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ON TORSIONAL RIGIDITY AND PRINCIPAL FREQUENCIES: AN INVITATION TO THE KOHLER–JOBIN REARRANGEMENT TECHNIQUE

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Abstract. We generalize to the p-Laplacian Δ_p a spectral inequality proved by M.-T. Kohler–Jobin. As a particular case of such a generalization, we obtain a sharp lower bound on the first Dirichlet eigenvalue of Δ_p of a set in terms of its p-torsional rigidity. The result is valid in every space dimension, for every $1 < p < \infty$ and for every open set with finite measure. Moreover, it holds by replacing the first eigenvalue with more general optimal Poincaré-Sobolev constants. The method of proof is based on a generalization of the rearrangement technique introduced by Kohler–Jobin.

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1. Introduction

1.1. Background and motivations

Given an open set $\Omega \subset \mathbb{R}^N$ with finite measure, we consider the following quantities

$$\lambda(\Omega) = \min_{u\in W^{1,2}_0(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\displaystyle\int_\Omega |u|^2\,\mathrm{d}x} \qquad \text{and} \qquad T(\Omega) = \max_{u\in W^{1,2}_0(\Omega)\setminus\{0\}} \frac{\left(\displaystyle\int_\Omega |u|\,\mathrm{d}x\right)^2}{\displaystyle\int_\Omega |\nabla u|^2\,\mathrm{d}x}.$$

The first one is called *principal frequency* of Ω and the second one is its *torsional rigidity*. Our terminology is a little bit improper, since the usual definition of torsional rigidity differs from our by a multiplicative factor. Since this factor will have no bearing in the whole discussion, we will forget about it. Another frequently used terminology for $\lambda(\Omega)$ is *first eigenvalue of the Dirichlet–Laplacian.* Indeed $\lambda(\Omega)$ coincides with the smallest real number λ such that the problem

$$-\Delta u = \lambda\, u \quad \text{in } \Omega, \qquad u = 0, \text{ on } \partial\Omega,$$

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has a nontrivial solution[2]. In [24] Pólya and Szegő conjectured that the ball should have the following isoperimetric-type property:

$$\textit{among sets with given torsional rigidity, balls minimize the principal frequency.} \tag{$\star$}$$

In other words, by taking advantage of the fact that

$$\lambda(t\,\Omega) = t^{-2}\,\lambda(\Omega) \qquad \text{and} \qquad T(t\,\Omega) = t^{N+2}\,T(\Omega), \qquad t>0,$$

they conjectured the validity of the following scaling invariant inequality

$$T(\Omega)^{\frac{2}{N+2}}\,\lambda(\Omega) \geq T(B)^{\frac{2}{N+2}}\,\lambda(B), \tag{1.1}$$

where B is any ball. We recall that among sets with given volume, balls were already known to minimize λ (the celebrated *Faber−Krahn inequality*) and maximize T (the so-called *Saint−Venant Theorem*). This means that the inequality conjectured by Pólya and Szegő *was not* a trivial consequence of existing inequalities. A proof of (1.1) was finally given by Kohler−Jobin in [19, 20], by using a sophisticated new rearrangement technique. The latter is indeed a general result which permits, given Ω and a *smooth* positive function $u \in W^{1,2}_0(\Omega)$, to construct a ball B having smaller torsional rigidity and a radially symmetric decreasing function $u^* \in W^{1,2}_0(B)$ such that

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x = \int_B |\nabla u^*|^2\,\mathrm{d}x \qquad \text{and} \qquad \int_\Omega |u|^q\,\mathrm{d}x \leq \int_B |u^*|^q\,\mathrm{d}x,$$

for every $q>1$. It is clear that once we have this result, the Pólya−Szegő conjecture is easily proven. Of course this also shows that ($\star$) is still true if we replace the principal frequency $\lambda(\Omega)$ by *any other optimal Poincaré−Sobolev constant*. In other words, balls minimize the quantity

$$\min_{u\in W^{1,2}_0(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\left(\displaystyle\int_\Omega |u|^q\,\mathrm{d}x\right)^{\frac{2}{q}}}, \qquad \text{where } 1<q<2^*=\frac{2\,N}{N-2}, \tag{1.2}$$

among sets with given torsional rigidity (see [18], Thm. 3). For some related studies on the quantities (1.2), we also mention the recent paper [6].

1.2. Aim of the paper

Unfortunately, the Kohler−Jobin's rearrangement technique seems not to be well-known, even among specialists. Then the goal of this paper is twofold: first of all, we try to revitalize interest in her methods and results. Secondly, we will extend the Kohler−Jobin inequality to more general "principal frequencies" associated with the nonlinear *p-Laplace operator*, defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right),$$

and to some anisotropic variants of it (Sect. 6). The main difficulty of this extension lies in the lack of regularity of solutions to equations involving Δ_p, indeed in general these are far from being analytic or C^∞, as required in [18–20]. We will show that the Kohler−Jobin technique can be extended to functions enjoying a mild regularity property (see Def. 3.1), which is indeed satisfied by solutions to a wide class of quasilinear equations (see Lem. 3.2). Also, we will simplify some arguments used in [18–20]. For example, in order to compare the L^q norms of the original function and its rearrangement, we will sistematically use Cavalieri's principle, as it is natural. Finally, we will not require smoothness hypotheses on Ω, which is another difference with the work of Kohler−Jobin.

[2]Here solutions are always understood in the *energy sense*, *i.e.* $u \in W^{1,2}_0(\Omega)$ and is a weak (then classical if $\partial\Omega$ is smooth enough) solution. It is well-known that by dropping the assumption $u \in W^{1,2}_0(\Omega)$ strange phenomena can be observed, like for example nontrivial harmonic functions being constantly 0 at the boundary $\partial\Omega$.

1.3. Notation

In order to clearly explain the contents of this work and the results here contained, we now proceed to introduce some required notation.

By $\Omega \subset \mathbb{R}^N$ we still denote an open set with finite measure, while $W_0^{1,p}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla u\|_{L^p(\Omega)}$. Throughout the whole paper we will always assume that $1 < p < \infty$. In this work we will consider the "first eigenvalues"

$$\lambda_{p,q}(\Omega) = \min_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\left(\displaystyle\int_\Omega |u|^q \,\mathrm{d}x\right)^{\frac{p}{q}}}, \tag{1.3}$$

where the exponent q is such that

$$\begin{cases} 1 < q < \dfrac{N\,p}{N-p}, & \text{if } 1 < p < N, \\ 1 < q < \infty, & \text{if } p \ge N. \end{cases} \tag{1.4}$$

Then the quantity $\lambda_{p,q}(\Omega)$ is always well-defined, thanks to Sobolev embeddings. Sometimes we will also refer to $\lambda_{p,q}(\Omega)$ as a *principal frequency*, in analogy with the linear case. Observe that a minimizer of the previous Rayleigh quotient is a nontrivial solution of

$$-\Delta_p u = \lambda\, \|u\|_{L^q(\Omega)}^{p-q}\, |u|^{q-2}\, u, \text{ in } \Omega \qquad u = 0, \text{ on } \partial\Omega, \tag{1.5}$$

with $\lambda = \lambda_{p,q}(\Omega)$. The two terms on both sides of (1.5) have the same homogeneity, then if u is solution, so is $t\,u$ for every $t \in \mathbb{R}$. Moreover, it is not difficult to see that if for a certain λ there exists a nontrivial solutions of (1.5), then we must have $\lambda \ge \lambda_{p,q}(\Omega)$. These considerations justify the name "first eigenvalue" for the quantity $\lambda_{p,q}(\Omega)$ (see [14] for a comprehensive study of these nonlinear eigenvalue problems).

The principal frequency $\lambda_{p,q}$ obeys the following scaling law

$$\lambda_{p,q}(t\,\Omega) = t^{N-p-\frac{p}{q}N}\, \lambda_{p,q}(\Omega),$$

then the general form of the previously mentioned *Faber–Krahn inequality* is

$$|B|^{\frac{p}{N}+\frac{p}{q}-1}\, \lambda_{p,q}(B) \le |\Omega|^{\frac{p}{N}+\frac{p}{q}-1}\, \lambda_{p,q}(\Omega), \tag{1.6}$$

with equality if and only if Ω is a ball. In other words, balls are the unique solutions to the problem

$$\min\{\lambda_{p,q}(\Omega) \,:\, |\Omega| \le c\}.$$

Properly speaking, the name *Faber–Krahn inequality* is usually associated with the particular case of $p = q$ in (1.6). Since the proof is exactly the same for all range of admissible p and q, this small abuse is somehow justified. The special limit case $q = 1$ deserves a distinguished notation, namely we will set

$$T_p(\Omega) = \frac{1}{\lambda_{p,1}(\Omega)} = \max_{v \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\left(\displaystyle\int_\Omega |v| \,\mathrm{d}x\right)^p}{\displaystyle\int_\Omega |\nabla v|^p \,\mathrm{d}x}.$$

In analogy with the case $p = 2$, we will call it the *p-torsional rigidity* of the set Ω. Of course, inequality (1.6) can now be written as

$$|\Omega|^{1-\frac{p}{N}-p}\, T_p(\Omega) \le |B|^{1-\frac{p}{N}-p}\, T_p(B). \tag{1.7}$$

For ease of completeness, we mention that inequalities (1.6) and (1.7) have been recently improved in [4,15], by means of a quantitative stability estimate. Roughly speaking, this not only says that balls are the unique sets for which equality can hold, but also that sets "almost" achieving the equality are "almost" balls.

