{\frac{\rho_2-\sigma_2}{\rho_2}}\right].
\end{aligned}$$

Also,

$$|y(s)|\le C_0\left[\langle\, x\,\rangle+\int_t^{\bar{t}}|u_2^\varepsilon(s)|^{\sigma_2}\mathrm{d}s\right]\le N(|x|),\qquad s\in[t,\bar{t}].$$

Hence, by the proved (5.21), we obtain

$$|V^+(\bar{t},y(\bar{t}))-V^+(\bar{t},x)|\le N(|x|\vee|y(\bar{t}))|y(\bar{t})-x|\le N(|x|)(\bar{t}-t)^{\frac{\rho_2-\sigma_2}{\rho_2}}.$$

Consequently,

$$V^+(t,x)-V^+(\bar{t},x)\le N(|x|)(\bar{t}-t)^{\frac{\rho_2-\sigma_2}{\rho_2}}+\varepsilon,$$

which yields

$$V^+(t,x)-V^+(\bar{t},x)\le N(|x|)(\bar{t}-t)^{\frac{\rho_2-\sigma_2}{\rho_2}}.$$

On the other hand,

$$V^+(t,x) \geq \inf_{u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|)]} \left\{ \int_t^{\bar t} g(s,y(s),u_1(s),0)\mathrm{d}s + V^+(\bar t, y(\bar t)) \right\}.$$

Hence, for any $\varepsilon > 0$, there exists a $u_1^\varepsilon(\cdot) \in \mathcal{U}_1[t,T;N(|x|)]$ such that

$$\begin{aligned} V^+(t,x)+\varepsilon &\geq \int_t^{\bar t} g(s,y(s),u_1^\varepsilon(s),0)\mathrm{d}s + V^+(\bar t,y(\bar t)) \\ &\geq -\int_t^{\bar t} L\,\langle\, y(s)\,\rangle^{\mu}\mathrm{d}s + c\int_t^{\bar t} |u_1^\varepsilon(s)|^{\rho_1}\mathrm{d}s + V^+(\bar t,x) - |V^+(\bar t,y(\bar t)) - V^+(\bar t,x)| \\ &\geq -\int_t^{\bar t} L\,\langle\, y(s)\,\rangle^{\mu}\mathrm{d}s + V^+(\bar t,x) - |V^+(\bar t,y(\bar t)) - V^+(\bar t,x)|. \end{aligned}$$

Now, in the current case, we have

$$\begin{aligned} |y(\bar t)-x| &\leq C\left[\langle\, x\,\rangle(\bar t - t) + \int_t^{\bar t}|u_1^\varepsilon(s)|^{\sigma_1}\mathrm{d}s\right] \leq C\left[\langle\, x\,\rangle(\bar t-t) + \left(\int_t^{\bar t}|u_1^\varepsilon(s)|^{\rho_1}\mathrm{d}s\right)^{\frac{\sigma_1}{\rho_1}}(\bar t-t)^{\frac{\rho_1-\sigma_1}{\rho_1}}\right] \\ &\leq C\left[\langle\, x\,\rangle(\bar t-t) + N(|x|)(\bar t-t)^{\frac{\rho_1-\sigma_1}{\rho_1}}\right]. \end{aligned}$$

Also,

$$|y(s)| \leq C_0\left[\langle\, x\,\rangle + \int_t^{\bar t}|u_1^\varepsilon(s)|^{\sigma_1}\mathrm{d}s\right] \leq N(|x|), \qquad s\in[t,\bar t].$$

Hence, by the proved (5.21), we obtain

$$|V^+(\bar t,y(\bar t)) - V^+(\bar t,x)| \leq N(|x|\vee|y(\bar t)|)|y(\bar t)-x| \leq N(|x|)(\bar t-t)^{\frac{\rho_1-\sigma_1}{\rho_1}}.$$

Consequently,

$$V^+(t,x) - V^+(\bar t,x) \geq -N(|x|)(\bar t-t)^{\frac{\rho_1-\sigma_1}{\rho_1}} - \varepsilon,$$

which yields

$$V^+(t,x) - V^+(\bar t,x) \geq -N(|x|)(\bar t-t)^{\frac{\rho_1-\sigma_1}{\rho_1}}.$$

Hence, we obtain the estimate (5.22) for $V^+(\cdot,\cdot)$. □

5.4. Characterization of the upper and lower value functions

Having the above preparations, we are now at the position to characterize the upper and the lower value functions of our differential game. Recall that in order Theorem 4.3 applies, we need the conditions (4.4)–(4.6) (for the maps $H(\cdot,\cdot,\cdot)$ and $h(\cdot)$ stated in (HJ) hold, and the upper and lower value functions have to be Lipschitz continuous in a particular form (see (4.7)). It is clear that the only thing that we need is the compatibility condition (4.4) for the numbers λ_i,ν_i appeared in (3.24) with the parameter μ appeared in (H2) and (H2)$'$. Let us now look at what we need here. From (3.24) (which is for the upper value function $V^+(\cdot,\cdot)$ only), and the similar set of conditions for lower value function $V^-(\cdot,\cdot)$, we should require:

