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ON SHAPE OPTIMIZATION PROBLEMS INVOLVING THE FRACTIONAL LAPLACIAN

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Abstract. Our concern is the computation of optimal shapes in problems involving $(-\Delta)^{1/2}$. We focus on the energy $J(\Omega)$ associated to the solution u_Ω of the basic Dirichlet problem $(-\Delta)^{1/2}u_\Omega = 1$ in Ω, $u = 0$ in Ω^c. We show that regular minimizers Ω of this energy under a volume constraint are disks. Our proof goes through the explicit computation of the shape derivative (that seems to be completely new in the fractional context), and a refined adaptation of the moving plane method.

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1. Introduction

This article is concerned with shape optimization problems involving the fractional laplacian. The typical example we have in mind comes from the following system:

$$\begin{aligned}(-\Delta)^{1/2}u &= 1 \text{ on } \Omega,\\ u &= 0 \text{ on } \Omega^c,\end{aligned} \tag{1.1}$$

set for a bounded open set Ω of $\mathbb{R}^2$. We wish to find the minimizers of the associated energy

$$J(\Omega) := \inf_{\substack{v\in H^{1/2}(\mathbb{R}^2),\\ v|_{\Omega^c}=0}} \left(\frac{1}{2}\left\langle(-\Delta)^{1/2}v, v\right\rangle_{H^{-1/2},H^{1/2}} - \int_{\mathbb{R}^2} v\right). \tag{1.2}$$

among open sets Ω with prescribed measure (and a smoothness assumption). Beyond this specific example, we wish to develop mathematical tools for shape optimization in the context of fractional operators.

Our original motivation comes from a drag reduction problem in microfluidics. Recent experiments, carried on liquids in microchannels, have suggested that drag is substantially lowered when the wall of the channel is water-repellent and rough [11,16]. The idea is that the liquid sticks to the bumps of the roughness, but may slip over its hollows, allowing for less friction at the boundary. Mathematically, one can consider Stokes equations

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for the liquid (variable $(x,z) = (x_1, x_2, z)$, velocity field $u = (v,w) = (v_1, v_2, w)$):

$$-\Delta u + \nabla p = 0, \quad \nabla \cdot u = 0, \quad z > 0, \tag{1.3}$$

set above a flat surface $\{z = 0\}$. This flat surface replaces the rough hydrophobic one, and is composed of areas on which the fluid satisfies alternately perfect slip and no-slip boundary conditions. In the simplest models, these areas of perfect slip and no-slip form a periodic pattern, corresponding to a periodic pattern of hollows and bumps. That means that the impermeability condition $w = 0$ at $\{z = 0\}$ is completed with mixed Dirichlet/Navier conditions:

$$\partial_z v = 0 \ \text{ in } \Omega, \quad v = 0 \text{ in } \Omega^c \tag{1.4}$$

where $\Omega \subset \mathbb{R}^2$ corresponds to the zones of perfect slip and Ω^c to the zones of no-slip. The whole issue is to design Ω so that the energy $J(\Omega)$ associated with this problem is minimal for a fixed fraction of slip area. Unfortunately, this optimization problem (Stokes operator, periodic pattern) is still out of reach. That is why we start with the simpler equations (1.1) (still difficult, and interesting on their own!). Note that they can be seen as a scalar version of (1.3)–(1.4), replacing the Stokes operator by the Laplacian, and the Navier by the Neumann condition. Using the classical characterization of $(-\Delta)^{1/2}$ as the Dirichlet-to-Neumann operator leads to system (1.1) and energy (1.2). Again, we stress that the methods developed in our paper may be useful in more elaborate contexts.

If the fractional laplacian in (1.1) is replaced by a standard laplacian (which leads to the classical Dirichlet energy), this problem is well-known and described in detail in the book [10] by Henrot and Pierre (see also [14]). In this case, one can show that a smooth domain Ω minimizing the Dirichlet energy under the constraint $|\Omega| = 1$ is a disc. A standard proof of this result has two main steps:

1. One computes the shape derivative associated with the Dirichlet energy for the laplacian. This leads to the following result: if Ω is a minimizer of the Dirichlet energy and u_Ω is the solution of the associated Euler–Lagrange equation, then $\partial_n u$ is constant on $\partial\Omega$.
2. One analyzes an overdetermined problem. More precisely, the idea is to prove that if there exists a function u solving

$$\begin{aligned} -\Delta u &= 1 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \\ \partial_n u &= c \text{ on } \partial\Omega, \end{aligned}$$

then all the connected components of Ω are discs. This second step is achieved thanks to the moving plane method.

Our goal in this article is to develop the same approach in the context of the fractional laplacian, showing radial symmetry of any smooth minimizer Ω of (1.2). Accordingly, we start with the computation of the shape derivative.

Theorem 1.1. *Let $f \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$, and let*

$$J_f(\Omega) := \inf_{\substack{v \in H^{1/2}(\mathbb{R}^2), \\ v|_{\Omega^c} = 0}} \left(\frac{1}{2} \left\langle (-\Delta)^{1/2} v, v \right\rangle_{H^{-1/2}, H^{1/2}} - \int_{\mathbb{R}^2} f v \right).$$

Let $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$, and let $(\phi_t)_{t\in\mathbb{R}}$ be the flow associated with ζ, namely $\dot{\phi}_t = \zeta(\phi_t), \quad \phi_0 = \mathrm{Id}$.

Let Ω be an open set with $\mathcal{C}^\infty$ boundary, and let $u_{\Omega,f}$ be the unique minimizer of $J_f(\Omega)$,

$$\begin{aligned} (-\Delta)^{1/2} u_{\Omega,f} &= f \quad \text{in } \Omega, \\ u_{\Omega,f} &= 0 \quad \text{in } \Omega^c. \end{aligned}$$

Then, denoting by $n(x)$ the outward pointing normal vector to $\partial\Omega$,

1. *For all $x \in \partial\Omega$, the limit*

$$\lim_{y \to x, y \in \Omega} \frac{u_{\Omega,f}(y)}{|(y-x)n(x)|^{1/2}}$$

exists; we henceforth denote it by $\partial_n^{1/2} u_{\Omega,f}(x)$.
2. *The function $\partial_n^{1/2} u_{\Omega,f}$ belongs to $\mathcal{C}^1(\partial\Omega)$.*
3. *There exists an explicit constant C_0, which does not depend on Ω, such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} J_f(\phi_t(\Omega))_{|t=0} = C_0 \int_{\partial\Omega} (\partial_n^{1/2} u_{\Omega,f})^2 \zeta \; n \, \mathrm{d}\sigma.$$

This theorem implies easily

Corollary 1.2. *Assume that there exists an open set $\Omega \subset \mathbb{R}^2$ such that Ω has $\mathcal{C}^\infty$ regularity, $|\Omega| = 1$, and*

$$J(\Omega) = \inf_{\substack{\Omega' \subset \mathbb{R}^2, \\ |\Omega'|=1}} J(\Omega').$$

Let u_Ω be the solution of the associated Euler-Lagrange equation (1.1), and let $n(x)$ be the outward pointing normal at $x \in \partial\Omega$. Then, $\partial_n^{1/2} u_\Omega$ exists, and there exists a constant $c_0 \geq 0$ such that

$$\partial_n^{1/2} u_\Omega(x) = c_0 \quad \forall x \in \partial\Omega.$$

Taking into account this extra condition on the fractional normal derivative, we can then determine the minimizing domain:

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a $\mathcal{C}^\infty$ open set such that the system*

$$\begin{aligned} (-\Delta)^{1/2} u &= 1 \text{ in } \Omega, \\ u &= 0 \text{ in } \Omega^c, \\ \partial_n^{1/2} u &= c_0 \text{ on } \partial\Omega. \end{aligned} \tag{1.5}$$

has at least one solution. Assume that Ω is connected. Then Ω is a disc.

Of course, Theorem 1.1 and Corollary 1.3 imply immediately the following:

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a $\mathcal{C}^\infty$ connected open set such that $|\Omega| = 1$, and*

$$J(\Omega) = \inf_{\substack{\Omega' \subset \mathbb{R}^2, \\ |\Omega'|=1}} J(\Omega').$$

Then Ω is a disc.

Let us make a few comments on our results. As regards Theorem 1.1, we stress that, due to the non-locality of the fractional laplacian, the shape derivative of J_f is hard to compute. In the case of the classical laplacian, it is obtained through integration by parts, which are completely unavailable in the present context. The idea is to bypass the nonlocality by using an asymptotic expansion of $u_{\Omega,f}(x)$ as the distance between x and $\partial\Omega$ goes to zero. Such asymptotic expansion follows from general results of [7], on the solutions of linear elliptic systems in domains with cracks. We insist that the proof of the theorem neither uses scalar arguments, nor Fourier-based calculations. In particular, we believe that its interet goes much beyond Corollary 1.4 (that is finding the minimizer of (1.2)). It is likely that it can be adapted to vectorial settings, or functionals with non-constant coefficients. Notice also that Theorem 1.1 and Corollary 1.2 do not require the connectedness of Ω.

For our special case $f = 1$, they imply that the value of the fractional normal derivative is the same on all the boundaries of the connected components of Ω.

As regards Theorem 1.3, it is deduced from an adaptation of the moving plane method to our fractional setting. Again, this is not straightforward, as the standard method relies heavily on the maximum principle and Hopf's Lemma, which are essentially local tools. To overcome our non-local problem, we use appropriate three-dimensional extensions of u, to which we can apply maximum principle methods in a classical context. Note however that we need to assume that Ω is connected: this hypothesis is precisely due to the nonlocality of the fractional laplacian.

We conclude this introduction by a brief review of related results. Let us first mention that the condition $\partial_n^{1/2} u = c_0$ appears in other problems related to the fractional laplacian. In [6], Caffarelli, Roquejoffre and Sire consider a minimization problem for another energy related to the fractional laplacian, and they prove that minimizers satisfy the condition $\partial_n^{1/2} u = c_0$ on $\partial\{u = 0\}$. However, we emphasize that the issues of the present paper and those of [6] are rather different. The problem addressed in [6] is essentially a free-boundary problem (*i.e.* Ω is not given, but is defined as $\{u > 0\}$), and therefore questions such as the regularity and the non-degeneracy of u, and the regularity of the free boundary, are highly non trivial and are at the core of the paper [6]. Here, our goal is not to investigate these questions, but rather to derive information on the shape of Ω, assuming *a priori* regularity. As mentioned before, we rely on article [7] by Costabel and co-authors, which provides asymptotic expansions for solutions of linear elliptic equations near cracks. As they derive accurate asymptotics, based on pseudo-differential calculus, they need the domain to be C^∞. In our context, only cruder information is needed (broadly, we need the first term in the expansion, and tangential regularity). Apart from these asymptotic expansions, the proofs of Theorems 1.1 and 1.3 use very little information on the regularity of $\partial\Omega$ (existence of tangent and normal vectors, boundedness and regularity of the curvature). Hence, it is likely that our C^∞ regularity requirement can be lowered.

As regards our adaptation of the moving plane method, it relates to other results on the proof of radial symmetry for minimizers of nonlocal functionals: see for instance [13] on local Riesz potentials

$$u(x) = \int_\Omega \frac{1}{|x-y|^{N-1}}$$

or [5] on the radial symmetry of solutions of nonlinear equations involving $A^{1/2}$, where A is the Dirichlet laplacian of a ball in $\mathbb{R}^n$. Note that in these two papers, "local" maximum principles are still available, which helps. Further references (notably to article [3]) will be provided in due course.

Let us eventually point out that more direct proofs of the final Corollary 1.4 might be available. For instance, in the case of the classical laplacian, one way to proceed is to consider the auxiliary problem: *minimize*

$$E(u) \;:=\; \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \;-\; \int_{\mathbb{R}^2} u$$

under the measure constraint: $|\{u > 0\}| = 1\}$. Crudely, any minimizer u of E provides a solution $\Omega_u = \{u > 0\}$ of the shape optimization problem, and vice versa. We refer once again to [10] for rigourous statements. In particular, showing that any minimizer of E is radial shows that any minimizing domain is radial (without *a priori* regularity assumption).

In the case of the fractional laplacian, a close context has been recently investigated by Lopes and Maris in [12]. Their result is the following:

Proposition 1.5. *Let $s \in (0,1)$ and assume that $F, G : \mathbb{R} \to \mathbb{R}$ are such that $u \mapsto F(u)$, $u \mapsto G(u)$ map $\dot{H}^s(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Assume that $u \in \dot{H}^s(\mathbb{R}^d)$ is a solution of the minimization problem*

$$\textit{Minimize } E(u) := \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi + \int_{\mathbb{R}^d} F(u) \quad \textit{under the constraint } \int_{\mathbb{R}^d} G(u) = \lambda.$$

Then u is radially symmetric.

Looking closely at their proof, it seems that their arguments can be extended as such to the functionals

$$F(u) := u, \quad G(u) := \mathbf{1}_{u>0}$$

although such F and G do not map $\dot H^{1/2}(\mathbb{R}^2)$ into $L^1(\mathbb{R}^2)$. This is likely to yield the radial symmetry of the minimizing domain for our shape functional J (without *a priori* regularity assumption). See Remark 4.5 for further discussion.

The plan of our paper is the following: Section 2 collects more or less standard results on the fractional laplacian, which will be used throughout the article. Special attention is paid to regularity properties of solutions of (1.1), that we deduce from regularity results for the Laplace equations in domains with cracks. Section 3 is devoted to the proof of Theorem 1.1 and Corollary 1.2. Finally, Section 4 contains the proof of Theorem 1.3.

2. Preliminaries

2.1. Reminders on the fractional laplacian

We remind here some basic knowledge about $(-\Delta)^{1/2}$, see for instance [15]. We start with

Definition 2.1. For any $f \in \mathcal{S}(\mathbb{R}^n)$, one defines $(-\Delta)^{1/2} f$ through the identity

$$\widehat{(-\Delta)^{1/2}} f(\xi) = |\xi| \hat f(\xi).$$

Note that $g = (-\Delta)^{1/2} f$ does not belong to $\mathcal{S}(\mathbb{R}^n)$ because of the singularity of $|\xi|$ at 0. Nevertheless, it is C^∞ and satisfies for all $k \in \mathbb{N}$:

$$\sup_{x\in\mathbb{R}^n} (1+|x|^{n+1})|g^{(k)}(x)| < +\infty$$

This allows for a definition of $(-\Delta)^{1/2}$ over a large subspace of $\mathcal{S}'(\mathbb{R}^n)$, by duality. We shall only retain

Proposition 2.2. *$(-\Delta)^{1/2}$ extends into a continuous operator from $H^{1/2}(\mathbb{R}^n)$ to $H^{-1/2}(\mathbb{R}^n)$.*

This is clear from the definition.

It is also well-known that $(-\Delta)^{1/2}$ can be identified with the Dirichlet-to-Neumann operator, in the following sense (writing $(x,z) \in \mathbb{R}^n \times \mathbb{R}$ the elements of $\mathbb{R}^{n+1}$):

Theorem 2.3. *Let $u \in H^{1/2}(\mathbb{R}^n)$. One has $(-\Delta)^{1/2} u = -\partial_z U|_{z=0}$, where U is the unique solution in*

$$\dot H^1(\mathbb{R}^{n+1}_+) \ := \ \left\{ U \in L^2_{loc}(\mathbb{R}^{n+1}_+), \quad \nabla U \in L^2(\mathbb{R}^{n+1}_+) \right\}$$

of

$$-\Delta_{x,z} U = 0 \ \text{ in } \mathbb{R}^{n+1}_+, \quad U|_{z=0} = u.$$

Let us remind that the normal derivative $\partial_z U|_{z=0} \in H^{-1/2}(\mathbb{R}^n)$ has to be understood in a weak sense: for all $\phi \in H^1(\mathbb{R}^{n+1}_+)$,

$$\langle \partial_z U|_{z=0}, \gamma\phi\rangle \ = \ -\int_{\mathbb{R}^{n+1}_+} \nabla U \cdot \nabla \phi$$

where γ is the trace operator (onto $H^{1/2}(\mathbb{R}^n)$). It coincides with the standard derivative whenever U is smooth.

We end this reminder with a formula for the fractional laplacian:

Theorem 2.4. *Let f satisfying $\int_{\mathbb{R}^n} \frac{|f(x)|}{1+|x|^{n+1}} \mathrm{d}x < +\infty$, with regularity $C^{1,\epsilon}$, $\epsilon > 0$, over an open set Ω. Then, $(-\Delta)^{1/2} f$ is continuous over Ω, and for all $x \in \Omega$, one has*

$$(-\Delta)^{1/2} f(x) = C_1 \,\mathrm{PV} \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+1}}\,\mathrm{d}y \tag{2.1}$$

$$= C_1 \int_{\mathbb{R}^n} \frac{f(x)-f(y) - \nabla f(x)\cdot(x-y)\mathbf{1}_{|x-y|\le C}}{|x-y|^{n+1}}\,\mathrm{d}y. \tag{2.2}$$

for some $C_1 = C_1(n)$ and any $C > 0$.

2.2. The Dirichlet problem for $(-\Delta)^{1/2}$. Regularity properties (2d case)

In view of system (1.1), a key point in our analysis is to know the behavior near the boundary of solutions to the following fractional Dirichlet problem:

$$\begin{aligned} (-\Delta)^{1/2}u = f \text{ on } \Omega, \\ u = 0 \text{ on } \Omega^c, \end{aligned} \tag{2.3}$$

where Ω is a smooth open set of $\mathbb{R}^2$ and $f \in C^\infty(\mathbb{R}^2)$. Note that in system (2.3), we prescribe the value of u not only at $\partial\Omega$, but in the whole Ω^c. This is reminiscent of the non-local character of $(-\Delta)^{1/2}$: remind that $u = U|_{z=0}$, where U satisfies the (3d) local problem

$$\begin{aligned} -\Delta U = 0 \;\text{ in }\; z > 0, \\ \partial_z U = -f \;\text{ over }\; \Omega \times \{0\}, \quad U = 0 \;\text{ over }\; \Omega^c \times \{0\}, \end{aligned} \tag{2.4}$$

whose mixed Robin/Dirichlet boundary condition must be specified over the whole plane $\{z = 0\}$.

