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REGULARITY PROPERTIES OF OPTIMAL TRANSPORTATION PROBLEMS ARISING IN HEDONIC PRICING MODELS

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Abstract. We study a form of optimal transportation surplus functions which arise in hedonic pricing models. We derive a formula for the Ma–Trudinger–Wang curvature of these functions, yielding necessary and sufficient conditions for them to satisfy **(A3w)**. We use this to give explicit new examples of surplus functions satisfying **(A3w)**, of the form $b(x, y) = H(x + y)$ where H is a convex function on $\mathbb{R}^n$. We also show that the distribution of equilibrium contracts in this hedonic pricing model is absolutely continuous with respect to Lebesgue measure, implying that buyers are fully separated by the contracts they sign, a result of potential economic interest.

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1. Introduction

Fix Borel probability measures μ and ν on smooth manifolds X and Y, respectively, and a smooth surplus function $b : X \times Y \to \mathbb{R}$. Monge's optimal transportation problem is to find the map $F : X \to Y$, pushing μ forward to ν, which maximizes the transportation surplus:

$$\int_X b(x, F(x))\mathrm{d}\mu.$$

Assuming some regularity on μ (say absolute continuity with respect to local coordinates), sufficient conditions on b for the existence and uniqueness of the minimizer F were found by Levin [25], building on results of Gangbo and McCann [21], Caffarelli [5] and Gangbo [20] (condition **(A1)** in the next section). The regularity, or smoothness, of F is currently a very hot area of research. Ma *et al.* [30] found a condition **(A3s)** on b under which the optimizer must be smooth as long as the marginals μ and ν are smooth and bounded above and below. Trudinger and Wang then proved that a weaker version of this hypothesis, **(A3w)**, is in fact sufficient for the regularity of F [35,36] and Loeper showed that, even for rougher marginals, **(A3w)** is both necessary and sufficient for the continuity of F [27]. This framework generalizes and unifies a series of earlier regularity results obtained by Caffarelli [2,3], Urbas [37], Delanoe [8,9] and Wang [39]. Since then, many interesting results on regularity have been obtained; see, for example, [13, 15–19, 22–24, 26, 28, 29]. An interesting line of current research is to find examples of surplus functions satisfying **(A3w)** and **(A3s)**.

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One main goal in the present paper is to determine when **(A3w)** holds for a particular class of surplus functions. Explicitly, we are interested in functions of the form:

$$b(x, y) = \sup_{z \in Z} h(x, z) + g(y, z). \tag{1.1}$$

Our motivation in studying surplus functions of this form comes partially from mathematical economics. A recent paper by Chiappori *et al.* [7] demonstrated that finding an equilibrium in a certain hedonic pricing model is equivalent to solving an optimal transportation problem with a surplus function of this form (see also Ekeland [11,12] and Carlier and Ekeland [6] for another approach to this problem). We briefly review this model now. Imagine X parameterizes a population of consumer types who are looking to buy some specific good (say houses). The different directions in the manifold X may represent various characteristics which differentiate among types (for example, age, income, family size, etc.) and $\mathrm{d}\mu(x)$ represents the relative frequency of types with characteristics $x \in X$. Suppose now that Y parameterizes a space of sellers looking to produce and sell the same good (say companies looking to build customized houses). Again, the different directions in Y represent characteristics differentiating seller types from each other (for example, the size of the company and the location of its headquarters) and $d\nu(y)$ their relative frequencies. Suppose that Z represents the space of available goods that can potentially be produced (for example, the collection of all houses that can feasibly be built, characterized by their size, location, etc.). The space Z is often referred to as the space of contracts. The functions $h(x, z)$ and $g(y, z)$ represent the preference of consumer x to buy house z and the preference of seller y to build house z, respectively. The result of Chiappori *et al.* implies that the equilibrium coupling of buyers to sellers is also a solution to the optimal transportation problem with marginals μ and ν and surplus $b(x, y)$ given by equation (1.1). Despite their relevance in mathematical economics, however, optimal transportation problems with surplus functions of this form do not seem to have been studied systematically in the literature.

The **(A3w)** condition is that a certain tensor – the Ma–Trudinger–Wang curvature, defined in the next section – should be nonnegative on a certain set of vectors. One of our goals here is to find a formula for the Ma–Trudinger–Wang curvature of b in terms of g and h. This will then yield necessary and sufficient conditions on g and h in order for b to satisfy **(A3w)**.

