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POINTWISE ESTIMATES AND RIGIDITY RESULTS FOR ENTIRE SOLUTIONS OF NONLINEAR ELLIPTIC PDE'S

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Abstract. We prove pointwise gradient bounds for entire solutions of pde's of the form

$$\mathscr{L}u(x) = \psi(x, u(x), \nabla u(x)),$$

where $\mathscr{L}$ is an elliptic operator (possibly singular or degenerate). Thus, we obtain some Liouville type rigidity results. Some classical results of J. Serrin are also recovered as particular cases of our approach.

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1. Introduction

A long-established topic in partial differential equations is the study of entire (*i.e.* reasonably smooth and defined in the whole of $\mathbb{R}^n$) solutions of the equation

$$\mathscr{L}u(x) = \psi(x, u(x), \nabla u(x)) \qquad \text{for any } x \in \mathbb{R}^n. \tag{1.1}$$

In order to obtain regularity and rigidity results, and keeping in mind the important physical applications covered by such models, one considers the case in which $\mathscr{L}$ is an elliptic operator (possibly with singularities or degeneracies). Typical examples are: the Laplacian operator, in which

$$\mathscr{L}u = \Delta u, \tag{1.2}$$

the p-Laplacian, in which

$$\mathscr{L}u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \tag{1.3}$$

and the mean curvature operator, namely

$$\mathscr{L}u = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\cdot$$

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Of course, the equation in (1.3) boils down to the one in (1.2) when $p = 2$. The main feature of all these operators is that they induce some kind of regularity on the solutions, typically in Hölder and Sobolev spaces. On the other hand, a more recent regularity approach focuses on pointwise estimates, to wit, for instance, the gradient of the solution is not only bounded with respect to some Hölder or Sobolev norm, but at any point too. These type of pointwise bounds give a very good local control on the solutions and they can be used to obtain further rigidity and symmetry properties.

As far as we know, the idea of dealing with pointwise bounds may be traced back to [1], where it was introduced the idea to look at the equation (or the variational inequality) satisfied by the gradient of the solution, and to deduce universal bounds from that, *via* the Maximum Principle. This classical approach was used in [15], where new and powerful ideas were introduced in order to prove that entire solutions, under suitable assumptions, need to be constant, thus obtaining important extensions of the so-called Liouville Theorem for harmonic functions.

The original idea of [1] has then been extended and modified in several ways (see, among the others, [2,7,13,14,16]) and used for a detailed classification of entire solutions in many cases of interest. In particular, instead of looking at the variational inequality satisfied by the gradient only, it has become relevant to look at a more general variational inequality involving the analogue of the Hamiltonian (or Lagrangian) function in the dynamical system framework. The pointwise estimates obtained in this way are therefore somewhat more precise, since they better take into account the particular features of the nonlinearities involved, and they may be seen as the generalization of the conservation of energy principle to the PDE setting.

The case in which the solution is not entire, but defined on a proper domain of $\mathbb{R}^n$ has been recently studied in [3,8,9]. In this case, the geometry of the domain (and, in particular, its mean curvature) turn out to play an important role. The situation arising for an elliptic PDE in a compact Riemannian manifold has been considered in [10].

The scope of this paper is to deal with entire solutions of equation (1.1) in a unified framework and under very general assumptions (in particular, we comprise, at the same time, possibly singular and degenerate operators, and gradient dependence in the nonlinearity), obtaining both pointwise gradient estimates and rigidity results, and also recovering some classical results (such as the ones in [15]) as particular cases. For this, we consider the following PDE in divergence form:

$$\operatorname{div}\left(\Phi'\left(|\nabla u|^2\right)\nabla u\right) = f(u) + g(x,u,\nabla u). \tag{1.4}$$

Here $u \in C^1(\mathbb{R}^n)$, $f \in C^1(\mathbb{R})$, $g \in C^1(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$, and the notation $g := g(x,u,\zeta)$ with $(x,u,\zeta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ will be often used. The PDE in (1.4) is intended in the weak sense and we suppose that Φ satisfies the possibly singular or degenerate elliptic conditions listed below.

We recall that pde's with gradient dependence, of which (1.4) comprises a quite general setting, are a classical topic of research, see *e.g.* [5], and in fact, even ode's with gradient dependence have been the object of intense study, see [4,12] and references therein. Differently from many classical approaches, here we obtain pointwise estimates on the gradient of the solution, and not only estimates in the L^∞-norm.

We assume that $\Phi \in C^{3,\alpha}_{\text{loc}}\big((0,+\infty)\big) \cap C\big([0,+\infty)\big)$ for some $\alpha \in (0,1)$, and that $\Phi(0) = 0$.

