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BOUNDARY OBSERVABILITY INEQUALITIES FOR THE 3D OSEEN–STOKES SYSTEM AND APPLICATIONS

Sérgio S. Rodrigues[1]

Abstract. Controllability properties for the Navier–Stokes system are closely related to observability properties for the adjoint Oseen–Stokes system; boundary observability inequalities are derived, for that adjoint system, that will be appropriate to deal with suitable constrained controls, like finite-dimensional controls supported in a given subset of the boundary. As an illustration, new boundary controllability results for the Oseen–Stokes system are derived. Finally, some further plausible consequences of the derived inequalities, concerning the Navier–Stokes system, are discussed.

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a connected bounded domain located locally on one side of its smooth boundary $\Gamma = \partial\Omega$, and let $I \subseteq \mathbb{R}$ be a nonempty open interval. The Navier–Stokes system, in $I \times \Omega$, controlled through the boundary reads

$$\partial_t u + \langle u \cdot \nabla \rangle u - \nu \Delta u + \nabla p_u + h = 0, \quad \operatorname{div} u = 0, \quad u|_\Gamma = \gamma + \zeta \tag{1.1}$$

where ζ is a control taking values in a suitable subspace of square-integrable functions in Γ whose support, in x, is contained in the closure $\overline{\Gamma_{\mathrm{c}}}$ of a given open subset $\Gamma_{\mathrm{c}} \subseteq \Gamma$. Furthermore, as usual, $u = (u_1, u_2, u_3)$ and p_u, defined for $(t,\, x_1,\, x_2,\, x_3) \in I \times \Omega$, are the unknown velocity field and pressure of the fluid, $\nu > 0$ is the viscosity, the operators ∇ and Δ are respectively the well known gradient and Laplacean in the space variables $(x_1,\, x_2,\, x_3)$, $\langle u \cdot \nabla \rangle v$ stands for $(u \cdot \nabla v_1,\, u \cdot \nabla v_2,\, u \cdot \nabla v_3)$, $\operatorname{div} u := \partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3$, and h and γ are fixed functions.

It turns out that (local) controllability properties to trajectories for system (1.1) are often related to observability inequalities for the time-backward "adjoint" Oseen–Stokes system

$$-\partial_t q + \mathcal{B}^*(\hat{u}) q - \nu \Delta q + \nabla p_q + f = 0, \quad \operatorname{div} q = 0, \quad q|_\Gamma = 0, \tag{1.2}$$

where $\hat{u}$ is a given reference (desired) trajectory of (1.1) (with $\zeta = 0$), f is a suitable force, and $\mathcal{B}^*(\hat{u})$ is the formal adjoint to $\mathcal{B}(\hat{u}) : v \mapsto \langle \hat{u} \cdot \nabla \rangle v + \langle v \cdot \nabla \rangle \hat{u}$. We refer the reader to the works [5, 19] for the case of

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[1] Johann Radon Institute for Computational and Applied Mathematics (RICAM), ÖAW, Altenbergerstraße 69, 4040 Linz, Austria. sergio.rodrigues@oeaw.ac.at

internal controls, and to [16] for a procedure to obtain boundary controllability results from internal ones. See also [12, 13, 15, 17] and references therein.

We are particularly interested in the case where the reference solution $\hat{u}$ is nonstationary (*i.e.*, $\hat{u} = \hat{u}(t)$ depends on time), a situation that often can occur in real world applications, as in the case suitable (say non-gradient) external forces (h and γ) depend on time. Moreover, for applications purposes it is often required that the control obeys some general constraints like, for example, to be feedback, finite-dimensional and supported in a given (small) open subset. It turns out that with these constraints on the boundary control, the procedure in [16] is (or may be) no longer sufficient to derive the wanted boundary controllability results.