In particular, our work here will produce sufficient conditions on g and h which will ensure that the equilibrium assignment of sellers to buyers in the hedonic pricing problem is continuous. In addition, this result should be of independent interest to mathematicians working in optimal transportation, as at present there are fairly few known examples of surplus functions satisfying **(A3w)**. Our result here yields new examples of such functions. In particular, we show that for special choices of the preference functions h and g, we obtain $b(x, y) = H(x+y)$, where H is a convex function on $\mathbb{R}^n$ [2]. Our work here yields a new formula for the Ma–Trudinger–Wang curvature in terms of the Legendre dual H^* of H; this formula splits into two terms, one of which is always positive. Therefore, for b to satisfy **(A3w)**, it suffices that the other term (which has a simple form) is non-negative. In this sense, it is simpler to verify **(A3w)** from this formula than the original formula of Ma, Trudinger and Wang.

In a recent paper by Figalli *et al.*, a variant of **(A3w)**, called **(B3w)**, emerged as a central concept in another type of economic problem [14]. This condition also asserts the positivity of the Ma–Trudinger–Wang curvature, but on a larger set of vectors than **(A3w)**. Let us mention here that, as we derive a formula for the Ma–Trudinger–Wang curvature, our method also yields necessary and sufficient conditions on g and h so that b satisfies **(B3w)**.

Another issue of economic interest that does not seem to have received much attention is the structure of the distribution μ_Z of *signed contracts*, which is a measure on the space Z of feasible contracts; we define μ_Z precisely in section 4. Economically, we can interpret the support of μ_Z as the set of contracts that are executed in equilibrium. In fact, the original version of the hedonic pricing problem, due to Ekeland, is equivalent to finding the measure on Z which minimizes the sum of the transportation costs to μ and ν [11]. This formulation

[2] Surplus functions of this form are reminiscent of a class of functions studied by Gangbo and McCann [21], who were interested in minimizing the transportation cost for cost functions of the form $l(|x - y|)$, where $l \geq 0$ is strictly concave.

is in fact equivalent to the formulation of Chiappori *et al.* and μ_Z coincides with the minimizing measure in Ekeland's formulation [6,7].

While Ekeland proved existences and uniqueness of an optimal measure in his formulation of the problem, it is natural to ask about the structure of this measure, and this issue is related to a number of questions of economic interest. For instance, does *bunching*, the phenomena where a positive fraction of buyers choose the same good in equilibrium, occur? This would correspond to point masses in μ_Z. On the other hand, is there local *separation of types* by contracts; that is, do all pairs x and $\overline{x}$ of nearby buyers choose distinct goods in equilibrium? The optimal transportation formulation of the hedonic pricing problem has already proven useful in resolving structural questions about the optimal coupling of buyers to sellers and it is reasonable to expect it to be helpful in addressing the structure of μ_Z as well.

Our main result in this direction is that, assuming twist and non-degeneracy conditions on g and h, defined in the next section, and equality of the dimensions of X, Y and Z, the function $x \mapsto z(x, F(x))$ mapping each buyer to the contract he signs is locally invertible almost everywhere. This implies local separation of types, and also the absolute continuity of μ_Z (which, in turn, implies global separation of types under stronger assumptions).

In the next section, we recall various structural hypotheses on the function b which arise in the regularity theory of optimal transportation, **(A3w)** and **(A3s)** being the most important. In Section 3, we derive conditions on g and h that ensure b will satisfy these hypotheses. In the final section, we study the structure of the distribution of contracts that gets signed in equilibrium.

It is a pleasure to thank Robert McCann for suggesting this problem to me and for useful discussions during the course of this work.

2. Assumptions and definitions

We will assume that the domains X, Y and Z can all be smoothly embedded in larger manifolds, such that their closures $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ are compact. We will also make several assumptions on the preference functions g and h.

1. The smooth manifolds X, Y, and Z all share the same dimension, which we will denote by n.
2. $h \in C^2(\overline{X} \times \overline{Z})$ and $g \in C^2(\overline{Y} \times \overline{Z})$.
3. For each (x, y) the supremum is attained by a unique $z(x, y) \in Z$.
4. For all (x, y), the $n \times n$ matrix $D^2_{zz}h(x, z(x, y)) + D^2_{zz}g(y, z(x, y))$ is non-singular.

Economically, $z(x, y)$ is the contract that maximizes the total utility of agents x and y; if, in equilibrium, x and y choose to conduct business with each other, $z(x, y)$ is the contract they sign.

Assumption (1) on the dimensions is not entirely necessary. Most of the literature on optimal transportation deals with equal dimensional spaces; one exception is a recent paper by the present author on the regularity of optimal transportation when the dimensions differ [34]. For economic applications, however, it may be desirable to allow these dimensions to differ. The dimensions of X, Y and Z may represent the number of characteristics used in the model to differentiate among buyers, sellers and contracts, respectively, and it is certainly possible that these will not coincide. For simplicity, we assume here that the dimensions are all equal, but note that it is straightforward to extend the analysis to the case $\dim(X) \geq \dim(Z) \geq \dim(Y)$, using the extensions of the conditions **(A0)**–**(A3s)** found in [33,34].

