ESAIM: COCV 20 (2014) 174–189
DOI: 10.1051/cocv/2013060

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

UNIFORM STABILIZATION OF SOME DAMPED SECOND ORDER EVOLUTION EQUATIONS WITH VANISHING SHORT MEMORY

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Abstract. We consider a damped abstract second order evolution equation with an additional vanishing damping of Kelvin–Voigt type. Unlike the earlier work by Zuazua and Ervedoza, we do not assume the operator defining the main damping to be bounded. First, using a constructive frequency domain method coupled with a decomposition of frequencies and the introduction of a new variable, we show that if the limit system is exponentially stable, then this evolutionary system is uniformly − with respect to the calibration parameter − exponentially stable. Afterwards, we prove uniform polynomial and logarithmic decay estimates of the underlying semigroup provided such decay estimates hold for the limit system. Finally, we discuss some applications of our results; in particular, the case of boundary damping mechanisms is accounted for, which was not possible in the earlier work mentioned above.

Mathematics Subject Classification. 93D15, 35L10, 35Q74, 37L15, 74K20.

Received June 1, 2012. Revised December 28, 2012.
Published online December 23, 2013.

1. Introduction and statements of main results

Let H be a Hilbert space, and let A be an unbounded coercive operator on H with $A = A^*$. Denote $(.,.)$, the scalar product on H, and $|.|$, the corresponding norm on H. Set $V = D(A^{\frac{1}{2}})$, and for every $v \in V$, set $||v|| = |A^{\frac{1}{2}}v|$. Denote by V' the topological dual space of V, and let $B : V \to V'$ be a nonnegative operator, *viz.* $\langle Bv, v\rangle \geq 0$ for all v in V, where $\langle , \rangle$ denotes the duality product between V and V'. Throughout the paper, it is assumed that H is the pivot space, and that the embeddings $D(A) \subset V \subset H \subset V'$ are compact and dense.

For each $\varepsilon > 0$, consider the following abstract second order evolution equation

$$\begin{aligned} &y_{\varepsilon,tt} + Ay_\varepsilon + By_{\varepsilon,t} + \varepsilon Ay_{\varepsilon,t} = 0, \quad t \in \mathbb{R}, \\ &y_\varepsilon(0) = y^0, \quad y_{\varepsilon t}(0) = y^1. \end{aligned} \tag{1.1}$$

The associated limit system is

$$\begin{aligned} &y_{,tt} + Ay + By_{,t} = 0, \quad t \in \mathbb{R}, \\ &y(0) = y^0, \quad y_t(0) = y^1. \end{aligned} \tag{1.2}$$

Keywords and phrases. Second order evolution equation, Kelvin–Voigt damping, hyperbolic equations, stabilization, boundary dissipation, localized damping, plate equations, elasticity equations, frequency domain method, resolvent estimates.

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Article published by EDP Sciences

If $y^0 \in V$, and $y^1 \in H$, the energy of (1.1) is given for every $t \geq 0$ by

$$E_\varepsilon(t) = \frac{1}{2}(|y_{\varepsilon,t}(t)|^2 + |A^{\frac{1}{2}} y_\varepsilon(t)|^2), \tag{1.3}$$

and it is a nonincreasing function of the time variable as

$$E_\varepsilon(t) = E_\varepsilon(s) - \int_s^t \{\langle B y_{\varepsilon,t}(r), y_{\varepsilon t}(r)\rangle + \varepsilon |A^{\frac{1}{2}} y_{\varepsilon,t}(r)|^2\} \,\mathrm{d}r, \quad \forall 0 \leq s < t < +\infty. \tag{1.4}$$

System (1.1) is motivated by the study of the uniform stabilization of the finite differences or traditional finite element space discretization of the wave equation. Indeed, for such approximation schemes, the energy of the damped wave equation does not decay uniformly with respect to the mesh size; for that to happen, a suitably calibrated vanishing viscosity has to be introduced into the system, *e.g.* [15, 56, 57].

The main question that will be dealt with in this paper is the following: Given that the decay of the energy of the limit system (1.2) is exponential, polynomial, or logarithmic, is the exponential/polynomial/logarithmic decay of the energy E_ε, as $t \to \infty$, uniform with respect to ε? In other words, given $\varepsilon_0 > 0$, do there exist positive constants M and λ such that

$$E_\varepsilon(t) \leq M \mathrm{e}^{-\lambda t} E_\varepsilon(0), \quad \forall t \geq 0, \quad \forall 0 < \varepsilon \leq \varepsilon_0, \quad \forall (y^0, y^1) \in V \times H, \tag{1.5}$$

or

$$E_\varepsilon(t) \leq \frac{M||(y^0, y^1)||^2_{D(\mathcal{A}_\varepsilon)}}{(1+t)^\lambda}, \quad \forall t \geq 0, \quad \forall 0 < \varepsilon \leq \varepsilon_0, \quad \forall (y^0, y^1) \in D(\mathcal{A}_\varepsilon), \tag{1.6}$$

or

$$E_\varepsilon(t) \leq \frac{M||(y^0, y^1)||^2_{D(\mathcal{A}_\varepsilon)}}{[\log(2+t)]^2}, \quad \forall t \geq 0, \quad \forall 0 < \varepsilon \leq \varepsilon_0, \quad \forall (y^0, y^1) \in D(\mathcal{A}_\varepsilon), \tag{1.7}$$

where $D(\mathcal{A}_\varepsilon)$ is defined below.

This work was motivated by one of the questions tackled in [14]; indeed the authors of [14] consider, among other things, (1.1) with $B = C^*C$, where C is a bounded operator, *viz.* $C \in \mathcal{L}(H)$. Assuming that the limit system (1.2) is exponentially stable, using an appropriate decomposition of the solution along high and low frequencies, and the fact that C is bounded, they prove (1.5). One may argue that the additional viscoelastic damping makes the stabilization problem much easier, which is true for ε fixed, but then the decay rate is not uniform with respect to ε, and overdamping may occur as shown in [14]. What makes the study of this stabilization problem interesting is the requirement that the decay rate be uniform with respect to ε as ε goes to zero. It is to be noted that in [14], the uniform energy decay estimate (1.5) critically relies on the following two facts:

i) the limit system is exponentially stable;

ii) the damping operator C is bounded.

Consequently if one of those two facts fails, then the method developed in [14], and which is based on Proposition 1 in [19] that establishes an equivalence between observability and stabilization for second order evolution equations with bounded damping operators, becomes inoperative; in particular, the case where the damping operator B is unbounded is left as an open problem therein. It is the intent of the author of the present paper to propose a solution to that open problem; the method that will be developed below to address that problem and which is based on the resolvent estimates will enable us to deal not only with the case where the limit system is exponentially stable, as in [14], but also to deal with situations where the limit system is polynomially or logarithmically stable only; this may happen even for some bounded operators B, *e.g.* [16, 31, 37, 39, 42, 49, 54]. At this stage, it is worth mentioning that the present work as well as [14] are closely related to the earlier works [56, 57] where the uniform stabilization of the finite differences space semi-discretization of the wave equation is discussed; in those two papers, the addition of a well-calibrated viscoelastic damping is the key element for the uniform exponential decay of the energy. Indeed, it is shown in [56, 57] that without that additional damping, the discrete system fails to be uniformly exponentially stable. But as will be seen below, and as it was already observed in [14], the presence of the viscoelastic damping in (1.1) makes the study of the stabilization problem at hand more intricate; in fact, the authors of [14] had to rely on a judicious decomposition of the solution along high and low frequencies in order to prove that the perturbed system is uniformly exponentially stable. In the present work, where the decay estimates will be established through estimates of the resolvent along the imaginary axis, we will decompose the axis into two portions; for the unbounded portion, the extra viscoelastic damping will be enough to get the necessary estimates, while for the bounded portion, we will rely on the introduction of a new variable and the fact that the limit resolvent satisfies the corresponding estimate.