In reference [5], an internal stabilizing finite-dimensional feedback controller was found for the case of non-stationary reference solutions. Then, one question arises: can we find a similar boundary controller? We can, for example, see that from the internal result and from the procedure in [16] we cannot guarantee that the obtained boundary control is finite-dimensional. Also, the methods used in the particular case of a stationary reference solution, in [2–4, 6, 25], use some (spectral-like) properties of the (time-independent) Oseen–Stokes operator $u \mapsto \nu\Delta u - \mathcal{B}(\hat{u})u - \nabla p_u$ and/or of its "adjoint" $q \mapsto \nu\Delta q - \mathcal{B}^*(\hat{u})q - \nabla p_q$, which seem to give us no hint for the nonstationary case. A more promising idea to obtain a positive answer is to adapt the procedure in [5] to the boundary control case, even if we can realize that the adaptation is not straightforward because of new difficulties we will encounter, namely some regularity issues and the "tighter" compatibility conditions relating the solution and the control. In other words, we need to develop first some tools in order to be able to adapt the procedure to the boundary control case.

One of the main ingredients in [5] is a suitable internal truncated observability inequality for system (1.2), where the truncation is closely related to the finite-dimensional control space; this inequality was derived by truncating the "observed space" in a well-known observability inequality we find in reference [19].

The work [5] and the relation between observability inequalities for the adjoint Oseen–Stokes system (1.2) and controllability properties for the Navier–Stokes system (1.1) are the main motivations of this paper. We establish appropriate observability inequalities for (1.2) to deal with boundary control problems for (1.1), in particular to deal with the constraints on the finite dimension and on the support of the boundary controls ζ. To give an idea, from the results we will derive in Section 4.2, we can conclude that the solution of (1.2), in the case $f = 0$ and $I \times \Omega = (a,\, b) \times \Omega$, satisfies

$$|q(a)|^2_{L^2(\Omega,\,\mathbb{R}^3)} \leq \overline{C}_{[|\hat{u}|_{\mathcal{W}}]} |P^{\Gamma}_M \chi (p_q \mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{L^2((a,\,b),\,L^2(\Gamma,\,\mathbb{R}^3))},$$

where $\overline{C}_{[|\hat{u}|_{\mathcal{W}}]}$ is some constant depending on the norm of the reference solution $\hat{u}$ in an appropriate Banach space $\mathcal{W}$, $P^{\Gamma}_M : L^2(\Gamma,\,\mathbb{R}^3) \to L^2_M(\Gamma,\,\mathbb{R}^3)$ is a projection onto an M-dimensional space $L^2_M(\Gamma,\,\mathbb{R}^3)$, and $\chi : \Gamma \to \mathbb{R}$ is an a priori given smooth function. This inequality can be related to control problems for system (1.1) where the controls take their values in the "adjoint" finite dimensional space $\chi L^2_M(\Gamma,\,\mathbb{R}^3) = \chi P^{\Gamma}_M L^2(\Gamma,\,\mathbb{R}^3)$, that is, $\zeta : (a,\, b) \to \chi L^2_M(\Gamma,\,\mathbb{R}^3)$; the support of the controls is then necessarily contained in that of χ.

As an illustration, we use the derived observability inequalities, to obtain a new controllability result: let $\{e_i \mid i \in \mathbb{N}_0\}$ be the eigenvector fields of the Stokes operator, forming an orthogonal basis for the subspace $H \subset L^2(\Omega,\,\mathbb{R}^3)$ of solenoidal vector fields. Then, we can construct a family $\{\chi\Psi_n \mid n \in \mathbb{N}_0\} \subset C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}^3))$ such that for any given $N \in \mathbb{N}_0$, there is a positive integer $M_{N,\,|\hat{u}|_{\mathcal{W}}}$ depending on the pair $(N,\,|\hat{u}|_{\mathcal{W}})$ with the following property: for any given $v_0 \in H$, there is $\kappa(v_0) \in \mathbb{R}^{M_{N,\,|\hat{u}|_{\mathcal{W}}}}$, such that the control $\zeta = \chi\sum_{n=1}^{M_{N,\,|\hat{u}|_{\mathcal{W}}}} \kappa_n\Psi_n$, drives the Oseen–Stokes system