Definition 2.1. We say b is (x, y)-twisted if for all $x \in \overline{X}$, the mapping $y \mapsto D_x b(x, y)$ is injective on $\overline{Y}$. We say b is (y, x)-twisted if for all $y \in \overline{Y}$, the mapping $x \mapsto D_y b(x, y)$ is injective on $\overline{X}$. We say b is bi-twisted if it is both (x, y) and (y, x)-twisted.

If b is (x, y)-twisted, the map $y \mapsto D_x b(x, y)$ is invertible on its range and we will denote its inverse by $b\text{-}\exp_x(\cdot)$.

Definition 2.2. We say b is non-degenerate if for all (x, y) the matrix of mixed, second order partials $D^2_{xy}b(x, y)$ is non-singular.

We will use analogous terminology for g and h; for example, we will say that g is (y,z)-twisted if $z \mapsto D_y g(y,z)$ is injective.

The first three regularity conditions formulated by Ma, Trudinger and Wang are:

(A0) $b \in C^4(\overline{X} \times \overline{Y})$.
(A1) b is bi-twisted.
(A2) b is non-degenerate.

For sufficiently regular marginals μ and ν, **(A1)** implies the existence and uniqueness of a maximizer F, [5, 20, 21, 25]. The condition **(A2)**, in turn, implies that the graph of F is contained in an n-dimensional Lipschitz submanifold of the product $X \times Y$ [32].

Our next definition concerns the structure of the domain Y.

Definition 2.3. We say Y is b-convex if for all x the set $D_x b(x,Y) \subseteq T^*_x X$ is convex.

Ma *et al.* showed that the b-convexity of Y is necessary for the continuity of the map F for arbitrary smooth marginals μ and ν [30]. Assuming this condition, as well as, **(A0)**–**(A2)** they showed that under, **(A3s)**, which we define below, the optimal map F is smooth. Loeper then showed that the weakening **(A3w)** of this condition is necessary and sufficient for the continuity of F [27].

To formulate **(A3w)** and **(A3s)**, we will need the following definition.

Definition 2.4. Assume **(A0)**–**(A2)** hold and that Y is b-convex. Let $x \in X$ and $y \in Y$. Choose tangent vectors $v \in T_x X$ and $u \in T_y Y$. Set $q = D_x b(x,y) \in T^*_x X$ and $p = (D^2_{xy} b(x,y)) \cdot u \in T^*_x X$. For any smooth curve $\beta(s)$ in X with $\beta(0) = x$ and $\frac{\mathrm{d}\beta}{\mathrm{d}s}(0) = v$, we define the Ma–Trudinger–Wang curvature of b at x and y, in the directions v and u by:

$$MTW^b_{(x,y)}\langle v,u\rangle := \frac{\partial^4}{\partial s^2 \partial t^2} b(\beta(s), b\text{-}\exp_x(tp+q)).$$

A local coordinates expression for $MTW^b_{(x,y)}\langle v,u\rangle$ was first introduced by Ma *et al.* [30]. The formulation above is due to Loeper [27], who showed that $MTW^b_{(x,y)}\langle v,u\rangle$ is invariant under smooth changes of coordinates and, when, $-b(x,y)$ is the quadratic cost on a Riemannian manifold, it is equal to the sectional curvature along the diagonal. For general smooth surplus functions, Kim and McCann [23] showed that it is the sectional curvature of certain null planes, corresponding to a certain pseudo-Riemannian metric.

We can now state the final regularity conditions:

(A3w) For all $(x,y) \in X \times Y$, $v \in T_x X$ and $u \in T_y Y$ such that $v^T \cdot D^2_{xy} b \cdot u = 0$ we have $MTW^b_{(x,y)}\langle v,u\rangle \geq 0$.
(A3s) For all $(x,y) \in X \times Y$, $v \in T_x X$ and $u \in T_y Y$ such that $v^T \cdot D^2_{xy} b \cdot u = 0$ and $v,u \neq 0$ we have $MTW^b_{(x,y)}\langle v,u\rangle > 0$.

In subsequent sections, we will often refer to the curve $t \mapsto b\text{-}\exp_x(tp+q) \in Y$ as a *b-segment* in Y. We also note here that the conditions **(B3w)** and **(B3s)** found in [14] are equivalent to **(A3w)** and **(A3s)**, respectively, without the orthogonality condition $v^T \cdot D^2_{xy} b \cdot u = 0$.

