ESAIM: COCV 19 (2013) 486–515
DOI: 10.1051/cocv/2012018

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

Γ-LIMITS OF CONVOLUTION FUNCTIONALS

Luca Lussardi[1] and Annibale Magni[2]

Abstract. We compute the Γ-limit of a sequence of non-local integral functionals depending on a regularization of the gradient term by means of a convolution kernel. In particular, as Γ-limit, we obtain free discontinuity functionals with linear growth and with anisotropic surface energy density.

Mathematics Subject Classification. 49Q20, 49J45, 49M30.

Received December 24, 2011.
Published online January 23, 2013.

1. Introduction

As it is well known, many variational problems which are recently under consideration, arising for instance from image segmentation, signal reconstruction, fracture mechanics and liquid crystals, involve a *free discontinuity set* (according to a terminology introduced in [19]). This means that the variable function u is required to be smooth outside a surface K, depending on u, and both u and K enter the structure of the functional, which takes the form given by

$$\mathcal{F}(u,K)=\int_{\Omega\setminus K}\phi(|\nabla u|)\,\mathrm{d}x+\int_{K\cap\Omega}\theta(|u^+-u^-|,\nu_K)\,\mathrm{d}\mathcal{H}^{n-1},$$

being Ω an open subset of $\mathbb{R}^n$, K is a $(n-1)$-dimensional compact subset of $\mathbb{R}^n$, $|u^+-u^-|$ the jump of u across K, ν_K the normal direction to K, while ϕ and θ given positive functions, whereas $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

The classical weak formulation for such problems can be obtained considering K as the set of the discontinuities of u and thus working in the space of functions with bounded variation. More precisely, the aforementioned weak form of $\mathcal{F}$ takes on $BV(\Omega)$ the general form

$$\mathcal{F}(u)=\int_{\Omega}\phi(|\nabla u|)\,\mathrm{d}x+\int_{S_u}\theta(|u^+-u^-|,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}+c_0|D^cu|(\Omega), \tag{1.1}$$

where $Du=\nabla u\,\mathcal{L}^n+(u^+-u^-)\mathcal{H}^{n-1}+D^cu$ is the decomposition of the measure derivative of u in its absolutely continuous, jump and Cantor part, respectively, S_u denotes the set of discontinuity points of u, and ν_u is a choice of the unit normal at S_u.

Keywords and phrases. Free discontinuities, Γ-convergence, anisotropy.

[1] Dipartimento di Matematica e Fisica "N. Tartaglia", Università Cattolica del Sacro Cuore, via dei Musei 41, 25121 Brescia, Italy. l.lussardi@dmf.unicatt.it

[2] Mathematisches Institut Abt. für Reine Mathematik, Albert-Ludwigs Universität Freiburg, Eckerstrasse 1, 79104 Freiburg im Breisgau, Germany. annibale.magni@math.uni-freiburg.de

Article published by EDP Sciences

The main difficulty in the actual minimization of $\mathcal{F}$ comes from the surface integral

$$\int_{S_u} \theta(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1},$$

which makes it necessary to use suitable approximations guaranteeing the convergence of minimum points and naturally leads to Γ-convergence.

As pointed out in [10], it is not possible to obtain a variational approximation for $\mathcal{F}$ by the typical integral functionals

$$\mathcal{F}_\varepsilon(u) = \int_\Omega f_\varepsilon(\nabla u)\,\mathrm{d}x$$

defined on some Sobolev spaces. Indeed, when considering the lower semicontinuous envelopes of these functionals, we would be lead to a convex limit, which conflicts with the non-convexity of $\mathcal{F}$.

Heuristic arguments suggest that, to get rid of the difficulty, we have to prevent that the effect of *large* gradients is concentrated on *small* regions. Several approximation methods fit this requirements. For instance in [7, 12, 24] the case where the functionals $\mathcal{F}_\varepsilon$ are restricted to finite elements spaces on regular triangulations of size ε is considered. In [1, 2, 23] the implicit constraint on the gradient through the addition of a higher order penalization is investigated. Moreover, it is important to mention the Ambrosio and Tortorelli approximation (see [3, 4]) of the Mumford–Shah functional *via* elliptic functionals.

The study of non-local models, where the effect of a large gradient is spread onto a set of size ε, was first introduced by Braides and Dal Maso in order to approximate the Mumford–Shah functional (see [10] and also [11, 13–16]) by means of the family

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\biggl(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u|^2\,\mathrm{d}y\biggr)\mathrm{d}x, \quad u \in H^1(\Omega), \tag{1.2}$$

where, for instance, $f(t) = t \wedge 1/2$ and $B_\varepsilon(x)$ denotes the ball of centre x and radius ε. A variant of the method proposed in [10] has been used in [22] to deal with the approximation of a functional $\mathcal{F}$ of the form (1.1), with ϕ having linear growth and θ independent on the normal ν_u (see also [20, 21]). More precisely, in [22] the Γ-limit of the family

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\biggl(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u|\,\mathrm{d}y\biggr)\mathrm{d}x, \quad u \in W^{1,1}(\Omega),$$

for a suitable concave function f, is computed.

In [25] (see also [13]) the case of an anisotropic variant of (1.2) has been considered. In particular it is proven that the family

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\bigl(\varepsilon|\nabla u|^p * \rho_\varepsilon\bigr)\mathrm{d}x, \quad u \in H^1(\Omega), \quad p > 1,$$

Γ-converges to an anisotropic version of the Mumford–Shah functional.

In this paper we investigate the Γ-convergence of the family

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f_\varepsilon\bigl(\varepsilon|\nabla u| * \rho_\varepsilon\bigr)\mathrm{d}x, \quad u \in W^{1,1}(\Omega),$$

where the family $(f_\varepsilon)_{\varepsilon>0}$ satisfies some conditions. The main difficulty to overcome is the estimate from below for the lower Γ-limit in terms of the surface part, while the contribution arising from the volume and Cantor parts has been treated along the same line of the argument already exploited in [25]. The estimate from above has been achieved by density and relaxation arguments. We prove that the Γ-limit, in the strong L^1-topology, is given by

$$\mathcal{F}(u) = \int_\Omega \phi(|\nabla u|)\,\mathrm{d}x + \int_{S_u} \theta(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + c_0|D^c u|(\Omega),$$

where $\phi(t) \sim \frac{1}{\varepsilon} f_\varepsilon(\varepsilon t)$, as $\varepsilon \to 0^+$, is a convex and non-decreasing function with $\phi(0) = 0$ and with $\phi(t)/t \to c_0 > 0$ as $t \to +\infty$; moreover,

$$\theta(s,\nu) = \inf\left\{\liminf_{j\to+\infty} \frac{1}{\varepsilon_j}\int_{Q_\nu} f(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x : (u_j) \in W_\nu^{0,s},\ \varepsilon_j \to 0^+\right\},$$

being f the uniform limit, on compact subsets of $[0,+\infty)$, of f_ε, $W_\nu^{a,b}$ the space of all sequences on the cylinder

$$Q_\nu = \{x \in \mathbb{R}^n : |\langle x,\nu\rangle| \le 1, \text{ the orthogonal projection of } x \text{ onto } \nu^\perp \text{ belongs to the unit ball}\},$$

which converge, shrinking onto the interface, to the function that jumps from a to b around the origin (see Sect. 3.1 for details).

In Section 7 we have been able to show that the method used in [22] to write θ in a more explicit form works only if $n = 1$. In the case $n > 1$ such an argument does not work. Let us briefly discuss the reason. Without loss of generality we can suppose $\nu = \mathbf{e}_1$. Let $P_C^\perp$ be the orthogonal projection of C onto $\{x_1 = 0\}$. Denote by X the space of all functions $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}\times P_C^\perp)$ which are non-decreasing in the first variable and such that there exist $\xi_0 < \xi_1$ with $v(x) = 0$ if $x_1 < \xi_0$ and $v(x) = s$ if $x_1 > \xi_1$. Then, exploiting the same argument as in [22], we have $\theta(s,\mathbf{e}_1) \ge \inf_X G$, where

$$G(v) = \int_{-\infty}^{+\infty} f\left(\int_{C(s\mathbf{e}_1)} \partial_1 v(z)\rho(z - t\mathbf{e}_1)\,\mathrm{d}z\right)\mathrm{d}t.$$

The estimate $\theta(s,\mathbf{e}_1) \ge \inf_X G$ turns out to be optimal if $\inf_X G = \inf_Y G$, where Y is the space of all functions $v \in X$ such that v depends only on the first variable. This is due to the fact that proving the inequality $\theta(s,\mathbf{e}_1) \ge \inf_X G$ we lose control on all the derivatives $\partial_i v$ for any $i = 2,\dots,n$. In the case $C = B_1$ and $\rho = \frac{1}{\omega_n}\chi_{B_1}$, treated in [22], one is able to prove that $\inf_X G = \inf_Y G$ computing directly $\inf_X G$ by a discretization argument (see Prop. 5.7 in [22]). In general, $\inf_X G = \inf_Y G$ does not hold. Indeed proceeding at first as in the proof of Proposition 5.6 in [22], one is able to show that for any $C \subset \mathbb{R}^2$ open, bounded, convex and symmetrical set (*i.e.* $C = -C$) and for $\rho = \frac{1}{|C|}\chi_C$, it holds

$$\inf_Y G = \int_{-h_1}^{h_1} f\left(\frac{s}{|C|}\mathcal{H}^1(C \cap \{z_1 = t\}\right)\mathrm{d}t. \tag{1.3}$$

Now if C is the parallelogram $C = \{(x,y) \in \mathbb{R}^2 : -2 \le y \le 2,\ x - 1 \le y \le x + 1\}$ applying (1.3), we get

$$\inf_Y G = 2f\left(\frac{2s}{|C|}\right) + 2\int_0^2 f\left(\frac{sr}{|C|}\right)\mathrm{d}r.$$

If we compute G on the function w given by

$$w(x,y) = \begin{cases} 0 \text{ if } y > x - 1 \\ s \text{ if } y \le x - 1, \end{cases}$$

(to do this we notice that the functional G makes sense also on $BV_{\mathrm{loc}}(\mathbb{R}\times(-2,2))$ writing $D_1 v$ instead of $\partial_1 v\,\mathrm{d}z$) we obtain

$$G(w) = 2f\left(\frac{4s}{|C|}\right).$$

If f is strictly concave then

$$G(w) < 2f\left(\frac{2s}{|C|}\right) + 2f\left(\frac{2s}{|C|}\right) < 2f\left(\frac{2s}{|C|}\right) + 2\int_0^2 f\left(\frac{sr}{|C|}\right)\mathrm{d}r = \inf_Y G.$$

By a density argument we deduce that $\inf_X G < \inf_Y G$.

As a conclusion, it seems that for a generic anisotropic convolution kernel ρ_ε the expression for θ can not be further simplified when $n > 1$.

2. Notation and preliminaries

We will denote by $L^p(\Omega)$ and by $W^{k,p}(\Omega)$, for $k \in \mathbb{N}$, $k \geq 1$, and for $1 \leq p \leq +\infty$, respectively the classical Lebesgue and Sobolev spaces on Ω. The Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$ will be denoted by $|A|$, whereas the Hausdorff measure of A of dimension $m < n$ will be denoted by $\mathcal{H}^m(A)$. The ball centered in x with radius r will be denoted by $B_r(x)$, while B_r stands for $B_r(0)$; moreover, we will use the notation $\mathbb{S}^{n-1}$ for the boundary of B_1 in $\mathbb{R}^n$. The volume of the unit ball in $\mathbb{R}^n$ will be denoted by ω_n, with the convention $\omega_0 = 1$. Finally $\mathcal{A}(\Omega)$ denotes the set of all open subsets of Ω.

2.1. Functions of bounded variation

For a thorough treatment of BV functions we refer the reader to [5]. Let Ω be an open subset of $\mathbb{R}^n$. We recall that the space $BV(\Omega)$ of real *functions of bounded variation* is the space of the functions $u \in L^1(\Omega)$ whose distributional derivative is representable by a measure in Ω, *i.e.*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, \mathrm{d}x = -\int_\Omega \varphi \,\mathrm{d}D_i u, \quad \forall \varphi \in C_c^\infty(\Omega),\ \forall i = 1, \ldots, n,$$

for some $\mathbb{R}^n$-valued measure $Du = (D_1 u, \ldots, D_n u)$ on Ω. We say that u has *approximate limit* at $x \in \Omega$ if there exists $z \in \mathbb{R}$ such that

$$\lim_{r \to 0^+} \fint_{B_r(x)} |u(y) - z|\, \mathrm{d}y = 0.$$

The set S_u where this property fails is called *approximate discontinuity set* of u. The vector z is uniquely determined for any point $x \in \Omega \backslash S_u$ and is called the *approximate limit* of u at x and denoted by $\tilde{u}(x)$. We say that x is an *approximate jump point* of the function $u \in BV(\Omega)$ if there exist $a, b \in \mathbb{R}$ and $\nu \in \mathbb{S}^{n-1}$ such that $a \neq b$ and

$$\lim_{r \to 0^+} \fint_{B_r^+(x,\nu)} |u(y) - a|\, \mathrm{d}y = 0, \quad \lim_{r \to 0^+} \fint_{B_r^-(x,\nu)} |u(y) - b|\, \mathrm{d}y = 0, \tag{2.1}$$

where $B_r^+(x,\nu) = \{y \in B_r(x) :\ \langle y - x, \nu \rangle > 0\}$ and $B_r^-(x,\nu) = \{y \in B_r(x) :\ \langle y - x, \nu \rangle < 0\}$. The set of approximate jump points of u is denoted by J_u. The triplet (a, b, ν), which turns out to be uniquely determined up to a permutation of a and b and a change of sign of ν, is usually denoted by $(u^+(x), u^-(x), \nu_u(x))$. On $\Omega \setminus S_u$ we set $u^+ = u^- = \tilde{u}$. It turns out that for any $u \in BV(\Omega)$ the set S_u is countably $(n-1)$-rectifiable and $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$. Moreover,

$$Du \llcorner J_u = (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner J_u$$

and $\nu_u(x)$ gives the approximate normal direction to S_u for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$.

For a function $u \in BV(\Omega)$ let $Du = D^a u + D^s u$ be the Lebesgue decomposition of Du into absolutely continuous and singular part. We denote by ∇u the density of $D^a u$; the measures $D^j u := D^s u \llcorner J_u$ and $D^c u := D^s u \llcorner (\Omega \setminus S_u)$ are called the *jump part* and the *Cantor part* of the derivative, respectively. It holds $Du = \nabla u \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner J_u + D^c u$. Let us recall the following important compactness theorem in BV (see Thm. 3.23 and Prop. 3.21 in [5]):

Theorem 2.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Every sequence (u_h) in $BV(\Omega)$ which is bounded in $BV(\Omega)$ admits a subsequence converging in $L^1(\Omega)$ to a function $u \in BV(\Omega)$.*

We say that a function $u \in BV(\Omega)$ is a *special function of bounded variation*, and we write $u \in SBV(\Omega)$, if $|D^c u|(\Omega) = 0$. We say that a function $u \in L^1(\Omega)$ is a *generalized function of bounded variation*, and we write $u \in GBV(\Omega)$, if $u^T := (-T) \vee u \wedge T$ belongs to $BV(\Omega)$ for every $T \geq 0$. If $u \in GBV(\Omega)$, the function ∇u given by

$$\nabla u = \nabla u^T \qquad \text{a.e. on } \{|u| \leq T\} \tag{2.2}$$

turns out to be well-defined. Moreover, the set function $T \mapsto S_{u^T}$ is monotone increasing; therefore, if we set $S_u = \bigcup_{T>0} J_{u^T}$, for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$ we can consider the functions of T given by $(u^T)^-(x)$, $(u^T)^+(x)$, $\nu_{u^T}(x)$. It turns out that

$$u^-(x) = \lim_{T\to+\infty} (u^T)^-(x), \quad u^+(x) = \lim_{T\to+\infty} (u^T)^+(x), \quad \nu_u(x) = \lim_{T\to+\infty} \nu_{u^T}(x) \tag{2.3}$$

are well-defined for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$ Finally, for a function $u \in GBV(\Omega)$, let $|D^c u|$ be the supremum, in the sense of measures, of $|D^c u^T|$ for $T > 0$. It can be proved that for any Borel subset B of Ω

$$|D^c u|(B) = \lim_{T\to+\infty} |D^c u^T|(B). \tag{2.4}$$

2.2. Slicing

In order to obtain the estimate from below of the lower Γ-limit (see next paragraph) we need some basic properties of one-dimensional sections of BV-functions. We first introduce some notation. Let $\xi \in \mathbb{S}^{n-1}$, and let $\xi^\perp$ be the vector subspace orthogonal to ξ. If $y \in \xi^\perp$ and $E \subseteq \mathbb{R}^n$ we set $E_{\xi,y} = \{t \in \mathbb{R} : \ y + t\xi \in E\}$. Moreover, for any given function $u\colon \Omega \to \mathbb{R}$ we define $u_{\xi,y}\colon \Omega_{\xi,y} \to \mathbb{R}$ by $u_{\xi,y}(t) = u(y + t\xi)$. For the results collected in the following theorem see [5], Section 3.11.