$$\partial_t v + \mathcal{B}(\hat{u})v - \nu\Delta v + \nabla p_v = 0, \quad \operatorname{div} v = 0, \quad v|_{\Gamma} = \zeta,$$

from $v(a) = v_0 \in H$, at time $t = a$, to a vector field $v(b) \in H$, at time $t = b$, with $(v(b),\, e_i)_{L^2(\Omega,\,\mathbb{R}^3)} = 0$ for all $i \leq N$. Roughly speaking, there is a control ζ, that can be "realized" by $M_{N,\,|\hat{u}|_{\mathcal{W}}}$ constants, driving the (first N) less stable Stokes modes to zero. Further, the mapping $v_0 \mapsto \zeta(v_0)$ is linear and continuous.

The rest of the paper is organized as follows. In Section 2, we introduce the functional spaces arising in the theory of the Navier–Stokes equations, set up our problem, and recall some well-known facts. In Sections 3 and 4 we derive some boundary observability inequalities including some appropriate to deal with finite-dimensional controls supported in a given subset of the boundary. In Section 5 we illustrate/recall how the observability inequalities can be used to obtain controllability results, deriving two new controllability results. In Section 6 we give some remarks and discuss some further plausible consequences of the derived inequalities and of the controllability results derived in Section 5; namely, the boundary versions of the internal results in [5] and [29] concerning, respectively, the stabilization to a nonstationary solution of the Navier–Stokes equations and a property of the stochastic version of the same equations. Finally, the Appendix gathers some auxiliary results used in the main text.

Notation. We write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$. We denote by $\Omega \subset \mathbb{R}^3$ a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. Given a vector function $u : (t, x_1, x_2, x_3) \mapsto u(t, x_1, x_2, x_3) \in \mathbb{R}^3$, defined in an open subset of $\mathbb{R} \times \Omega$, its partial time derivative $\frac{\partial u}{\partial t}$ will be denoted by $\partial_t u$. Also the partial spatial derivatives $\frac{\partial u}{\partial x_i}$ will be denoted by $\partial_{x_i} u$.

Given a Banach space X and an open subset $O \subset \mathbb{R}^n$, let us denote by $L^p(O, X)$, with either $p \in [1, +\infty)$ or $p = \infty$, the Bochner space of measurable functions $f : O \to X$, and such that $|f|_X^p$ is integrable over O, for $p \in [1, +\infty)$, and such that $\operatorname{ess\,sup}_{x \in O} |f(x)|_X < +\infty$, for $p = \infty$. In the case $X = \mathbb{R}$ we recover the usual Lebesgue spaces, and $L^p(O, \mathbb{R}^k) \sim L^p(O, \mathbb{R})^k$, $k \in \mathbb{N}_0$. By $W^{s,p}(O, \mathbb{R}^k)$, for $s \in \mathbb{R}$, denote the usual Sobolev space of order s. In the case $p = 2$, as usual, we denote $H^s(O, \mathbb{R}^k) := W^{s,2}(O, \mathbb{R}^k)$. Recall that $H^0(O, \mathbb{R}^k) = L^2(O, \mathbb{R}^k)$. For each $s > 0$, we recall also that $H^{-s}(O, \mathbb{R}^k)$ stands for the dual space of $H_0^s(O, \mathbb{R}^k) =$ closure of $\{f \in C^\infty(O, \mathbb{R}^k) \mid \operatorname{supp}(f) \subset O\}$ in $H^s(O, \mathbb{R}^k)$. Notice that $H^{-s}(O, \mathbb{R}^k)$ is a space of distributions.

For a normed space X, we denote by $|\cdot|_X$ the corresponding norm, by X' its dual, and by $\langle \cdot, \cdot \rangle_{X',X}$ the duality between X' and X. The dual space is endowed with the usual dual norm: $|f|_{X'} := \sup\{\langle f, x\rangle_{X',X} \mid x \in X \text{ and } |x|_X = 1\}$.

