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$\mathcal{C}^{1,\beta}$ REGULARITY FOR DIRICHLET PROBLEMS ASSOCIATED TO FULLY NONLINEAR DEGENERATE ELLIPTIC EQUATIONS

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Abstract. We prove Hölder regularity of the gradient, up to the boundary for solutions of some fully-nonlinear, degenerate elliptic equations, with degeneracy coming from the gradient.

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1. Introduction

In a recent paper Imbert and Silvestre [14] have proved that the solutions of

$$|\nabla u|^\alpha F(D^2u) = f(x) \text{ in } \Omega \subset \mathbb{R}^N \tag{1.1}$$

have first derivative which are Hölder continuous in the interior of Ω when $\alpha \geq 0$, F is uniformly elliptic and f is continuous.

Results concerning regularity of solutions have an intrinsic interest which doesn't need to be explained. When $\alpha = 0$, it is known (see *e.g.* Evans [11], Cabré, Caffarelli [7–9]) that u is $C^{1,\beta}$ for some $\beta \in (0,1)$. But for solutions of (1.1) with $\alpha > -1$ the question of the continuity of the gradient was open and it was naturally raised, in [5]. In fact, the question raised concerned precisely regularity near the boundary. Let us recall that context. The values

$$\mu^+ = \sup\left\{\mu, \exists\phi > 0 \text{ in } \Omega, |\nabla\phi|^\alpha F(D^2\phi) + \mu\phi^{1+\alpha} \leq 0 \quad \text{in } \Omega\right\},$$

$$\mu^- = \sup\left\{\mu, \exists\psi < 0 \text{ in } \Omega, |\nabla\psi|^\alpha F(D^2\psi) + \mu|\psi|^\alpha\psi \geq 0 \quad \text{in } \Omega\right\}$$

are generalised principal eigenvalues in the sense that there exists a non trivial solution to the Dirichlet problem

$$|\nabla\phi|^\alpha F(D^2\phi) + \mu^\pm|\phi|^\alpha\phi = 0 \quad \text{in} \quad \Omega,\ \phi = 0 \quad \text{on} \quad \partial\Omega,$$

with constant sign.

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The main scope in [5] was to prove the simplicity of these principal eigenvalues. The main difficulty comes from the fact that the strong comparison principle holds only in open subsets of Ω where the gradient is away from zero (in the viscosity sense). It is well known that the Hopf Lemma guarantees that this is true on $\partial\Omega$. So that the continuity of the gradient up to the boundary would imply that, in a neighbourhood of it, the strong comparison principle holds, which is exactly what is needed to prove that the eigenvalues are simple.

In that same paper, we proved that if $\alpha \in (-1, 0)$ the solutions of the Dirichlet problem associated to (1.1) are indeed $C^{1,\beta}$ and we raised the problem of whether that regularity would hold also for $\alpha \geq 0$ *i.e.* when the operator is degenerate elliptic.

[14] was a first answer in that direction, very much inspired by that breakthrough, we wanted to complete the work. We want to point out that a difficulty specific to this type of equation is due to the fact that the difference of two solutions is not a sub- or super solution to another equation, which would provide regularity.

Recall that F is uniformly elliptic if there exists $\Lambda \geq \lambda > 0$ such that for any symmetric matrices M and N

$$F(M) + \mathcal{M}^-_{\lambda,\Lambda}(N) \leq F(M+N) \leq F(M) + \mathcal{M}^+_{\lambda,\Lambda}(N) \tag{1.2}$$

here we have denoted the Pucci operators $\mathcal{M}^-_{\lambda,\Lambda}(N) = \lambda tr(N^+) + \Lambda tr(N^-)$ and $\mathcal{M}^+_{\lambda,\Lambda}(M) = \lambda tr(M^-) + \Lambda tr(M^+)$. In the rest of the paper we shall drop the indices λ and Λ of the Pucci operators.

We now state the main result of the paper in its generality.

Theorem 1.1. *Suppose that Ω is a bounded $\mathcal{C}^2$ domain of $\mathbb{R}^N$ and $\alpha \geq 0$. Suppose that F is uniformly elliptic and that $h \in \mathcal{C}(\overline{\Omega})$. Let $f \in \mathcal{C}(\overline{\Omega})$ and $\varphi \in \mathcal{C}^{1,\beta_o}(\partial\Omega)$. For any u, viscosity solution of*

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(x)\cdot\nabla u) = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

there exist $\beta = \beta(\lambda, \Lambda, \|f\|_\infty, N, \Omega, \|h\|_\infty, \beta_o)$ and $C = C(\beta)$ such that

$$\|u\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} \leq C\left(\|\varphi\|_{\mathcal{C}^{1,\beta_o}(\partial\Omega)} + \|u\|_\infty + \|f\|_\infty^{\frac{1}{1+\alpha}}\right).$$

For radial solutions, and a more general class of operators, this was proved in [6], with the optimal Hölder's coefficient $\inf(\frac{1}{1+\alpha}, \beta_o)$. In a recent paper the result of Imbert and Silvestre was improved by Araujo, Ricarte, Teixeira in [1].

The novelty with respect to the paper of Imbert and Silvestre is two folded, on one hand we have added the lower order term $h(x)\cdot\nabla u|\nabla u|^\alpha$, and on the other hand we go all the way to the boundary.

The proof follows the scheme of the one in [14], but requires new tools. In particular in Section 2, we give some *a priori* Lipschitz and Hölder estimates in the presence of boundary conditions on one part of the boundary. These are important because the proof of Theorem 1.1 requires that sequence of bounded solutions do converge to a solution of a limit equation.

The main tool is an "improvement of flatness lemma". In the proof of Lemma 3.3, we need to use regularity estimates for a limit equation with boundary terms. The novelty is in guaranteeing that even in the presence of the lower order terms and of the boundary term the limit equations are sufficiently "good" and the new terms behave well, see in particular the Claim in the proof of Lemma 3.3.

Let us make a final remark:

Following Caffarelli's famous technique, the results here obtained could be generalised to an operator depending on x *i.e.* of the form $|\nabla u|^\alpha F(x, D^2u)$; in particular by imposing conditions on

$$\beta(x) = \sup_{M\in S} \frac{F(x,M) - F(x,0)}{|M|}.$$

But we would further need to impose conditions that guarantee that u, a solution of

$$|\nabla u|^\alpha F(x, D^2u) = 0 \text{ in } \Omega \Rightarrow \; F(x, D^2u) = 0 \text{ in } \Omega.$$

For the sake of clarity we have chosen not to treat the case where there is a dependence on x.

After this paper was submitted for publication, Silvestre and Sirakov have posted an interesting related paper [18].

2. Local Hölder and Lipschitz estimates up to the boundary.

Throughout the paper, the notation $B_r(x)$ indicates the euclidian ball of radius r and centre x, we will write B_r when no ambiguity arises.

It is a classical fact that in order to prove that u is $\mathcal{C}^{1,\beta}$ at x_o, it is enough to prove that there exists some constant C such that, for all $r < 1$, there exists $p_r \in \mathbb{R}^N$, such that $\mathrm{osc}_{B_r(x_o)}(u(x) - p_r \cdot x) \le Cr^{1+\beta}$.

As a consequence, u is $\mathcal{C}^{1,\beta}$ in some bounded open set B if there exists a constant C_β such that for all $x \in B$ and $r < 1$, there exists $p_{r,x}$ such that

$$\operatorname*{osc}_{B_r(x)} (u(y) - p_{r,x} \cdot y) \le C_\beta r^{1+\beta}.$$

This will be used in the whole paper.

We begin by stating the following comparison theorem which will be needed later, proved in [3] under stronger conditions on h, later improved to h continuous and bounded in [17].