Before stating our main results, we will recast (1.1) as a first-order system. To this end, introduce the Hilbert space on the field $\mathbb{C}$ of complex numbers $\mathcal{H} = V \times H$, equipped with the norm

$$||(u,v)||_1^2 \doteq ||(u,v)||_{\mathcal{H}}^2 = |A^{\frac{1}{2}}u|^2 + |v|^2. \tag{1.8}$$

Let $\mathcal{A}_\varepsilon$ be the unbounded operator given by

$$\mathcal{A}_\varepsilon = \begin{pmatrix} 0 & I \\ -A & -B - \varepsilon A \end{pmatrix} \tag{1.9}$$

with:

$$D(\mathcal{A}_\varepsilon) = \Big\{(u,v) \in V \times V; A(u+\varepsilon v) + Bv \in H\Big\}.$$

System (1.1) may now be recast as

$$\begin{aligned} &Z_{\varepsilon t} = \mathcal{A}_\varepsilon Z_\varepsilon \\ &Z_\varepsilon(0) = \begin{pmatrix} y^0 \\ y^1 \end{pmatrix}. \end{aligned} \tag{1.10}$$

We denote by $\mathcal{A}_0$ the unbounded limit operator with domain

$$D(\mathcal{A}_0) = \Big\{(u,v) \in V \times V; Au + Bv \in H\Big\}.$$

It will be assumed in the sequel that

$$\exists \lambda_0 > 0 : |u| \leq \lambda_0 |A^{\frac{1}{2}}u|, \qquad \forall u \in V, \tag{1.11}$$

and

$$\exists \mu_0 > 0 : \langle Bu, u \rangle \leq \mu_0^2 ||u||^2, \qquad \forall u \in V. \tag{1.12}$$

We can now state our main results:

Theorem 1.1. *Let the operators A and B be given as above. Assume that the limit operator $\mathcal{A}_0$ generates a C_0 semigroup of contractions $(S_0(t))_{t\geq 0}$ on the Hilbert space $\mathcal{H}$ which is exponentially stable. Let $\varepsilon_0 > 0$ be an arbitrary constant. There exist positive constants M and λ, that eventually depend on λ_0, μ_0, and ε_0 only, such that the energy decay estimate (1.5) holds for every solution of (1.1).*

Theorem 1.2. *Let the operators A and B be given as above. Assume that the limit operator $\mathcal{A}_0$ satisfies $i\mathbb{R} \subset \rho(\mathcal{A}_0)$, where $\rho(\mathcal{A})$ denotes the resolvent of $\mathcal{A}$. Suppose that $\mathcal{A}_0$ generates a C_0 semigroup of contractions $(S_0(t))_{t\geq 0}$ on the Hilbert space $\mathcal{H}$, which is polynomially stable, viz., there are positive constants M_0 and α_0 such*

$$||S_0(t)Z^0||_{\mathcal{H}} \leq \frac{M_0 ||Z^0||_{D(\mathcal{A}_0)}}{(1+t)^{\alpha_0}}, \quad \forall t \geq 0, \quad Z^0 \in D(\mathcal{A}_0). \tag{1.13}$$

Set $\varepsilon_0 = 1$. There exist positive constants M and λ, that eventually depend on λ_0 and μ_0 only, such that the energy decay estimate (1.6) holds for every solution of (1.1).

Theorem 1.3. *Let the operators A and B be given as above. Assume that $\mathcal{A}_0$ generates a C_0 semigroup of contractions $(S_0(t))_{t\geq 0}$ on the Hilbert space $\mathcal{H}$. Suppose that there exists a positive constant C_0 such that for every $\lambda \in \mathbb{C}$ with $\Re\lambda \in [-\frac{e^{-C_0|\Im\lambda|}}{C_0}, 0]$ and $|\Im\lambda| \geq 1$, one has the resolvent estimate*

$$||(\lambda\mathcal{I} - \mathcal{A}_0)^{-1}||_{\mathcal{L}(\mathcal{H})} \leq C_0 e^{C_0|\Im\lambda|}. \tag{1.14}$$

Set $\varepsilon_0 = 1$. There exists a positive constant M, that eventually depends on λ_0 and μ_0 only, such that the energy decay estimate (1.7) holds for every solution of (1.1).

Remark 1.4. It is known that when the resolvent estimate (1.14) holds, then the semigroup $(S_0(t))_{t\geq 0}$ is logarithmically stable [16, 17, 31]; there exists a positive constant M_0 such that for every positive integer k,

$$||S_0(t)Z^0||_{\mathcal{H}} \leq \frac{M_0||Z^0||_{D(\mathcal{A}_0^k)}}{[\log(2+t)]^k}, \quad \forall t \geq 0, \quad Z^0 \in D(\mathcal{A}_0^k). \tag{1.15}$$

The rest of this paper is organized as follows: in Section 2, we recall some important preliminary results relating the energy decay estimates to resolvent estimates. Section 3 deals with the proofs of Theorems 1.1−1.3, while in Section 4 we discuss several applications of our results and some final remarks.

2. Some technical lemmas

Lemma 2.1 [20, 44]**.** *Let $\mathcal{A}$ be the generator of a bounded C_0 semigroup $(S(t))_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. Then $(S(t))_{t\geq 0}$ is exponentially stable if and only if:*
i) *$i\mathbb{R} \subset \rho(\mathcal{A})$, and*
ii) *$\sup\{||(ib - \mathcal{A})^{-1}||;\ b \in \mathbb{R}\} < \infty$, where $\rho(\mathcal{A})$ denotes the resolvent of $\mathcal{A}$.*

Lemma 2.2 [7]**.** *Let $\mathcal{A}$ be the generator of a bounded C_0 semigroup $(S(t))_{t\geq 0}$ on a Hilbert space $\mathcal{H}$ such that $i\mathbb{R} \subset \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ denotes the resolvent of $\mathcal{A}$. Then $(S(t))_{t\geq 0}$ is polynomially stable,* viz.*, there are positive constants M and α that are independent of the initial data such*

$$||S(t)Z^0||_{\mathcal{H}} \leq \frac{M||Z^0||_{D(\mathcal{A})}}{(1+t)^{\frac{1}{\alpha}}}, \quad \forall t \geq 0, \quad Z^0 \in D(\mathcal{A}). \tag{2.1}$$

if and only if

$$\exists C_0 > 0 : ||(ib - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} \leq C_0|b|^{\alpha}, \forall b \in \mathbb{R} \text{ with } |b| \geq 1.$$

Weaker versions of Lemma 2.2 may be found in [5, 6, 35].

3. Proofs of Theorems 1.1, 1.2, and 1.3

As indicated in the introduction, the energy decay estimates will be derived from resolvent estimates. For that derivation, we will rely on Lemma 2.1 for the case of Theorem 1.1 and Lemma 2.2 for the case of Theorem 1.2. First, we shall prove that $\mathcal{A}_\varepsilon$ generates a C_0 semigroup of contractions $(S_\varepsilon(t))_{t\geq 0}$, then we shall show that $i\mathbb{R} \subset \rho(\mathcal{A}_\varepsilon)$.
We have:

- the operator $\mathcal{A}_\varepsilon$ is dissipative as:

$$\Re\left(\mathcal{A}_\varepsilon Z, Z\right) = -\varepsilon|A^{\frac{1}{2}}v|^2 - \langle Bv, v\rangle \leq 0, \quad \forall Z = (u, v) \in \mathcal{D}(\mathcal{A}_\varepsilon).$$

- $\mathcal{I} - \mathcal{A}_\varepsilon$ is onto, by Lax–Milgram Lemma, ($\mathcal{I}$ denotes the identity operator).