Let X and Y be normed spaces, and let Z be a Hausdorff topological space. Suppose that both inclusions $X \subseteq Z$ and $Y \subseteq Z$ are continuous; then the Cartesian product $X \times Y$, the intersection $X \cap Y$ and the sum $X + Y$ are supposed to be endowed with the norms $|(a, b)|_{X \times Y} := \left(|a|_X^2 + |b|_Y^2\right)^{\frac{1}{2}}$; $|a|_{X \cap Y} := |(a, a)|_{X \times Y}$; and $|a|_{X+Y} := \inf_{(a^X, a^Y) \in X \times Y} \left\{|(a^X, a^Y)|_{X \times Y} \mid a = a^X + a^Y\right\}$, respectively. We can show that, if X and Y are endowed with a scalar product, then also $X \times Y$, $X \cap Y$, and $X + Y$ are. In the case we know that $X \cap Y = \{0\}$, we say that $X + Y$ is a direct sum and we write $X \oplus Y$ instead.

Given an open interval $I \subseteq \mathbb{R}$ and two Banach spaces X, Y, then we write $W(I, X, Y) := \{f \in L^2(I, X) \mid \partial_t f \in L^2(I, Y)\}$, where the derivative $\partial_t f$ is taken in the sense of distributions. This space is endowed with the natural norm $|f|_{W(I, X, Y)} := \left(|f|^2_{L^2(I, X)} + |\partial_t f|^2_{L^2(I, Y)}\right)^{\frac{1}{2}}$. In the case $X = Y$ we write $H^1(I, X) := W(I, X, X)$. Again, if X and Y are endowed with a scalar product, then also $W(I, X, Y)$ is. The space of continuous linear mappings from X into Y will be denoted by $\mathcal{L}(X \to Y)$.

If $\bar{I} \subset \mathbb{R}$ is a closed bounded interval, $C(\bar{I}, X)$ stands for the space of continuous functions $f : \bar{I} \to X$ with the norm $|f|_{C(\bar{I}, X)} = \max_{t \in \bar{I}} |f(t)|_X$.

$\overline{C}_{[a_1, \ldots, a_k]}$ denotes a nonnegative function of nonnegative variables a_j that increases in each of its arguments.

C, C_i, $i = 1, 2, \ldots$, stand for unessential positive constants.

2. Preliminaries

2.1. Functional spaces

Let $\Omega \subset \mathbb{R}^3$ be a connected bounded domain of class C^∞ located locally on one side of its boundary $\Gamma = \partial\Omega$, with $\int_\Gamma \mathrm{d}\Gamma < +\infty$. Due to the incompressibility condition, $\operatorname{div} u = 0$, some important subspaces in the study of the systems (1.1) and (1.2) are the Lebesgue and Sobolev subspaces

$$L^r_{\mathrm{div}}(\Omega, \mathbb{R}^3) := \{u \in L^r(\Omega, \mathbb{R}^3) \mid \operatorname{div} u = 0 \text{ in } \Omega\}, \quad 1 \le r \le +\infty,$$
$$H^s_{\mathrm{div}}(\Omega, \mathbb{R}^3) := \{u \in H^s(\Omega, \mathbb{R}^3) \mid \operatorname{div} u = 0 \text{ in } \Omega\}, \quad s \ge 0.$$

The incompressibility condition allows us to define the trace of $u \cdot \mathbf{n}$ on the boundary $\varGamma = \partial\varOmega$, where $\mathbf{n}$ is the unit outward normal vector to the boundary $\varGamma$, and then to write

$$H := \{u \in L^2_{\mathrm{div}}(\varOmega, \mathbb{R}^3) \mid u \cdot \mathbf{n} = 0 \text{ on } \varGamma\}, \quad H_{\mathrm{c}} := \{u \in L^2_{\mathrm{div}}(\varOmega, \mathbb{R}^3) \mid u \cdot \mathbf{n} = 0 \text{ on } \varGamma \setminus \overline{\varGamma_{\mathrm{c}}}\},$$