Theorem 2.1. [3,17] *In the hypothesis of Theorem 1.1, let u and v be respectively $\mathcal{C}(\overline{\Omega})$ solutions of*

$$|\nabla u|^\alpha (F(D^2u) + h(x)\cdot \nabla u) \le f \quad \text{in } \Omega$$

and

$$|\nabla v|^\alpha (F(D^2v) + h(x)\cdot \nabla v) \ge g \quad \text{in } \Omega$$

with f and g continuous and bounded such that $f < g$.

If $u \ge v$ on $\partial\Omega$ then $u \ge v$ in Ω.

In order to prove Hölder and Lipschitz estimates we fix a few notations concerning Ω and F. We suppose, without loss of generality, that at $0 \in \partial\Omega$, the interior normal is e_N. By the implicit function theorem, there exist a ball $B = B_R(0)$ in $\mathbb{R}^N$, and $D' \subset B'_R(0)$ ball of $\mathbb{R}^{N-1}$ and $a \in \mathcal{C}^2(D')$, such that $a(0) = 0$, $\nabla a(0) = 0$ and, for $y = (y', y_N)$,

$$\Omega \cap B \subset \{y_N > a(y'), y' \in D'\}, \text{ and } \partial\Omega \cap B = \{y_N = a(y'), y' \in D'\}\,.$$

We shall also act as if F be positively homogeneous of degree 1 *i.e.* such that for any $t > 0$, $F(tM) = tF(M)$. Observe though that, if this doesn't hold, when necessary it is enough to replace $F(M)$ by $G_t(M) = t^{-1}F(tM)$; this operator satisfies (1.2) with the same constants as F and the results are unchanged.

In the lemma below we have supposed, for simplicity, that B is the unit ball centred at the origin.

Lemma 2.2. *Let $\varphi \in \mathcal{C}^{1,\beta_o}$. Let $a \in \mathcal{C}^2(D')$ such that $a(0) = 0$ and $\nabla a(0) = 0$. Let d be the distance to the hyper surface $\{y_N = a(y')\}$.*

Then, for all $r < 1$ and for all $\gamma < 1$, there exists δ_o depending on $\|f\|_\infty$, λ, Λ, $\|h\|_\infty$, Ω, r and Lipφ, *such that for all $\delta < \delta_o$, if u is a solution of*

$$\begin{cases} |\nabla u|^\alpha (F(D^2 u) + h(y) \cdot \nabla u) = f & \text{in } B \cap \{y_N > a(y')\} \\ u = \varphi & \text{on } \ B \cap \{y_N = a(y')\} \end{cases} \tag{2.1}$$

such that $\operatorname{osc} u \leq 1$ *then it satisfies*

$$|u(y', y_N) - \varphi(y')| \leq \frac{6}{\delta} \frac{d(y)}{1 + d(y)^\gamma} \quad \textit{in } B_r(0) \cap \{y_N > a(y')\}.$$

Proof. We write the details of the proof for $\varphi = 0$. Note that then $\|u\|_\infty \leq 1$. The changes to bring in the case $\varphi \neq 0$ will be given at the end of the proof, the detailed calculation being left to the reader.

It is sufficient to consider the set where $d(y) < \delta$ since the assumption $\|u\|_\infty \leq 1$ implies the result elsewhere.

We begin by choosing $\delta < \delta_1$, such that on $d(y) < \delta_1$ the distance is $\mathcal{C}^2$ and satisfies $|D^2 d| \leq C_1$. We shall also later choose δ smaller depending of $(\lambda, \Lambda, \|f\|_\infty, \|h\|_\infty, N)$.

In order to use the comparison principle we want to construct w a super solution of

$$|\nabla w|^\alpha (\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) + h(y) \cdot \nabla w) < -\|f\|_\infty, \text{ in } B \cap \{y_N > a(y'),\ d(y) < \delta\} \tag{2.2}$$

such that $w \geq u$ on $\partial(B \cap \{y_N > a(y'),\ d(y) < \delta\})$.

The candidate is

$$w(y) = \begin{cases} \dfrac{2}{\delta} \dfrac{d(y)}{1 + d^\gamma(y)} & \text{for } |y| < r \\ \dfrac{2}{\delta} \dfrac{d(y)}{1 + d^\gamma(y)} + \dfrac{1}{(1-r)^3}(|y| - r)^3 & \text{for } |y| \geq r. \end{cases}$$

In order to prove the boundary condition, let us observe that,

on $\{d(y) = \delta\}$, $w \geq \frac{2}{\delta} \frac{\delta}{1+\delta^\gamma} \geq 1 \geq u$,

on $\{|y| = 1\} \cap \{d(y) < \delta\}$, $w \geq \frac{1}{(1-r)^3}(1-r)^3 \geq u$ and finally

on $B \cap \{y_N = a(y')\}$, $w \geq 0 = u$.

We need to check that w is a super solution. For that aim, we compute

$$\nabla w = \begin{cases} \dfrac{2}{\delta} \dfrac{1 + (1-\gamma) d^\gamma}{(1 + d^\gamma)^2} \nabla d & \text{when } |y| < r \\ \dfrac{2}{\delta} \dfrac{1 + (1-\gamma) d^\gamma}{(1 + d^\gamma)^2} \nabla d + \dfrac{y}{|y|} \dfrac{3}{(1-r)^3}(|y| - r)^2 & \text{if } |y| > r. \end{cases}$$

Note that $|\nabla w| \geq \frac{1}{4\delta}$ as soon as $\delta \leq \frac{1-r}{12}$. By construction w is $\mathcal{C}^2$ and

$$D^2 w = -\left(\frac{2\gamma d^{\gamma-1}}{\delta}\right) \frac{(1+\gamma) + (1-\gamma) d^\gamma}{(1 + d^\gamma)^3} \nabla d \otimes \nabla d + \frac{2}{\delta} \frac{1 + (1-\gamma) d^\gamma}{(1 + d^\gamma)^2} D^2 d + H(y)$$

where $\|H(y)\| \leq \frac{6}{(1-r)^2} + \frac{3(N-1)}{r(1-r)}$.

Standard computations that use (1.2) imply that w satisfies (2.2), when $\delta < \min(\delta_1, \frac{1-r}{12})$ is small enough that the two following inequalities hold

$$\lambda(\gamma\delta^{\gamma-2})\frac{(1+\gamma)}{(1+\delta^\gamma)^3} > 2\Lambda\left(\frac{6}{(1-r)^2}+\frac{3(N-1)}{r(1-r)}+\frac{2C_1}{\delta}\right)+\frac{4\|h\|_\infty}{\delta},$$

$$\frac{\lambda}{2^{2+2\alpha}}(\gamma\delta^{\gamma-(2+\alpha)})\frac{(1+\gamma)}{(1+\delta^\gamma)^3} > \|f\|_\infty.$$

By the comparison principle, Theorem 2.1, $u \le w$ in $B\cap\{y_N > a(y')\}\cap\{d(y)<\delta\}$.

Furthermore the desired lower bound on u is easily deduced by considering $-w$ in place of w in the previous computations and restricting to $B_r\cap\{y_N > a(y')\}$. This ends the case $\varphi\equiv 0$.

When $\varphi \not\equiv 0$, let ψ be a solution of

$$\begin{cases}\mathcal{M}^+_{\lambda,\Lambda}(D^2\psi)=0 & \text{in } B\cap\{y_N>a(y')\}\\ \psi=\varphi & \text{on } B\cap\{y_N=a(y')\}.\end{cases}$$

It is well known that ψ is $\mathcal{C}^{1,\beta_o}(B\cap\{y_N\ge a(y')\})\cap\mathcal{C}^2(B\cap\{y_N>a(y')\})$. Furthermore, we can choose ψ such that $\|\psi\|_\infty \le \|\varphi\|_\infty \le 1$, $\|\nabla\psi\|_\infty \le c\|\nabla\varphi\|_\infty$, for some constant c which depends on $\lambda, \Lambda, N, \Omega$, see [9].