It is useful to recall that the proof of (1.6) and (1.7) is based on the use of the *Schwarz symmetrization*. The latter consists in associating to each positive function $u \in W^{1,p}_0(\Omega)$ a radially symmetric decreasing function $u^\# \in W^{1,p}_0(\Omega^\#)$, where $\Omega^\#$ is the ball centered at the origin such that $|\Omega^\#| = |\Omega|$. The function $u^\#$ is *equimeasurable* with u, that is

$$|\{x\,:\,u(x)>t\}| = |\{x\,:\,u^\#(x)>t\}|, \qquad \text{for every } t \geq 0,$$

so that $\|u\|_{L^q} = \|u^\#\|_{L^q}$ for every $q \geq 1$. More important, by using the Coarea Formula and by exploiting the convexity of $t \mapsto t^p$ and the isoperimetric inequality, one can obtain the celebrated *Pólya-Szegő principle*

$$\int_{\Omega^\#} |\nabla u^\#|^p \,\mathrm{d}x \leq \int_\Omega |\nabla u|^p \,\mathrm{d}x.$$

The reader is referred to [16], Chapter 2 or [17], Chapters 1 and 2, for more details on the Schwarz symmetrization and the Pólya–Szegő principle.

1.4. Main result

In order to describe the Kohler−Jobin technique and illustrate its range of applicability, in this paper we will consider the following shape optimization problem

$$\min\{\lambda_{p,q}(\Omega)\,:\,T_p(\Omega) \leq c\}, \tag{1.8}$$

in the same spirit as conjecture ($\star$) recalled at the beginning. Again by taking into account the homogeneities of the quantities involved, the previous problem is the same as

$$\min T_p(\Omega)^{\alpha(p,q,N)}\,\lambda_{p,q}(\Omega), \qquad \text{where} \;\; \alpha(p,q,N) = \frac{\dfrac{p}{N} + \dfrac{p}{q} - 1}{\dfrac{p}{N} + p - 1}\cdot$$

We point out that the previous shape functional can be written as follows

$$T_p(\Omega)^{\alpha(p,q,N)}\,\lambda_{p,q}(\Omega) = \left(|\Omega|^{1-\frac{p}{N}-p}\,T_p(\Omega)\right)^{\alpha(p,q,N)}\;|\Omega|^{\frac{p}{N}+\frac{p}{q}-1}\,\lambda_{p,q}(\Omega),$$

i.e. the product of two functionals which are maximized and minimized by balls, respectively.

By suitably extending the Kohler−Jobin technique, we prove the following inequality, which represents the main result of this paper.

Theorem 1.1. *Let $1 < p < \infty$ and q be an exponent verifying (1.4). For every $\Omega \subset \mathbb{R}^N$ open set with finite measure, we have*

$$T_p(\Omega)^{\alpha(p,q,N)}\,\lambda_{p,q}(\Omega) \geq T_p(B)^{\alpha(p,q,N)}\,\lambda_{p,q}(B), \tag{1.9}$$

where B is any ball. Equality can hold if and only if Ω itself is a ball.

In other words, the only solutions to the shape optimization problem (1.8) are given by balls having p-torsional rigidity equal to c.

Remark 1.2. We observe that the whole family of inequalities (1.6) can now be derived by using (1.7) and (1.9). Indeed, we have

$$\begin{aligned}|\Omega|^{\frac{p}{N}+\frac{p}{q}-1}\,\lambda_{p,q}(\Omega) &= \left(|\Omega|^{\frac{p}{N}+\frac{p}{q}-1}\,T_p(\Omega)^{-\alpha(p,q,N)}\right)\,\left(T_p(\Omega)^{\alpha(p,q,N)}\,\lambda_{p,q}(\Omega)\right)\\ &= \left(|\Omega|^{1-\frac{p}{N}-p}\,T_p(\Omega)\right)^{-\alpha(p,q,N)}\,\left(T_p(\Omega)^{\alpha(p,q,N)}\,\lambda_{p,q}(\Omega)\right)\\ &\geq \left(|B|^{1-\frac{p}{N}-p}\,T_p(B)\right)^{-\alpha(p,q,N)}\,\left(T_p(B)^{\alpha(p,q,N)}\,\lambda_{p,q}(B)\right)\\ &= |B|^{\frac{p}{N}+\frac{p}{q}-1}\,\lambda_{p,q}(B).\end{aligned}$$

This implies that the Saint-Venant inequality (1.7) permits to improve the lower bounds on the principal frequencies $\lambda_{p,q}$ provided by the Faber−Krahn inequality, since we can now infer

$$\lambda_{p,q}(\Omega) \geq \left(\frac{T_p(B)}{T_p(\Omega)}\right)^{\alpha(p,q,N)}\,\lambda_{p,q}(B),$$

and the term $(T_p(B)/T_p(\Omega))^{\alpha(p,q,N)}$ is greater than $(|B|/|\Omega|)^{p/N+p/q-1}$ coming from (1.6). In the same spirit, we also point out that (1.9) can be written as

$$\frac{|\Omega|^{\frac{p}{N}+\frac{p}{q}-1}\,\lambda_{p,q}(\Omega)}{|B|^{\frac{p}{N}+\frac{p}{q}-1}\,\lambda_{p,q}(B)} - 1 \geq \left(\frac{T_p(B)\,|B|^{1-\frac{p}{N}-p}}{T_p(\Omega)\,|\Omega|^{1-\frac{p}{N}-p}}\right)^{\alpha(p,q,N)} - 1.$$

The previous inequality assures that in order to prove a quantitive stability estimate for (1.6), one only has to consider the case $q = 1$, *i.e.* it is sufficient to prove a stability estimate for the p-torsional rigidity. For inequalities (1.6) with $p = 2$, this idea has been crucially exploited in the recent paper [4].

1.5. Plan of the paper

In Section 2 we collect some basic facts we will need througout the whole paper. In the subsequent section we introduce and characterize the *modified torsional rigidity* of a set, which will be the main tool needed to define the Kohler−Jobin symmetrization technique. The latter is described in the crucial Proposition 4.1, which occupies the whole Section 4 and represents the core of the paper. Finally, in Section 5 we give the proof of Theorem 1.1 and draw some consequences. In particular, we will see that Theorem 1.1 is connected with a limit case of functional inequalities of interpolation type. The paper is concluded by Section 6, where we discuss the extension of the Kohler−Jobin procedure to general *anisotropic Dirichlet integrals*, *i.e.* to quantities like

$$\int_\Omega \|\nabla u\|^p\,\mathrm{d}x, \qquad u \in W^{1,p}_0(\Omega),$$

where $\|\cdot\|$ is a strictly convex C^1 norm. In this case as well we can prove the analogous of Theorem 1.1. For ease of exposition, we preferred to treat this kind of generalization in a separate section, so to neatly present the Kohler−Jobin rearrangement avoiding unnecessary technicalities.

2. Preliminaries

The first result we need is very simple, but quite useful in the sequel. It follows at once from Young's inequality.

Lemma 2.1. *Let $A, B > 0$ and $p > 1$, then we have*

$$A\,t - B\,\frac{t^p}{p} \leq \frac{p-1}{p}\,\left(\frac{A^p}{B}\right)^{\frac{1}{p-1}}, \qquad \textit{for every } t \geq 0, \tag{2.1}$$

and equality sign in (2.1) *holds if and only if*

$$t = \left(\frac{A}{B}\right)^{\frac{1}{p-1}}.$$

We define the strictly concave functional

$$\mathfrak{F}_p(u) = \int_\Omega u\,\mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x, \qquad u \in W_0^{1,p}(\Omega),$$

and we denote by $u_\Omega \in W_0^{1,p}(\Omega)$ its unique maximizer. Observe that u_Ω is the unique weak solution of

$$-\Delta_p u = 1, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{2.2}$$

i.e. u_Ω verifies

$$\int_\Omega \langle |\nabla u_\Omega|^{p-2}\,\nabla u_\Omega, \nabla\varphi\rangle\,\mathrm{d}x = \int_\Omega \varphi\,\mathrm{d}x, \quad \text{for every } \varphi \in W_0^{1,p}(\Omega),$$

The next result collects some equivalent definitions for the p-torsional rigidity. In what follows, we denote by $W^{-1,p'}(\Omega)$ the dual space of $W_0^{1,p}(\Omega)$.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set with finite measure. Let us denote by 1_Ω the characteristic function of Ω and set $p' = p/(p-1)$. Then $T_p(\Omega)$ can be equivalently characterized as*

$$T_p(\Omega) = \|1_\Omega\|^p_{W^{-1,p'}(\Omega)}, \tag{2.3}$$

$$T_p(\Omega) = \left(\int_\Omega u_\Omega\,\mathrm{d}x\right)^{p-1} = \left(p' \max_{u\in W_0^{1,p}(\Omega)} \mathfrak{F}_p(u)\right)^{p-1}, \tag{2.4}$$

and also

$$T_p(\Omega) = \left(\min_{V\in L^{p'}(\Omega;\mathbb{R}^N)} \left\{\int_\Omega |V|^{p'}\,\mathrm{d}x \,:\, -\mathrm{div}\,V = 1 \text{ in } \Omega\right\}\right)^{p-1}. \tag{2.5}$$

where the divergence constraint is intended in distributional sense, i.e.

$$\int_\Omega \langle V, \nabla\varphi\rangle\,\mathrm{d}x = \int_\Omega \varphi\,\mathrm{d}x \qquad \textit{for every } \varphi \in W_0^{1,p}(\Omega).$$

Proof. For the first characterization, we just observe that by definition of dual norm we have

$$\|1_\Omega\|_{W^{-1,p'}(\Omega)} = \sup_{v\in W_0^{1,p}(\Omega)} \left\{\int_\Omega v\,\mathrm{d}x \,:\, \|v\|_{W_0^{1,p}(\Omega)} = 1\right\},$$

which immediately gives (2.3), since

$$\max_{v\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\left|\displaystyle\int_\Omega v\,\mathrm{d}x\right|^p}{\displaystyle\int_\Omega |\nabla v|^p\,\mathrm{d}x} = \max_{v\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\left(\displaystyle\int_\Omega |v|\,\mathrm{d}x\right)^p}{\displaystyle\int_\Omega |\nabla v|^p\,\mathrm{d}x} = T_p(\Omega).$$

By testing the equation (2.2) with $\varphi = u_\Omega$, we obtain

$$\int_\Omega |\nabla u_\Omega|^p\,\mathrm{d}x = \int_\Omega u_\Omega\,\mathrm{d}x,$$

so that the maximal value of $\mathfrak{F}_p$ is given by

$$\max_{u\in W_0^{1,p}(\Omega)} \mathfrak{F}_p(u) = \int_\Omega u_\Omega \,\mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla u_\Omega|^p \,\mathrm{d}x = \frac{p-1}{p}\int_\Omega u_\Omega \,\mathrm{d}x.$$