$$\begin{cases} \frac{\sigma_1\mu}{\rho_1}\leq 1, \quad \frac{\sigma_2\mu}{\rho_2}\leq 1, \quad \frac{(\mu-1)\sigma_1}{\rho_1-\sigma_1}\leq 1, \quad \frac{(\mu-1)\sigma_2}{\rho_2-\sigma_2}\leq 1, \\ \frac{(\mu-1)\sigma_1\rho_2}{\rho_1(\rho_2-\sigma_2)}\leq 1, \quad \frac{(\mu-1)\sigma_2\rho_1}{\rho_2(\rho_1-\sigma_1)}\leq 1, \\ \frac{\sigma_1\sigma_2\mu}{\rho_1\rho_2}+\frac{(\mu-1)\sigma_1}{\rho_1}\leq 1, \qquad \frac{\sigma_1\sigma_2\mu}{\rho_1\rho_2}+\frac{(\mu-1)\sigma_2}{\rho_2}\leq 1, \\ \frac{\sigma_1\sigma_2}{\rho_1\rho_2}+\frac{(\mu-1)\sigma_2(\sigma_1+\rho_1)}{\rho_1\rho_2}\leq 1, \qquad \frac{\sigma_1\sigma_2}{\rho_1\rho_2}+\frac{(\mu-1)\sigma_1(\sigma_2+\rho_2)}{\rho_1\rho_2}\leq 1, \\ \frac{(\mu-1)\sigma_1\sigma_2}{\rho_1(\rho_2-\sigma_2)}+\frac{(\mu-1)\sigma_2}{\rho_2}\leq 1, \qquad \frac{(\mu-1)\sigma_1\sigma_2}{\rho_2(\rho_1-\sigma_1)}+\frac{(\mu-1)\sigma_1}{\rho_1}\leq 1. \end{cases} \tag{5.24}$$

We now have the following proposition.

Proposition 5.7. *Let*

$$\mu\sigma_i \le \rho_i, \qquad i = 1, 2. \tag{5.25}$$

Then all the inequalities in (5.24) *hold.*

Proof. First of all, we have that

$$\frac{\mu\sigma_i}{\rho_i} \le 1 \qquad \Longleftrightarrow \qquad \frac{(\mu-1)\sigma_i}{\rho_i - \sigma_i} \le 1.$$

Thus, under (5.25), the last two inequalities in the first line of (5.24) hold. Next, by the above equivalence and $\mu \ge 1$,

$$\frac{(\mu-1)\sigma_1\rho_2}{\rho_1(\rho_2-\sigma_2)} \le \frac{(\mu-1)\rho_2}{\mu(\rho_2-\sigma_2)} \le 1,$$

and

$$\frac{(\mu-1)\sigma_2\rho_1}{\rho_2(\rho_1-\sigma_1)} \le \frac{(\mu-1)\rho_1}{\mu(\rho_1-\sigma_1)} \le 1.$$

Thus, the inequalities in the second line of (5.24) hold. Now, for the third line, we have

$$\frac{\sigma_1\sigma_2\mu}{\rho_1\rho_2} + \frac{(\mu-1)\sigma_1}{\rho_1} \le \frac{\sigma_1}{\rho_1} + \frac{(\mu-1)\sigma_1}{\rho_1} = \frac{\mu\sigma_1}{\rho_1} \le 1,$$

and

$$\frac{\sigma_1\sigma_2\mu}{\rho_1\rho_2} + \frac{(\mu-1)\sigma_2}{\rho_2} \le \frac{\sigma_2}{\rho_2} + \frac{(\mu-1)\sigma_2}{\rho_2} = \frac{\mu\sigma_2}{\rho_2} \le 1.$$

This shows that the inequalities in the third line of (5.24) hold. We now look at the fourth line. It is seen that

$$\frac{\sigma_1\sigma_2}{\rho_1\rho_2} + \frac{(\mu-1)\sigma_2(\sigma_1+\rho_1)}{\rho_1\rho_2} \le \frac{\sigma_1\sigma_2}{\rho_1\rho_2} + \frac{(\mu-1)\sigma_2(\sigma_1+\mu\sigma_1)}{\rho_1\rho_2} = \frac{\mu^2\sigma_1\sigma_2}{\rho_1\rho_2} \le 1,$$

and

$$\frac{\sigma_1\sigma_2}{\rho_1\rho_2} + \frac{(\mu-1)\sigma_1(\sigma_2+\rho_2)}{\rho_1\rho_2} \le \frac{\sigma_1\sigma_2}{\rho_1\rho_2} + \frac{(\mu-1)\sigma_1(\sigma_2+\mu\sigma_2)}{\rho_1\rho_2} = \frac{\mu^2\sigma_1\sigma_2}{\rho_1\rho_2} \le 1.$$