We define

$$a_{ij}(\sigma) := 2\Phi''\left(|\sigma|^2\right)\sigma_i\sigma_j + \Phi'\left(|\sigma|^2\right)\delta_{ij}, \tag{1.5}$$

and we suppose that one of the following conditions is satisfied:

Assumption (A). There exist $p > 1$, $a \geqslant 0$ and c_1, $c_2 > 0$ such that for any σ, $\xi \in \mathbb{R}^n \setminus \{0\}$

$$c_1(a+|\sigma|)^{p-2} \leqslant \Phi'\left(|\sigma|^2\right) \leqslant c_2(a+|\sigma|)^{p-2} \tag{1.6}$$

and

$$c_1(a+|\sigma|)^{p-2}|\xi|^2 \leqslant \sum_{i,j=1}^n a_{ij}(\sigma)\xi_i\xi_j \leqslant c_2(a+|\sigma|)^{p-2}|\xi|^2. \tag{1.7}$$

Assumption (B). $\Phi \in C^1\big([0,+\infty)\big)$, and there exist c_1, $c_2 > 0$ such that for any $\sigma \in \mathbb{R}^n$

$$c_1(1+|\sigma|)^{-1} \leqslant \Phi'\left(|\sigma|^2\right) \leqslant c_2(1+|\sigma|)^{-1} \tag{1.8}$$

and

$$c_1(1+|\sigma|)^{-1}|\xi'|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(\sigma)\xi_i\xi_j \leqslant c_2(1+|\sigma|)^{-1}|\xi'|^2, \tag{1.9}$$

for any $\xi' = (\xi, \xi_{n+1}) \in \mathbb{R}^{n+1}$ which is orthogonal to $(-\sigma, 1) \in \mathbb{R}^{n+1}$.

The above Assumptions (A) and (B) are classical: they agree, for instance, with the ones of [2], and examples of functional satisfying the above conditions are the p-Laplacian (with $p \in (1,+\infty)$) and the mean curvature operators – which correspond to the cases

$$\Phi(r) := \frac{2}{p} r^{p/2} \text{ and } \Phi(r) := 2\sqrt{1+r} - 2,$$

respectively.

The simplest case of our result can be stated in the following way:

Theorem 1.1. *Let $u \in C^1(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ be a weak solution in the whole of $\mathbb{R}^n$ of one of the following equations: either*

$$\begin{cases} \operatorname{div}\left(\Phi'\left(|\nabla u|^2\right)\nabla u\right) = h(u) + c(x)\cdot\nabla u \\ \text{with } h \in C^1(\mathbb{R}),\ c \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \\ h' \geqslant 0 \text{ and } \left\{\frac{\partial c_i}{\partial x_j}\right\}_{\{i,j=1,\dots,n\}} \text{ a nonnegative matrix,} \end{cases} \tag{1.10}$$

or

$$\begin{cases} \operatorname{div}\left(\Phi'\left(|\nabla u|^2\right)\nabla u\right) = f(u) + g(\nabla u) \\ \text{with } f \in C^1(\mathbb{R}),\ g \in C^1(\mathbb{R}^n) \text{ and } fg \leqslant 0. \end{cases} \tag{1.11}$$

Let $F := 0$ if (1.10) holds, and F be a primitive[3] of f with $F \geqslant 0$ if (1.11) holds.

Then,

$$2\Phi'\left(|\nabla u(x)|^2\right)|\nabla u(x)|^2 - \Phi\left(|\nabla u(x)|^2\right) \leqslant 2F(u(x)) \qquad \text{for any } x \in \mathbb{R}^n. \tag{1.12}$$

More generally, we prove the following results:

Theorem 1.2. *Let $u \in C^1(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ be a weak solution of (1.4) in the whole of $\mathbb{R}^n$, with $f \in C^1(\mathbb{R})$, $g \in C^1(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$.*

Let F be a primitive of f, with $F \geqslant 0$. Let

$$\mathscr{R}(x) := -\frac{2f(u)\,g(x,u,\nabla u)\,|\nabla u|^2}{\Phi'(|\nabla u|^2)} + 2|\nabla u|^2\nabla_x g(x,u,\nabla u)\cdot\nabla u + 2|\nabla u|^4 g_u(x,u,\nabla u) \tag{1.13}$$

and assume that

$$\mathscr{R}(x) \geqslant 0 \text{ for any } x \in \mathbb{R}^n. \tag{1.14}$$

Then,

$$2\Phi'\left(|\nabla u(x)|^2\right)|\nabla u(x)|^2 - \Phi\left(|\nabla u(x)|^2\right) \leqslant 2F(u(x)) \qquad \text{for any } x \in \mathbb{R}^n. \tag{1.15}$$

[3] We observe that, since u is bounded, we can find a primitive of f that is non-negative in the range of u.

Theorem 1.3. *Let the assumptions of Theorem 1.2 hold.*

Also, if Assumption (A) holds with $p \geqslant 2$, assume that for any $\mu \in \{F = 0\}$ we have

$$F(r) = O(|r - \mu|^p). \tag{1.16}$$

Then, if there exists $x_o \in \mathbb{R}^n$ for which $F(u(x_o)) = 0$, we have that $u(x) = u(x_o)$ for any $x \in \mathbb{R}^n$.

Theorem 1.2 is a pointwise estimate on the gradient of the solution and Theorem 1.3 is a rigidity result of Liouville type. When $g := 0$, similar results have been obtained by [13] in the semilinear case $\Phi(r) := r$ and by [2] under Assumptions (A) or (B). Then, Theorems 1.2 and 1.3 here may be seen as extensions of the works of [2, 13] to the case of more complicated nonlinearities involving $g(x, u, \nabla u)$.