We now prove that the last quantity coincides with $T_p(\Omega)$. Let $v_0 \in W_0^{1,p}(\Omega)$ be a function achieving the supremum in the definition of $T_p(\Omega)$, *i.e.*

$$T_p(\Omega) = \frac{\left(\displaystyle\int_\Omega |v_0| \,\mathrm{d}x\right)^p}{\displaystyle\int_\Omega |\nabla v_0|^p \,\mathrm{d}x}\cdot$$

We notice that v_0 can be taken to be positive. It is not difficult to see that if we set

$$\lambda_0 = \left[\frac{\displaystyle\int_\Omega v_0 \,\mathrm{d}x}{\displaystyle\int_\Omega |\nabla v_0|^p \,\mathrm{d}x}\right]^{\frac{1}{p-1}}.$$

then the function $w_0 = \lambda_0\, v_0 \in W_0^{1,p}(\Omega)$ maximizes $\mathfrak{F}_p$. Indeed, by using Lemma 2.1 and the definition of p-torsional rigidity, for every $v \in W_0^{1,p}(\Omega)$ we get

$$\begin{aligned}\int_\Omega v \,\mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x &\le \max_{\lambda\ge 0}\left[\lambda\int_\Omega |v| \,\mathrm{d}x - \frac{\lambda^p}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x\right] \\ &= \frac{p-1}{p}\left[\frac{\left(\displaystyle\int_\Omega |v| \,\mathrm{d}x\right)^p}{\displaystyle\int_\Omega |\nabla v|^p \,\mathrm{d}x}\right]^{\frac{1}{p-1}} \le \frac{p-1}{p}\, T_p(\Omega)^{\frac{1}{p-1}},\end{aligned}$$

and equality holds in the previous chain of inequalities if $v = \lambda_0\, v_0$. This finally shows that

$$T_p(\Omega) = \left(\frac{p}{p-1}\max_{u\in W_0^{1,p}(\Omega)} \mathfrak{F}_p(u)\right)^{p-1} = \left(\int_\Omega u_\Omega \,\mathrm{d}x\right)^{p-1},$$

thus concluding the proof of (2.4).

The characterization (2.5) is a consequence of the equality

$$\max_{u\in W_0^{1,p}(\Omega)}\left\{\int_\Omega u \,\mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x\right\} = \min_{V\in L^{p'}(\Omega;\mathbb{R}^N)}\left\{\int_\Omega |V|^{p'} \,\mathrm{d}x \,:\, -\mathrm{div}\, V = 1 \text{ in } \Omega\right\},$$

which in turn follows from a standard duality result in Convex Analysis, for which the reader is referred to [8], Proposition 5, page 89. We also recall that the unique vector field V_Ω minimizing the problem on the right-hand side has the form $V_\Omega = |\nabla u_\Omega|^{p-2}\,\nabla u_\Omega$. □

Remark 2.3 (torsional rigidity of a ball)**.** For a ball $B_R(x_0)$ having radius R and center x_0, one can verify that

$$u_{B_R(x_0)}(x) = \frac{\left(R^{\frac{p}{p-1}} - |x-x_0|^{\frac{p}{p-1}}\right)_+}{\beta_{N,p}}, \qquad \text{where} \quad \beta_{N,p} = \frac{p}{p-1}\, N^{\frac{1}{p-1}},$$

is the unique solution of (2.2). Then we get

$$T_p(B_R(x_0)) = \left(\int_{B_R(x_0)} u_{B_R(x_0)}\, \mathrm{d}x\right)^{p-1} = \left[\frac{\omega_N}{\beta_{N,p}}\, \frac{p}{N(p-1)+p}\right]^{p-1} R^{N(p-1)+p}, \tag{2.6}$$

where ω_N is the measure of the N-dimensional unit ball. In what follows, we will set for simplicity

$$\gamma_{N,p} = \left[\frac{\omega_N}{\beta_{N,p}}\, \frac{p}{N(p-1)+p}\right]^{p-1}, \tag{2.7}$$

which just coincides with the p-torsional rigidity of the unit ball in $\mathbb{R}^N$.

We recall some regularity properties of our eigenfunctions, *i.e.* functions achieving the minimal value $\lambda_{p,q}(\Omega)$. These are collected below.

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set with finite measure. Let $u \in W^{1,p}_0(\Omega)$ be a first eigenfunction relative to $\lambda_{p,q}(\Omega)$,* i.e. *a nontrivial solution to*

$$-\Delta_p u = \lambda_{p,q}(\Omega)\, \|u\|^{p-q}_{L^q(\Omega)}\, |u|^{q-2}\, u. \tag{2.8}$$

Then we have $u \in C^1(\Omega) \cap L^\infty(\Omega)$.

Proof. Observe that since the equation is $(p-1)$-homogeneous, we can always scale a solution u in such a way that $\|u\|_{L^q(\Omega)} = 1$. Then the L^∞ bound follows in a standard way by means of a Moser's iteration argument. The C^1 result is a consequence of the by now classical results in [7]. Of course should the boundary of Ω be smooth enough, then this result would be global (see [21]). □

At last, we will need the following particular version of the one-dimensional area formula.

Lemma 2.5. *Let $A > 0$ and $\psi \in \mathrm{Lip}_{loc}([0,A))$ such that $\psi'(t) > 0$ for almost every $t \in [0,A]$. We also set $\sup_{[0,A]} \psi = M$. Let $\varphi = \psi^{-1}$ be its inverse function, then we have the change of variable formula*

$$\int_{\psi(0)}^{M} F(\varphi(t))\, \varphi'(t)\, \mathrm{d}t = \int_0^A F(s)\, \mathrm{d}s,$$

for any non-negative Borel function F.

Proof. The statement is known to be true if $\psi \in \mathrm{Lip}([0,A])$, see [1], Example 3.4.5 and Theorem 3.4.6. To prove it under our slightly weaker hypotheses, we just have to use an approximation argument. For every $\varepsilon > 0$ sufficiently small, we have

$$\int_{\psi(0)}^{\psi(A-\varepsilon)} F(\varphi(t))\, \varphi'(t)\, \mathrm{d}t = \int_0^{A-\varepsilon} F(s)\, \mathrm{d}s,$$

since $\psi \in \mathrm{Lip}([0, A-\varepsilon])$ and $\psi' > 0$ almost everywhere. It is now sufficient to let ε go to 0 and observe that all the functions involved are positive. □

3. The modified torsional rigidity

Definition 3.1. Given an open set $\Omega \subset \mathbb{R}^N$ having finite measure, we will say that $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a *reference function for* Ω if $u \geq 0$ in Ω and

$$t \mapsto \frac{\mu(t)}{\displaystyle\int_{\{u=t\}} |\nabla u|^{p-1}\, \mathrm{d}\mathcal{H}^{N-1}} \in L^\infty([0,M]), \tag{3.1}$$

where $M = \|u\|_{L^\infty(\Omega)}$ and μ denotes the *distribution function* of u, *i.e.* the function defined by

$$\mu(t) = |\{x \in \Omega \,:\, u(x) > t\}|, \qquad t \in [0,M].$$

We will denote by $\mathcal{A}_p(\Omega)$ the set of all reference functions for Ω, *i.e.*

$$\mathcal{A}_p(\Omega) = \Big\{ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \,:\, u \geq 0 \text{ and } (3.1) \text{ holds} \Big\}.$$

We will see in a while the importance of condition (3.1). Firstly, let us consider a particular class of functions which verify it. The next result is somehow classical, related computations can be found in [26].

Lemma 3.2. *Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a positive function such that*

$$-\Delta_p u = f(x,u), \qquad \text{in } \Omega,$$

in a weak sense, where $f : \Omega \times [0,\infty) \to [0,\infty)$ verifies:

(i) *$t \mapsto f(x,t)$ is increasing, for almost every $x \in \Omega$;*
(ii) *for every $t \geq 0$, there exists $L = L(t)$ such that we have $\sup_{x\in\Omega} f(x,t) \leq L$;*
(iii) *for every $t > 0$, we have $\inf_{x\in\Omega} f(x,t) > 0$.*

We set $M = \|u\|_{L^\infty(\Omega)}$, then we have

$$\int_{\{x\,:\,u(x)=t\}} |\nabla u|^{p-1}\, \mathrm{d}\mathcal{H}^{N-1} = \int_{\{x\,:\,u(x)>t\}} f(x,u)\, \mathrm{d}x, \qquad \text{for a.e. } t \in [0,M]. \tag{3.2}$$

In particular, we get $u \in \mathcal{A}_p(\Omega)$, i.e. *$u$ verifies* (3.1).

Proof. By using test functions of the form $(u-s)_+$ in the equation solved by u, we get

$$\begin{aligned}\int_{\{x\,:\,u(x)>s\}} |\nabla u|^p\, \mathrm{d}x &= \int_\Omega \langle |\nabla u|^{p-2}\, \nabla u, \nabla (u-s)_+ \rangle\, \mathrm{d}x \\ &= \int_\Omega f(x,u)\, (u-s)_+\, \mathrm{d}x.\end{aligned}$$

On the other hand, by Coarea Formula we have

$$\int_{\{x\,:\,u(x)>s\}} |\nabla u|^p\, \mathrm{d}x = \int_s^M \left(\int_{\{x\,:\,u(x)=\tau\}} |\nabla u|^{p-1}\, \mathrm{d}\mathcal{H}^{N-1} \right) \mathrm{d}\tau.$$

By taking first $s = t$ and then $s = t + h$ with $h > 0$ and subtracting, we then get

$$\frac{1}{h} \int_t^{t+h} \left(\int_{\{x\,:\,u(x)=\tau\}} |\nabla u|^{p-1}\, \mathrm{d}\mathcal{H}^{N-1} \right) \mathrm{d}\tau = \int_\Omega f(x,u)\, \frac{(u-t)_+ - (u-t-h)_+}{h}\, \mathrm{d}x.$$

By passing to the limit on both sides, we conclude the proof of (3.2).