We were not able to find direct references for regularity properties of problem (2.3), although the $C^{0,1/2}$ regularity of u and the existence of $\partial_n^{1/2} u$ are evoked in [4, 6, 9]. In particular, we could not collect information on transverse and tangential regularity of the solution u near $\partial\Omega \times \{0\}$. We shall use results for the Laplace equation in domains with cracks, in the following way. Let $\eta \in C_c^\infty(\mathbb{R}^3)$, odd in z, with $\theta(x, z) = z$ for x in a neighborhood of $\bar{\Omega}$ and $|z| \leq 1$. Then, $V(x, z) := U(x, z) + \eta(x, z)f(x)$ satisfies

$$\begin{aligned} -\Delta V = F \;\text{ in }\; z > 0, \\ \partial_z V = 0 \;\text{ over }\; \Omega \times \{0\}, \quad U = 0 \;\text{ over }\; \Omega^c \times \{0\}, \end{aligned} \tag{2.5}$$

where $F := -\Delta_{x,z}(\eta f)$ is smooth, odd in z, and compactly supported in $\mathbb{R}^3$. We then extend V to $\{z < 0\}$ by the formula $V(x, z) := -V(x, -z)$, $z < 0$. In this way, we obtain the system

$$-\Delta V = F \;\text{ in }\; \mathbb{R}^3 \setminus (\Omega \times \{0\}) \tag{2.6}$$

$$\partial_z V = 0 \;\text{ at }\; (\Omega \times \{0\})^\pm \tag{2.7}$$

which corresponds to a Laplace equation outside a "crack" $\Omega \times \{0\}$ with Neumann boundary condition on each side of the crack. We can now use regularity results for the laplacian in singular domains, such as those of [7]. First, note that V is C^∞ away from $\Omega \times \{0\}$, by standard elliptic regularity. Let now Γ be a connected component of $\partial\Omega$, and φ a truncation function such that $\varphi = 1$ in a neighborhood of Γ, with $\operatorname{Supp}\varphi \cap (\partial\Omega \setminus \Gamma) = \emptyset$. Then, φV still satisfies a system of type (2.6), with Ω replaced by $\operatorname{Supp}\varphi$, and F by $F - [\Delta, \varphi]V$ (which is still smooth). We can then apply [7], Theorem A.4.3, which leads to the following

Theorem 2.5. *Let Γ be a connected component of $\partial\Omega$. We denote by (r, θ) polar coordinates in the planes normal to Γ and centered on Γ, and by s the arc-length on Γ, so that*

$$\mathbb{R}^3 \setminus ((\operatorname{Supp}\varphi \cap \Omega) \times \{0\}) = \{(s, r, \theta),\; s \in (0, L),\; r > 0,\; \theta \in (-\pi, \pi)\}.$$

Then, the solution V of (2.6) has the following asymptotic expansion, as $r \to 0$: for any integer $K \geq 0$,

$$V = \sum_{k=0}^{K} r^{1/2+k} \Psi^k(s, \theta) \;+\; U_{\mathrm{reg},K} \;+\; U_{\mathrm{rem},K} \tag{2.8}$$

where:

- *the coefficients Ψ^k are regular over $[0, L] \times [-\pi, \pi]$;*
- *$U_{reg,K}$ is regular over $\mathbb{R}^3$;*

• *the remainder* $U_{\mathrm{rem},K}$ *satisfies* $\partial^\beta U_{\mathrm{rem},K} = o(r^{K-|\beta|+1/2})$ *for all* $\beta \in \mathbb{N}^3$.

Setting

$$\psi^k(s) := \Psi^k(s,\pi),\ u_{\mathrm{rem},K}(x) := U_{\mathrm{rem},K}(x,0^+),\ u_{\mathrm{reg},K}(x) := U_{\mathrm{reg},K}(x,0^+),$$

we can get back to the solution u of (2.3) and obtain the asymptotic expansion

$$u = \sum_{k=0}^{K} r^{1/2+k}\psi^k(s) \ + \ u_{\mathrm{reg},K} \ + \ u_{\mathrm{rem},K}. \tag{2.9}$$

Such formulas will be at the core of the next sections.

3. Shape derivative of the energy $\mathsf{J}(\Omega)$

This section is devoted to the proof of Theorem 1.1 and Corollary 1.2. The proof of Theorem 1.1 is rather technical, although it follows a simple intuition. Therefore let us first explain how Corollary 1.2 is derived.

Assume that Theorem 1.1 holds. Consider the application

$$\mathcal{J} : \zeta \in \mathcal{C}^1_b(\mathbb{R}^2,\mathbb{R}^2) \mapsto J((I+\zeta)\Omega),$$

where

$$\mathcal{C}^1_b(\mathbb{R}^2,\mathbb{R}^2) = \mathcal{C}^1 \cap W^{1,\infty}(\mathbb{R}^2,\mathbb{R}^2).$$

We recall that $\mathcal{C}^1_b$ equipped with the norm $\|\cdot\|_{W^{1,\infty}}$ is a Banach space.

We first claim that $\mathcal{J}$ is differentiable at $\zeta = 0$. This follows from the differentiability of the application

$$\zeta \in \mathcal{C}^1_b(\mathbb{R}^2,\mathbb{R}^2) \mapsto v_\zeta \in H^{1/2}(\mathbb{R}^2),$$

where $v_\zeta = u_\zeta \circ (I+\zeta)$ and $u_\zeta = u_{(I+\zeta)\Omega}$ is the solution of the Euler–Lagrange equation associated with $(I+\zeta)\Omega$. The proof goes along the same lines as the one of Lemma 3.1 below. Furthermore, the variational formulation of the Euler–Lagrange equation implies (see formula (3.9))

$$J((I+\zeta)\Omega) = -\frac{1}{2}\int_{\mathbb{R}^2} v_\zeta \det(I+\nabla\zeta).$$

The differentiability of $\mathcal{J}$ follows.

Using Theorem 1.1, we then identify the differential of $\mathcal{J}$ at $\zeta = 0$. Indeed, if $(\phi_t)_{t\in\mathbb{R}}$ is the flow associated with $\zeta \in \mathcal{C}^\infty_0(\mathbb{R}^2)$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} J(\phi_t(\Omega))\right)_{|t=0}$$

is the Gâteaux derivative of $\mathcal{J}$ at point 0 in the direction ζ. We infer that

$$\mathrm{d}\mathcal{J}(0)\zeta = C_0 \int_{\partial\Omega} \zeta\cdot n (\partial_n^{1/2} u_\Omega)^2 \,\mathrm{d}\sigma$$

for all $\zeta \in \mathcal{C}^\infty_0(\mathbb{R}^2,\mathbb{R}^2)$, and by density for all $\zeta \in \mathcal{C}^1_b(\mathbb{R}^2,\mathbb{R}^2)$.

Now, assume that Ω is a bounded domain with $\mathcal{C}^\infty$ boundary, which minimizes J under the constraint $|\Omega| = 1$. In other words, $\zeta = 0$ is a minimizer of $\mathcal{J}(\zeta)$ in the Banach space $\mathcal{C}^1_b$ under the constraint $V(\zeta) := |(I+\zeta)(\Omega)| = 1$. According to the theorem of Lagrange multipliers, there exists $\lambda \in \mathbb{R}$ such that

$$\forall \zeta \in \mathcal{C}^1_b(\mathbb{R}^2,\mathbb{R}^2), \quad (d\mathcal{J}(0) + \lambda \mathrm{d}V(0))\,\zeta = 0. \tag{3.1}$$

It is proved in [10] that

$$\mathrm{d}V(0)\zeta = \int_{\partial\Omega} \zeta \cdot n \, \mathrm{d}\sigma.$$

Thus (3.1) becomes

$$\exists \lambda \in \mathbb{R},\ \forall \zeta \in \mathcal{C}^1_b(\mathbb{R}^2, \mathbb{R}^2), \qquad \int_{\partial\Omega} (\zeta \cdot n) \left((\partial_n^{1/2} u_\Omega)^2 + \frac{\lambda}{C_0} \right) \mathrm{d}\sigma = 0.$$

Since ζ is arbitrary, we infer that $\partial_n^{1/2} u_\Omega$ is constant on Ω. Moreover, since $u_\Omega \geq 0$ on Ω by the maximum principle, the constant is positive. This completes the proof of Corollary 1.2.

We now turn to the proof of Theorem 1.1. In the case of the classical laplacian, the shape derivative of the Dirichlet energy is well-known and is proved in the book by Henrot et Pierre [10]. Let us recall the main steps of the derivation, which will be useful in the case of the fractional laplacian. Let

$$I_f(\Omega) := \inf_{u \in H^1_0(\Omega)} \left(\frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega f u \right),$$

For all $\zeta \in \mathcal{C}^1 \cap W^{1,\infty}(\mathbb{R}^2)$, consider the flow ϕ_t associated with ζ. Then for all $t \in \mathbb{R}$, ϕ_t is a diffeomorphism of $\mathbb{R}^2$. We recall the following properties, which hold for all $t \in \mathbb{R}$:

$$\begin{aligned} \dot{\phi}_0 = \zeta, \quad |\det(\nabla \phi_t(x))| =: j(t,x) = \exp\left(\int_0^t (\operatorname{div} \zeta)(\phi_s(x))\, \mathrm{d}s \right) \\ \frac{\mathrm{d}\phi_t^{-1}}{\mathrm{d}t}_{|t=0} = -\zeta, \quad \left|\det(\nabla \phi_t^{-1})\right| = j(-t,x). \end{aligned} \tag{3.2}$$

The last line merely expresses the fact that $\phi_t^{-1} = \phi_{-t}$, for all $t \in \mathbb{R}$.

For $t \geq 0$, let $\Omega_t = \phi_t(\Omega)$, and let w_t be the solution of the associated Euler–Lagrange equation, namely

$$\begin{aligned} -\Delta w_t &= f \quad \text{in } \Omega_t, \\ w_t &= 0 \text{ on } \partial\Omega_t. \end{aligned}$$

Eventually, we define $z_t := w_t \circ \phi_t$.

It is proved in [10] that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} I_f(\Omega_t) \right)_{|t=0} = -\frac{1}{2} \int_{\partial\Omega} (\zeta \cdot n)(\partial_n w_0)^2 \mathrm{d}\sigma.$$

Indeed, since w_t solves the Euler–Lagrange equation, we have

$$I_f(\Omega_t) = -\frac{1}{2} \int_{\Omega_t} f w_t = -\frac{1}{2} \int_\Omega z_t(y) f \circ \phi_t(y) j(t,y)\, \mathrm{d}y.$$

Differentiating $z_t = w_t \circ \phi_t$ with respect to t, we obtain

$$\dot{z}_t = \dot{w}_t + \dot{\phi}_t \cdot \nabla w_t,$$

and thus in particular

$$\dot{w}_0 + \zeta \cdot n\, \partial_n w_0 = 0 \text{ on } \partial\Omega.$$

The Euler–Lagrange equation yields

$$-\Delta \dot{w}_0 = 0 \quad \text{in } \Omega.$$

Gathering all the terms and using the fact that $w_{0|\partial\Omega} = 0$, we deduce that

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t} I_f(\Omega_t)\right)_{|t=0} &= -\frac{1}{2}\int_\Omega (f(\dot{w}_0 + \zeta\cdot\nabla w_0) + (\zeta\cdot\nabla f)w_0 + f\mathrm{div}\,\zeta w_0)\\ &= -\frac{1}{2}\int_\Omega (-\dot{w}_0 \Delta w_0 + \mathrm{div}\,(\zeta f w_0))\\ &= \frac{1}{2}\left(\int_\Omega w_0\Delta\dot{w}_0 + \int_{\partial\Omega}\dot{w}_0\partial_n w_0 \mathrm{d}\sigma\right)\\ &= -\frac{1}{2}\int_{\partial\Omega}(\zeta\cdot n)(\partial_n w_0)^2\mathrm{d}\sigma.\end{aligned}$$

Therefore the shape derivative of I_f is similar to the one of J_f, the fractional derivative being merely replaced by a classical derivative.

Unfortunately, the proof in the case of the classical laplacian can only partially be transposed to the fractional laplacian. Indeed, several integration by parts play a crucial role in the computation, and cannot be used in the framework of the fractional laplacian.

We use therefore a different method to estimate $\mathrm{d}J_f(\Omega_t)/\mathrm{d}t$. The main steps of the proof are as follows:

1. As above, we introduce $u_t = u_{\Omega_t,f}$ and $v_t = u_t\circ\phi_t$. We derive regularity properties and asymptotic expansions for v_t, from which we deduce a decomposition of u_0 and $\dot{u}_0$.
2. In order to avoid the singularities of $\dot{u}_0$ near $\partial\Omega$, we introduce a truncation function χ_k supported in Ω, and vanishing in the vicinity of the boundary. Using the integral form of the fractional laplacian, we then derive an integral formula for an approximation of $\mathrm{d}J_f(\Omega_t)/\mathrm{d}t$ involving u_0, $\dot{u}_0$ and χ_k.
3. Keeping only the leading order terms in the decomposition of u_0 and $\dot{u}_0$, we obtain an expression of $\mathrm{d}J_f(\Omega_t)/\mathrm{d}t$ in terms of $\partial_n^{1/2}u_0$, and we prove that this expression is independent of the choice of the truncation function χ_k.
4. We then evaluate the contributions of the remainder terms to the integral formula, and we prove that they all vanish as $k\to\infty$.

Most of the technicalities are contained in steps 3 and 4. However, some more or less formal calculations – performed at the end of step 2 – lead relatively easily to the desired result. Before tackling the core of the proof, let us introduce some notation:

- We denote by $\Gamma_1,\dots,\Gamma_N$ the connected components of $\partial\Omega$, and we parametrize each Γ_i by its arc-length s. We denote by L_i the length of Γ_i.
- The number $r\in\mathbb{R}$ stands for the (signed) distance to the boundary of Ω. More precisely, $|r|$ is the distance to the boundary, and $r>0$ inside Ω, $r<0$ outside Ω;
- We denote by U_t the three dimensional extension of u_t in the half-space, *i.e.* the function such that

$$-\Delta U_t = 0 \text{ on } \mathbb{R}^3_+,\quad U_t|_{z=0} = u_t.$$

  The three-dimensional function V_t is then defined by

$$V_t(x,z) = U_t(\phi_t(x),z)\quad x\in\mathbb{R}^2, z\in\mathbb{R},$$

  so that $V_t|_{z=0} = v_t$.
- Derivatives with respect to t are denoted with a dot.

3.1. Regularity of $\mathbf{u_0}$ and $\dot{\mathbf{u}}_\mathbf{0}$ and expansions

We start with the following lemma

Lemma 3.1. *For all t in a neighbourhood of zero,*

$$\begin{aligned} v_t \in H^{1/2}(\mathbb{R}^2), \quad \dot{v}_t \in H^{1/2}(\mathbb{R}^2),\\ u_t \in H^{1/2}(\mathbb{R}^2), \quad \dot{u}_t \in H^{-1/2}(\mathbb{R}^2). \end{aligned} \tag{3.3}$$

The proof is rather close to the one of Theorem 5.3.2 in [10]. In order to keep the reading as fluent as possible, the details are postponed to the end of the section. The idea is to prove that V_t solves a three-dimensional elliptic equation with smooth coefficients. The implicit function theorem then implies that $t \mapsto V_t \in H^1$ is $\mathcal{C}^1$ in a neighbourhood of $t = 0$. Since v_t is the trace of V_t, $t \mapsto v_t \in H^{1/2}$ is also a $\mathcal{C}^1$ application. Eventually, the chain-rule formula entails that $\dot{u}_t \in H^{-1/2}$.

We also derive asymptotic formulas for V_t and v_t in terms of r: we rely on the results in the paper by Costabel, Dauge and Duduchava [7], and we use the notations of paragraph 2.2 (see also Thm. 2.5 of the present paper). We claim that there exists $\psi_0, \psi_1, \psi_2 \in \mathcal{C}^1(\partial\Omega)$, $u_1, u_2 \in W^{1,\infty}(\mathbb{R}^2)$ such that

$$u_0 = \sqrt{r_+}\psi_0(s) + u_1(x), \tag{3.4}$$

$$\dot{u}_0 = \mathbf{1}_{r>0}\frac{1}{\sqrt{r}}\psi_1(s) + \sqrt{r_+}\psi_2(s) + u_2(x), \tag{3.5}$$

where

$$\psi_0(s) = \partial_n^{1/2} u_0(s), \tag{3.6}$$

$$\psi_1(s) = \frac{1}{2}\zeta \cdot n(s)\psi_0(s). \tag{3.7}$$

Moreover, there exists $\delta > 0$ such that

$$\dot{u}_t \in L^\infty((-\delta, \delta), L^1(\mathbb{R}^2)). \tag{3.8}$$

These decompositions and the regularity result (3.8) will be proved at the end of the section, after the Proof of Lemma 3.1.

3.2. An integral formula for an approximation of $\mathrm{d}J_f(\Omega_t)/\mathrm{d}t$

We first use the Euler–Lagrange equation (1.1) in order to transform the expression defining $J_f(\Omega_t)$. Classically, we prove that the unique minimizer u of the energy

$$\frac{1}{2}\left\langle (-\Delta)^{1/2} v, v \right\rangle_{H^{-1/2}, H^{1/2}} - \int_{\mathbb{R}^2} f v$$

in the class $\{v \in H^{1/2}(\mathbb{R}^2),\ v_{|\Omega_t^c} = 0\}$ satisfies

$$\left\langle (-\Delta)^{1/2} u, v \right\rangle_{H^{-1/2}, H^{1/2}} - \int_{\mathbb{R}^2} f v = 0 \quad \forall v \in H^{1/2}(\mathbb{R}^2) \text{ s.t. } v_{|\Omega_t^c} = 0.$$

Choosing $v \in \mathcal{C}_0^\infty(\Omega_t)$, we infer that $(-\Delta)^{1/2} u = f$ in Ω. Since $u_{|\Omega_t^c} = 0$, we infer that $u = u_{\Omega_t, f} = u_t$, and in particular

$$\langle (-\Delta)^{1/2} u_t, u_t \rangle = \int_{\Omega_t} f u_t.$$

The identity above yields

$$J_f(\Omega_t) = -\frac{1}{2}\int_{\mathbb{R}^2} f u_t = -\frac{1}{2}\int_{\Omega_t} f u_t.$$

Changing variables in the integral, we obtain

$$J_f(\Omega_t) = -\frac{1}{2}\int_\Omega v_t(y) f\circ\phi_t(y) j(t,y)\,\mathrm{d}y. \tag{3.9}$$

Since $t\mapsto v_t\in H^{1/2}$ is differentiable, $t\mapsto J_f(\Omega_t)$ is $\mathcal{C}^1$ for t close to zero, and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}J_f(\Omega_t)\right)_{|t=0} = -\frac{1}{2}\int_\Omega (\dot v_0 f + \zeta\cdot\nabla f v_0 + v_0 f \mathrm{div}\,\zeta).$$

Now, for $k\in\mathbb{N}$ large enough, we define $\chi_k\in\mathcal{C}_0^\infty(\mathbb{R}^2)$ by $\chi_k(x)=\chi(kr)$, where $\chi\in\mathcal{C}^\infty(\mathbb{R})$ and $\chi(\rho)=0$ for $\rho\le 1$, $\chi(\rho)=1$ for $\rho\ge 2$. Then, since $u_0=v_0$,

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}J_f(\Omega_t)\right)_{|t=0} &= -\frac{1}{2}\lim_{k\to\infty}\int_{\mathbb{R}^2}\chi_k(\dot v_0 f + v_0\,\mathrm{div}\,(\zeta f))\\ &= -\frac{1}{2}\lim_{k\to\infty}\left(\int_{\mathbb{R}^2}\chi_k u_0\,\mathrm{div}\,(\zeta f) + \left[\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\chi_k f v_t\right]_{|t=0}\right)\\ &= -\frac{1}{2}\lim_{k\to\infty}\left(\int_{\mathbb{R}^2}\chi_k u_0\,\mathrm{div}\,(\zeta f) + \left[\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}u_t(f\chi_k)\circ\phi_t^{-1}j(-t,\cdot)\right]_{|t=0}\right).\end{aligned}$$

For fixed k and for t in a neighbourhood of zero, there exists a compact set K_k such that $K_k\Subset\Omega$ and $\mathrm{Supp}\,\chi_k\circ\phi_t^{-1}\subset K_k$. Since $\dot u_t\in L_t^\infty(L_x^1)$ according to (3.8), we can use the chain rule and write

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}u_t(f\chi_k)\circ\phi_t^{-1}j(-t,\cdot)\right]_{|t=0} = \int_{\mathbb{R}^2}(\dot u_0 f\chi_k - u_0 f\zeta\cdot\nabla\chi_k - u_0\chi_k\zeta\cdot\nabla f - u_0\chi_k f\mathrm{div}\,\zeta).$$

Using the decomposition (3.4) together with the definition of χ_k, we deduce that

$$\left|\int_{\mathbb{R}^2}u_0 f\zeta\cdot\nabla\chi_k\right| \le C\int_{\mathbb{R}}\sqrt{r_+}k|\chi'(kr)|\,\mathrm{d}r \le \frac{C}{\sqrt{k}}.$$