Remark 1.4. We observe that condition (1.14), although artificial at first glance, comprises many cases of interest, such as:

- $f := 0$, $g(x, u, \zeta) := h(u) + c(x) \cdot \zeta$, with $h' \geqslant 0$ and $\{\frac{\partial c_i}{\partial x_j}\}_{\{i,j=1,\dots,n\}}$ a nonnegative matrix;
- $g = g(\zeta)$ and $fg \leqslant 0$.

These cases, which are the ones presented in Theorem 1.1, may also be seen as extensions of some of the results of [15] to the case in which the operator is not uniformly elliptic and in divergence form, and techniques we use are different (see also Appendix A).

Remark 1.5. We notice that, in its generality, Theorem 1.2 is new, to the best of our knowledge, even in the semilinear case $\Phi(r) := r$. See also [3, 9, 10] for related works on proper domains and on manifolds.

Example 1.6. If condition (1.14) is dropped, then (1.15) may not hold, as the following example shows. Let $n := 1$, $u(x) := \arctan(x)$, $\Phi(r) := r$, $f := 0$, $F := 0$ and $g(x, u, \zeta) := -2x/(1 + x^2)^2$. Then we see that $u'' = f + g$, hence (1.4) is satisfied, that $\mathscr{R} = 2|u'(x)|^2 g'(x) u'(x) = 4(3x^2 - 1)/(1 + x^2)^3$, hence (1.14) does not hold, and (1.15) is violated because u is not constant.

Example 1.6 also shows that (1.15) is quite a strong estimate, since the gradient of the solution is estimated (at all points, and not only in the average) by something that depends only on f, not on g.

The Proof of Theorems 1.2 and 1.3 will be based on a long and delicate computation, detailed in Section 2, and on the proofs of the results of [2], as discussed in Section 3.

2. P-function computations

Let

$$\Lambda(s) := 2s\Phi''(s) + \Phi'(s), \tag{2.1}$$

and

$$d_{ij}(\sigma) := \frac{a_{ij}(\sigma)}{\Lambda(|\sigma|^2)}. \tag{2.2}$$

We define

$$P(u, x) := 2\Phi'\left(|\nabla u(x)|^2\right)|\nabla u(x)|^2 - \Phi\left(|\nabla u(x)|^2\right) - 2F(u(x)). \tag{2.3}$$

The main idea driving the following computation comes from some classical works such as [2, 14, 16] in which it is shown that some suitable P-function solves an elliptic PDE (at least at nonsingular points). The Maximum Principle then provides an estimate on P, which, in turn, would give the desired result.

With this goal in mind, we pursue the following result (which, for $g := 0$ reduces to formula (2.7) of [2]):

Lemma 2.1. *Let Ω be an open subset of $\mathbb{R}^n$. Let $u \in C^1(\Omega)$ be a solution of (1.4) in Ω, with $\nabla u \neq 0$ in Ω. Let*

$$B_i(x) := -2\frac{f(u)}{\Lambda(|\nabla u|^2)}\left(1+\frac{|\nabla u|^2\Phi''(|\nabla u|^2)}{\Phi'(|\nabla u|^2)}\right)\frac{\partial u}{\partial x_i} - \frac{|\nabla u|^2}{\Lambda(|\nabla u|^2)}\frac{\partial g}{\partial p_i}(x,u,\nabla u).$$

Then,

$$\sum_{ij}|\nabla u|^2\frac{\partial}{\partial x_j}\left(d_{ij}(\nabla u)\frac{\partial P}{\partial x_i}\right) + \sum_i B_i\frac{\partial P}{\partial x_i} \geqslant \frac{|\nabla P|^2}{2\Lambda(|\nabla u|^2)} + \mathscr{R} \qquad \text{weakly in } \Omega. \tag{2.4}$$

Proof. First, we remark that, by our assumptions, the map $r \mapsto 2\Phi'(r)r - \Phi(r)$ is invertible. We call Ψ its inverse. Notice that

$$\Psi\big(P(u,x) + 2F(u(x))\big) = |\nabla u(x)|^2. \tag{2.5}$$

Moreover, by the definition of Ψ and (2.1),

$$1 = \frac{\mathrm{d}}{\mathrm{d}r}\left(\Psi\Big(2\Phi'(r)r - \Phi(r)\Big)\right) = \Psi'\Big(2\Phi'(r)r - \Phi(r)\Big)\,\Lambda(r),$$

hence

$$\Psi'\Big(2\Phi'(|\nabla u|^2)|\nabla u|^2 - \Phi(|\nabla u|^2)\Big) = \frac{1}{\Lambda(|\nabla u|^2)}. \tag{2.6}$$

Now, differentiating (2.3) and recalling (2.1), we see that

$$\frac{\partial P}{\partial x_i} = 2\Lambda(|\nabla u|^2)\sum_k \frac{\partial^2 u}{\partial x_i \partial x_k}\frac{\partial u}{\partial x_k} - 2f(u)\frac{\partial u}{\partial x_i} \tag{2.7}$$

hence, recalling (2.2),

$$\begin{aligned}\sum_{ij}\frac{\partial}{\partial x_j}\left(d_{ij}(\nabla u)\frac{\partial P}{\partial x_i}\right) =& -2\sum_{ij}\frac{\partial}{\partial x_j}\left(f(u)d_{ij}(\nabla u)\frac{\partial u}{\partial x_i}\right) + 2\sum_{ijk}\frac{\partial}{\partial x_j}\left(a_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i\partial x_k}\right)\frac{\partial u}{\partial x_k}\\ &+ 2\sum_{ijk}a_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_k}\frac{\partial^2 u}{\partial x_j\partial x_k}.\end{aligned} \tag{2.8}$$