We now define

$$w(y)=\begin{cases}\dfrac{8}{\delta}\dfrac{d(y)}{1+d^\gamma(y)}+\psi(y) & \text{for } |y|<r\\ \dfrac{8}{\delta}\dfrac{d(y)}{1+d^\gamma(y)}+\dfrac{1}{(1-r)^3}(|y|-r)^3+\psi(y) & \text{for } |y|>r.\end{cases}$$

Computations similar to the case $\varphi = 0$ imply that

$$w\ge u \text{ on } \partial(B\cap\{y_N>a(y')\}\cap\{d(y)<\delta\}).$$

Furthermore choosing δ small enough, we can ensure that

$$|\nabla w|^\alpha(\mathcal{M}^+_{\lambda,\Lambda}(D^2w)+h(x)\cdot\nabla w) < -\|f\|_\infty.$$

For the lower bound, we replace w by $2\psi - w$. This ends the proof of Lemma 2.2. □

Using this estimate together with an argument due to Ishii and Lions [16], one finally gets the Hölder regularity of the solution, which can be stated as follows with the same hypothesis on a, and f as above:

Proposition 2.3. *Let φ be a Lipschitz continuous function. Suppose that u satisfies (2.1).*

For all $r<1$, and for all $\gamma<1$, u is γ Hölder continuous on $B_r\cap\{y_N>a(y')\}$, with some Hölder's constant depending on $(r,\lambda,\Lambda,a,N,|f\|_\infty,\|h\|_\infty,\mathrm{Lip}\varphi)$.

Remark 2.4. In the absence of boundary conditions, the solutions are Hölder continuous inside B_r for any r such that $B_r \subset\subset B$. We do not give the proof which follows the lines in the proof below, it is sufficient to cancel in it the dependence on φ. This will be used in the proof of the interior improvement of flatness lemma with additional lower terms.

In the following proof we shall use directly the definition of viscosity solutions, so, in particular, in order to fix the notations we state the definition of semi-jets:

Definition 2.5. Let S^{2n} denote the symmetric $2n\times 2n$ matrices. For any continuous function g we define the intrinsic semi-jets by:

$$J^{2,+}g(x)=\left\{(p,X)\in\mathbb{R}^N\times S^N,\ g(x+h)\le g(x)+p\cdot h+\frac12\langle Xh,h\rangle+o(h^2)\ \forall h\in\mathbb{R}^N\right\},$$

$$J^{2,-}g(x)=\left\{(p,X)\in\mathbb{R}^N\times S^N,\ g(x+h)\ge g(x)+p\cdot h+\frac12\langle Xh,h\rangle+o(h^2)\ \forall h\in\mathbb{R}^N\right\}.$$

and the definition of the closed semi jets

Definition 2.6.

$$\overline{J^{2,+}}g(x) = \{(p, X) \in \mathbb{R}^N \times S^N,\ \exists\, x_n,\ \exists (p_n, X_n) \in J^{2,+}g(x_n), g(x_n) \to g(x), \text{and } (p_n, X_n) \to (p, X)\}$$

and analogous definition for $\overline{J^{2,-}}g(x)$

Proof of Proposition 2.3. The proof relies on arguments similar to those in [2,3,14]. Let $1 > r_1 > r$. Without loss of generality we can suppose that $\operatorname{osc} u \leq 1$. Let $x_o \in B_r \cap \{y_N > a(y')\}$ and Φ be defined as

$$\Phi(x, y) = u(x) - u(y) - M|x - y|^\gamma - L|x - x_o|^2 - L|y - x_o|^2.$$

The scope is to prove that for L and M independent of x_o, chosen large enough,

$$\Phi(x, y) \leq 0 \text{ on } (B_{r_1} \cap \{y_N > a(y')\})^2. \tag{2.3}$$

This will imply that u is γ-Hölder continuous on $B_r \cap \{y_N > a(y')\}$ by taking $x = x_o$, and letting x_o vary.

To prove (2.3), we begin by observing that the inequality holds on the boundary. First we treat the points where $y_N = a(y')$. According to Lemma 2.2 there exists M_o such that for $x \in B_{r_1} \cap \{y_N > a(y')\}$,

$$|u(x) - \varphi(x')| \leq M_o d(x, \partial\Omega).$$

Then, using $|x' - y'| \leq |x - y|$, one has

$$\begin{aligned} |u(x', x_N) - u(y', a(y'))| &\leq |u(x', x_N) - u(x', a(x'))| + |u(x', a(x')) - u(y', a(y'))| \\ &\leq M_o d(x, \partial\Omega) + \mathrm{Lip}_\varphi |x' - y'| \\ &\leq M_o |x - (y', a(y'))| + \mathrm{Lip}_\varphi |x - (y', a(y'))|. \end{aligned}$$

So, if M is chosen greater than $M_o + \mathrm{Lip}\varphi$, then we have obtained that $\Phi(x, y) \leq 0$ on $(B_{r_1} \cap \{y_N = a(y')\})^2$.

In order to satisfy the required estimate on the rest of the boundary, it is enough to choose $L > \frac{4}{(r_1 - r)^2}$ and to recall that the oscillation of u is bounded by 1.

In the sequel we will choose L large and $M > \frac{4L^2 N}{\gamma(1-\gamma)}$. Suppose by contradiction that $\Phi(x, y) > 0$ for some $(x, y) \in B_{r_1} \cap \{y_N > a(y')\}$. Then there exists $(\bar{x}, \bar{y})$ such that

$$\Phi(\bar{x}, \bar{y}) = \sup_{\overline{B_{r_1}}} (\Phi(x, y)) > 0.$$

Clearly $\bar{x} \neq \bar{y}$. Furthermore the hypothesis on L forces $\bar{x}$ and $\bar{y}$ to be in $B_{\frac{r_1 + r}{2}} \cap \{y_N > a(y')\}$. Then, as in [2], for all small $\epsilon > 0$ depending on the norm of $Q := D^2(M|x - y|^\gamma)$, using Ishii's Lemma [13], there exist X and Y such that

$$(\gamma M(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{\gamma - 2} + 2L(\bar{x} - x_o), X) \in \overline{J^{2,+}}u(\bar{x})$$

$$(\gamma M(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{\gamma - 2} - 2L(y - x_o), -Y) \in \overline{J^{2,-}}u(\bar{y})$$

with

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \begin{pmatrix} Q & -Q \\ -Q & Q \end{pmatrix} + (2L + \epsilon) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

In the sequel, since we assumed that $\frac{L}{M} \leq \frac{C}{L}$ one also has $\frac{L+\epsilon}{M} \leq \frac{C}{L}$ and then we drop ϵ for simplicity.