Theorem 2.2. *Let $u \in BV(\Omega)$. Then $u_{\xi,y} \in BV(\Omega_{\xi,y})$ for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \xi^\perp$. For such values of y we have $u'_{\xi,y}(t) = \langle \nabla u(y + t\xi), \xi\rangle$ for a.e. $t \in \Omega_{\xi,y}$ and $J_{u_{\xi,y}} = (J_u)_{\xi,y}$, where $u'_{\xi,y}$ denotes the absolutely continuous part of the measure derivative of $u_{\xi,y}$. Moreover, for every open subset A of Ω we have*

$$\int_{\xi^\perp} |D^c u_{\xi,y}|(A_{\xi,y})\, \mathrm{d}\mathcal{H}^{n-1}(y) = |\langle D^c u, \xi\rangle|(A).$$

2.3. Γ-convergence

For the general theory see [9, 18]. Let (X, d) be a metric space. Let $(\mathcal{F}_j)$ be a sequence of functions $X \to \overline{\mathbb{R}}$. We say that $(\mathcal{F}_j)$ *Γ-converges*, as $j \to +\infty$, to $\mathcal{F}\colon X \to \overline{\mathbb{R}}$, if for all $u \in X$ we have:

(a) for every sequence (u_j) converging to u it holds

$$\mathcal{F}(u) \le \liminf_{j\to+\infty} \mathcal{F}_j(u_j);$$

(b) there exists a sequence (u_j) converging to u such that

$$\mathcal{F}(u) \ge \limsup_{j\to+\infty} \mathcal{F}_j(u_j).$$

The *lower* and *upper* Γ-limits of $(\mathcal{F}_j)$ in $u \in X$ are defined as

$$\mathcal{F}'(u) = \inf\Big\{\liminf_{j\to+\infty} \mathcal{F}_j(u_j) : \ u_j \to u\Big\}, \quad \mathcal{F}''(u) = \inf\Big\{\limsup_{j\to+\infty} \mathcal{F}_j(u_j) : \ u_j \to u\Big\}$$

respectively. We extend this definition of convergence to families depending on a real parameter. Given a family $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ of functions $X \to \overline{\mathbb{R}}$, we say that it Γ-converges, as $\varepsilon \to 0$, to $\mathcal{F}\colon X \to \overline{\mathbb{R}}$ if for every positive infinitesimal sequence (ε_j) the sequence $(\mathcal{F}_{\varepsilon_j})$ Γ-converges to F. If we define the lower and upper Γ-limits of $(\mathcal{F}_\varepsilon)$ as

$$\mathcal{F}'(u) = \inf\Big\{\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\Big\}, \quad \mathcal{F}''(u) = \inf\Big\{\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\Big\}$$

respectively, then $(\mathcal{F}_\varepsilon)$ Γ-converges to $\mathcal{F}$ in u if and only if $\mathcal{F}'(u) = \mathcal{F}''(u) = \mathcal{F}(u)$. It turns out that both $\mathcal{F}'$ and $\mathcal{F}''$ are lower semicontinuous on X. In the estimate of $\mathcal{F}'$ we shall use the following immediate consequence of the definition:

$$\mathcal{F}'(u) = \inf\left\{\liminf_{j\to+\infty} \mathcal{F}_{\varepsilon_j}(u_j) : \ \varepsilon_j \to 0^+, \ u_j \to u\right\}.$$

It turns out that the infimum is attained.

An important consequence of the definition of Γ-convergence is the following result about the convergence of minimizers (see *e.g.* [18], Cor. 7.20):

Theorem 2.3. *Let $\mathcal{F}_j\colon X \to \overline{\mathbb{R}}$ be a sequence of functions which Γ-converges to some $\mathcal{F}\colon X \to \overline{\mathbb{R}}$; assume that $\inf_{v\in X}\mathcal{F}_j(v) > -\infty$ for every j. Let (σ_j) be a positive infinitesimal sequence, and for every j let $u_j \in X$ be a σ_j-minimizer of $\mathcal{F}_j$,* i.e.

$$\mathcal{F}_j(u_j) \le \inf_{v\in X}\mathcal{F}_j(v) + \sigma_j.$$

Assume that $u_j \to u$ for some $u \in X$. Then u is a minimum point of $\mathcal{F}$, and

$$\mathcal{F}(u) = \lim_{j\to+\infty}\mathcal{F}_j(u_j).$$

Remark 2.4. The following property is a direct consequence of the definition of Γ-convergence: if $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}$ then $\mathcal{F}_\varepsilon + \mathcal{G} \xrightarrow{\Gamma} \mathcal{F} + \mathcal{G}$ whenever $\mathcal{G}\colon X \to \overline{\mathbb{R}}$ is continuous.

2.4. Supremum of measures

In order to prove the Γ-liminf inequality we recall the following useful tool, which can be found in [8].

Lemma 2.5. *Let Ω be an open subset of $\mathbb{R}^n$ and denote by $\mathcal{A}(\Omega)$ the family of its open subsets. Let λ be a positive Borel measure on Ω, and $\mu\colon \mathcal{A}(\Omega) \to [0,+\infty)$ a set function which is superadditive on open sets with disjoint compact closures,* i.e. *if $A, B \subset\subset \Omega$ and $\overline{A}\cap\overline{B} = \emptyset$, then*

$$\mu(A\cup B) \ge \mu(A) + \mu(B).$$

Let $(\psi_i)_{i\in I}$ be a family of positive Borel functions. Suppose that

$$\mu(A) \ge \int_A \psi_i \,\mathrm{d}\lambda \quad \textit{for every } A \in \mathcal{A}(\Omega) \textit{ and } i \in I.$$

Then

$$\mu(A) \ge \int_A \sup_i \psi_i \,\mathrm{d}\lambda \quad \textit{for every } A \in \mathcal{A}(\Omega).$$

2.5. A density result

The right bound for the upper Γ-limit from above will be first obtained for a suitable dense subset of $SBV(\Omega)$. More precisely, let $\mathcal{W}(\Omega)$ be the space of all functions $w \in SBV(\Omega)$ such that

(a) $\mathcal{H}^{n-1}(\overline{S}_w \setminus S_w) = 0$;
(b) $\overline{S}_w$ is the intersection of Ω with the union of a finite member of $(n-1)$-dimensional simplexes;
(c) $w \in W^{k,\infty}(\Omega\setminus\overline{S}_w)$ for every $k \in \mathbb{N}$.

Theorem 3.1 in [17] gives us the density property of $\mathcal{W}(\Omega)$ we need; here

$$SBV^2(\Omega) = \{u \in SBV(\Omega) : |\nabla u| \in L^2(\Omega),\ \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

Theorem 2.6. *Assume that $\partial\Omega$ is Lipschitz. Let $u \in SBV^2(\Omega)\cap L^\infty(\Omega)$. Then there exists a sequence (w_h) in $\mathcal{W}(\Omega)$ such that $w_h \to u$ strongly in $L^1(\Omega)$, $\nabla w_h \to \nabla u$ strongly in $L^2(\Omega,\mathbb{R}^n)$, with $\limsup_{h\to+\infty}\|w_h\|_\infty \le \|u\|_\infty$ and such that*

$$\limsup_{h\to+\infty}\int_{S_{w_h}}\psi(w_h^+, w_h^-, \nu_{w_h})\,\mathrm{d}\mathcal{H}^{n-1} \le \int_{S_u}\psi(u^+, u^-, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every upper semicontinuous function ψ such that $\psi(a,b,\nu) = \psi(b,a,-\nu)$ whenever $a, b \in \mathbb{R}$ and $\nu \in \mathbb{S}^{n-1}$.

2.6. A relaxation result

To conclude this section we prove a relaxation result which will be used in the sequel. Recall that given X be a topological space and $\mathcal{F}\colon X \to \mathbb{R} \cup \{\pm\infty\}$, the *relaxed functional* of $\mathcal{F}$, denoted by $\overline{\mathcal{F}}$, is the largest lower semicontinuous functional which is smaller than F.

Theorem 2.7. *Let $\phi\colon [0,+\infty) \to [0,+\infty)$ be a convex, non-decreasing and lower semicontinuous function with $\phi(0) = 0$ and with*

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = c \in (0,+\infty).$$

Let $\theta\colon [0,+\infty) \times \mathbb{S}^{n-1} \to [0,+\infty)$ be a lower semicontinuous function such that $\theta(s,\nu) \le c's$ for any $(s,\nu) \in [0,+\infty) \times \mathbb{S}^{n-1}$, for some $c' > 0$. For any $A \in \mathcal{A}(\Omega)$ let

$$\mathcal{F}(u,A) = \begin{cases} \displaystyle\int_A \phi(|\nabla u|)\,\mathrm{d}x + \int_{S_u\cap A} \theta(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} & \text{if } u \in SBV^2(\Omega) \cap L^\infty(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Then the relaxed functional of $\mathcal{F}$ with respect to the strong L^1-topology satisfies

$$\overline{\mathcal{F}}(u) \le \int_\Omega \phi(|\nabla u|)\,\mathrm{d}x + \int_{S_u} \theta(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + c|D^c u|(\Omega)$$

for any $u \in BV(\Omega)$.

Proof. Combining a standard convolution argument with a well known relaxation result (see, for instance, Thm. 5.47 in [5]) we can say that the relaxed functional of

$$\mathcal{G}(u,A) = \begin{cases} \displaystyle\int_A \phi(|\nabla u|)\,\mathrm{d}x & \text{if } u \in C^1(\overline{\Omega}) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

is given by

$$\overline{\mathcal{G}}(u,A) = \begin{cases} \displaystyle\int_A \phi(|\nabla u|)\,\mathrm{d}x + c|D^s u|(A) & \text{if } u \in BV(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Since $C^1(\overline{\Omega}) \subseteq SBV^2(\Omega) \cap L^\infty(\Omega)$ then we get $\mathcal{F}(u,A) \le \mathcal{G}(u,A)$. Hence for any $A \in \mathcal{A}(\Omega)$ and for any $u \in BV(\Omega)$

$$\overline{\mathcal{F}}(u,A) \le \int_A \phi(|\nabla u|)\,\mathrm{d}x + c|D^s u|(A).$$

We can now conclude using the fact that for every $u \in BV(\Omega)$ the set function $\overline{\mathcal{F}}(u,\cdot)$ is the trace on $\mathcal{A}(\Omega)$ of a regular Borel measure μ. This can be proven exactly along the same line of Proposition 3.3 in [6]. Hence

$$\overline{\mathcal{F}}(u) = \mu(\Omega) = \mu(\Omega \setminus S_u) + \mu(\Omega \cap S_u) \le \int_\Omega \phi(|\nabla u|)\,\mathrm{d}x + c|D^c u|(\Omega) + \int_{S_u} \theta(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$$

which is what we wanted to prove. $\square$

3. Statement of the main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $\phi\colon [0,+\infty) \to [0,+\infty)$ be a convex and non-decreasing function with $\phi(0) = 0$ and

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = c_0 \in (0,+\infty). \tag{3.1}$$

For any $\varepsilon > 0$ let $f_\varepsilon\colon [0,+\infty) \to [0,+\infty)$ be such that:

(A1) f_ε is non-decreasing, continuous, with $f_\varepsilon(0) = 0$.

(A2) It holds
$$\lim_{(\varepsilon,t)\to(0,0)} \frac{f_\varepsilon(t)}{\varepsilon\phi\left(\frac{t}{\varepsilon}\right)} = 1.$$

(A3) f_ε converges uniformly on the compact subsets of $[0,+\infty)$ to a concave function f.

Example 3.1. Given f and ϕ as above, a possible choice for f_ε satisfying A1–A3 is given by

$$f_\varepsilon(t) = \begin{cases} \varepsilon\phi\left(\frac{t}{\varepsilon}\right) & \text{if } 0 \le t \le t_\varepsilon \\ f(t - t_\varepsilon) + \varepsilon\phi\left(\frac{t_\varepsilon}{\varepsilon}\right) & \text{if } t > t_\varepsilon \end{cases}$$

where $t_\varepsilon \to 0$, and $t_\varepsilon/\varepsilon \to +\infty$. The only non-trivial assumption to verify is A2. Since $\varepsilon/t\phi(t/\varepsilon) \to c_0$ as $(\varepsilon,t) \to (0,0)$, with $t \ge t_\varepsilon$, the check amounts to verify that

$$\lim_{\substack{(\varepsilon,t)\to(0,0)\\ t\ge t_\varepsilon}} \frac{f(t-t_\varepsilon) + \varepsilon\phi\left(\frac{t_\varepsilon}{\varepsilon}\right)}{t} = c_0.$$

This follows immediately from $f(t - t_\varepsilon)/(t - t_\varepsilon) \to c_0$ and $\varepsilon/t_\varepsilon\phi(t_\varepsilon/\varepsilon) \to c_0$ as $(\varepsilon,t) \to (0,0)$, and $t \ge t_\varepsilon$.

Let $C \subset \mathbb{R}^n$ be open, bounded, and connected with $0 \in C$. Let $\rho\colon C \to (0,+\infty)$ be a continuous and bounded convolution kernel with

$$\int_C \rho \,\mathrm{d}x = 1.$$

For any $\varepsilon > 0$ and for any $x \in \mathbb{R}^n$ we will denote by $C_\varepsilon(x)$ the set $x + \varepsilon C$. For any $x \in \varepsilon C$ let

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)\cdot$$

We consider the family $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ of functionals $L^1(\Omega) \to [0,+\infty]$ defined by

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega f_\varepsilon(\varepsilon|\nabla u| * \rho_\varepsilon)\,\mathrm{d}x & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases} \tag{3.2}$$

where, for any $x \in \Omega$,

$$|\nabla u| * \rho_\varepsilon(x) = \int_{C_\varepsilon(x)\cap\Omega} |\nabla u(y)|\rho_\varepsilon(y - x)\,\mathrm{d}y \tag{3.3}$$

is a regularization by convolution of $|\nabla u|$ by means of the kernel ρ_ε.

Remark 3.2. Notice that with the choice $C = B_1$ and $\rho = \frac{1}{\omega_n}\chi_{B_1}$ we get

$$|\nabla u| * \rho_\varepsilon(x) = ⨍_{B_\varepsilon(x)\cap\Omega} |\nabla u|\,\mathrm{d}y$$

and thus the family $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ reduces to the case already investigated in [20–22].

In order to prove the Γ-convergence of $\mathcal{F}_\varepsilon$ it is convenient to introduce a localized version of $\mathcal{F}_\varepsilon$: more precisely, for each $A \in \mathcal{A}(\Omega)$ we set

$$\mathcal{F}_\varepsilon(u, A) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_A f_\varepsilon(\varepsilon|\nabla u| * \rho_\varepsilon)\,\mathrm{d}x & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{3.4}$$

Clearly, $\mathcal{F}_\varepsilon\big(\cdot, \Omega\big)$ coincides with the functional $\mathcal{F}_\varepsilon$ defined in (3.2). The lower and upper Γ-limits of $\big(\mathcal{F}_\varepsilon(\cdot, A)\big)$ will be denoted by $\mathcal{F}'(\cdot, A)$ and $\mathcal{F}''(\cdot, A)$, respectively.

3.1. The anisotropy

In this paragraph we define the surface density

$$\theta\colon [0, +\infty) \times \mathbb{S}^{n-1} \to [0, +\infty)$$

which will appear in the expression of the Γ-limit of $\mathcal{F}_\varepsilon$.