To prove that u satisfies (3.1), it is sufficient to observe that

$$\int_{\{x\,:\,u(x)>t\}} f(x,u)\,\mathrm{d}x \geq \left(\inf_{y\in\Omega} f(y,t)\right) \int_{\{x\,:\,u(x)>t\}} \mathrm{d}x = \left(\inf_{y\in\Omega} f(y,t)\right) \mu(t),$$

thanks to the monotonicity of f, so that

$$0 \leq \frac{\mu(t)}{\displaystyle\int_{\{u=t\}} |\nabla u|^{p-1}\,\mathrm{d}\mathcal{H}^{N-1}} \leq \frac{1}{\displaystyle\inf_{y\in\Omega} f(y,t)},$$

which means that the quantity we are considering stays bounded whenever t is away from 0, for example if $t \geq M/2$. On the other hand if $t < M/2$ we just notice that $f \geq 0$ implies

$$\int_{\{x\,:\,u(x)>t\}} f(x,u)\,\mathrm{d}x \geq \int_{\{x\,:\,u(x)>M/2\}} f(x,u)\,\mathrm{d}x,$$

and then again we may proceed as before, to obtain that $\int_{\{u=t\}} |\nabla u|^{p-1}$ is bounded from below by a strictly positive constant, for $t < M/2$. This and the fact that $\mu(t) \leq |\Omega|$ allow to conclude. □

Remark 3.3. As an example of function f satisfying the hypotheses of the previous Lemma, we may consider $f(x,t) = g(x)\,t^m$, with $m > 0$ and g positive measurable function such that $c_2 \geq g(x) \geq c_1 > 0$, for almost every $x \in \Omega$.

The next counterexample shows that smooth functions may fail to verify (3.1).

Example 3.4. Let us take $B = \{x\,:\,|x| < 1\}$ and a smooth radial function u such that

$$u(x) = (1-|x|)^\alpha \qquad \text{for } |x| \simeq 1,$$

where $\alpha > 1$. We then have

$$\int_{\{u=t\}} |\nabla u|^{p-1}\,\mathrm{d}\mathcal{H}^{N-1} \simeq t^{\frac{(\alpha-1)\,(p-1)}{\alpha}}\,(1-t^{1/\alpha})^{N-1}, \qquad t \simeq 0,$$

and

$$\mu(t) = \{x\,:\,(1-|x|)^\alpha > t\} = \{x\,:\,|x| < 1 - t^{1/\alpha}\} \simeq (1-t^{1/\alpha})^N, \qquad t \simeq 0.$$

This implies that the ratio of the two quantities is unbounded for t approaching 0. Moreover, this ratio behaves like $t^{\frac{(\alpha-1)\,(1-p)}{\alpha}}$, which may even fail to be merely integrable.

We introduce the following set of Lipschitz functions

$$\mathcal{L} = \Big\{ g \in \mathrm{Lip}\,([0,M])\,:\,g(0) = 0 \Big\},$$

then for every reference function $u \in \mathcal{A}_p(\Omega)$, we clearly have $g \circ u \in W^{1,p}_0(\Omega)$ whenever $g \in \mathcal{L}$. Taking advantage of the equivalent formulations of $T_p(\Omega)$ provided by Proposition 2.2, we define the *modified p-torsional rigidity of Ω according to u* by

$$T_{\mathrm{p,mod}}(\Omega;u) := \left(\frac{p}{p-1}\,\sup_{g\in\mathcal{L}} \mathfrak{F}_p(g\circ u)\right)^{p-1}, \tag{3.3}$$

and notice that since we restricted the class of admissible functions, we decreased the value of $T_p(\Omega)$, *i.e.*

$$T_{\mathrm{p,mod}}(\Omega;u) \leq T_p(\Omega).$$

The key point is that for every $u \in \mathcal{A}_p(\Omega)$ the modified torsional rigidity is well-defined, *i.e.* the supremum is attained in the class $\mathcal{L}$. Moreover it can be fully characterized in terms of the distribution function of u and of the Coarea factor $\int_{\{u=t\}} |\nabla u|^{p-1}$.

Proposition 3.5. *Let $u \in \mathcal{A}_p(\Omega)$ be a reference function for Ω. Then the function g_0 defined by*

$$g_0(t) = \int_0^t \left(\frac{\mu(\tau)}{\displaystyle\int_{\{u=\tau\}} |\nabla u|^{p-1} \, \mathrm{d}\mathcal{H}^{N-1}} \right)^{\frac{1}{p-1}} \mathrm{d}\tau, \tag{3.4}$$

(uniquely) achieves the supremum in (3.3). *The modified torsional rigidity is then given by*

$$T_{\mathrm{p,mod}}(\Omega; u) = \left[\int_0^M \frac{\mu(t)^{\frac{p}{p-1}}}{\left(\displaystyle\int_{\{u=t\}} |\nabla u|^{p-1} \, \mathrm{d}\mathcal{H}^{N-1} \right)^{\frac{1}{p-1}}} \, \mathrm{d}t \right]^{p-1}. \tag{3.5}$$

Proof. We first observe that the value of $T_{\mathrm{mod}}(\Omega; u)$ remains unchanged if we restrict the optimization to positive non-decreasing functions. Indeed, let $g \in \mathcal{L}$ be admissible and let us set

$$\widetilde{g}(t) = \int_0^t |g'(\tau)| \, \mathrm{d}\tau, \qquad t \in [0, M],$$

then this is non-decreasing by construction and obviously $g \in \mathcal{L}$. It satisfies

$$\int_\Omega |\nabla g \circ u(x)|^p \, \mathrm{d}x = \int_\Omega |\nabla \widetilde{g} \circ u(x)|^p \, \mathrm{d}x, \qquad \text{since} \quad \widetilde{g}'(t) = |g'(t)|, \quad \text{for a.e. } t \in [0, M].$$

We also notice that we have $\widetilde{g}(t) \geq g(t)$, so that we can simply infer

$$\int_\Omega g(u(x)) \, \mathrm{d}x \leq \int_\Omega \widetilde{g}(u(x)) \, \mathrm{d}x,$$

which finally implies

$$\mathfrak{F}_p(g \circ u) \leq \mathfrak{F}_p(\widetilde{g} \circ u).$$

We then observe that for every positive non-decreasing $g \in \mathcal{L}$, we have

$$\begin{aligned}
\mathfrak{F}_p(g \circ u) &= \int_\Omega g(u(x)) \, \mathrm{d}x - \frac{1}{p} \int_\Omega g'(u(x))^p \, |\nabla u(x)|^p \, \mathrm{d}x \\
&= \int_0^M \left[g'(t) \, \mu(t) \, \mathrm{d}t - \frac{g'(t)^p}{p} \left(\int_{\{u=t\}} |\nabla u|^{p-1} \, \mathrm{d}\mathcal{H}^{N-1} \right) \right] \mathrm{d}t,
\end{aligned}$$

where we used Cavalieri's principle in the first integral and Coarea Formula in the second one. By using again Lemma 2.1, we then get

$$\begin{aligned}
\mathfrak{F}_p(g \circ u) &\leq \int_0^M \max_{s \geq 0} \left[s \, \mu(t) - \frac{s^p}{p} \left(\int_{\{u=t\}} |\nabla u|^{p-1} \, \mathrm{d}\mathcal{H}^{N-1} \right) \right] \mathrm{d}t \\
&= \frac{p-1}{p} \int_0^M \frac{\mu(t)^{\frac{p}{p-1}}}{\left(\displaystyle\int_{\{u=t\}} |\nabla u|^{p-1} \, \mathrm{d}\mathcal{H}^{N-1} \right)^{\frac{1}{p-1}}} \, \mathrm{d}t,
\end{aligned} \tag{3.6}$$

and equality holds if and only if

$$g'(t) = \left(\frac{\mu(t)}{\displaystyle\int_{\{u=t\}} |\nabla u|^{p-1}\, \mathrm{d}\mathcal{H}^{N-1}} \right)^{\frac{1}{p-1}}, \qquad \text{for a.e. } t \in [0, M].$$

Observe that the latter is an L^∞ function on $[0, M]$, since u satisfies (3.1) by hypothesis. This means the function g_0 defined by

$$g_0(t) = \int_0^t \left(\frac{\mu(\tau)}{\displaystyle\int_{\{u=\tau\}} |\nabla u|^{p-1}\, \mathrm{d}\mathcal{H}^{N-1}} \right)^{\frac{1}{p-1}} \mathrm{d}t,$$

belongs to $\mathcal{L}$ and is thus the unique maximizer of $\mathfrak{F}_p$. In particular

$$\left(\frac{p}{p-1}\, \mathfrak{F}_p(g_0 \circ u) \right)^{p-1} = T_{\mathrm{mod}}(\Omega; u).$$

Finally, the previous equation and (3.6) show the validity of the expression (3.5). $\square$

Remark 3.6. The previous result generalizes [19], Lemme 1. Observe that our proof runs similarly to that in [19], but the use of Cavalieri's principle and Lemma 2.1 permitted some simplifications.

Remark 3.7. If $\Omega \subset \mathbb{R}^2$ is a convex polygon and we take as reference function the distance d_Ω from $\partial\Omega$, the corresponding modified p-torsional rigidity has been recently considered in [13], in connection with a conjecture by Pólya and Szegő.

The following result is an isoperimetric inequality for the modified torsional rigidity. This fact will be crucially exploited in the next section, in order to define our spherical rearrangement.

Proposition 3.8. *Let $\Omega \subset \mathbb{R}^N$ be an open set with finite measure and $u \in \mathcal{A}_p(\Omega)$ be a reference function. If $B \subset \mathbb{R}^N$ is a ball such that*

$$T_p(B) = T_{\mathrm{p,mod}}(\Omega; u),$$

then we have

$$|B| \le |\Omega|.$$

We have equality if and only if Ω is a ball and u coincides with a (translate of a) radially symmetric function.