Finally, for the fifth line, we have (making use of the inequalities in the second line of (5.24))

$$\frac{(\mu-1)\sigma_1\sigma_2}{\rho_1(\rho_2-\sigma_2)} + \frac{(\mu-1)\sigma_2}{\rho_2} = \frac{\sigma_2}{\rho_2}\left[\frac{(\mu-1)\sigma_1\rho_2}{\rho_1(\rho_2-\sigma_2)} + \mu - 1\right] \le \frac{\sigma_2\mu}{\rho_2} \le 1,$$

and

$$\frac{(\mu-1)\sigma_1\sigma_2}{\rho_2(\rho_1-\sigma_1)} + \frac{(\mu-1)\sigma_1}{\rho_1} = \frac{\sigma_1}{\rho_1}\left[\frac{(\mu-1)\sigma_2\rho_1}{\rho_2(\rho_1-\sigma_1)} + \mu - 1\right] \le \frac{\sigma_1\mu}{\rho_1} \le 1.$$

This completes the proof. □

With the above result, we have the following theorem.

Theorem 5.8. *Let* (H1)′–(H3)′ *hold. Then* $V^{\pm}(\cdot,\cdot)$ *are the unique viscosity solution to the upper and lower HJI equations* (1.3), *respectively. Further, if the Isaacs' condition holds:*

$$H^+(t,x,p) = H^-(t,x,p), \qquad \forall (t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n, \tag{5.26}$$

then

$$V^+(t,x) = V^-(t,x), \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n. \tag{5.27}$$

6. Remarks on the existence of viscosity solutions to HJ equations

We have seen that under (H1)–(H3), the upper and lower Hamiltonians can be well-defined and the corresponding upper and lower HJI equations can be well-formulated. Moreover, we have proved the uniqueness of the viscosity solutions to the upper and lower HJI equations within a suitable class of locally Lipschitz continuous functions. On the other hand, we have introduced a little stronger hypotheses (H1)′–(H3)′ to obtain the upper and lower value functions $V^{\pm}(\cdot,\cdot)$ being well-defined so that the corresponding upper and lower HJI equations have viscosity solutions. In another word, weaker conditions ensure the uniqueness of viscosity solutions to the upper and lower HJI equations, and stronger conditions seem to be needed for the existence. There are some general existence results of viscosity solutions for the first order HJ equations in the literature, see [3,7,14,18,23]. A natural question is whether the conditions that we assumed for the existence of viscosity solutions are sharp (or close to be necessary). In this section, we present a simple situation which tells us that our conditions are sharp in some sense.

We consider the following one-dimensional controlled linear system:

$$\begin{cases} \dot y(s) = Ay(s) + B_1u_1(s) + B_2u_2(s), \qquad s\in[t,T], \\ y(t) = x, \end{cases} \tag{6.1}$$

with the performance functional:

$$J(t,x;u_1(\cdot),u_2(\cdot)) = \int_t^T \big[Qy(s)^2 + R_1u_1(s)^2 - R_2u_2(s)^2\big]\,\mathrm{d}s + Gy(T)^2, \tag{6.2}$$

where $A, B_1, B_2, A, R_1, R_2, G \in \mathbb{R}$. We assume that

$$R_1, R_2 > 0. \tag{6.3}$$

Note that in the current case,

$$\sigma_1 = \sigma_2 = 1, \quad \mu = \rho_1 = \rho_2 = 2.$$

Thus,

$$\mu\sigma_i = \rho_i, \qquad i = 1, 2,$$

which violates (5.5). In the current case, we have

$$\begin{aligned} H^{\pm}(t,x,p) = H(t,x,p) &= \inf_{u_1}\sup_{u_2}\,[pf(t,x,u_1,u_2) + g(t,x,u_1,u_2)] \\ &= Apx + Qx^2 + \inf_{u_1}\big[R_1u_1^2 + pB_1u_1\big] - \inf_{u_2}\big[R_2u_2^2 - pB_2u_2\big] \\ &= Apx + Qx^2 + \left(\frac{B_2^2}{4R_2} - \frac{B_1^2}{4R_1}\right)p^2. \end{aligned} \tag{6.4}$$

Consequently, the upper and lower HJI equation have the same form:

$$\begin{cases} V_t(t,x) + AxV_x(t,x) + Qx^2 + \left(\frac{B_2^2}{4R_2} - \frac{B_1^2}{4R_1}\right)V_x(t,x)^2 = 0, \quad (t,x)\in[0,T]\times\mathbb{R}, \\ V(T,x) = Gx^2, \qquad x\in\mathbb{R}. \end{cases} \tag{6.5}$$

If the above HJI equation has a viscosity solution, by the uniqueness, the solution has to be of the following form:

$$V(t,x) = p(t)x^2, \qquad (t,x)\in[0,T]\times\mathbb{R}, \tag{6.6}$$

where $p(\cdot)$ is the solution to the following Riccati equation:

$$\begin{cases} \dot p(t) + 2Ap(t) + Q + \left(\frac{B_2^2}{R_2} - \frac{B_1^2}{R_1}\right)p(t)^2 = 0, \qquad t\in[0,T], \\ p(T) = G. \end{cases} \tag{6.7}$$

In another word, the solvability of (6.5) is equivalent to that of (6.7).