Given $\nu \in \mathbb{S}^{n-1}$ and $a, b \in \mathbb{R}$ let us denote by $u_\nu^{a,b}$ the function $\mathbb{R}^n \to \mathbb{R}$ given by

$$u_\nu^{a,b}(x) = \begin{cases} a & \text{if } \langle x, \nu\rangle < 0 \\ b & \text{if } \langle x, \nu\rangle \geq 0. \end{cases}$$

For any $x \in \mathbb{R}^n$ and any $\nu \in \mathbb{S}^{n-1}$ let $P_\nu^\perp(x)$ be the orthogonal projection of x onto the subspace $\nu^\perp = \{x \in \mathbb{R}^n : \langle x, \nu\rangle = 0\}$. We define the cylinder

$$Q_\nu = \{x \in \mathbb{R}^n : |\langle x, \nu\rangle| \leq 1,\ P_\nu^\perp(x) \in B_1 \cap \nu^\perp\}.$$

Given $\Omega' \subset \mathbb{R}^n$ with $Q_\nu \subset\subset \Omega'$ denote by $W_\nu^{a,b}$ the space of all sequences (u_j) in $W^{1,1}_{\mathrm{loc}}(\Omega')$ such that $u_j \to u_\nu^{a,b}$ in $L^1(\Omega')$, and such that there exist two positive infinitesimal sequences $(a_j), (b_j)$ with $u_j(x) = a$ if $\langle x, \nu\rangle < -a_j$ and $u_j = b$ if $\langle x, \nu\rangle > b_j$. Let

$$\theta(s, \nu) = \frac{1}{\omega_{n-1}} \inf\left\{\liminf_{j\to+\infty} \frac{1}{\varepsilon_j}\int_{Q_\nu} f(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x : (u_j) \in W_\nu^{0,s},\ \varepsilon_j \to 0^+\right\}. \tag{3.5}$$

Notice that $\theta(s, \nu)$ does not depend on the choice of Ω'. Let us collect some easy properties of θ which immediately descend from the definition.

Lemma 3.3. *The following properties hold:*

$$\theta \text{ is continuous.} \tag{3.6}$$

$$\theta(s, \nu) = \theta(s, -\nu), \quad \forall s \geq 0, \quad \forall \nu \in \mathbb{S}^{n-1}. \tag{3.7}$$

$$\begin{aligned} &\inf\left\{\liminf_{j\to+\infty} \frac{1}{\varepsilon_j}\int_{Q_\nu} f(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x : (u_j) \in W_\nu^{0,s},\ \varepsilon_j \to 0^+\right\} \\ &\qquad = \inf\left\{\liminf_{j\to+\infty} \frac{1}{\varepsilon_j}\int_{Q_\nu} f(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x : (u_j) \in W_\nu^{a,b},\ \varepsilon_j \to 0^+\right\} \\ &\qquad \textit{whenever } |a - b| = s. \end{aligned} \tag{3.8}$$

Moreover, for any $x_0 \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and $s \geq 0$ we have

$$\theta(s, \nu) = \frac{1}{\omega_{n-1}} \inf\left\{\liminf_{j\to+\infty} \frac{1}{\varepsilon_j}\int_{x_0+Q_\nu} f(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x : (u_j(\cdot - x_0)) \in W_\nu^{0,s},\ \varepsilon_j \to 0^+\right\}. \tag{3.9}$$

3.2. Main results

We are now in position to state the main result of the paper.

Theorem 3.4. *Let $\mathcal{F}_\varepsilon$ be as in (3.2), with f_ε satisfying conditions A1–A3. Then $\mathcal{F}_\varepsilon$ Γ-converges, with respect to the strong L^1-topology, as $\varepsilon \to 0$, to $\mathcal{F}\colon L^1(\Omega) \to [0, +\infty]$ given by*

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega \phi(|\nabla u|)\,\mathrm{d}x + \int_{S_u} \theta(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + c_0|D^c u|(\Omega) & \text{if } u \in GBV(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Remark 3.5. Notice that for any $u \in GBV(\Omega)$ the expression $\theta(|u^+ - u^-|, \nu_u)$ turns out to be well defined $\mathcal{H}^{n-1}$-a.e. $x \in S_u$, since (3.7) holds.

The proof of Theorem 3.4 will descend combining Proposition 5.10 (the Γ-liminf inequality) with Proposition 6.3 (the Γ-limsup inequality).

As a typical consequence of a Γ-convergence result, we are able to prove a result of convergence of minima by means of the following compactness result for equibounded (in energy) sequences, which will be proved in Section 4.

Theorem 3.6. *Let (ε_j) be a positive infinitesimal sequence, and let (u_j) be a sequence in $L^1(\Omega)$ such that $||u_j||_\infty \le M$, and such that $\mathcal{F}_{\varepsilon_j}(u_j) \le M$ for some positive constant M independent of j. Then the sequence (u_j) converges, up to a subsequence, in $L^1(\Omega)$ to a function $u \in BV(\Omega)$.*

Theorem 3.7. *Let (ε_j) be a positive infinitesimal sequence and let $g \in L^\infty(\Omega)$. For every $u \in L^1(\Omega)$ and $j \in \mathbb{N}$ let*

$$\mathcal{I}_j(u) = \mathcal{F}_{\varepsilon_j}(u) + \int_\Omega |u - g|\,\mathrm{d}x, \quad \mathcal{I}(u) = \mathcal{F}(u) + \int_\Omega |u - g|\,\mathrm{d}x.$$

For every j let $u_j \in L^1(\Omega)$ be such that

$$\mathcal{I}_j(u_j) \le \inf_{L^1(\Omega)} \mathcal{I}_j + \varepsilon_j.$$

Then the sequence (u_j) converges, up to a subsequence, to a minimizer of $\mathcal{I}$ in $L^1(\Omega)$.

Proof. Since $g \in L^\infty(\Omega)$ and since $\mathcal{F}_{\varepsilon_j}$ decreases by truncation, we can assume that (u_j) is equibounded in $L^\infty(\Omega)$; for instance $||u_j||_\infty \le ||g||_\infty$. Applying Theorem 3.6 there exists $u \in BV(\Omega)$ such that (up to a subsequence) $u_j \to u$ in $L^1(\Omega)$. By Theorem 2.3, since $(\mathcal{I}_j)$ Γ-converges to $\mathcal{I}$ (see Thm. 3.4 and Rem. 2.4), u is a minimum point of $\mathcal{I}$ on $L^1(\Omega)$. □

4. Compactness

In this section we prove Theorem 3.6. Let us first recall a useful technical Lemma which can be found in [10], Proposition 4.1. Actually such a proposition has been proved for $|\nabla u|^2$, but, up to simple modifications, the same proof works for $|\nabla u|$.

For every $A \in \mathcal{A}(\Omega)$ and $\sigma > 0$ we set

$$A_\sigma = \{x \in A : \ d(x, \partial A) > \sigma\}.$$

Lemma 4.1. *Let $g\colon [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function such that*

$$\lim_{t \to 0} \frac{g(t)}{t} = c$$

for some $c > 0$. Let $A \in \mathcal{A}(\Omega)$ with $A \subset\subset \Omega$, and let $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$. For any $\delta > 0$ and for any $\varepsilon > 0$ sufficiently small, there exists a function $v \in SBV(A) \cap L^\infty(A)$ such that

$$(1-\delta)\int_A |\nabla v|\,\mathrm{d}x \le \frac{1}{\varepsilon}\int_A g\bigg(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u|\,\mathrm{d}y\bigg)\,\mathrm{d}x,$$

$$\mathcal{H}^{n-1}(S_v \cap A_{6\varepsilon}) \le \frac{c'}{\varepsilon}\int_A g\bigg(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u|\,\mathrm{d}y\bigg)\,\mathrm{d}x,$$

$$\|v\|_{L^\infty(A)} \le \|u\|_{L^\infty(A)}$$

$$\|v-u\|_{L^1(A_{6\varepsilon})} \le c'\|u\|_{L^\infty(A)}\int_A g\bigg(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u|\,\mathrm{d}y\bigg)\,\mathrm{d}x,$$

where c' is a constant depending only on n, δ and g.

Proof of Theorem 3.6. Let $A \in \mathcal{A}(\Omega)$ with $A \subset\subset \Omega$ and ∂A smooth. Let $r > 0$ such that $B_r \subset C$, and let $m = \inf_{B_r} \rho > 0$. Then for any $x \in A$ we have $B_{r\varepsilon_j}(x) \subset C_{\varepsilon_j}(x)$ and thus for j sufficiently large,

$$|\nabla u_j| * \rho_{\varepsilon_j}(x) = \int_{C_{\varepsilon_j}(x)} |\nabla u_j(y)|\rho_{\varepsilon_j}(y-x)\,\mathrm{d}y \ge \frac{m}{\varepsilon_j^n}\int_{B_{r\varepsilon_j}(x)} |\nabla u_j(y)|\,\mathrm{d}y = mr^n\omega_n \fint_{B_{r\varepsilon_j}(x)} |\nabla u_j(y)|\,\mathrm{d}y$$

for any $x \in A$. Fix $\delta > 0$. By A2 there exist $t_\delta > 0$ and j_δ such that $f_{\varepsilon_j}(t) \ge (1-\delta)\varepsilon_j\phi(t/\varepsilon_j)$ for any $t \in [0, t_\delta]$ and $j > j_\delta$. Let $\alpha, \beta \in \mathbb{R}$, with $\alpha > 0$ and $\beta < 0$, be such that $\phi(t) \ge \alpha t + \beta$ everywhere. Then, since f_{ε_j} is non-decreasing, we have $f_{\varepsilon_j}(t) \ge g^\delta_{\varepsilon_j}(t)$ for any $t \ge 0$, being

$$g^\delta_{\varepsilon_j}(t) = \begin{cases} (1-\delta)\alpha t + \varepsilon_j\beta & \text{if } t \in [0, t_\delta] \\ (1-\delta)\alpha t_\delta + \varepsilon_j\beta & \text{if } t > t_\delta. \end{cases}$$

Therefore, letting $h_\delta(t) = g^\delta_{\varepsilon_j}(t) - \varepsilon_j\beta$, we have

$$\mathcal{F}_{\varepsilon_j}(u_j, A) \ge \frac{1}{\varepsilon_j}\int_A h_\delta(|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x + \beta|A| \ge \frac{1}{\varepsilon_j}\int_A h_\delta\bigg(mr^n\omega_n\varepsilon_j \fint_{B_{r\varepsilon_j}(x)} |\nabla u_j|\,\mathrm{d}y\bigg)\,\mathrm{d}x + \beta|A|. \tag{4.1}$$

Let $\eta_j = r\varepsilon_j$ and $g_{\delta,m,r}(t) = \frac{1}{r}g_\delta(mr^{n-1}\omega_n t)$. Notice that, by construction,

$$\lim_{t\to 0}\frac{g_{\delta,m,r}(t)}{t}$$

exists and is finite. Then inequality (4.1) becomes

$$\mathcal{F}_{\varepsilon_j}(u_j, A) - \beta|A| \ge \frac{1}{\eta_j}\int_\Omega g_{\delta,r,m}\bigg(\eta_j \fint_{B_{\eta_j}(x)} |\nabla u_j|\,\mathrm{d}y\bigg)\,\mathrm{d}x.$$

Applying Lemma 4.1 we find a sequence (v_j) in $SBV(A)$ and a constant C independent of A such that $\|v_j\|_{BV(A)} \le C$ and $\|v_j\|_{L^\infty(A)} \le C$. Moreover,

$$\|v_j - u_j\|_{L^1(A)} \to 0. \tag{4.2}$$

Hence, by Theorem 2.1, the sequence (v_j) converges, up to a subsequence not relabeled, to some $u \in BV(A)$, with $\|u\|_{BV(A)} \le C$. By (4.2) also u_j converges to u in $L^1(A)$. The arbitrariness of A and a diagonal argument allow to find a subsequence (u_{j_k}) which converges in $L^1_{\mathrm{loc}}(\Omega)$ to a function $u \in BV_{\mathrm{loc}}(\Omega)$, and the uniform bound of $\|u_j\|_{L^\infty(\Omega)}$ implies the convergence is strong in $L^1(\Omega)$. □

5. The Γ-liminf inequality

In this section we will prove that for any $u \in L^1(\Omega)$ the inequality

$$\mathcal{F}(u) \leq \liminf_{j\to+\infty} \mathcal{F}_{\varepsilon_j}(u_j)$$

holds for any $u_j \to u$ in $L^1(\Omega)$. First we will investigate two particular situations.

5.1. A preliminary estimate from below in terms of the volume and Cantor parts

In this paragraph we will take into account a simpler family of functionals. Let $\alpha, \beta > 0$ and let $g\colon [0,+\infty) \to [0,+\infty)$ given by $g(t) = \alpha t \wedge \beta$. Let $\mathcal{G}_\varepsilon\colon L^1(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty]$ be defined by

$$\mathcal{G}_\varepsilon(u,A) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_A g(\varepsilon|\nabla u| * \rho_\varepsilon)\,\mathrm{d}x & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

We wish to estimate from below the lower Γ-limit $\mathcal{G}'(\cdot, A)$ in terms of the volume and the Cantor parts of Du. To this sake, we apply a slicing procedure, so that at first we will establish a suitable one-dimensional inequality. The idea of the proof is the same as in [25], where the superlinear growth case is treated.

Let $m \in \mathbb{N}$ odd, let A be an open interval in $\mathbb{R}$, and let (ε_j) be a positive infinitesimal sequence. Let $A_j = \{x \in \varepsilon_j\mathbb{Z} : x \in A\}$. For any $j \in \mathbb{N}$ and for any $x \in A_j$ we define the interval

$$I_j(x) = \left[x - \frac{m\varepsilon_j}{2}, x + \frac{m\varepsilon_j}{2}\right].$$

Lemma 5.1. *Let $\alpha', \beta' > 0$ and let $h_j\colon [0,+\infty) \to [0,+\infty)$ given by $h_j(t) = \alpha' t \wedge \frac{\beta'}{\varepsilon_j}$. Let $u \in BV(A)$ and let $u_j \to u$ in $L^1(A)$ with $u_j \in W^{1,1}(A)$ for any $j \in \mathbb{N}$. Then*

$$\liminf_{j\to+\infty} \varepsilon_j \sum_{x\in A_j} h_j\left(⨍_{I_j(x)} |u_j'|\,\mathrm{d}y\right) \geq \alpha' \int_A |u'|\,\mathrm{d}y + \alpha'|D^c u|(A). \tag{5.1}$$

Proof. For any $j \in \mathbb{N}$ and $i = 0, \dots, m-1$ let $A_j^i = (i\varepsilon_j + m\varepsilon_j\mathbb{Z}) \cap A$. Obviously A_j is the disjoint union of A_j^i for $i \in \{0, \dots, m-1\}$. Then

$$\sum_{x\in A_j} h_j\left(⨍_{I_j(x)} |u_j'|\,\mathrm{d}y\right) \geq \frac{1}{m}\sum_{i=0}^{m-1}\sum_{x\in A_j^i} m h_j\left(⨍_{I_j(x)} |u_j'|\,\mathrm{d}y\right).$$

Now let

$$\overline{A_j^i} = \left\{ x \in A_j^i :\ ⨍_{I_j(x)} |u_j'|\,\mathrm{d}x \leq \frac{\beta'}{\alpha'\varepsilon_j}\right\}$$

and let $v_j \in SBV(A)$ given by

$$v_j(x) = \begin{cases} u_j(x) & \text{if } x \in \bigcup_{y\in\overline{A_j^i}} I_j(y) \\ 0 & \text{otherwise in } A. \end{cases}$$

Hence

$$\sum_{x\in A_j^i} m\varepsilon_j h_j\left(⨍_{I_j(x)} |u_j'|\,\mathrm{d}y\right) \geq \sum_{x\in\overline{A_j^i}} m\varepsilon_j h_j\left(⨍_{I_j(x)} |u_j'|\,\mathrm{d}y\right) = \alpha' \sum_{x\in\overline{A_j^i}} \int_{I_j(x)} |u_j'|\,\mathrm{d}y = \alpha' \int_A |v_j'|\,\mathrm{d}y.$$