Proof. We just observe that $T_p(B) = T_{\mathrm{p,mod}}(\Omega; u) \le T_p(\Omega)$, then by using (1.7) we get

$$1 \le \frac{T_p(\Omega)}{T_p(B)} \le \left(\frac{|\Omega|}{|B|} \right)^{\frac{p}{N}+p-1},$$

which proves the first assertion.

As for equality cases, if $|B| = |\Omega|$ by appealing to the equality cases in (1.7) we can surely infer that Ω has to be a ball. Moreover, in this case we also have

$$T_{\mathrm{p,mod}}(\Omega; u) = T_p(\Omega).$$

We now recall that the function achieving the p-torsional rigidity is unique, up to a renormalization, and that such a function has to be radial for a ball (see Rem. 2.3). This implies that for the optimal g_0 achieving $T_{\mathrm{p,mod}}(\Omega; u)$, we must have that $g_0 \circ u$ is radial as well which finally implies that u has to be radial.

On the other hand, it is easily seen that if Ω itself is a ball and u is radial, then $T_{\mathrm{p,mod}}(\Omega; u) = T_p(\Omega)$ and the equality $T_p(\Omega) = T_p(B)$ implies $|\Omega| = |B|$, since two balls have the same p-torsional rigidity if and only if they share the same radius. $\square$

Remark 3.9. The previous result can be rewritten in scaling invariant form as

$$|B|^{1-\frac{p}{N}-p}\, T_p(B) \geq |\Omega|^{1-\frac{p}{N}-p}\, T_{\mathrm{p,mod}}(\Omega; u),$$

with equality if and only if Ω is a ball and u is a radial function. This result generalizes the first inequality[3] appearing in [19], Lemme 2. See also [18], Corollary 1.

4. The Kohler–Jobin rearrangement technique

We are now going to describe the Kohler–Jobin rearrangement for $W_0^{1,p}$ functions that satisfy property (3.1). We recalled that in the classical Schwarz symmetrization it is the measure of superlevel sets which plays the leading role in the rearrangement procedure. Now, it is their p-torsional rigidity which will do the job. This is natural, since we are dealing with a shape optimization problem with a constraint on the torsional rigidity, rather than on the measure of admissible sets.

Proposition 4.1. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set having finite measure. Given a reference function $u \in \mathcal{A}_p(\Omega)$, let B be the ball centered at the origin such that*

$$T_{\mathrm{p,mod}}(\Omega; u) = T_p(B).$$

Then there exists a radially symmetric decreasing function $u^* \in W_0^{1,p}(B)$ such that

$$\int_B |\nabla u^*|^p \,\mathrm{d}x = \int_\Omega |\nabla u|^p \,\mathrm{d}x \qquad \text{and} \qquad \int_B f(u^*) \,\mathrm{d}x \geq \int_\Omega f(u) \,\mathrm{d}x, \tag{4.1}$$

for every function $f : [0,\infty) \to [0,\infty)$ C^1 strictly convex and such that $f(0) = 0$.

Proof. In order to simplify the notation, in the whole proof we will use the notation

$$\mathcal{T} = T_{\mathrm{p,mod}}(\Omega; u).$$

For every $t \in [0, M]$, we will also set

$$\Omega_t = \{x \in \Omega \,:\, u(x) > t\} \qquad \text{and} \qquad T(t) = T_{\mathrm{p,mod}}(\Omega_t; (u-t)_+),$$

i.e. the latter is the modified torsional rigidity of Ω_t according to the function $(u-t)_+$. Since we have

$$|\{x \,:\, (u(x)-t)_+ > s\}| = \mu(t+s),$$

from (3.5) we can infer the explicit expression of this modified torsion, *i.e.*

$$\begin{aligned} T(t) &= \left[\int_0^{M-t} \frac{\mu(s+t)^{\frac{p}{p-1}}}{\left(\int_{\{(u-t)_+=s\}} |\nabla u|^{p-1}\,\mathrm{d}\mathcal{H}^{N-1}\right)^{\frac{1}{p-1}}}\,\mathrm{d}s\right]^{p-1} \\ &= \left[\int_t^{M} \frac{\mu(\tau)^{\frac{p}{p-1}}}{\left(\int_{\{u=\tau\}} |\nabla u|^{p-1}\,\mathrm{d}\mathcal{H}^{N-1}\right)^{\frac{1}{p-1}}}\,\mathrm{d}\tau\right]^{p-1}. \end{aligned}$$

[3] The reader should notice that when $p = 2$, our definition of torsional rigidity differs from that in [19] by a multiplicative factor 4.

Observe that from the previous expression we obtain that $T \in \mathrm{Lip}_{loc}([0,M))$, such that $T'(t) < 0$ almost everywhere on $[0,M]$, since we have

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t) = -(p-1) \frac{\mu(t)^{\frac{p}{p-1}}}{\left(\displaystyle\int_{\{u=t\}} |\nabla u|^{p-1} \, \mathrm{d}\mathcal{H}^{N-1} \right)^{\frac{1}{p-1}}} \, T(t)^{\frac{p-2}{p-1}}, \qquad \text{for a.e. } t.$$

This is useful, since we are going to write the L^p norm of ∇u in terms of the "variable" $T(t)$. More precisely, we first observe that applying the Coarea Formula and then introducing a change of variable $\varphi : [a,b] \to [0,M]$, we have

$$\int_\Omega |\nabla u|^p \, \mathrm{d}x = \int_a^b \varphi'(\tau) \left(\int_{\{u=\varphi(\tau)\}} |\nabla u|^{p-1} \, \mathcal{H}^{N-1} \right) \mathrm{d}\tau. \tag{4.2}$$

As function φ we just take $\varphi : [0,\mathcal{T}] \to [0,M]$ defined by the inverse function

$$\varphi(\tau) = T^{-1}(\tau), \qquad \tau \in [0,\mathcal{T}],$$

then the derivative $\varphi'(\tau)$ is obviously given by

$$\varphi'(\tau) = -\frac{1}{p-1} \frac{\left(\displaystyle\int_{\{u=\varphi(\tau)\}} |\nabla u|^{p-1} \, \mathrm{d}\mathcal{H}^{N-1} \right)^{\frac{1}{p-1}}}{\mu(\varphi(\tau))^{\frac{p}{p-1}}} \, \tau^{\frac{2-p}{p-1}}.$$

Also observe that φ satisfies the hypotheses of Lemma 2.5, then we finally obtain

$$\int_\Omega |\nabla u|^p \, \mathrm{d}x = (p-1)^{p-1} \int_0^{\mathcal{T}} \tau^{p-2} \, \mu(\varphi(\tau))^p \, (-\varphi'(\tau))^p \, \mathrm{d}\tau. \tag{4.3}$$

The finiteness of the integral in the left-hand side and the previous identity justify the convergence of the right-hand side in (4.3), also for $1 < p < 2$.

We now define a radially symmetric decreasing function $u^* \in W_0^{1,p}(B)$. As before, the idea is to prescribe the values of u^* by using the torsional rigidity of its superlevel sets. For every $\tau \in [0,\mathcal{T}]$, let $R(\tau)$ be the unique radius such that the ball $B_{R(\tau)} = \{x \, : \, |x| < R(\tau)\} \subset B$ has torsional rigidity τ, *i.e.* by using formula (2.6) we have

$$R(\tau) = \left(\frac{\tau}{\gamma_{N,p}} \right)^{\frac{1}{N(p-1)+p}}, \qquad \tau \in [0,\mathcal{T}],$$

where the constant $\gamma_{N,p}$ is defined in (2.7). Then we introduce the change of variable $\psi : [0,\mathcal{T}] \to [0,+\infty)$ and we set

$$u^*(x) = \psi(\tau), \qquad \text{if } |x| = R(\tau).$$

In other words, u^* attains the value $\psi(\tau)$ on the boundary of a ball whose torsional rigidity coincides with τ. Of course, the function u^* will be completely determined, once we will specify the function ψ.

We can write the Dirichlet integral of u^* as before, that is using τ as variable. This yields

$$\int_B |\nabla u^*|^p \, \mathrm{d}x = -\int_0^{\mathcal{T}} \psi'(\tau) \left(\int_{\{u^*=\psi(\tau)\}} |\nabla u^*|^{p-1} \, \mathrm{d}\mathcal{H}^{N-1} \right) \mathrm{d}\tau.$$

We then observe that for u^* by construction we have

$$|\nabla u^*(x)| = (N(p-1)+p) \, \gamma_{N,p} \, (-\psi'(\tau)) \, R(\tau)^{N(p-1)+p-1}, \qquad \text{if } |x| = R(\tau),$$

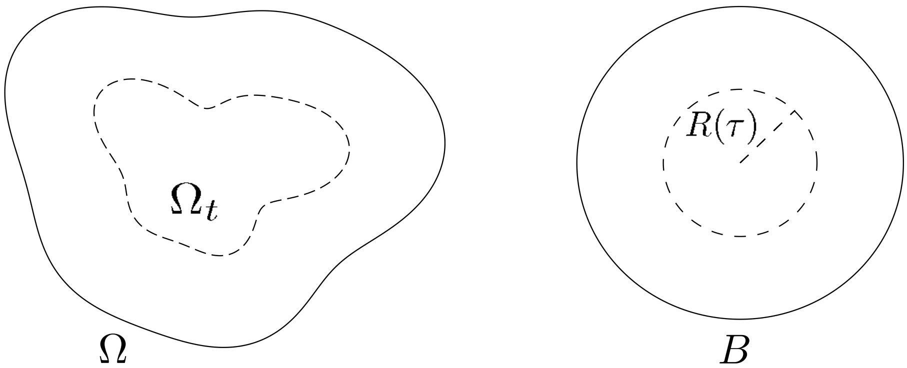

FIGURE 1. The construction of Proposition 4.1: the set Ω_t has modified torsional rigidity equal to τ. On the circle having torsional rigidity equal to τ, we set u^* to be equal to the value $\psi(\tau)$ defined through (4.5).

so that after some (tedious) computations we get

$$\int_{\{u^*=\psi(\tau)\}} |\nabla u^*|^{p-1}\,\mathrm{d}\mathcal{H}^{N-1} = (p-1)^{p-1}\,\tau^{p-2}\,\mu_*(\psi(\tau))^p\,(-\psi'(\tau))^p,$$

i.e. we can infer again

$$\int_B |\nabla u^*|^p\,\mathrm{d}x = (p-1)^{p-1}\int_0^{\mathcal{T}} \tau^{p-2}\,\mu_*(\psi(\tau))^p\,(-\psi'(\tau))^p\,\mathrm{d}\tau, \tag{4.4}$$

where μ_* is the distribution function of u^*.

