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EXACT NULL INTERNAL CONTROLLABILITY FOR THE HEAT EQUATION ON UNBOUNDED CONVEX DOMAINS*

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Abstract. The liner parabolic equation $\frac{\partial y}{\partial t} - \frac{1}{2}\,\Delta y + F\cdot\nabla y = \mathbb{1}_{\mathcal{O}_0} u$ with Neumann boundary condition on a convex open domain $\mathcal{O} \subset \mathbb{R}^d$ with smooth boundary is exactly null controllable on each finite interval if $\mathcal{O}_0$ is an open subset of $\mathcal{O}$ which contains a suitable neighbourhood of the recession cone of $\overline{\mathcal{O}}$. Here, $F : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded, C^1-continuous function, and $F = \nabla g$, where g is convex and coercive.

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1. Introduction

We are concerned here with the exact null controllability of the linear parabolic equation with a drift term,

$$\begin{aligned}
&\frac{\partial y}{\partial t} - \frac{1}{2}\,\Delta y + F(x)\cdot\nabla y = \mathbb{1}_{\mathcal{O}_0} u \ \text{ in } \ (0,T)\times\mathcal{O},\\
&\frac{\partial y}{\partial \nu} = 0 \ \text{ on } \ (0,T)\times\partial\mathcal{O},\\
&y(0) = y_0(x), \ x\in\mathcal{O},
\end{aligned} \tag{1.1}$$

where $\mathcal{O}$ is an open and convex set in $\mathbb{R}^d$ (eventually unbounded), $d \geq 1$, $\mathcal{O}_0$ is an open subset of $\mathcal{O}$ and $F : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-continuous, coercive and bounded mapping of gradient type.

The main result, Theorem 2.2, amounts to saying that under suitable conditions on $\mathcal{O}$, system (1.1) is exactly null controllable *via* a controller $u \in L^2_{\text{loc}}(\mathcal{O})$.

There is already an extensive literature on exact null controllability with internal controller for general linear parabolic equations with Dirichlet and Neumann boundary conditions on bounded domains.

The first result is due to Lebeau and Robbiano [13] and refer to exact null internal controllability of the heat equation with Dirichlet homogeneous conditions. Later on this result was extended to general linear parabolic equations with smooth coefficients by Fursikov and Yu. Imanuvilov [12]. The extension to parabolic equations with discontinuous coefficients is due to Le Rousseau and Robbiano [15]. (On these lines, see also [9, 14]). The

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exact null controllability of semilinear parabolic equations with superlinear nonlinearity was established in [2,11]. For other results on these lines, we mention the works [1,8,10], and refer to Zhang's survey [21] for more recent results on exact controllability of parabolic equations. As regards the case where $\mathcal{O}$ is unbounded, which is the main objective of this work, this remained largely open with notable exception of the works [18,22]. In some special cases, ($\mathcal{O}$ half-space), the boundary controllability was also discussed in [16,17].

The assumption we impose upon F and $\mathcal{O}$ implies the existence of an invariant probability measure $\mathrm{d}\mu=\rho \mathrm{d}x$, $\rho\in L^1(\mathcal{O})$ for the parabolic operator

$$Ny=-\frac{1}{2}\,\Delta y+F\cdot\nabla y,\ D(N)=\left\{y\in W^{2,2}(\mathcal{O});\ \frac{\partial y}{\partial\nu}=0\right\} \tag{1.2}$$

and the existence of a controller u which steers y_0 into origin follows by a Carleman type inequality for N in the space $L^2(\mathcal{O};\mathrm{d}\mu)$. In fact, this represents the main novelty of the method: the use of an observation (Carleman) inequality in an $L^2(\mathcal{O};\mathrm{d}\mu)$ space with respect to an invariant measure μ for the linear parabolic operator N. As a matter of fact, as seen later on, μ is an invariant measure for the flow determined by the stochastic reflection problem

$$\begin{aligned}&dX+F(X)\mathrm{d}t+N_K(X)\mathrm{d}t\ni dW,\ t\geq 0,\\ &X(0)=x,\end{aligned} \tag{1.3}$$

where W is a Brownian process in $\mathbb{R}^d$, $K=\overline{\mathcal{O}}$ and N_K is the normal cone to K. (See [4,5,7] for existence theory).

Notation. $L^p(\mathcal{O}),\ W^{k,p}(\mathcal{O}),\ L^1_{\mathrm{loc}}(\mathcal{O}),\ W^{k,p}_{\mathrm{loc}}(\mathcal{O}),\ k\geq 1,\ 1\leq p\leq\infty$ are standard Lebesgue and Sobolev spaces on $\mathcal{O}$. If μ is a probability measure on $\mathbb{R}^d$, we denote by $L^p(\mathbb{R}^d;\mu)$ and $W^{k,p}(\mathbb{R}^d;\mu)$, respectively, the spaces

$$L^p(\mathbb{R}^d;\mu)=\left\{u\in L^p_{\mathrm{loc}}(\mathbb{R}^d);\int|u|^p\mathrm{d}\mu<\infty\right\}$$

$$W^{k,p}(\mathbb{R}^d;\mu)=\left\{u\in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d);\int\sum_{j=1}^{k}|D^ju|_d^p\mathrm{d}\mu<\infty\right\},$$

where $D^ju=\left\{\frac{\partial^j u}{\partial x_k^j},\ k=1,\dots,d\right\}$. Similarly, there are defined the spaces $L^p(\mathcal{O};\mu)$, $W^{k,p}(\mathcal{O};\mu)$. We also set $D_j=\frac{\partial}{\partial x_j}$, $D^2_{ij}=\frac{\partial^2}{\partial x_i\partial x_j}$ and $\nabla u=\left\{\frac{\partial u}{\partial x_j}\right\}_{j=1}^{d}$. By $C^k_b(\mathbb{R}^d)$, respectively $C^k_b(\overline{\mathcal{O}})$, $k=0,1,2$, we denote the space of all k-differentiable functions on $\mathbb{R}^d$, respectively $\overline{\mathcal{O}}$, with uniformly continuous and bounded derivatives of order k. We denote by $|\cdot|_d$ the Euclidean norm of $\mathbb{R}^d$ and by ν the outward normal to the boundary $\partial\mathcal{O}$ of $\mathcal{O}$. By $\mathbb{1}_{\mathcal{O}_0}$ we denote the characteristic function of the subset $\mathcal{O}_0$. If Y is a Banach space, we denote by $L^p(0,T;Y)$ the space of all L^p-Bochner integrable functions $u:(0,T)\to Y$. By $C([0,T];Y)$, we denote the space of all Y-valued continuous functions on $[0,T]$. Given a closed convex set $K\subset\mathbb{R}^d$, the recession cone of K is defined by (see [19,20])

$$\mathrm{recc}(K)=\{y\in\mathbb{R}^d;\ x+\lambda y\in K,\ \forall x\in K,\ \forall\lambda\geq 0\}$$

or, equivalently,

$$\mathrm{recc}(K)=\bigcap_{\lambda>0}\lambda(K-y),\ \forall y\in K.$$

If K is bounded, then $\mathrm{recc}(K)=\{0\}$, but otherwise $\mathrm{recc}(K)$ is an unbounded set (cone).