Observe that since we can suppose, without loss of generality, that

$$\varepsilon_j \sum_{x\in A_j} h_j\biggl(\fint_{I_j(x)} |u_j'|\,\mathrm{d}y\biggr) \le M$$

for some $M \ge 0$, we deduce that

$$M \ge \varepsilon_j \sum_{x\in A_j\setminus\bigcup_{i=0}^{m-1}\overline{A_j^i}} h_j\biggl(\fint_{I_j(x)} |u_j'|\,\mathrm{d}y\biggr) = \varepsilon_j \frac{\beta'}{\varepsilon_j}\sharp\biggl(A_j \setminus \bigcup_{i=0}^{m-1}\overline{A_j^i}\biggr)$$

from which necessarily we have

$$\varepsilon_j\sharp\biggl(A_j \setminus \bigcup_{i=0}^{m-1}\overline{A_j^i}\biggr) \to 0, \quad \text{as } j\to+\infty.$$

This implies that $||u_j - v_j||_{L^1(A)} \to 0$ as $j \to +\infty$. Therefore, $v_j \to u$ in $L^1(A)$. Finally, by the superadditivity of the lim inf and by the lower semicontinuity of the total variation, we get

$$\begin{aligned}
\liminf_{j\to+\infty} \varepsilon_j \sum_{x\in A_j} h_j\biggl(\fint_{I_j(x)} |u_j'|\,\mathrm{d}y\biggr) &\ge \frac{1}{m}\sum_{i=0}^{m-1} \liminf_{j\to+\infty} \sum_{x\in\overline{A_j^i}} m\varepsilon_j h_j\biggl(\fint_{I_j(x)} |u_j'|\,\mathrm{d}y\biggr)\\
&\ge \alpha' \liminf_{j\to+\infty}\int_A |v_j'|\,\mathrm{d}y \ge \alpha'|Du|(A)\\
&\ge \alpha'\int_A |u'|\,\mathrm{d}y + \alpha'|D^c u|(A)
\end{aligned}$$

which ends the proof. □

Now, by applying the slicing Theorem 2.2, we will reduce the n-dimensional inequality to the one-dimensional inequality 5.1. Fix $\xi \in \mathbb{S}^{n-1}$ and $\delta \in (0,1)$; consider an orthonormal basis $\{\mathbf{e}_i\}$ with $\mathbf{e}_n = \xi$. Let

$$Q_\delta^\xi = \biggl\{x \in \mathbb{R}^n : |\langle x, \mathbf{e}_i\rangle| \le \frac{\delta}{2},\, i = 1,\dots,n\biggr\}, \quad Q_\delta^\xi(x) = x + Q_\delta^\xi$$

and the lattice $Z_\delta^\xi = \{x \in \mathbb{R}^n : \langle x, \mathbf{e}_i\rangle \in \delta\mathbb{Z},\, i = 1,\dots,n\}$. In what follows we will denote by $g_j(t) = \frac{1}{\varepsilon_j} g(\varepsilon_j t)$; in particular it holds $g_j(t) = \alpha t \wedge \frac{\beta}{\varepsilon_j}$ and

$$\mathcal{G}_{\varepsilon_j}(u, A) = \int_A g_j(|\nabla u| * \rho_{\varepsilon_j})\,\mathrm{d}x, \quad u \in W^{1,1}(\Omega).$$

Finally fix $A \in \mathcal{A}(\Omega)$ and let $A_\delta^\xi = \{x \in Z_\delta^\xi : Q_\delta^\xi(x) \subset A\}$. The following Lemma is a standard easy application of the mean value theorem (see also Lem. 4.2 in [10]).

Lemma 5.2. *Let $u \in W^{1,1}(\Omega)$. Then there exists $\tau \in Q_\delta^\xi$ such that*

$$\mathcal{G}_{\varepsilon_j}(u, A) \ge \sum_{x\in A_\delta^\xi} \delta^n g_j(|\nabla u| * \rho_{\varepsilon_j}(x + \tau)).$$

Proof. We have

$$\mathcal{G}_{\varepsilon_j}(u, A) \ge \sum_{x\in A_\delta^\xi} \int_{Q_\delta^\xi(x)} g_j(|\nabla u| * \rho_{\varepsilon_j}(y))\,\mathrm{d}y = \int_{Q_\delta^\xi} \sum_{x\in A_\delta^\xi} g_j(|\nabla u| * \rho_{\varepsilon_j}(y + x))\,\mathrm{d}y.$$

Applying the mean value theorem we get

$$\int_{Q^\xi_\delta} \sum_{x\in A^\xi_\delta} g_j(|\nabla u| * \rho_{\varepsilon_j}(y+x))\,\mathrm{d}y = \sum_{x\in A^\xi_\delta} g_j(|\nabla u| * \rho_{\varepsilon_j}(\tau+x))$$

for some $\tau \in Q^\xi_\delta$, which concludes the proof. □

We are in position to apply the slicing procedure.

Proposition 5.3. *Let $u \in BV(\Omega)$ and $A \in \mathcal{A}(\Omega)$. Then*

$$\mathcal{G}'(u,A) \geq \alpha \int_A |\nabla u|\,\mathrm{d}x \quad \textit{and} \quad \mathcal{G}'(u,A) \geq \alpha |D^c u|(A).$$

Proof. Fix $\xi \in \mathbb{S}^{n-1}$. For any $\eta > 0$ let P^ξ_η be the union of the squares $Q^\xi_\eta(y_i) \subset C$ with $y_i \in Z^\xi_\eta$ for $i = 1,\dots,m$, for some $m \in \mathbb{N}$ depending on η and ξ. Let ρ_η be a non-negative constant function on the squares $Q^\xi_\eta(y_i)$ with $0 < \rho_\eta \leq \rho$ and such that

$$c_\eta = \int_C \rho_\eta \,\mathrm{d}x \to 1, \quad \text{as } \eta \to 0.$$

Let $c_i = \rho_\eta(y_i)$; then we can rewrite c_η as $c_\eta = \sum_{i=1}^m c_i \eta^n$. Let $P^\xi_{\eta\varepsilon_j}$ be the union of the squares $Q^\xi_{\eta\varepsilon_j}(y_i) \subseteq C_{\varepsilon_j}$, with $y_i \in Z^\xi_{\eta\varepsilon_j}$, for $i = 1,\dots,m$. Let $A^\xi_j = A^\xi_{\eta\varepsilon_j}$; applying Lemma 5.2, since we can suppose, without loss of generality, that $u_j \in W^{1,1}(\Omega)$, there exists $\tau_j \in Q^\xi_{\eta\varepsilon_j}$ such that

$$\mathcal{G}_{\varepsilon_j}(u_j,A) \geq \sum_{x\in A^\xi_j} (\eta\varepsilon_j)^n g_j(|\nabla u_j| * \rho_{\varepsilon_j}(x+\tau_j)).$$

Let $B \subset\subset A$, and, for any j sufficiently large, let $v_j(y) = u_j(y+\tau_j)$. Then we get $v_j \in W^{1,1}(B)$ and $v_j \to u$ in $L^1(B)$. Thus

$$\mathcal{G}_{\varepsilon_j}(u_j,A) \geq \sum_{x\in B^\xi_j} (\eta\varepsilon_j)^n g(|\nabla v_j| * \rho_{\varepsilon_j}(x))$$

being $B^\xi_j = \{x \in Z^\xi_{\eta\varepsilon_j} : Q^\xi_{\eta\varepsilon_j} \subseteq B\}$. Now, for each $x \in B^\xi_j$, we estimate the term $|\nabla v_j| * \rho_{\varepsilon_j}(x)$; we have, for j large enough,

$$\begin{aligned}|\nabla v_j| * \rho_{\varepsilon_j}(x) &= \int_{C_{\varepsilon_j}} |\nabla v_j(y+x)|\rho_{\varepsilon_j}(y)\,\mathrm{d}y \geq \frac{1}{\varepsilon_j^n}\int_{P^\xi_{\eta\varepsilon_j}} |\nabla v_j(y+x)|\rho_\eta\left(\frac{y}{\varepsilon_j}\right)\mathrm{d}y\\ &\geq \frac{1}{\varepsilon_j^n}\sum_{i=1}^m c_i \int_{Q^\xi_{\eta\varepsilon_j}(y_i)} |\nabla v_j(y+x)|\,\mathrm{d}y = \sum_{i=1}^m \frac{c_i\eta^n}{c_\eta} \fint_{Q^\xi_{\eta\varepsilon_j}(y_i)} c_\eta |\nabla v_j(y+x)|\,\mathrm{d}y.\end{aligned}$$

Since $\sum_{i=1}^m \frac{c_i\eta^n}{c_\eta} = 1$ and since g_j is concave we get, for every $x \in B^\xi_j$,

$$g_j(|\nabla v_j| * \rho_{\varepsilon_j}(x)) \geq \sum_{i=1}^m \frac{c_i\eta^n}{c_\eta} g_j\left(c_\eta \fint_{Q^\xi_{\eta\varepsilon_j}(y_i)} |\nabla v_j(y+x)|\,\mathrm{d}y\right).$$

Thus, reordering the terms, we deduce that

$$\mathcal{G}_{\varepsilon_j}(u_j,A) \geq \sum_{x\in D^\xi_j} (\eta\varepsilon_j)^n g_j\left(c_\eta \fint_{Q^\xi_{\eta\varepsilon_j}(x)} |\nabla v_j|\,\mathrm{d}z\right)$$

for any $D \subset\subset B$ and j sufficiently large, being, as usual, $D_j^\xi = \{x \in Z^\xi_{\eta\varepsilon_j} : Q^\xi_{\eta\varepsilon_j} \subseteq D\}$. For convenience we can suppose $\nabla v_j = 0$ on

$$\mathbb{R}^n \setminus \bigcup_{Q^\xi_{\eta\varepsilon_j} \subseteq D} Q^\xi_{\eta\varepsilon_j}.$$

Let $\langle\xi\rangle$ be the one-dimensional space generated by ξ. Let us denote by $Z^{\xi_\parallel}_{\eta\varepsilon_j}$ and by $Z^{\xi_\perp}_{\eta\varepsilon_j}$ the orthogonal projections of $Z^\xi_{\eta\varepsilon_j}$ respectively on $\langle\xi\rangle$ and $\xi^\perp$. Then

$$\mathcal{G}_{\varepsilon_j}(u_j, A) \geq \sum_{x \in Z^\xi_{\eta\varepsilon_j}} (\eta\varepsilon_j)^n g_j\left(c_\eta \fint_{Q^\xi_{\eta\varepsilon_j}(x)} |\nabla v_j| \,\mathrm{d}z\right) \geq \sum_{x_\perp \in Z^{\xi_\perp}_{\eta\varepsilon_j}} \sum_{x_\parallel \in Z^{\xi_\parallel}_{\eta\varepsilon_j}} (\eta\varepsilon_j)^n g_j\left(c_\eta \fint_{Q^\xi_{\eta\varepsilon_j}(x_\perp + x_\parallel)} |\nabla v_j| \,\mathrm{d}z\right)$$

where $x = x_\parallel + x_\perp$ turns out to be the unique decomposition of any $x \in Z^\xi_{\eta\varepsilon_j}$ with $x_\parallel \in Z^{\xi_\parallel}_{\eta\varepsilon_j}$ and $x_\perp \in Z^{\xi_\perp}_{\eta\varepsilon_j}$. Moreover, denoting by $Q^{\xi_\parallel}_{\eta\varepsilon_j}$ and by $Q^{\xi_\perp}_{\eta\varepsilon_j}$ the projections of $Q^\xi_{\eta\varepsilon_j}$ respectively on $\langle\xi\rangle$ and on $\xi^\perp$, applying Jensen's inequality we deduce that

$$\begin{aligned}
\mathcal{G}_{\varepsilon_j}(u_j, A) &\geq \sum_{x_\perp \in Z^{\xi_\perp}_{\eta\varepsilon_j}} \sum_{x_\parallel \in Z^{\xi_\parallel}_{\eta\varepsilon_j}} (\eta\varepsilon_j)^n g_j\left(c_\eta \fint_{Q^{\xi_\perp}_{\eta\varepsilon_j}(x_\perp)} \fint_{Q^{\xi_\parallel}_{\eta\varepsilon_j}(x_\parallel)} |\langle\nabla v_j(z_\perp + z_\parallel), \xi\rangle| \,\mathrm{d}z_\parallel \,\mathrm{d}z_\perp\right) \\
&\geq \sum_{x_\perp \in Z^{\xi_\perp}_{\eta\varepsilon_j}} \sum_{x_\parallel \in Z^{\xi_\parallel}_{\eta\varepsilon_j}} (\eta\varepsilon_j)^n \fint_{Q^{\xi_\perp}_{\eta\varepsilon_j}(x_\perp)} g_j\left(c_\eta \fint_{Q^{\xi_\parallel}_{\eta\varepsilon_j}(x_\parallel)} |\langle\nabla v_j(z_\perp + z_\parallel), \xi\rangle| \,\mathrm{d}z_\parallel\right) \mathrm{d}z_\perp \\
&\geq \sum_{x_\perp \in Z^{\xi_\perp}_{\eta\varepsilon_j}} \int_{Q^{\xi_\perp}_{\eta\varepsilon_j}(x_\perp)} \sum_{x_\parallel \in Z^{\xi_\parallel}_{\eta\varepsilon_j}} \eta\varepsilon_j g_j\left(c_\eta \fint_{Q^{\xi_\parallel}_{\eta\varepsilon_j}(x_\parallel)} |\langle\nabla v_j(z_\perp + z_\parallel), \xi\rangle| \,\mathrm{d}z_\parallel\right) \mathrm{d}z_\perp \\
&\geq \int_{\xi^\perp} \sum_{x_\parallel \in Z^{\xi_\parallel}_{\eta\varepsilon_j}} \eta\varepsilon_j g_j\left(c_\eta \fint_{Q^{\xi_\parallel}_{\eta\varepsilon_j}(x_\parallel)} |\langle\nabla v_j(z_\perp + z_\parallel), \xi\rangle| \,\mathrm{d}z_\parallel\right) \mathrm{d}z_\perp.
\end{aligned}$$

For any $\sigma > 0$ small let $D_\sigma = \{x \in D : d(x, \partial D) > \sigma\}$ and $D^{x_\perp}_\sigma = \{x \in D_\sigma : x = x_\perp + x_\parallel \xi,\ x_\parallel \in \mathbb{R}\}$, for $x_\perp \in \xi^\perp$. For j sufficiently large, $v_j(x_\perp + \cdot) \in W^{1,1}(D^{x_\perp}_\sigma)$. Furthermore, $v_j \to u$ in $L^1(D^{x_\perp}_\sigma)$ for a.e. $x_\perp \in \xi^\perp$. Let $h_j(t) = g_j(c_\eta t)$; then, by the very definition of g, it is easy to see that $h_j(t) = \alpha c_\eta t \wedge \frac{\beta}{\varepsilon_j}$. We are in position to apply Lemma 5.1 with choice $\alpha' = \alpha c_\eta$ and $\beta' = \beta$. Thus

$$\begin{aligned}
\liminf_{j\to+\infty} &\sum_{x_\parallel \in Z^{\xi_\parallel}_{\eta\varepsilon_j}} \eta\varepsilon_j g_j\left(c_\eta \fint_{Q^{\xi_\parallel}_{\eta\varepsilon_j}(x_\parallel)} |\langle\nabla v_j(z_\perp + z_\parallel), \xi\rangle| \,\mathrm{d}z_\parallel\right) \\
&= \liminf_{j\to+\infty} \sum_{x_\parallel \in Z^{\xi_\parallel}_{\eta\varepsilon_j}} \eta\varepsilon_j h_j\left(\fint_{Q^{\xi_\parallel}_{\eta\varepsilon_j}(x_\parallel)} |\langle\nabla v_j(z_\perp + z_\parallel), \xi\rangle| \,\mathrm{d}z_\parallel\right) \\
&\geq \alpha c_\eta \int_{D^{z_\perp}_\sigma} |\langle\nabla u(z_\perp + z_\parallel), \xi\rangle| \,\mathrm{d}z_\parallel + \alpha c_\eta |\langle D^c u(z_\perp + \cdot), \xi\rangle|(D^{z_\perp}_\sigma).
\end{aligned}$$