3. Regularity properties of b

The aim of this section is to understand when b satisfies certain regularity properties, namely **(A1)**, which ensures the existence and uniqueness of the optimal map F, **(A2)**, ensuring the rectifiability of the graph of F, b-convexity of Y and **(A3w)**/**(A3s)**, governing the regularity of the optimal map.

First, we verify some simple facts.

Lemma 3.1. *The map $z(x,y)$ is continuously differentiable and b is C^2 smooth. For all x, y we have the following formulae:*

$$D_x b(x,y) = D_x h(x, z(x,y)) \tag{3.1}$$

$$D_y b(x,y) = D_y g(y, z(x,y)) \tag{3.2}$$

$$D_x z(x,y) = -M^{-1}(x,y) D^2_{zx} h(x,z) \tag{3.3}$$

$$D_y z(x,y) = -M^{-1}(x,y) D^2_{zy} g(y,z) \tag{3.4}$$

$$D^2_{xy} b(x,y) = -D^2_{xz} h(x,z) M^{-1}(x,y) D^2_{zy} g(y,z) \tag{3.5}$$

where $M(x,y) := D^2_{zz} h(x, z(x,y)) + D^2_{zz} g(y, z(x,y))$.

Proof. Note that, as $z(x,y)$ maximizes $z \mapsto h(x,z) + g(y,z)$, we have

$$D_z h(x, z(x,y)) + D_z g(y, z(x,y)) = 0. \tag{3.6}$$

As $D^2_{zz} h(x, z(x,y)) + D^2_{zz} g(y, z(x,y))$ is non-singular by assumption, the Implicit Function Theorem now implies that $z(x,y)$ is C^1. This, in turn, implies that $b(x,y) = h(x, z(x,y)) + g(y, z(x,y))$ is at least C^1. Now note that $b(x,y) - h(x,z) + g(y,z) \geq 0$, for all z, with equality when $z = z(x,y)$, which implies (3.1) and (3.2). Differentiating (3.6) with respect to x and y, respectively, yields (3.3) and (3.4). Differentiating (3.1) with respect to y and using (3.4) yields (3.5). $\square$

We will first prove results about **(A0)**–**(A2)**. These proofs are based on similar arguments, found in [7, 34]; we recreate them here for the reader's convenience.

Corollary 3.2. *If h and g satisfy **(A0)**, then b satisfies **(A0)** as well.*

Proof. Using (3.6) and the implicit function theorem we find that $z(x,y)$ is C^3; equations (3.1) and (3.2) together with the chain rule now imply the desired result. $\square$

Proposition 3.3. *If h and g are non-degenerate, then b is non-degenerate. If h is (x,z)-twisted and g is (z,y)-twisted, then b is (x,y)-twisted.*

Proof. The non-degeneracy part of the proposition follows immediately from (3.5).

Assume that h is (x,z) twisted and g is (z,y) twisted. Suppose we have $D_x b(x, y_0) = D_x b(x, y_1)$; we need to show $y_0 = y_1$. By (3.1), we have $D_x h(x, z(x, y_0)) = D_x h(x, z(x, y_1))$ and so by the twistedness of h we have $z(x, y_0) = z(x, y_1)$. Now, for $i = 0, 1$ we have $D_z h(x, z(x, y_i)) + D_z g(y_i, z(x, y_i)) = 0$ by 3.6. Therefore, as $z(x, y_0) = z(x, y_1)$,

$$D_z g(y_0, z(x, y_0)) = -D_z h(x, z(x, y_0)) = -D_z h(x, z(x, y_1)) = D_z g(y_1, z(x, y_1)).$$

Again using $z(x, y_0) = z(x, y_1)$, the equality $D_z g(y_0, z(x, y_0)) = D_z g(y_1, z(x, y_1))$ and the twistedness of g imply that $y_0 = y_1$ as desired. $\square$

Of course, an analogous result holds if h is (z,x)-twisted and g is (y,z)-twisted and so we immediately obtain the following.

Corollary 3.4. *If h and g satisfy **(A1)**, then so does b. If h and g satisfy **(A2)**, then so does b.*

Next, we consider the b-convexity condition.

Proposition 3.5. *Assume Z is h-convex and for all h-segments z_t at x, $-D_z h(x, z_t)$ is in the domain of $g - \exp_{z_t}(\cdot)$. Then the domain Y is b-convex.*

Proof. Let $D_x b(x, y_i) = p_i$, for $i = 0, 1$. For all $t \in [0, 1]$, we must show that there is some $y_t \in Y$ such that $D_x b(x, y_t) = t(p_1 - p_0) + p_0 := p_t$. Let $z_i = z(x, y_i)$; then $D_x h(x, z_i) = D_x b(x, y_i) = p_i$ for $i = 0, 1$, by 3.1 in Lemma 3.1. The given conditions imply the existence of a z_t such that $D_x h(x, z_t) = p_t$ and a y_t such that $D_z h(x, z_t) + D_z g(y_t, z_t) = 0$. Therefore, $z_t = z(x, y_t)$, and so $D_x b(x, y_t) = D_x h(x, z_t) = p_t$, as desired. □

We now provide a formula for the Ma–Trudinger–Wang curvature in terms of g and h.