Denote by p_K the Minkowski functional (gauge) associated with the closed convex set K, that is,

$$p_K(x)=\inf\left\{\lambda\geq 0;\ \frac{1}{\lambda}x\in K\right\},\ \forall x\in\mathbb{R}^d. \tag{1.4}$$

We recall that p_K is subadditive, positively homogeneous and, if $\overset{\circ}{K} \neq \emptyset$, then

$$\overset{\circ}{K} = \{x \in \mathbb{R}^d;\ p_K(x) < 1\},\ \partial K = \{x \in \mathbb{R}^d;\ p_K(x) = 1\}. \tag{1.5}$$

(Here, $\overset{\circ}{K}$ is the interior of K and ∂K is its boundary).

Assume further that $0 \in \overset{\circ}{K}$. Then

$$\mathrm{recc}(K) = \{x \in K;\ p_K(x) = 0\}. \tag{1.6}$$

2. The main result

Assumption 2.1.

(i) $\mathcal{O}$ is an open convex set with C^2-boundary $\partial\mathcal{O}$, $0 \in \mathcal{O}$.
(ii) $F = \nabla g$, where $g \in C^2(\mathbb{R}^d)$ is convex and

$$\sup\{|F(x)|_d + \|DF(x)\|;\ x \in \mathbb{R}^d\} < \infty \tag{2.1}$$

$$g(x) \geq \alpha_1 |x|_d + \alpha_2,\ \forall x \in \mathbb{R}^d, \tag{2.2}$$

where $\alpha_1 > 0$ and $\alpha_2 \in \mathbb{R}$. Here, DF stands for the differential of $F : \mathbb{R}^d \to \mathbb{R}^d$ and $\|\cdot\|$ is the norm in $L(\mathbb{R}^d, \mathbb{R}^d)$.
(iii) $\mathcal{O}_0$ is an open subset of $\mathcal{O}$ which contains an open subset $\mathcal{O}_1$ such that $\overline{\mathcal{O}}_1 \subset \mathcal{O}_0$ and

$$\inf\{|\nabla p_{\overline{\mathcal{O}}}(x)|_d;\ x \in \overline{\mathcal{O}} \setminus \mathcal{O}_1\} = \gamma > 0. \tag{2.3}$$

We note that, by Lemma A.2, $p_{\overline{\mathcal{O}}} \in C^2(\overline{\mathcal{O}} \setminus \mathrm{recc}(\overline{\mathcal{O}}))$ and so (2.3) makes sense.

Taking into account (1.6), we see by (iii) that $\mathrm{recc}(\overline{\mathcal{O}}) \subset \mathcal{O}_1 \subset \mathcal{O}_0$.

Consider the function $\rho : \mathbb{R}^d \to \mathbb{R}^+$

$$\rho(x) = \begin{cases} \exp(-2g(x)) \left(\int_{\mathcal{O}} \exp(-2g(x))\mathrm{d}x\right)^{-1}, & x \in \overline{\mathcal{O}}, \\ 0 & x \in \mathbb{R}^d \setminus \overline{\mathcal{O}}. \end{cases} \tag{2.4}$$

A simple calculation shows that, if $g \in C^2(\mathbb{R}^d)$, then ρ is a solution to the Neumann problem

$$\begin{aligned} &\frac{1}{2}\,\Delta\rho + \mathrm{div}\,(F\rho) = 0 \text{ in } \mathcal{O}, \\ &\frac{1}{2}\,\frac{\partial\rho}{\partial\nu} + (F\cdot\nu)\rho = 0 \text{ on } \partial\mathcal{O}. \end{aligned} \tag{2.5}$$

(Otherwise, (2.5) holds in the weak distributional sense).

We consider the probability measure μ defined by

$$\mathrm{d}\mu = \rho\,\mathrm{d}x \tag{2.6}$$

and consider in the space $L^2(\mathcal{O};\mu)$ the operator

$$\begin{aligned} Ny &= -\frac{1}{2}\,\Delta y + F\cdot\nabla y,\ y \in D(N), \\ D(N) &= \left\{y \in W^{2,2}(\mathcal{O};\mu);\ \frac{\partial y}{\partial\nu} = 0 \text{ on } \partial\mathcal{O}\right\}. \end{aligned} \tag{2.7}$$

In fact, N is the Kolmogorov operator associated with the stochastic reflection equation (1.3). (See [4,5]). By Lemma A.1 in Appendix, the operator N is m-accretive in $L^2(\mathcal{O};\mu)$ and so it generates a C_0-semigroup of contractions e^{-tN} in $L^2(\mathcal{O};\mu)$.

In terms of the process $X(t,x)$ defined by equation (1.3),

$$\mathrm{e}^{-tN}y(x) = \mathbb{E}\,y(X(t,x)),\ \forall x \in \mathbb{R}^d,$$

where $\mathbb{E}$ is the expectation. In particular, this implies that problem (1.1), or equivalently

$$\frac{\mathrm{d}y}{\mathrm{d}t} + Ny = \mathbb{1}_{\mathcal{O}_0}u,\ t \in [0,T],\ y(0) = y_0, \tag{2.8}$$

has for each $y_0 \in L^2(\mathcal{O};\mu)$ and all $u \in L^2(0,T;L^2(\mathcal{O};\mu))$, a unique mild solution $y^u \in C([0,T];L^2(\mathcal{O};\mu))$, that is,

$$y^u(t) = \mathrm{e}^{-tN}y_0 + \int_0^t \mathrm{e}^{-(t-s)N}(\mathbb{1}_{\mathcal{O}_0}u)(s)\mathrm{d}s,\ t \in [0,T]. \tag{2.9}$$

Now, we can formulate the main controllability result.

Theorem 2.2. *Under assumptions* (i)-(iii)*, for each* $0 < T < \infty$ *and all* $y_0 \in L^2(\mathcal{O};\mu)$ *there is at least one controller* $u \in L^2(0,T;L^2(\mathcal{O};\mu))$ *such that* $y^u(T) \equiv 0$.

Theorem 2.2 will be proved in Section 3 *via* the Carleman inequality for the backward dual equation associated with (2.8). It should be said that in equation (1.1) (respectively (2.7)) the coefficient $\frac{1}{2}$ in front of Δ was taken for the sake of symmetry only. Of course, one can replace it by any constant $a > 0$. As a matter of fact, the operator $\frac{1}{2}\Delta$ can be replaced by any second order elliptic operator with constant coefficients.

In the following, we discuss the form of $\mathcal{O}_0$ arising in (iii) in some special cases.

Assume that

$$\mathcal{O} = \{(x', x_d) \in \mathbb{R}^d;\ x_d > \phi(x') - b\}, \tag{2.10}$$

where $b > 0$ and $\phi \in C^2(\mathbb{R}^{d-1})$ is a convex function satisfying $\phi(0) = 0$ and

$$\phi(u) \geq a|u|_{d-1}^m,\ \forall u \in \mathbb{R}^d, \tag{2.11}$$

where $a > 0$ and $1 \leq m < \infty$. (We have always such a local representation of $\mathcal{O}$.)

It is readily seen that

$$\mathrm{recc}(\overline{\mathcal{O}}) = \{(0, x_d);\ x_d \geq 0\} \qquad \text{for } m > 1,$$
$$\mathrm{recc}(\overline{\mathcal{O}}) = \{(x', x_d);\ x_d \geq a|x'|_{d-1}\} \text{ for } m = 1.$$

(Here, $x' = x_1, \ldots, x_{d-1})$).