Notice also that

$$\int_{\mathbb{R}^2}(-u_0\chi_k\zeta\cdot\nabla f - u_0\chi_k f\mathrm{div}\,\zeta) = -\int_{\mathbb{R}^2}u_0\chi_k\mathrm{div}\,(f\zeta).$$

We now focus on the term involving $\dot u_0$; since $(-\Delta)^{1/2}u_0=f$ on the support of χ_k, we have

$$\begin{aligned}\int_{\mathbb{R}^2}\dot u_0 f\chi_k &= \int_{\mathbb{R}^2}\dot u_0\chi_k(-\Delta)^{1/2}u_0\\ &= \int_{\mathbb{R}^2}u_0(-\Delta)^{1/2}(\dot u_0\chi_k).\end{aligned}$$

Notice also that $\chi_k(-\Delta)^{1/2}(\dot u_0)=0$. Indeed, $\dot u_t$ is smooth on K_k for t small enough (see for instance (3.28) below). Hence for $x\in K_k$, the integral formula (2.2) makes sense and we have, using (3.8),

$$\begin{aligned}(-\Delta)^{1/2}\dot u_0(x) &= C_1\int_{\mathbb{R}^2}\frac{\dot u_0(x)-\dot u_0(y)-\nabla\dot u_0(x)\cdot(x-y)\mathbf{1}_{|x-y|\le C}}{|x-y|^3}\,\mathrm{d}y\\ &= C_1\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\frac{u_t(x)-u_t(y)-\nabla u_t(x)\cdot(x-y)\mathbf{1}_{|x-y|\le C}}{|x-y|^3}\,\mathrm{d}y\right]_{|t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(f(x))=0.\end{aligned}$$

Eventually, we obtain

$$\int_{\mathbb{R}^2} \dot{u}_0 f \chi_k = \int_{\mathbb{R}^2} u_0 \left[(-\Delta)^{1/2}, \chi_k\right] \dot{u}_0.$$

Gathering all the terms, we infer eventually

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} J_f(\Omega_t)\right)_{|t=0} = -\frac{1}{2} \lim_{k\to\infty} \int_{\mathbb{R}^2} u_0 \left[(-\Delta)^{1/2}, \chi_k\right] \dot{u}_0.$$

Let us now express the right-hand side in terms of the kernel of the fractional laplacian. Using the integral formula (2.2) together with the expansion (3.5), we infer that

$$\left[(-\Delta)^{1/2}, \chi_k\right] \dot{u}_0(x) = C_1 \int_{\mathbb{R}^2} \frac{\dot{u}_0(y)(\chi_k(x) - \chi_k(y)) - \dot{u}_0(x)\nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le C}}{|x-y|^3} \mathrm{d}y.$$

The value of the integral above is independent of the constant C. Therefore the shape derivative of the energy J_f is given by

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} J_f(\Omega_t)_{|t=0} = -\frac{C_1}{2} \lim_{k\to\infty} I_k, \quad \text{where} \\ &I_k := \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{u_0(x)}{|x-y|^3} \left\{\dot{u}_0(y)(\chi_k(x)-\chi_k(y)) - \dot{u}_0(x)\nabla\chi_k(x)\cdot(x-y))\mathbf{1}_{|x-y|\le C}\right\} \mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{3.10}$$

The next step is to compute the asymptotic value of I_k as $k\to\infty$. This part is rather technical, and involves several error estimates. However, the intuition leading the calculations is simple: first, the main order is obtained when u_0 and $\dot{u}_0$ are replaced by the leading terms in their respective developments. Moreover, because of the truncation χ_k, the integral is concentrated on the boundary $\partial\Omega$. All these claims will be fully justified in the next paragraph.

If we follow these guidelines, we end up with

$$I_k \approx \mathrm{PV} \int_{T_\delta\times T_\delta} \frac{u_0(x)\dot{u}_0(y)}{|x-y|^3} (\chi_k(x) - \chi_k(y))\,\mathrm{d}x\,\mathrm{d}y,$$

where T_δ is a tubular neighbourhood of $\partial\Omega$ of width $\delta \ll 1$ (see (3.11)). If we change cartesian coordinates for local ones, and if we neglect the curvature of $\partial\Omega$ – which is legitimate if δ is small – we are led to

$$I_k \approx \mathrm{PV} \int_{[0,\delta]^2} \int_{(0,L)^2} \frac{\psi_0(s)\psi_1(s)}{((s-s')^2 + (r-r')^2)^{3/2}} \sqrt{\frac{r}{r'}} (\chi(kr) - \chi(kr')\,\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}r\,\mathrm{d}r'.$$

For simplicity, we have assumed that $\partial\Omega$ only has one connected component, of length L. Integrating first with respect to s', and changing variables by setting $\rho = kr$, $\rho' = kr'$, we obtain eventually

$$\begin{aligned} I_k &\approx 2C \left(\int_0^L \psi_0\psi_1\right) \mathrm{PV} \int_0^\infty \int_0^\infty \sqrt{\frac{\rho}{\rho'}} \frac{\chi(\rho) - \chi(\rho')}{|\rho-\rho'|^2} \mathrm{d}\rho\,\mathrm{d}\rho' \\ &\approx C \left(\int_0^L \psi_0\psi_1\right) \int_0^\infty \int_0^\infty \frac{\chi(\rho)-\chi(\rho')}{\sqrt{\rho\rho'}(\rho-\rho')} \mathrm{d}\rho\,\mathrm{d}\rho', \end{aligned}$$

where

$$C = \int_0^\infty \frac{\mathrm{d}z}{(1+z^2)^{3/2}}.$$

There remains to prove that the value of the integral in the right-hand side does not depend on χ, which we do at the end of the next paragraph, and the formula of Theorem 1.1 is proved.

Of course, the above calculation is very sketchy, and careful justification must be given at every step. But the general direction follows these formal arguments.

3.3. Asymptotic value of $\mathsf{I_k}$

We now evaluate the integral I_k defined in (3.10). There are two main ideas:

- We prove that the domain of integration can be restricted to a tubular neighbourhood of $\partial\Omega$ (see Lem. 3.2).
- Since the integral I_k is bilinear in $u_0, \dot u_0$, we replace u_0 and $\dot u_0$ by their expansions in (3.4), (3.5). The leading term is obtained when u_0 and $\dot u_0$ are replaced by the first terms in their respective developments. We prove in the next subsection that all other terms vanish as $k \to \infty$.

We begin with the following Lemma (of which we postpone the proof):

Lemma 3.2. *For $\delta > 0$, let*

$$T_\delta := \{x \in \mathbb{R}^2,\ d(x,\partial\Omega) < \delta\}. \tag{3.11}$$

Choose $C = \frac{\delta}{2}$ in the definition of I_k (3.10). Then there exists a constant C_δ such that for $k > 5/\delta$,

$$\left| \int_{(T_\delta\times T_\delta)^c} \frac{u_0(x)}{|x-y|^3} \left\{ \dot u_0(y)(\chi_k(x)-\chi_k(y)) - \dot u_0(x)\nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2}\right\} \mathrm{d}x\,\mathrm{d}y \right| \le \frac{C_\delta}{\sqrt{k}}.$$

We henceforth focus our attention on the value of the integral on $T_\delta \times T_\delta$. Replacing u_0 and $\dot u_0$ by the first terms in the expansions (3.4), (3.5), we define

$$I_k^\delta := \int_{T_\delta\times T_\delta} \frac{\psi_0(s)\sqrt{r_+}}{|x-y|^3} \left[\psi_1(s')\frac{\mathbf{1_{r'>0}}}{\sqrt{r'}}(\chi(kr)-\chi(kr')) \right.$$
$$\left. -\psi_1(s)\frac{\mathbf{1_{r>0}}}{\sqrt{r}} k\chi'(kr) n(s)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2} \right] \mathrm{d}x\,\mathrm{d}y. \tag{3.12}$$

There is a slight abuse of notation in the integral above, since we use simultaneously cartesian and local coordinates. In order to be fully rigorous, r, r', s, s' should be replaced by $r(x),\ r(y),\ s(x), s(y)$ respectively. However, the first step of the proof will be to express the integral I_k^δ in local coordinates, and therefore we will avoid these heavy notations.

In fact, the computation of the limit of I_k^δ is much more technical than the estimation of all other quadratic remainder terms, which we will achieve in the next subsection. We now prove the following:

Lemma 3.3. *There exists an explicit constant C_2, independent of χ and of Ω, such that*

$$\lim_{\delta\to0}\lim_{k\to\infty} I_k^\delta = C_2 \sum_{i=1}^N \int_0^{L_i} \psi_0(s)\psi_1(s)\,\mathrm{d}s.$$

Proof. Throughout the proof, different types of error terms will appear, which we have gathered in Lemma 3.4 below. We will therefore refer to (3.19), (3.20), (3.21), (3.22) to classify the different types of error terms.

Several preliminary simplifications are necessary:

- We use local coordinates instead of cartesian ones, *i.e.* we change x into (s,r) and y into (s',r'). Since $|r|, |r'| \le \delta$, the jacobian of this change of coordinates is, for $\delta > 0$ small enough,

$$|1 + r\kappa(s)|\ |1 + r'\kappa(s')| = (1 + r\kappa(s))\,(1 + r'\kappa(s'))\,,$$

  where κ is the algebraic curvature of $\partial\Omega$. We refer to the Appendix for a simple proof. Notice that this jacobian is always bounded.

- We write $T_\delta = \cup_{i=1}^N T_\delta^i$, where
$$T_\delta^i = \{x \in \mathbb{R}^2,\ \mathrm{d}(x, \mathit{\Gamma}_i) < \delta\}.$$
If δ is small enough, $T_\delta^i \cap T_\delta^j = \emptyset$ for $i \neq j$, and $|x-y|$ is bounded from below for $x \in T_\delta^i$, $y \in T_\delta^j$ by a constant independent of δ. Hence, for $i \neq j$,
$$\begin{aligned}
&\left| \int_{T_\delta^i \times T_\delta^j} \frac{\psi_0(s)\sqrt{r_+}}{|x-y|^3} \left[\psi_1(s') \frac{\mathbf{1}_{\mathbf{r}'>\mathbf{0}}}{\sqrt{r'}} (\chi(kr) - \chi(kr')) - \psi_1(s) \frac{\mathbf{1}_{\mathbf{r}>\mathbf{0}}}{\sqrt{r}} k\chi'(kr) n(s) \cdot (x-y) \mathbf{1}_{|x-y| \leq \delta/2} \right] \mathrm{d}x\, \mathrm{d}y \right| \\
&\leq C \int_{T_\delta^i \times T_\delta^j} \mathbf{1}_{\mathbf{r},\mathbf{r}'>\mathbf{0}} \frac{\sqrt{r}}{\sqrt{r'}} \,\mathrm{d}x\, \mathrm{d}y \\
&\leq C \int_{(0,\delta)^2} \frac{\sqrt{r}}{\sqrt{r'}} \,\mathrm{d}r\, \mathrm{d}r' \\
&\leq C\delta^2.
\end{aligned}$$
Therefore, in the integral defining I_k^δ, we replace the domain of integration $T_\delta \times T_\delta$ by $\cup_{i=1}^N T_\delta^i \times T_\delta^i$, and this introduces an error term of order δ^2.
- In local coordinates, we write T_δ^i as $(0, L_i) \times (-\delta, \delta)$. We replace the jacobian
$$(1 + r\kappa(s))\,(1 + r'\kappa(s'))$$
by $(1 + r\kappa(s))^2$. Since $s, s' \in (0, L_i)$ (*i.e.* x and y belong to the neighbourhood of the same connected component of $\partial\Omega$), this change introduces error terms bounded by (3.19), (3.21), which vanish as $k \to \infty$ and $\delta \to 0$. More details will be given in the fourth step for similar error terms; we refer to Lemma 3.4.
- We evaluate $|x-y|$ in local coordinates for $x, y \in T_\delta^i$. We have
$$\begin{aligned} x &= p(s) - rn(s), \\ y &= p(s') - r'n(s'), \end{aligned}$$
where $p(s) \in \mathbb{R}^2$ is the point of $\partial\Omega$ with arc-length s. Since the boundary $\mathit{\Gamma}_i$ is $\mathcal{C}^\infty$,
$$p(s) - p(s') = (s - s')\tau(s) + O(|s-s'|^2),$$
where $\tau(s)$ is the unit tangent vector at $p(s) \in \partial\Omega$. Using the Frenet-Serret formulas, we also have
$$n(s) - n(s') = -(s-s')\kappa(s)\tau(s) + O(|s-s'|^2),$$
where κ is the curvature of $\mathit{\Gamma}_i$. Gathering all the terms, we infer that
$$x - y = (s-s')(1 + \kappa(s)r)\tau(s) + (r - r')n(s) + O(|s-s'|^2 + |r-r'|^2), \tag{3.13}$$
so that
$$|x-y|^2 = (s-s')^2(1+\kappa(s)r)^2 + (r-r')^2 + O\left(|s-s'|^3 + |r-r'|^3\right) \tag{3.14}$$
and
$$|x-y|^{-3} = \left((s-s')^2(1+\kappa(s)r)^2 + (r-r')^2\right)^{-3/2} (1 + O(|s-s'| + |r-r'|))\,.$$
In particular, there exists a constant C such that
$$\frac{1}{|x-y|^3} \leq \frac{C}{((s-s')^2 + (r-r')^2)^{3/2}},$$
and replacing $|x-y|^{-3}$ by $\left((s-s')^2(1+\kappa(s)r)^2 + (r-r')^2\right)^{-3/2}$ generates yet another error term bounded by (3.19), (3.21).

- We replace the factor $\psi_1(s')$ by $\psi_1(s)$; this also leads to an error term of the type (3.19).
- Using (3.13), we infer that

$$n(s)\cdot(x-y) = r - r' + O(|s-s'|^2 + |r-r'|^2).$$

The second term in the right-hand side of the above equality gives rise to an error term of the type (3.21).
- The last preliminary step is to replace the indicator function $\mathbf{1}_{|x-y|\le\delta/2}$ in (3.12) by a quantity depending on s, s', r, r'. Using the asymptotic development (3.14) above, it can be easily proved that there exists a constant c such that

$$\left|\mathbf{1}_{|x-y|\le\delta/2} - \mathbf{1}_{(s-s')^2(1+\kappa(s)r)^2+(r-r')^2\le\delta^2/4}\right| \le \mathbf{1}_{(1-c\delta)\frac{\delta}{2}\le|x-y|\le(1+c\delta)\frac{\delta}{2}}.$$

Therefore the substitution between the two indicator functions yields an error term bounded by

$$\begin{aligned}\sum_{i=1}^N \int_{T_\delta^i\times T_\delta^i} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^2}|\nabla\chi_k(x)|\mathbf{1}_{(1-c\delta)\frac{\delta}{2}\le|x-y|\le(1+c\delta)\frac{\delta}{2}} &\le \sum_{i=1}^N\int_{T_\delta^i}|\nabla\chi_k(x)|\,\mathrm{d}x\int_{\mathbb{R}^2}\frac{\mathrm{d}z}{|z|^2}\mathbf{1}_{(1-c\delta)\frac{\delta}{2}\le|z|\le(1+c\delta)\frac{\delta}{2}}\\ &\le C\ln\left(\frac{1+c\delta}{1-c\delta}\right)\le C\delta.\end{aligned}$$

As a consequence, at this stage, we have proved that

$$\begin{aligned}I_k^\delta = \sum_{i=1}^N\int_0^\delta\int_{-\delta}^\delta\int_{(0,L_i)^2} &\frac{\psi_0(s)\psi_1(s)\sqrt{r}(1+r\kappa(s))^2}{\left((s-s')^2(1+\kappa(s)r)^2+(r-r')^2\right)^{3/2}}\\ &\times\left[\frac{\mathbf{1_{r'>0}}}{\sqrt{r'}}(\chi(kr)-\chi(kr')) - \frac{1}{\sqrt{r}}k\chi'(kr)(r-r')\mathbf{1}_{(s-s')^2(1+\kappa r)^2+(r-r')^2\le\delta^2/4}\right]\mathrm{d}s\,\mathrm{d}s'\mathrm{d}r'\,\mathrm{d}r\\ &+O\left(\frac{\ln k}{\sqrt{k}}+\delta|\ln\delta|\right).\end{aligned}\tag{3.15}$$

We now evaluate the right-hand side of the above identity. We first prove that the term involving $\chi'(kr)$ does not contribute to the limit, due to symmetry properties of the integral. This was expected, since this term had a vanishing integral in the beginning; its only role was to ensure the convergence of I_k. We then focus on the term involving $\chi(kr)-\chi(kr')$, and we prove that its asymptotic value is independent of χ.

First, since the integral (3.15) is convergent, we have

$$(3.15) = \sum_{i=1}^N \lim_{\epsilon\to 0}\int_0^\delta\int_{-\delta}^\delta\int_{(0,L_i)^2}\mathbf{1}_{|r-r'|\ge\epsilon}\ \cdots$$

Therefore, up to the introduction of a truncation, we can separate the two terms of (3.15). We have in particular

$$\begin{aligned}\int_{-\delta}^\delta &\frac{\psi_0(s)\psi_1(s)\mathbf{1}_{|r-r'|\ge\epsilon}(1+r\kappa(s))^2}{\left((s-s')^2(1+\kappa(s)r)^2+(r-r')^2\right)^{3/2}}k\chi'(kr)(r-r')\mathbf{1}_{(s-s')^2(1+\kappa r)^2+(r-r')^2\le\delta^2/4}\mathrm{d}r' =\\ &-\int_{-\delta-r}^{\delta-r}\frac{\psi_0(s)\psi_1(s)\mathbf{1}_{|\xi|\ge\epsilon}(1+r\kappa(s))^2}{\left((s-s')^2(1+\kappa(s)r)^2+\xi^2\right)^{3/2}}k\chi'(kr)\xi\ \mathbf{1}_{(s-s')^2(1+\kappa r)^2+\xi^2\le\delta^2/4}\mathrm{d}\xi.\end{aligned}\tag{3.16}$$

Notice that the above integral only bears on the values of ξ such that $|\xi|\le\delta/2$. On the other hand, for all r such that $\chi'(kr)\ne 0$, we have $1/k\le r\le 2/k$, so that if $k>4/\delta$,

$$\begin{aligned}\delta - r &\ge \delta-\frac{2}{k}>\frac{\delta}{2},\\ -\delta-r &< -\delta < -\frac{\delta}{2}.\end{aligned}$$

Hence the integral (3.16) is in fact equal to

$$\psi_0(s)\psi_1(s)(1+r\kappa(s))^2 k\chi'(kr)\int_{-\delta/2}^{\delta/2}\frac{\xi\mathbf{1}_{|\xi|\geq\epsilon}\mathbf{1}_{(s-s')^2(1+\kappa r)^2+\xi^2\leq\delta^2/4}}{((s-s')^2(1+\kappa(s)r)^2+\xi^2)^{3/2}}\,\mathrm{d}\xi.$$

Since the integrand is odd in ξ, the integral is identically zero for all $\epsilon>0$ and for all δ, k such that $k\delta>4$.