Our claim is that Riccati equation (6.7) is not always solvable for any $T > 0$. To state our result in a relatively neat way, let us rewrite equation (6.7) as follows:

$$\begin{cases} \dot{p} + \alpha p + \beta p^2 + \gamma = 0, \\ p(T) = g, \end{cases} \tag{6.8}$$

with

$$\alpha = 2A, \qquad \beta = \frac{B_2^2}{R_2^2} - \frac{B_1^2}{R_1^2}, \qquad \gamma = Q, \qquad g = G.$$

Note that β could be positive, negative, or zero. We have the following result.

Proposition 6.1. *Riccati equation (6.8) admits a solution on $[0, T]$ for any $T > 0$ if and only if one of the following holds:*

$$\alpha^2 - 4\beta\gamma \geq 0, \quad 2\beta g + \alpha - \sqrt{\alpha^2 - 4\beta\gamma} \leq 0. \tag{6.9}$$

The proof is elementary and straightforward. For reader's convenience, we provide a proof in the appendix.

It is clear that there are a lot of cases for which the Riccati equation is not solvable. For example,

$$\alpha = \beta = \gamma = 1,$$

which violates (6.9). Also, the case

$$\alpha = 0, \quad \beta = -1, \quad \gamma = 1, \quad g = -2,$$

which also violates (6.9). For the above two cases, Riccati equation (6.8) does not have a global solution on $[0, T]$ for some $T > 0$. Correspondingly we have some two-person zero-sum differential game with unbounded controls for which the coercivity condition (5.5) fails and the upper and lower value functions could not be defined on the whole time interval $[0, T]$, or equivalently, the corresponding upper/lower HJI equation have no viscosity solutions on $[0, T]$.

Acknowledgements. The authors would like to thank the referee for informing the authors several important references in the field, especially some most recent papers. Also, some comments made by Professor Y. Hu (of University of Rennes 1, France) on the previous version are really appreciated.

Appendix A

Proof of Theorem 5.4. We only prove (5.19). The other can be proved similarly. Since $N(|x|)$ and $\bar{t}$ are fixed, for notational simplicity, we denote below that

$$\widetilde{\mathcal{U}}_1 = \mathcal{U}_1[t, \bar{t}; N(|x|)], \qquad \widetilde{\mathcal{A}}_2 = \mathcal{A}_2[t, \bar{t}; N(|x|)].$$

Denote the right hand side of (5.19) by $\widehat{V}^+(t, x)$. For any $\varepsilon > 0$, there exists an $\alpha_2^\varepsilon \in \widetilde{\mathcal{A}}_2$ such that

$$\widehat{V}^+(t, x) - \varepsilon < \inf_{u_1(\cdot) \in \widetilde{\mathcal{U}}_1} \left\{ \int_t^{\bar{t}} g(s, y(s), u_1(s), \alpha_2^\varepsilon[u_1(\cdot)](s)) \mathrm{d}s + V^+(\bar{t}, y(\bar{t})) \right\}.$$

By the definition of $V^+(\bar{t}, y(\bar{t}))$, there exists an $\bar{\alpha}_2^\varepsilon \in \mathcal{A}_2[\bar{t}, T]$ such that

$$V^+(\bar{t}, y(\bar{t})) - \varepsilon < \inf_{\bar{u}_1(\cdot) \in \mathcal{U}_1[\bar{t}, T]} J(\bar{t}, y(\bar{t}); \bar{u}_1(\cdot), \bar{\alpha}_2^\varepsilon[\bar{u}_1(\cdot)]).$$

Now, we define an extension $\widehat{\alpha}_2^\varepsilon \in \mathcal{A}_2[t, T]$ of $\alpha_2^\varepsilon \in \mathcal{A}_2[\bar{t}, T]$ as follows: For any $u_1(\cdot) \in \mathcal{U}_1[t, T]$,

$$\widehat{\alpha}_2^\varepsilon[u_1(\cdot)](s) = \begin{cases} \alpha_2^\varepsilon[u_1(\cdot)](s), & s \in [t, \bar{t}), \\ \bar{\alpha}_2^\varepsilon[u_1(\cdot)\big|_{[\bar{t}, T]}](s), & s \in [\bar{t}, T]. \end{cases}$$

Since $\alpha_2^\varepsilon \in \widetilde{\mathcal{A}}_2$, we have

$$\int_t^{\bar t} |\widehat\alpha^\varepsilon[u_1(\cdot)](s)|^{\rho_2}\mathrm{d}s = \int_t^{\bar t} |\alpha_2^\varepsilon[u_1(\cdot)](s)|^{\rho_2}\mathrm{d}s \le N(|x|).$$