We have

Proposition 2.3. *Let* $\eta > 0$ *be the solution to the equation*

$$x_d = \eta\phi\left(\frac{x'}{\eta}\right) - b\eta. \tag{2.12}$$

Then, $\eta = p_{\overline{\mathcal{O}}}$ *and any set* $\mathcal{O}_0 = \overline{\mathcal{O}} \setminus G_\alpha^\varepsilon$,

$$G_\alpha^\varepsilon := \{(x', x_d) \in \overline{\mathcal{O}};\ |x'|_{d-1} \leq \alpha p_{\overline{\mathcal{O}}}(x) - \varepsilon\}, \alpha > 0,\ 0 < \varepsilon < b,$$

satisfies (iii).

Proof. By (1.4), we see that, for each $x \in \overline{\mathcal{O}}$, $p_{\overline{\mathcal{O}}}(x) = \eta(x)$ is the unique positive solution to (2.12). We have

$$\frac{\partial \eta}{\partial x_2} = \frac{\eta}{\eta\phi\left(\frac{x'}{\eta}\right) - \nabla\phi\left(\frac{x'}{\eta}\right)\cdot x' - b}$$

and, since ϕ and $\nabla\phi$ are bounded on bounded sets, we infer that, for each $\alpha > 0$,

$$\left|\frac{\partial \eta}{\partial x_2}\right| \geq \gamma(\alpha) > 0, \quad \text{for } |x'|_{d-1} \leq \alpha\eta,$$

which implies that $\inf\{|\nabla p_{\overline{\mathcal{O}}}(x)|;\ x \in \overline{\mathcal{O}} \setminus \mathcal{O}_1\} > 0$, where $\mathcal{O}_1 = \{(x', x_d);\ |x'|_{d-1} > \alpha p_{\overline{\mathcal{O}}}(x)\} \subset \mathcal{O}_0$. □

Example 2.4. Let $\phi(u) = a|u|_{d-1}^m$, where $a > 0$ and $m \geq 2$. Then (2.12) reduces to

$$x_d = a\eta^{1-m}|x'|_{d-1}^m - b\eta. \tag{2.13}$$

Equivalently,

$$by^m + \frac{x_d}{|x'|_{d-1}^{m-1}}\ y^{m-1} - a = 0, \tag{2.14}$$

where $y = \frac{x_d}{|x'|_{d-1}}\cdot$ A simple analysis of equation (2.14) reveals that, for each $\alpha > 0$,

$$y \geq \alpha \quad \text{if} \quad 0 < x_d \leq \zeta(\alpha)|x'|_d^{m-1}.$$

Then, by Proposition (2.3), it follows that, for each $\gamma > 0$, $0 < \varepsilon < b$, and

$$G_\gamma^\varepsilon = \{x'x_d) \in \mathcal{O};\ x_d \leq \gamma|x'|_d^{m-1} - \varepsilon\}, \tag{2.15}$$

the set $\mathcal{O}_0 \subset \overline{\mathcal{O}} \setminus G_\gamma^\varepsilon$ satisfies (iii).

Then, Theorem 2.2 implies that

Corollary 2.5. *Let $\mathcal{O} = \{(x', x_d); x_d > a|x'|_d^m - b\}$ for $a, b > 0$, $2 \leq m < \infty$. Then, we may take $\mathcal{O}_0$, any set of the form*

$$\{(x', x_d);\ x_d > \gamma|x'|_d^{m-1} - \varepsilon\}, \tag{2.16}$$

where $\gamma > 0$ and $0 < \varepsilon < b$.

In particular, it follows by Theorem 2.2 that (1.1) is exactly null controllable with controllers $v = \mathbb{1}_{\mathcal{O}_0} u$ in any set $\mathcal{O}_0$ of the form (2.16). At finite distance, this set can be taken as close as we want of the recession cone $\{(0, x_d);\ x_d \geq 0\}$.

Remark 2.6. The conclusion of Corollary 2.5 remains true if $\mathcal{O}$ is of the form (2.10) away from origin, that is,

$$\phi(u) = a|u|_{d-1}^m \text{ for } |u|_{d-1} \geq \lambda > a,\ 1 < m < \infty,\ a > 0,\ b > 0. \tag{2.17}$$

Indeed, the calculation in Example 2.4 shows that (2.15) holds because only the values $|x'|_{d-1} + x_d$ large enough are relevant. This extends to the case $m = 1$, where $p_{\overline{\mathcal{O}}}(x) = b^{-1}(a|x'|_{d-1} - x_d)^-$ for $|x'|_{d-1} \geq \lambda > 0$, and so

$$\mathcal{O}_0 = \{(x', x_d);\ x_d - a|x'|_{d-1} \geq -\varepsilon\}, \tag{2.18}$$

where $\varepsilon > 0$ is arbitrarily small.

Remark 2.7. In [18], L. Miller proved the exact controllability of the heat equation in an unbounded domain $\mathcal{O}$ (or, more generally, in a Riemannian compact manifold) *via* a controller u with the support in a subset $\mathcal{O}_0$ which is the nonempty interior of a compact set K such that $K \cap \overline{\mathcal{O}}_0 \cap \partial\mathcal{O} = \emptyset$. Theorem 2.2 is not a consequence of this result and the methods used in [18] are not applicable because (iii) does not necessarily imply that $\mathcal{O} \setminus \mathcal{O}_0$ is a bounded set. For instance, in the case (2.17) with $m = 1$, the controllability set $\mathcal{O}_0$ is given by (2.18) and $\mathcal{O} \setminus \mathcal{O}_0$ is unbounded.

As communicated us C. Lefter, by the inversion transformation $\widetilde{x} \overset{\phi}{=} \frac{x}{|x|_d^2}$ one obtains, by Theorem 2.2, an exact controllability result for the singular parabolic equation

$$\frac{\partial y}{\partial t} - |\widetilde{x}|_d^4 \Delta y + \ell.o.t. = \mathbf{1}_{\widetilde{\mathcal{O}}_0} u \ \text{ in } \widetilde{\mathcal{O}} = \phi(\mathcal{O}),$$

where $\widetilde{\mathcal{O}}_0 = \phi(\mathcal{O}_0)$.

3. Proof of Theorem 2.2

Denote by N^* the dual operator of N in the space $L^2(\mathcal{O};\mu)$, that is,

$$\langle N^* p, y\rangle_{L^2(\mathcal{O};\mu)} = \langle p, Ny\rangle_{L^2(\mathcal{O};\mu)},$$

for all $y \in D(N)$ and $p \in D(N^*)$. A simple calculation involving (2.5) and (2.7) shows that

$$\begin{aligned} N^* p &= -\frac{1}{2}\,\Delta p - F\cdot\nabla p - \nabla(\log\rho)\cdot\nabla p,\\ D(N^*) &= \left\{ p \in W^{2,2}(\mathcal{O});\, \frac{\partial p}{\partial n} = 0 \ \text{ on } \ \partial\mathcal{O} \right\}. \end{aligned} \tag{3.1}$$

Moreover, taking into account that $F = \nabla g = -\frac{1}{2}\nabla(\log\rho)$, we see by (3.1) that $N^* = N$, *i.e.*, N is self-adjoint.