Taking into account Theorem 2.2 and Fatou's lemma we conclude that

$$\liminf_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(u_j, A) \geq c_\eta \alpha \int_{D_\sigma} |\langle\nabla u(z), \xi\rangle| \,\mathrm{d}z + c_\eta \alpha |\langle D^c u, \xi\rangle|(D_\sigma).$$

Since $c_\eta \to 1$ as $\eta \to 0$, let $\sigma \to 0$ and $D \nearrow A$. Then

$$\mathcal{G}'(u, A) \geq \alpha \int_A |\langle\nabla u(z), \xi\rangle| \,\mathrm{d}z \quad \text{and} \quad \mathcal{G}'(u, A) \geq \alpha |\langle D^c u, \xi\rangle|(A) \tag{5.2}$$

for any $\xi \in \mathbb{S}^{n-1}$. From the first inequality, using the superadditivity of $\mathcal{G}'$ and Lemma 2.5 we easily deduce that

$$\mathcal{G}'(u, A) \geq \alpha \int_A |\nabla u| \, \mathrm{d}z.$$

Now if $\psi_\xi = \left\langle \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}, \xi \right\rangle$ the second inequality in (5.2) can be rewritten as

$$\mathcal{G}'(u, A) \geq \alpha \int_A |\psi_\xi| \, \mathrm{d}|D^c u|.$$

Another application of Lemma 2.5 yields

$$\mathcal{G}'(u, A) \geq \alpha \int_A \sup_{\xi \in \mathbb{S}^{n-1}} |\psi_\xi| \, \mathrm{d}|D^c u| \geq \alpha \int_A \left| \sup_{\xi \in \mathbb{S}^{n-1}} \psi_\xi \right| \mathrm{d}|D^c u| = \alpha |D^c u|(A).$$

This concludes the proof. □

5.2. A preliminary estimate in terms of the surface part

In this section we will consider the family of functionals $L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ given by

$$\mathcal{E}_\varepsilon(u, A) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_A h(\varepsilon |\nabla u| * \rho_\varepsilon) \, \mathrm{d}x & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

where $h\colon [0, +\infty) \to [0, +\infty)$ is a non-decreasing concave function with $h(0) = 0$ and with

$$\lim_{t \to 0} = \frac{h(t)}{t} = c' > 0.$$

The aim of this section is to estimate from below the lower Γ-limit of $\mathcal{E}_\varepsilon$ in terms of a surface integral; to do this the main idea, as in [22], is to estimate from below the Radon–Nikodym derivative of the lower Γ-limit $\mathcal{E}'$ with respect to the Hausdorff measure $\mathcal{H}^{n-1}$ by means of a blow-up argument around a jump point; then the result follows applying Besicovitch's differentiation theorem in a standard way.

Given $x_0 \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and $a, b \in \mathbb{R}$, when considering $\mathcal{E}'$ for the blow up $u^{\nu,a,b}_{x_0} = u^{a,b}_\nu(\cdot - x_0)$ (see Sect. 3.1 for the definition of $u^{a,b}_\nu$) on a unit ball B_1 as below (or on a cylinder Q_ν as in the sequel), we will assume as Ω any set Ω' strictly containing B_1 (or Q_ν): the lower Γ-limit of $\mathcal{E}_\varepsilon(\cdot, A)$ does not change by replacing Ω with any $\Omega' \supset\supset A$.

For every $A \in \mathcal{A}(\Omega)$ let $\mathcal{E}'_-(\cdot, A)$ be the inner regular envelope of $\mathcal{E}'$, *i.e.*

$$\mathcal{E}'_-(\cdot, A) = \sup\{\mathcal{E}'(\cdot, B) : B \in \mathcal{A}(\Omega), B \subset\subset A\}.$$

Proposition 5.4. *Let $u \in BV(\Omega)$ and let $x_0 \in J_u$. Then*

$$\liminf_{\varrho \to 0} \frac{\mathcal{E}'_-(u, B_\varrho(x))}{\varrho^{n-1}} \geq \mathcal{E}'(u_{x_0}^{\nu_u(x_0), u^+(x_0), u^-(x_0)}, B_1(x_0)).$$

Proof. Let $\delta \in (0,1)$. Then $\mathcal{E}'_-\big(u, B_\varrho(x_0)\big) \geq \mathcal{E}'\big(u, B_{\delta\varrho}(x_0)\big)$ for every $\varrho > 0$. Thus

$$\liminf_{\varrho \to 0} \frac{\mathcal{E}'_-(u, B_\varrho(x_0))}{\varrho^{n-1}} \geq \delta^{n-1} \liminf_{r \to 0} \frac{\mathcal{E}'(u, B_r(x_0))}{r^{n-1}} \cdot \tag{5.3}$$

Let us now estimate the lower limit in the right-hand side. Without loss of generality we can assume $x_0 = 0$; moreover, for the sake of simplicity, we will denote by u_0 the function $u_0^{\nu_u(0), u^+(0), u^-(0)}$.

Let (r_k) be a decreasing infinitesimal sequence; for every $k \in \mathbb{N}$ there exists $u_j \in W^{1,1}(\Omega)$ such that $u_j \to u$ in $L^1(\Omega)$ and

$$\liminf_{j\to+\infty} \mathcal{E}_{\varepsilon_j}(u_j, B_{r_k}) \le \mathcal{E}'(u, B_{r_k}) + \frac{r_k^{n-1}}{2k}\cdot$$

Let $\bar{j} = j(k)$ be such that $\varepsilon_{\bar{j}}/r_k \le 1/k$ and

$$\mathcal{E}_{\varepsilon_{\bar{j}}}(u_{\bar{j}}, B_{r_k}) \le \mathcal{E}'(u, B_{r_k}) + \frac{r_k^{n-1}}{k},$$

$\|u_{\bar{j}} - u\|_{L^1(\Omega)} \le \frac{1}{k}$ and such that

$$\int_{B_2} |u_{\bar{j}}(r_k x) - u(r_k x)|\,\mathrm{d}x \le \frac{1}{k}\cdot$$

Let $v_k = u_{j(k)}$. We can suppose that the sequence $j(k)$ is increasing, and we set $\sigma_k = \varepsilon_{j(k)}$. Hence, $v_k \to u$ in $L^1(\Omega)$,

$$\mathcal{E}_{\sigma_k}(v_k, B_{r_k}) \le \mathcal{E}'(u, B_{r_k}) + \frac{r_k^{n-1}}{k} \tag{5.4}$$

and

$$\int_{B_2} |v_k(r_k x) - u(r_k x)|\,\mathrm{d}x \le \frac{1}{k}\cdot \tag{5.5}$$

Inequality (5.4) gives

$$\liminf_{k\to+\infty} \frac{\mathcal{E}'(u, B_{r_k})}{r_k^{n-1}} \ge \liminf_{k\to+\infty} \frac{\mathcal{E}_{\sigma_k}(v_k, B_{r_k})}{r_k^{n-1}}$$

while from (5.5) we get

$$\int_{B_2} |v_k(r_k x) - u_0(r_k x)|\,\mathrm{d}x \le \frac{1}{k} + \int_{B_2} |v(r_k x) - u_0(r_k x)|\,\mathrm{d}x \to 0$$

as $k \to +\infty$. Let $w_k(t) = v_k(r_k t)$. Then $w_k \to u_0$ in $L^1(B_2)$; moreover, for every $x \in B_{r_k}$ we have, setting $y = r_k t$ and observing that $|\nabla w_k(t)| = r_k |\nabla v_k(r_k t)|$,

$$\begin{aligned}|\nabla v_k| * \rho_{\sigma_k}(x) &= \int_{C_{\sigma_k}(x)} |\nabla v_k(y)| \rho_{\sigma_k}(y - x)\,\mathrm{d}y = \frac{1}{\sigma_k^n} \int_{C_{\sigma_k}(x)} |\nabla v_k(y)| \rho\left(\frac{y - x}{\sigma_k}\right) \mathrm{d}y \\ &= \frac{r_k^{n-1}}{\sigma_k^n} \int_{C_{\sigma_k/r_k}(x/r_k)} |\nabla w_k(t)| \rho\left(\frac{t}{\sigma_k/r_k} - \frac{x}{\sigma_k}\right) \mathrm{d}t.\end{aligned}$$

Therefore, setting $x = r_k z$, we obtain

$$\begin{aligned}\frac{\mathcal{E}_{\sigma_k}(v_k, B_{r_k})}{r_k^{n-1}} &= \frac{1}{r_k^{n-1}\sigma_k} \int_{B_{r_k}} h(\sigma_k |\nabla v_k| * \rho_{\sigma_k}(x))\,\mathrm{d}x \\ &= \frac{1}{r_k^{n-1}\sigma_k^n} \int_{B_{r_k}} h\left(\frac{r_k^{n-1}}{\sigma_k^{n-1}} \int_{C_{\sigma_k/r_k}(x/r_k)} |\nabla w_k(t)| \rho\left(\frac{t}{\sigma_k/r_k} - \frac{x}{\sigma_k}\right) \mathrm{d}t\right) \mathrm{d}x \\ &= \frac{1}{\sigma_k/r_k} \int_{B_1} h\left(\frac{\sigma_k}{r_k} \frac{r_k^n}{\sigma_k^n} \int_{C_{\sigma_k/r_k}(z)} |\nabla w_k(t)| \rho\left(\frac{t - z}{\sigma_k/r_k}\right) \mathrm{d}t\right) \mathrm{d}z \\ &= \frac{1}{\sigma_k/r_k} \int_{B_1} h\left(\frac{\sigma_k}{r_k} |\nabla w_k| * \rho_{\sigma_k/r_k}(z)\right) \mathrm{d}z.\end{aligned}$$

Since $\sigma_k/r_k \to 0$, and $w_k \to u_0$ in $L^1(B_2)$, by the arbitrariness of (r_k) and the definition of $\mathcal{E}'$, we conclude combining (5.3) with the arbitrariness of $\delta \in (0,1)$. $\square$

Now we estimate from below $\mathcal{E}'(u_{x_0}^{\nu,a,b}, B_1(x_0))$. Without loss of generality, we can assume $x_0 = 0$ and $\nu = \mathbf{e}_1$; we will denote, for the sake of simplicity, by $u^{a,b}$ the function $u_0^{\mathbf{e}_1,a,b}$. In order to estimate from below $\mathcal{E}'(u^{a,b}, B_1)$ first we need to consider the problem on a suitable cylinder.

Recall that (see Sect. 3.1) $Q_{\mathbf{e}_1} = \{x \in \mathbb{R}^n : |x_1| < 1,\, P_{\mathbf{e}_1}^{\perp}(x) \in B_1 \cap \mathbf{e}_1^{\perp}\}$, being $P_{\mathbf{e}_1}^{\perp}(x)$ the orthogonal projection of x onto the subspace $\mathbf{e}_1^{\perp}$; for simplicity of notation we will use Q instead of $Q_{\mathbf{e}_1}$.

Lemma 5.5. *For any A open subset of Q there exist a positive infinitesimal sequence (ε_j) and a sequence u_j in $W^{1,1}(\Omega')$ converging to $u^{a,b}$ in $L^1(\Omega')$ such that*

$$\lim_{j\to+\infty} \mathcal{E}_{\varepsilon_j}(u_j, A) = \mathcal{E}'(u^{a,b}, A) \tag{5.6}$$

and such that

$$u_j(x) = a, \quad \text{if } x_1 \le -a_j \quad \text{and} \quad u_j(x) = b, \quad \text{if } x_1 \ge b_j \tag{5.7}$$

for some positive infinitesimal sequences (a_j) and (b_j).

Proof. We divide the proof in two steps.

Step 1. Fix $A \in \mathcal{A}(Q)$ with $A \subset\subset Q$, $\varepsilon, \sigma > 0$ sufficiently small. Let φ given by

$$\varphi(x) = \begin{cases} 0 & x_1 \le -2\varepsilon - \sigma \\ \text{affine} & -2\varepsilon - \sigma < x_1 < -2\varepsilon \\ 1 & x_1 \ge -2\varepsilon. \end{cases}$$

Obviously we have $|\nabla\varphi| \le \frac{1}{\sigma}$. Let

$$A_\varepsilon = \{x \in \mathbb{R}^n : x_1 < -2\varepsilon - k_1\varepsilon - \sigma\}, \quad B_\varepsilon = \{x \in \mathbb{R}^n : x_1 > -2\varepsilon + \varepsilon k_2\}$$

$$S_\varepsilon = \{x \in \mathbb{R}^n : -2\varepsilon - \varepsilon k_1 - \sigma < x_1 < -2\varepsilon + \varepsilon k_2\}$$

where $k_1 = \sup_{x\in C}\langle x, \mathbf{e}_1\rangle$ and $k_2 = -\inf_{x\in C}\langle x, \mathbf{e}_1\rangle$. Let $u_1, u_2 \in W^{1,1}(\Omega')$ and $v = \varphi u_1 + (1-\varphi)u_2$. Then

$$\mathcal{E}_\varepsilon(v, A) = \frac{1}{\varepsilon}\int_{A\cap A_\varepsilon} h(\varepsilon|\nabla u_2| * \rho_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{A\cap B_\varepsilon} h(\varepsilon|\nabla u_1| * \rho_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{A\cap S_\varepsilon} h(\varepsilon|\nabla v| * \rho_\varepsilon)\,\mathrm{d}x.$$

Taking into account the subadditivity of h we get

$$\begin{aligned}\frac{1}{\varepsilon}\int_{A\cap S_\varepsilon} h(\varepsilon|\nabla v| * \rho_\varepsilon)\,\mathrm{d}x \le{}& \frac{1}{\varepsilon}\int_{A\cap S_\varepsilon} h(\varepsilon(\varphi|\nabla u_1|) * \rho_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{A\cap S_\varepsilon} h(\varepsilon((1-\varphi)|\nabla u_2|) * \rho_\varepsilon)\,\mathrm{d}x \\ &+ \frac{1}{\varepsilon}\int_{A\cap S_\varepsilon} h(\varepsilon(|\nabla\varphi||u_1 - u_2|) * \rho_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

Then

$$\mathcal{E}_\varepsilon(v, A) \le \mathcal{E}_\varepsilon(u_1, A \cap (B_\varepsilon \cup S_\varepsilon)) + \mathcal{E}_\varepsilon(u_2, A \cap (A_\varepsilon \cup S_\varepsilon)) + \frac{c'}{\sigma}\int_{A\cap S_\varepsilon} |u_1 - u_2| * \rho_\varepsilon\,\mathrm{d}x$$

where we have used $h(t) \le c't$ for each $t \ge 0$.

Step 2. Now let (ε_j) be a positive infinitesimal sequence and let (v_j) be a sequence in $W^{1,1}(\Omega')$ such that $v_j \to u^{a,b}$ in $L^1(\Omega')$ and

$$\lim_{j\to+\infty} \mathcal{E}_{\varepsilon_j}(v_j, A) = \mathcal{E}'(u^{a,b}, A).$$

Choosing $u_1 = v_j$ and $u_2 = a$ we have, since $\mathcal{E}_{\varepsilon_j}(u_2, A) = 0$,

$$\mathcal{E}_{\varepsilon_j}(\varphi v_j + (1-\varphi)u_2, A) \le \mathcal{E}_{\varepsilon_j}(v_j, A) + \frac{c'}{\sigma}\int_{\{x_1<0\}} |v_j - u_2| * \rho_{\varepsilon_j}\,\mathrm{d}x.$$

By standard properties of the convolution,

$$\int_{\{x_1<0\}} |v_j - u_2| * \rho_{\varepsilon_j}\,\mathrm{d}x \le ||v_j - u_2||_{L^1(\{x_1<0\})} \to 0$$

as $j \to +\infty$. Therefore, by a diagonal argument, if $\sigma_h \to 0$ we can find $j_h \to +\infty$ be such that

$$\lim_{h\to+\infty} \frac{1}{\sigma_h}\int_{\{x_1<0\}} |v_{j_h} - u_2| * \rho_{\varepsilon_{j_h}}\,\mathrm{d}x = 0.$$

Thus

$$\limsup_{h\to+\infty} \mathcal{E}_{\varepsilon_{j_j}}(\varphi v_{j_h} + (1-\varphi)u_2, A) \le \limsup_{h\to+\infty} \mathcal{E}_{\varepsilon_{j_h}}(v_{j_h}, A) = \mathcal{E}'(u^{a,b}, A).$$

Setting

$$u_{j_h} = \begin{cases} a & x_1 \le -2\varepsilon_{j_h} - \sigma_h \\ v_{j_h} & x_1 \ge 0 \end{cases}$$

we easily have $u_{j_h} \to u^{a,b}$ in $L^1(\Omega')$ and $u_{j_h} = a$ for $x_1 \le -a_j$ for a suitable positive infinitesimal sequence (a_j). With the same argument one can prove that $u_{j_h} = b$ for $x_1 \ge b_j$ for another suitable positive infinitesimal sequence (b_j). Thus (u_{j_h}) is optimal and (5.7) hold. □

Proposition 5.6. *We have* $\mathcal{E}'(u^{a,b}, B_1) \ge \mathcal{E}'(u^{a,b}, Q)$.