Theorem 3.6. *Let $x \in X$, $z \in Z$, $v \in T_x X$ and z_t be a b-segment at x. Set $y_t = g\text{-}\exp_{z_t}(-D_z h(x, z_t))$. Then, for $u = \dot{y_0}$, $MTW^b_{(x,y_0)}\langle v, u\rangle$ is given by*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} v^T \cdot \Big(D^2_{xx}h(x, z_t) - D^2_{xz}h(x, z_t) \cdot \big(D^2_{zz}h(x, z_t) + D^2_{zz}g(y_t, z_t)\big)^{-1} \cdot D^2_{zx}h(x, z_t)\Big) \cdot v. \tag{3.7}$$

To prove this result, we will need a simple lemma about the Ma–Trudinger–Wang curvature. This lemma is well known, and the resulting formula for the Ma–Trudinger–Wang curvature appears in, for example, [22], but it does not seem to be proven explicitly in the literature, so we provide a proof below.

Lemma 3.7. *Let y_t be a b-segment at x: $D_x b(x, y_t) = tp + q$. Then, for $u = \dot{y_0}$,*

$$MTW^b_{(x,y_0)}\langle v, u\rangle = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} v^T \cdot D^2_{xx}b(x, y_t) \cdot v.$$

Proof. Letting x_s be a curve such that $x_0 = x$ and $\dot{x_0} = v$, we have

$$\begin{aligned}
MTW^b_{(x,y_0)}\langle v, u\rangle &= \frac{\mathrm{d}^4}{\mathrm{d}t^2\mathrm{d}s^2}\Big|_{t=0,s=0} b(x_s, y_t) \\
&= \frac{\mathrm{d}^3}{\mathrm{d}t^2\mathrm{d}s}\Big|_{t=0,s=0} D_x b(x_s, y_t) \cdot \dot{x_s} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0,s=0} (\dot{x_s}^T \cdot D^2_{xx}b(x_s, y_t) \cdot \dot{x_s} + D_x b(x_s, y_t) \cdot \ddot{x_s}) \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} (v^T \cdot D^2_{xx}b(x, y_t) \cdot v) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} (D_x b(x, y_t) \cdot \ddot{x_0}) \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} (v^T \cdot D^2_{xx}b(x, y_t) \cdot v) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} ((pt + q) \cdot \ddot{x_0}) \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} (v^T \cdot D^2_{xx}b(x, y_t) \cdot v).
\end{aligned}$$

□

We can now prove Proposition 3.6.

Proof. In light of the preceding lemma, it suffices to show:

$$D^2_{xx}b(x, y_t) = D^2_{xx}h(x, z_t) - D^2_{xz}h(x, z_t) \cdot \big(D^2_{zz}h(x, z_t) + D^2_{zz}g(y_t, z_t)\big)^{-1} \cdot D^2_{zx}h(x, z_t).$$

By Lemma 3.1,

$$D_x b(x, y) = D_x h(x, z(x, y)).$$

Differentiating this equation with respect to x, noting that $z(x, y_t) = z_t$ and using the formula in Lemma 3.1 for $D_x z(x, y)$ yields the desired result. □

Equation (3.7) for the Ma–Trudinger–Wang curvature naturally splits into two terms:

$$MTW^b_{(x,y_0)}\langle v, u\rangle = A + B$$

where

$$A = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} v^T \cdot \Big(D^2_{xx}h(x,z_t)\Big) \cdot v,\ \ B = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} v_t^T \cdot (M_t)^{-1} \cdot v_t.$$

Here $M_t = D^2_{zz}h(x,z_t) + D^2_{zz}g(y_t,z_t)$, and $v_t = D^2_{zx}h(x,z_t)\cdot v$. The first term, A, is exactly the Ma–Trudinger–Wang curvature of h. The second term B, can be further refined using the chain rule and the formulae for the derivatives of the matrix M_t^{-1}:

$$(\dot{M_t^{-1}}) = -M_t^{-1}\dot{M}_tM_t^{-1},\ (\ddot{M_t^{-1}}) = -M_t^{-1}\ddot{M}_tM_t^{-1} + 2M_t^{-1}\dot{M}_tM_t^{-1}\dot{M}_tM_t^{-1}.$$