As it is well known, for the exact controllability of (2.8) we need the observability inequality

$$\begin{aligned} \|p(0)\|^2_{L^2(\mathcal{O};\mu)} &\le C\int_0^T \mathrm{d}t \int_{\mathcal{O}_0} |p(x,t)|^2 \mathrm{d}\mu\\ &= C\int_0^T\int_{\mathcal{O}_0} \rho(x)|p(x,t)|^2 \mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{3.2}$$

for any solution p to the backward equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} - N^* p = 0,\ \ t \in (0,T),$$

or, equivalently (recall that $N^* = N$),

$$\begin{aligned} &\frac{\partial p}{\partial t} + \frac{1}{2}\,\Delta p - F\cdot\nabla p = 0 \ \text{ in } \ (0,T)\times\mathcal{O},\\ &\frac{\partial p}{\partial \nu} = 0 \qquad\qquad\qquad \text{ on } \ (0,T)\times\partial\mathcal{O}. \end{aligned} \tag{3.3}$$

To get (3.2), we prove first a Carleman-type inequality for solutions p to equation (3.3). To this end, proceeding as in [12], we consider an open set $\mathcal{O}_1$, $\overline{\mathcal{O}}_1 \subset \mathcal{O}_0$ as in assumption (iii), and set

$$\begin{aligned} \alpha(t,x) &= \frac{\mathrm{e}^{-\lambda\psi(x)} - \mathrm{e}^{2\lambda\|\psi\|_{C(\overline{\mathcal{O}})}}}{t(T-t)},\\ \varphi(t,x) &= \frac{\mathrm{e}^{-\lambda\psi(x)}}{t(T-t)},\ \ x \in \overline{\mathcal{O}},\ \ t \in (0,T), \end{aligned}$$

where ψ is the function given by Lemma 3.1 below.

Lemma 3.1. *There is $\psi \in C^2(\overline{\mathcal{O}})$ such that*

$$\psi(x) > 0, \ \forall x \in \mathcal{O}, \ \psi(x) = 0, \ \forall x \in \partial\mathcal{O}, \tag{3.4}$$

$$|\nabla\psi(x)|_d \geq \gamma > 0, \ \forall x \in \overline{\mathcal{O}} \setminus \mathcal{O}_1, \tag{3.5}$$

$$\sup\{|\nabla\psi(x)|_d + |D^2_{x_ix_j}\psi(x)|; \ i,j = 1,\dots,d\} < \infty. \tag{3.6}$$

Proof. Lemma 3.1 was established in [12] for general bounded open sets $\mathcal{O}$, but the arguments used there does not apply to the present case. Let $\mathcal{O}_2$ be an open subset of $\mathcal{O}_1$ such that $\overline{\mathcal{O}}_2 \subset \mathcal{O}_1$ and $\text{dist}(\partial\mathcal{O}_2, \partial\mathcal{O}_1) > 0$ is sufficiently small. Then, consider a function $\mathcal{X} \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leq \mathcal{X} \leq 1$, $\mathcal{X} = 0$ on $\overline{\mathcal{O}} \setminus \mathcal{O}_1$ and $\mathcal{X} = 1$ in $\overline{\mathcal{O}}_2$. (This function can be constructed in a standard way *via* mollifiers technique). Then, we set $\psi = 1 - (1-\mathcal{X})p_{\overline{\mathcal{O}}}$. Taking into account (1.4), (2.3) and assumption (iii), we see that ψ satisfies (3.4), (3.5) (because $\nabla\psi = -\nabla p_{\overline{\mathcal{O}}}$ on $\overline{\mathcal{O}} \setminus \mathcal{O}_1$). Moreover, by Lemma A.2 in Appendix, (3.6) follows, too. □

The following Carleman inequality is exactly of the same form as that given in [12].

Proposition 3.2. *There are $\lambda_0 > 0$ and a function $s_0 : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for $\lambda \geq \lambda_0$ and $s \geq s_0(\lambda)$,*

$$\int_0^T \int_{\mathcal{O}} \mathrm{e}^{2s\alpha}(s^3\varphi^3p^2 + s\varphi|\nabla p|^2)\,\mathrm{d}\mu\,\mathrm{d}t \leq C_\lambda s^3 \int_0^T \int_{\mathcal{O}_0} \mathrm{e}^{2s\alpha}\varphi^3p^2\mathrm{d}\mu\,\mathrm{d}t, \tag{3.7}$$

for all the solutions p to (3.3).

By (3.7), we obtain estimate (3.2). Namely,

Corollary 3.3. *The observability inequality (3.2) holds for all the solutions p to (3.3).*

Proof. We note first that

$$\langle p, Np\rangle_{L^2(\mathcal{O};\mu)} = \frac{1}{2}\int_{\mathcal{O}} |\nabla p|^2\mathrm{d}\mu,$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} p^2(t,x)\mathrm{d}\mu - \int_{\mathcal{O}} |\nabla p(x,t)|^2\mathrm{d}\mu = 0, \ \forall t \geq 0.$$

We have, for all $0 \leq \tau \leq t < \infty$,

$$\int_{\mathcal{O}} p^2(x,\tau)\mathrm{d}\mu + \int_\tau^t \int_{\mathcal{O}} |\nabla p(x,\theta)|^2\mathrm{d}\mu\,d\theta = \int_{\mathcal{O}} p^2(x,t)\mathrm{d}\mu, \tag{3.8}$$

and, therefore,

$$\int_{\mathcal{O}} p^2(x,0)\mathrm{d}\mu \leq \int_{\mathcal{O}} p^2(x,t)\mathrm{d}\mu \leq \gamma(t)\int_{\mathcal{O}} \mathrm{e}^{2s\alpha}\varphi^3p^2\mathrm{d}\mu, \tag{3.9}$$

where

$$\gamma(t) = \sup\{\mathrm{e}^{-2s\alpha}\varphi^{-3}(x,t); \ x \in \mathcal{O}\} \leq C\exp\left(\tfrac{\mu s}{t(T-t)}\right), \ \mu = 2\mathrm{e}^{2\lambda}\|\psi\|_{C(\overline{\mathcal{O}})}.$$

Integrating (3.9) on $(t_1,t_2) \subset (0,T)$, we obtain by (3.7) the desired inequality (3.2) with a suitable constant C. □

We note also that (3.8) implies that $p \in W^{1,2}(0,T;L^2(\mathcal{O};\mu))$. Without loss of generality, we may assume in the following that $p \in L^2(0,T;W^{2,2}(\mathcal{O};\mu))$. (Taking into account the structure of the domain $D(N)$ this happens if $p(T) \in D(N)$ which, without any loss of generality, we may assume).

Also, for the sake of simplicity, we shall prove Proposition 3.2 for the equation

$$\begin{aligned} &\frac{\partial p}{\partial t} + \Delta p - F\cdot\nabla p = 0 \ \text{ in } \ (0,T)\times\mathcal{O},\\ &\frac{\partial p}{\partial \nu} = 0 \qquad\qquad\qquad\quad \text{on } \ (0,T)\times\partial\mathcal{O}, \end{aligned} \tag{3.10}$$

because (3.3) is obtained from (3.10) by rescaling the time t.

Proof of Proposition 3.2. Since the proof is similar to that given in [12], it will be outlined only by emphasizing however on the differences which arise here due to the presence of $\mathrm{d}\mu = \rho\,\mathrm{d}x$ instead of the Lebesgue measure $\mathrm{d}x$. However, in the proof we follow [2,3].