We are finally ready to define ψ: we impose

$$\begin{cases} (-\psi'(\tau))\,\mu_*(\psi(\tau)) = (-\varphi'(\tau))\,\mu(\varphi(\tau)) \\ \psi(\mathcal{T}) = 0. \end{cases} \tag{4.5}$$

By recalling that by construction we have

$$\mu_*(\psi(\tau)) = \omega_N\,R(\tau)^N = \omega_N\,\left(\frac{\tau}{\gamma_{N,p}}\right)^{\frac{N}{N(p-1)+p}},$$

the change of variable ψ is explicitely given by

$$\psi(\tau) = \frac{\gamma_{N,p}^{\frac{N}{N(p-1)+p}}}{\omega_N}\int_\tau^{\mathcal{T}} (-\varphi'(s))\,\mu(\varphi(s))\,s^{-\frac{N}{N(p-1)+p}}\,\mathrm{d}s, \qquad \tau\in[0,\mathcal{T}].$$

By using the information (4.5) in (4.3) and (4.4), we immediately obtain

$$\int_B |\nabla u^*|^p\,\mathrm{d}x = \int_\Omega |\nabla u|^p\,\mathrm{d}x,$$

as desired.

As for the integrals of u, first of all we observe that thanks to Proposition 3.8 we have

$$\mu_*(\psi(\tau)) \le \mu(\varphi(\tau)), \qquad \text{for every } \tau\in[0,M],$$

since the torsional rigidity of the ball $\{x \in B : u^*(x) > \psi(\tau)\}$ is equal to the modified torsional rigidity of $\{x \in \Omega : u(x) > \varphi(\tau)\}$. Then (4.5) implies

$$-\psi'(\tau) \geq -\varphi'(\tau), \qquad \text{for a.e. } \tau \in [0, \mathcal{T}],$$

thus integrating we get

$$\psi(\tau) = -\int_\tau^{\mathcal{T}} \psi'(s)\,\mathrm{d}s \geq -\int_\tau^{\mathcal{T}} \varphi'(s)\,\mathrm{d}s = \varphi(\tau), \qquad \tau \in [0, \mathcal{T}], \tag{4.6}$$

since $\psi(\mathcal{T}) = \varphi(\mathcal{T}) = 0$. Once again Cavalieri's principle and a change of variable gives

$$\begin{aligned}
\int_\Omega f(u)\,\mathrm{d}x &= \int_0^M f'(t)\,\mu(t)\,\mathrm{d}t \\
&= \int_0^{\mathcal{T}} f'(\varphi(\tau))\,(-\varphi'(\tau))\,\mu(\varphi(\tau))\,\mathrm{d}\tau \\
&\leq \int_0^{\mathcal{T}} f'(\psi(\tau))\,(-\psi'(\tau))\,\mu_*(\psi(\tau))\,\mathrm{d}\tau = \int_B f(u^*)\,\mathrm{d}x,
\end{aligned}$$

where we exploited (4.5), (4.6) and the strict convexity of f. This finally concludes the proof. $\square$

Remark 4.2 (equality cases)**.** Observe that in the previous construction we have

$$\mu_*(\psi(\tau)) = \mu(\varphi(\tau)),$$

if and only if the superlevel set $\{x \in \Omega : u(x) > \varphi(\tau)\}$ is a ball and $(u - \varphi(\tau))_+$ is radial, thanks to the equality cases in Proposition 3.8. This implies that equality holds in (4.6) for almost every $\tau \in [0, \mathcal{T}]$ if and only if Ω is a ball and u is a radial function. This finally gives that for $f : [0, \infty) \to [0, \infty)$ strictly convex and such that $f(0) = 0$, we have

$$\int_\Omega f(u)\,\mathrm{d}x = \int_B f(u^*)\,\mathrm{d}x,$$

if and only if Ω is a ball and u is a radial function.

Remark 4.3 (assumptions on f)**.** Observe that the strict convexity f is not really necessary for (4.1) to hold, the argument still works with an f convex. But in this case the identification of equality cases is lost. On the other hand, the condition $f(0) = 0$ is vital, since one has

$$\int_\Omega f(u)\,\mathrm{d}x \leq \int_B f(u^*)\,\mathrm{d}x + f(0)\,\Big[|\Omega| - |B|\Big],$$

and by construction we have $|\Omega| \geq |B|$. If $f(0) > 0$, the latter inequality does not permit to say that the integral $\int_\Omega f(u)\,\mathrm{d}x$ is increased after the rearrangement.

5. Proof of Theorem 1.1

Let $v_\Omega \in W_0^{1,p}(\Omega)$ be a function such that

$$\frac{\|\nabla v_\Omega\|^p_{L^p(\Omega)}}{\|v_\Omega\|^p_{L^q(\Omega)}} = \lambda_{p,q}(\Omega).$$

Since the value of the Rayleigh quotient on the left is unchanged if we replace a function v_Ω by its modulus $|v_\Omega|$, we can assume that $v_\Omega \geq 0$. Moreover, the function v_Ω solves equation (2.8), then by Proposition 2.4

and Lemma 3.2, we get immediately that v_Ω is a reference function for Ω, *i.e.* $v_\Omega \in \mathcal{A}_p(\Omega)$. Accordingly, the modified torsional rigidity $T_{\mathrm{p,mod}}(\Omega; v_\Omega)$ is well-defined. Let us simply set for brevity

$$T_\Omega = T_{\mathrm{p,mod}}(\Omega; v_\Omega),$$

and recall that $T_\Omega \le T_p(\Omega)$. We then take B the ball centered at the origin such that

$$T_\Omega = T_p(B),$$

thanks to Proposition 4.1 we can construct $v_\Omega^* \in W_0^{1,p}(B)$ such that

$$\int_B |v_\Omega^*|^q \,\mathrm{d}x \ge \int_\Omega |v_\Omega|^q \,\mathrm{d}x \qquad \text{and} \qquad \int_B |\nabla v_\Omega^*|^p \,\mathrm{d}x = \int_\Omega |\nabla v_\Omega|^p \,\mathrm{d}x.$$

Using this and the definition of $\lambda_{p,q}(B)$, we then obtain

$$\begin{aligned} T_p(B)^{\alpha(p,q,N)}\, \lambda_{p,q}(B) &\le T_\Omega^{\alpha(p,q,N)}\, \frac{\|\nabla v_\Omega^*\|^p_{L^p(B)}}{\|v_\Omega^*\|^p_{L^q(B)}} \\ &\le T_\Omega^{\alpha(p,q,N)}\, \frac{\|\nabla v_\Omega\|^p_{L^p(\Omega)}}{\|v_\Omega\|^p_{L^q(\Omega)}} \le T_p(\Omega)^{\alpha(p,q,N)}\, \lambda_{p,q}(\Omega), \end{aligned} \tag{5.1}$$

which concludes the proof of (1.9).

As for equality cases, we observe that if equality holds in (1.9), then equality holds everywhere in (5.1). In particular we get

$$\frac{\displaystyle\int_B |\nabla v_\Omega^*|^p \,\mathrm{d}x}{\left(\displaystyle\int_B |v_\Omega^*|^q \,\mathrm{d}x\right)^{\frac{p}{q}}} = \frac{\displaystyle\int_\Omega |\nabla v_\Omega|^p \,\mathrm{d}x}{\left(\displaystyle\int_\Omega |v_\Omega|^q \,\mathrm{d}x\right)^{\frac{p}{q}}}.$$

Thanks to Proposition 4.1 and Remark 4.2, we can finally conclude that Ω has to be a ball. This concludes the Proof of Theorem 1.1.

Remark 5.1 (Moser–Trudinger sharp constant)**.** In the conformal case $p = N$, one may wonder what can be said for the best constant in the *Moser–Trudinger inequality* (see [23, 27]), such a constant being defined by

$$MT_N(\Omega) = \sup\left\{\int_\Omega \exp\left(c_N\, |u|^{\frac{N}{N-1}}\right) \,:\, \|\nabla u\|_{W_0^{1,N}(\Omega)} \le 1\right\}, \qquad c_N = N\,(N\,\omega_N)^{\frac{1}{N-1}}.$$

It is immediate to see that this quantity is maximized by balls among sets having given volume, *i.e.*

$$\frac{MT_N(\Omega)}{|\Omega|} \le \frac{MT_N(B)}{|B|}, \tag{5.2}$$

the proof consisting of a straightfoward application of the Schwarz rearrangement. This time, *it is not clear* that balls still maximize with a constraint on the $N-$torsional rigidity. The reason lies in the fact that the function $f(t) = \exp(c_N\, t^{N'})$ verifies $f(0) > 0$. Then by taking an optimal function v_Ω for $MT_N(\Omega)$ (see [12, 22] for the existence of such a function) and applying Proposition 4.1, we would obtain (see Rem. 4.3)

$$MT_N(\Omega) \le MT_N(B_\Omega) + |\Omega| - |B_\Omega|,$$

where B_Ω is a ball such that $T_N(B_\Omega) = T_{\mathrm{N,mod}}(\Omega; v_\Omega)$. In particular B_Ω is smaller than a ball $B^\star$ having the same torsional rigidity as Ω. In order to show that the ball is still a maximizer, it would sufficies to verify that for every set Ω there holds

$$MT_N(B_\Omega) + |\Omega| - |B_\Omega| \le MT_N(B^\star).$$

The previous is in turn equivalent to
$$\frac{|\Omega|-|B_\Omega|}{|B^\star|-|B_\Omega|}\le \frac{M\,T_N(B^\star)}{|B^\star|},$$
an estimate which seems difficult to check, since the measure of B_Ω depends in an intricate way on Ω.