There remains to investigate the first term in (3.15), namely

$$\int_{(0,\delta)^2}\int_{(0,L_i)^2}\frac{\psi_0(s)\psi_1(s)(1+r\kappa(s))^2(\chi(kr)-\chi(kr'))}{((s-s')^2(1+\kappa(s)r)^2+(r-r')^2)^{3/2}}\sqrt{\frac{r}{r'}}\mathbf{1}_{|r-r'|\geq\epsilon}\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}r'\,\mathrm{d}r. \tag{3.17}$$

We first symmetrize the integral by exchanging the roles of r and r'. We have

$$\sqrt{\frac{r}{r'}}(1+r\kappa(s))^2-\sqrt{\frac{r'}{r}}(1+r'\kappa(s))^2=\frac{r-r'}{\sqrt{rr'}}(1+r\kappa(s))^2+O\left(\sqrt{\frac{r'}{r}}|r-r'|\right),$$

$$\frac{1}{((s-s')^2(1+\kappa(s)r)^2+(r-r')^2)^{3/2}}-\frac{1}{((s-s')^2(1+\kappa(s)r')^2+(r-r')^2)^{3/2}}=O\left(\frac{|r-r'|}{((s-s')^2+(r-r')^2)^{3/2}}\right).$$

The term

$$\int_{(0,\delta)^2}\int_{(0,L_i)^2}|\psi_0(s)\psi_1(s)(\chi(kr)-\chi(kr'))|\sqrt{\frac{r'}{r}}\frac{|r-r'|}{((s-s')^2+(r-r')^2)^{3/2}}\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}r'\,\mathrm{d}r$$

is bounded by an error term of the type (3.19), and is therefore $O(\ln k/\sqrt{k})$ for all $\epsilon>0$. As a consequence,

$$\lim_{\epsilon\to0}(3.17)=\frac{1}{2}\int_{(0,\delta)^2}\int_{(0,L_i)^2}\frac{\psi_0(s)\psi_1(s)(1+r\kappa(s))^2(\chi(kr)-\chi(kr'))}{((s-s')^2(1+\kappa(s)r)^2+(r-r')^2)^{3/2}}\frac{r-r'}{\sqrt{rr'}}\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}r'\,\mathrm{d}r+O\left(\frac{\ln k}{\sqrt{k}}\right). \tag{3.18}$$

We now compute the integral with respect to $s'\in(0,L_i)$. Setting

$$\phi(\xi):=\int_0^\xi\frac{\mathrm{d}z}{(1+z^2)^{3/2}},$$

we have

$$\int_0^{L_i}\frac{\mathrm{d}s'}{((s-s')^2(1+\kappa(s)r)^2+(r-r')^2)^{3/2}}=\frac{1}{(r-r')^2(1+\kappa(s)r)}\left(\phi\left(\frac{(L_i-s)(1+\kappa(s)r)}{|r-r'|}\right)+\phi\left(\frac{s(1+\kappa(s)r)}{|r-r'|}\right)\right).$$

Inserting this formula into the integral above and changing variables, we obtain

$$\begin{aligned}(3.18)=\frac{1}{2}\int_{(0,k\delta)^2}\int_0^{L_i}&\psi_0(s)\psi_1(s)\frac{\chi(\rho)-\chi(\rho')}{(\rho-\rho')\sqrt{\rho\rho'}}\left(1+\kappa(s)\frac{\rho}{k}\right)\\&\times\left[\phi\left(\frac{k(L_i-s)(1+\kappa(s)\rho/k)}{|\rho-\rho'|}\right)+\phi\left(\frac{ks(1+\kappa(s)\rho/k)}{|\rho-\rho'|}\right)\right]\mathrm{d}s\,\mathrm{d}\rho\,\mathrm{d}\rho'.\end{aligned}$$

Since

$$\int_0^\infty\int_0^\infty\left|\frac{\chi(\rho)-\chi(\rho')}{(\rho-\rho')\sqrt{\rho\rho'}}\right|\mathrm{d}\rho\,\mathrm{d}\rho'<\infty,$$

and $0\leq\phi(\xi)\leq\phi(+\infty)<\infty\quad\forall\xi>0$, using Lebesgue's theorem, we infer that for all $\delta > 0$,

$$\lim_{k\to\infty} (3.18) = \phi(+\infty)\left(\int_0^{L_i} \psi_0(s)\psi_1(s)\mathrm{d}s\right)\int_0^\infty\int_0^\infty \frac{\chi(\rho)-\chi(\rho')}{(\rho-\rho')\sqrt{\rho\rho'}}\mathrm{d}\rho\,\mathrm{d}\rho'.$$

There only remains to prove that the integral involving χ is in fact independent of χ. We have

$$\begin{aligned} I_0 \quad &:= \int_0^\infty\int_0^\infty \frac{\chi(\rho)-\chi(\rho')}{(\rho-\rho')\sqrt{\rho\rho'}}\mathrm{d}\rho\,\mathrm{d}\rho' \\ &= \int_0^\infty\int_0^\infty\int_0^1 \chi'(\tau\rho+(1-\tau)\rho')\frac{\mathrm{d}\tau\,\mathrm{d}\rho\,\mathrm{d}\rho'}{\sqrt{\rho\rho'}} \\ &\underset{z=\rho'/\rho}{=} \int_0^\infty\int_0^\infty\int_0^1 \chi'(\rho(\tau+(1-\tau)z))\frac{\mathrm{d}\tau\,\mathrm{d}\rho\,\mathrm{d}z}{\sqrt{z}} \\ &= \int_0^\infty\int_0^1 \frac{1}{\tau+(1-\tau)z}\frac{\mathrm{d}\tau\,\mathrm{d}z}{\sqrt{z}} \\ &= \int_0^\infty \frac{\ln z}{(z-1)\sqrt{z}}\mathrm{d}z. \end{aligned}$$

Gathering all the terms, we infer that

$$\lim_{k\to\infty} I_k^\delta = I_0\phi(+\infty)\sum_{i=1}^N \int_0^{L_i} \psi_0\psi_1 + O\left(\delta|\ln\delta|\right).$$

Passing to the limit as $\delta\to 0$, we obtain the result announced in Lemma 3.3. ☐

3.4. Evaluation of remainder terms in the integral $\mathbf{I_k}$

▹ We start with the proof of Lemma 3.2. The idea is to divide $\mathbb{R}^2\times\mathbb{R}^2 \setminus T_\delta\times T_\delta$ into subdomains and to evaluate the contribution of every subdomain.

- For $(x,y)\in(T_\delta^c\cap\Omega)^2$, $\chi_k(x)=\chi_k(y)=1$ and $\nabla\chi_k(x)=0$ for $k>\delta^{-1}$, so that the contribution of $T_\delta^c\cap\Omega\times T_\delta^c\cap\Omega$ is zero.
- For $x\in T_\delta^c\cap\Omega^c$, $u_0(x)=0$: the contribution of $T_\delta^c\cap\Omega^c\times\mathbb{R}^2$ is zero.
- For $x\in T_\delta^c\cap\Omega, y\in T_\delta^c\cap\Omega^c$, $\dot u_0(y)=0$ and $\nabla\chi_k(x)=0$, so that the contribution of $T_\delta^c\cap\Omega\times T_\delta^c\cap\Omega^c$ is zero.
- For $x\in T_\delta^c\cap\Omega$, $y\in T_\delta$, $\chi_k(x)=1$ and $\nabla\chi_k(x)=0$, so that the contribution of the sub-domain is

$$\begin{aligned} &\left|\int_{T_\delta^c\cap\Omega\times T_\delta} \frac{u_0(x)}{|x-y|^3}\left\{\dot u_0(y)(\chi_k(x)-\chi_k(y)) - \dot u_0(x)\nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2}\right\}\mathrm{d}x\,\mathrm{d}y\right| \\ &\le C\int_{T_\delta^c\cap\Omega\times T_\delta} \frac{1}{|x-y|^3}|\dot u_0(y)|(1-\chi_k(y))\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

  Since $1-\chi_k$ is supported in $\Omega^c\cup\{d(y,\partial\Omega)\le 2/k\}$ and $|\dot u_0(y)|\le C\frac{\mathbf{1}_\Omega}{\sqrt{d(y,\partial\Omega)}}$, the right-hand side is bounded by

$$\frac{C}{|\delta-\frac{2}{k}|^3}\int_0^{2/k}\frac{\mathrm{d}r}{\sqrt{r}} \le \frac{C_\delta}{\sqrt{k}}.$$

- For $x\in T_\delta$, $y\in T_\delta^c$, the integral

$$\int_{T_\delta\times T_\delta^c} \frac{u_0(x)\dot u_0(x)}{|x-y|^3}\nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2}\,\mathrm{d}x\,\mathrm{d}y$$

is in fact supported in

$$(\{\mathrm{d}(x,\partial\Omega) \leq 2/k\} \times \{d(y,\partial\Omega) > \delta\}) \cap \{|x-y| \leq \delta/2\}.$$

It is easily seen that for k large enough (say $k > 5/\delta$) this set is empty, and therefore the integral is zero.

- There only remains

$$\int_{T_\delta \times T_\delta^c} \frac{u_0(x)}{|x-y|^3} \dot{u}_0(y)(\chi_k(x) - \chi_k(y))\, \mathrm{d}x\, \mathrm{d}y.$$

Using the same kind of estimates as for the domain $T_\delta^c \cap \Omega \times T_\delta$, it can be proved that this term is bounded by $C_\delta k^{-3/2}$.

▷ We now estimate the remainder terms on $T_\delta \times T_\delta$. We use the following lemma:

Lemma 3.4. *Let $k \geq 1$ and $\delta > 0$ such that $k\delta > 5$.*
Then the following estimates hold: for $1 \leq i \leq N$,

$$\int_{T_\delta^i \times T_\delta^i} \mathbf{1}_{r,r'>0} \frac{\sqrt{r}}{\sqrt{r'}(\sqrt{r}+\sqrt{r'})} \frac{1}{|x-y|^2} |\chi_k(x) - \chi_k(y)|\, \mathrm{d}x\, \mathrm{d}y = O\left(\frac{\ln k}{\sqrt{k}}\right), \tag{3.19}$$

$$\int_{T_\delta^i \times T_\delta^i} \frac{\sqrt{r_+}}{|x-y|^3} \left|\chi_k(x) - \chi_k(y) - \nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\leq\delta/2}\right|\, \mathrm{d}x\, \mathrm{d}y = O\left(\sqrt{\delta} + \frac{\ln k}{\sqrt{k}}\right), \tag{3.20}$$

$$\int_{T_\delta^i \times T_\delta^i} \frac{|\nabla\chi_k(x)|}{|x-y|} \mathrm{d}x\, \mathrm{d}y = O(\delta|\ln\delta|), \tag{3.21}$$

$$\int_{T_\delta^i \times T_\delta^i} \mathbf{1}_{r>0,r'<0} \frac{\sqrt{r}}{|x-y|^3} |\chi_k(x) - \chi_k(y)|\, \mathrm{d}x \mathrm{d}y = O\left(\frac{1}{k^{1/2}}\right). \tag{3.22}$$

Proof. Throughout the proof, we use the following facts:

- The jacobian of the change of variables $(x,y) \to (s,r,s',r')$ is

$$(1 + r\kappa(s))\,(1 + r'\kappa(s')) \leq C;$$

- Using the expansion (3.14), we infer that there exists a constant C such that

$$\frac{1}{|x-y|} \leq \frac{C}{((s-s')^2 + (r-r')^2)^{1/2}}.$$

- For all $\alpha \geq 1$, there exists a constant C_α such that for all $s \in (0, L_i)$, for all $r, r' \in (-\delta,\delta)$ with $r \neq r'$,

$$\int_0^{L_i} \frac{\mathrm{d}s'}{((s-s')^2 + (r-r')^2)^{\alpha/2}} \leq C_\alpha \begin{cases} |r-r'|^{1-\alpha} & \text{if } \alpha > 1, \\ |\ln|r-r'|\,| & \text{if } \alpha = 1. \end{cases} \tag{3.23}$$

Indeed, for $\alpha > 1$

$$\int_0^{L_i} \frac{\mathrm{d}s'}{((s-s')^2 + (r-r')^2)^{\alpha/2}} \leq \int_{\mathbb{R}} \frac{\mathrm{d}z}{(z^2 + (r-r')^2)^{\alpha/2}} \underset{\xi = z/|r-r'|}{\leq} |r-r'|^{1-\alpha} \int_{\mathbb{R}} \frac{\mathrm{d}\xi}{(1+\xi^2)^{\alpha/2}},$$

while for $\alpha = 1$

$$\int_0^{L_i} \frac{\mathrm{d}s'}{((s-s')^2+(r-r')^2)^{1/2}} = \int_{-\frac{s}{|r-r'|}}^{\frac{L_i-s}{|r-r'|}} \frac{\mathrm{d}\xi}{(1+\xi^2)^{1/2}}$$
$$\leq \int_{-\frac{L_i}{|r-r'|}}^{\frac{L_i}{|r-r'|}} \frac{\mathrm{d}\xi}{(1+\xi^2)^{1/2}}$$
$$\leq 2\ln\left(\frac{L_i}{|r-r'|}\right).$$

We now tackle the proof of (3.19)–(3.22).

1. *Proof of* (3.19)*:* we split the domain $(T_\delta \times T_\delta) \cap \{r, r' \geq 0\}$ into four subdomains:
   - $0 \leq r, r' \leq 3/k$;
   - $0 \leq r' \leq 3/k$ and $3/k < r < \delta$;
   - $0 \leq r \leq 3/k$ and $3/k < r' < \delta$;
   - $3/k < r, r' < \delta$.

   Since $\chi_k(x) = 1$ for $r > 2/k$, it is easily seen that $\chi_k(x) - \chi_k(y) = 0$ on the last subdomain. Moreover, we obviously have

$$\int_{s,s'\in\Gamma_i}\int_{0\leq r\leq 3/k,\ 3/k<r'<\delta} \frac{\sqrt{r}}{\sqrt{r'}(\sqrt{r}+\sqrt{r'})}\frac{1}{|x-y|^2}|\chi_k(x)-\chi_k(y)|\,\mathrm{d}x\,\mathrm{d}y \leq$$
$$\int_{s,s'\in\Gamma_i}\int_{0\leq r'\leq 3/k,\ 3/k<r<\delta} \frac{\sqrt{r}}{\sqrt{r'}(\sqrt{r}+\sqrt{r'})}\frac{1}{|x-y|^2}|\chi_k(x)-\chi_k(y)|\,\mathrm{d}x\,\mathrm{d}y.$$

   Therefore we focus on the estimates of the integral on the first two subdomains above. First, since χ_k is a Lispchitz function with a Lipschitz constant of order k, we have

$$\begin{aligned}
&\int_{s,s'\in\Gamma_i}\int_{0\leq r,r'\leq 3/k} \frac{\sqrt{r}}{\sqrt{r'}(\sqrt{r}+\sqrt{r'})}\frac{1}{|x-y|^2}|\chi_k(x)-\chi_k(y)|\,\mathrm{d}x\,\mathrm{d}y\\
&\leq Ck\int_{s,s'\in\Gamma_i}\int_{0\leq r,r'\leq 3/k} \frac{\sqrt{r}}{\sqrt{r'}(\sqrt{r}+\sqrt{r'})}\frac{1}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y\\
&\leq Ck\int_{0\leq r,r'\leq 3/k}\int_{(0,L_i)^2} \frac{\sqrt{r}}{\sqrt{r'}(\sqrt{r}+\sqrt{r'})}\frac{1}{((s-s')^2+(r-r')^2)^{1/2}}\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}r\,\mathrm{d}r'\\
&\leq Ck\int_{0\leq r,r'\leq 3/k} \frac{\sqrt{r}}{\sqrt{r'}(\sqrt{r}+\sqrt{r'})}|\ln|r-r'||\,\mathrm{d}r\,\mathrm{d}r'\\
&\leq Ck\left(\int_0^{3/k}\frac{\mathrm{d}r'}{\sqrt{r'}}\right)\left(\int_{-3/k}^{3/k}|\ln|z||\,\mathrm{d}z\right)\\
&\leq Ck\times\frac{1}{\sqrt{k}}\times\frac{\ln k}{k}.
\end{aligned}$$

As for the second subdomain, since $\chi_k(x) = 1$ for $r > 2/k$, in fact, the domain of integration in r' is only $r' < 2/k$. Since χ_k is bounded by 1, we have

$$\begin{aligned}
&\int_{s,s'\in\Gamma_i}\int_{0\le r'\le 3/k,\ 3/k<r<\delta} \frac{\sqrt{r}}{\sqrt{r'}(\sqrt{r}+\sqrt{r'})}\frac{1}{|x-y|^2}|\chi_k(x)-\chi_k(y)|\,\mathrm{d}x\,\mathrm{d}y\\
&\le C\int_{0\le r'\le 2/k,\ 3/k<r<\delta}\int_{(0,L_i)^2}\frac{\sqrt{r}}{\sqrt{r'}(\sqrt{r}+\sqrt{r'})}\frac{1}{(s-s')^2+(r-r')^2}\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}r\,\mathrm{d}r'\\
&\le C\int_{0\le r'\le 2/k,\ 3/k<r<\delta}\frac{\sqrt{r}}{\sqrt{r'}(\sqrt{r}+\sqrt{r'})}\frac{1}{|r-r'|}\mathrm{d}r\,\mathrm{d}r'\\
&\le C\left(\int_0^{2/k}\frac{\mathrm{d}r'}{\sqrt{r'}}\right)\left(\int_{1/k}^{\delta}\frac{\mathrm{d}z}{z}\right)\\
&\le C\frac{\ln k}{\sqrt{k}}.
\end{aligned}$$

2. *Proof of* (3.20)*:* we use the same type of domain decomposition as above; the only difference lies in the fact that r' may take negative values.
   - On the subdomain $r \ge 3/k, r' \ge 3/k$, the integral is identically zero;
   - On the subdomain $3/k < r < \delta$, $-\delta < r' < 3/k$, we have $\nabla\chi_k(x) = 0$ and as soon as $r' \ge 2/k$,

$$\chi_k(x) - \chi_k(y) - \nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2} = 0.$$

Therefore

$$\begin{aligned}
&\int_{\substack{3/k<r<\delta,-\delta<r'<3/k,\\ s,s'\in\Gamma_i}} \frac{\sqrt{r_+}}{|x-y|^3}\left|\chi_k(x)-\chi_k(y)-\nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2}\right|\,\mathrm{d}x\,\mathrm{d}y\\
&\le C\int_{3/k<r<\delta,-\delta<r'<2/k}\int_{(0,L_i)^2}\frac{\sqrt{r_+}}{((s-s')^2+(r-r')^2)^{3/2}}\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}r\,\mathrm{d}r'\\
&\le C\int_{3/k<r<\delta,-\delta<r'<2/k}\frac{\sqrt{r}}{|r-r'|^2}\,\mathrm{d}r\,\mathrm{d}r'\\
&\le C\int_{3/k}^{\delta}\frac{\sqrt{r}}{r-\frac{2}{k}}\mathrm{d}r.
\end{aligned}$$

Using the inequality

$$\sqrt{r} \le \sqrt{r-\frac{2}{k}} + \sqrt{\frac{2}{k}},$$

we infer eventually that the integral is bounded by $C\left(\sqrt{\delta} + \frac{\ln k}{\sqrt{k}}\right)$.
   - On the subdomain $0 \le r \le 3/k$, $|r'| \ge 3/k$, we use the bound

$$|\chi_k(x) - \chi_k(y) - \nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2}| \le Ck|x-y|,$$

so that

$$\begin{aligned}
&\int_{\substack{0\le r\le 3/k, 3/k<|r'|<\delta,\\ s,s'\in\Gamma_i}} \frac{\sqrt{r}}{|x-y|^3}\left|\chi_k(x)-\chi_k(y)-\nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2}\right|\,\mathrm{d}x\,\mathrm{d}y\\
&\le Ck\int_{0\le r\le 3/k, 3/k<|r'|<\delta}\frac{\sqrt{r}}{|r-r'|}\mathrm{d}r\,\mathrm{d}r'\\
&\le -Ck\int_0^{3/k}\sqrt{r}\ln\left(\frac{3}{k}-r\right)\,\mathrm{d}r\\
&\le Ck\times\frac{1}{\sqrt{k}}\times\frac{\ln k}{k}.
\end{aligned}$$