Differentiating $v_t^T \cdot (M_t)^{-1} \cdot v_t$ twice, applying these formulae and using the symmetry of M_t and its derivatives yields:

$$\begin{aligned} B = \Big(&- 2\ddot{v}_t{}^T \cdot M_t^{-1} \cdot v_t - 2\dot{v}_t{}^T \cdot M_t^{-1} \cdot \dot{v}_t + 4\dot{v}_t{}^T \cdot M_t^{-1}\dot{M}_tM_t^{-1} \cdot v_t \\ &+\ v_t^T \cdot M_t^{-1}\ddot{M}_tM_t^{-1} \cdot v_t - 2v_t^T \cdot M_t^{-1}\dot{M}_tM_t^{-1}\dot{M}_tM_t^{-1} \cdot v_t\Big)|_{t=0}. \end{aligned} \tag{3.8}$$

Now, note that the maximality of $z \mapsto h(x_t,z)+g(y_t,z)$ at $z_t = z(x,y_t)$ implies that M_t, the second derivative of this map, is negative semi-definite. Assumption 4 at the beginning of Section 2 asserts that M_t is non-singular, and therefore it is negative-definite. The second and last terms in (3.8) are therefore non-negative, due to the negative definiteness of M_t^{-1} and the symmetry of M_t^{-1} and $\dot{M}_t$.

3.1. Convex functions of the sum

For appropriate forms of the functions h and g, $b(x,y) = H(x+y)$ becomes an arbitrary convex function of the sum. Understanding when functions of this form have positive cross curvature is interesting in its own right. In this setting, the present approach – essentially doing calculations on H^* rather than H – has a distinct advantage; b-segments for H correspond to ordinary line segments for the dual variables. That is, instead of evaluating H along a b-segment, we evaluate H^* along a line.

Proposition 3.8. *Take $X = Y = Z = \mathbb{R}^n$ and set $h(x,z) = x\cdot z - H^*(z)$, where H^* is the Legendre transform of some smooth, uniformly convex function $H : \mathbb{R}^n \to \mathbb{R}$, and $g(y,z) = y\cdot z$. Then $b(x,y) = H(x+y)$ and $MTW^b_{(x,y)}(v,u)$ is given by:*

$$\sum_{i,j,k,l} -\frac{\partial^4 H^*}{\partial z^i\partial z^j\partial z^k\partial z^l}p_kp_lw_iw_j + 2\sum_{i,j,k,l,a,r}\frac{\partial^3 H^*}{\partial z^i\partial z^l\partial z^k}\frac{\partial^3 H^*}{\partial z^r\partial z^j\partial z^a}\frac{\partial^2 H}{\partial z^l\partial z^r}w_iw_jp_ap_k \tag{3.9}$$

where $p = D^2H(x+y)\cdot u$, $w = D^2H(x+y)\cdot v$.

Proof. The curve y_t is given by $DH(x+y_t) = tp+q$, or $y_t = DH^*(tp+q)-x$, and z_t is given by $z_t = D_xh(x,z_t) = tp+q$. Note that $A = 0$, as $D^2_{xx}h(x,z_t) = 0$. Turning to B, note that as $D^2_{xz}h(x,z_t) = I$, we have $v_t = v$, and $M_t = -D^2H^*(tp+q)$, so that $\dot{M}_t = -D^3H^*(tp+q)\cdot p$, or, in matrix notation, $(\dot{M}_t)_{ij} = -\sum_k \frac{\partial^3 H^*}{\partial z^i\partial z^j\partial z^k}p_k$. Similarly, $(\ddot{M}_t)_{ij} = -\sum_{k,l}\frac{\partial^4 H^*}{\partial z^i\partial z^j\partial z^k\partial z^l}p_kp_l$.

Now, the first three terms in (3.8) vanish. The final two terms are exactly the desired expression. □

The formula (3.9) for the Ma–Trudinger–Wang curvature does not seem much simpler than the original formula of Ma *et al.* [30], but it has an important advantage; the second term is always non-negative, because of the symmetry of mixed partials and the positive definiteness of D^2H. It is therefore straightforward to find examples of functions H^* so that $H(x+y)$ satisfies ($A3w$). We obtain immediately, for example, that on any domain where the fourth order derivatives of H^* are small compared to its third order derivatives and the second order derivatives of H, b satisfies **(A3s)** (in fact, it satisfies the stronger condition **(B3s)**). In this sense, it is easier to check **(A3w)** and **(A3s)** of $H(x+y)$ using formula (3.9) than using the traditional formula, which involves H but not its dual H^*. We work out some explicit examples below.