Also, (1.5) gives that

$$\frac{\partial a_{ij}}{\partial \sigma_\ell}(\sigma) = \frac{\partial a_{\ell j}}{\partial \sigma_i}(\sigma) \tag{2.9}$$

and, by (1.4),

$$\sum_{ij}a_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(u) + g(x,u,\nabla u). \tag{2.10}$$

Therefore, by (2.9) and (2.10), for any fixed k,

$$\sum_{ijk}\frac{\partial}{\partial x_j}\left(a_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i\partial x_k}\right) = \sum_{ijk}\frac{\partial}{\partial x_k}\left(a_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j}\right) \tag{2.11}$$

$$= f'(u)\frac{\partial u}{\partial x_k} + g_{x_k} + g_u\frac{\partial u}{\partial x_k} + \sum_i g_{p_i}\frac{\partial^2 u}{\partial x_i\partial x_k}, \tag{2.12}$$

where the notation $g = g(x,u,\zeta)$ has been used.

From (2.8) and (2.11), we gather

$$\sum_{ij} \frac{\partial}{\partial x_j}\left(d_{ij}(\nabla u)\frac{\partial P}{\partial x_i}\right) = 2f'(u)\sum_k \frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_k} + 2\sum_{ijk} a_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_k}\frac{\partial^2 u}{\partial x_j \partial x_k} \tag{2.13}$$

$$- 2\sum_{ij} f'(u) d_{ij}(\nabla u)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - 2f(u)\sum_{ij}\frac{\partial}{\partial x_j}\left(d_{ij}(\nabla u)\frac{\partial u}{\partial x_i}\right)$$

$$+ 2\sum_k \left[g_{x_k} + g_u \frac{\partial u}{\partial x_k} + \sum_i g_{p_i}\frac{\partial^2 u}{\partial x_i \partial x_k}\right]\frac{\partial u}{\partial x_k}. \tag{2.14}$$

Furthermore, from (2.1) and (2.2), we obtain

$$f'(u)\sum_k \frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_k} - \sum_{ij} f'(u) d_{ij}(\nabla u)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} = f'(u)\left[|\nabla u|^2 - \frac{\Phi'(|\nabla u|^2)|\nabla u|^2 + 2\Phi''(|\nabla u|^2)|\nabla u|^4}{\Lambda(|\nabla u|^2)}\right]$$

$$= f'(u)\left[|\nabla u|^2 - |\nabla u|^2\right] = 0. \tag{2.15}$$

Plugging this into (2.13), we conclude that

$$\sum_{ij} \frac{\partial}{\partial x_j}\left(d_{ij}(\nabla u)\frac{\partial P}{\partial x_i}\right) = 2\sum_{ijk} a_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_k}\frac{\partial^2 u}{\partial x_j \partial x_k} - 2f(u)\sum_{ij}\frac{\partial}{\partial x_j}\left(d_{ij}(\nabla u)\frac{\partial u}{\partial x_i}\right) \tag{2.16}$$

$$+ 2\sum_k \left[g_{x_k} + g_u \frac{\partial u}{\partial x_k} + \sum_i g_{p_i}\frac{\partial^2 u}{\partial x_i \partial x_k}\right]\frac{\partial u}{\partial x_k}. \tag{2.17}$$

Also, from (2.2) and (2.10),

$$\sum_{ij} d_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{f(u) + g(x,u,\nabla u)}{\Lambda(|\nabla u|^2)},$$

and so (2.16) becomes

$$\sum_{ij} \frac{\partial}{\partial x_j}\left(d_{ij}(\nabla u)\frac{\partial P}{\partial x_i}\right) = 2\sum_{ijk} a_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_k}\frac{\partial^2 u}{\partial x_j \partial x_k} - 2f(u)\sum_{ij}\frac{\partial}{\partial x_j} d_{ij}(\nabla u)\frac{\partial u}{\partial x_i}$$

$$- 2\frac{f(u)\left[f(u) + g(x,u,\nabla u)\right]}{\Lambda(|\nabla u|^2)} + 2\sum_k \left[g_{x_k} + g_u \frac{\partial u}{\partial x_k} + \sum_i g_{p_i}\frac{\partial^2 u}{\partial x_i \partial x_k}\right]\frac{\partial u}{\partial x_k}\cdot \tag{2.18}$$