- On the subdomain $0\le r\le 3/k, |r'|\le 3/k$, we use a Taylor–Lagrange expansion for the function χ_k, which yields

$$|\chi_k(x)-\chi_k(y)-\nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2}|\le Ck^2|x-y|^2,$$

so that

$$\begin{aligned}
&\int_{\substack{0\le r\le 3/k, |r'|\le 3/k,\\ s,s'\in\Gamma_i}} \frac{\sqrt{r_+}}{|x-y|^3}\left|\chi_k(x)-\chi_k(y)-\nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2}\right|\,\mathrm{d}x\,\mathrm{d}y\\
&\le Ck^2\int_{0\le r\le 3/k, |r'|\le 3/k}\sqrt{r}\,|\ln|r-r'||\,\mathrm{d}r\,\mathrm{d}r'\\
&\le Ck^2\frac{\ln k}{k}k^{-3/2}.
\end{aligned}$$

3. *Proof of* (3.21)*:* this term is easier to estimate than (3.19), (3.20). We merely have

$$\begin{aligned}
\int_{T^i_\delta\times T^i_\delta}\frac{|\nabla\chi_k(x)|}{|x-y|}\mathrm{d}x\,\mathrm{d}y &\le C\int_{(-\delta,\delta)^2}k|\chi'(kr)|\;|\ln|r-r'|\,|\,\mathrm{d}r\,\mathrm{d}r'\\
&\le C\int_{-2\delta}^{2\delta}|\ln|z|\,|\,\mathrm{d}z\\
&\le C\delta|\ln\delta|.
\end{aligned}$$

4. *Proof of* (3.22)*:* since $\chi_k(y)=0$ if $r'<0$, we have

$$\begin{aligned}
&\int_{T^i_\delta\times T^i_\delta}\mathbf{1}_{\mathbf{r}>\mathbf{0},\mathbf{r'}<\mathbf{0}}\frac{\sqrt{r}}{|x-y|^3}|\chi_k(x)-\chi_k(y)|\,\mathrm{d}x\mathrm{d}y\\
&=\int_{T^i_\delta\times T^i_\delta}\mathbf{1}_{\mathbf{r}>\mathbf{0},\mathbf{r'}<\mathbf{0}}\frac{\sqrt{r}}{|x-y|^3}\chi_k(x)\,\mathrm{d}x\mathrm{d}y\\
&\le C\int_{-\delta}^0\int_0^\delta\frac{\sqrt{r}}{|r-r'|^2}\chi(kr)\,\mathrm{d}r\,\mathrm{d}r'\\
&\le C\int_0^{2/k}\sqrt{r}\left(\frac{1}{r}-\frac{1}{r+\delta}\right)\,\mathrm{d}r\\
&\le\frac{C}{\sqrt{k}}.
\end{aligned}$$

□

We now address the rest of the proof of Theorem 1.2. With the help of Lemma 3.4, the estimation of the remainder terms in I_k is immediate. We set

$$\begin{aligned} w_0(x) &:= \mathbf{1}_{r>0}\frac{\psi_1(s)}{\sqrt{r}}, \\ w_1(x) &:= \sqrt{r_+}\psi_2(s) + u_2(x), \end{aligned}$$

so that $\dot u_0 = w_0 + w_1$. We write

$$\begin{aligned} \dot u_0(y)(\chi_k(x)-\chi_k(y)) - \dot u_0(x)\nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2} &= \dot u_0(x)\left[\chi_k(x)-\chi_k(y)-\nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2}\right] \\ &\quad + (\dot u_0(y)-\dot u_0(x))(\chi_k(x)-\chi_k(y)). \end{aligned}$$

There exists a constant C such that

$$|u_1(x)\dot u_0(x)| + |\sqrt{r_+}\psi_0(s)w_1(x)| \le C\sqrt{r_+} \quad \forall x\in\mathbb{R}^2,$$

so that

$$\begin{aligned} &\left|\int_{T_\delta\times T_\delta}\frac{u_1(x)\dot u_0(x)}{|x-y|^3}\left[\chi_k(x)-\chi_k(y)-\nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2}\right]\,\mathrm{d}x\,\mathrm{d}y\right| + \\ &\qquad\left|\int_{T_\delta\times T_\delta}\frac{\sqrt{r_+}\psi_0(s)w_1(x)}{|x-y|^3}\left[\chi_k(x)-\chi_k(y)-\nabla\chi_k(x)\cdot(x-y)\mathbf{1}_{|x-y|\le\delta/2}\right]\,\mathrm{d}x\,\mathrm{d}y\right| \le (3.20). \end{aligned}$$

On the other hand, simple calculations show that

$$\begin{aligned} |\dot u_0(x)-\dot u_0(y)| &\le C|x-y|\left(1+\frac{\mathbf{1}_{r,r'>0}}{\sqrt{rr'}(\sqrt{r}+\sqrt{r'})}\right) \\ &\quad + C\mathbf{1}_{rr'<0}\frac{1}{\sqrt{r_+}+\sqrt{r'_+}}, \\ |w_1(x)-w_1(y)| &\le C|x-y|\left(1+\frac{\mathbf{1}_{r,r'>0}}{\sqrt{r}+\sqrt{r'}}\right) \\ &\quad + C\mathbf{1}_{rr'<0}(\sqrt{r_+}+\sqrt{r'_+}). \end{aligned}$$

As a consequence,

$$\begin{aligned} |u_1(x)||\dot u_0(x)-\dot u_0(y)| &\le C|x-y|\left(r_+ + \mathbf{1}_{r,r'>0}\frac{\sqrt{r}}{\sqrt{r'}(\sqrt{r}+\sqrt{r'})}\right) + C\mathbf{1}_{\mathbf{r>0,r'<0}}\sqrt{r}, \\ \left|\sqrt{r_+}\psi_0(s)\right|\,|w_1(x)-w_1(y)| &\le C|x-y|\left(\sqrt{r_+}+\mathbf{1}_{r,r'>0}\frac{\sqrt{r}}{\sqrt{r}+\sqrt{r'}}\right) + C\mathbf{1}_{\mathbf{r>0,r'<0}}r, \end{aligned}$$

so that

$$\begin{aligned} &\left|\int_{T_\delta\times T_\delta}\frac{u_1(x)}{|x-y|^3}(\dot u_0(x)-\dot u_0(y))(\chi_k(x)-\chi_k(y))\,\mathrm{d}x\,\mathrm{d}y\right| + \\ &\qquad\left|\int_{T_\delta\times T_\delta}\frac{\sqrt{r_+}\psi_0(s)}{|x-y|^3}(|w_1(x)-w_1(y))(\chi_k(x)-\chi_k(y))\,\mathrm{d}x\,\mathrm{d}y\right| \le (3.19)+(3.22). \end{aligned}$$

Gathering all the terms, we deduce that

$$\lim_{\delta\to0}\lim_{k\to\infty} I_k = C_2\sum_{i=1}^N\int_0^{L_i}\psi_0\psi_1,$$

where C_2 is an explicit positive constant. Moreover, using formulas (3.6), (3.7), we have

$$\psi_0(s)\psi_1(s) = -\frac{1}{2}(\partial_n^{1/2}u_0(s))^2\zeta\cdot n(s),$$

where $\zeta(x) = \dot{\phi}_0(x)$. Eventually, we deduce that

$$\frac{\mathrm{d}J_f(\Omega_t)}{\mathrm{d}t}_{|t=0} = -\frac{C_1C_2}{4}\sum_{i=1}^{N}\int_0^{L_i}(\partial_n^{1/2}u_0(s))^2\zeta\cdot n(s)\mathrm{d}s = -\frac{C_1C_2}{4}\int_{\partial\Omega}(\partial_n^{1/2}u_0(s))^2\zeta\cdot n(s)\mathrm{d}\sigma(s).$$

Therefore Theorem 1.1 is proved, with $C_0 := -(C_1C_2)/4$. Notice that C_0 does not depend on Ω.

3.5. Proofs of Lemma 3.1 and formulas (3.4), (3.5)

- The proof of (3.3) follows closely the one of Theorem 5.3.2 in [10]. The only differences come from the mixed boundary conditions and the fact that ϕ_t only affects horizontal variables.
  The key point is to prove that V_t is differentiable with respect to t with values in $H^1(\mathbb{R}^3_+)$. To that end, observe that V_t is the solution of the elliptic problem

$$\begin{aligned} -\mathrm{div}\,(A_t\nabla V_t) &= 0 \text{ in } \mathbb{R}^3_+,\\ \partial_z V_t &= f\circ\phi_t \text{ on } \Omega\times\{0\},\\ V_t &= 0 \text{ on } \Omega^c\times\{0\}, \end{aligned} \tag{3.24}$$

  where

$$A_t = \det(\nabla\phi_t)\begin{pmatrix}(\nabla\phi_t)^{-1}(\nabla\phi_t^T)^{-1} & 0\\ 0 & 1\end{pmatrix}.$$

  (Notice that $\nabla\phi_t$ is a 2×2 matrix.)
  Indeed, (3.24) is easily proved by writing the variational formulation associated with the equation on U_t and performing changes of variables. Since the latter are strictly identical to the ones of [10], we skip the proof.
  For further reference, we also write the system derived by lifting the Neumann boundary condition. We set $V_t(x,z) = \tilde{V}_t(x,z) + f(\phi_t(x))\eta(\phi_t(x),z)$, where $\eta\in\mathcal{C}_0^\infty(\mathbb{R}^3)$ is such that $\eta(x,z) = z$ for x in a neighbourhood of Ω and $|z|\le 1$. Then $\tilde{V}_t$ solves

$$\begin{aligned} -\mathrm{div}\,(A_t\nabla\tilde{V}_t) &= |\det(\nabla\phi_t)|(\Delta(f\eta))(\phi_t(x),z) \quad \text{on } \mathbb{R}^3_+,\\ \partial_z\tilde{V}_t &= 0 \text{ on } \Omega\times\{0\}, \quad \tilde{V}_t = 0 \text{ on } \Omega^c\times\{0\}. \end{aligned} \tag{3.25}$$

  Let

$$\mathcal{V} := \left\{V\in L^2_{\mathrm{loc}}(\mathbb{R}^3_+),\ \nabla V\in L^2(\mathbb{R}^3_+),\ V = 0 \text{ on } \Omega^c\times\{0\}\right\}.$$

  Then according to the Hardy inequality in $\mathbb{R}^3$, there exists a constant C_H such that for all $V\in\mathcal{V}$, $V(1+|x|^2+|z|^2)^{-1/2}\in L^2(\mathbb{R}^3_+)$ and

$$\int_{\mathbb{R}^3_+}\frac{|V(x,z)|^2}{1+|x|^2+|z|^2}\,\mathrm{d}x\,\mathrm{d}z \le C_H\int_{\mathbb{R}^3_+}|\nabla V|^2.$$

  This inequality is usually stated in the whole space, but a simple symmetry argument shows that it remains true in the half-space. Therefore $\|\nabla V\|_{L^2}$ is a norm on the Hilbert space $\mathcal{V}$.
  Now, for t in a neighbourhood of zero and $\tilde{V}\in\mathcal{V}$, define the linear form $F(t,\tilde{V})\in\mathcal{V}'$ by

$$\forall W\in\mathcal{V}, \langle W, F(t,\tilde{V})\rangle = \int_{\mathbb{R}^3_+}(A_t\nabla\tilde{V})\cdot\nabla W - \int_{\mathbb{R}^3_+}|\det(\nabla\phi_t)|(\Delta(f\eta))(\phi_t(x,z))W(x,z)\,\mathrm{d}x\,\mathrm{d}z.$$

Notice that $F(t,\tilde{V})=0$ is the variational formulation associated with equation (3.25). We then claim that the operator

$$F:(t,\tilde{V})\in\mathbb{R}\times\mathcal{V}\mapsto F(t,\tilde{V})\in\mathcal{V}'$$

has $\mathcal{C}^1$ regularity for t small enough. Indeed, $t\mapsto A_t\in L^\infty(\mathbb{R}^2,\mathcal{M}_3)$ is $\mathcal{C}^\infty$ for t in a neighbourhood of zero. On the other hand, for $(A,\tilde{V})\in L^\infty(\mathbb{R}^2,\mathcal{M}_3)\times\mathcal{V}$, W, let

$$a(A,\tilde{V}):W\in\mathcal{V}\mapsto\int_{\mathbb{R}^3_+}(A\nabla\tilde{V})\cdot\nabla W-\int_{\mathbb{R}^3_+}|\det(\nabla\phi_t)|(\Delta(f\eta))(\phi_t(x,z))W(x,z)\,\mathrm{d}x\,\mathrm{d}z.$$

Then the application

$$(A,V)\in L^\infty(\mathbb{R}^2,\mathcal{M}_3)\times\mathcal{V}\mapsto a(A,V)\in\mathcal{V}',$$

has $\mathcal{C}^1$ regularity since it is the sum of a bilinear and continuous function and a constant term (with respect to A,V).

Let $\tilde{V}_0\in\mathcal{V}$ be the solution of (3.25) for $t=0$, *i.e.*

$$\begin{aligned}-\Delta\tilde{V}_0&=\Delta(f\eta),\\ \partial_z\tilde{V}_0=0\text{ on }\Omega\times\{0\},&\quad\tilde{V}_0=0\text{ on }\Omega^c\times\{0\}.\end{aligned}$$

Now, for $W\in\mathcal{V}$, $d_VF(0,\tilde{V}_0)W$ is the linear form

$$W'\in\mathcal{V}\mapsto\int_{\mathbb{R}^3_+}\nabla W'\cdot\nabla W,$$

i.e. the scalar product on $\mathcal{V}$. Therefore the differential

$$d_VF(0,\tilde{V}_0):\mathcal{V}\to\mathcal{V}'$$

is an isomorphism. The implicit function theorem implies that there exists a $\mathcal{C}^1$ function $t\mapsto\tilde{V}(t)$ in a neighbourhood of zero such that

$$F(t,\tilde{V}(t))=0.$$

Uniqueness for equation (3.25) yields $\tilde{V}_t=\tilde{V}(t)$. We infer immediately that $t\mapsto V_t$ is $\mathcal{C}^1$ in a neighbourhood of zero. We now use the following trace result: for any $U\in\mathcal{V}$,

$$\|U|_{z=0}\|_{H^{1/2}(\mathbb{R}^2)}\le C(\Omega)\|\nabla U\|_{L^2(\mathbb{R}^3_+)}.\tag{3.26}$$

Indeed, it is well known that if $U\in\mathcal{V}$, the following estimates hold (with constants which do not depend on Ω)

$$\begin{aligned}\|U|_{z=0}\|_{\dot{H}^{1/2}(\mathbb{R}^2)}&\le C\|\nabla U\|_{L^2(\mathbb{R}^3_+)},\\ \|U|_{z=0}\|_{L^4(\mathbb{R}^2)}&\le C\|\nabla U\|_{L^2(\mathbb{R}^3_+)}.\end{aligned}$$

There only remains to prove that the L^2 norm of $U|_{z=0}$ is bounded by $\|\nabla U\|_{L^2}$. Since $U|_{z=0}=0$ on Ω^c, we have

$$\begin{aligned}\|U|_{z=0}\|_{L^2(\mathbb{R}^2)}&\le|\Omega|^{1/2}\|U|_{z=0}\|_{L^4(\mathbb{R}^2)}\\ &\le C(\Omega)\|\nabla U\|_{L^2(\mathbb{R}^3_+)}.\end{aligned}$$

Therefore (3.26) is proved. As a consequence, $t\mapsto v_t\in H^{1/2}(\mathbb{R}^2)$ is $\mathcal{C}^1$ in a neighbourhood of zero. The result on $\dot{u}_t$ follows almost immediately by differentiating the formula

$$u_t=v_t\circ\phi_t^{-1}$$

with respect to t. The only difficulty comes from the fact that v_t is *a priori* not smooth enough with respect to x in order to use the chain rule. However, we can write

$$u_t - u_0 = (v_t - v_0)\circ\phi_t^{-1} + \left(v_0\circ\phi_t^{-1} - v_0\right).$$

Since $\dot v_t \in \mathcal{C}([-\delta,\delta], H^{1/2}(\mathbb{R}^2))$ for $\delta > 0$ small enough, the first term is $O(t)$ in $L^2(\mathbb{R}^2)$. As for the second one, if v_0 were smooth, say $v_0 \in H^1(\mathbb{R}^2)$, then we could write

$$\frac{\mathrm{d}}{\mathrm{d}t} v_0\circ\phi_t^{-1} = \left(\frac{\mathrm{d}\phi_t^{-1}}{\mathrm{d}t}\right)\cdot\nabla v_0\circ\phi_t^{-1}.$$

It can be easily checked that the right-hand side is bounded in $H^{-1/2}$ by $C\|v_0\|_{H^{1/2}}$, where C is a constant depending only on ϕ_t. Therefore if $v_0 \in H^1$, $t > 0$,

$$\|v_0\circ\phi_t^{-1} - v_0\|_{H^{-1/2}(\mathbb{R}^2)} \le Ct\|v_0\|_{H^{1/2}(\mathbb{R}^2)}.$$

By density, this inequality remains true for all $v_0 \in H^{1/2}$. We conclude that $\dot u_t$ belongs to $\mathcal{C}([-\delta,\delta], H^{-1/2}(\mathbb{R}^2))$.