Consequently, the operator $\mathcal{A}_\varepsilon$ generates a C_0 semigroup of contractions on $\mathcal{H}$ by Lumer−Phillips theorem [41]; note that $\overline{D(\mathcal{A}_\varepsilon)} = \mathcal{H}$, by [41], Theorem 4.6, page 16.

We now observe that the operator $\mathcal{A}_\varepsilon$ does not have a compact resolvent even though $\mathcal{A}_0$ might have one as we will see in the examples that are discussed later on. This is due to the fact that the extra viscoelastic damping has the same order as the principal operator A, thereby precluding the embedding of $D(\mathcal{A}_\varepsilon)$ into $\mathcal{H}$ to be compact. Next, we note that $0 \in \rho(\mathcal{A}_\varepsilon)$. Let $b \in \mathbb{R}$ with $b \neq 0$, the assertion about the resolvent will be established once we prove: i) $\mathrm{Ker}(ib - \mathcal{A}_\varepsilon) = \{\mathbf{0}\}$ and ii) $R(ib - \mathcal{A}_\varepsilon) = \mathcal{H}$, where $Ker(B)$ stands for the kernel of the operator B and $R(B)$ stands for the range of B.

Proof of i). Let b be a nonzero real number and let $Z = (u, v) \in D(\mathcal{A}_\varepsilon)$ with $\mathcal{A}_\varepsilon Z = ibZ$, we shall prove that $Z = \mathbf{0}$. The equation $\mathcal{A}_\varepsilon Z = ibZ$ easily yields $\Re(\mathcal{A}_\varepsilon Z, Z) = -\varepsilon|A^{\frac{1}{2}}v|^2 - \langle Bv, v\rangle = 0$; from which one derives $v = 0$, thanks to (1.11), and then $u = 0$. Hence $Z = \mathbf{0}$.

Proof of ii). For this proof, we borrow some ideas from [36]. Let b be a nonzero real number, and let $U = (f, g) \in \mathcal{H}$. We shall show that there exists $Z = (u, v) \in D(\mathcal{A}_\varepsilon)$ such that $ibZ - \mathcal{A}_\varepsilon Z = U$, which may be recast as:

$$\begin{aligned} &ibu - v = f \\ &ibv + A(u + \varepsilon v) + Bv = g. \end{aligned} \tag{3.1}$$

We may use the first equation in (3.1) to eliminate v in the second one, thereby getting

$$-b^2u + (1 + ib\varepsilon)Au + ibBu = g + ibf + \varepsilon Af + Bf \in V'. \tag{3.2}$$

If we set $A_{b\varepsilon} = (1 + ib\varepsilon)A + ibB : V \longrightarrow V'$, then Lax–Milgram theorem shows that $A_{b\varepsilon}$ is an isomorphism. Further, one checks that $A_{b\varepsilon}^{-1}$ is compact as $A_{b\varepsilon}^{-1}(V') = V$, and the embedding $V \subset H$ is compact. We may rewrite (3.2) as

$$u - b^2A_{b\varepsilon}^{-1}u = A_{b\varepsilon}^{-1}(g + ibf + \varepsilon Af + Bf). \tag{3.3}$$

Thanks to the Fredholm alternative *e.g.* [8], Theorem VI.6, page 92, solving (3.3) in H amounts to showing that the equation $u - b^2A_{b\varepsilon}^{-1}u = 0$ has the unique solution $u = 0$, or equivalently that $u = 0$ is the unique solution of the equation $-b^2u + (1 + ib\varepsilon)Au + ibBu = 0$. Taking the duality product $V' - V$ of u and both sides of the latter equation, we get: $-b^2|u|^2 + (1 + ib\varepsilon)|A^{\frac{1}{2}}u|^2 + ib\langle Bu, u\rangle = 0$, so that taking the imaginary parts, and keeping in mind that $b \neq 0$, one finds: $\varepsilon|A^{\frac{1}{2}}u|^2 + \langle Bu, u\rangle = 0$; from which one derives $u = 0$, thanks to (1.11). Hence ii) holds. Therefore, combining i), ii) and the closed graph theorem, one derives $i\mathbb{R} \subset \rho(\mathcal{A}_\varepsilon)$. One may now invoke the stability theorem in [3] to conclude that the semigroup $(S_\varepsilon(t))_{t\geq 0}$ is strongly stable. The Proofs of Theorems 1.1−1.3 that follow now will quantify that strong stability according to the stability property satisfied by the limit system.

3.1. Proof of Theorem 1.1

According to Lemma 2.1, it remains to show that one has:

$$\sup\{||(ib - \mathcal{A}_\varepsilon)^{-1}||_{\mathcal{L}(\mathcal{H})};\ b \in \mathbb{R}\} < \infty, \quad \forall 0 < \varepsilon \leq \varepsilon_0. \tag{3.4}$$

To this end, let $U = (f, g) \in \mathcal{H}$. We shall prove that there exists a constant $C > 0$ such that for every $b \in \mathbb{R}$, and every ε with $0 < \varepsilon \leq \varepsilon_0$, if $Z = \begin{pmatrix} u \\ v \end{pmatrix} \in D(\mathcal{A}_\varepsilon)$ satisfies

$$(ib - \mathcal{A}_\varepsilon)\,Z = U, \tag{3.5}$$

then

$$||Z||_{\mathcal{H}} \leq C||U||_{\mathcal{H}}. \tag{3.6}$$

Normally Z should depend on ε and b, but for simplicity sake, that dependence is omitted. Here and in the sequel, C is a generic constant that may eventually depend on λ_0, μ_0, and ε_0 only.

Denoting by $(.,.)_1$, the inner product in $\mathcal{H}$, and by $||.||_1$, the corresponding norm, as introduced in 3.59, we derive from (3.5):

$$((ib - \mathcal{A}_\varepsilon)\, Z, Z)_1 = (U, Z)_1,$$

so that taking the real parts, we get

$$\varepsilon |A^{\frac{1}{2}} v|^2 + \langle Bv, v\rangle \leq ||U||_1 ||Z||_1. \tag{3.7}$$

Now (3.5) is equivalent to:

$$\begin{aligned} &ibu - v = f \\ &ibv + A(u + \varepsilon v) + Bv = g. \end{aligned} \tag{3.8}$$

It follows from the first equation in (3.8), and (3.7):

$$\varepsilon b^2 |A^{\frac{1}{2}} u|^2 \leq 2\{\varepsilon |A^{\frac{1}{2}} v|^2 + \varepsilon |A^{\frac{1}{2}} f|^2\} \leq 2(||U||_1 ||Z||_1 + \varepsilon_0 ||U||_1^2). \tag{3.9}$$

At this stage, we note that if $\underline{\varepsilon b^2 > 1}$, then one derives from (3.9):

$$|A^{\frac{1}{2}} u|^2 \leq 2(||U||_1 ||Z||_1 + \varepsilon_0 ||U||_1^2). \tag{3.10}$$