Moreover, making use of (1.5), (2.1) and (2.2), we obtain

$$\sum_{ij}\frac{\partial}{\partial x_j} d_{ij}(\nabla u)\frac{\partial u}{\partial x_i} = \sum_{ij}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}\frac{2\Phi''(|\nabla u|^2)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \Phi'(|\nabla u|^2)\delta_{ij}}{2|\nabla u|^2\Phi''(|\nabla u|^2) + \Phi'(|\nabla u|^2)}$$

$$= \sum_{ij}\frac{\partial u}{\partial x_i}\frac{\left[4\Phi'''\sum_k \frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_j \partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + 2\Phi''\frac{\partial^2 u}{\partial x_j \partial x_i}\frac{\partial u}{\partial x_j} + 2\Phi''\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_j^2} + 2\Phi''\sum_k \frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_j \partial x_k}\delta_{ij}\right]}{2|\nabla u|^2\Phi'' + \Phi'}$$

$$- \sum_{ij}\frac{\partial u}{\partial x_i}\frac{(2\Phi''\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \Phi'\delta_{ij})\left[6\Phi'' + 4|\nabla u|^2\Phi'''\right]\sum_k \frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_j \partial x_k}}{(2|\nabla u|^2\Phi'' + \Phi')^2}$$

$$= 2\frac{\Phi''(|\nabla u|^2)}{\Lambda(|\nabla u|^2)}\left(|\nabla u|^2\Delta u - \sum_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)\cdot \tag{2.19}$$

Also, from (1.5) and (2.10),

$$f(u)+g(x,u,\nabla u)=\sum_{ij}\left(2\Phi''(|\nabla u|^2)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}+\Phi'(|\nabla u|^2)\delta_{ij}\right)\frac{\partial^2 u}{\partial x_i\partial x_j}$$
$$=2\Phi''(|\nabla u|^2)\sum_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}+\Phi'(|\nabla u|^2)\Delta u,$$

from which we obtain

$$\Delta u=\frac{f(u)+g(x,u,\nabla u)}{\Phi'(|\nabla u|^2)}-2\frac{\Phi''(|\nabla u|^2)}{\Phi'(|\nabla u|^2)}\sum_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\cdot$$

Therefore, recalling also (2.7), we write (2.19) as

$$\sum_{ij}\frac{\partial}{\partial x_j}d_{ij}(\nabla u)\frac{\partial u}{\partial x_j}=2\frac{\Phi''(|\nabla u|^2)}{\Phi'(|\nabla u|^2)\Lambda(|\nabla u|^2)}\left[(f+g)|\nabla u|^2-\Lambda(|\nabla u|^2)\sum_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right]$$
$$=-\frac{\Phi''(|\nabla u|^2)}{\Phi'(|\nabla u|^2)\Lambda(|\nabla u|^2)}\sum_i\frac{\partial P}{\partial x_i}\frac{\partial u}{\partial x_i}+\frac{2g\Phi''(|\nabla u|^2)|\nabla u|^2}{\Phi'(|\nabla u|^2)\,\Lambda(|\nabla u|^2)}\cdot \tag{2.20}$$

Thus, exploiting (2.18),

$$\sum_{ij}\frac{\partial}{\partial x_j}\left(d_{ij}(\nabla u)\frac{\partial P}{\partial x_i}\right)=2f\frac{\Phi''(|\nabla u|^2)}{\Phi'(|\nabla u|^2)\Lambda(|\nabla u|^2)}\sum_i\frac{\partial P}{\partial x_i}\frac{\partial u}{\partial x_i}+2\sum_{ijk}a_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial^2 u}{\partial x_j\partial x_k}-2\frac{f\left[f+g\right]}{\Lambda(|\nabla u|^2)}$$
$$+2\sum_k\left[g_{x_k}+g_u\frac{\partial u}{\partial x_k}+\sum_i g_{p_i}\frac{\partial^2 u}{\partial x_i\partial x_k}\right]\frac{\partial u}{\partial x_k}-\frac{4fg\Phi''(|\nabla u|^2)|\nabla u|^2}{\Phi'(|\nabla u|^2)\,\Lambda(|\nabla u|^2)}\cdot \tag{2.21}$$

Now we set

$$z_k=\sum_i\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial u}{\partial x_i},$$

and we use Schwarz Inequality to see that

$$|z_k|\leqslant\sqrt{\sum_i\left(\frac{\partial^2 u}{\partial x_i\partial x_k}\right)^2}\sqrt{\sum_i\left(\frac{\partial u}{\partial x_i}\right)^2}$$

and so

$$\sum_{ijk}\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_j\partial x_k}\frac{\partial u}{\partial x_j}=\sum_k z_k^2\leqslant\sum_k\left(\sum_i\left(\frac{\partial^2 u}{\partial x_i\partial x_k}\right)^2\right)\left(\sum_i\left(\frac{\partial u}{\partial x_i}\right)^2\right)=|\nabla u|^2\sum_{ik}\left(\frac{\partial^2 u}{\partial x_i\partial x_k}\right)^2\cdot$$

This and (1.5) give that

$$\sum_{ijk}a_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial^2 u}{\partial x_j\partial x_k}\geqslant\sum_{ijk}\frac{\Phi'}{|\nabla u|^2}\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_j\partial x_k}\frac{\partial u}{\partial x_j}+2\Phi''\sum_{ijk}\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_j\partial x_k}\frac{\partial u}{\partial x_j}$$
$$=\frac{\Lambda}{|\nabla u|^2}\sum_{ijk}\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_j\partial x_k}\frac{\partial u}{\partial x_j}\cdot \tag{2.22}$$