Let us denote $q_x = \gamma M(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{\gamma - 2} + 2L(\bar{x} - x_o)$, and $q_y = \gamma M(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{\gamma - 2} - 2L(\bar{y} - x_o)$. Since $\gamma < 1$, as soon as $2L(r + r_1) \leq \frac{\gamma}{2} M r_1^{\gamma - 1}$, we get that $2L|\bar{x} - x_o| \leq \frac{\gamma M}{2} |\bar{x} - \bar{y}|^{\gamma - 1}$ *i.e.*

$$2\gamma M|\bar{x} - \bar{y}|^{\gamma - 1} \geq (|q_x|,\ |q_y|) \geq \frac{1}{2}\gamma M|\bar{x} - \bar{y}|^{\gamma - 1}.$$

Since $|q_x - q_y| \leq 4L$, by the mean value theorem and using a constant $\kappa < \alpha$ if $\alpha < 1$ and $\kappa = 1$ if $\alpha \geq 1$:

$$\begin{aligned} ||q_x|^\alpha - |q_y|^\alpha| &\leq \alpha |q_x - q_y|^\kappa 2^{|\alpha-1|} |\gamma M|\bar{x}-\bar{y}|^{\gamma-1}|^{1-\kappa} |\gamma M|\bar{x}-\bar{y}|^{\gamma-1}|^{\alpha-1} \\ &\leq C(M\gamma|\bar{x}-\bar{y}|^{\gamma-1})^{\alpha-\kappa} L^\kappa = o\left(\left(M|\bar{x}-\bar{y}|^{(\gamma-1)}\right)^\alpha\right). \end{aligned}$$

We now treat the terms concerning the second order derivative. The previous inequalities can also be written as

$$\begin{pmatrix} X - 2LI & 0 \\ 0 & Y - 2LI \end{pmatrix} \leq \begin{pmatrix} Q & -Q \\ -Q & Q \end{pmatrix}.$$

We prove, in what follows, that $L = o(|\mathrm{tr}(X+Y)|)$, that there exist constants c and C such that $|X|, |Y| \leq C|\mathrm{tr}(X+Y)|$ and that

$$cM|\bar{x}-\bar{y}|^{\gamma-2} \leq |\mathrm{tr}(X+Y)| \leq CM|\bar{x}-\bar{y}|^{\gamma-2}.$$

Indeed, let

$$P := \frac{(\bar{x}-\bar{y} \otimes \bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^2} \leq I.$$

Using $4L - (X+Y) \geq 0$, $(I-P) \geq 0$ and the properties of the symmetric matrices, one has

$$\mathrm{tr}(X+Y-4L) \leq \mathrm{tr}(P(X+Y-4L)).$$

Remarking in addition that $X+Y-4L \leq 4Q$, one sees that $\mathrm{tr}(X+Y-4L) \leq 4\mathrm{tr}(PQ)$. But $\mathrm{tr}(PQ) = \gamma M(\gamma-1)|\bar{x}-\bar{y}|^{\gamma-2} < 0$, hence

$$|\mathrm{tr}(X+Y-4L)| \geq 4\gamma M(1-\gamma)|\bar{x}-\bar{y}|^{\gamma-2}. \tag{2.4}$$

Furthermore, by Lemma III.1 of [16], there exists a universal constant C such that

$$\begin{aligned} |X|, |Y|, |X-2L|, |Y-2L| &\leq C(|\mathrm{tr}(X+Y-4L)| + |Q|^{\frac{1}{2}} |\mathrm{tr}(X+Y-4L)|^{\frac{1}{2}}) \\ &\leq C|\mathrm{tr}(X+Y-4L)| \\ &\leq C|\mathrm{tr}(X+Y)|, \end{aligned}$$

since $|Q|$ and $|\mathrm{tr}(X+Y-4L)|$ are of the same order, and $\frac{L}{M} = o(1)$. This will yield the required estimates.

For some positive constants c_2, c_3, since u is both a sub- and a super- solution of (2.1), using the uniform ellipticity of F and the assumptions on h:

$$\begin{aligned} f(\bar{x}) &\leq |q_x|^\alpha (F(X) + h(\bar{x}) \cdot q_x) \\ &\leq |q_y|^\alpha (F(X) + h(\bar{x}) \cdot q_x) \\ &\quad + o(M\gamma|\bar{x}-\bar{y}|^{\gamma-1})^\alpha (\Lambda|X| + \|h\|_\infty \left(\gamma M|\bar{x}-\bar{y}|^{\gamma-1} + 2L\right)) \\ &\leq |q_y|^\alpha (F(-Y) + h(\bar{y}) \cdot q_x + 4\|h\|_\infty L) + \\ &\quad + |q_y|^\alpha \mathrm{tr}(X+Y)\Lambda + o\left(M^{1+\alpha}|\bar{x}-\bar{y}|^{(\gamma-1)\alpha+\gamma-2}\right) \\ &\leq M^\alpha c_2 |\bar{x}-\bar{y}|^{(\gamma-1)\alpha} \mathrm{tr}(X+Y) + o\left(M^{1+\alpha}|\bar{x}-\bar{y}|^{(\gamma-1)\alpha+\gamma-2}\right) + f(\bar{y}) \\ &\leq -c_3 M^{1+\alpha}|\bar{x}-\bar{y}|^{(\gamma-1)\alpha+\gamma-2} + f(\bar{y}). \end{aligned}$$

This is clearly false as soon as L (and then M) is large enough and it ends the proof. □

Compactness near the boundary is a natural consequence of Proposition 2.3.

Corollary 2.7. *Suppose that* (u_n) *is a bounded sequence of continuous functions which satisfy*

$$\begin{cases} |\nabla u_n|^\alpha (F(D^2 u_n) + h(y)\cdot \nabla u_n) = f_n & \text{in } B\cap \{y_N > a(y')\} \\ u_n = \varphi & \text{on } B\cap\{y_N = a(y')\}\end{cases}$$

and suppose that (f_n) *converges simply to some continuous function* f*. Then for all* $r<1$*, one can extract from* (u_n) *a subsequence which converges uniformly, on* $\overline{B_r\cap\{y_N > a(y')\}}$*, to a solution of*

$$\begin{cases} |\nabla u|^\alpha (F(D^2 u) + h(y)\cdot \nabla u) = f & \text{in } B\cap \{y_N > a(y')\} \\ u = \varphi & \text{on } B\cap\{y_N = a(y')\}.\end{cases}$$

Remark 2.8. In the absence of boundary conditions, the analogous result holds, in the sense that one can extract from $(u_n)_n$ a subsequence which converges uniformly on every $B_r \subset\subset B$ to u a solution of

$$|\nabla u|^\alpha (F(D^2 u) + h(y)\cdot\nabla u) = f \text{ in } B.$$

When we shall treat, in the improvement of flatness lemma up to the boundary, the case where the boundary is locally straight, we shall need the following Lipschitz estimate's near the boundary for some different but related equation.

Proposition 2.9. *(Lipschitz estimates for large* p'*s) Let* φ *be a Lipschitz continuous function. Suppose that* $h_N\equiv 0$*. Assume that* u *solves*

$$\begin{cases} |pe_N + \nabla u|^\alpha (F(D^2 u) + h(y)\cdot \nabla u) = f & \text{in } B_1(x)\cap \{y_N > 0\} \\ u = \varphi & \text{on } B_1(x)\cap\{y_N = 0\}\end{cases}$$

with $\operatorname{osc}_{B_1(x)\cap\{y_N>0\}} u \le 1$ *and* $\|f\|_{L^\infty(B_1(x)\cap\{y_N>0\})}\le \epsilon_o<1$*. Then, for all* $r<1$*, there exists* b_o *depending on* $(\lambda,\Lambda,N,\alpha,r,\epsilon_o,\mathrm{Lip}\varphi)$*, such that if* $|p|>\frac{1}{b_o}$*,* u *is Lipschitz continuous in* $B_r(x)\cap\{y_N>0\}$ *with some Lipschitz constant depending on* $(\lambda,\Lambda,N,\alpha,r,\epsilon_o,\mathrm{Lip}\varphi)$*.*

Remark 2.10. In the absence of boundary conditions, the solutions in some ball B are Lipschitz in B_r, for any r such that $B_r\subset\subset B$ with Lipschitz constant independent of p.

This will be used in the proof of the interior improvement of flatness lemma with lower order terms.