As a straightforward consequence of Theorem 1.1 we get the following functional inequality of interpolation type, with sharp constant. In what follows we will denote by $W^{1,p}_0(\mathbb{R}^N)$ the completion of $C^\infty_0(\mathbb{R}^N)$ with respect to the norm $\|\nabla u\|_{L^p}$.

Corollary 5.2. *Let $1<p<\infty$ and q be an exponent satisfying (1.4). We still denote*
$$\alpha(p,q,N)=\frac{\dfrac{p}{N}+\dfrac{p}{q}-1}{\dfrac{p}{N}+p-1},$$
and $p'=p/(p-1)$, then for every $u\in W^{1,p}_0(\mathbb{R}^N)\cap C(\mathbb{R}^N)$ we have
$$\|u\|_{L^q}\le KJ(p,q,N)\,\left\|1_{\{|u|>0\}}\right\|^{\alpha(p,q,N)}_{W^{-1,p'}}\,\|\nabla u\|_{L^p}, \tag{5.3}$$
where the sharp constant $KJ(p,q,N)$ is given by
$$KJ(p,q,N)=\Big(T(B_1)^{\alpha(p,q,N)}\,\lambda_{p,q}(B_1)\Big)^{-\frac{1}{p}},$$
and B_1 is the N-dimensional unit ball centered at the origin. Equality in (5.3) holds if and only if u has the form
$$u(x)=c\,U\left(\frac{x-x_0}{s}\right)\qquad \text{for some } (x_0,c,s)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^+,$$
where $U\in W^{1,p}_0(B_1)$ is the (unique) function solving
$$\begin{cases}-\Delta_p U=\lambda_{p,q}(B_1)\,U^{q-1}, & \text{in } B_1,\\[2mm] \qquad\|U\|_{L^q}=1 \qquad \text{and} & U>0.\end{cases}$$

Proof. It is sufficient to observe that (1.9) implies
$$T_p(B_1)^{\alpha(p,q,N)}\,\lambda_{p,q}(B_1)\le T_p(\{|u|>0\})^{\alpha(p,q,N)}\,\frac{\displaystyle\int_{\mathbb{R}^N}|\nabla u|^p\,\mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N}|u|^q\,\mathrm{d}x\right)^{\frac{p}{q}}},$$
then we use the characterization (2.3) of the p-torsional rigidity. Equality cases easily follow from those in (1.9). □

Remark 5.3. For simplicity, we stated the previous result for functions in $W^{1,p}_0(\mathbb{R}^N)\cap C(\mathbb{R}^N)$. This assures that $\{|u|>0\}$ is an open set. However, by appealing to the theory of quasi-open sets and of Sobolev spaces in a capacitary sense (see [5]), one could state the previous inequality for general functions in $W^{1,p}_0(\mathbb{R}^N)$

Remark 5.4. Observe that (5.3) is a limit case of the following family of interpolation inequalities
$$\|u\|_{L^q}\le C\,\|\nabla u\|^{\vartheta+\frac{1-\vartheta}{s}}_{L^p}\,\left\||u|^{s-2}\,u\right\|^{\frac{1-\vartheta}{s}}_{W^{-1,p'}}, \tag{5.4}$$

where $1 < s < q < p^*$ and the parameter ϑ is such that

$$\vartheta = \frac{p^*}{q}\,\frac{q-s}{p^*-s}\cdot$$

Here p^* denotes the usual Sobolev embedding exponent, *i.e.* $p = Np/(N-p)$ (let us confine ourselves to the case $1 < p < N$, for simplicity). The proof of (5.4) simply follows by combining the *Gagliardo–Nirenberg–Sobolev inequality*

$$\|u\|_{L^q} \le C\,\|\nabla u\|_{L^p}^{\vartheta}\,\|u\|_{L^s}^{1-\vartheta},$$

and the estimate

$$\|u\|_{L^s} = \left(\int |u|^s\,\mathrm{d}x\right)^{\frac{1}{s}} = \left(\int |u|^{s-2}\,u\,u\,\mathrm{d}x\right)^{\frac{1}{s}} \le \left\||u|^{s-2}\,u\right\|_{W^{-1,p'}}^{\frac{1}{s}}\,\|\nabla u\|_{L^p}^{\frac{1}{s}},$$

which is a plain consequence of the definition of dual norm. By formally taking the limit for s converging to 1 in (5.4), one ends up with (5.3).

6. The case of general norms

In this last section, we will see how to adapt the Kohler−Jobin rearrangement to the case of *anisotropic* principal frequencies and torsional rigidities. The reader could find useful to consult [25] for the basic facts about convex bodies needed below.

Let $1 < p < \infty$ and q still satisfying (1.4), we consider the quantities

$$\lambda_{p,q}^K(\Omega) = \min_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \|\nabla u\|_K^p\,\mathrm{d}x}{\left(\displaystyle\int_\Omega |u|^q\,\mathrm{d}x\right)^{\frac{p}{q}}} \qquad \text{and} \qquad T_p^K(\Omega) = \max_{v\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\left(\displaystyle\int_\Omega |v|\,\mathrm{d}x\right)^{p}}{\displaystyle\int_\Omega \|\nabla v\|_K^p\,\mathrm{d}x},$$

where K is a C^1 centro-symmetric[4] bounded strictly convex set and

$$\|x\|_K = \min\{\lambda \ge 0\,:\, x\in\lambda K\}, \qquad x\in\mathbb{R}^N,$$

i.e. $\|\cdot\|_K$ is the norm having K as unit ball. Of course by taking $K = B$ the Euclidean ball, we are back to the quantities considered in the previous sections. An interesting particular case of these anisotropic variants is when K is the unit ball of the ℓ^p norm, in this case we have

$$\int_\Omega \|\nabla u\|_K^p\,\mathrm{d}x = \sum_{i=1}^N \int_\Omega |u_{x_i}|^p\,\mathrm{d}x.$$

The Faber−Krahn inequality for the corresponding first eigenvalue $\lambda_{p,q}^K(\Omega)$ has been derived in [3], by using the *convex symmetrization* introduced in [2]. The latter is just a variant of the Schwarz symmetrization, where balls are replaced by rescaled copies of the *polar body* K^* defined by

$$K^* = \left\{\xi\in\mathbb{R}^N\,:\,\sup_{x\in K}\langle\xi,x\rangle \le 1\right\}.$$

In other words, given a positive function $u\in W_0^{1,p}(\Omega)$, we can construct a new Sobolev function $u^\#$ supported on a scaled copy $\widetilde{K}^*$ of K^*, such that

$$\{x\,:\,u^\#(x) > t\}\ \text{ is homothetic to } K^* \text{ for every } t,$$

[4]This means that $x\in K$ implies that $-x\in K$ as well.

and u and $u^\#$ are equimeasurable. Moreover, for the convex symmetrization as well we have the Pólya–Szegő principle, *i.e.*

$$\int_\Omega \|\nabla u\|_K^p \,\mathrm{d}x \geq \int_{\widetilde{K}^*} \|\nabla u^\#\|_K^p \,\mathrm{d}x. \tag{6.1}$$

We refer the reader to [2], Theorem 3.1 for the proof. The equality cases are investigated in [9], Theorem 5.1 and [10], Theorem 1.

We also recall the *Wulff inequality*

$$|\Omega|^{-\frac{N-1}{N}} \int_{\partial\Omega} \|\nu_\Omega\|_K \,\mathrm{d}\mathcal{H}^{N-1} \geq |K^*|^{-\frac{N-1}{N}} \int_{\partial K^*} \|\nu_{K^*}\|_K \,\mathrm{d}\mathcal{H}^{N-1}, \tag{6.2}$$

where ν_Ω is the outer normal versor[5] to $\partial\Omega$. Equality holds in the previous if and only if $\Omega = x_0 + s\,K^*$, for some $x_0 \in \mathbb{R}^N$ and $s > 0$. Inequality (6.2) is nothing but a generalization of the classical isoperimetric one and it is of course an essential ingredient of (6.1). The boundary integral appearing in (6.2) is called *anisotropic perimeter* and for K^* we have the simple formula

$$\int_{\partial K^*} \|\nu_{K^*}\|_K \,\mathrm{d}\mathcal{H}^{N-1} = N\,|K^*|, \tag{6.3}$$

as in the Euclidean case. A good reference for (6.2) is the recent paper [11], where stability issues are addressed as well.

By suitably adapting the rearrangement technique of Kohler−Jobin, one can obtain the following generalization of Theorem 1.1.

Theorem 6.1. *Let $1 < p < \infty$ and q satisfying (1.4). Then*

$$T_p^K(\Omega)^{\alpha(p,q,N)}\,\lambda_{p,q}^K(\Omega) \geq T_p^K(K^*)^{\alpha(p,q,N)}\,\lambda_{p,q}^K(K^*), \tag{6.4}$$

and equality holds if and only if Ω coincides with the polar body K^*, up to translations and dilations.

The proof is exactly the same as in the Euclidean case, it is sufficient to use Proposition 6.5 below, which is nothing but the anisotropic counterpart of Proposition 4.1. In the remaining part of the section, we list the main changes needed for the definition of the anisotropic Kohler−Jobin rearrangement and for the proof of its properties.

First of all, we need the expression of the p-torsional rigidity of the "unit ball" K^*.

Lemma 6.2. *The unique solution to the problem*

$$\max_{u\in W_0^{1,p}(K^*)} \int_{K^*} u\,\mathrm{d}x - \frac{1}{p}\int_{K^*} \|\nabla u\|_K^p \,\mathrm{d}x, \tag{6.5}$$

is given by

$$u_{K^*}(x) = \frac{1 - \|x\|_{K^*}^{\frac{p}{p-1}}}{\beta_{N,p}}, \qquad \textit{where} \quad \beta_{N,p} = \frac{p}{p-1}\,N^{\frac{1}{p-1}}.$$

In particular we have

$$T_p^K(K^*) = \left[\frac{|K^*|}{\beta_{N,p}}\,\frac{p}{N(p-1)+p}\right]^{p-1}. \tag{6.6}$$

[5]Here we are a little bit vague about the smoothness assumptions on Ω, since we will not really need this result in what follows. We just mention that (6.2) is naturally settled in the class of set having finite perimeter in the De Giorgi sense.