Moreover, by (2.7),

$$\sum_{ijk} \frac{\partial^2 u}{\partial x_i \partial x_k} \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_j \partial x_k} \frac{\partial u}{\partial x_j} = \frac{1}{4\Lambda^2} \sum_k \left(\frac{\partial P}{\partial x_k} + 2f \frac{\partial u}{\partial x_k} \right)^2$$

and so (2.22) becomes

$$\sum_{ijk} a_{ij}(\nabla u) \frac{\partial^2 u}{\partial x_i \partial x_k} \frac{\partial^2 u}{\partial x_j \partial x_k} \geqslant \frac{1}{4|\nabla u|^2 \Lambda} \sum_k \left(\frac{\partial P}{\partial x_k} + 2f \frac{\partial u}{\partial x_k} \right)^2 = \frac{|\nabla P|^2}{4|\nabla u|^2 \Lambda} + \frac{f \sum_i \frac{\partial u}{\partial x_i} \frac{\partial P}{\partial x_i}}{|\nabla u|^2 \Lambda} + \frac{f^2}{\Lambda} \cdot$$

By substituting this in (2.21), we obtain

$$\sum_{ij} \frac{\partial}{\partial x_j} \left(d_{ij}(\nabla u) \frac{\partial P}{\partial x_i} \right) - \frac{2f}{|\nabla u|^2 \Lambda} \left(1 + \frac{|\nabla u|^2 \Phi''}{\Phi'} \right) \sum_i \frac{\partial P}{\partial x_i} \frac{\partial u}{\partial x_i} \geqslant \frac{|\nabla P|^2}{2|\nabla u|^2 \Lambda} - \frac{2fg}{\Phi'}$$
$$+ 2 \sum_k \left[g_{x_k} + g_u \frac{\partial u}{\partial x_k} + \sum_i g_{p_i} \frac{\partial^2 u}{\partial x_i \partial x_k} \right] \frac{\partial u}{\partial x_k} \cdot \qquad (2.23)$$

Now, we use (2.5), according to which, for i fixed,

$$2 \sum_k \frac{\partial^2 u}{\partial x_i \partial x_k} \frac{\partial u}{\partial x_k} = \Psi'(P + 2F) \left(\frac{\partial P}{\partial x_i} + 2f \frac{\partial u}{\partial x_i} \right) \cdot$$

Accordingly,

$$2 \sum_{ki} g_{p_i} \frac{\partial^2 u}{\partial x_i \partial x_k} \frac{\partial u}{\partial x_k} = \Psi'(P + 2F) \sum_i g_{p_i} \left(\frac{\partial P}{\partial x_i} + 2f \frac{\partial u}{\partial x_i} \right) \cdot$$

Plugging this in (2.23) and recalling (2.6) we obtain the desired result. □

3. Proof of Theorems 1.2 and 1.3

We observe that, since $u \in C^1(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$, our assumptions on f and g imply that $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ and the family of all the translations of u is relatively compact in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, for a suitable $\alpha \in (0, 1)$, both under assumptions (A) and (B) (see, for instance, [6, 11, 17]).

Since $\mathscr{R} \geqslant 0$, we know by Lemma 2.1 that

$$\sum_{ij} \frac{\partial}{\partial x_j} \left(d_{ij}(\nabla u) \frac{\partial P}{\partial x_i} \right) + \frac{\nabla B \cdot \nabla P}{|\nabla u|^2} \geqslant 0$$

weakly in $\{\nabla u \neq 0\}$.

Then, we can repeat the arguments in the proofs of Theorems 1.6 and 1.8 of [2] (see pp. 1464–1466 there) and obtain our Theorems 1.2 and 1.3: we give the arguments in full detail for the facility of the reader.

We begin with the Proof of Theorem 1.2. For this, we consider the family $\mathscr{S}$ of all the translations of u, namely

$$\mathscr{S} := \{ v : \mathbb{R}^n \to \mathbb{R} \text{ s.t. } \exists p \in \mathbb{R}^n \text{ s.t. } v(x) = u(x + p) \ \forall x \in \mathbb{R}^n \} \cdot$$

By the above observation, we have that

$$\mathscr{S} \text{ is relatively compact in } C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n). \qquad (3.1)$$

Recalling the notation in (2.3), we also define

$$P_o := \sup_{\substack{v \in \mathscr{S} \\ x \in \mathbb{R}^n}} P(v, x) \cdot$$

We claim that

$$P_o \leqslant 0. \tag{3.2}$$

To prove this, we assume by contradiction that

$$P_o > 0 \tag{3.3}$$

and we take sequences $v_k \in \mathscr{S}$ and $x_k \in \mathbb{R}^n$ such that

$$\lim_{k \to +\infty} P(v_k, x_k) = P_o. \tag{3.4}$$

Let $w_k(x) := v_k(x + x_k)$. Then $w_k \in \mathscr{S}$ and $P(w_k, 0) = P(v_k, x_k)$. We conclude that

$$\lim_{k \to +\infty} P(w_k, 0) = P_o.$$

By (3.1) (and up to a subsequence) we may suppose that w_k converges to some w in $C^{1,\alpha}_{\rm loc}(\mathbb{R}^n)$. By construction, $w \in \mathscr{S}$ and $P(w_k, 0)$ converges to $P(w, 0)$. Thus, by (3.4),