We set $z = \mathrm{e}^{s\alpha}p$ and note that z is solution to the equation

$$\begin{aligned} &z_t + \Delta z + (\lambda^2 s^2\varphi^2|\nabla\psi|^2 - \lambda^2 s\varphi|\nabla\psi|^2)z - (s\lambda\varphi\nabla\psi - F)\cdot\nabla z\\ &\qquad -(F\cdot\nabla\psi)z + (s\alpha_t + \lambda s\varphi\Delta\psi)z = 0 \ \text{ in } \ Q=(0,T)\times\mathcal{O},\\ &\frac{\partial z}{\partial\nu} = 0 \ \text{ on } \ \Sigma = (0,T)\times\partial\mathcal{O}, \ z(x,0) = (x,T) = 0, \ \forall x\in\mathcal{O}. \end{aligned} \tag{3.11}$$

$z(x,T)=0$. We set

$$\begin{aligned} X(t)z &= -2(s\lambda^2\varphi|\nabla\psi|^2 z - s\lambda\varphi\nabla z\cdot\nabla\psi),\\ B(t)z &= -\Delta z - (\lambda^2 s^2\varphi^2|\nabla\psi|^2 + s\lambda^2\varphi|\nabla\psi|^2)z + s\alpha_t z. \end{aligned}$$

(Here, $z_t = \frac{\partial z}{\partial t}, \ \alpha_t = \frac{\partial\alpha}{\partial t}$). Then, we rewrite (3.11) as

$$z_t + X(t)z - B(t)z = Z(t)z \text{ in } Q, \ \frac{\partial z}{\partial\nu} = 0 \text{ in } \Sigma, \ z(x,0)\equiv z(x,T)\equiv 0, \tag{3.12}$$

where $Z(t)z = -(s\lambda\varphi\Delta\psi - F\cdot\nabla\psi)z + F\cdot\nabla z$. Multiplying (3.12) by $B(t)z$ and, integrating on $\mathcal{O}$, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}}(B(t)z)(x,t)z(x,t)\mathrm{d}\mu = \int_{\mathcal{O}}((B(t)z)(x,t)z_t(x,t)\\ &+(B(t)z_t)(x,t)z(x,t))\mathrm{d}\mu + \int_{\mathcal{O}}(B_t(t)z)(x,t)z(x,t)\mathrm{d}\mu\\ &= 2\int_{\mathcal{O}}(B(t)z)(x,t)(B(t)z(x,t) - X(t)z(x,t) + Z(t)z(x,t))\mathrm{d}\mu\\ &+\int_{\mathcal{O}}(B_t z)(x,t)z(x,t)\mathrm{d}\mu. \end{aligned}$$

This yields

$$\begin{aligned} 2\int_Q (B(t)z(x,t))^2\mathrm{d}\mu\,\mathrm{d}t + 2\int_Q (B(t)z)(x,t)\cdot Z(t)z\mathrm{d}\mu\,\mathrm{d}t + 2Y\\ \le -\int_Q (B_t(t)z)(x,t)z(x,t)\mathrm{d}\mu\,\mathrm{d}t, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} Y = -2\int_Q (s\lambda^2\varphi|\nabla\psi|^2 z - s\lambda\varphi\nabla z\cdot\nabla\psi)\\ \cdot(\Delta z + (\lambda^2 s^2\varphi^2|\nabla\psi|^2 + s\lambda^2\varphi|\nabla\psi|^2 - s\alpha_t)z\mathrm{d}\mu\,\mathrm{d}t. \end{aligned} \tag{3.14}$$

We note that

$$\int_Q (B_t(t)z)z\mathrm{d}\mu\,\mathrm{d}t = -\int_Q z^2(2\lambda^2 s^2\varphi\varphi_t|\nabla\psi|^2 + s\lambda^2\varphi_t|\nabla\psi|^2 - s\alpha_{tt})\mathrm{d}\mu\,\mathrm{d}t.$$

We set $\gamma(\lambda)=\exp(2\lambda\|\psi\|_{C(\overline{\mathcal{O}})})$. Then, we get

$$\left|\int_Q (B_t(t)z)z\mathrm{d}\mu\,\mathrm{d}t\right| \le C(\lambda^2 s^2+s\gamma(\lambda))\int_Q \varphi^3 z^2\mathrm{d}\mu\,\mathrm{d}t + Cs\lambda\int_Q \varphi|\nabla z|^2\mathrm{d}\mu\,\mathrm{d}t. \tag{3.15}$$

(Here and everywhere in the sequel, C is a positive constant independent of s,λ,z and g). Note also that, since F is bounded, we have

$$\left|2\int_Q B(t)z)(Z(t)z)\mathrm{d}\mu\,\mathrm{d}t\right| \le \int_0^T\int_{\mathcal{O}} |B(t)z|^2\mathrm{d}\mu\,\mathrm{d}t + C\int_0^T\int_{\mathcal{O}}(s^2\lambda^2|\varphi z|^2+|\nabla z|^2)\mathrm{d}\mu\,\mathrm{d}t,$$

and so, by (3.13) and (3.15), we see that

$$Y\le C(s^2\lambda^2+s\gamma(\lambda))\int_Q \varphi^3z^2\mathrm{d}\mu\,\mathrm{d}t + C\int_Q s\lambda\varphi|\nabla z|^2\mathrm{d}\mu\,\mathrm{d}t, \tag{3.16}$$

while, by (3.14), we obtain that, for $s,\lambda\ge\lambda_0$ sufficiently large,

$$Y\ge -2s\int_Q(\lambda^2\varphi|\nabla\psi|^2z+\lambda\varphi\nabla\psi\cdot\nabla z)(\Delta z+(\lambda^2s^2\varphi^2|\nabla\psi|^2+s\lambda^2\varphi|\nabla\psi|^2)z)\mathrm{d}\mu\,\mathrm{d}t - CD_1(s,\lambda,z), \tag{3.17}$$

where

$$D_1(s,\lambda,z)=s^2\gamma(\lambda)\lambda^2\int_Q\varphi^3z^2\mathrm{d}\mu\,\mathrm{d}t + s\lambda\int_Q\varphi|\nabla z|^2\mathrm{d}\mu\,\mathrm{d}t.$$

By Green's formula and (2.1), (2.5), (A.12), we have

$$\begin{aligned}-\int_Q z\Delta z\varphi|\nabla\psi|^2\mathrm{d}\mu\,\mathrm{d}t &= \int_Q\varphi|\nabla z|^2|\nabla\psi|^2\mathrm{d}\mu\,\mathrm{d}t + \frac12\int_Q(\nabla\rho\cdot\nabla z^2)\varphi|\nabla\psi|^2\mathrm{d}x\,\mathrm{d}t + \int_Q z\nabla z\cdot\nabla(\varphi|\nabla\psi|^2)\mathrm{d}\mu\,\mathrm{d}t\\ &\ge \frac34\int_Q\varphi|\nabla\psi|^2|\nabla z|^2\mathrm{d}\mu\,\mathrm{d}t - C(\lambda^2+1)\int_Q\varphi^2z^2\mathrm{d}\mu\,\mathrm{d}t.\end{aligned} \tag{3.18}$$