Proof. Fix $\delta \in (0,1)$. Let (u_j) be given by the previous Lemma, applied with $A = B_1$. Then $u_j(x) = a$ if $x_1 \le -a_j$, and $u_j(x) = b$ if $x_1 \ge b_j$, where (a_j) and (b_j) are suitable positive infinitesimal sequences. Let $S_j = (-a_j, b_j)\times\mathbb{R}^{n-1}$. For j sufficiently large, we have $\delta Q \cap S_j \subset\subset B_1$, from which $\mathcal{E}_{\varepsilon_j}\big(u_j, \delta Q \cap B_1\big) = \mathcal{E}_{\varepsilon_j}\big(u_j, \delta Q\big)$. Then

$$\mathcal{E}_{\varepsilon_j}(u_j, B_1) \ge \mathcal{E}_{\varepsilon_j}(u_j, B_1 \cap \delta Q) = \mathcal{E}_{\varepsilon_j}(u_j, \delta Q). \tag{5.8}$$

Let $v_j(x) = u_j(\delta x)$. Then by a simple scaling argument we have $\mathcal{E}_{\varepsilon_j}(u_j, \delta Q) = \delta^{n-1}\mathcal{E}_{\varepsilon_j/\delta}(v_j, Q)$. Passing to the limit in (5.8) we get

$$\mathcal{E}'(u^{a,b}, B_1) \ge \delta^{n-1} \liminf_{j\to+\infty} \mathcal{E}_{\varepsilon_j/\delta}(v_j, Q) \ge \delta^{n-1}\mathcal{E}'(u^{a,b}, Q).$$

We conclude by taking the limit as $\delta \to 1^-$. □

Now, by an application of the Besicovitch's Differentiation Theorem, we are able to prove the correct estimate from below for the lower Γ-limit of $\mathcal{E}_{\varepsilon_j}$. In order to apply such a Theorem, let us consider the set function $\mathcal{E}'_-(u, \cdot)$. It is well known that an increasing set function $\alpha\colon \mathcal{A}(\Omega) \to [0, +\infty]$ which satisfies $\alpha(\emptyset) = 0$, which is subadditive, superadditive and inner regular, can be extended to a Borel measure on Ω (for instance see [18], Thm. 14.23). This result can be applied to $\mathcal{E}'_-(u, \cdot)$, the subadditivity of $\mathcal{E}'_-(u, \cdot)$ being the only condition which is not easy to prove, but it can be recovered as in the proof of Proposition 4.3 and Theorem 4.6 of [13]; these results are established in the case $p > 1$, but the same arguments work if $p = 1$.

Denote by μ_u the Borel measure on Ω which extends $\mathcal{E}'_-(u, \cdot)$.

Lemma 5.7. *Let* $u \in BV(\Omega)$*. Then* μ_u *is a finite measure.*

Proof. Let (u_h) be a sequence in $L^1(\Omega)$ converging weakly* converging to u in $BV(\Omega)$. By definition

$$|Du_h| * \rho_\varepsilon(x) = \int_{C_\varepsilon(x)\cap\Omega} \rho_\varepsilon(x-y)\,\mathrm{d}|Du_h|(y).$$

Since $Du_h \overset{*}{\rightharpoonup} Du$ as measures, by Fatou's lemma and taking into account that f is non-decreasing and continuous, we get

$$\liminf_{h\to+\infty} \frac{1}{\varepsilon}\int_\Omega h\big(\varepsilon|Du_h| * \rho_\varepsilon\big)\,\mathrm{d}x \geq \frac{1}{\varepsilon}\int_\Omega h\big(\varepsilon \liminf_{h\to+\infty}|Du_h| * \rho_\varepsilon\big)\,\mathrm{d}x \geq \frac{1}{\varepsilon}\int_\Omega h(\varepsilon|Du| * \rho_\varepsilon)\,\mathrm{d}x. \tag{5.9}$$

Now let $u \in BV(\Omega)$ and let (u_h) be a sequence in $L^1(\Omega)$ strictly converging to u. In particular, $|Du_h| \to |Du|$ weakly* as measures (see, for instance, Prop. 3.15 in [5]). Note that that $D^c u$ vanishes on the sets with finite $\mathcal{H}^{n-1}$ measure. Moreover, if S is σ-finite with respect to $\mathcal{H}^{n-1}$, then $\{x \in \Omega : \mathcal{H}^{n-1}(S \cap \partial C_\varepsilon(x)) > 0\}$ is at most countable. Then (see, for instance, Prop. 1.62 in [5]), we have

$$\lim_{h\to+\infty} |Du_h| * \rho_\varepsilon(x) = |Du| * \rho_\varepsilon(x), \quad \text{a.e. } x \in \Omega.$$

Applying the dominated convergence theorem, we obtain

$$\lim_{h\to+\infty} \frac{1}{\varepsilon}\int_\Omega h\left(\varepsilon|Du_h| * \rho_\varepsilon\right)\,\mathrm{d}x = \frac{1}{\varepsilon}\int_\Omega h\left(\varepsilon|Du| * \rho_\varepsilon\right)\,\mathrm{d}x. \tag{5.10}$$

Combining (5.9) with (5.10) and taking into account that $\mathcal{E}'_-$ is lower semicontinuous, we have

$$\mathcal{E}'_-(u) \leq \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_\Omega h\left(\varepsilon|Du| * \rho_\varepsilon\right)\,\mathrm{d}x.$$

Notice that there exists $\gamma > 0$ such that $|C_\varepsilon(x) \cap \Omega| \leq \gamma\varepsilon^n$ for any $x \in \Omega$. Denoting by $M = \sup_C \rho$ and taking into Fubini's Theorem, we get that for sufficiently small ε,

$$\begin{aligned}\int_\Omega h\left(\varepsilon|Du| * \rho_\varepsilon\right)\,\mathrm{d}x &\leq c'\int_\Omega\int_{C_\varepsilon(x)\cap\Omega} \rho_\varepsilon(y-x)\,\mathrm{d}|Du|(y)\,\mathrm{d}x = c'\int_\Omega\int_\Omega \rho_\varepsilon(y-x)\chi_{C_\varepsilon(x)}\,\mathrm{d}x\,\mathrm{d}|Du|(y)\\ &\leq c'M\int_\Omega\int_\Omega \frac{|C_\varepsilon(x)\cap\Omega|}{\varepsilon^n}\,\mathrm{d}|Du|(y) \leq c'M\gamma|Du|(\Omega)\end{aligned}$$

and this yields the conclusion. □

Proposition 5.8. *Let $u \in BV(\Omega)$ and $A \in \mathcal{A}(\Omega)$. Then*

$$\mathcal{E}'(u, A) \geq \int_{S_u\cap A} \psi(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1},$$

where

$$\psi(s,\nu) = \frac{1}{\omega_{n-1}} \inf\left\{\liminf_{j\to+\infty} \frac{1}{\varepsilon_j}\int_{Q_\nu} h(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x : (u_j) \in W^{0,s}_\nu,\ \varepsilon_j \to 0^+\right\}.$$

Proof. For every $k \in \mathbb{N}$ let $S_k = \{x \in S_u : \ |u^+(x) - u^-(x)| > 1/k\}$. Clearly we have $\mathcal{H}^{n-1}(S_k) < +\infty$; let $\lambda_k = \mathcal{H}^{n-1} \llcorner S_k$. Applying the Besicovitch's differentiation theorem we deduce that the limit

$$g(x) = \lim_{\varrho\to 0} \frac{\mu_u(B_\varrho(x))}{\lambda_k(B_\varrho(x))}$$

exists and is finite for λ_k-a.e. $x \in \Omega$, and is λ_k-measurable. Moreover, the Radon–Nikodym decomposition of μ_u is given by $\mu_u = g\lambda_k + \mu^s$, with $\mu^s \perp \lambda_k$. By rectifiability for $\mathcal{H}^{n-1}$-a.e. $x \in S_k$ we get

$$\lim_{\varrho \to 0} \frac{\lambda_k(B_\varrho(x))}{\omega_{n-1}\varrho^{n-1}} = 1.$$

Thus, for $\mathcal{H}^{n-1}$-a.e. $x_0 \in S_k$ we have, applying Propositions 5.4, 5.6 and taking into account (5.7),

$$\begin{aligned} g(x_0) &= \lim_{\varrho\to 0} \frac{\mu_u(B_\varrho(x_0))}{\omega_{n-1}\varrho^{n-1}} = \liminf_{\varrho\to 0} \frac{\mathcal{E}'_-(u, B_\varrho(x_0))}{\omega_{n-1}\varrho^{n-1}} \\ &\geq \frac{1}{\omega_{n-1}} \inf\left\{ \liminf_{j\to+\infty} \frac{1}{\varepsilon_j}\int_{x_0+Q_\nu} h(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x : (u_j(\cdot - x_0)) \in W^{u^+(x_0),u^-(x_0)}_{\nu_u(x_0)},\ \varepsilon_j \to 0^+ \right\}. \end{aligned}$$

Taking into account (3.8) and (3.9) (which obviously hold for h instead of f) we get

$$\begin{aligned} &\inf\left\{ \liminf_{j\to+\infty} \frac{1}{\varepsilon_j}\int_{x_0+Q_\nu} h(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x : (u_j(\cdot - x_0)) \in W^{u^+(x_0),u^-(x_0)}_{\nu_u(x_0)},\ \varepsilon_j \to 0^+ \right\} \\ &\quad = \psi(|u^+(x_0) - u^-(x_0)|, \nu_u(x_0)). \end{aligned}$$

Since μ^s is non-negative, we deduce that

$$\mathcal{E}'_-(u, A) \geq \int_A \psi(|u^+ - u^-|, \nu_u)\,\mathrm{d}\lambda_k = \int_{S_k\cap A} \psi(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1}.$$

By considering the supremum for $k \in \mathbb{N}$ we easily obtain

$$\mathcal{E}'_-(u, A) \geq \int_{S_u\cap A} \psi(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$$

and the conclusion follows by definition of $\mathcal{E}'_-$. □

5.3. Proof of the Γ-liminf inequality

We are ready to prove the Γ-liminf inequality for the family $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$. The main step of the proof consists in combining Proposition 5.3 with Proposition 5.8 and then using a supremum of measures argument.

Lemma 5.9. *Let μ be as in Lemma 2.5. Let λ_1, λ_2 be mutually singular Borel measures, and ψ_1, ψ_2 positive Borel functions. Assume that*

$$\mu(A) \geq \int_A \psi_i\,\mathrm{d}\lambda_i$$

for every $A \in \mathcal{A}(\Omega)$ and $i = 1, 2$. Then it holds

$$\mu(A) \geq \int_A \psi_1\,\mathrm{d}\lambda_1 + \int_A \psi_2\,\mathrm{d}\lambda_2$$

for every $A \in \mathcal{A}(\Omega)$.

Proof. Let $E \subseteq \Omega$ be such that $\lambda_1(\Omega \setminus E) = 0$ and $\lambda_2(E) = 0$. Then we can suppose that $\psi_1 = 0$ on $\Omega \setminus E$ and $\psi_2 = 0$ on E. Then $\max\{\psi_1, \psi_2\} = \psi_1 + \psi_2$. We conclude by applying the lemma 2.5 with the choice $\lambda = \lambda_1 + \lambda_2$. □

Proposition 5.10. *Let $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$. Then*

$$\mathcal{F}'(u, A) \geq \int_A \phi(|\nabla u|)\,\mathrm{d}x + \int_{S_u\cap A} \theta(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + c_0|D^c u|(A).$$

Proof. First notice that we can suppose $u \in GBV(\Omega)$. Indeed, if $(\mathcal{F}_{\varepsilon_j}(u_j))$ is bounded and $u_j \to u$ in $L^1(\Omega)$ then $u \in GBV(\Omega)$: it suffices to apply Theorem 3.6 to $u_j^T = -T \vee u_j \wedge T$, hence we get $u^T \in BV(\Omega)$ which means $u \in GBV(\Omega)$.

Now the key point of the proof is the construction of a suitable family of functions below f_{ε_j}.

Step 1. Let $\delta \in (0,1)$. We claim that there exists $t_\delta > 0$ and for any $h \in \mathbb{N}$ and for any $\varepsilon > 0$ there exist $c_h^\delta > 0$, $d_h^\delta < 0$ and $g_h^\delta \colon [t_\delta, +\infty) \to \mathbb{R}$ such that if we let

$$f_\varepsilon^{h,\delta}(t) = \begin{cases} c_h^\delta t + \varepsilon d_h^\delta & \text{if } t \in [0, t_\delta] \\ c_h^\delta t_\delta + \varepsilon d_h^\delta + g_h^\delta(t) & \text{if } t > t_\delta \end{cases}$$

we have:

$$\sup_h (c_h^\delta t + d_h^\delta) = (1-\delta)\phi(t), \quad \forall t \geq 0 \tag{5.11}$$

$$f_\varepsilon(t) \geq f_\varepsilon^{h,\delta}(t),\ \forall t \geq 0,\ \forall h \in \mathbb{N}, \text{ for } \varepsilon \text{ sufficiently small,} \tag{5.12}$$

$$f_\varepsilon^{h,\delta} \text{ is continuous, non-decreasing and concave for any } \varepsilon > 0 \text{ and any } h \in \mathbb{N}, \tag{5.13}$$

$$f_\varepsilon^{h,\delta} - \varepsilon d_h^\delta \text{ converges to } (1-\delta)f \text{ uniformly on compact sets of } [0,+\infty) \text{ as } h \to +\infty. \tag{5.14}$$

First of all we point out that

$$\lim_{t \to 0} \frac{f(t)}{t} = c_0. \tag{5.15}$$

Indeed, by A2 for any $\sigma \in (0,1)$ there exist $t_\sigma, \varepsilon_\sigma > 0$ such that $f_\varepsilon(t) \leq (1+\sigma)\varepsilon\phi(t/\varepsilon)$ for each $t \in [0, t_\sigma]$ and for each $\varepsilon \in (0, \varepsilon_\sigma]$. Since $\phi(s) \leq c_0 s$ for any $s \geq 0$, we have $f_\varepsilon(t)/t \leq (1+\sigma)c_0$. By A3 the previous inequality reduces to $f(t)/t \leq (1+\sigma)c_0$. On the other hand there exist $t'_\sigma, \varepsilon'_\sigma > 0$ such that $f_\varepsilon(t) \geq (1-\sigma)\varepsilon\phi(t/\varepsilon)$ for each $t \in [0, t'_\sigma]$ and for each $\varepsilon \in (0, \varepsilon'_\sigma]$. Since $\phi(s) \geq c_0 s - q$, for a suitable $q > 0$, we have $f_\varepsilon(t)/t \geq (1-\sigma)(c_0 t - \varepsilon q)$. We thus get $f(t)/t \geq (1-\sigma)c_0$. By the arbitrariness of $\sigma > 0$ we have (5.15).

Formula (5.15) is useful in order to construct the family $(f_\varepsilon^{h,\delta})$ as follows. By A2 there exists $t_\delta > 0$ such that $f_\varepsilon(t) \geq (1-\delta)\varepsilon\phi(t/\varepsilon)$ for each $t \in [0, t_\delta]$ and for each ε sufficiently small. Fix $h \in \mathbb{N}$ with $h > 0$ and let $(\ell_h)_{h \in \mathbb{N}}$ be a family of affine functions such that $\sup_h \ell_h(t) = \phi(t)$ for any $t \geq 0$ (recall that ϕ is convex); we let $\ell_h(t) = c_h t + d_h$. Let $c_h^\delta = (1-\delta)c_h$ and $d_h^\delta = (1-\delta)d_h$. Then (5.11) holds and we obtain $f_\varepsilon(t) \geq c_h^\delta t + \varepsilon d_h^\delta$ for all $t \in [0, t_\delta]$. Now it is easy to conclude the construction of $f_\varepsilon^{h,\delta}$ in such a way (5.12)–(5.14) hold: for instance connecting the graphic of the affine piece with a suitable rotation and truncation of the graph of f (see also (5.15)).