This means that $\widehat\alpha_2^\varepsilon \in \widetilde{\mathcal{A}}_2$. Consequently,

$$\begin{aligned}
V^+(t,x) &\ge \inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1} J(t,x;u_1(\cdot),\widehat\alpha_2^\varepsilon[u_1(\cdot)])\\
&= \inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1}\left\{\int_t^{\bar t} g(s,y(s),u_1(s),\alpha_2^\varepsilon[u_1(\cdot)](s))\mathrm{d}s + J(\bar t,y(\bar t);u_1(\cdot)\big|_{[\bar t,T]},\bar\alpha_2^\varepsilon[u_1(\cdot)\big|_{[\bar t,T]})\right\}\\
&\ge \inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1}\left\{\int_t^{\bar t} g(s,y(s),u_1(s),\alpha_2^\varepsilon[u_1(\cdot)](s))\mathrm{d}s + \inf_{\bar u_1(\cdot)\in\mathcal{U}_1[\bar t,T]} J(\bar t,y(\bar t);\bar u_1(\cdot),\bar\alpha_2^\varepsilon[\bar u_1(\cdot))\right\}\\
&\ge \inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1}\left\{\int_t^{\bar t} g(s,y(s),u_1(s),\alpha_2^\varepsilon[u_1(\cdot)](s))\mathrm{d}s + V^+(\bar t,y(\bar t))\right\} - \varepsilon \ge \widehat V^+(t,x) - 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\widehat V^+(t,x) \le V^+(t,x).$$

On the other hand, for any $\varepsilon > 0$, there exists an $\alpha_2^\varepsilon \in \widetilde{\mathcal{A}}_2$ such that

$$V^+(t,x) - \varepsilon < \inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1} J(t,x;u_1(\cdot),\alpha_2^\varepsilon[u_1(\cdot)]).$$

Also, by definition of $\widehat V^+(t,x)$,

$$\widehat V^+(t,x) \ge \inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1}\left\{\int_t^{\bar t} g(s,y(s),u_1(s),\alpha_2^\varepsilon[u_1(\cdot)](s))\mathrm{d}s + V^+(\bar t,y(\bar t))\right\}.$$

Thus, there exists a $u_1^\varepsilon(\cdot) \in \widetilde{\mathcal{U}}_1$ such that

$$\widehat V^+(t,x) + \varepsilon \ge \int_t^{\bar t} g(s,y(s),u_1^\varepsilon(s),\alpha_2^\varepsilon[u_1^\varepsilon(\cdot)](s))\mathrm{d}s + V^+(\bar t,y(\bar t)).$$

Now, for any $\bar u_1(\cdot) \in \mathcal{U}_1[\bar t,T]$, define a particular extension $\widetilde u_1(\cdot) \in \mathcal{U}_1[t,T]$ by the following:

$$\widetilde u_1(s) = \begin{cases} u_1^\varepsilon(s), & s \in [t,\bar t),\\ \bar u_1(s), & s \in [\bar t,T]. \end{cases}$$

Namely, we patch $u_1^\varepsilon(\cdot)$ to $\bar u_1(\cdot)$ on $[t,\bar t)$. Since

$$\int_t^{\bar t} |\widetilde u_1(s)|^{\rho_1}\mathrm{d}s = \int_t^{\bar t} |u_1^\varepsilon(s)|^{\rho_1}\mathrm{d}s \le N(|x|),$$

we see that $\widetilde u_1(\cdot) \in \widetilde{\mathcal{U}}_1$. Next, we define a restriction $\bar\alpha_2^\varepsilon \in \mathcal{A}[\bar t,T]$ of $\alpha_2^\varepsilon \in \widetilde{\mathcal{A}}_2$, as follows:

$$\bar\alpha_2^\varepsilon[\bar u_1(\cdot)] = \alpha_2^\varepsilon[\widetilde u_1(\cdot)].$$

For such an $\bar\alpha_2^\varepsilon$, we have

$$V^+(\bar t,y(\bar t)) \ge \inf_{\bar u_1(\cdot)\in\mathcal{U}_1[\bar t,T]} J(\bar t,y(\bar t),\bar u_1(\cdot),\bar\alpha_2^\varepsilon[\bar u_1(\cdot)]).$$

Hence, there exists a $\bar{u}_1^\varepsilon(\cdot) \in \mathcal{U}_1[\bar{t}, T]$ such that

$$V^+(\bar{t}, y(\bar{t})) + \varepsilon > J(\bar{t}, y(\bar{t}), \bar{u}_1^\varepsilon(\cdot), \bar{\alpha}_2^\varepsilon[\bar{u}_1^\varepsilon(\cdot)]).$$

Then we further let

$$\widetilde{u}_1^\varepsilon(s) = \begin{cases} u_1^\varepsilon(s), & s \in [t, \bar{t}), \\ \bar{u}_1^\varepsilon(s), & s \in [\bar{t}, T]. \end{cases}$$