$$P(w, 0) = P_o.$$

So, if we define

$$\mathscr{W} := \{x \in \mathbb{R}^n \text{ s.t. } P(w, x) = P_o\},$$

we have that $0 \in \mathscr{W}$ and so

$$\mathscr{W} \neq \varnothing. \tag{3.5}$$

Also, by continuity, we see that

$$\mathscr{W} \text{ is closed.} \tag{3.6}$$

We aim to show that

$$\mathscr{W} \text{ is open.} \tag{3.7}$$

For this, let $\zeta \in \mathscr{W}$. Then $\nabla w(\zeta) \neq 0$, otherwise (2.3) would give that $0 \geqslant -2F(w(\zeta)) = P(w, \zeta) = P_o$, which would be in contradiction with (3.3). Therefore, there exist ϱ, $\kappa > 0$ such that $|\nabla w(x)| \geqslant \kappa$ for any $x \in B_\varrho(\zeta)$. Then, by (2.4) and (1.14), we have that

$$\sum_{ij} |\nabla u|^2 \frac{\partial}{\partial x_j}\left(d_{ij}(\nabla u) \frac{\partial P}{\partial x_i}\right) + \sum_i B_i \frac{\partial P}{\partial x_i} \geqslant 0 \qquad \text{weakly in } B_\varrho(\zeta).$$

Therefore, by Maximum Principle (recall that $P(w, \zeta) = P_o \geqslant P(w, x)$ for any $x \in \mathbb{R}^n$), it follows that $P(w, x) = P_o$ for any $x \in B_\varrho(\zeta)$.

This establishes (3.7). Now, by (3.5)–(3.7), we infer that $\mathscr{W} = \mathbb{R}^n$, that is

$$P(w, x) = P_o \text{ for any } x \in \mathbb{R}^n. \tag{3.8}$$

On the other hand, since w is bounded, by following the gradient lines we find a sequence of points η_j such that

$$\lim_{j \to +\infty} \nabla w(\eta_j) = 0\cdot$$

By using this in (3.8), we obtain

$$0 \geqslant \limsup_{j \to +\infty} -2F(w(\eta_j)) = \limsup_{j \to +\infty} P(w, \eta_j) = P_o,$$

which is in contradiction with (3.3).

This proves (3.2), from which Theorem 1.2 follows at once.

Now we prove Theorem 1.3. For this, we take x_o as in the statement of Theorem 1.3 and we define

$$\mathscr{V} := \{x \in \mathbb{R}^n \text{ s.t. } u(x) = u(x_o)\}.$$

Notice that $\mathscr{V} \neq \varnothing$ and V is closed. We claim that

$$\mathscr{V} \text{ is also open.} \tag{3.9}$$

From this, we would obtain that $\mathscr{V} = \mathbb{R}^n$, which is the thesis of Theorem 1.3. So we focus on the proof of (3.9). For this, we fix $y \in \mathscr{V}$ and $\omega \in \mathrm{S}^{n-1}$ and, for any $t \in \mathbb{R}$, we define

$$\varphi(t) := u(y + t\omega) - u(x_o).$$

We aim to prove that there exist positive c and C for which

$$|\varphi'(t)| \leqslant C\,|\varphi(t)| \text{ for all } t \in (-c, c). \tag{3.10}$$

For this scope, we define $q := p$ if Assumption (A) holds with $p \geqslant 2$, and $q := 2$ otherwise, and we introduce the functions

$$[0, +\infty) \ni s \mapsto \Psi(s) := 2s\Phi'(s) - \Phi(s)$$

and

$$G(s) := \Psi(s) - \epsilon s^{q/2}.$$

The parameter $\epsilon > 0$ will be chosen conveniently small with respect to $M := \|u\|_{W^{1,\infty}(\mathbb{R}^n)}$ and to the structural constants given in either Assumption (A) or (B).

Now we take $s \in (0, M^2]$ and $\sigma := (\sqrt{s}, 0, \ldots, 0) \in \mathbb{R}^n$ and we use (2.1) and (1.5), and either (1.7) or (1.9), to see that

$$\begin{aligned}
\Lambda(s) &= 2s\Phi''(s) + \Phi'(s) \\
&= |\sigma|^{-2} \sum_{ij} a_{ij}(\sigma)\sigma_i\sigma_j \\
&\geqslant \begin{cases} c_1(a + |\sigma|)^{p-2} & \text{if Assumption (A) holds and } p \geqslant 2, \\[2mm] \dfrac{c_1}{(a+|\sigma|)^{2-p}} & \text{if Assumption (A) holds and } p \in (1,2), \\[2mm] \dfrac{c_1}{1+|\sigma|} & \text{if Assumption (B) holds} \end{cases} \\
&\geqslant \begin{cases} c_1\,|\sigma|^{p-2} & \text{if Assumption (A) holds and } p \geqslant 2, \\[2mm] \dfrac{c_1}{(a+M)^{2-p}} & \text{if Assumption (A) holds and } p \in (1,2), \\[2mm] \dfrac{c_1}{1+M} & \text{if Assumption (B) holds} \end{cases} \\
&\geqslant \begin{cases} \dfrac{\epsilon p}{2}\, s^{(p/2)-1} & \text{if Assumption (A) holds and } p \geqslant 2, \\[2mm] \epsilon & \text{if Assumption (A) holds and } p \in (1,2), \\[2mm] \epsilon & \text{if Assumption (B) holds} \end{cases} \\
&= \frac{\epsilon q}{2} s^{(q/2)-1}.
\end{aligned} \tag{3.11}$$

as long as ϵ is small enough.