Taking the inner product in H of v with both sides of the first equation in (3.8), it follows

$$|v|^2 \leq |(ibu, v)| + |(f, v)| \leq |(ibu, v)| + C||U||_1 ||Z||_1. \tag{3.11}$$

Taking the duality product $V' - V$ of u with both sides of the second equation in (3.8), we derive, thanks to (3.10), (3.7), Cauchy−Schwarz inequality, and (1.12):

$$\begin{aligned} |(ibv, u)| &\leq |A^{\frac{1}{2}} u|^2 + \varepsilon |A^{\frac{1}{2}} u||A^{\frac{1}{2}} v| + |\langle Bv, u\rangle| + |(g, u)| \\ &\leq C\left\{||U||_1 ||Z||_1 + ||U||_1^2 + ||U||_1^{\frac{3}{2}} ||Z||_1^{\frac{1}{2}} + \sqrt{\langle Bv, v\rangle}\sqrt{\langle Bu, u\rangle}\right\} \\ &\leq C\left\{||U||_1 ||Z||_1 + ||U||_1^2 + ||U||_1^{\frac{3}{2}} ||Z||_1^{\frac{1}{2}} + ||U||_1^{\frac{1}{2}} ||Z||_1^{\frac{3}{2}}\right\}. \end{aligned} \tag{3.12}$$

Reporting (3.12) in (3.11), we find

$$|v|^2 \leq C\left\{||U||_1 ||Z||_1 + ||U||_1^2 + ||U||_1^{\frac{3}{2}} ||Z||_1^{\frac{1}{2}} + ||U||_1^{\frac{1}{2}} ||Z||_1^{\frac{3}{2}}\right\}. \tag{3.13}$$

Combining (3.10) and (3.13), we derive

$$||Z||_1^2 \leq C\left\{||U||_1 ||Z||_1 + ||U||_1^2 + ||U||_1^{\frac{3}{2}} ||Z||_1^{\frac{1}{2}} + ||U||_1^{\frac{1}{2}} ||Z||_1^{\frac{3}{2}}\right\}, \tag{3.14}$$

so that using Young inequality, (3.6) easily follows from (3.14), provided that $\varepsilon b^2 > 1$. We now turn to the case where $\underline{\varepsilon b^2 \leq 1}$. This case is a little bit trickier; first, we will have to make appropriate change of variables, then use the resolvent assumption on the limit system to derive (3.6). To this end, set $w = u + \varepsilon v$, $\hat{Z} = \begin{pmatrix} w \\ v \end{pmatrix}$, and $\hat{U} = \begin{pmatrix} f + ib\varepsilon v \\ g \end{pmatrix}$. We note that $\hat{Z}$ lies in the domain of the limit operator $D(\mathcal{A}_0)$, and $\hat{U} \in \mathcal{H}$; the energy space is the same for the perturbed and limit systems. With those notations, (3.8) becomes

$$\begin{aligned} &ibw - v = f + ib\varepsilon v \\ &ibv + Aw + Bv = g, \end{aligned} \tag{3.15}$$

or equivalently

$$(ib - \mathcal{A}_0)\hat{Z} = \hat{U}. \tag{3.16}$$

Thanks to the exponential stability assumption on the limit system, and Lemma 2.1, we know that there exists a positive constant C such that for every real number b, one has $||(ib - \mathcal{A}_0)^{-1}||_{\mathcal{L}(\mathcal{H})} \leq C$. Consequently, it follows from (3.16):

$$||\hat{Z}||_1 \leq C||\hat{U}||_1. \tag{3.17}$$

Now, one checks that

$$\begin{aligned} ||\hat{U}||_1^2 &= |A^{\frac{1}{2}}(f + i\varepsilon bv)|^2 + |g|^2 \\ &\leq 2|A^{\frac{1}{2}}f|^2 + \varepsilon^2 b^2 |A^{\frac{1}{2}}v|^2 + |g|^2 \\ &\leq 2||U||_1^2 + 2\varepsilon|A^{\frac{1}{2}}v|^2, \text{ since } \varepsilon b^2 \leq 1 \\ &\leq C(||U||_1^2 + ||U||_1||Z||_1), \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} ||Z||_1^2 &= |A^{\frac{1}{2}}u|^2 + |v|^2 \\ &\leq 2|A^{\frac{1}{2}}w|^2 + 2\varepsilon^2|A^{\frac{1}{2}}v|^2 + |v|^2 \\ &\leq 2||\hat{Z}||_1^2 + 2\varepsilon_0||U||_1||Z||_1. \end{aligned} \tag{3.19}$$

The combination of (3.17), (3.18) and (3.19) yields

$$||Z||_1^2 \leq C(||U||_1^2 + ||U||_1||Z||_1), \tag{3.20}$$

from which, one derives (3.6) with the help of Cauchy−Schwarz inequality. This completes the proof of (3.6), and that of Theorem 1.1.

3.2. Proof of Theorem 1.2

First, we note that the assumptions on the limit operator $\mathcal{A}_0$, and Lemma 2.2 show that there are two positive constants C_0 and $\alpha = 1/\alpha_0$ such that:

$$||(ib - \mathcal{A}_0)^{-1}||_{\mathcal{L}(\mathcal{H})} \leq C_0|b|^\alpha, \forall b \in \mathbb{R} \text{ with } |b| \geq 1. \tag{3.21}$$

To prove Theorem 1.1, we distinguished two cases: the case $\varepsilon b^2 > 1$, and the case $\varepsilon b^2 \leq 1$. One might be tempted to use exactly the same two cases in the proof of Theorem 1.2, but then the decay rate would be much weaker than that of the limit system. If one wants to get the same decay rate as in the limit system, then the threshold must involve the exponent α found in (3.21). Let $b \in \mathbb{R}$ with $|b| \geq 1$. Let $U \in \mathcal{H}$ and $Z \in D(\mathcal{A}_\varepsilon)$ satisfy (3.5). We shall prove that there exists a positive constant C such that

$$||Z||_1 \leq C|b|^\alpha||U||_1, \forall b \in \mathbb{R} \text{ with } |b| \geq 1, \quad \forall 0 < \varepsilon \leq 1. \tag{3.22}$$

Case $\boldsymbol{\varepsilon|b|^{2+\alpha} > 1}$. It follows from (3.9)

$$\varepsilon|b|^{2+\alpha}|A^{\frac{1}{2}}u|^2 \leq 2|b|^\alpha\{\varepsilon|A^{\frac{1}{2}}v|^2 + \varepsilon|A^{\frac{1}{2}}f|^2\} \leq 2|b|^\alpha(||U||_1||Z||_1 + ||U||_1^2), \tag{3.23}$$

from which one derives

$$|A^{\frac{1}{2}}u|^2 \leq C|b|^\alpha(||U||_1||Z||_1 + ||U||_1^2). \tag{3.24}$$

Proceeding as in the proof of Theorem 1.1, one gets

$$|v|^2 \leq C|b|^\alpha(||U||_1||Z||_1 + ||U||_1^2) + C||U||_1^{\frac{1}{2}}||Z||_1^{\frac{3}{2}}. \tag{3.25}$$

The combination of (3.24) and (3.25) yields

$$||Z||_1^2 \leq C|b|^\alpha(||U||_1||Z||_1 + ||U||_1^2) + C||U||_1^{\frac{1}{2}}||Z||_1^{\frac{3}{2}}. \tag{3.26}$$