- We end this section by proving the asymptotic expansions (3.4), (3.5). We apply the results of Chapter C in [7] to the function V_t. We start by extending V_t on $\mathbb{R}^3\setminus\Omega\times\{0\}$ by setting

$$V_t(x,z) = -V_t(x,-z) \quad x\in\mathbb{R}^2,\ z<0.$$

Without any loss of generality, we assume that the function $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ defined in the Proof of Lemma 3.1 is odd with respect to z.
Then the extended function V_t satisfies

$$\begin{aligned} V_t &= \tilde V_t + f(\phi_t(x))\eta(\phi_t(x), z),\\ -\mathrm{div}\,(A_t\nabla\tilde V_t) &= |\det(\nabla\phi_t)|\Delta(f\eta)(\phi_t(x),z) \quad \text{on } \mathbb{R}^3\setminus\Omega\times\{0\}\\ \partial_z\tilde V_t &= 0 \quad \text{on } \Omega\times\{0\}. \end{aligned}$$

By elliptic regularity, $\tilde V_t \in \mathcal{C}^\infty(K)$ for any compact set $K\subset\mathbb{R}^3\setminus\Omega\times\{0\}$ such that $K\cap\partial\Omega=\emptyset$. For every connected component Γ_i of $\partial\Omega$, we introduce a truncation function $\varphi_i\in\mathcal{C}_0^\infty(\mathbb{R}^2)$ such that

$$\begin{aligned} \varphi_i &\equiv 1 \text{ on a neighbourhood of } \Gamma_i,\\ \mathrm{Supp}\,\varphi_i &\cap (\partial\Omega\setminus\Gamma_i) = \emptyset. \end{aligned}$$

Since the support of $\nabla\varphi_i$ is separated from the zones where $\tilde V_t$ has singularities, we infer that for all t, $\varphi_i(x)\tilde V_t(x,z)$ solves a boundary value problem which is elliptic of order two in the sense of Agmon *et al.* (see [1,2] and the definitions in Chapter 7 of [8]), with homogeneous boundary conditions of order one and with $\mathcal{C}^\infty$ right-hand side. Moreover, we have proved in Lemma 3.1 that $V_t \in H^1_{\mathrm{loc}}(\mathbb{R}^3\setminus\Omega\times\{0\})$. Therefore we can apply Corollary C.6.5 in [7]. As in paragraph 2.2, we denote by (r,θ) polar coordinates in the planes normal to Γ_i and centered on Γ_i, and by s the arc-length on Γ_i, so that

$$\mathbb{R}^3\setminus(\mathrm{Supp}\,\varphi_i\cap\Omega\times\{0\}) = W_\pi = \{(s,r,\theta),\ s\in(0,L_i),\ r>0,\ \theta\in(-\pi,\pi)\}.$$

We deduce that for all i, there exist functions $\tilde\psi_i^0(t;s,\theta)$, $\tilde\psi_i^1(t;s,\theta)$, which are smooth with respect to $s,\theta\in[-\pi,\pi]$, such that

$$\varphi_i(x)\tilde V_t(x,z) = \tilde\psi_i^0(t;s,\theta)r^{1/2} + \tilde\psi_i^1(t;s,\theta)r^{3/2} + u_{\mathrm{reg}}(t;x,z) + u_{\mathrm{rem}}(t;x,z), \tag{3.27}$$

with $u_{\rm reg}(t;\cdot) \in \mathcal{C}^\infty(\mathbb{R}^3)$ for all t and $u_{rem}(t;\cdot) \in \mathcal{C}^2(\overline{W_\pi})$ with $\partial^\beta u_{rem}(t;x,z) = o(r^{3/2-|\beta|})$ for all t and for any multi-index β. Setting

$$V_i^1(t;x,z) := \tilde{\psi}_i^1(t;s,\theta)r^{3/2} + u_{\rm reg}(t;x,z) + u_{\rm rem}(t;x,z) + \eta(\phi_t(x),z),$$

we have $V_i^1 \in \mathcal{C}^1(\overline{W_\pi})$, $V_i^1(t;x,z) = O(r)$ for all t, and

$$V_t(x,z) = \tilde{\psi}^0(t;s,\theta)r^{1/2} + V_i^1(t;x,z)$$

for x in a neighbourhood of Γ_i. Since

$$u_0(x) = \lim_{z\to 0, z>0} V_0(x,z),$$

we obtain decomposition (3.4) with

$$\psi^0(s) := \tilde{\psi}_i^0(0,s,\pi) \text{ for } s \in \Gamma_i, u_1(x) = \lim_{z\to 0^+} V_i^1(0;x,z) \text{ for } x \in \operatorname{Supp}\varphi_i.$$

Differentiating (3.5) requires regularity results with respect to t on the terms of the decomposition (3.27). Using for instance Theorem C.6.2 in [7], or looking precisely at the details of the proof in Section C of [7], it is easily proved that the functions $\tilde{\psi}^0$, $\tilde{\psi}^1$, $u_{\rm reg}$ and u_{rem} are differentiable with respect to t, and that their t-derivatives are smooth with respect to x. Denoting by $s(x)$, $r(x)$ the local coordinates of a point $x \in \mathbb{R}^2$, we have

$$u_t(x) = \tilde{\psi}^0(t, s(\phi_t^{-1}(x)),\pi)(r(\phi_t^{-1}(x)))^{1/2} + V^1(t,\phi_t^{-1}(x),0^+)$$

so that

$$\begin{aligned}\dot{u}_t = &\left(\partial_t\tilde{\psi}^0 + \partial_t(\phi_t^{-1}(x))\cdot\nabla s(\phi_t^{-1}(x))\partial_s\tilde{\psi}^0\right)(t,s(\phi_t^{-1}(x)),\pi)(r(\phi_t^{-1}(x)))^{1/2}\\ &+\frac{1}{2}\partial_t(\phi_t^{-1}(x))\cdot\nabla r(\phi_t^{-1}(x))\frac{1}{r(\phi_t^{-1}(x))^{1/2}}\\ &+\partial_t V^1(t,\phi_t^{-1}(x),0^+) + \partial_t(\phi_t^{-1}(x))\cdot\nabla V^1(t,\phi_t^{-1}(x),0^+).\end{aligned} \tag{3.28}$$

Therefore $\dot{u}_t \in L_t^\infty(L_x^1)$, which proves (3.8). We recall that $\phi_0 = \phi_0^{-1} = \mathrm{Id}$, and that $\zeta = \dot{\phi}_0$. Notice also that since n is the outward pointing normal, $\nabla r = -n$. We infer

$$\dot{u}_0 = \frac{\zeta\cdot n}{2\sqrt{r}} + \psi_2(s)\sqrt{r} + u_2(x),$$

where

$$\begin{aligned}\psi_2(s) &= \left(\partial_t\tilde{\psi}^0 + \zeta\cdot\tau\partial_s\tilde{\psi}^0\right)(0,s,\pi),\\ u_2(x) &= \partial_t V^1(0,x,0^+) + \zeta\cdot\nabla V^1(0,x,0^+).\end{aligned}$$

Thus (3.5) is proved.

4. Radial symmetry by the moving plane method

The aim of this section is to prove Theorem 1.3. This theorem extends the well-known theorem of Serrin [14] on the classical laplacian, for which (1.5) is replaced by

$$\begin{cases} -\Delta u = 1, & x \in \Omega,\\ u = 0, & x \in \partial\Omega,\\ \partial_n u = c_0, & x \in \partial\Omega \end{cases} \tag{4.1}$$

The proof of Serrin uses the celebrated moving plane method, and we will adapt it to our fractional setting.

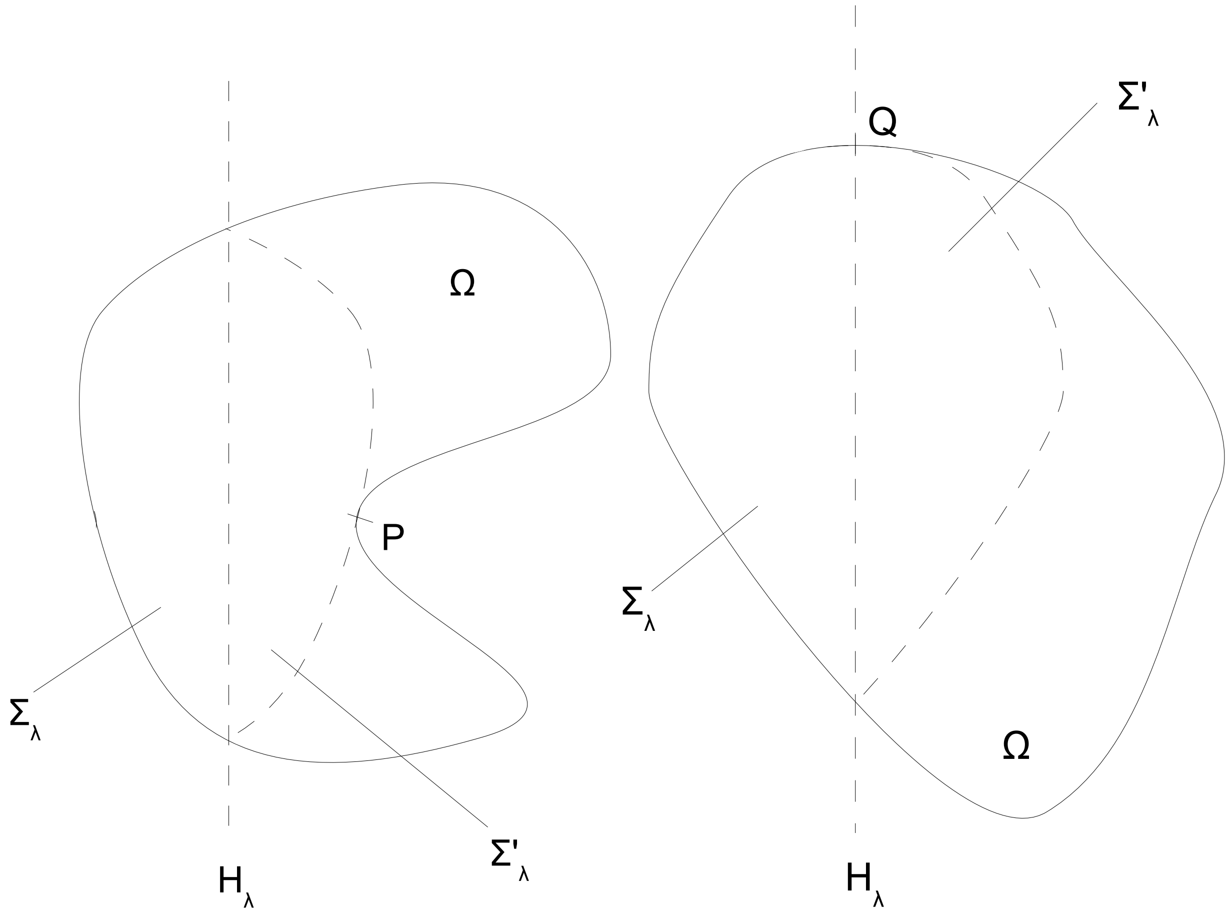

Figure 1. Configuration 1 (on the left) and configuration 2 (on the right).

4.1. Reminders on the moving plane method

We remind here the main arguments of Serrin's proof. The starting point is the introduction of a family of hyperplanes (lines in our 2d case), say $H_\lambda := \{x \in \mathbb{R}^2, x_1 = \lambda\}$, parametrized by $\lambda \in \mathbb{R}$. We also define, for any function u defined on $\mathbb{R}^2$, $R_\lambda u(x_1, x_2) := u(2\lambda - x_1, x_2)$ the reflection of u with respect to H_λ. For λ small enough, H_λ does not intersect the domain Ω. Increasing λ, that is moving H_λ from left to right, one reaches a first contact position, corresponding to

$$\lambda_0 := \inf\{\lambda, H_\lambda \cap \Omega \neq \emptyset\}.$$

Up to a translation, we can always assume that $\lambda_0 = 0$. Thus, for $\lambda > 0$, we can consider the cap $\Sigma_\lambda := \Omega \cap \{x_1 < \lambda\}$ and its reflection Σ'_λ with respect to H_λ. Note that, for $\lambda > 0$ small enough, Σ'_λ is non-empty and included in Ω. As λ increases, it remains included in Ω at least until one of the following two geometric configurations is reached (see Fig. 1):

1. Σ'_λ is internally tangent to the boundary of Ω at some point P not on H_λ.
2. H_λ is orthogonal to the boundary of Ω at some point Q.

Let Λ be the first value of λ for which configuration 1 or 2 holds. Note that we may have $\Sigma'_\lambda \subset \Omega$ for some $\lambda > \Lambda$, but this will be irrelevant in the proof. The main point in Serrin's proof is to show that one has $R_\Lambda u = u$

inside Σ_Λ, where u is the solution of (4.1). This fact yields easily the symmetry of Ω with respect to H_Λ. As the direction of H_Λ was chosen arbitrarily, it will follow that for any direction, there is an axis of symmetry for Ω with that direction. This property implies that Ω is a disk.

To show the identity $R_\Lambda u = u$ inside Σ_Λ, one argues by contradiction through three main steps. Assume that the equality does not hold. Then

- *Step 1.* One shows that the function $w := u - R_\Lambda u$ is positive inside Σ'_Λ.
- *Step 2.* One obtains an upper bound for w near the tangency point P (configuration 1) or Q (configuration 2). As the normal derivative of u is constant along $\partial\Omega$, one has $w = \partial_n w = 0$ at P or Q. For configuration 2, one can show furthermore that the second derivatives cancel at Q: $\partial_{ij} w = 0$ for all $i, j = 1, 2$. Denoting by r the distance at the tangency point, these properties imply that

$$w = O(r^2) \text{ at } P \text{ (configuration 1), or } w = O(r^3) \text{ at } Q \text{ (configuration 2).} \tag{4.2}$$

- *Step 3.* One shows a lower bound which contradicts (4.2). For configuration 1, this lower bound is obtained easily. Indeed, one knows from Step 1 that $w > 0$ inside Σ'_Λ. Hence, Hopf's lemma implies $\partial_n w > 0$ at P (where n is the inward normal), which shows that w should grow at least linearly with r inside the domain. For configuration 2, Σ'_Λ is not regular enough at Q to apply Hopf's lemma. However, one can still show that

$$\partial_s w > 0 \text{ or } \partial_s^2 w > 0 \text{ at } Q$$

  for any direction $\boldsymbol{s}$ entering Σ'_Λ non-tangentially: see [14], Lemma 3.1. It follows that w should grow at least like r^2 inside the domain. In both cases, we reach the targeted contradiction. We refer to [14] for all details.

Our ambition is to transpose this scheme of proof to problem (1.5). Due to the nonlocal character of $(-\Delta)^{1/2}$, it requires many modifications:

- In the case of the Laplacian, Step 1 follows from a simple application of the strong maximum principle, as $w \geq 0$ on $\partial\Sigma'_\Lambda$. In our fractional (and therefore nonlocal) setting, such use of the maximum principle is impossible: it would require that $w \geq 0$ on the whole $\mathbb{R}^2 \setminus \Sigma'_\Lambda$, which is not true ($w$ is an odd function). The appropriate treatment of Step 1 will be addressed in paragraph 4.2.
- In the case of the Laplacian, Steps 2 and 3 rely on the regularity of u up to the boundary. In the case of $(-\Delta)^{1/2}$, we only have $C^{0,1/2}$ regularity of u, which implies substantial changes. Loosely:
  - the upper bounds will follow from the asymptotic expansion (2.9) of u near the boundary;
  - the lower bounds will follow from the construction of refined subsolutions.

Additionally, we emphasize that the constant c_0 in (1.5) is necessarily strictly positive. Indeed, $c_0 \geq 0$ by the maximum principle. The fact that $c_0 > 0$ is a consequence of the Hopf Lemma for the fractional laplacian. In the present context, we may present a self-contained proof: if $c_0 = 0$, we jump to paragraph 4.3. The arguments developed there (for the function w instead of u) allow us to conclude that $u \equiv 0$. This is excluded by the equation $(-\Delta)^{1/2} u = 1$. Therefore we restrict our analysis to $c_0 > 0$.

4.2. Positivity of w

We shall first prove that if non-identically 0, $\check{w} := R_\Lambda u - u$ is positive inside Σ_Λ (which amounts to achieving Step 1). We shall rely on ideas developed by M. Birkner, J. Lopez–Mimbela and A. Wakolbinger in article [3]. They show there the radial symmetry of solutions of some semilinear fractional problems

$$(-\Delta)^\alpha u + F(u) = 0, \quad x \in B(0,1), \quad u = 0 \text{ in } \mathbb{R}^n \setminus B(0,1)$$

set in the unit ball $B(0,1)$ of $\mathbb{R}^n$. Their proof is based on an adaptation of the moving plane method, and its baseline can be used to show the positivity of w. Nevertheless, several changes are needed, and simplifications of the original arguments can be made, as we now describe.

First of all, we introduce the set

$$\mathcal{L} := \{\lambda \in]0, \Lambda[, \quad s.t. \quad \forall 0 < \gamma \le \lambda, \quad R_\gamma u - u \ge 0 \text{ on } \Sigma_\gamma, \text{ and } \partial_1 u > 0 \text{ on } H_\gamma \cap \Omega\}.$$

The main point is to show that $\mathcal{L} =]0, \Lambda[$. We proceed in several steps:

- *Statement 1:* $]0, \epsilon] \subset \mathcal{L}$ *for* $\epsilon > 0$ *small enough.*

To prove that $]0, \epsilon[\subset \mathcal{L}$, it is enough to show that $\partial_1 u > 0$ for all $x \in \Omega$ with $0 < x_1 < \epsilon$. From the condition $\partial_n^{1/2} u = c_0 > 0$, and the expansion (2.8), we know that

$$\partial_r u = \frac{1}{2} c_0 \, r^{-1/2} + O(1), \quad \partial_s u = O(1)$$

as the distance r to the boundary (measured inside the domain) goes to zero. It follows that

$$\partial_1 u = \partial_1 r \, \partial_r u + \partial_1 s \, \partial_s u = \frac{1}{2} c_0 \, (\partial_1 r) \, r^{-1/2} + O(1), \quad r \to 0.$$

Moreover, for $0 < \lambda < \Lambda$, for all inward normal vectors n along $\partial\Omega \cap \{x_1 < \lambda\}$, one has $e_1 \cdot n > 0$, where $e_1 = (1, 0)$. It follows that $\partial_1 r$ has a positive lower bound for $0 < x_1 < \epsilon$ small enough. The statement follows.

- *Statement 2:* $\mathcal{L}$ *is an open subset of* $]0, \Lambda[$.

The point is to show that if $\lambda \in \mathcal{L}$, $[\lambda, \lambda + \epsilon[\subset \mathcal{L}$ for $\epsilon > 0$ small enough. This statement will be deduced from the following:

Lemma 4.1. *Let* $\lambda < \Lambda$. *Then*

(i) *For all* $x \in \partial\Omega \cap \{x_1 < \lambda\}$, $R_\lambda u(x) - u(x) > 0$.
(ii) *Assume that* $R_\lambda u - u \ge 0$ *on* Σ_λ. *Then,* $R_\lambda u - u > 0$ *on* Σ_λ. *In particular, one has* $R_\lambda u - u > 0$ *on* Σ_λ *for any* $\lambda \in \mathcal{L}$.

Proof. We remind that $u > 0$ in Ω by the maximum principle.

(i) As $\lambda < \Lambda$, for all $x = (x_1, x_2) \in \partial\Omega \cap \{x_1 < \lambda\}$, $x_\lambda := (2\lambda - x_1, x_2) \in \Omega$. In particular:

$$R_\lambda u(x) - u(x) = u(x_\lambda) > 0.$$

(ii) We assume that $\check{w} := R_\lambda u - u \ge 0$ on Σ_λ. Note that on $\{x_1 < \lambda\} \setminus \Sigma_\lambda$, one has $\check{w} = R_\lambda u \ge 0$, so that $\check{w} \ge 0$ on the whole half-space $\{x_1 < \lambda\}$. Note also that by (i) and the continuity of u, the fonction $\check{w}$ is not identically zero in Σ_λ. We want to show that $\check{w} > 0$ in Σ_λ. We introduce the harmonic extension $\check{W}$ of $\check{w}$, defined on $\mathbb{R}^3_+$. We remind that

$$\Delta \check{W} = 0 \text{ on } \mathbb{R}^3_+, \quad \check{W}|_{z=0} = \check{w} \text{ on } \mathbb{R}^2.$$

Moreover, as $(-\Delta)^{1/2} u = 1$ in Ω, $\partial_z \check{W}|_{z=0} = 0$ in Σ_λ. We can express $\check{W}$ with the Poisson kernel:

$$\check{W}(x_1, x_2, z) = \frac{z}{2\pi} \int_{\mathbb{R}^2} \frac{\mathrm{d}t_1 \mathrm{d}t_2}{((x_1 - t_1)^2 + (x_2 - t_2)^2 + z^2)^{1/2}} \check{w}(t_1, t_2).$$

As $R_\lambda \check{w} = -\check{w}$, this integral formula can be written

$$\check{W}(x_1, x_2, z) = \frac{z}{2\pi} \int_{\{t_1 < \lambda\}} \mathrm{d}t_1 \mathrm{d}t_2 \left(\frac{1}{((x_1 - t_1)^2 + (x_2 - t_2)^2 + z^2)^{1/2}} - \frac{1}{(x_1 + (t_1 - 2\lambda))^2 + (x_2 - t_2)^2 + z^2)^{1/2}} \right) \check{w}(t_1, t_2).$$

As $\check{w} \ge 0$ on $\{t_1 < \lambda\}$ and not identically zero, we deduce from this formula that $\check{W} > 0$ for $x_1 < \lambda$, $z > 0$.