Again, $\widetilde{u}_1^\varepsilon(\cdot) \in \widetilde{\mathcal{U}}_1$, and therefore,

$$\begin{aligned} \widehat{V}^+(t,x) + \varepsilon &\geq \int_t^{\bar{t}} g(s, y(s), u_1^\varepsilon(s), \alpha_2^\varepsilon[u_1^\varepsilon(\cdot)](s)) \mathrm{d}s + V^+(\bar{t}, y(\bar{t})) \\ &\geq \int_t^{\bar{t}} g(s, y(s), u_1^\varepsilon(s), \alpha_2^\varepsilon[u_1^\varepsilon(\cdot)](s)) \mathrm{d}s + J(\bar{t}, y(\bar{t}), \bar{u}_1^\varepsilon(\cdot), \bar{\alpha}_2^\varepsilon[\bar{u}_1^\varepsilon(\cdot)]) - \varepsilon \\ &= J(t, x; \widetilde{u}_1^\varepsilon(\cdot), \alpha_2^\varepsilon[\widetilde{u}_1^\varepsilon(\cdot)]) - \varepsilon \geq \inf_{u_1(\cdot) \in \widetilde{\mathcal{U}}_1[t,T]} J(t, x; u_1(\cdot), \alpha_2^\varepsilon[u_1(\cdot)]) - \varepsilon \geq V^+(t,x) - 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\widehat{V}^+(t,x) \geq V^+(t,x).$$

This completes the proof. □

Proof of Proposition 6.1. Recall that we are considering the following Riccati equation:

$$\begin{cases} \dot{p} + \alpha p + \beta p^2 + \gamma = 0, \\ p(T) = g. \end{cases}$$

Case 1. $\beta = 0$. The Riccati equation reads

$$\begin{cases} \dot{p} + \alpha p + \gamma = 0, \\ p(T) = g. \end{cases}$$

This is an initial value problem for a linear equation, which admits a unique global solution $p(\cdot)$ on $[0, T]$.

Case 2. $\beta \neq 0$. Then Riccati equation reads

$$\begin{cases} \dot{p} + \beta \left[\left(p + \frac{\alpha}{2\beta} \right)^2 + \frac{4\beta\gamma - \alpha^2}{4\beta^2} \right] = 0, \\ p(T) = g. \end{cases}$$

Let

$$\kappa = \frac{\sqrt{|\alpha^2 - 4\beta\gamma|}}{2|\beta|} \geq 0.$$

There are three subcases.

Subscase 1. $\alpha^2 - 4\beta\gamma = 0$. The Riccati equation becomes

$$\begin{cases} \dot{p} + \beta \left(p + \frac{\alpha}{2\beta} \right)^2 = 0, \\ p(T) = g. \end{cases}$$

Therefore, in the case

$$2\beta g + \alpha = 0,$$

we have that $p(t) \equiv -\frac{\alpha}{2\beta}$ is the (unique) global solution on $[0,T]$. Now, let

$$2\beta g + \alpha \neq 0.$$

Then we have

$$\frac{\mathrm{d}p}{(p+\frac{\alpha}{2\beta})^2} = -\beta \mathrm{d}t,$$

which leads to

$$\frac{1}{p(t)+\frac{\alpha}{2\beta}} = \frac{1}{g+\frac{\alpha}{2\beta}} - \beta(T-t) = \frac{2\beta - \beta(2\beta g+\alpha)(T-t)}{2\beta g+\alpha}.$$

Thus,

$$p(t) = -\frac{\alpha}{2\beta} + \frac{2\beta g+\alpha}{2\beta - \beta(2\beta g+\alpha)(T-t)},$$

which is well-defined on $[0,T]$ if and only if

$$2 - (2\beta g+\alpha)(T-t) \neq 0, \qquad t \in [0,T].$$

This is equivalent to the following:

$$(2\beta g+\alpha)T < 2.$$

The above is true for all $T > 0$ if and only if

$$2\beta g + \alpha \leq 0.$$

Subcase 2. $\alpha^2 - 4\beta\gamma < 0$. The Riccati equation is

$$\dot{p} + \beta\left[\left(p+\frac{\alpha}{2\beta}\right)^2 + \kappa^2\right] = 0.$$

Hence,

$$\frac{\mathrm{d}p}{(p+\frac{\alpha}{2\beta})^2+\kappa^2} = -\beta \mathrm{d}t,$$

which results in

$$\frac{1}{\kappa}\tan^{-1}\left[\frac{1}{\kappa}\left(p(t)+\frac{\alpha}{2\beta}\right)\right] = -\beta t + C.$$

By the terminal condition,

$$C = \beta T + \frac{1}{\kappa}\tan^{-1}\left[\frac{1}{\kappa}\left(g+\frac{\alpha}{2\beta}\right)\right].$$

Consequently,

$$\tan^{-1}\left[\frac{1}{\kappa}\left(p(t)+\frac{\alpha}{2\beta}\right)\right] = \kappa\beta(T-t) + \tan^{-1}\left[\frac{1}{\kappa}\left(g+\frac{\alpha}{2\beta}\right)\right].$$

Then

$$p(t) = \frac{\alpha}{2\beta} + \kappa\tan\left\{\kappa\beta(T-t) + \tan^{-1}\left(\frac{2\beta g+\alpha}{2\kappa\beta}\right)\right\}.$$

The above is well-defined for $t \in [0,T]$ if and only if

$$-\frac{\pi}{2} < \tan^{-1}\frac{2\beta g+\alpha}{2\kappa\beta} + \kappa\beta T < \frac{\pi}{2},$$

which is true for all $T > 0$ if and only if $\beta = 0$.