Note also that

$$\begin{aligned}&2s\lambda\int_Q\varphi\nabla z\cdot\nabla\psi(\lambda^2s^2\varphi^2|\nabla\psi|^2+\ s\lambda^2\varphi|\nabla\psi|^2z\mathrm{d}\mu\,\mathrm{d}t\\ &=\ s\lambda\int_Q \operatorname{div}(\rho z^2\nabla\psi(\lambda^2s^2\varphi^3+s\lambda^2\varphi^2)|\nabla\psi|^2)\mathrm{d}x\,\mathrm{d}t\\ &\quad - s\lambda\int_{\mathcal{O}} z^2(\nabla\rho\cdot\nabla\psi)(s^2\lambda^2\varphi^3+s\lambda^2\varphi^2)|\nabla\psi|^2\mathrm{d}x\,\mathrm{d}t\\ &\quad - s\lambda\int_Q z^2\operatorname{div}(\nabla\psi(\lambda^2s^2\varphi^3+s\lambda^2\varphi^2)|\nabla\psi|^2)\mathrm{d}\mu\,\mathrm{d}t\\ &\ge s\lambda\int_\Sigma\rho z^2(\lambda^2s^2\varphi^3+s\lambda^2\varphi^2)|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t\\ &\quad + \int_Q(3s^3\lambda^4\varphi^3+2s^2\lambda^4\varphi^2)|\nabla\psi|^4z^2\mathrm{d}\mu\,\mathrm{d}t\\ &\quad - C\int_Q(\lambda^3s^3\varphi^3+s^2\lambda^3\varphi^2)z^2\mathrm{d}\mu\,\mathrm{d}t.\end{aligned} \tag{3.19}$$

(Here, we used the fact that $|\nabla\rho|_d\le C\rho$.) By (3.14), (3.17)–(3.19), we get

$$Y\ge\int_Q(s^3\lambda^4\varphi^3|\nabla\psi|^4z^2+\frac32 s\lambda^2\varphi|\nabla\psi|^2|\nabla z|^2)\mathrm{d}\mu\,\mathrm{d}t - I_0(z) - 2s\lambda\int_Q\varphi(\nabla z\cdot\nabla\psi)\Delta z\mathrm{d}\mu\,\mathrm{d}t - CD(s,\lambda,z), \tag{3.20}$$

where

$$I_0(z) = s\lambda \int_\Sigma \rho z^2(\lambda^2 s^2 \varphi^3 + s\lambda^2\varphi^2)|\nabla\psi|^2 \frac{\partial\psi}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t \le 0,$$
$$D(s,\lambda,z) = \int_Q ((s^3\lambda^3\varphi^3 + s^2\lambda^2\gamma(\lambda)\varphi^2)z^2 + s\lambda\varphi|\nabla z|^2)\mathrm{d}\mu.$$

Next, by the Green formula it follows after some calculation that

$$\begin{aligned}-2s\lambda \int_Q \varphi(\nabla z\cdot\nabla\psi)\Delta z\,\mathrm{d}\mu\,\mathrm{d}t &= -2s\lambda\int_Q \nabla z\cdot\nabla(\varphi\nabla z\cdot\nabla\psi)\mathrm{d}\mu\,\mathrm{d}t\\ &= -s\lambda\int_\Sigma \rho\varphi|\nabla z|^2(\nabla\psi\cdot\nu)\mathrm{d}\sigma\,\mathrm{d}t + s\lambda\int_Q \varphi|\nabla\psi|^2|\nabla z|^2\mathrm{d}x\,\mathrm{d}t\\ &+ -\int_Q\left(2s\lambda^2\varphi(\nabla z\cdot\nabla\psi)^2 - s\lambda\varphi\left(|\nabla z|^2\Delta\psi - 2\sum_{i,j=1}^n z_{x_i}z_{x_j}D^2_{ij}\psi\right)\right)\mathrm{d}\mu\,\mathrm{d}t.\end{aligned}$$

Since $\frac{\partial\psi}{\partial\nu}\le 0$ and $\psi = 0$ on $\partial\mathcal{O}$, we have

$$\nu = -\frac{\nabla\psi}{|\nabla\psi|},\ (\nabla\psi\cdot\nabla z)(\nabla z\cdot\nu) = -(\nabla\psi\cdot\nabla z)^2|\nabla\psi|^{-1} \ \text{ on } \ \partial\mathcal{O}.$$

This yields

$$2s\lambda\int_Q \varphi\nabla z\cdot\nabla\psi\Delta z\mathrm{d}\mu\,\mathrm{d}t \ge \lambda^2\int_Q \varphi|\nabla\psi|^2|\nabla z|^2\mathrm{d}\mu\,\mathrm{d}t - CD(s,\lambda,z).$$

Then, by (3.17)–(3.20), we obtain that

$$s^3\lambda^4\int_Q \varphi^3|\nabla\psi|^4 z^2\mathrm{d}\mu\,\mathrm{d}t + s\lambda^2\int_Q \varphi|\nabla\psi|^2|\nabla z|^2\mathrm{d}\mu\,\mathrm{d}t \le CD(s,\lambda,z). \tag{3.21}$$

To get (3.21) we used the trace inequality

$$\int_{\partial\mathcal{O}} \rho|\theta|^2 d\tau \le C\int_{\mathcal{O}} \rho(|\theta|^2 + |\nabla\theta|^2)\mathrm{d}x\,\mathrm{d}t,$$

which clearly is true for each $\theta \in W^{1,2}_{\mathrm{loc}}(\mathcal{O}) \cap L^2(\mathcal{O};\mu)$, such that $|\nabla\rho| \in L^2(\mathcal{O};\mu)$.

Recalling the definition of $D(s,\lambda,z)$ and the fact that

$$|\nabla\psi(x)| \ge \gamma > 0,\ \forall z \in \overline{\mathcal{O}}\setminus\mathcal{O}_1,$$

it follows by (3.21) that there are $\lambda_0 > 0$ and $s_0 = s_0(\lambda)$ such that for $\lambda \ge \lambda_0$, $s \ge s_0(\lambda)$

$$\int_Q (s^3\lambda^4\varphi^3 z^2 + s\lambda^2\varphi|\nabla z|^2)\mathrm{d}\mu\,\mathrm{d}t \le C_\lambda\int_0^T\int_{\mathcal{O}_1}(s^3\lambda^4\varphi^3 z^2 + s\lambda\varphi|\nabla z|^2)\mathrm{d}\mu\,\mathrm{d}t.$$

Coming back to p, we get

$$\begin{aligned}&\int_Q \mathrm{e}^{2s\alpha}(s^3\lambda^4\varphi^3 p^2 + s\lambda^2\varphi|s\lambda\varphi p\nabla\psi + \nabla p|^2)\mathrm{d}\mu\,\mathrm{d}t\\ &\qquad\le C_\lambda\int_0^T\int_{\mathcal{O}_1}\mathrm{e}^{2s\alpha}(s^3\lambda^4\varphi^3 p^2 + s\lambda^2\varphi|s\lambda\varphi p\nabla\psi + \nabla p|^2)\mathrm{d}\mu\,\mathrm{d}t.\end{aligned}$$

Now, choose $\mathcal{X} \in C_b^2$ such that $\mathcal{X} = 1$ on $\overline{\mathcal{O}}_1$ and $\mathcal{X} = 0$ on $\mathcal{O} \setminus \mathcal{O}_0$. If we multiply (3.13) by $\mathcal{X}\varphi e^{2s\alpha}\rho p$ and integrate on q, we obtain that

$$\int_Q e^{2s\alpha}(s^3\varphi^3p^2 + s\varphi|\nabla p|^2)dx\,dt \le C_\lambda s^3 \int_0^T \int_{\mathcal{O}_1} e^{2s\alpha}\varphi^3p^2 d\mu\,dt,$$

for all $\lambda \ge \lambda_0$ sufficiently large and $s \ge s_0(\lambda)$. This completes the proof. □

Proof of Theorem 2.2. Theorem 2.2 follows by observability inequality by a standard general result: the observability of solutions to backward dual system implies exact null controllability, so nothing remains to be done. □

4. Remarks on the nongradient case

One might suspect that assumption (ii) can be weakened to

(ii)′ $F \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ satisfies

$$(Fu - Fv)\cdot(u - v) \ge -\gamma|u - v|_\alpha^2,\ \forall u, v \in \mathbb{R}^d, \tag{4.1}$$

$$Fu \cdot u \ge \alpha_2 + \alpha_1|u|_d^2,\ \forall u \in \mathbb{R}^d, \tag{4.2}$$

$$|F(u)| \le C(1 + |u|_d^m),\ \forall u \in \mathbb{R}^d, \tag{4.3}$$

where $\alpha_1 > 0,\ m \in \mathbb{N}$ and $\alpha_2, \gamma \in \mathbb{R}$.