This Proposition is a consequence of the following

Lemma 2.11. *Suppose that* φ *is Lipschitz continuous and that* $h_N\equiv 0$*. For all* $\gamma<1$*, for all* $r<1$*, there exists* $\delta=\delta(\|f\|_\infty,\lambda,\Lambda,r,\mathrm{Lip}\varphi)$*, such that for* $b<\frac{\delta}{4}$*, any solution* u *of*

$$\begin{cases} |e_N + b\nabla u|^\alpha (F(D^2 u) + h(y)\cdot \nabla u) = f & \text{in } B_1(x)\cap \{y_N > 0\} \\ u = \varphi & \text{on } B_1(x)\cap\{y_N = 0\}.\end{cases}$$

such that $\operatorname{osc}(u)\le 1$*, satisfies* $|u(y',y_N)-\varphi(y')|\le \frac{2}{\delta}\frac{y_N}{1+y_N^\gamma}$ *in* $B_r(x)\cap\{y_N>0\}$*.*

Proof of Lemma 2.11. Suppose for simplicity that $\varphi=0$. If $b=0$ the result is known by properties of solutions of $F(D^2u)+h(x)\cdot\nabla u = f$ which are zero on the boundary. So we assume in what follows that $b\neq 0$.

We proceed as in Lemma 2.2, replacing the distance of y to the boundary by y_N; so we consider

$$w(y)=\begin{cases} \dfrac{2}{\delta}\dfrac{y_N}{1+y_N^\gamma} & \text{for } y_N<\delta, |y'|<r \\ \dfrac{2}{\delta}\dfrac{y_N}{1+y_N^\gamma} + \dfrac{1}{(1-r)^3}(|y'|-r)^3 & \text{for } y_N<\delta, |y'|>r.\end{cases}$$

Similarly to the proof of Lemma 2.2, it is sufficient to consider the set where $y_N < \delta$, since the assumption $\operatorname{osc} u \leq 1$ implies $\|u\|_\infty \leq 1$, so the result holds elsewhere. Furthermore we only prove that $u \leq w$, the desired lower bound can be obtained by considering $-w$ in place of w.

In order for w to satisfy

$$|e_N + b\nabla w|^\alpha (\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) + h(y)\cdot \nabla w) \leq -\|f\|_\infty, \text{ in } B,$$

it is sufficient to choose δ such that

$$\left(\frac{1}{2}\right)^\alpha \lambda\gamma\delta^{\gamma-2}(1-\gamma)\frac{1}{(1+\delta^\gamma)^3} > \|f\|_\infty + 2\Lambda\left(\frac{6}{(1-r)^2} + \frac{3(N-1)}{r(1-r)} + \frac{4\|h\|_\infty}{\delta}\right),$$

$b < \frac{\delta}{4}$, and recall that $|\nabla w| \leq \frac{2}{\delta}$. Furthermore $w \geq u$ on $\partial(B \cap \{0 < y_N < \delta\})$.

Hence we can use the comparison principle in Theorem 2.1, for $\tilde{w}(x) = x_N + bw(x)$ and $\tilde{u}(x) = x_N + bu(x)$, with $b \neq 0$, (recall that $h_N \equiv 0$)which implies that $u \leq w$ in $B \cap \{y_N > 0\}$. Finally the desired estimate is obtained in $\{|y'| < r,\ y_N > 0\}$.

In the case $\varphi \not\equiv 0$, we take the function w as in the proof of Lemma 2.2, with d replaced by y_N. Requiring sufficient restriction on the smallness of δ give the result. □

We are now ready to give the

Proof of Proposition 2.9. The proof proceeds as in [14] so we just detail the differences. Recall that u is a solution of

$$|e_N + bDu|^\alpha (F(D^2u) + h(y)\cdot\nabla u) = \tilde{f}$$

with $b = \frac{1}{p}$ and $\tilde{f} = |p|^{-\alpha} f$.

Let $r < r_1 < 1$ and let $x_o \in B_{r_1}(x) \cap \{y_N > 0\}$, $L_2 = \frac{4}{(r_1-r)^2}$,

$$\psi(z,y) = u(z) - u(y) - L_1\omega(|z-y|) - L_2|z-x_o|^2 - L_2|y-x_o|^2$$

where $\omega(s) = s - \omega_o s^{\frac{3}{2}}$ if $s \leq s_o = \left(\frac{2}{3\omega_o}\right)^2$ and $\omega(s) = \omega(s_o)$ if $s \geq s_o$. We also require $L_1 > 3(\frac{2}{\delta} + \mathrm{Lip}_\varphi)$.

If we prove that $\psi(z,y) \leq 0$ in $B_{r_1}(x)$, since L_1 is independent of x_o, by choosing $z = x_o$ one gets

$$u(x_o) - u(y) \leq L_1|x_o - y| + L_2|x_o-y|^2;$$

next choosing $y = x_o$ for all $z \in B_{r_1}$,

$$u(z) - u(x_o) \leq L_1|x_o - z| + L_2|x_o - z|^2.$$

Finally, for $(x,y) \in B_r(x)$, $|u(x) - u(y)| \leq L_1|x-y| + L_2|x-y|^2$, which implies the desired result.

We begin to observe that if the supremum is achieved in $(\bar{x},\bar{y}) \in \overline{B_r(x)}$ then, with our choice of L_1, neither $\bar{x}$ nor $\bar{y}$ can belong to the part $\{z_N = 0\}$ according to Lemma 2.11.

The rest of the proof is as in [2] and [14], (see also the Proof of Prop. 2.3) as long as we choose δ small enough in order that

$$\left(\frac{1}{2}\right)^\alpha \lambda(\gamma\delta^{\gamma-2})\frac{(1+\gamma)}{2(1+\delta^\gamma)^3} > \|f\|_\infty + 2\Lambda\left(\frac{6}{(1-r)^2} + \frac{3(N-1)}{r(1-r)} + \frac{4\|h\|_\infty}{\delta}\right)$$

and such that $b < \frac{\delta}{4}$. Let us note that this implies that b is small enough depending on $(\lambda, \Lambda, N, \alpha, r, \epsilon_o, \|h\|_\infty, \mathrm{Lip}\varphi)$. □

As a corollary of this Lemma one has the following compactness result

Corollary 2.12. *Let φ be a Lipschitz continuous function. Suppose that (u_n) is a sequence of uniformly bounded continuous viscosity solutions of*

$$\begin{cases} |e_N + b_n \nabla u_n|^\alpha \left(F(D^2 u_n) + h\cdot\nabla u_n\right) = f_n & \text{in } B\cap\{y_N > a(y')\},\\ u_n = \varphi & \text{on } B\cap\{y_N = a(y')\}.\end{cases}$$

where $b_n \le b_o$, b_o is given above in Proposition 2.9. Suppose that f_n converges simply to some function f in Ω. Then for all $r < 1$, one can extract from (u_n, b_n) a subsequence which converges uniformly on $\overline{B_r \cap \{y_N > a(y')\}} \times \mathbb{R}$, and the limit $(u, \bar b)$ satisfies

$$\begin{cases} |e_N + \bar b \nabla u|^\alpha \left(F(D^2 u) + h\cdot\nabla u\right) = f & \text{in } B\cap\{y_N > a(y')\},\\ u = \varphi & \text{on } B\cap\{y_N = a(y')\}.\end{cases}$$

Remark 2.13. In the absence of boundary conditions, the conclusion is that the sequence (u_n) contains a subsequence which converges locally uniformly and up to a constant toward a solution of

$$|e_N + \bar b\nabla u|^\alpha \left(F(D^2u) + h\cdot\nabla u\right) = f \text{ in } B.$$

3. Proof of Theorem 1.1.

In fact Theorem 1.1 is an immediate consequence of the following local result up to the boundary, together with some argument of finite covering:

Theorem 3.1. *Suppose that F, h and f are as in Theorem 1.1 and φ is a function in $\mathcal{C}^{1,\beta_o}$. Let B be a ball in $\mathbb{R}^N$ and let a be a $\mathcal{C}^2$ function defined on $\mathbb{R}^{N-1}$ with $a(0) = 0$, $\nabla a(0) = 0$. There exists β such that for any u solution of*

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(x)\cdot\nabla u) = f & \text{in } B\cap\{x_N > a(x')\}\\ u = \varphi & \text{on } B\cap\{x_N = a(x')\},\end{cases}$$

u is $\mathcal{C}^{1,\beta}(\tilde B \cap \{x_N > a(x')\})$ for any $\tilde B \subset\subset B$.