Of course, from an economic perspective, formula (3.9) has another advantage. When solving the hedonic pricing problem, one is given the functions g and h (in the preceding example, $g(y,z) = y \cdot z$ and $h(x,z) = x \cdot z - H^*(z)$) but one needs to calculate $b(x,y) = \sup_{z \in Z} g(y,z) + h(x,z)$. In may then be desirable to find the Ma–Trudinger–Wang curvature without determining b. Our main result, Theorem 3.6, shows how to do this in general and the example illustrates this for special forms of g and h. Of course, given H^*, one must still compute D^2H to use (3.9), but as $D^2H(x+y) = [D^2H^*(z(x,y))]^{-1}$, one can do this without direct knowledge of H.

Example 3.9. Let $H^*(z) = \sum_{i=1}^n A_i z^i + \sum_{i,j=1}^n B_{ij} z^i z^j + \sum_{i,j,k=1}^n C_{ijk} z^i z^j z^k$, where B is a positive definite $n \times n$ matrix. Then in a neighbourhood U of the origin, H^* is convex. As the fourth order derivatives of H^* vanish, Proposition 3.8 immediately implies that the Legendre transform, $b(x,y) = H(x+y)$, satisfies **(B3w)** on $V := DH^*(U)$.

Example 3.10. Suppose $H^*(z) = \sum_{\alpha=0}^{\infty} \sum_{i_1,i_2,\ldots i_\alpha=1}^n A^\alpha_{i_1,i_2,\ldots i_\alpha} z^{i_1} z^{i_2} \ldots z^{i_\alpha}$ is an analytic function, expressed via it's Taylor series at the origin. Here $A^\alpha_{i_1,i_2,\ldots i_\alpha} = \frac{1}{\alpha!} \frac{\partial H^*}{\partial z^{i_1} \partial z^{i_2} \ldots \partial z^{i_\alpha}}(0)$ is a symmetric α-tensor. Assume that

1. $A^2_{i_1 i_2} = \delta_{i_1 i_2}$.
2. $A^4_{i_1 i_2 i_3 i_4} = -\delta_{i_1 i_2}\delta_{i_3 i_4} - \delta_{i_1 i_3}\delta_{i_2 i_4} - \delta_{i_1 i_4}\delta_{i_2 i_3}$

The first condition ensures that H^* is convex in some neighbourhood U of the origin. The second ensures that

$$\begin{aligned} -\sum_{i,j,k,l} \frac{\partial^4 H^*}{\partial z^i \partial z^j \partial z^k \partial z^l} p_k p_l w_i w_j &= \alpha! \cdot \sum_{i,j,k,l} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) p_k p_l w_i w_j \\ &= \alpha! \cdot [|p|^2 |w|^2 + 2(p \cdot w)^2] \\ &= > 0. \end{aligned}$$

Therefore, $H(x+y)$ satisfies **(B3s)** at $DH^*(0)$ (and therefore in a neighbourhood of $DH^*(0)$).

4. Structure of the distribution of equilibrium contracts

Here we study the structure of the distribution of contracts that get signed in equilibrium in the hedonic pricing problem. As in previous sections, we will assume for simplicity that $\dim(X) = \dim(Y) = \dim(Z) := n$, but remark that Proposition 4.1 may be extended to the case where $\dim(X) \geq \dim(Z) \geq \dim(Y)$ in a straightforward way.

Assuming that μ assigns zero mass to every set of Hausdorff dimension less than or equal to $n-1$, the (x,y)-twist condition implies the existence of a unique map $F : X \to Y$ solving Monge's optimal transportation problem; as b is C^2, this map is differentiable almost everywhere [5, 20, 21, 25]. In economic terms, this means that, in equilibrium almost every agent x conducts business with a unique agent $y := F(x)$; they sign the contract $z(x, F(x))$. The fact that it is optimal for each buyer x to conduct business with exactly one seller, $y = F(x)$, is sometimes referred to as *purity* of the equilibrium in the economics literature. We define the distribution of signed contracts μ_Z to be the push forward of μ by the map $x \mapsto z(x, F(x))$; our goal here is to investigate the structure of this measure. Let us emphasize again that this measure coincides with the solution to the optimal pricing problem in the formulation on Ekeland [11].

Economically, the support of this measure can be interpreted as the set of all contracts that are executed in equilibrium, while $\mathrm{d}\mu_Z(z)$ represents the relative frequency of contracts that are executed in equilibrium.