Step 2. Let $\delta \in (0,1)$ and let $(f_{\varepsilon_j}^{h,\delta})$ be the family constructed in Step 1. Let $\psi_h^\delta = f_{\varepsilon_j}^{h,\delta} - \varepsilon_j d_h^\delta$. Then we get

$$\mathcal{F}_{\varepsilon_j}(u, A) \geq \frac{1}{\varepsilon_j} \int_A \psi_h^\delta \left(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)\right) \, \mathrm{d}x + d_h^\delta |A| \tag{5.16}$$

for any $u \in W^{1,1}(\Omega)$ and $A \in \mathcal{A}(\Omega)$. Let A', A'' be open disjoint subsets of A such that $|A''| < \delta$, $S_u \subset A''$. Therefore,

$$\mathcal{F}_{\varepsilon_j}(u, A) \geq \frac{1}{\varepsilon_j} \int_{A'} \psi_h^\delta \left(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)\right) \, \mathrm{d}x + \frac{1}{\varepsilon_j} \int_{A''} \psi_h^\delta \left(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)\right) \, \mathrm{d}x + d_h^\delta |A'| + \delta d_h^\delta. \tag{5.17}$$

In particular we get

$$\mathcal{F}_{\varepsilon_j}(u, A) \geq \frac{1}{\varepsilon_j} \int_{A'} \psi_h^\delta \left(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)\right) \, \mathrm{d}x + d_h^\delta |A'|.$$

Notice that ψ_h^δ is linear in $[0,t_\delta]$. Applying Proposition 5.3 with the choice $g=\psi_h^\delta\wedge\psi_h^\delta(t_\delta)$ we obtain

$$\mathcal{F}'(u,A)\geq c_h^\delta\int_{A'}|\nabla u|\,\mathrm{d}x+c_h^\delta|D^cu|(A)+d_h^\delta|A'|=(1-\delta)\int_{A'}\ell_h(|\nabla u|)\,\mathrm{d}x+(1-\delta)c_h|D^cu|(A').$$

Since $\mathcal{F}'(u,\cdot)$ is a superadditive function on open sets of Ω with disjoint compact closures, by applying Lemma 2.5 and (5.11) we get, by the arbitrariness of A' and δ,

$$\mathcal{F}'(u,A)\geq\int_A\phi(|\nabla u|)\mathrm{d}x+c_0|D^cu|(A). \tag{5.18}$$

Now (5.17) implies also

$$\mathcal{F}_{\varepsilon_j}(u,A)\geq\frac{1}{\varepsilon_j}\int_{A''}\psi_h^\delta\left(\varepsilon_j|\nabla u|*\rho_{\varepsilon_j}(x)\right)\,\mathrm{d}x.$$

Applying now Proposition 5.8 with the choice $h=\psi_h^\delta$ we deduce that

$$\mathcal{F}'(u,A)\geq\int_{S_u\cap A''}\theta_h^\delta(|u^+-u^-|,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1},$$

being

$$\theta_h^\delta(s,\nu)=\frac{1}{\omega_{n-1}}\inf\left\{\liminf_{j\to+\infty}\frac{1}{\varepsilon_j}\int_{Q_\nu}\psi_h^\delta(\varepsilon_j|\nabla u_j|*\rho_{\varepsilon_j})\,\mathrm{d}x:(u_j)\in W_\nu^{0,s},\,\varepsilon_j\to0^+\right\}.$$

Using (5.14) and the arbitrariness of δ, it follows that $\theta_h^\delta\to\theta$ as $h\to+\infty$ and $\delta\to0$. Applying once more Lemma 2.5, by the arbitrariness of A'', we have

$$\mathcal{F}'(u,A)\geq\int_{S_u\cap A}\theta(|u^+-u^-|,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{5.19}$$

Applying Lemma 5.9 choosing $\lambda_1=\mathcal{L}^n$, $\lambda_2=\mathcal{H}^{n-1}\llcorner J_u$, $\lambda_3=|D^cu|$ and taking into account (5.18) and (5.19), we immediately obtain $\mathcal{F}'(u)\geq\mathcal{F}(u)$ for any $u\in BV(\Omega)$.

Let us now consider the case $u\in GBV(\Omega)$. Let (u_j) be a sequence in $W^{1,1}(\Omega)$ converging to u in $L^1(\Omega)$ and such that

$$\lim_{j\to+\infty}\mathcal{F}_{\varepsilon_j}(u_j)=\mathcal{F}'(u).$$

Define $u_j^T=(-T)\vee u_j\wedge T$, and $u^T=(-T)\vee u\wedge T$. Since $u_j^T\to u^T$ in $L^1(\Omega)$, and $u^T\in BV(\Omega)$, we have

$$\mathcal{F}'(u)=\liminf_{j\to+\infty}\mathcal{F}_{\varepsilon_j}(u_j)\geq\liminf_{j\to+\infty}\mathcal{F}_{\varepsilon_j}(u_j^T)\geq\mathcal{F}(u^T).$$

Applying (2.2)–(2.4) and taking into account the continuity of θ we obtain

$$\lim_{T\to+\infty}\left(\int_\Omega\phi(|\nabla u^T|)\,\mathrm{d}x+\int_{S_{u^T}}\theta\big(|(u^T)^+-(u^T)^-|,\nu_{u^T}\big)\,\mathrm{d}\mathcal{H}^{n-1}+c_0|D^cu^T|(\Omega)\right)=\mathcal{F}(u)$$

so we are done. □

6. The Γ-limsup inequality

In this section we will prove that $\mathcal{F}''(u)\leq\mathcal{F}(u)$ for any $u\in L^1(\Omega)$; since, by definition, $\mathcal{F}(u)=+\infty$ for any $u\in L^1(\Omega)\setminus GBV(\Omega)$, it is sufficient to consider the case $u\in GBV(\Omega)$.

Lemma 6.1. *Let (ε_j) be a positive infinitesimal sequence, $\nu \in \mathbb{S}^{n-1}$ and $s \geq 0$. Let $(u_j) \in W^{0,s}_\nu$ be such that*

$$\omega_{n-1}\theta(s,\nu) = \lim_{j\to+\infty} \frac{1}{\varepsilon_j}\int_{Q_\nu} f(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x.$$

Then for any $r > 0$ there exists a positive infinitesimal sequence σ_j and $(v_j) \in W^{0,s}_\nu$ such that for any $\sigma > 0$ it holds

$$\omega_{n-1}r^{n-1}\theta(s,\nu) = \lim_{j\to+\infty} \frac{1}{\sigma_j}\int_{rQ^\sigma_\nu} f(\sigma_j|\nabla v_j| * \rho_{\sigma_j})\,\mathrm{d}x,$$

where $Q^\sigma_\nu = \{x \in Q_\nu : |\langle x,\nu\rangle| < \sigma\}$.

Proof. Let $\sigma_j = r\varepsilon_j$ and $v_j(x) = u_j(rx)$. Then by the change of variables $x = rz$ and $y = rt$ we get

$$\begin{aligned}\frac{1}{\sigma_j}\int_{rQ_\nu} f(\sigma_j|\nabla v_j| * \rho_{\sigma_j})\,\mathrm{d}x &= \frac{r^n}{\sigma_j}\int_{Q_\nu} f\left(\frac{\sigma_j}{r}\int_{C_{\sigma_j/r}} |\nabla v_j(rz-rt)|\rho_{\sigma_j/r}(t)\,\mathrm{d}t\right)\mathrm{d}z\\ &= \frac{r^{n-1}}{\varepsilon_j}\int_{Q_\nu} f\left(\varepsilon_j\int_{C_{\varepsilon_j}} |\nabla u_j(z-t)|\rho_{\varepsilon_j}(t)\,\mathrm{d}t\right)\mathrm{d}z.\end{aligned}$$

Passing to the limit as $j \to +\infty$ we get

$$\lim_{j\to+\infty} \frac{1}{\sigma_j}\int_{rQ_\nu} f(\sigma_j|\nabla v_j| * \rho_{\sigma_j})\,\mathrm{d}x = r^{n-1}\theta(s,\nu).$$

Since the transition set of the optimal sequence (u_j) shrinks onto the interface (see (5.7) or the definition of $W^{0,s}_\nu$) we deduce that

$$\lim_{j\to+\infty} \frac{1}{\sigma_j}\int_{rQ_\nu} f(\sigma_j|\nabla v_j| * \rho_{\sigma_j})\,\mathrm{d}x = \lim_{j\to+\infty} \frac{1}{\sigma_j}\int_{rQ^\sigma_\nu} f(\sigma_j|\nabla v_j| * \rho_{\sigma_j})\,\mathrm{d}x$$

for any $\sigma > 0$, hence we conclude. □

Proposition 6.2. *For any $u \in \mathcal{W}(\Omega)$ it holds $\mathcal{F}''(u) \leq \mathcal{F}(u)$.*

Proof. By the very definition of $\mathcal{W}(\Omega)$ (see Sect. 2.5) the set S_u is contained in the union of a finite collection $K_1,\ldots,K_m$ of $(n-1)$-dimensional simplexes; it will not be restrictive to assume $m = 1$ and $K = K_1 \subseteq \{x \in \mathbb{R}^n :\ x_1 = 0\}$. Fix $h \in \mathbb{N}$, $h \geq 1$. Let $\Omega_h = \{x \in \Omega \setminus K : d(x,K) > 1/h\}$. Let S be the relative boundary of K; obviously it holds $\mathcal{H}^{n-1}(S) = 0$. Let $K_h = \{x \in K : d(x,S) > 1/h\}$. Let $k \in \mathbb{N}$, $k \geq 1$, $x_1,\ldots,x_k \in K_h$ and $r \geq 0$ be such that $B_r(x_i)$ are pairwise disjoint, $B_r(x_i) \cap \{x_1 = 0\} \subseteq K_h$ for any $i = 1,\ldots,k$ and

$$\mathcal{H}^{n-1}\left(K_h \setminus \left(\bigcup_{i=1}^k B_r(x_i) \cap \{x_1 = 0\}\right)\right) < \frac{1}{h}. \tag{6.1}$$

Let $Q_h = \{x \in rQ_{\mathbf{e}_1} : |x_1| < 1/h\}$ and $Q_h(x) = x + Q_h$ for any $x \in \mathbb{R}^n$. Moreover, let $Q^+_h = Q_h \cap \{x_1 > 0\}$ and $Q^-_h = Q_h \cap \{x_1 < 0\}$. At this point we divide the proof in two steps.

Step 1. Take a function $v \in \mathcal{W}(\Omega)$ with $S_v \subseteq K$ and such that v is constant in any $x_i + Q^+_h$ and in any $x_i + Q^-_h$. Denote by v^+_i the value of v in $x_i + Q^+_h$ and by v^-_i the value of v in $x_i + Q^-_h$. We claim that

$$\mathcal{F}''(v) \leq \int_\Omega \phi(|\nabla v|)\,\mathrm{d}x + \sum_{i=1}^k \int_{K\cap B_r(x_i)} \theta(|v^+_i - v^-_i|,\mathbf{e}_1)\,\mathrm{d}\mathcal{H}^{n-1} + c|Dv|(\Omega'_h), \tag{6.2}$$

for some $c > 0$, where

$$\Omega'_h = \Omega \setminus \left(\Omega_h \cup \bigcup_{i=1}^{k} (x_i + Q_h) \right).$$

Let (ε_j) be a positive infinitesimal sequence and let $\delta \in (0,1)$. Accordingly to Lemma 6.1, let us define $v_j \in \mathcal{W}(\Omega)$ be such that we have

$$\lim_{j \to +\infty} \mathcal{F}_{\sigma_j}(v_j, x_i + \delta Q_h) = (\delta r)^{n-1} \theta(|v_i^+ - v_i^-|, \mathbf{e}_1), \tag{6.3}$$

where $\sigma_j = r\varepsilon_j$. Otherwise in Ω we set $v_j = v$. Then, using the same argument as in the proof of Lemma 5.7, we deduce that

$$\frac{1}{\sigma_j} \int_\Omega f_{\sigma_j}(\sigma_j |\nabla v_j| * \rho_{\sigma_j}) \, \mathrm{d}x \le \mathcal{F}_{\sigma_j}(v, \Omega_h) + \sum_{i=1}^{k} \mathcal{F}_{\sigma_j}(v_j, x_i + \delta Q_h) + c|Dv|(\Omega'_{h,\delta}), \tag{6.4}$$

being

$$\Omega'_{h,\delta} = \Omega \setminus \left(\Omega_h \cup \bigcup_{i=1}^{k} (x_i + \delta Q_h) \right).$$

The first term on the right-hand side of (6.4) is given by

$$\frac{1}{\sigma_j} \int_{\Omega_h} f_{\sigma_j}(\sigma_j |\nabla v| * \rho_{\sigma_j}) \, \mathrm{d}x.$$

By standard properties of the convolution we have $|\nabla v| * \rho_{\sigma_j} \to |\nabla v|$ in $L^1(\Omega)$ and a.e. in Ω. From A2 we deduce that

$$\lim_{\varepsilon \to 0} \frac{f_\varepsilon(\varepsilon t_\varepsilon)}{\varepsilon} = \phi(t) \tag{6.5}$$

whenever $t_\varepsilon \to t$, for each $t \ge 0$. By the dominated convergence theorem we get

$$\lim_{j \to +\infty} \frac{1}{\sigma_j} \int_{\Omega_h} f_{\sigma_j}(\sigma_j |\nabla v| * \rho_{\sigma_j}) \, \mathrm{d}x = \int_{\Omega_h} \phi(|\nabla v|) \, \mathrm{d}x \le \int_\Omega \phi(|\nabla v|) \, \mathrm{d}x.$$

Passing to the limsup in (6.4), using (6.3) and using the arbitrariness of $\delta \in (0,1)$ we get (6.2).