Case $\varepsilon|b|^{2+\alpha} \leq 1$. Let $\hat{U}$ and $\hat{Z}$ be given as in the proof of Theorem 1.1. Then $\hat{U}$ and $\hat{Z}$ satisfy (3.16), so that using (3.21), we find

$$||\hat{Z}||_1 \leq C_0|b|^\alpha||\hat{U}||_1. \tag{3.27}$$

Thanks to (3.18) and (3.7), one has

$$\begin{aligned} ||\hat{U}||_1^2 &\leq 2|A^{\frac{1}{2}}f|^2 + \varepsilon^2 b^2|A^{\frac{1}{2}}v|^2 + |g|^2 \\ &\leq 2||U||_1^2 + 2\varepsilon|b|^2||U||_1||Z||_1 \\ &\leq 2(||U||_1^2 + |b|^{-\alpha}||U||_1||Z||_1), \text{ since } \varepsilon|b|^{2+\alpha} \leq 1. \end{aligned} \tag{3.28}$$

Proceeding as in the proof of Theorem 1.1, and using (3.27) and (3.28), we get the estimate (keeping in mind that $|b| \geq 1$)

$$\begin{aligned} ||Z||_1^2 &\leq 2||\hat{Z}||_1^2 + 2||U||_1||Z||_1 \\ &\leq 2C_0^2 b^{2\alpha}||\hat{U}||_1^2 + 2||U||_1||Z||_1 \\ &\leq 4C_0^2 b^{2\alpha}||U||_1^2 + (4C_0 + 2)|b|^\alpha||U||_1||Z||_1. \end{aligned} \tag{3.29}$$

Combining (3.26) and (3.29), then applying the Cauchy−Schwarz inequality, one derives (3.22). Using Lemma 2.2, we obtain the claimed energy estimate, which completes the proof of Theorem 1.2.

3.3. Proof of Theorem 1.3

Thanks to our resolvent hypothesis, we already know that there exists a positive constant C_0 such that for every $\lambda \in \mathbb{C}$ with $\Re\lambda \in [-\frac{e^{-C_0|\Im\lambda|}}{C_0}, 0]$ and $|\Im\lambda| \geq 1$, one has the resolvent estimate

$$||(\lambda\mathcal{I} - \mathcal{A}_0)^{-1}||_{\mathcal{L}(\mathcal{H})} \leq C_0 e^{C_0|\Im\lambda|}. \tag{3.30}$$

We shall find a positive constant K_0 such that for every $\lambda \in \mathbb{C}$ with $\Re\lambda \in [-\frac{e^{-K_0|\Im\lambda|}}{K_0}, 0]$ and $|\Im\lambda| \geq 1$, one has the resolvent estimate

$$||(\lambda\mathcal{I} - \mathcal{A}_\varepsilon)^{-1}||_{\mathcal{L}(\mathcal{H})} \leq K_0 e^{K_0|\Im\lambda|}, \quad \forall 0 < \varepsilon \leq 1. \tag{3.31}$$

Once (3.31) is established, the claimed decay estimate follows as in the proof of Theorem 3 in [9]. So it remains to prove (3.31). To this end, let $U = (f, g) \in \mathcal{H}$, and let $L_0 > C_0$, with C_0 as in (3.30). Let $\lambda \in \mathbb{C}$ with

$$\Re\lambda \in [-\frac{e^{-L_0|\Im\lambda|}}{L_0}, 0] \text{ and } |\Im\lambda| \geq 1. \tag{3.32}$$

As in the proof of Theorem 1.1 above, introduce $Z = (u, v) \in D(\mathcal{A}_\varepsilon)$ such that

$$\lambda Z - \mathcal{A}_\varepsilon Z = U, \tag{3.33}$$

which is equivalent to

$$\begin{aligned} &\lambda u - v = f \\ &\lambda v + A(u + \varepsilon v) + Bv = g. \end{aligned} \tag{3.34}$$

The inner product of Z with both sides of (3.33) yields

$$\lambda||Z||_1^2 - (\mathcal{A}_\varepsilon Z, Z)_1 = (U, Z)_1,$$

so that taking the real parts, we derive

$$\varepsilon|A^{\frac{1}{2}}v|^2 + \langle Bv, v\rangle \le ||U||_1||Z||_1 + |\Re\lambda|||Z||_1^2. \tag{3.35}$$

Applying the operator $A^{\frac{1}{2}}$ to both sides of the first equation in (3.34), and using (3.35), we obtain

$$\varepsilon|\lambda|^2|A^{\frac{1}{2}}u|^2 \le 2\{\varepsilon|A^{\frac{1}{2}}v|^2 + \varepsilon|A^{\frac{1}{2}}f|^2\} \le 2(||U||_1||Z||_1 + |\Re\lambda|||Z||_1^2 + ||U||_1^2). \tag{3.36}$$

As in the proof of Theorem 1.1, we note that if $\underline{\varepsilon|\lambda|^2 > 1}$, then one derives from (3.36):

$$|A^{\frac{1}{2}}u|^2 \le 2(||U||_1||Z||_1 + |\Re\lambda|||Z||_1^2 + ||U||_1^2). \tag{3.37}$$

Taking the inner product in H of v with both sides of the first equation in (3.34), it follows

$$|v|^2 \le |(\lambda u, v)| + |(f, v)| \le |(\lambda u, v)| + C||U||_1||Z||_1. \tag{3.38}$$

Taking the duality product $V' - V$ of u with both sides of the second equation in (3.34), we derive, thanks to (3.37), (3.35), Cauchy−Schwarz inequality, and (1.12):

$$\begin{aligned}|(\lambda v, u)| &\le |A^{\frac{1}{2}}u|^2 + \varepsilon|A^{\frac{1}{2}}u||A^{\frac{1}{2}}v| + |\langle Bv, u\rangle| + |(g, u)|\\ &\le 2|A^{\frac{1}{2}}u|^2 + \varepsilon|A^{\frac{1}{2}}v|^2 + \sqrt{\langle Bv, v\rangle}\sqrt{\langle Bu, u\rangle} + C||U||_1||Z||_1\\ &\le C\left\{||U||_1||Z||_1 + ||U||_1^2\right\} + ||U||_1^{\frac{1}{2}}||Z||_1^{\frac{3}{2}} + (3|\Re\lambda| + |\Re\lambda|^{\frac{1}{2}})||Z||_1^2.\end{aligned} \tag{3.39}$$

Reporting (3.39) in (3.38), we find

$$|v|^2 \le C\left\{||U||_1||Z||_1 + ||U||_1^2\right\} + ||U||_1^{\frac{1}{2}}||Z||_1^{\frac{3}{2}} + (3|\Re\lambda| + |\Re\lambda|^{\frac{1}{2}})||Z||_1^2. \tag{3.40}$$

Combining (3.37) and (3.40), we derive

$$||Z||_1^2 \le C\left\{||U||_1||Z||_1 + ||U||_1^2\right\} + ||U||_1^{\frac{1}{2}}||Z||_1^{\frac{3}{2}} + (5|\Re\lambda| + |\Re\lambda|^{\frac{1}{2}})||Z||_1^2. \tag{3.41}$$

Choosing L_0 large enough in (3.32), it follows that $5|\Re\lambda| + |\Re\lambda|^{\frac{1}{2}} \le 1/4$; combining that with Young inequality, one finds

$$||Z||_1^2 \le C||U||_1^2. \tag{3.42}$$

We got (3.42) by assuming $\varepsilon|\lambda|^2 > 1$. We now investigate the case $\varepsilon|\lambda|^2 \le 1$. Let $w = u + \varepsilon v$, and $\check{f} = f + \varepsilon\lambda v$, and set $\check{Z} = (w, v)$ and $\check{U} = (\check{f}, g)$. One easily checks that $\check{Z} \in D(\mathcal{A}_0)$, and $\check{U} \in \mathcal{H}$ satisfy the equation

$$(\lambda - \mathcal{A}_0)\check{Z} = \check{U},$$

so that using (3.30), one gets

$$||\check{Z}||_1 \le C_0 \mathrm{e}^{C_0|\Im\lambda|}||\check{U}||_1. \tag{3.59}$$