Subcase 3. $\alpha^2 - 4\beta\gamma > 0$. The Riccati equation becomes

$$\dot{p} + \beta\left[\left(p + \frac{\alpha}{2\beta}\right)^2 - \kappa^2\right] = 0.$$

If

$$(2\beta g + \alpha - 2\kappa\beta)(2\beta g + \alpha + 2\kappa\beta) \equiv 4\beta^2\left(g + \frac{\alpha}{2\beta} - \kappa\right)\left(g + \frac{\alpha}{2\beta} + \kappa\right) = 0, \tag{A.1}$$

then one of the following

$$p(t) \equiv -\frac{\alpha}{2\beta} \pm \kappa, \qquad t \in [0, T],$$

is the unique global solution to the Riccati equation. We now let

$$(2\beta g + \alpha - 2\kappa\beta)(2\beta g + \alpha + 2\kappa\beta) \equiv 4\beta^2\left(g + \frac{\alpha}{2\beta} - \kappa\right)\left(g + \frac{\alpha}{2\beta} + \kappa\right) \neq 0.$$

Then

$$\frac{\mathrm{d}p}{(p + \frac{\alpha}{2\beta})^2 - \kappa^2} = -\beta \mathrm{d}t.$$

Hence,

$$\frac{1}{2\kappa}\ln\left|\frac{p(t) + \frac{\alpha}{2\beta} - \kappa}{p(t) + \frac{\alpha}{2\beta} + \kappa}\right| = -\beta t + \widetilde{C},$$

which implies

$$\frac{p(t) + \frac{\alpha}{2\beta} - \kappa}{p(t) + \frac{\alpha}{2\beta} + \kappa} = C\mathrm{e}^{-2\kappa\beta t},$$

with

$$C = \mathrm{e}^{2\kappa\beta T}\frac{g + \frac{\alpha}{2\beta} - \kappa}{g + \frac{\alpha}{2\beta} + \kappa} = \mathrm{e}^{2\kappa\beta T}\frac{2\beta g + \alpha - 2\kappa\beta}{2\beta g + \alpha + 2\kappa\beta}.$$

Then

$$\frac{p(t) + \frac{\alpha}{2\beta} - \kappa}{p(t) + \frac{\alpha}{2\beta} + \kappa} = \mathrm{e}^{2\kappa\beta(T-t)}\frac{2\beta g + \alpha - 2\kappa\beta}{2\beta g + \alpha + 2\kappa\beta}.$$

Consequently,

$$p(t) + \frac{\alpha}{2\beta} - \kappa = \mathrm{e}^{2\kappa\beta(T-t)}\frac{2\beta g + \alpha - 2\kappa\beta}{2\beta g + \alpha + 2\kappa\beta}\left[p(t) + \frac{\alpha}{2\beta} + \kappa\right].$$

Thus, $p(\cdot)$ globally exists on $[0, T]$ if and only if

$$\mathrm{e}^{2\kappa\beta(T-t)}\frac{2\beta g + \alpha - 2\kappa\beta}{2\beta g + \alpha + 2\kappa\beta} - 1 \neq 0, \qquad \forall t \in [0, T],$$

which is equivalent to

$$\psi(t) \equiv \mathrm{e}^{2\kappa\beta(T-t)}(2\beta g + \alpha - 2\kappa\beta) - (2\beta g + \alpha + 2\kappa\beta) \neq 0, \qquad \forall t \in [0, T].$$

Since $\psi'(t)$ does not change sign on $[0, T]$, the above is equivalent to the following:

$$0 < \psi(0)\psi(T) = \left[\mathrm{e}^{2\kappa\beta T}(2\beta g + \alpha - 2\kappa\beta) - (2\beta g + \alpha + 2\kappa\beta)\right](-4\kappa\beta),$$

which is equivalent to

$$\left[\mathrm{e}^{2\kappa\beta T}(2\beta g + \alpha - 2\kappa\beta) - (2\beta g + \alpha + 2\kappa\beta)\right]\beta < 0.$$