To conclude on the positivity of $\breve{w}$, we assume *a contrario* that $\breve{w}(x^*) = 0$ for some $x^* = (x_1, x_2) \in \Sigma_\lambda$. The function $\breve{W}$ is smooth up to the boundary in the vicinity of $(x_1^*, x_2^*, 0)$, and $0 = \breve{W}(x_1^*, x_2^*, 0) < \breve{W}(x, z)$ for all $(x, z) \in \mathbb{R}^3_+$. By Hopf's lemma, we should have $\partial_z \breve{W}(x_1^*, x_2^*, 0) > 0$ hence reaching a contradiction. This concludes the proof. □

Back to Statement 2: we argue by contradiction, as in [3], paragraph 4.2. We assume that there is $\lambda \in \mathcal{L}$ and a decreasing sequence $(\lambda_n) \subset]0, \Lambda[\setminus\mathcal{L}$ converging to λ. Up to the extraction of a subsequence, we can assume that there exists a sequence (γ_n) such that $\lambda < \gamma_n < \lambda_n$ and

1. either there exists a sequence (x_n), $x_n \in \Sigma_{\gamma_n}$ for all n, such that $x_n \to x^*$ and $u(x_n) > R_{\gamma_n} u(x_n)$ for all n.
2. or there exists a sequence (x_n), $x_n \in H_{\gamma_n} \cap \Omega$ for all n, such that $x_n \to x^*$ and $\partial_1 u(x_n) \leq 0$ for all n.

We first consider case 1: since $\lambda \in \mathcal{L}$, we have $R_\lambda u - u \geq 0$ on Σ_λ. As u is continuous, we get $u(x^*) - R_\lambda u(x^*) \geq 0$. By Lemma 4.1, we cannot have $x^* \in \Sigma_\lambda \cup (\partial\Omega \cap \{x_1 < \lambda\})$. Hence, $x^* \in H_\lambda \cap \overline{\Omega}$. If $x^* \in H_\lambda \cap \partial\Omega$, one has $\partial_n^{1/2} u(x^*) = c_0 > 0$. Moreover, as $\lambda < \Lambda$, $e_1 \cdot n^* > 0$ for n^* the inward normal vector at x^*: indeed, since $\lambda < \Lambda$, configuration 2 has not been met, and therefore $e_1 \cdot n(y)$ does not vanish for all $y \in \partial\Omega$ such that $y_1 \leq \lambda$. Since $e_1 \cdot n(y) = 1$ when $y = (0, y_2) \in \partial\Omega$, we infer $e_1 \cdot n^* > 0$.

Reasoning as in the proof of Statement 1, we obtain that u is strictly increasing with x_1 in the vicinity of x^* (inside the domain). Hence, for n large enough, we get

$$R_{\gamma_n} u(x_n) - u(x_n) > 0$$

in contradiction with the assumption $u(x_n) > R_{\gamma_n} u(x_n)$ for all n. If $x^* \in H_\lambda \cap \Omega$, then $\partial_1 u(x^*) > 0$ (because $\lambda \in \mathcal{L}$). This means again that u is strictly increasing with x_1 in the vicinity of x^*, which yields the same contradiction as before.

It remains to consider case 2. With the same reasoning as in case 1, we get that $x^* \in H_\lambda \cap \overline{\Omega}$, and that u is strictly increasing with x_1 in the vicinity of x^*. This contradicts the assumption $\partial_1 u(x_n) \leq 0$ for all n.

- *Statement 3:* $\mathcal{L} =]0, \Lambda[$.

From the previous statements on $\mathcal{L}$, we know that $\mathcal{L} =]0, \Lambda_{\max}[$ for some $\Lambda_{\max} \leq \Lambda$. Again, we shall argue by contradiction and assume that $\Lambda_{\max} < \Lambda$. For all $\lambda < \Lambda_{\max}$, one has $R_\lambda u - u > 0$ in Σ_λ, so that by continuity of u, $R_{\Lambda_{\max}} u - u \geq 0$ in $\Sigma_{\Lambda_{\max}}$. Lemma 4.1 implies in turn that

$$R_{\Lambda_{\max}} u - u > 0 \ \text{ in } \ \Sigma_{\Lambda_{\max}}.$$

We want to show that $\partial_1 u > 0$ on $H_{\Lambda_{\max}} \cap \Omega$. This would imply that $\Lambda_{\max} \in \mathcal{L}$, so the contradiction.

As $\breve{w} := R_{\Lambda_{\max}} u - u$ is odd with respect to $H_{\Lambda_{\max}}$, it is enough to prove that $\partial_1 \breve{w} < 0$ on $H_{\Lambda_{\max}} \cap \Omega$. We introduce its harmonic extension, as in the proof of Lemma 4.1: it satisfies

$$\Delta \breve{W} = 0 \ \text{ on } \ \mathbb{R}^3_+, \quad \breve{W}|_{z=0} = \breve{w} \ \text{ on } \ \mathbb{R}^2, \quad \partial_z \breve{W}|_{z=0} = 0 \ \text{ on } \ \Sigma_{\Lambda_{\max}}.$$

We extend $\breve{W}$ to the lower half-space by setting

$$\breve{W}(x_1, x_2, z) \ := \ \breve{W}(x_1, x_2, -z) \quad \text{ for } \ z < 0.$$

As $\partial_z \breve{W}|_{z=0} = 0$ on $\Sigma_{\Lambda_{\max}} \times \{0\}$, this extension is harmonic through $\Sigma_{\Lambda_{\max}}$, that is outside $(\mathbb{R}^2 \setminus \Sigma_{\Lambda_{\max}}) \times \{0\}$. Notice also that as in the proof of Lemma 4.1, $\breve{w} \geq 0$ on $\{x_1 < \Lambda_{\max}\}$, and therefore $\breve{W} \geq 0$ on $\{x_1 < \Lambda_{\max}\}$. Let $x^* \in H_{\Lambda_{\max}} \cap \Omega$. $\breve{W}$ is smooth and harmonic near $(x_1^*, x_2^*, 0)$. Moreover, one has

$$0 = \breve{w}(x^*) = \breve{W}(x_1^*, x_2^*, 0) < \breve{W}(x, z), \quad \text{ say for all } \ (x, z) \in \mathcal{O} := \Sigma_{\Lambda_{\max}} \times \mathbb{R}$$

Applying Hopf's lemma in $\mathcal{O}$ (for which $(x_1^*, x_2^*, 0)$ is a boundary point), we obtain that $\partial_1 \breve{W}(x_1^*, x_2^*, 0) < 0$ that is $\partial_1 \breve{w}(x^*) < 0$, as expected.

As a conclusion of our analysis, we obtain that $R_\lambda u - u > 0$ on Σ_λ for all $\lambda < \Lambda$. By continuity of u, we obtain that $\breve{w} := R_\Lambda u - u \geq 0$ on Σ_Λ. From there, there are two possibilities:

- either $\breve{w} = 0$ on Σ_Λ. It follows easily that Ω is symmetric with respect to H_Λ. As explained in paragraph 4.1, the direction of H_Λ being arbitrary, it follows that Ω is a disk.

- or $\check{w}(x) > 0$ for some point x in Σ_Λ. But then, following exactly the proof of point ii) in Lemma 4.1, we obtain that $\check{w} > 0$ on Σ_Λ (or $w := u - R_\Lambda u > 0$ on Σ'_Λ).

The rest of the section aims at excluding the second possibility. *From now on, we assume that $w > 0$ on Σ'_Λ, and will establish contradictory lower and upper bounds on w.*

First of all, we state a lemma, to be used in both configurations 1 and 2.

Lemma 4.2. *Let W be the harmonic extension of w to $\mathbb{R}^3_+$. For all $R > 0$ sufficiently large there exists $c_R > 0$ such that*

$$W(x,z) \;\geq\; c_R\,(x_1 - \Lambda)\,z, \qquad \textit{for all } \;\Lambda \leq x_1 \leq R, \quad -R \leq x_2 \leq R, \quad 0 \leq z \leq R.$$

Proof. Let $R > 0$ large, and $Q_R :=]\Lambda, R[\times] - R, R[\times]0, R[$. As W and $W' := x \mapsto (x_1 - \Lambda)\,z$ are both harmonic functions on Q_R, we can use the maximum principle to compare them: we just have to show that $W \geq c_R W'$ on ∂Q_R for $c_R > 0$. We take R large enough so that the vertical sides of the cube (other than the one supported by $\{x_1 = \Lambda\}$) do not intersect $\overline{\Omega} \times \{0\}$. Then, we distinguish between the different sides:

- On $\partial Q_R \cap \{z = 0\}$ or $\partial Q_R \cap \{x_1 = \Lambda\}$, one has $W \geq 0$ (remind that $w \geq 0$ on $\{x_1 \geq \Lambda\}$), and $W' = 0$. The inequality is clear.
- Let $\Gamma_{1,R} := \partial Q_R \cap \{x_1 = R\}$. Let $\epsilon > 0$. As W is continous and > 0 on the compact set $\Gamma_{1,R} \cap \{z \geq \epsilon\}$, one can find $c = c_\epsilon > 0$ such that $W \;\geq\; c\,W'$. To have the same inequality on the whole $\Gamma_{1,R}$, it is enough to show that
$$\liminf_{\substack{(x,z)\in\Gamma_{1,R},\\ z\to 0^+}} \frac{W(x,z)}{W'(x,z)} = \frac{1}{R-\Lambda} \liminf_{\substack{(x,z)\in\Gamma_{1,R},\\ z\to 0^+}} \frac{W(x,z)}{z} > 0. \tag{4.3}$$
This is a consequence of Hopf's lemma: W is positive harmonic on Q_R, satisfies $W = 0$ on $\Gamma_{1,R} \cap \{z = 0\}$. It follows that $\partial_z W > 0$ on $\Gamma_{1,R} \cap \{z = 0\}$. Note that $\partial_z W$ exists thanks to our choice of R, away from Ω. It is furthermore continuous, so that it is bounded from below by a positive constant. Inequality (4.3) follows.
- On $\Gamma_{3,R} := \partial Q_R \cap \{z = R\}$, one can proceed exactly as in the case of $\Gamma_{1,R}$: use the positivity of W away from $\{x_1 = \Lambda\}$, and use Hopf's lemma to get $\partial_1 W \geq c > 0$ at $\Gamma_{3,R} \cap \{x_1 = \Lambda\}$.
- We now turn to $\Gamma_{2,R} = \partial Q_R \cap \{x_2 = \pm R\}$. Away from the edge $x_1 = \Lambda, z = 0$, we can as before use the positivity of W and Hopf's lemma to obtain $W \geq cW'$. It then remains to handle the vicinity of $X = (\Lambda, \pm R, 0)$. The main point is to show that
$$\liminf_{\substack{(x,z)\in\Gamma_{2,R}\\ (x,z)\to X}} \frac{W(x,z)}{(x_1 - \Lambda)\,z} > 0. \tag{4.4}$$
We can not apply Hopf's lemma: W is harmonic in the dihedra $\{x_1 > \Lambda, z > 0\}$, but it does not satisfy the interior sphere condition at X. As $W(x,z) = 0$ both for $x_1 = \Lambda$ and $z = 0$, one has
$$\begin{cases} \partial_z^k W(x,z) = 0 & \text{for } x_1 = \Lambda,\ \forall k \in \mathbb{N} \\ \partial_1^k W(x,z) = 0 & \text{for } z = 0,\ \forall k \in \mathbb{N}. \end{cases} \tag{4.5}$$
((x,z) in the vicinity of X). Hence, to prove (4.4), it is enough to show that
$$\partial_1 \partial_z W(x,z) \geq c > 0, \quad (x,z) \in B(X,\delta) \cap \{x_1 \geq \Lambda, z \geq 0\},$$
for small enough δ. By continuity of $\partial_1\partial_z W$, it is enough to show that $\partial_1\partial_z W(X) > 0$. This positivity condition follows straightforwardly from a famous lemma of Serrin, see [14], Lemma 3.1, p. 308. In our context, it reads: *for any vector s entering the region $\{x_1 > \Lambda,\ z > 0\}$, we have $\partial_s W(X) > 0$ or $\partial_s^2 W(X) > 0$.* Taking $s = e_1 + e_3$, we obtain that
$$(\partial_1 + \partial_z) W(X) > 0, \quad \text{ or } \;(\partial_1 + \partial_z)^2 W(X) > 0.$$

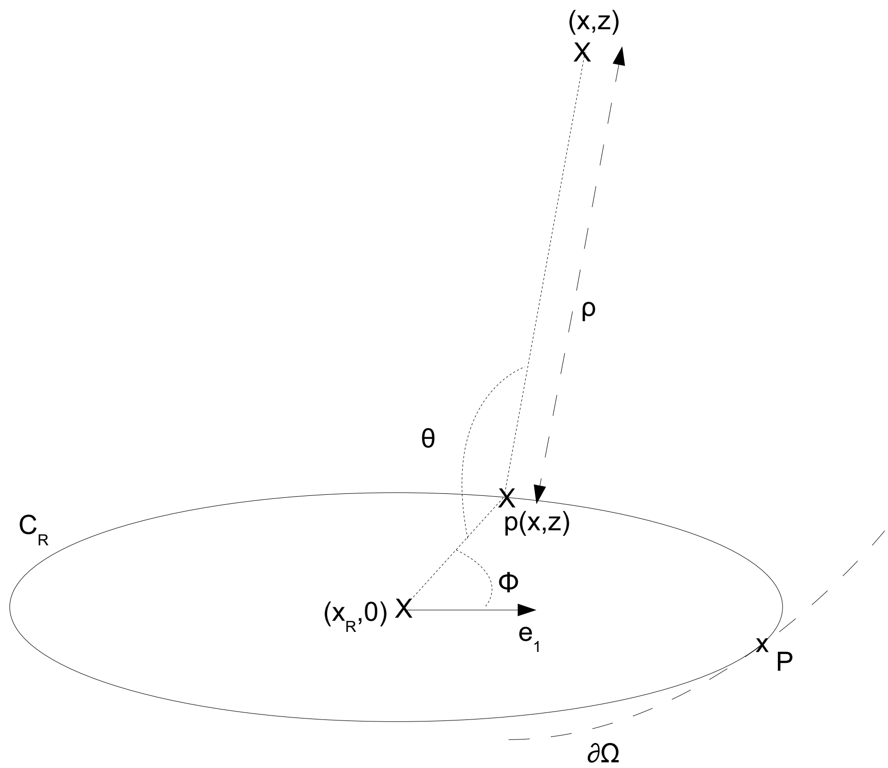

FIGURE 2. The coordinates (ρ, θ, ϕ).

Combining this statement with (4.5), we see that the first condition is not realized, and that the second one amounts to $\partial_1 \partial_z W(X) > 0$ as expected. □

4.3. Contradictory bounds on w: configuration 1

Lower bound

We start with configuration 1, that is when Σ'_Λ is internally tangent to $\partial\Omega$ at a point P not in H_Λ. We introduce a small open disk $D_R \subset \Omega$ of radius R, also tangent to $\partial\Omega$ at P. We take R small enough so that $\partial D_R \cap \partial\Omega = \{P\}$. Let x_R be the center of D_R, and let $\phi \in [0, 2\pi]$ parametrizing ∂D_R. Finally, for any x in D_R, denote by $\rho = \rho(x)$ the distance between x and the circle ∂D_R. One has $x = x_R + ((R-\rho)\cos\phi, (R-\rho)\sin\phi)$. The aim of this paragraph is to prove the following

Proposition 4.3. *Let $\beta > 1/2$. There exists $c_\beta > 0$, $\rho_\beta > 0$ such that for all $x \in D_R$ with $0 < \rho(x) < \rho_\beta$,*

$$w(x) \geq c_\beta\, \rho^\beta.$$

First, we shall extend the 2d coordinate system (ρ, ϕ) to a 3d coordinate system (ρ, ϕ, θ) in the $R/2$-neighborhood of $C_R := \partial D_R \times \{0\}$ (see Fig. 2). Any $(x, z) = (x_1, x_2, z)$ in this neighborhood has a unique projection $p(x, z)$ on C_R, which reads:

$$p(x, z) = (x_R, 0) + (R\cos\phi, R\sin\phi, 0).$$

We introduce $\rho = |(x,z) - p(x,z)|$ the distance between (x,z) and C_R, and $\theta \in [-\pi, \pi[$ the oriented angle between the vector $(x_R, 0) - p(x,z)$ and $(x,z) - p(x,z)$. Then, one can write

$$\begin{aligned}(x,z) &= p(x,z) + (-\rho\cos\theta\cos\phi, -\rho\sin\theta\sin\phi, \rho\sin\theta) \\ &= (x_R, 0) + ((R - \rho\cos\theta)\cos\phi, (R-\rho\sin\theta)\sin\phi, \rho\sin\theta).\end{aligned}$$

The triplet (ρ, θ, ϕ) defines a system of orthogonal curvilinear coordinates in the $R/2$ neighborhood of C_R. For $\theta = 0$, it matches the 2d coordinates (ρ, ϕ) introduced above.

To prove Proposition 4.3, we shall rely again on the harmonic extension W of w. It satisfies

$$\Delta W = 0 \text{ on } \mathbb{R}^3_+, \quad W|_{z=0} = w \text{ on } \mathbb{R}^2, \quad \partial_z W|_{z=0} = 0 \text{ on } \Sigma'_\Lambda.$$

We can extend W to $\mathbb{R}^3$ into an even function of z (still denoted W). Thanks to the last condition, it is harmonic through Σ'_Λ. We will show that for $\beta > \frac{1}{2}$,

$$W(x,z) \geq c_\beta\, \rho^\beta \cos(\theta/2), \quad \forall \rho = \rho(x) < \delta \text{ small enough.} \tag{4.6}$$

This will yield Proposition 4.3 for $\theta = 0$. This lower bound on W will follow from the maximum principle. We introduce $U_\delta := \{x, \rho(x) < \delta\} \setminus \{\theta = \pi\}$. The open set U_δ is obtained by removing from the δ-neighborhood of C_R the part of the plane $\{z = 0\}$ outside D_R. Let W_β given in (ρ, θ, ϕ) coordinates by $W_\beta(x,z) = \rho^\beta \cos(\theta/2)$. We will show that for δ >0 small enough, one has:

(i) $\Delta W_\beta \geq 0 \, (= \Delta W)$ in U_δ.
(ii) $W \geq c_\beta W_\beta$ on ∂U_δ, for some $c_\beta > 0$.

The lower bound (4.6) will follow.

Proof of (i). One can check that the Laplacian in coordinates (ρ, θ, ϕ) reads

$$\Delta f = \frac{1}{\rho(R - \rho\cos\theta)}\left(\frac{\partial}{\partial\rho}\left(\rho(R-\rho\cos\theta)\frac{\partial}{\partial\rho}f\right) + \frac{\partial}{\partial\theta}\left(\frac{R-\rho\cos\theta}{\rho}\frac{\partial}{\partial\theta}f\right) + \frac{\partial}{\partial\phi}\left(\frac{\rho}{R-\rho\cos\theta}\frac{\partial}{\partial\phi}f\right)\right). \tag{4.7}$$

This implies that

$$\begin{aligned}\Delta W_\beta &= \frac{1}{\rho(R-\rho\cos\theta)}\Big(\Big((\beta^2 - 1/4)\rho^{\beta-1}(R-\rho\cos\theta) - \beta\rho^\beta\cos\theta\Big)\cos(\theta/2) - \frac{1}{2}\rho^\beta\sin\theta\sin(\theta/2)\Big) \\ &= \frac{1}{\rho(R-\rho\cos\theta)}\big((\beta^2 - 1/4)\rho^{\beta-1}(R-\rho\cos\theta) - \beta\rho^\beta\cos\theta - \rho^\beta\sin^2(\theta/2)\big)\cos(\theta/2) \\ &= \frac{1}{\rho(R-\rho\cos\theta)}\big((\beta^2 - 1/4)\, R\,\rho^{\beta-1} + O(\rho^\beta)\big)\cos(\theta/2) \geq 0\end{aligned}$$

for δ (and $\rho < \delta$) small enough. Note that we used the identity $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ in the second line.