Theorem 3.1 is proved *via* the following two "improvement of flatness" lemma and their consequences.

Lemma 3.2. *There exist $\epsilon_o \in]0,1[$ and $\rho \in]0,1[$ depending on $(\alpha, \|h\|_\infty, \lambda, \Lambda, N)$ such that for any $p \in \mathbb{R}^N$ and for any viscosity solution u of*

$$|p + \nabla u|^\alpha \left(F(D^2u) + h(y)\cdot(\nabla u + p)\right) = f \ \textit{in } B_1$$

such that $\operatorname{osc}_{B_1} u \le 1$ and $\|f\|_{L^\infty(B_1)} \le \epsilon_o$, there exists $p' \in \mathbb{R}^N$ such that

$$\operatorname*{osc}_{B_\rho}(u - p'\cdot x) \le \frac{1}{2}\rho.$$

and

Lemma 3.3. *For any $a \in \mathcal{C}^2$, such that $a(0) = 0$ and $\nabla a(0) = 0$, there exist $\epsilon_o > 0$ and ρ which depend on $(\alpha, \lambda, \Lambda, N, \|D^2 a\|_\infty, \|h\|_\infty, \|\varphi\|_{\mathcal{C}^{1,\beta_o}})$ such that for any $p \in \mathbb{R}^N$ and u a viscosity solution of*

$$\begin{cases} |p + \nabla u|^\alpha (F(D^2u) + h(y)\cdot(\nabla u + p)) = f & \text{in } B\cap\{y_N > a(y')\}\\ u + p\cdot y = \varphi & \text{on } B\cap\{y_N = a(y')\},\end{cases}$$

the following holds: for all $x \in B$ such that $B_1(x) \subset B$, if $\operatorname{osc}_{B_1(x)\cap\{y_N > a(y')\}} u \le 1$, and $\|f\|_{L^\infty(B_1(x)\cap\{y_N>a(y')\})} \le \epsilon_o$ then there exists $q_{x,\rho} \in \mathbb{R}^N$ such that

$$\operatorname*{osc}_{B_\rho(x)\cap\{y_N > a(y')\}} (u(y) - q_{x,\rho}\cdot y) \le \frac{\rho}{2}\cdot$$

Suppose that these Lemmata have been proved and let us derive the following one.

Lemma 3.4. *Suppose that ρ and $\epsilon_o \in [0,1]$ and B are as in Lemma 3.3 and suppose that u is a viscosity solution of*

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(y)\cdot\nabla u) = f \text{ in } B\cap\{y_N > a(y')\} \\ u = \varphi \qquad\qquad\qquad\qquad\qquad\qquad \text{on } B\cap\{y_N = a(y')\} \end{cases} \tag{3.1}$$

with $\operatorname{osc} u \le 1$ *and* $\|f\|_\infty \le \epsilon_o$*, then, there exists* $\beta \in]0,1[$*, such that for all* k *and for all* $x \in B$ *such that* $B_1(x) \subset B$*, there exists* $p_k \in \mathbb{R}^N$ *for which*

$$\operatorname*{osc}_{B_{r_k}(x)\cap\{y_N>a(y')\}} (u(y) - p_k\cdot y) \le r_k^{1+\beta}$$

where $r_k = \rho^k$*.*

Remark 3.5. Of course, in the absence of boundary condition we obtain the interior regularity: Suppose that ρ and $\epsilon_o \in [0,1]$ are as in Lemma 3.2 and suppose that u is a viscosity solution of

$$|\nabla u|^\alpha (F(D^2u) + h(\cdot)\cdot\nabla u) = f \text{ in } B_1$$

with $\operatorname{osc} u \le 1$ and $\|f\|_\infty \le \epsilon_o$, then there exists $\beta \in]0,1[$ such that for all k there exists $p_k \in \mathbb{R}^N$ such that

$$\operatorname*{osc}_{B_{r^k}} (u(x) - p_k\cdot x) \le r_k^{1+\beta}.$$

Proof of Lemma 3.4. As in [14] we use a recursive argument.

We first remark that one can assume that $\varphi(x') = \partial_i\varphi(x') = 0$ for $i = 1,\dots,N-1$.

Indeed, let u be a solution of (3.1). Then $v(y) := u(y) - u(x) - \nabla\varphi(x')\cdot(y'-x')$ satisfies

$$\begin{cases} |\nabla v + q|^\alpha (F(D^2v) + h(y)\cdot(\nabla v + q)) = f \quad \text{in } B_1(x)\cap\{y_N > a(y')\} \\ v(y', a(y')) = \phi(y') \qquad\qquad\qquad\qquad\qquad \text{on } B_1(x)\cap\{y_N = a(y')\} \end{cases}$$

where $q = (\nabla\varphi(x'), 0)$ and $\phi(y') = \varphi(y') - \varphi(x') - \nabla\varphi(x')\cdot(y'-x')$ which satisfies $\phi(x') = \partial_i\phi(x') = 0$ for $i = 1,\dots,N-1$.

So the result obtained for v would easily transfer to u.

Choose β small enough in order that $\rho^\beta > \frac{1}{2}$, and define $r_k = \rho^k$.

We can start the recursive argument. For $k = 0$, taking $p_o = 0$ yields the desired inequality. Suppose that p_k exists we now construct p_{k+1}.

Let $\varphi_k(y') = r_k^{-1-\beta}\varphi(x' + r_k(y'-x'))$, which satisfies, for $\beta < \beta_o$, with the above choice of φ,

$$\|\varphi_k\|_{\mathcal{C}^{1,\beta}(B_1(x'))} \le \|\varphi\|_{\mathcal{C}^{1,\beta}}.$$

We consider

$$u_k(y) = r_k^{-1-\beta}\left(u(r_k(y-x)+x) - p_k\cdot(r_k(y-x)+x)\right).$$

u_k is well defined on $B_1(x)\cap\{y_N > a_k(y')\}$, where $a_k(y') = x_N(1-\frac{1}{r_k}) + \frac{a(r_k(y'-x')+x')}{r_k}$.

It is immediate to see that u_k is a solution of

$$\begin{cases} |p_k r_k^{-\beta} + \nabla u_k|^\alpha (F(D^2u_k) + h_k\cdot(p_k r_k^{-\beta} + \nabla u_k)) = f_k \quad \text{in } B_1(x)\cap\{y_N > a_k(y')\} \\ u_k + r_k^{-\beta}p_k\cdot y = r_k^{-\beta}(r_k-1)p_k\cdot x + \varphi_k(y') \qquad\qquad \text{on } B_1(x)\cap\{y_N = a_k(y')\} \end{cases}$$

with $f_k(y) = r_k^{1-\beta(1+\alpha)}f(r_k(y-x)+x)$ and $h_k(y) = r_k h(r_k(y-x)+x)$.

Observe that if $y \in B_1(x)\cap\{y_N = a_k(y')\}$, then

$$\left|\frac{a(r_k(y'-x')+x') - x_N}{r_k}\right| \le 1$$

but this implies, using the mean value's theorem and $|y'-x'|<1$, that

$$|x_N - a(x')| \leq r_k(1+|\nabla a|).$$

So if x is not on the boundary *i.e.* $\{x_N > a(x')\}$ then, for k sufficiently large, $B_1(x) \subset \{y_N > a_k(y')\}$, and in that case we don't have to worry about the boundary terms.