Indeed, in this case, it follows that there is a positive function $\rho \in L^1(\mathbb{R}^d) \cap L^p_{\text{loc}}(\mathbb{R}^d)$ for $p < \frac{d}{d-p}$ such that (Prop. 2.2 in [2])

$$\begin{aligned} \frac{1}{2}\,\Delta\rho + \operatorname{div}(\rho F) &= 0 \quad \text{in} \quad \mathcal{O}, \\ \frac{1}{2}\,\frac{\partial\rho}{\partial n} + (F\cdot\nu)\rho &= 0 \quad \text{on} \quad \partial\mathcal{O}, \end{aligned} \tag{4.4}$$

and $\rho = 0$ on $\mathcal{O}^c$. Moreover, the operator N defined by (2.7) is m-accretive in $L^2(\mathcal{O}; \mu)$, where $d\mu = \rho\,dx$ (Thm. 5.2 in [2]), and one has

$$\sqrt{\rho} \in W^{1,2}(\mathcal{O}, dx),\ \int_{\mathcal{O}} \frac{|\nabla\rho|^2}{\rho}\,dx < \infty. \tag{4.5}$$

As in the gradient case $F = \nabla g$ treated above, the exact null controllabili8ty of (2.8) is equivalent with the observability inequality (3.2) for the dual backward system

$$\frac{dp}{dt} - N^*p = 0.$$

(Note that in this case N is no longer self-adjoint). The major difficulty to obtain a Carleman inequality in this case is the form of the drift term $\nabla(\log\rho)\cdot p$ arising in the form of N^* (see (3.1)) and also the integral term $\int_{\mathcal{O}} z^2(\nabla\rho\cdot\nabla\psi)(s^2\lambda^2\varphi^3 + s\lambda^2\varphi^2)|\nabla\psi|^2 dx\,dt$ arising in (3.19). In order to have a convenient estimate for both terms, we need a condition of the form $|\nabla\rho|_d \le C\rho$ on $\mathcal{O}$, which automatically holds in the gradient case, but is no longer implied by (4.5). One might expect, however, that under additional conditions on F this condition holds. Taking into account that ρ is the solution to (4.4), this means to find conditions on F such that $|\nabla\log\rho| \in L^\infty(\mathcal{O})$.

Appendix A.

Lemma A.1. *The operator N defined by (2.7) is m-accretive in $L^2(\mathcal{O};\mu)$.*

Proof. This result was proved under more general conditions in [4] (see, also [5]), so here the proof will be only outlined. One must prove that, for each $f \in L^2(\mathcal{O};\mu)$ and $\lambda > 0$, the equation

$$\lambda y - \frac{1}{2}\,\Delta y + F\cdot\nabla y = f \ \text{ in } \ \mathcal{O},\ \frac{\partial y}{\partial n} = 0 \ \text{ on } \ \partial\mathcal{O}, \tag{A.1}$$

has a solution $\varphi \in D(N)$ and

$$\|y\|_{L^2(\mathcal{O};\mu)} \le \frac{1}{\lambda}\ \|f\|_{L^2(\mathcal{O};\mu)},\ \forall\lambda > 0. \tag{A.2}$$

To this end, we assume first that $f \in C_b^1(\mathbb{R}^d)$, that is, f and ∇f are uniformly continuous and bounded. Then, consider the elliptic equation in $\mathbb{R}^d$

$$\lambda y_\varepsilon - \frac{1}{2}\ \Delta y_\varepsilon + (F_\varepsilon + \beta_\varepsilon)\cdot\nabla y_\varepsilon = f, \tag{A.3}$$

where $F_\varepsilon = F(I+\varepsilon F)^{-1} = \frac{1}{\varepsilon}\left(I - (I+\varepsilon F)^{-1}\right)$ and $\beta_\varepsilon(x) = \frac{1}{\varepsilon}\left(x - P_{\overline{\mathcal{O}}}(x)\right) = \frac{1}{2\varepsilon}\,\nabla d^2_{\overline{\mathcal{O}}}(x)$, $\forall x \in \mathbb{R}^d$. Here, $P_{\overline{\mathcal{O}}}$ is the projection of convex closed set $\overline{\mathcal{O}}$ and $d_{\overline{\mathcal{O}}}(x)$ is the distance from x to $\overline{\mathcal{O}}$. For $\lambda > 0$ sufficiently large, (A.3) has a unique solution $y_\varepsilon \in C_b^2(\mathbb{R}^d)$ (see, *e.g.*, [6]). In fact, y_ε is given by the probabilistic formula

$$y_\varepsilon(x) = \mathbb{E}\int_0^\infty \mathrm{e}^{-\lambda t} f(X_\varepsilon(t,x))\mathrm{d}t,\ x \in \mathbb{R}^d, \tag{A.4}$$

where X_ε is the solutions to the stochastic equation

$$\begin{aligned}&dX_\varepsilon + F_\varepsilon(X_\varepsilon)\mathrm{d}t + \beta_\varepsilon(X_\varepsilon)\mathrm{d}t = dW_t,\ t \ge 0,\\ &X_\varepsilon(0) = x,\end{aligned} \tag{A.5}$$

which approximates the stochastic reflection problem (1.3) for $\varepsilon \to 0$.

We set

$$\rho_\varepsilon(x) = \exp\left(-2g_\varepsilon(x) - \frac{d^2_{\overline{\mathcal{O}}}(x)}{\varepsilon}\right),\ \forall x \in \mathbb{R}^d,$$

where $\nabla g_\varepsilon = F_\varepsilon$. It is readily seen that $\beta_\varepsilon \to \rho$ for $\varepsilon \to 0$. Moreover, by (A.4) we see that

$$\|y_\varepsilon\|_{C_b^1(\mathbb{R}^d)} \le \frac{1}{\lambda-\gamma}\ \|f\|_{C_b^1(\mathbb{R}^d)},\ \forall\lambda > 0,$$

and, since $X_\varepsilon \to X$ (the solution to (1.3)), we have that, for $\varepsilon \to 0$,

$$y_\varepsilon(x) \to y(x) = \mathbb{E}\int_0^\infty \mathrm{e}^{-\lambda t} f(X(t,x))\mathrm{d}t,\ \forall x \in \mathbb{R}^d. \tag{A.6}$$

Now, taking into account that

$$\frac{1}{2}\,\Delta\rho_\varepsilon + \operatorname{div}(\rho_\varepsilon(F_\varepsilon + \beta_\varepsilon)) = 0 \ \text{ in } \ \mathbb{R}^d. \tag{A.7}$$

We have, by (4.3),

$$\lambda\int_{\mathbb{R}^d} y_\varepsilon\psi\rho_\varepsilon\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d}(\nabla y_\varepsilon\cdot\nabla\psi)\rho_\varepsilon\mathrm{d}x = \int_{\mathbb{R}^d} f\psi\rho_\varepsilon\mathrm{d}x,\ \forall\psi \in C_b^1(\mathbb{R}^d). \tag{A.8}$$