Proof of (ii). We must distinguish between different zones.

- On $\partial U_\delta \cap \{\theta = \pm\pi\}$, one has $W \geq 0, W_\beta = 0$, so that the inequality is clear.
- The compact set $\partial U_\delta \cap \{\theta = 0\}$ is included in Σ'_Λ, so that $W > 0$ there, and as W is continuous, it is even bounded from below by a positive constant. In particular, one can find $c > 0$ such that $W \geq cW_\beta$.
- For any $\epsilon > 0$, we also know that $W > 0$ on the compact set $\partial U_\delta \cap \{|z| \geq \epsilon\}$. We obtain again that $W \geq cW_\beta$ for some $c = c_\epsilon > 0$.
- It remains to show that for $\epsilon > 0$ small enough, one has

$$W \geq cW_\beta \text{ on } \partial U_\delta \cap \{0 < |z| \leq \epsilon\}.$$

Due to the fact that W is non-negative in $\mathbb{R}^3$ and even in the variable z, it is enough to show that

$$\liminf_{\substack{(x,z)\in\partial U_\delta,\\ z\to 0^+}} |W(x,z)/W_\beta(x,z)| > 0.$$

To show such a property, we argue by contradiction. Assume that we can find a sequence (x^k, z^k) satisfying:

$$(x^k, z^k) \subset \mathbb{R}^3_+ \cap \partial U_\delta, \quad z^k \to 0^+, \quad \left|W(x^k, z^k)/W_\beta(x^k, z^k)\right| \to 0.$$

Up to the extraction of a subsequence, we can assume that $x^k, z^k \to X$ for some $X \in \partial U_\delta$ with $X = (X_1, X_2, 0)$. There are two possibilities:

- either $W(X) > 0$. As W_β is bounded, this yields easily a contradiction.
- or $W(X) = 0$. In particular, X is outside C_R, which implies in turn $W_\beta(X) = 0$. More generally, $W(x_1, x_2, 0) = 0$ for all (x_1, x_2) close to (X_1, X_2). Moreover, as W_β is smooth near X, we obtain

$$|W_\beta(x,z)| = O(z), \quad \text{as } z \to 0^+, \quad x \text{ in the vicinity of } X.$$

From there, we get that

$$\limsup_{k\to\infty} |W(x^k, z^k)|/z^k = 0 \tag{4.8}$$

As x_1^k remains away from Λ, this limit contradicts Lemma 4.2.

Upper bound

We now turn to an upper bound for w. Let $x \in \Sigma'_\Lambda$. For x close to P, we can write in a unique way

$$x = x^* + r\,n, \quad \text{resp. } x = \bar{x}^* + \bar{r}\,\bar{n}, \quad r, \bar{r} > 0,$$

where $x^* \in \partial\Omega$, resp. $\bar{x}^* \in \partial\Sigma'_\Lambda$, and $n = n(x^*)$, resp. $\bar{n} = \bar{n}(x^*)$, refers to the inward normal to $\partial\Omega$ at x, resp. to the inward normal to $\partial\Sigma'_\Lambda$ at $\bar{x}$. We now use the results of Costabel *et al.* [7], and the fact that $\partial_n^{1/2} u = c_0$ at the boundary. Following (2.9), we get the expansions

$$u(x) = c_0 r^{1/2} + O(r), \quad R_\Lambda u(x) = c_0 \bar{r}^{1/2} + O(\bar{r}), \quad r, \bar{r} \to 0.$$

Now, *we remark that the inward normal vectors n and $\bar{n}$ coincide at P*. The expansions above then lead to the upper bound

$$w(P + r\,n) = O(r), \quad r > 0 \text{ small enough.} \tag{4.9}$$

Furthermore, the coordinate r along the normal at P coincides with the coordinate ρ from Proposition 4.3. Applying this proposition with $\beta = 3/4$, we get

$$w(P + r\,n) \geq c\,r^{3/4}, \quad r > 0 \text{ small enough}$$

Comparison between this lower bound and the upper bound (4.9) gives a contradiction.

4.4. Contradictory bounds on w: configuration 2

In this paragraph, we investigate configuration 2, in which H_Λ is orthogonal to $\partial\Omega$ at some point Q.

Lower bound

We shall prove the following

Proposition 4.4. *Let* $\boldsymbol{\nu} = (1, \nu)$ *be a vector entering* Σ'_Λ *at* Q*. Let* $\beta > 1/2$*. There exists* $c_\beta > 0$ *such that*

$$w(Q + t\boldsymbol{\nu}) \ \geq \ c_\beta \, t^{1+\beta}, \qquad \textit{for } t \textit{ small enough.}$$

To prove this proposition, we consider again a small disk D_R, of radius R, tangent to $\partial\Omega$ at Q. Exactly as in the proof of Proposition 4.3, we introduce the 3d coordinates (ρ, θ, ϕ) in a $R/2$-neighborhood of $C_R = \partial D_R \times \{0\}$. We also introduce the 3d harmonic extension W of w, even with respect to the variable z. Our aim is to prove a lower bound on W which implies Proposition 4.4, namely

$$W(x, z) \geq c_\beta \, \rho^\beta \cos(\theta/2) \cos\phi, \quad \forall -\pi/2 \leq \phi \leq \pi/2, \quad \forall \theta \in [-\pi, \pi], \quad \forall \rho = \rho(x) < \rho_\beta \tag{4.10}$$

with $\rho_\beta > 0$ small enough. Therefore, let $W_\beta(x, z) = \rho^\beta \cos(\theta/2) \cos\phi$. Comparison between W and W_β will come from the maximum principle. The main change with respect to configuration 1 is that $W = 0$ on the hyperplane $\{x = (x_1, x_2, z),\, x_1 = \Lambda\}$ (corresponding to $\phi = \pm\pi/2$). We have to restrict to the open set

$$U_\delta^+ = U_\delta \cap \{x_1 > \Lambda\} = \{\rho < \delta,\ x_1 > \Lambda,\ \theta \neq \pm\pi\}.$$

We shall prove that

(a) $\Delta W_\beta \geq 0 \,(= \Delta W)$ on U_δ^+.
(b) $W \geq c_\beta W_\beta$ on ∂U_δ^+, $\delta > 0$ small.

Proof of (a). We use again formula (4.7). It gives

$$\begin{aligned} \Delta W_\beta &= \frac{1}{\rho(R - \rho\cos\theta)} \Big(\big((\beta^2 - 1/4)\, R\, \rho^{\beta-1} + O(\rho^\beta)\big) \cos(\theta/2) \cos\phi \\ &\quad - \frac{\rho^{\beta+1}}{R - \rho\cos\theta} \cos(\theta/2)\cos\phi \Big) \\ &= \frac{1}{\rho(R - \rho\cos\theta)} \big((\beta^2 - 1/4)\, R\, \rho^{\beta-1} + O(\rho^\beta)\big) \cos(\theta/2)\cos\phi \geq 0. \end{aligned}$$

Proof of (b). Let $\epsilon > 0$. Away from H_Λ, that is over $\partial U_\delta^+ \cap \{x_1 > \Lambda + \epsilon\}$, we can perform exactly the same analysis as in the proof of Proposition 4.3, point ii). Thus,

$$W \ \geq \ c\, W_\beta, \qquad \text{over } \ \partial U_\delta^+ \cap \{x_1 > \Lambda + \epsilon\} \ \text{ for some } \ c = c_\epsilon > 0.$$

It remains to treat the vicinity of H_Λ inside ∂U_δ^+. For $x_1 = \Lambda$ (that is $\phi = \pm\pi/2$), one has $W(x, z) = W_\beta(x, z) = 0$. Also, for $\theta = \pm\pi$, one has $W(x, z) \geq 0, W_\beta(x, z) = 0$. Everywhere else, $W_\beta(x, z) > 0$ and $W(x, z) > 0$. Hence, it remains to show that

$$\liminf_{\substack{(x,z)\in\partial U_\delta^+ \setminus \{|\theta|=\pi\}, \\ x_1 \to \Lambda^+}} |W(x, z)/W_\beta(x, z)| > 0.$$

Again, we argue by contradiction: we assume that there is a sequence

$$(x^k, z^k) \subset \partial U_\delta^+ \setminus \{\theta = \pi\}, \quad x_1^k \to \Lambda^+, \quad \big|W(x^k, z^k)/W_\beta(x^k, z^k)\big| \to 0.$$

Up to a subsequence, we can assume that $(x^k, z^k) \to X$ for some $X = (\mathit{\Lambda}, X_2, Z) \in \partial U_\delta^+$. As W and W_β are even in z, we can also assume that $z^k \geq 0$ for all k. There are several cases:

$(X_1, X_2) = (\mathit{\Lambda}, X_2) \in \Omega$. W is harmonic and positive on $\{(x,z), |(x,z) - X| \leq \epsilon, x_1 > \mathit{\Lambda}\}$, $\epsilon > 0$ small enough. Moreover, for all (x,z) in this set, we have $W(x,z) > 0 = W(\mathit{\Lambda}, x_2, z)$. We can apply Hopf's lemma, which yields $\partial_1 W(\mathit{\Lambda}, x_2, z) \geq C > 0$. Hence,

$$\liminf_{k\to\infty} \frac{W(x^k, z^k)}{|x_1^k - \mathit{\Lambda}|} > 0$$

which yields in turn

$$\liminf_{k\to\infty} \frac{W(x^k, z^k)}{|W_\beta(x^k, z^k)|} > 0.$$

Indeed, $|W_\beta(x^k, z^k)| = |W_\beta(x^k, z^k) - W_\beta(\mathit{\Lambda}, x_2^k, z^k)| = O(|x_1^k - \mathit{\Lambda}|)$ due to the regularity of W_β near X. Thus, we reach a contradiction.

$(X_1, X_2) = (\mathit{\Lambda}, X_2) \notin \Omega$, $Z \neq 0$. We are still in a situation where we can apply Hopf's lemma to W near X, which yields the same contradiction as above.

$(X_1, X_2) = (\mathit{\Lambda}, X_2) \notin \Omega$, $Z = 0$. Although W is harmonic and positive on the dihedra $\{x_1 > \mathit{\Lambda}, z > 0\}$, we can not apply Hopf's lemma because the dihedra does not satisfy the interior sphere condition at X. Nevertheless, thanks to Lemma 4.2, we get that for some $c' > 0$, for all k large enough

$$W(x^k, z^k) \;\geq\; c\,(x_1^k - \mathit{\Lambda}) z^k \;\geq\; c' \left(\frac{\pi}{2} - |\phi^k|\right)(\pi - \theta^k) \;\geq\; c''\, W_\beta(x_k, z^k).$$

This ends the proof of (4.10), and so the proof of Proposition 4.4.

Upper bound

We now look for an upper bound for w along the ray $Q + t\boldsymbol{\nu}$, $\boldsymbol{\nu} = (1, \nu)$, $t > 0$ small. We show

$$w(Q + t\boldsymbol{\nu}) = O(t^2), \qquad \text{as } t \text{ goes to } 0^+. \tag{4.11}$$

Of course, such an upper bound leads to a contradiction with Proposition 4.4 (for $1/2 < \beta < 1$).

As in configuration 1, this bound follows from the behaviour of u near $\partial\Omega$, as given by (2.9). We remind that for x near Q,

$$u(x) \;=\; c_0\, r^{1/2} \;+\; C_1(x_1)\, r^{3/2} \;+\; u_{\text{reg}}(x)$$

where

- $r = r(x)$ is a normal coordinate in the vicinity of $\partial\Omega$ (which corresponds to $r = 0$).
- c_0 is the constant fractional normal derivative at $\partial\Omega$, C_1 is a smooth function of x_1.
- u_{reg} is C^2 near $\partial\Omega$, and vanishes at $\partial\Omega$.

We can then write

$$w(x) = c_0 w_{1/2}(x) + C_1 w_{3/2}(x) + w_{\text{reg}}(x) + O(|x_1 - \mathit{\Lambda}| r^{3/2})$$

with $C_1 := C_1(\mathit{\Lambda})$,

$$w_{1/2}(x) \;:=\; r^{1/2}(x) - r^{1/2}(\check{x}), \quad w_{3/2}(x) \;:=\; r^{3/2}(x) - r^{3/2}(\check{x}), w_{\text{reg}}(x) \;:=\; u_{\text{reg}}(x) - u_{\text{reg}}(\check{x}),$$

where $\check{x} := (2\mathit{\Lambda} - x_1, x_2)$. Note that $w_{1/2}$, $w_{3/2}$, w_{reg} vanish at Q.

Let us first prove that $\nabla w_{\text{reg}}(Q) = 0$. The idea is the same as in the paper of Serrin [14]. Clearly, $w_{\text{reg}}(\mathit{\Lambda}, \cdot) = 0$, so that $\partial_2 w_{\text{reg}}(\mathit{\Lambda}, \cdot) = 0$. In particular, $\partial_2 w_{\text{reg}}(Q) = 0$. We then have to show that $\partial_1 w_{\text{reg}}(Q)$. Near Q, we can write $\partial\Omega$ as a graph: $x_2 = \psi(x_1)$, with $\psi'(\mathit{\Lambda}) = 0$. Moreover,

$$u_{\text{reg}}(x_1, \psi(x_1)) = 0 \;\Rightarrow\; \partial_1 u_{\text{reg}} + \psi' \partial_2 u_{\text{reg}} = 0,$$

so that $\partial_1 u_{\text{reg}}(\mathit{\Lambda}, 0) = 0$, and from there, $\partial_1 w_{\text{reg}}(Q) = 2\,\partial_1 u_{\text{reg}}(Q) = 0$. It follows that $w_{\text{reg}}(Q + t\boldsymbol{\nu}) = O(t^2)$ for $t \to 0^+$.

We still have to control w_α, $\alpha = 1/2$ or $3/2$. For x close enough to $\partial\Omega$, we denote $p = p(x)$ its orthogonal projection on $\partial\Omega$. Near Q, it reads $p = (p_1, \psi(p_1))$, with

$$\begin{pmatrix} x_1 - p_1 \\ x_2 - \psi(p_1) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \psi'(p_1) \end{pmatrix} = 0. \tag{4.12}$$

Then, $r(x)^2 = (x_1 - p_1)^2 + (x_2 - \psi(p_1))^2 = (1 + \psi'(p_1)^2)(x_2 - \psi(p_1))^2$, and

$$w_\alpha(x) = (1 + \psi'(p_1)^2)^{\alpha/2} |x_2 - \psi(p_1)|^\alpha - (1 + \psi'(\check{p}_1)^2)^{\alpha/2} |x_2 - \psi(\check{p}_1)|^\alpha$$

with $\check{p} := p(\check{x})$. One may then Taylor expand $w_\alpha(Q + t\boldsymbol{\nu})$ with respect to t ($x_1 = \mathit{\Lambda} + t$, $x_2 = \psi(\mathit{\Lambda}) + \nu t$). Indeed, relation (4.12):

$$\mathit{\Lambda} + t - p_1 + (\psi(\mathit{\Lambda}) + \nu t - \psi(p_1))\psi'(p_1) = 0,$$
$$\text{resp.} 2\mathit{\Lambda} - (\mathit{\Lambda} + t) - \check{p}_1 + (\psi(\mathit{\Lambda}) + \nu t - \psi(\check{p}_1))\psi'(\check{p}_1) = 0$$

leads to $p_1 = \mathit{\Lambda} + t + O(t^2)$, resp. $\check{p}_1 = \mathit{\Lambda} - t + O(t^2)$. Then, since $\psi'(\mathit{\Lambda}) = 0$,

$$\psi(p_1) = \psi(\mathit{\Lambda}) + \frac{\psi''(\mathit{\Lambda})}{2} t^2 + O(t^3), \quad \psi(\check{p}_1) = \psi(\mathit{\Lambda}) + \frac{\psi''(\mathit{\Lambda})}{2} t^2 + O(t^3),$$

so that

$$|x_2 - \psi(p_1)|^\alpha = |\nu t|^\alpha \left(1 - \frac{\alpha\psi''(\mathit{\Lambda})}{2\nu} t + O(t^2)\right),$$
$$\text{resp. } |x_2 - \psi(\check{p}_1)|^\alpha = |\nu t|^\alpha \left(1 - \frac{\alpha\psi''(\mathit{\Lambda})}{2\nu} t + O(t^2)\right)$$

and

$$(1 + \psi'(p_1)^2)^{\alpha/2} = 1 + \frac{\alpha}{2}\psi''(\mathit{\Lambda}) t^2 + O(t^3), \qquad \text{resp. } (1 + \psi'(\check{p}_1)^2)^{\alpha/2} = 1 + \frac{\alpha}{2}\psi''(\mathit{\Lambda}) t^2 + O(t^3).$$

Combining previous Taylor expansions yields: $w_\alpha(Q + t\boldsymbol{\nu}) = O(t^{2+\alpha})$ as $t \to 0^+$. Together with the estimate $w_{\text{reg}}(Q + t\boldsymbol{\nu}) = O(t^2)$, it implies the upper bound (4.11).

Remark 4.5. If the domain Ω is not connected, our method can still be applied, provided the connected components of Ω can be "ordered", in the following sense: for every direction of the moving hyperplanes H_λ, for all $\lambda < \mathit{\Lambda}$, H_λ intersects at most one connected component of Ω. In this case, it can be checked that our arguments remain valid. In particular, if Ω has radial symmetry, we infer that Ω is a disc.

Appendix

This appendix is devoted to the calculation of the jacobian of the change of variables $x = (x_1, x_2) \to (s, r)$, where x belongs to a tubular neighbourhood T_δ of $\partial\Omega$, s is the arc-length, and $r = d(x, \partial\Omega)(2\mathbf{1}_{\mathbf{x}\in\Omega} - 1)$. Using the same notation as in Section 3, we write

$$x = p(s) - rn(s),$$

where $p(s) \in \mathbb{R}^2$ is the point of $\partial\Omega$ with arc-length s, and $n(s)$ is the outward pointing normal. We deduce that

$$\frac{\partial x}{\partial s} = \tau(s) - r\frac{\mathrm{d}n}{\mathrm{d}s} = (1 + \kappa(s)r)\tau(s),$$
$$\frac{\partial x}{\partial r} = -n(s).$$

Let θ be the oriented angle between e_1 and n. Then, up to a change of the orientation of the curve,

$$n(s) = \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix}, \quad \tau(s) = \begin{pmatrix} -\sin\theta(s) \\ \cos\theta(s) \end{pmatrix}.$$

We infer that the jacobian matrix of the change of variables is

$$\begin{pmatrix} \dfrac{\mathrm{d}x_1}{\mathrm{d}s} & \dfrac{\mathrm{d}x_1}{\mathrm{d}r} \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}s} & \dfrac{\mathrm{d}x_2}{\mathrm{d}r} \end{pmatrix} = \begin{pmatrix} -(1+\kappa(s)r)\sin\theta(s) & -\cos\theta(s) \\ (1+\kappa(s)r)\cos\theta(s) & -\sin\theta(s) \end{pmatrix},$$

and therefore the jacobian of the change of variables is $|1+\kappa(s)r|$.

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