A direct computation gives that, $|\nabla a_k(y')| = |\nabla a(r_k(y'-x')+x')|$, while, $D^2a_k(y') = r_k D^2 a(r_k(y'-x')+x')$, and hence $\|D^2 a_k\|_\infty = r_k \|D^2 a\|_\infty \leq \|D^2 a\|_\infty$.

Furthermore, as long as $\beta < \frac{1}{1+\alpha}$,

$$\operatorname*{osc}_{B_1(x)\cap\{y_N>a_k(y')\}} u_k \leq 1,\ \|f_k\|_{L^\infty(B_1(x)\cap\{y_N>a_k(y')\})} \leq \epsilon_o \text{ and } \|h_k\|_\infty \leq \|h\|_\infty.$$

Hence, using Lemma 3.3 with obvious changes, there exists $q_{k+1} \in \mathbb{R}^N$ such that

$$\operatorname*{osc}_{B_\rho(x)\cap\{y_N>a_k(y')\}} (u_k(y) - q_{k+1}\cdot y) \leq \frac{\rho}{2}\cdot$$

Defining $p_{k+1} = p_k + q_{k+1} r_k^\beta$, with the assumptions on β and ρ, one gets:

$$\operatorname*{osc}_{B_{r_{k+1}}(x)\cap\{y_N>a(y')\}} (u(y) - p_{k+1}\cdot y) \leq \frac{\rho}{2} r_k^{1+\beta} \leq r_{k+1}^{1+\beta},$$

since the oscillation is invariant by translation. This is the desired conclusion. □

There remains to prove the improvement of flatness lemmata. We start by the interior case with lower order terms.

Proof of Lemma 3.2. Suppose by contradiction that there exist a sequence of functions $(f_n)_n$ whose norm goes to zero, a sequence of $(p_n)_n \in \mathbb{R}^N$ and a sequence of functions $(u_n)_n$ with $\operatorname{osc} u_n \leq 1$, solutions of

$$|p_n + \nabla u_n|^\alpha \left(F(D^2 u_n) + h(y)\cdot(\nabla u_n + p_n)\right) = f_n, \tag{3.2}$$

such that, for all $q \in \mathbb{R}^N$, and any $\rho \in (0,1)$,

$$\operatorname*{osc}_{B_\rho}(u_n(y) - q\cdot y) > \frac{\rho}{2}\cdot \tag{3.3}$$

Let us suppose first that $(p_n)_n$ is bounded then, up to subsequences, it converges to p_∞. Considering $v_n(y) = u_n(y) + p_n \cdot y$ and using the compactness Remark 2.8, we can extract form $(v_n)_n$ a subsequence converging to a limit v_∞, which satisfies

$$|\nabla v_\infty|^\alpha (F(D^2 v_\infty) + h(x)\cdot\nabla v_\infty) = 0.$$

Remark next that the solutions of such an equation are solutions of

$$F(D^2 v_\infty) + h(x)\cdot\nabla v_\infty = 0 \tag{3.4}$$

as it is the case for $h=0$ (see [14]). But, passing to the limit in (3.3) gives that $\operatorname{osc}_{B_\rho}(v_\infty - (q-p_\infty)\cdot x) > \frac{\rho}{2}$. This contradicts the regularity results known for solutions of equation (3.4), (see [20]) and it ends the case where the sequence $(p_n)_n$ is bounded.

In the case where $(p_n)_n$ is unbounded, take a subsequence such that $\frac{p_n}{|p_n|}$ converges to some p_∞.

Claim. There exist $q_\infty \in \mathbb{R}^N$ and a subsequence $\sigma(n)$ such that for any $r<1$,

$$\lim_{n\to\infty} h(y)\cdot p_{\sigma(n)} = h(y)\cdot q_\infty$$

uniformly in B_r.

We postpone the proof of that claim and we end the Proof of Lemma 3.2.

We now divide the equation (3.2) by $|p_n|^\alpha$ and get, with $e_n = \frac{p_n}{|p_n|}$ and $a_n = \frac{1}{|p_n|}$,

$$|a_n \nabla u_n + e_n|^\alpha \left(F(D^2 u_n) + h(y)\cdot(\nabla u_n + p_n)\right) = \frac{f_n}{|p_n|^\alpha}\cdot$$

Using Remark 2.13 and the claim, a subsequence of $u_{\sigma(n)}$ converges to u_∞ a solution of the limit equation

$$F(D^2 u_\infty) + h(y)\cdot(\nabla u_\infty + q_\infty) = 0.$$

On the other hand, $\operatorname{osc}(u_\infty - q'\cdot x) > \frac{1}{2}\rho$, which contradicts the regularity of solutions of such equations.

Proof of the Claim. Let $V \subset \mathbb{R}^N$ be the space generated by $h(B_r)$. Let $q_n = \Pi_V p_n$ be the projection of p_n on V, hence $h(x)\cdot p_n = h(x)\cdot q_n$.

Let $\bar{p}$ be such that for a subsequence $\frac{p_n}{|p_n|} \to \bar{p}$, and let

$$v_n := \frac{u_n + p_n\cdot x}{|p_n|}\cdot$$

Clearly v_n converges to $v_\infty = \bar{p}\cdot x$ and it is a solution of

$$|\nabla v_n|^\alpha (F(D^2 v_n) + h(x)\cdot\nabla v_n) = \frac{f_n}{|p_n|^{1+\alpha}}$$

Using compactness of bounded sequences one obtain by passing to the limit in the equation that v_∞ must be a solution of

$$|\nabla v_\infty|^\alpha (F(D^2 v_\infty) + h(x)\cdot\nabla v_\infty) = 0,$$

hence we have obtained

$$h(x)\cdot\bar{p} = 0$$

Let us note that this implies that $\frac{|q_n|}{|p_n|} \to 0$. Indeed suppose that $h(x_i)_{1\le i\le k}$ generate V, let E_j be an orthonormal basis such that

$$[E_1, \dots E_k] = [h(x_1), \dots h(x_k)]$$

then there exist A_j^i such that $E_j = \sum_i A_j^i h(x_i)$ and $q_n = \sum (q_n\cdot E_i)E_i = \sum_{ij}(q_n\cdot h(x_i))A_j^i$ and then

$$\frac{|q_n|}{|p_n|} \le C\frac{\sum_{1\le i\le k}|(q_n\cdot h(x_i))|}{|p_n|} = C\sum_{1\le i\le k}\left|\frac{p_n}{|p_n|}\cdot h(x_i))\right| \to 0.$$

Suppose by contradiction that the sequence $(q_n)_n$ goes to infinity in norm.

Define $w_n = \frac{u_n + q_n\cdot x}{|q_n|}$. Divide the equation by $|p_n - q_n|^\alpha |q_n|$, then w_n is a solution of

$$\left|\frac{p_n - q_n}{|p_n - q_n|} + \frac{|q_n|}{|p_n - q_n|}\nabla w_n\right|^\alpha \left(F(D^2 w_n) + h(y)\cdot\nabla w_n\right) = \frac{f_n}{|p_n - q_n|^\alpha |q_n|}\cdot$$

Now, the sequence w_n which is bounded converges to $w_\infty(x) = \bar{q}\cdot x$ for some $\bar{q}$ of norm 1.

Using the compactness of $(w_{\sigma(n)})_n$, proceeding as above, since $|p_n - q_n|^\alpha |q_n|$ diverges and since $\frac{q_n}{|p_n - q_n|} \to 0$, we get that it converges to a solution of the limit equation

$$F(D^2 w_\infty) + h(y)\cdot\nabla w_\infty = 0$$

which implies

$$h(y)\cdot\bar{q} = 0 \quad \text{for all} \quad y \in B_r.$$

This means that $\bar q$ is both in V and $V^\perp$, hence $\bar q = 0$, which is a contradiction and the sequence $(q_n)_n$ is bounded. This gives the claim, indeed, up to a subsequence, q_n converges to q_∞ and

$$\lim_{n\to\infty} h(y)\cdot p_{\sigma(n)} = \lim_{n\to\infty} h(y)\cdot q_{\sigma(n)} = h(y)\cdot q_\infty.$$

This ends the proof. □

We now want to prove Lemma 3.3. In the case of a non straight boundary it requires the following technical proposition, which is probably known, but for which we give a proof at the end of the section since we could not locate one in the literature.