Step 2. For any $i = 1, \ldots, k$ let

$$u_i^+ = \fint_{B_r(x_i) \cap K} u^+ \, \mathrm{d}\mathcal{H}^{n-1}, \quad u_i^- = \fint_{B_r(x_i) \cap K} u^- \, \mathrm{d}\mathcal{H}^{n-1}$$

and

$$u_i(x) = \begin{cases} u_i^+ & \text{if } (x_i)_1 - x_1 > 0 \\ u_i^- & \text{if } (x_i)_1 - x_1 \le 0, \end{cases} \qquad x \in B_r(x_i).$$

For any $h \in \mathbb{N}$, $h \ge 1$, let $u_h = u_i$ on $Q_h(x_i)$ and $u_h = u$ otherwise in Ω. Applying Step 1 with the choice $v = u_h$ we get

$$\mathcal{F}''(u_h) \le \int_\Omega \phi(|\nabla u|) \, \mathrm{d}x + \sum_{i=1}^{k} \int_{K \cap B_r(x_i)} \theta(|u_i^+ - u_i^-|, \mathbf{e}_1) \, \mathrm{d}\mathcal{H}^{n-1} + c|Du|(\Omega'_h).$$

Now $|\Omega'_h| \to 0$. Furthermore, taking into account (6.1) we deduce that $\mathcal{H}^{n-1}(S_u \cap \Omega'_h) \to 0$ as $h, k \to +\infty$. Hence $|Du|(\Omega'_h) \to 0$ as $h, k \to +\infty$. Exploiting the uniform continuity of the traces of u and the continuity of θ, we also get

$$\sum_{i=1}^{k} \int_{K \cap B_r(x_i)} \theta(|u_i^+ - u_i^-|, \mathbf{e}_1) \, \mathrm{d}\mathcal{H}^{n-1} \overset{h,k\to+\infty}{\longrightarrow} \int_{S_u} \theta(|u^+ - u^-|, \mathbf{e}_1) \, \mathrm{d}\mathcal{H}^{n-1}$$

and the lower semicontinuity of $\mathcal{F}''$ yields the conclusion. □

Proposition 6.3. *Let $u \in GBV(\Omega)$. Then it holds $\mathcal{F}''(u) \leq \mathcal{F}(u)$.*

Proof. First let $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$. We can apply Theorem 2.6, choosing

$$\psi(a, b, \nu) = \theta(|a - b|, \nu)$$

(see (3.6) and (3.7)). Then there exists a sequence $w_j \to u$ in $L^1(\Omega)$, with $w_j \in \mathcal{W}(\Omega)$, such that $\nabla w_j \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^n)$ and

$$\limsup_{j\to+\infty} \int_{S_{w_j}} \theta(|w_j^+ - w_j^-|, \nu_{w_j}) \, \mathrm{d}\mathcal{H}^{n-1} \leq \int_{S_u} \theta(|u^+ - u^-|, \nu_u) \, \mathrm{d}\mathcal{H}^{n-1}. \tag{6.6}$$

By the lower semicontinuity of $\mathcal{F}''$ and by Proposition 6.2 we deduce that, applying the dominated convergence theorem and (6.6),

$$\mathcal{F}''(u) \leq \liminf_{j\to+\infty} \mathcal{F}''(w_j) \leq \int_\Omega \phi(|\nabla u|) \, \mathrm{d}x + \int_{S_u} \theta(|u^+ - u^-|, \nu_u) \, \mathrm{d}\mathcal{H}^{n-1}.$$

Using relaxation Theorem 2.7 we get

$$\mathcal{F}''(u) \leq \int_\Omega \phi(|\nabla u|) \, \mathrm{d}x + \int_{J_u} \theta(|u^+ - u^-|, \nu_u) \, \mathrm{d}\mathcal{H}^{n-1} + c_0 |D^c u|(\Omega)$$

for each $u \in BV(\Omega)$. Finally, let $u \in GBV(\Omega)$ and, for any $T > 0$, $u^T = -T \vee u \wedge T$. Then $u^T \in BV(\Omega)$ for each $T > 0$ and $u^T \to u$ in $L^1(\Omega)$ as $T \to +\infty$. Taking into account (2.2)–(2.4) we obtain, exploiting again the lower semicontinuity of $\mathcal{F}''$ and the continuity of θ,

$$\begin{aligned}\mathcal{F}''(u) &\leq \limsup_{T\to+\infty} \left(\int_\Omega \phi(|\nabla u^T|) \, \mathrm{d}x + \int_{S_{u^T}} \theta(|(u^T)^+ - (u^T)^-|, \nu_{u^T}) \, \mathrm{d}\mathcal{H}^{n-1} + c_0 |D^c u^T|(\Omega) \right) \\ &= \int_\Omega \phi(|\nabla u|) \, \mathrm{d}x + \int_{S_u} \theta(|u^+ - u^-|, \nu_u) \, \mathrm{d}\mathcal{H}^{n-1} + c_0 |D^c u|(\Omega)\end{aligned}$$

which is what we wanted to prove. □

7. Computation of θ in the one-dimensional case

In this section we are able to give an explicit formula for θ if $n = 1$ along the same line of the discretization argument used in [22].

Let $n = 1$, then we can set $\Omega = (a, b)$, $C = I$ to be an open interval around 0, $\rho \colon I \to (0, +\infty)$ continuous and bounded with

$$\int_I \rho \, \mathrm{d}t = 1.$$

For any $\varepsilon > 0$ let $\rho_\varepsilon(t) = 1/\varepsilon \rho(t/\varepsilon)$ and $I_\varepsilon(x) = x + \varepsilon I$.

Theorem 7.1. *It holds*

$$\theta(s) = \int_{-\infty}^{+\infty} f(s\rho(t))\,\mathrm{d}t.$$

Proof. In the one-dimensional setting the expression for θ given by (3.5) reads

$$\theta(s) = \inf\left\{\liminf_{j\to+\infty} \frac{1}{\varepsilon_j}\int_{-1}^{1} f(\varepsilon_j|u_j'|*\rho_{\varepsilon_j})\,\mathrm{d}t : (u_j)\in W^{0,s},\ \varepsilon_j\to 0^+\right\},$$

being $W^{0,s}$ the space of all sequences (u_j) in $W^{1,1}_{\mathrm{loc}}(\Omega')$, $(-1,1)\subset\Omega'$, such that $u_j\to s\chi_{(0,+\infty)}$ in $L^1(\Omega')$, and such that there exist two positive infinitesimal sequences $(a_j),(b_j)$ with $u_j(t)=0$ if $t<-a_j$ and $u_j=s$ if $t>b_j$. Let $(u_j)\in W^{0,s}$ and

$$v_j(t) = \int_{-1}^{t} (u_j'(r))^+\,\mathrm{d}r.$$

Moreover, let $w_j = 0\vee v_j\wedge s$. Then $(w_j)\in W^{0,s}$ and by the change of variables $y=\varepsilon_j z$ and $t=\varepsilon_j r$ we get

$$\begin{aligned}
\frac{1}{\varepsilon_j}\int_{-1}^{1} f(\varepsilon_j|u_j'|*\rho_{\varepsilon_j})\,\mathrm{d}t &\geq \frac{1}{\varepsilon_j}\int_{-1}^{1} f\left(\int_{I_{\varepsilon_j}} w_j'(t+y)\rho\left(\frac{y}{\varepsilon_j}\right)\right)\mathrm{d}t\\
&= \frac{1}{\varepsilon_j}\int_{-1}^{1} f\left(\varepsilon_j\int_I w_j'(t+\varepsilon_j z)\rho(z)\right)\mathrm{d}t\\
&= \int_{-1/\varepsilon_j}^{1/\varepsilon_j} f\left(\varepsilon_j\int_I w_j'(\varepsilon_j r+\varepsilon_j z)\rho(z)\right)\mathrm{d}r\\
&= \int_{-1/\varepsilon_j}^{1/\varepsilon_j} f\left(\int_I \tilde{w}_j'(r+z)\rho(z)\right)\mathrm{d}r,
\end{aligned}$$

where $\tilde{w}_j(t) = w_j(\varepsilon_j t)$. Since $(w_j)\in W^{0,s}$ then the previous inequality becomes

$$\frac{1}{\varepsilon_j}\int_{-1}^{1} f(\varepsilon_j|u_j'|*\rho_{\varepsilon_j})\,\mathrm{d}t \geq \int_{-\infty}^{+\infty} f\left(\int_I \tilde{w}_j'(t+z)\rho(z)\,\mathrm{d}z\right)\mathrm{d}t.$$

Denoting by X the space of all functions $v\in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ which are non-decreasing and such that there exist $\xi_0<\xi_1$ with $v(t)=0$ if $t<t_0$ and $v=s$ if $t>t_1$, we are led to solve the minimization problem $\inf_X G$, being

$$G(v) = \int_{-\infty}^{+\infty} f\left(\int_{I(t)} v'(x)\rho(x-t)\,\mathrm{d}x\right)\mathrm{d}t, \quad v\in X.$$

By a simple regularization argument it is not restrictive to assume $f\in C^2(0,+\infty)$ and f strictly concave. For each $k\in\mathbb{N}$, with $k\geq 1$, we now consider a discrete version G_k of G defined on the space of the functions on $\mathbb{R}$ which are constant on each interval of the form

$$J_i^k = \left[\frac{i}{k},\frac{i+1}{k}\right), \quad i\in\mathbb{Z}.$$

We define X_k as the set of the functions $v\colon\mathbb{R}\to[0,s]$, such that:

(a) v is constant on any J_i^k; denote by v^i the value of v on J_i^k;
(b) $v^i\leq v^{i+1}$ for any $i\in\mathbb{Z}$;
(c) $v^i=0$ if $i<i_0$ and $v^i=s$ if $i>i_1$ for some $i_0<i_1$.

Let $I^k = \{i \in \mathbb{Z} : J_i^k \subset I\}$. Finally, let $G_k \colon X_k \to \mathbb{R}$ be defined by

$$G_k(v) = \frac{1}{k} \sum_{i\in\mathbb{Z}} f\Big(\sum_{h\in I^k} (v^{i+h+1} - v^{i+h})\rho_h^k \Big), \quad \rho_h^k = \fint_{J_h^k} \rho(z)\,\mathrm{d}z.$$

Obviously G_k admit minimizers on X_k. We claim that each minimizer of G_k on X_k takes only the values 0 and s.

Let v be a minimizer of G_k on X_k. Suppose, by contradiction, that there exists $i_0 \in \mathbb{Z}$ with $v^{i_0} = c \in (0, s)$. We can assume that for a suitable $r \in \mathbb{N}$ it holds

$$v^{i_0-1} < c, \quad c = v^{i_0} = v^{i_0+1} = \cdots = v^{i_0+r}, \quad v^{i_0+r+1} > c.$$

Given $t \in \mathbb{R}$ sufficiently small, we define $v_t \in X_k$ letting $v_t^{i_0+l} = c + t$, if $0 \le l \le r$, and $v_t = v$ otherwise. It is easy to see that for some $\alpha_i^k, \beta_i^k \neq 0$ which do not depend on t, we have

$$G_k(v_t) = \frac{1}{k} \sum_{i\in J} f(\alpha_i^k + t\beta_i^k)$$

for some finite set $J \subset \mathbb{Z}$. The function $t \mapsto G_k(v_t)$ is twice continuously differentiable in $t = 0$, due to the smoothness of f and we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} G_k(v_t)\Big|_{t=0} = \frac{1}{k} \sum_{i\in J} f''(\alpha_i^k)(\beta_i^k)^2 < 0$$

by the strict concavity of f. This contradicts the fact that v is a minimizer for G_k on X_k.

Since G_k is invariant under translation, we have already shown that

$$\min_{X_k} G_k = G_k(\hat{v})$$

where $v = s\chi_{(0,+\infty)}$. Since

$$G_k(\hat{v}) = \frac{1}{k} \sum_{i\in\mathbb{Z}} f(s\rho_{-i}^k).$$

by the definition of the Riemann integral as the limit of the Riemann sums, we deduce that

$$\liminf_{k\to+\infty} \min_{X_k} G_k \ge \int_{-\infty}^{+\infty} f(s\rho(t))\,\mathrm{d}t.$$

Given $\sigma > 0$ let $v_\sigma \in X$ be such that $\inf_X G \ge G(v_\sigma) - \sigma$. Let $w_\sigma \colon \mathbb{R} \to [0, s]$ given by

$$w_\sigma(t) = w_\sigma^i = \fint_{J_i^k} v_\sigma(r)\,\mathrm{d}r, \quad t \in J_i^k.$$

Notice that $w_\sigma \in X_k$. Let k be sufficiently large such that $G(v_\sigma) \ge G_k(w_\sigma) - \sigma$. Hence

$$G(v_\sigma) \ge \liminf_{k\to+\infty} \min_{X_k} G_k - \sigma \ge \int_{-\infty}^{+\infty} f(s\rho(t))\,\mathrm{d}t - \sigma.$$

By the arbitrariness of $\sigma > 0$ we obtain

$$\theta(s) \ge \inf_X G \ge \int_{-\infty}^{+\infty} f(s\rho(t))\,\mathrm{d}t.$$

If we let

$$u_j(t) = \begin{cases} 0 & \text{if } \le -\varepsilon_j \\ \frac{s}{\varepsilon_j} t + s & \text{if } t \in (-\varepsilon_j, 0) \\ s & \text{if } t \ge 0 \end{cases}$$

for $\varepsilon_j \to 0^+$, we have $(u_j) \in W^{0,s}$ and a straightforward computation shows that

$$\lim_{j\to+\infty} \frac{1}{\varepsilon_j} \int_{-1}^{1} f(\varepsilon_j |u_j'| * \rho_{\varepsilon_j}) \,\mathrm{d}t = \int_{-\infty}^{+\infty} f(s\rho(t)) \,\mathrm{d}t$$

and this yields the conclusion. □

Remark 7.2. Observe that when $I = (-1,1)$ and $\rho = \frac{1}{2}\chi_{(-1,1)}$ we get

$$\theta(s) = 2f\left(\frac{s}{2}\right).$$

Hence we recover the case investigated in [21].

Acknowledgements. A. Magni have been supported by the research group "Forschergruppe 718 der Deutschen Forschungsgemeinschaft: Analysis and Stochastics in Complex Physical Systems". Both the authors would like to thank the Biomathematic group of the University of Dortmund, which made the collaboration possible.

References

[1] R. Alicandro and M.S. Gelli, Free discontinuity problems generated by singular perturbation: the n-dimensional case. *Proc. R. Soc. Edinb. Sect. A* **130** (2000) 449–469.

[2] R. Alicandro, A. Braides, and M.S. Gelli, Free-discontinuity problems generated by singular perturbation. *Proc. R. Soc. Edinburgh Sect. A* **6** (1998) 1115–1129.

[3] L. Ambrosio and V.M. Tortorelli, Approximation of functionals depending on jumps by elliptic functionals *via* Γ-convergence. *Commut. Pure Appl. Math.* **XLIII** (1990) 999–1036.

[4] L. Ambrosio and V.M. Tortorelli, On the approximation of free discontinuity problems. *Boll. Unione Mat. Ital. B (7)* **VI** (1992) 105–123.

[5] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford University Press (2000).

[6] G. Bouchitté, A. Braides and G. Buttazzo, Relaxation results for some free discontinuity problems. *J. Reine Angew. Math.* **458** (1995) 1–18.

[7] B. Bourdin and A. Chambolle, Implementation of an adaptive finite-element approximation of the Mumford–Shah functional. *Numer. Math.* **85** (2000) 609–646.

[8] A. Braides, Approximation of free-discontinuity problems. *Lect. Notes Math.* **1694** (1998).

[9] A. Braides, *Γ-convergence for beginners*. Oxford University Press (2002).

[10] A. Braides and G. Dal Maso, Non-local approximation of the Mumford–Shah functional. *Calc. Var.* **5** (1997) 293–322.

[11] A. Braides and A. Garroni, On the non-local approximation of free-discontinuity problems. *Commut. Partial Differ. Equ.* **23** (1998) 817–829.

[12] A. Chambolle and G. Dal Maso, Discrete approximation of the Mumford–Shah functional in dimension two. *ESAIM: M2AN* **33** (1999) 651–672.

[13] G. Cortesani, Sequence of non-local functionals which approximate free-discontinuity problems. *Arch. R. Mech. Anal.* **144** (1998) 357–402.

[14] G. Cortesani, A finite element approximation of an image segmentation problem. *Math. Models Methods Appl. Sci.* **9** (1999) 243–259.

[15] G. Cortesani and R. Toader, Finite element approximation of non-isotropic free-discontinuity problems. *Numer. Funct. Anal. Optim.* **18** (1997) 921–940.

[16] G. Cortesani and R. Toader, Nonlocal approximation of nonisotropic free-discontinuity problems. *SIAM J. Appl. Math.* **59** (1999) 1507–1519.

[17] G. Cortesani and R. Toader, A density result in *SBV* with respect to non-isotropic energies. *Nonlinear Anal.* **38** (1999) 585–604.

[18] G. Dal Maso, *An Introduction to Γ-Convergence*. Birkhäuser, Boston (1993).

[19] E. De Giorgi, Free discontinuity problems in calculus of variations, in *Frontiers in pure and applied mathematics*, edited by R. Dautray. A collection of papers dedicated to Jacques-Louis Lions on the occasion of his sixtieth birthday, Paris 1988. North-Holland Publishing Co., Amsterdam (1991) 55–62.

[20] L. Lussardi, An approximation result for free discontinuity functionals by means of non-local energies. *Math. Methods Appl. Sci.* **31** (2008) 2133–2146.

[21] L. Lussardi and E. Vitali, Non local approximation of free-discontinuity functionals with linear growth: the one dimensional case. *Ann. Mat. Pura Appl.* **186** (2007) 722–744.

[22] L. Lussardi and E. Vitali, Non local approximation of free-discontinuity problems with linear growth. *ESAIM: COCV* **13** (2007) 135–162.

[23] M. Morini, Sequences of singularly perturbed functionals generating free-discontinuity problems. *SIAM J. Math. Anal.* **35** (2003) 759–805.

[24] M. Negri, The anisotropy introduced by the mesh in the finite element approximation of the Mumford–Shah functional. *Numer. Funct. Anal. Optim.* **20** (1999) 957–982.

[25] M. Negri, A non-local approximation of free discontinuity problems in SBV and SBD. *Calc. Var.* **25** (2006) 33–62.