Note when (A.1) holds, the above it true. In the case $\beta > 0$, the above reads

$$e^{2\kappa\beta T}(2\beta g + \alpha - 2\kappa\beta) < 2\beta g + \alpha + 2\kappa\beta,$$

which is true for all $T > 0$ if and only if

$$2\beta g + \alpha - 2\kappa\beta \leq 0. \tag{A.2}$$

Finally, if $\beta < 0$, then

$$\begin{aligned} 0 &< e^{2\kappa\beta T}(2\beta g + \alpha - 2\kappa\beta) - (2\beta g + \alpha + 2\kappa\beta) \\ &= e^{-2\kappa|\beta|T}(-2|\beta|g + \alpha + 2\kappa|\beta|) - (-2|\beta|g + \alpha - 2\kappa|\beta|) \\ &= e^{-2\kappa|\beta|T}\Big[-(2|\beta|g - \alpha - 2\kappa|\beta|) + e^{2\kappa|\beta|T}(2|\beta|g - \alpha + 2\kappa|\beta|)\Big], \end{aligned}$$

which is true for all $T > 0$ if and only if

$$0 \leq 2|\beta|g - \alpha + 2\kappa|\beta| = -(2\beta g + \alpha - 2\kappa|\beta|).$$

Thus,

$$2\beta g + \alpha - 2\kappa|\beta| \leq 0.$$

which has the same form as (A.2). This completes the proof. □

References

[1] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations.* Birkhäuser, Boston (1997).

[2] M. Bardi and F. Da Lio, On the Bellman equation for some unbounded control problems. *NoDEA* **4** (1997) 491–510.

[3] G. Barles, Existence results for first order Hamilton-Jacobi equations. *Ann. Inst. Henri Poincaré* **1** (1984) 325–340.

[4] S. Biton, Nonlinear monotone semigroups and viscosity solutions. *Ann. Inst. Henri Poincaré* **18** (2001) 383–402.

[5] M.G. Crandall and P.L. Lions, Viscosity solutions of Hamilton-Jacobi equations. *Trans. AMS* **277** (1983) 1–42.

[6] M.G. Crandall and P.L. Lions, On existence and uniqueness of solutions of Hamilton-Jacobi equations. *Nonlinear Anal.* **10** (1986) 353–370.

[7] M.G. Crandall and P.L. Lions, Remarks on the existence and uniqueness of unbounded viscosity solutions of Hamilton-Jacobi equations. *Ill. J. Math.* **31** (1987) 665–688.

[8] F. Da Lio, On the Bellman equation for infinite horizon problems with unblounded cost functional. *Appl. Math. Optim.* **41** (2000) 171–197.

[9] F. Da Lio and O. Ley, Uniqueness results for second-order Bellman-Isaacs equations under quadratic growth assumptions and applications. *SIAM J. Control Optim.* **45** (2006) 74–106.

[10] F. Da Lio and O. Ley, Convex Hamilton-Jacobi equations under superlinear growth conditions on data. *Appl. Math. Optim.* **63** (2011) 309–339.

[11] R.J. Elliott and N.J. Kalton, The existence of value in differential games. Amer. Math. Soc., Providence, RI. *Memoirs of AMS* **126** (1972).

[12] L.C. Evans and P.E. Souganidis, Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. *Indiana Univ. Math. J.* **5** (1984) 773–797.

[13] W.H. Fleming and P.E. Souganidis, On the existence of value functions of two-players, zero-sum stochastic differential games. *Indiana Univ. Math. J.* **38** (1989) 293–314.

[14] A. Friedman and P.E. Souganidis, Blow-up solutions of Hamilton-Jacobi equations. *Commun. Partial Differ. Equ.* **11** (1986) 397–443.

[15] M. Garavello and P. Soravia, Optimality principles and uniqueness for Bellman equations of unbounded control problems with discontinuous running cost. *NoDEA* **11** (2004) 271–298.

[16] H. Ishii, Uniqueness of unbounded viscosity solutions of Hamilton-Jacobi equations. *Indiana Univ. Math. J.* **33** (1984) 721–748.

[17] H. Ishii, Representation of solutions of Hamilton-Jacobi equations. *Nonlinear Anal.* **12** (1988) 121–146.
[18] P.L. Lions, *Generalized Solutions of Hamilton-Jacobi equations*. Pitman, London (1982).
[19] P.L. Lions and P.E. Souganidis, Differential games, optimal conrol and directional derivatives of viscosity solutions of Bellman's and Isaacs' equations. *SIAM J. Control Optim.* **23** (1985) 566–583.
[20] W. McEneaney, A uniqueness result for the Isaacs equation corresponding to nonlinear H_∞ control. *Math. Control Signals Syst.* **11** (1998) 303–334.
[21] F. Rampazzo, Differential games with unbounded versus bounded controls. *SIAM J. Control Optim.* **36** (1998) 814–839.
[22] P. Soravia, Equivalence between nonlinear $\mathcal{H}_\infty$ control problems and existence of viscosity solutions of Hamilton-Jacobi-Isaacs equations. *Appl. Math. Optim.* **39** (1999) 17–32.
[23] P.E. Souganidis, Existence of viscosity solution of Hamilton-Jacobi equations. *J. Differ. Equ.* **56** (1985) 345–390.
[24] J. Yong, Zero-sum differential games involving impulse controls. *Appl. Math. Optim.* **29** (1994) 243–261.
[25] Y. You, Syntheses of differential games and pseudo-Riccati equations. *Abstr. Appl. Anal.* **7** (2002) 61–83.