Now on the one hand, one has the estimate

$$\begin{aligned}||\check{U}||_1^2 &\le 2|A^{\frac{1}{2}}f|^2 + \varepsilon^2|\lambda|^2|A^{\frac{1}{2}}v|^2 + |g|^2\\ &\le 2||U||_1^2 + 2\varepsilon|\lambda|^2||U||_1||Z||_1 + 2\varepsilon|\lambda|^2|\Re\lambda|||Z||_1^2\\ &\le 2(||U||_1^2 + ||U||_1||Z||_1 + |\Re\lambda|||Z||_1^2), \text{ since } \varepsilon|\lambda|^2 \le 1.\end{aligned} \tag{3.60}$$

On the other hand, one has, thanks to 3.59, the estimate

$$\begin{aligned}||Z||_1^2 &\le 2||\check{Z}||_1^2 + 2\varepsilon^2|A^{\frac{1}{2}}v|^2\\ &\le 2C_0^2\mathrm{e}^{2C_0|\Im\lambda|}||\check{U}||_1^2 + 2||U||_1||Z||_1 + 2|\Re\lambda|||Z||_1^2.\end{aligned} \tag{3.61}$$

Accounting for (3.60) in (3.61), we find

$$||Z||_1^2 \leq 4C_0^2 \mathrm{e}^{2C_0|\Im\lambda|}(||U||_1^2 + ||U||_1||Z||_1 + |\Re\lambda|||Z||_1^2) \\ + 2||U||_1||Z||_1 + 2|\Re\lambda|||Z||_1^2. \tag{3.62}$$

Thanks to Young inequality, one derives the estimates

$$4C_0^2 \mathrm{e}^{2C_0|\Im\lambda|}||U||_1||Z||_1 \leq \frac{1}{4}||Z||_1^2 + 16C_0^4 \mathrm{e}^{4C_0|\Im\lambda|}||U||_1^2. \tag{3.63}$$

and

$$2||U||_1||Z||_1 \leq \mathrm{e}^{2L_0|\Im\lambda|}||U||_1^2 + \mathrm{e}^{-2L_0|\Im\lambda|}||Z||_1^2. \tag{3.64}$$

We also note that for large enough L_0, one has the estimates

$$4\mathrm{e}^{2C_0|\Im\lambda|}|\Re\lambda| \leq \frac{1}{16}, \quad \mathrm{e}^{-2L_0|\Im\lambda|} \leq \frac{1}{16}, \text{ and } 2|\Re\lambda| \leq \frac{1}{16}. \tag{3.65}$$

Reporting (3.63)–(3.65) in (3.62), we obtain, with some algebra

$$||Z||_1^2 \leq \frac{16}{9}\left(20C_0^4 \mathrm{e}^{4C_0|\Im\lambda|} + \mathrm{e}^{2L_0|\Im\lambda|}\right)||U||_1^2. \tag{3.66}$$

At this stage, we note that C_0 is a large constant (*cf. e.g.* [9, 31]); this explains replacing $C_0^2\mathrm{e}^{2C_0|\Im\lambda|}$ with $C_0^4\mathrm{e}^{4C_0|\Im\lambda|}$ to get $20C_0^4\mathrm{e}^{4C_0|\Im\lambda|}$. Therefore, choosing $L_0 \geq 8\sqrt{10}C_0^2/3$, it holds

$$||Z||_1^2 \leq L_0^2 \mathrm{e}^{2L_0|\Im\lambda|}||U||_1^2. \tag{3.67}$$

Hence, it suffices to choose $K_0 = L_0$.

4. Applications

In this section we shall discuss some examples of application of our theorems. Throughout this section, Ω denotes a bounded domain in $\mathbb{R}^N$ with smooth enough boundary, subscripts following a comma stand for differentiation, and we use the Einstein summation convention on repeated indices. Further, ∂_i stands for $\partial/\partial x_i$, $|u|_r$ denotes $||u||_{L^r(\Omega)}$ for $1 \leq r \leq +\infty$. We assume that the boundary Γ of Ω satisfies: $\Gamma = \Gamma_c \cup \Gamma_u$, with $\bar{\Gamma}_c \cap \bar{\Gamma}_u = \emptyset$, where Γ_c stands for the controlled portion of Γ and $meas(\Gamma_c) > 0$, while Γ_u corresponds to the uncontrolled portion, and $meas(\Gamma_u) > 0$, for simplification purposes.

4.1. The wave equation with boundary damping

Consider the damped wave equation:

$$\begin{cases} u_{,tt} - \partial_i(b_{ij}(x)\partial_j u) = 0 \text{ in } \Omega \times (0,\infty) \\ b_{ij}(x)\partial_j u\nu_i + u_t = 0 \text{ on } \Sigma_c, \quad u = 0 \text{ on } \Sigma_u = \partial\Omega \times (0,T) \\ u(0) = u^0; \quad u_{,t}(0) = u^1 \text{ in } \Omega, \end{cases} \tag{4.1}$$

where here and in the sequel, ν denotes the unit vector pointing into the exterior of Ω, the coefficients $(b_{ij})_{i,j}$, satisfy:

$$b_{ij} \in C^1(\bar{\Omega}); \quad b_{ij} = b_{ji}, \quad \forall i,\ j = 1,2,\ldots,N, \tag{4.2}$$

and

$$\exists a_0 > 0 : b_{ij}(x)z_i z_j \geq a_0 z_i z_i, \quad \forall (x,z) \in \bar{\Omega} \times \mathbb{R}^N. \tag{4.3}$$

If one sets $V = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_u\}$, then one can show that for $(u^0, u^1) \in V \times L^2(\Omega)$, one has $u \in C([0,\infty); V) \cap C^1([0,\infty); L^2(\Omega))$.

For every $t \in [0,T]$, set

$$E(t) = \frac{1}{2}\int_\Omega \{|u_t(x,t)|^2 + (b_{ij}(x)\partial_j u(x,t)\partial_i u(x,t)\}\,\mathrm{d}x.$$

The energy E is a nonincreasing function of the time variable, as we have the dissipation law:

$$E(s) = E(t) + \int_s^t \int_{\Gamma_c} |u_{,t}(\gamma,t)|^2\,\mathrm{d}\gamma\mathrm{d}r, \quad \forall 0 \le s < t < \infty. \tag{4.4}$$

It is now well known that if the boundary of Ω is C^∞, the coefficients $b_{ij} \in C^\infty(\Omega)$, and Γ_c satisfies the geometric control condition of Bardos–Lebeau–Rauch [4]: *there exists a time $T > 0$ such that every ray of geometric optics meets $\Gamma_c \times (0,T)$*, then the energy E satisfies the exponential decay estimate:

$$E(t) \le M\mathrm{e}^{-\lambda t}E(0), \quad \forall t \ge 0. \tag{4.5}$$

Many other authors proved the exponential decay estimates under various conditions on the boundary of Ω, the coefficients b_{ij} and the feedback control support Γ_c, *e.g.* [10,13,21,22,24,25,27,29,30,37,38,45–48,59–61].