where $\varGamma_{\mathrm{c}}$ is an open subset of $\varGamma$. Some spaces of more regular vector fields we find throughout the paper are

$$\begin{gathered} V := \{u \in H^1_{\mathrm{div}}(\varOmega, \mathbb{R}^3) \mid u = 0 \text{ on } \varGamma\}, \quad V_{\mathrm{c}} := \{u \in H^1_{\mathrm{div}}(\varOmega, \mathbb{R}^3) \mid u = 0 \text{ on } \varGamma \setminus \overline{\varGamma_{\mathrm{c}}}\}, \\ \mathrm{D}(L) := V \cap H^2(\varOmega, \mathbb{R}^3). \end{gathered} \tag{2.1}$$

The spaces $H^s_{\mathrm{div}}(\varOmega, \mathbb{R}^3)$ are endowed with the scalar product inherited from $H^s(\varOmega, \mathbb{R}^3)$; the spaces H and H_{c} with that inherited from $L^2(\varOmega, \mathbb{R}^3)$; the spaces V and V_{c} with that inherited from $H^1(\varOmega, \mathbb{R}^3)$; and $\mathrm{D}(L)$ with that inherited from $H^2(\varOmega, \mathbb{R}^3)$. Notice that if $\varPi$ is the orthogonal projection in $L^2(\varOmega, \mathbb{R}^3)$ onto H, it is well known that $\mathrm{D}(L)$ coincides with the domain $\{u \in V | Lu \in H\}$ of the Stokes operator $L := -\nu\varPi\varDelta$. That is the reason for the notation.

Next, fix a constant $\sigma > \frac{6}{5}$. For any pair of real numbers a, b, with $a < b$, we introduce the Banach spaces $\mathcal{W}^{(a,\,b)|\mathrm{wk}}$ and $\mathcal{W}^{(a,\,b)|\mathrm{st}}$ of the measurable vector functions $u = (u_1, u_2, u_3)$, defined in $(a,\, b) \times \varOmega$, satisfying

$$\begin{aligned} |u|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}} &:= \left(|u|^2_{L^\infty((a,\,b),\,L^\infty_{\mathrm{div}}(\varOmega,\mathbb{R}^3))} + |\partial_t u|^2_{L^2((a,\,b),\,L^\sigma(\varOmega,\mathbb{R}^3))}\right)^{\frac{1}{2}} < \infty \\ |u|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}} &:= \left(|u|^2_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}} + |\nabla u|^2_{L^2((a,\,b),\,L^3(\varOmega,\mathbb{R}^9))}\right)^{\frac{1}{2}} < \infty. \end{aligned} \tag{2.2}$$

Remark 2.1. The lower bound $\frac{6}{5}$ for σ is motivated from the results in [12,27].

Now, we recall that, in [14], the set of traces $u|_\varGamma$ at the boundary $\varGamma$ of the elements u in the space $W((a,\,b),\, H^s_{\mathrm{div}}(\varOmega,\, \mathbb{R}^3),\, H^{s-2}(\varOmega,\, \mathbb{R}^3))$ is completely characterized, for each $s > \frac{1}{2}$, with $s \notin \left\{\frac{3}{2},\, \frac{5}{2}\right\}$. Denoting that trace space by $G^s_{\mathrm{av}}((a,\,b),\,\varGamma)$, we have that $v \mapsto v|_\varGamma$ is continuous:

$$\left|v|_\varGamma\right|_{G^s_{\mathrm{av}}((a,\,b),\,\varGamma)} \leq C_1 |w|_{W((a,\,b),\,H^s_{\mathrm{div}}(\varOmega,\mathbb{R}^3),\,H^{s-2}(\varOmega,\mathbb{R}^3))}$$

and, there is a continuous extension $E_s : G^s_{\mathrm{av}}((a,\,b),\,\varGamma) \to W((a,\,b),\, H^s_{\mathrm{div}}(\varOmega,\,\mathbb{R}^3),\, H^{s-2}(\varOmega,\,\mathbb{R}^3))$ such that

$$(E_s w)|_\varGamma = w \text{ and } |E_s w|_{W((a,\,b),\,H^s_{\mathrm{div}}(\varOmega,\mathbb{R}^3),\,H^{s-2}(\varOmega,\mathbb{R}^3))} \leq C_2 |w|_{G^s_{\mathrm{av}}((a,\,b),\,\varGamma)}.$$