Proof. The uniqueness of the solution for (6.5) simply follows by the strict concavity of the functional[6]. For simplicity, let us now introduce the notation

$$H(x) = \|x\|_K \qquad \text{and} \qquad H_*(x) = \|x\|_{K^*},$$

and observe that the Euler–Lagrange equation for problem (6.5) is

$$\int_{K^*} H^{p-1}(\nabla u)\,\langle \nabla H(\nabla u), \nabla\varphi\rangle\,\mathrm{d}x = \int_{K^*} \varphi\,\mathrm{d}x, \qquad \varphi \in W_0^{1,p}(K^*).$$

By inserting the function u_{K^*} defined above and using the relations (see [25])

$$H(\nabla H_*(x)) = 1 \qquad \text{and} \qquad \nabla H(\nabla H_*(x)) = \frac{x}{H_*(x)}, \quad x \in K^*, \tag{6.7}$$

we then obtain that u_{K^*} solves this equation and thus is the p-torsion function.

In order to compute the exact value of $T_p^K(K^*)$ we can use the following trick. First of all, by using the expression of u_{K^*} we have

$$\int_{K^*} u_{K^*}\,\mathrm{d}x = \frac{|K^*|}{\beta_{N,p}} - \frac{1}{\beta_{N,p}}\int_{K^*} H_*(x)^{\frac{p}{p-1}}\,\mathrm{d}x.$$

On the other hand, by using u_{K^*} as a test function in the Euler-Lagrange equation and appealing to (6.7), we obtain

$$\int_{K^*} u_{K^*}\,\mathrm{d}x = \int_{K^*} H(\nabla u)^p\,\mathrm{d}x = \left(\frac{p}{p-1}\,\frac{1}{\beta_{N,p}}\right)^p \int_{K^*} H_*(x)^{\frac{p}{p-1}}\,\mathrm{d}x.$$

By comparing the two previous expressions we can compute the value of $\int_{K^*} H_*(x)^{\frac{p}{p-1}}\,\mathrm{d}x$. This finally gives the desired expression for $T_p^K(K^*)$, since

$$T_p^K(K^*) = \left(\int_{K^*} u_{K^*}\,\mathrm{d}x\right)^{p-1},$$

as in Proposition 2.2. □

We can still characterize the modified torsional rigidity in terms of the distribution function and of the (anisotropic) Coarea factor. This is the content of the next result.

Proposition 6.3. *Let $u \in \mathcal{A}_p(\Omega)$ be a reference function for Ω. Then the modified torsional rigidity is given by*

$$T^K_{\mathrm{p,mod}}(\Omega; u) = \left[\int_0^M \frac{\mu(t)^{\frac{p}{p-1}}}{\left(\displaystyle\int_{\{u=t\}} \|\nabla u\|_K^{p-1}\left\|\frac{\nabla u}{|\nabla u|}\right\|_K \mathrm{d}\mathcal{H}^{N-1}\right)^{\frac{1}{p-1}}}\,\mathrm{d}t\right]^{p-1}. \tag{6.8}$$

Proof. It is sufficient to use the Coarea Formula in the following form

$$\int_\Omega \|\nabla u\|_K^p\,\mathrm{d}x = \int_0^M \int_{\{u=t\}} \|\nabla u\|_K^{p-1}\left\|\frac{\nabla u}{|\nabla u|}\right\|_K \mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t.$$

[6] Here enters the assumption of strict convexity on K.

Also observe that since in $\mathbb{R}^N$ all norms are equivalent, if u is a reference function we also have that

$$t \mapsto \frac{\mu(t)}{\displaystyle\int_{\{u=t\}} \|\nabla u\|_K^{p-1} \left\|\frac{\nabla u}{|\nabla u|}\right\|_K \mathrm{d}\mathcal{H}^{N-1}} \in L^\infty([0,M]).$$

These two modifications permit to conclude the proof as before. □

Proposition 6.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set having finite measure and $u \in \mathcal{A}_p(\Omega)$ a reference function. If $\widetilde{K}^* \subset \mathbb{R}^N$ is a scaled copy of K^* such that*

$$T_p^K(\widetilde{K}^*) = T^K_{\mathrm{p,mod}}(\Omega; u),$$

then we have

$$|\widetilde{K}^*| \le |\Omega|.$$

We have equality if and only if Ω is a translated and scaled copy of K^* and u coincides with a (translate of a) function of the form $x \mapsto h(\|x\|_{K^*})$.

Proof. This is a consequence of the Saint-Venant inequality for the anistropic torsional rigidity, *i.e.*

$$|\Omega|^{1-\frac{p}{N}-p}\, T_p^K(\Omega) \le |K^*|^{1-\frac{p}{N}-p}\, T_p^K(K^*),$$

which in turn follows from the Pólya-Szegő principle (6.1) for the convex rearrangement. Thanks to the result of [9,10], equality is attained if and only if $\Omega = x + s\,K^*$, then the proof of the second part of the statement is as in Proposition 3.8. □

Finally, we define the Kohler−Jobin rearrangement, which still keeps the Dirichlet integral fixed and increases the L^q norms of a function. This permits to prove Theorem 6.1.

Proposition 6.5. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set having finite measure. Given a reference function $u \in \mathcal{A}_p(\Omega)$, let $\widetilde{K}^*$ be a scaled copy of the polar body K^* such that*

$$T^K_{\mathrm{p,mod}}(\Omega; u) = T_p^K(\widetilde{K}^*).$$

Then there exists a function $u^* \in W_0^{1,p}(\widetilde{K}^*)$ such that its superlevel sets are scaled copies of K^* and

$$\int_{\widetilde{K}^*} |\nabla u^*|^p \,\mathrm{d}x = \int_\Omega |\nabla u|^p \,\mathrm{d}x \qquad \text{and} \qquad \int_{\widetilde{K}^*} f(u^*)\,\mathrm{d}x \ge \int_\Omega f(u)\,\mathrm{d}x, \tag{6.9}$$

for every function $f : [0,\infty) \to [0,\infty)$ strictly convex and such that $f(0) = 0$.

Proof. The proof runs exactly as in Proposition 4.1, up to some relevant changes that we list below. First of all, we observe that the modified torsional rigity of Ω_t according to $(u-t)_+$ is now given by

$$T^K(t) = \left[\int_t^M \frac{\mu(\tau)^{\frac{p}{p-1}}}{\left(\displaystyle\int_{\{u=t\}} \|\nabla u\|_K^{p-1} \left\|\frac{\nabla u}{|\nabla u|}\right\|_K \mathrm{d}\mathcal{H}^{N-1}\right)^{\frac{1}{p-1}}} \,\mathrm{d}\tau\right]^{p-1}, \qquad t \in [0,M].$$

Then we can infer again

$$\int_\Omega \|\nabla u\|^p \,\mathrm{d}x = (p-1)^{p-1} \int_0^{\mathcal{T}} \tau^{p-2}\, \mu(\varphi(\tau))^p\, (-\varphi'(\tau))^p \,\mathrm{d}\tau, \tag{6.10}$$

where now φ is the inverse function of T^K. As before, we define the new "radial" function

$$u^*(x) = \psi(\tau), \qquad \text{if } \|x\|_{K^*} = R(\tau),$$

where for every $\tau \in [0, T]$ the "radius" $R(\tau)$ is such that

$$T_p^K(R(\tau)\, K^*) = \tau.$$

In other words $R(\tau)\, K^*$ is the unique scaled copy of the polar body K^* having torsional rigidity equal to τ. Observe that from the previous we have the relation

$$R(\tau) = \left(\frac{\tau}{T_p^K(K^*)}\right)^{\frac{1}{N\,(p-1)+p}},$$

and $T_p^K(K)^*$ is the constant depending only on K, N and p given by (6.6). By construction we get

$$\|\nabla u^*(x)\|_K = (N(p-1)+p)\, T_p^K(K^*)\, (-\psi'(\tau))\, R(\tau)^{N(p-1)+p-1}, \qquad \text{if } \|x\|_{K^*} = R(\tau),$$

so that, with a small abuse of notation, we obtain

$$\int_{\{u^*=\psi(\tau)\}} \|\nabla u\|_K^{p-1} \left\|\frac{\nabla u^*}{|\nabla u^*|}\right\|_K \mathrm{d}\mathcal{H}^{N-1} = \|\nabla u^*(R(\tau))\|_K^{p-1} \int_{\{u^*=\psi(\tau)\}} \|\nu\|_K \, \mathrm{d}\mathcal{H}^{N-1}.$$

Here ν is the outer normal to the set $\{u^* > \psi(\tau)\}$, the latter being $R(\tau)\, K^*$. Then the integral on the right-hand side is nothing but the anisotropic perimeter of this set, which is homothetic to K^*. By (6.3) we can infer

$$\int_{\{u^*=\psi(\tau)\}} \|\nu\|_K \, \mathrm{d}\mathcal{H}^{N-1} = N\, R(\tau)^{N-1}\, |K^*|.$$

By keeping everything together, we get

$$\begin{aligned}\int_{\{u^*=\psi(\tau)\}} \|\nabla u\|_K^{p-1} \left\|\frac{\nabla u^*}{|\nabla u^*|}\right\|_K \mathrm{d}\mathcal{H}^{N-1} &= (N(p-1)+p)^{p-1}\, T_p^K(K^*)^{p-1}\, (-\psi'(\tau))^{p-1} \\ &\quad \times\, R(\tau)^{(N+1)(p-1)^2+N-1}\, N\, |K^*|,\end{aligned}$$

so that after some some simplifications we obtain

$$\int_B |\nabla u^*|^p \, \mathrm{d}x = (p-1)^{p-1} \int_0^{\mathcal{T}} \tau^{p-2}\, \mu_*(\psi(\tau))^p\, (-\psi'(\tau))^p \, \mathrm{d}\tau.$$

Then we can define once again ψ through (4.5). The resulting function u^* has the desired properties, the proof being exactly the same as in Proposition 4.1. □

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