Now, consider the perturbed problem, with all parameters as above:

$$\begin{cases} u_{\varepsilon,tt} - \partial_i(b_{ij}(x)\partial_j u_\varepsilon) - \varepsilon\partial_i(b_{ij}(x)\partial_j u_{\varepsilon,t}) = 0 \text{ in } \Omega \times (0,\infty) \\ b_{ij}(x)\partial_j(u_\varepsilon + \varepsilon u_{\varepsilon,t})\nu_i + u_{\varepsilon,t} = 0 \text{ on } \Sigma_c = \Gamma_c \times (0,\infty), \\ u_\varepsilon = 0 \text{ on } \Sigma_u = \Gamma_u \times (0,\infty) \\ u_\varepsilon(0) = u^0; \quad u_{\varepsilon,t}(0) = u^1 \text{ in } \Omega. \end{cases} \tag{4.6}$$

We are going to apply Theorem 1.1 to System (4.6) to prove that its energy given by

$$E_\varepsilon(t) = \frac{1}{2}\int_\Omega \{|u_{\varepsilon,t}(x,t)|^2 + (b_{ij}(x)\partial_j u_\varepsilon(x,t)\partial_i u_\varepsilon(x,t)\}\,\mathrm{d}x,$$

and which satisfies, for all $0 \le s < t < \infty$, the dissipation law

$$E_\varepsilon(s) = E_\varepsilon(t) + \int_s^t \int_{\Gamma_c} |u_{\varepsilon,t}(\gamma,t)|^2\,\mathrm{d}\gamma\mathrm{d}r + \varepsilon\int_s^t\int_\Omega b_{ij}(x)\partial_j u_{\varepsilon,t}(x,t)\partial_i u_{\varepsilon t}(x,t)\,\mathrm{d}x\mathrm{d}s, \tag{4.7}$$

decays exponentially, uniformly with respect to the perturbation parameter ε. To this end, set $H = L^2(\Omega)$, $V = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_u\}$, $Au = -\partial_i(b_{ij}\partial_j u)$ with $D(A) = \{u \in V; Au \in H\}$, $\langle Bu, v\rangle = \int_{\Gamma_c} u\bar{v}\,\mathrm{d}\gamma$, for all $u,\ v \in V$. Then according to the hypotheses on the coefficients, the operator A is coercive with $A^* = A$. The operator B is nonnegative, well-defined on V with values in the dual space V' thanks to Riesz representation theorem. If we also set $y_\varepsilon = u_\varepsilon$, then (4.6) may be recast as the abstract equation (1.1). Further, it can be shown that the operators A and B satisfy (1.11) and (1.12) respectively. Therefore, if Γ_c satisfies the Bardos–Lebeau–Rauch geometric control condition (GCC), then the semigroup generated by the limit operator $\mathcal{A}_0$ is exponentially stable thanks to (4.5) and Lemma 2.1; applying Theorem 1.1, one derives that the perturbed energy E_ε decays exponentially, uniformly with respect to the perturbation parameter ε, as the time variable t goes to infinity. On a different note, it can be shown that $\mathcal{A}_0$ has a compact resolvent, *e.g.* [22], Lemmas 7.7, 7.8, but the perturbed operator $\mathcal{A}_\varepsilon$ does not.

Now, if Γ_c does not satisfy (GCC), then the exponential decay (4.5) for the energy E fails. In this case, it is known that for smoother initial data, and under certain conditions on Γ_c, polynomial decay estimates for the energy E hold when $Au = -\Delta u$, [43], while logarithmic decay estimates for the energy E hold for any Γ_c with a nonzero measure, *e.g.* [16,32]. In the former case, applying Theorem 1.2, one derives uniform polynomial decay estimates for the energy E_ε, and in the latter case, the application of Theorem 1.3 provides a uniform logarithmic decay for the energy E_ε.

4.2. The elasticity equations with boundary damping

Consider the damped elasticity system

$$\begin{cases} y_{i,tt} - \sigma_{ij,j} = 0 \text{ in } \Omega \times (0,\infty) \\ \sigma_{ij}\nu_j + y_{i,t} = 0 \text{ on } \Gamma_c \times (0,\infty) \quad y_i = 0 \text{ on } \Gamma_u \times (0,\infty) \\ y_i(0) = y_i^0, \quad y_{i,t}(0) = y_i^1, \quad i = 1,\ 2,\ \ldots,\ N, \end{cases} \tag{4.8}$$

where the elasticity stress tensor (σ_{ij}) is given by

$$\sigma_{ij} = \sigma_{ij}(y) = a_{ijkl}\varepsilon_{kl}$$

with (ε_{kl}) defined by

$$\varepsilon_{kl} = \varepsilon_{kl}(y) = \frac{1}{2}(y_{k,l} + y_{l,k})$$

is the strain tensor. The a_{ijkl} are the elasticity coefficients. They satisfy the symmetry properties

$$a_{ijkl} = a_{jilk} = a_{klij}, \quad \forall i,j,k,l.$$

Throughout the paper we assume that the a_{ijkl} depend on the space variable x but not on time, and that they are continuously differentiable, and satisfy the ellipticity condition

$$\exists a_0 > 0 : a_{ijkl}u_{ij}u_{kl} \geq a_0 u_{ij}u_{kl} \tag{4.9}$$

for all second order symmetric tensors (u_{ij}).
Under the above assumptions on the coefficients, and for all i, $(y_i^0, y_i^1) \in V \times L^2(\Omega)$, it is well-known that System (4.8) has a unique weak solution $y \in \mathcal{C}([0,\infty); V^N) \cap \mathcal{C}^1([0,\infty); [L^2(\Omega)]^N)$.

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \{|y_{,t}(x,t)|^2 + (\sigma_{ij}\varepsilon_{ij})(x,t)\}\,\mathrm{d}x, \quad \forall t \geq 0. \tag{4.10}$$

The energy E is a nonincreasing function of the time variable t and its derivative satisfies

$$E'(t) = -\int_{\Gamma_c} |y_{,t}(\gamma,t)|^2\,\mathrm{d}\gamma, \quad \forall t \geq 0. \tag{4.11}$$

It is well-known that if Γ_c is an appropriate portion of the boundary, and some structural constraints are imposed on the coefficients a_{ijkl}, then the energy E satisfies an exponential decay estimate of type (4.5), *e.g.* [2,23,26,37,40,48]. We shall now show that the perturbed system

$$\begin{cases} y_{\varepsilon i,tt} - \sigma_{ij,j}(y_\varepsilon) - \varepsilon\sigma_{ij,j}(y_{\varepsilon,t}) = 0 \text{ in } \Omega \times (0,\infty) \\ \sigma_{ij}(y_\varepsilon + \varepsilon y_{\varepsilon,t})\nu_j + y_{\varepsilon i,t} = 0 \text{ on } \Gamma_c \times (0,\infty) \quad y_{\varepsilon i} = 0 \text{ on } \Gamma_u \times (0,\infty) \\ y_{\varepsilon i}(0) = y_i^0, \quad y_{\varepsilon i,t}(0) = y_i^1, \quad i = 1,\ 2,\ \ldots,\ N, \end{cases} \tag{4.12}$$

where the elasticity stress tensor (σ_{ij}) is now given by

$$\sigma_{ij}(y_\varepsilon) = a_{ijkl}\varepsilon_{kl}(y_\varepsilon),$$

is uniformly exponentially stable. To this end, set $H = [L^2(\Omega)]^N$ and $V = \{u \in [H^1(\Omega)]^N; u = 0 \text{ on } \Gamma_u\}$. If we set $Au = -\sigma_{ij,j}(u)$, with $D(A) = \{u \in V; Au \in H\}$. Define the operator B by $\langle Bu, v\rangle = \int_{\Gamma_c} u \cdot \bar{v}\,d\Gamma$, then B is nonnegative, well-defined according to Riesz representation theorem, and one can check that A and B satisfy (1.11) and (1.12) respectively. Moreover (4.12) may be recast as (1.1). On the other hand, knowing that for an appropriate portion of the boundary, the limit system is exponentially stable, it follows from Theorem 1.1 that the perturbed system is uniformly, with respect to the perturbation parameter, exponentially stable.