Proposition 4.1. *Assume that h is (x,z)-twisted, g is (z,y)-twisted and that both g and h are non-degenerate. At any point where the map $x \mapsto z(x, F(x))$ is differentiable, it's derivative has full rank.*

Proof. Fix a point x_0 where F is well defined and differentiable. Set $y_0 = F(x_0)$ and $z_0 = z(x_0, y_0)$. It is well known that there is a function $u : X \to \mathbb{R}$, twice differentiable almost everywhere, such that for all x where u is differentiable we have

$$Du(x) = D_x b(x, F(x)).$$

Wherever F is differentiable, u is twice differentiable, and we have

$$D^2u(x) = D^2_{xx}b(x, F(x)) + D^2_{xy}b(x, F(x))DF(x). \tag{4.1}$$

In particular, the preceding equality holds at $x = x_0$. Set $P_0 = D^2u(x_0) - D^2_{xx}b(x_0, y_0)$; it is well known that $P_0 \geq 0$. Rearranging (4.1) implies

$$DF(x_0) = \big(D^2_{xy}b(x_0, y_0)\big)^{-1}P_0. \tag{4.2}$$

Now, the derivative of $x \mapsto z(x, F(x))$ at x_0 is

$$D_x z(x_0, y_0) + D_y z(x_0, y_0)DF(x_0)$$

Now, using the formulae in Lemma 3.1 together with (4.2) we see that this is equal to

$$\begin{aligned}&-M^{-1}(x_0, y_0) \cdot D^2_{zx}h(x_0, z_0) + \big(D^2_{xz}h(x_0, z_0)\big)^{-1} \cdot P_0 = \\ &\Big(-M^{-1}(x_0, y_0) + \big(D^2_{xz}h(x_0, z_0)\big)^{-1} \cdot P_0 \cdot \big(D^2_{zx}h(x_0, z_0)\big)^{-1}\Big) \cdot D^2_{zx}h(x_0, z_0)\end{aligned} \tag{4.3}$$

Now, as $M(x_0, y_0) < 0$, we have $-M^{-1}(x_0, y_0) > 0$. Also, $P_0 \geq 0$, hence

$$\big(D^2_{xz}h(x_0, z_0)\big)^{-1}P_0\big(D^2_{zx}h(x_0, z_0)\big)^{-1} = \big(D^2_{xz}h(x_0, z_0)\big)^{-1}P_0\Big(\big(D^2_{xz}h(x_0, z_0)\big)^{-1}\Big)^T \geq 0.$$

Therefore, the term in brackets in (4.3) is positive definite and thus invertible, which, as $D^2_{zx}h(x_0, z_0)$ is invertible, implies the invertibility of (4.3). □

It is interesting to note that this Proposition 4.1 immediately implies the local separation of types by contracts: choose any x where F is differentiable. Then there is a neighborhood U of x on which $x \mapsto z(x, F(x))$ is injective; that is, no two buyers on U will choose the same contract in equilibrium. This should be compared with the separation of buyers by sellers; in the case where g and h are bi-twisted, then so is b by Corollary 3.4. Assuming that ν is absolutely continuous with respect to local coordinates, this implies that the map F is invertible almost everywhere. This means that almost no two buyers x and $\overline{x}$ conduct business with the same seller y; that is, sellers fully separate buyers.

This proposition, combined with the change of variables formula immediately implies that a large set of contracts gets signed in equilibrium:

Corollary 4.2. *Assume the conditions in Proposition 4.1. If in addition μ is absolutely continuous with respect to Lebesgue measure, then the distribution μ_Z of signed contracts is absolutely continuous with respect to Lebesgue measure.*

Proof. The result follows easily from a sufficiently general change of variables formula (for example, [38]) Theorem 11.1. □

Finally, we show that the preceding Corollary actually implies *global* separation of types if we assume in addition that h is (z, x) twisted. This means that the map $x \mapsto z(x, F(x))$ is globally injective almost everywhere, so that buyers of different types almost always buy goods of different types.

Corollary 4.3. *Assume the conditions in Proposition 4.1, and that h is (z, x) twisted. If μ is absolutely continuous with respect to Lebesgue measure, then the map $z(x, F(x))$ is invertible almost everywhere.*

Proof. It is proven in [6] that the map $x \mapsto z(x, F(x))$ is in fact the optimal map between μ and μ_Z; that is, $(Id, z(\cdot, F(\cdot)))_{\#}\mu$ is the unique maximizer of

$$\int_{X \times Z} h(x, z) \mathrm{d}\gamma$$

among measures γ on $X \times Z$ with marginals μ and μ_Z. Now, as μ_Z is absolutely continuous with respect to Lebesgue measure h is (z, x) twisted, a standard result in optimal transport implies that this optimal measure is concentrated on the graph of a function $G : Z \to X$ [5, 20, 21, 25]. It is then straightforward to verify that $z(G(z), F(G(z))) = z$ almost everywhere, implying the almost everywhere injectivity of $x \mapsto z(x, F(x))$. □

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