Similarly,

$$\lambda \int_{\mathcal{O}} y_\varepsilon \psi \rho_\varepsilon \mathrm{d}x + \frac{1}{2} \int_{\mathcal{O}} (\nabla y_\varepsilon \cdot \nabla \psi) \rho_\varepsilon \mathrm{d}x - \frac{1}{2} \int_{\partial \mathcal{O}} \frac{\partial y_\varepsilon}{\partial \nu} \rho_\varepsilon \,\mathrm{d}x = \int_{\mathcal{O}} f \psi \rho_\varepsilon \mathrm{d}x, \ \forall \psi \in C_b^1(\mathbb{R}^d).$$

This yields

$$\frac{1}{2} \int_{\partial \mathcal{O}} \frac{\partial y_\varepsilon}{\partial \nu} \rho_\varepsilon \psi \mathrm{d}x = - \int_{\mathcal{O}^c} (\lambda y_\varepsilon - f) \psi \rho_\varepsilon \mathrm{d}x - \frac{1}{2} \int_{\mathcal{O}^c} \nabla y_\varepsilon \cdot \nabla \psi \rho_\varepsilon \,\mathrm{d}x.$$

Then, letting $\varepsilon \to 0$, we see by (A.8) that y is the solution to (A.1) (in the sense of distributions). More precisely, we have

$$\lambda \int_{\mathcal{O}} y \psi \rho \mathrm{d}x + \frac{1}{2} \int_{\partial \mathcal{O}} (\nabla y \cdot \nabla \psi) \rho \,\mathrm{d}x = \int_{\mathcal{O}} f \psi \rho \mathrm{d}x,$$
$$\int_{\partial \mathcal{O}} \frac{\partial y}{\partial n} \rho \psi \,\mathrm{d}\sigma = 0, \ \forall \psi \in C_b^1(\overline{\mathcal{O}}).$$

We also have the estimate (see [2])

$$\|\nabla y_\varepsilon\|_{L^2(\mathcal{O};\mu)} + \|D^2 y_\varepsilon\|_{L^2(\mathcal{O};\mu_\varepsilon)} \le C, \ \forall \varepsilon > 0, \tag{A.9}$$

where $\mu_\varepsilon = \rho_\varepsilon \mathrm{d}x$. Then, letting $\varepsilon \to 0$, we infer that $y \in W^{2,2}(\mathcal{O}; \mu)$ and satisfies equation (A.1), a.e., on $\mathbb{R}^d$.

Multiplying (A.3) by $\rho_\varepsilon y_\varepsilon$ and integrating on $\mathbb{R}^d$, we see by (A.5) that

$$\|y_\varepsilon\|_{L^2(\mathbb{R}^d;\mu_\varepsilon)} \le \frac{1}{\lambda} \|f\|_{L^2(\mathbb{R}^d;\mu_\varepsilon)}, \tag{A.10}$$

and, letting ε tend to zero in (A.10), we obtain (A.2). By density (A.1), (A.2) extends to all $f \in L^2(\mathcal{O}; \mu)$, as claimed. □

Lemma A.2. *Let $\mathcal{O}$ be an open and convex set with C^2-boundary $\partial \mathcal{O}$ and $0 \in \mathcal{O}$. Let $p_{\overline{\mathcal{O}}}$ be the gauge of $\overline{\mathcal{O}}$. Then*

$$p_{\overline{\mathcal{O}}} \in C^2(\overline{\mathcal{O}} \setminus \mathrm{recc}(\overline{\mathcal{O}})), \tag{A.11}$$

$$\sup\{|\nabla p_{\overline{\mathcal{O}}}(x)|_d; \ x \in \overline{\mathcal{O}} \setminus \mathrm{recc}(\overline{\mathcal{O}})\} < \infty, \tag{A.12}$$

$$\sup\{|D^2_{x_i x_j} p_{\overline{\mathcal{O}}}(x)|; \ x \in \overline{\mathcal{O}} \setminus \mathrm{recc}(\overline{\mathcal{O}})\} < \infty, \ i,j = 0,1,2\dots,d. \tag{A.13}$$

Proof. We set $\eta = p_{\overline{\mathcal{O}}}^{-1}$ on $\overline{\mathcal{O}} \setminus \mathrm{recc}(\overline{\mathcal{O}})$. Then, $\eta(x)x \in \partial \mathcal{O}$ and, since $\partial \mathcal{O}$ is of class C^2, it is locally represented as $x_d = \phi(x')$, where ϕ is convex and of class C^2. We have, therefore,

$$x_d \eta = \phi(\eta x'), \ \forall x \in \overline{\mathcal{O}} \setminus \mathrm{recc}(\overline{\mathcal{O}}),$$

and, since $\nabla \phi$ is monotone and of class C^1 in $\mathbb{R}^{d-1}$, $\alpha I + \nabla \phi$, is invertible for all $\alpha > 0$. So, we conclude that η is of class C^2 on $\overline{\mathcal{O}} \setminus \mathrm{recc}(\overline{\mathcal{O}})$.

Since $p_{\overline{\mathcal{O}}}$ is positively homogeneous, subadditive and $0 \in \mathcal{O}$, we have, for all $y \in \mathcal{O}$ with $|y|_d \le \varepsilon$ sufficiently small,

$$\nabla p_{\overline{\mathcal{O}}}(x) \cdot y \le p_{\mathcal{O}}(y), \ \forall x \in \overline{\mathcal{O}} \setminus \mathrm{recc}(\overline{\mathcal{O}}),$$

and this, clearly, implies (A.12).

Now, if we differentiate two times with respect to x the equation $p_{\overline{\mathcal{O}}}(\lambda x) = \lambda p_{\overline{\mathcal{O}}}(x)$, we obtain

$$\lambda D^2_{ij} p_{\overline{\mathcal{O}}}(\lambda x) = D^2_{ij} p_{\overline{\mathcal{O}}}(x), \ \forall x \in \overline{\mathcal{O}} \setminus \mathrm{recc}(\overline{\mathcal{O}}).$$

This yields

$$D^2_{ij} p_{\overline{\mathcal{O}}}(y) = \frac{1}{\lambda}\ D^2_{ij} p_{\overline{\mathcal{O}}}\left(\frac{y}{\lambda}\right),\ y \in \overline{\mathcal{O}} \setminus \mathrm{recc}(\overline{\mathcal{O}}),$$

and, for $\lambda = |y|_d/\varepsilon$, we get

$$D^2_{ij} p_{\overline{\mathcal{O}}}(y) = \frac{\varepsilon}{|y|_d}\ D^2 p_{\overline{\mathcal{O}}}\left(\varepsilon\ \frac{y}{|y|_d}\right),\ \forall y \in \overline{\mathcal{O}} \setminus \mathrm{recc}(\overline{\mathcal{O}}),$$

which, clearly, implies (A.13), as desired. □

Remark A.3. The condition $\partial\mathcal{O}$ of class C^2 is excessively restrictive and was imposed to have (A.11) and by construction in Lemma 3.1, ψ in $C^2(\overline{\mathcal{O}})$. However, as seen in the proof of Proposition 2.3 instead of (1.6), it suffices to have $D^2_{ij}\psi \in L^\infty(\mathcal{O})$ only and, therefore, it suffices to take $\partial\mathcal{O}$ of class $C^{1,\infty}$.

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