4.3. The Euler–Bernoulli equation with unbounded locally distributed damping

Let $a \in L^\infty(\Omega)$, be a nonnegative function satisfying:

$$\exists a_0 > 0 : a(x) \geq a_0, \quad \text{a.e. } x \in \omega, \tag{4.13}$$

where ω is an appropriate open set contained in Ω.
Consider the following damped Euler–Bernoulli equation

$$\begin{cases} w_{,tt} + \Delta^2 w - \operatorname{div}(a\nabla w_{,t}) = 0 \text{ in } \Omega \times (0,\infty) \\ w = \dfrac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma \times (0,\infty) \\ w(0) = y^0 \text{ in } \Omega \\ w_{,t}(0) = y^1 \text{ in } \Omega. \end{cases} \tag{4.14}$$

System (4.14) corresponds to the clamped plate equation with structural damping when $a \equiv 1$, and $N = 2$, [12]. Condition (4.13) ensures that the damping term $-\operatorname{div}(a\nabla w_{,t})$ is effective on the set ω.

Let $\{y^0, y^1\} \in H_0^2(\Omega) \times L^2(\Omega)$. System (4.14) is then well-posed in the space $H_0^2(\Omega) \times L^2(\Omega)$.

Introduce the energy

$$E(t) \equiv E(w;t) = \frac{1}{2}\int_\Omega \{|w_{,t}(x,t)|^2 + |\Delta w(x,t)|^2\}\,\mathrm{d}x, \quad \forall t \geq 0. \tag{4.15}$$

The energy E is a nonincreasing function of the time variable t and we have for almost every $t \geq 0$,

$$E'(t) = -\int_\Omega a(x)|\nabla w_{,t}(x,t)|^2\,\mathrm{d}x. \tag{4.16}$$

The decay estimates of the energy of plate equations with a locally distributed frictional damping of the form ay_t or $ag(y_t)$, for an appropriate nonlinear function g, are well-known, *e.g.* [1, 11, 18, 19, 34, 50, 52, 58, 62]. Concerning the system (4.14) with a locally distributed structural damping, it was recently shown in [55] that, if ω satisfies the geometric constraint described in [22, 33] or [34], then its energy, given by (4.15), satisfies an exponential decay estimate of type (4.5).

Introduce the perturbed system

$$\begin{cases} w_{\varepsilon,tt} + \Delta^2 w_\varepsilon - \operatorname{div}(a\nabla w_{\varepsilon,t}) + \varepsilon\Delta^2 w_{\varepsilon,t} = 0 \text{ in } \Omega \times (0,\infty) \\ w_\varepsilon = \dfrac{\partial w_\varepsilon}{\partial \nu} = 0 \text{ on } \Gamma \times (0,\infty) \\ w_\varepsilon(0) = y^0 \text{ in } \Omega \\ w_{\varepsilon,t}(0) = y^1 \text{ in } \Omega. \end{cases} \tag{4.17}$$

If we set $H = L^2(\Omega)$, $V = H_0^2(\Omega)$, $A = \Delta^2$ with clamped boundary conditions, $\langle Bu, v\rangle = \int_\Omega a(x)\nabla u \cdot \nabla \bar{v}\,\mathrm{d}x$, for all $u,\ v \in V$. Then, the operator A is coercive with $A^* = A$, and the operator B is nonnegative, well-defined on V with values in the dual space V'. If we also set $y_\varepsilon = w_\varepsilon$, then (4.17) may be recast as the abstract equation (1.1). Further, it can be shown that the operators A and B satisfy (1.11) and (1.12) respectively. On the other hand, we also know that the unbounded operator $\mathcal{A}_0$ associated with (4.14) generates an exponentially stable semigroup [55]. The application of Theorem 1.1 shows that the perturbed energy $E_\varepsilon = E(w_\varepsilon;.)$ decays exponentially, uniformly with respect to ε.

4.4. Final remarks

1) Although the perturbation $\varepsilon A y_{\varepsilon,t}$ is more relevant physically, given its viscoelastic character, one could have, from a purely mathematical viewpoint, used a more general operator $\hat{A}$ having properties similar to those of the operator A. But the operator $\hat{A}$ should further be required that $V = D(A^{\frac{1}{2}})$ be densely embedded in $D(\hat{A}^{\frac{1}{2}})$, so that $A^{-1}\hat{A}u$ lies in V for all u in $D(\hat{A}^{\frac{1}{2}})$; in particular that requirement is fulfilled by $\hat{A} = A^{\alpha}$ for any $\alpha \le 1$. If one were to use the perturbation εA^{α} instead, the interesting values of α would be those in the interval $(0, 1]$. Indeed, for $\alpha \le 0$ the perturbation is either compact ($\alpha < 0$), or it can be easily absorbed by the energy.
2) It was noted in the introduction that overdamping may occur, meaning that the exponential decay does not hold uniformly with respect to ε as ε goes to infinity. This fact was demonstrated in the case of a bounded damping operator B in [14]. It can also be established in the case of an unbounded damping operator B. For instance, if we choose $B = A$, which corresponds to the Kelvin–Voigt damping, then the operator $\mathcal{A}_0$ is known to generate an analytic semigroup; so, it can be shown that the semigroup generated by the operator $\mathcal{A}_\varepsilon$ is also analytic; the proof of this fact just follows the same algorithm devised above in the proof of the exponential decay of the semigroup. Now, we are going to show that a branch of the eigenvalues of the operator $\mathcal{A}\varepsilon$ behaves so badly as $\varepsilon \to \infty$ that exponential decay of the energy fails to be uniform as $\varepsilon \to \infty$. To this end, remember that the operator A is elliptic, self-adjoint, and has a compact resolvent; therefore the spectrum of the operator A is discrete and we assume that it is given by the increasing sequence $\{\mu_j^2; \mu_j > 0, \quad j \ge 1\}$. The j^{th} eigenvalue $\lambda_{j\varepsilon}$ of the operator $\mathcal{A}_\varepsilon$ (keep in mind that we have chosen $B = A$ here) satisfies the quadratic equation: $\lambda^2 + (1+\varepsilon)\mu_j^2\lambda + \mu_j^2 = 0$. Consequently, one has:
$$\lambda_{j\varepsilon}^{\pm} = \frac{(1+\varepsilon)\mu_j^2}{2}\left(-1 \pm \sqrt{1 - \frac{4}{(1+\varepsilon)^2\mu_j^2}}\right),$$
so that
$$\lambda_{j\varepsilon}^{+} \sim -\frac{1}{1+\varepsilon}, \quad \text{as } (1+\varepsilon)\mu_j \to \infty.$$
Therefore this branch of the spectrum of $\mathcal{A}_\varepsilon$ approaches zero as $\varepsilon \to \infty$, thereby precluding the uniform exponential decay of the energy.

Acknowledgements. The author thanks the anonymous referees for their valuable comments, and suggestions that helped to improve the presentation of this paper.

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