Furthermore, notice that, by either (1.6) or (1.8), we have that $G(0) = 0$. Also, by (2.1),

$$G'(s) = \Lambda(s) - \frac{\epsilon q}{2} s^{(q/2)-1}$$

for any $s > 0$ and therefore $G'(s) \geqslant 0$ for any $s \in (0, M^2]$, thanks to (3.11) (as long as ϵ is small enough). As a consequence, $G(s) \geqslant 0$ and therefore

$$\Psi(s) \geqslant \epsilon s^{q/2}$$

for any $s \in (0, M^2]$. By taking $s := |\nabla u(y + t\omega)|^2$ here above and using Theorem 1.2, we obtain

$$\begin{aligned} |\varphi'(t)|^q = |\nabla u(y+t\omega)|^q &\leqslant \frac{1}{\epsilon}\Psi\big(|\nabla u(y+t\omega)|^2\big) \\ &\leqslant \frac{2}{\epsilon}F\big(u(y+t\omega)\big) = \frac{2}{\epsilon}\Big[F\big(u(y+t\omega)\big) - F\big(u(x_o)\big)\Big]. \end{aligned} \tag{3.12}$$

We observe that if r is sufficiently close to $u(x_o)$ then

$$\big|F(r) - F(u(x_o))\big| \leqslant C_o\big|r - u(x_o)\big|^q. \tag{3.13}$$

Indeed, since $F(u(x_o)) = 0 \leqslant F(r)$ for any r, then (3.13) follows from a second order Taylor expansion of F when $q = 2$, and it follows from (1.16) when $q = p$.

Then, we plug (3.13) into (3.12), and we get that

$$|\varphi'(t)|^q \leqslant \frac{2C_o}{\epsilon}\big|u(y+t\omega) - u(x_o)\big|^q = \frac{2C_o}{\epsilon}|\varphi(t)|^q$$

as long as t is small enough. This proves (3.10).

From (3.10) we obtain that the function $t \mapsto |\varphi(t)|^2 e^{-2Ct}$ is non-increasing for small t. Accordingly, $|\varphi(t)| \leqslant |\varphi(0)|\, e^{Ct} = 0$ for small t, that is $\varphi(t)$ vanishes identically (for small t, independently of ω). By varying ω, we obtain that u is constant in a small neighborhood of x_o. This proves (3.9) and thus Theorem 1.3. □

Appendix A. Recovering some important results of [15] as particular cases of our results

The purpose of this appendix is to recover some important results of [15]. More precisely, from (1) and (2) in [15], one has the following results:

Theorem A.1. *(p. 348 in [15]). Let $g \in C^1(\mathbb{R} \times \mathbb{R}^n)$. Let $u \in C^3(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ be a solution of*

$$\Delta u = g(u, \nabla u) \tag{A.1}$$

in $\mathbb{R}^n$, with

$$g_u \geqslant 0. \tag{A.2}$$

Then u is constant.

Theorem A.2. *(p. 349 in [15]). Let $\psi \in C^1(\mathbb{R})$. Let $u \in C^3(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ be a solution of*

$$(1 + |\nabla u|^2)\Delta u - u_i u_j u_{ij} = \psi(\nabla u) \tag{A.3}$$

in $\mathbb{R}^n$. Then u is constant.

We can obtain Theorems A.1 and A.2 directly from our Theorems 1.2 and 1.1, respectively (Thm. 1.3 could have also been used) *via* the following argument. For Theorem A.1, we take $f := 0$, $F := 0$ and $\Phi(r) := r$, hence the left hand side of (1.15) equals to $|\nabla u|^2$. In this case (1.4) agrees with (A.1) and g is independent of x. Also, by (1.13) and (A.2), $\mathscr{R} = 2|\nabla u|^4 g_u \geqslant 0$, so (1.14) holds true. Thus, since F vanishes identically, by (1.15), we have that ∇u vanishes identically too, giving a new proof of Theorem A.1.

As for Theorem A.2, we take $f := 0$, $F := 0$, $g(p) := \psi(p)/(1 + |p|^2)^{3/2}$ and $\Phi(r) := 2\sqrt{1+r} - 2$. Then (A.3) agrees with (1.11), and the left hand side of (1.12) is

$$\frac{2\left(\sqrt{1+|\nabla u|^2}-1\right)}{\sqrt{1+|\nabla u|^2}},$$

which is nonnegative, and it vanishes if and only if $\nabla u = 0$. Accordingly, since F vanishes identically, (1.12) gives that ∇u vanishes identically as well, giving a new proof of Theorem A.2. □

Remark A.3. We observe that Theorems 1.1 and 1.2 are more general than Theorems A.1 and A.2 since the former hold true for nonlinear elliptic operators as in (1.4).

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