Proposition 3.6. *Suppose that a is not identically zero in $B'_\delta(0)$ and $a(0) = \nabla a(0) = 0$. Suppose that $(p_n)_n$ is a sequence in $\mathbb{R}^N$ such that, for all n and for all $x \in B'_\delta(0)$,*

$$|p_n \cdot (x', a(x'))| \le C$$

for some constant C. Then $(p_n)_n$ is a bounded sequence.

Proof of Lemma 3.3. We assume first that $h = 0$.

Note that if $B_1(x) \cap \{y_N = a(y')\} = \emptyset$ then it is sufficient to use the result of [14]. So we now assume that $B_1(x) \cap \{y_N = a(y')\} \neq \emptyset$.

We argue by contradiction and suppose that for all n, there exist $x_n \in \overline{B}$ and $p_n \in \mathbb{R}^N$, $|f_n|_{L^\infty(\Omega)} \le \frac{1}{n}$ and u_n with $\mathrm{osc}(u_n) \le 1$ a solution of

$$\begin{cases} |p_n + \nabla u_n|^\alpha F(D^2 u_n) = f_n & \text{in } B \cap \{y_N > a(y')\} \\ u_n(y) + p_n \cdot y = \varphi(y') & \text{on } B \cap \{y_N = a(y')\} \end{cases} \tag{3.5}$$

such that for any $q \in \mathbb{R}^N$

$$\operatorname*{osc}_{B_\rho(x_n)} (u_n(y) - q\cdot y) > \frac{\rho}{2}\cdot \tag{3.6}$$

Extract from $(x_n)_n$ a subsequence which converges to $x_\infty \in \overline{B} \cap \{y_N \ge a(y')\}$.

We denote in the sequel $B_\infty = \cap_{n\ge N_1} B_1(x_n)$, which contains $B_\rho(x_\infty)$ as soon as N_1 is large enough.

The case where T is not straight. Observe that $u_n - u_n(x_n)$ satisfies the same equation as u_n, it has oscillation 1 and it is bounded, we can then suppose that the sequence (u_n) is bounded. This, together with the boundary condition implies that

$$|p_n \cdot (y', a(y'))| \le C$$

and Proposition 3.6, gives that (p_n) is bounded.

So, up to subsequences, u_n converges to some u_∞, uniformly on $\overline{B_r} \cap \{y_N \ge a(y')\}$ for every $B_r \subset\subset B_\infty$, (due to Cor. 2.7), and p_n converges to p_∞. Furthermore, (u_∞, p_∞) solves

$$\begin{cases} |p_\infty + \nabla u_\infty|^\alpha F(D^2 u_\infty) = 0 & \text{in } B_\infty \cap \{y_N > a(y')\} \\ u_\infty + p_\infty \cdot y = \varphi(y) & \text{on } B_\infty \cap \{y_N = a(y')\}. \end{cases}$$

Using Lemma 6 in [14] one gets that

$$\begin{cases} F(D^2 u_\infty) = 0 & \text{in } B_\infty \cap \{y_N > a(y')\} \\ u_\infty + p_\infty \cdot y = \varphi(y) & \text{on } B_\infty \cap \{y_N = a(y')\}. \end{cases} \tag{3.7}$$

On the other hand, by passing to the limit in (3.6), one obtains

$$\operatorname*{osc}_{B_\rho(x_\infty)\cap\{y_N>0\}} (u_\infty(y) - q\cdot y) \ge \frac{\rho}{2}\cdot$$

This contradicts the fact that, by known regularity results on uniformly elliptic operators, there exists ρ_1 and $q_1 = q_{x,\rho_1}$ so that any u, solution of (3.7) satisfies

$$\operatorname*{osc}_{B_{\rho_1}(x_\infty)}(u(y) - q_1 \cdot y) \leq \frac{\rho_1}{2}\cdot$$

The case where $T = \{y_N = 0\}$**.** Let $p_n = p'_n + p_n^N e_N$. The boundedness of u_n and the boundary condition imply that $(p'_n)_n$ is bounded. If $(p_n^N)_n$ is bounded just proceed as above. So we suppose that p_n^N is unbounded. Dividing (3.5) by $|p_n^N|^\alpha$ it becomes

$$\begin{cases} |\frac{p_n}{p_n^N} + \frac{1}{p_n^N}\nabla u_n|^\alpha F(D^2 u_n) = \frac{f_n}{|p_n^N|^\alpha} & \text{in } B_\infty \cap \{y_N > 0\} \\ u_n + p'_n \cdot y' = \varphi(y') & \text{on } B_\infty \cap \{y_N = 0\} \end{cases}$$

Denoting by p' the limit of a subsequence of p'_n, and u_∞ the limit of a subsequence of (u_n), one gets by passing to the limit and using Corollary 2.12 that

$$\begin{cases} F(D^2 u_\infty) = 0 & \text{in } B_\infty \cap \{y_N > 0\} \\ u_\infty + p' \cdot y = \varphi(y') & \text{on } B_\infty \cap \{y_N = 0\}. \end{cases}$$

Passing to the limit in (3.6) one gets that $\operatorname{osc}_{B_\rho(x_\infty)}(u_\infty - q \cdot y) > \frac{\rho}{2}$, a contradiction. This ends the proof for $h = 0$.

We briefly point out the differences in the case $h \neq 0$. It is sufficient to treat the case of a straight boundary, the other cases being as before. Indeed, we already know that if the boundary is not locally straight, $(p_n)_n$ is bounded.

In the case where the boundary is locally straight, say $T = \{y_N = 0\}$, the only possibly unbounded component is $p_n \cdot e_N$.

The Claim in the interior flatness Lemma 3.2 implies that for any open set $D \subset B_r(x) \cap \{y_N > 0\}$, $h(y) \cdot e_N = 0$ for any $y \in D$. By the arbitrariness of D, this implies that $h(y) \cdot e_N = 0$ in $B_r(x) \cap \{y_N > 0\}$. We can now apply Proposition 2.9 and end the proof as in the case $h = 0$.

We end the paper with the *proof of Proposition 3.6*.

Its thesis is proved if there exists N independent vectors, $V_1, \ldots, V_N$ such that $|p_n \cdot V_j| \leq C$ for $j = 1, \ldots, N$.

Since a is not identically 0 on $[-\delta, \delta]$ and $a(0) = 0$, $\nabla a(0) = 0$, then there exists i such that $a(te_i)$ is not constant for $t \in [-\delta, \delta]$. Hence, there exist $0 < |t_1| < \delta$ and $|t_2| < \delta$ such that the vectors

$$V_i := t_1 e_i + a(t_1 e_i) e_N \quad \text{and } V_N := t_2 e_i + a(t_2 e_i) e_N$$

are linearly independent. Indeed, since a is not constant there exists t_1 and e_i such that $a(t_1 e_i) \neq 0$. Suppose now that for any t_2, V_i and V_N are linearly dependent, this implies that

$$\frac{a(t_1 e_i)}{t_1} = \frac{a(t_2 e_i)}{t_2} = \partial_i a(0) = 0$$

a contradiction

Now, for $j \neq i$ and $j \neq N$ choose $V_j = t_1 e_j + a(t_1 e_j) e_N$. By constructions the V_j for $j = 1, \ldots, N$ are linearly independent and by hypothesis $|p_n \cdot V_j| \leq C$ this ends the proof. □

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