Moreover, from ([14], Sect. 2.2), we know that

$$G^s_{\mathrm{av}}((a,\,b),\,\varGamma) = G^s_{\mathbf{t}}((a,\,b),\,\varGamma) \oplus G^s_{\mathbf{n},\mathrm{av}}((a,\,b),\,\varGamma)\mathbf{n},$$

with $\begin{cases} G^s_{\mathbf{t}}((a,\,b),\,\varGamma) = L^2((a,\,b),\, H^{s-\frac{1}{2}}(\varGamma,\, T\varGamma)) \cap H^{r_{\mathbf{t},1}(s)}((a,\,b),\, H^{r_{\mathbf{t},2}(s)}(\varGamma,\, T\varGamma)) \\ G^s_{\mathbf{n},\mathrm{av}}((a,\,b),\,\varGamma) = L^2((a,\,b),\, H^{s-\frac{1}{2}}_{\mathrm{av}}(\varGamma,\,\mathbb{R})) \cap H^{r_{\mathbf{n},1}(s)}((a,\,b),\, H^{r_{\mathbf{n},2}(s)}_{\mathrm{av}}(\varGamma,\,\mathbb{R})) \end{cases}$; and where $H^r_{\mathrm{av}}(\varGamma,\,\mathbb{R}) := \{\varkappa \in H^r(\varGamma,\,\mathbb{R}) \mid \int_\varGamma \varkappa \,\mathrm{d}\varGamma = 0\}$, and $r_{\mathbf{t},1}(s)$, $r_{\mathbf{t},2}(s)$, $r_{\mathbf{n},1}(s)$, $r_{\mathbf{n},2}(s)$ are constants, in $\mathbb{R}$, given by

$$(r_{\mathbf{t},1}(s),\, r_{\mathbf{t},2}(s)) = \begin{cases} \left(1,\, s - \frac{5}{2}\right) & \text{if } \frac{5}{2} < s \\ \left(\frac{2s-1}{4},\, 0\right) & \text{if } 2 \leq s < \frac{5}{2} \\ \left(\frac{2s-1}{2s},\, \frac{(s-2)(2s-1)}{2s}\right) & \text{if } \frac{1}{2} < s \leq 2,\ s \neq \frac{3}{2} \end{cases},$$

$$(r_{\mathbf{n},1}(s),\, r_{\mathbf{n},2}(s)) = \begin{cases} \left(1,\, s - \frac{5}{2}\right) & \text{if } \frac{3}{2} < s,\ s \neq \frac{5}{2} \\ \left(\frac{2s+1}{4},\, -1\right) & \text{if } 1 \leq s < \frac{3}{2} \\ \left(\frac{2s+1}{2s+2},\, \frac{2s^2-3(s+1)}{2s+2}\right) & \text{if } \frac{1}{2} < s \leq 1 \end{cases}.$$

The space $G^s_{\mathrm{av}}((a,\,b),\,\varGamma)$, if nothing is said in contrary, is supposed to be endowed with the scalar product $(u,\,v)_{G^s_{\mathrm{av}}((a,\,b),\,\varGamma)} = (u^{\mathbf{t}} + u^{\mathbf{n}}\mathbf{n},\, v^{\mathbf{t}} + v^{\mathbf{n}}\mathbf{n})_{G^s_{\mathrm{av}}((a,\,b),\,\varGamma)} := (u^{\mathbf{t}},\, v^{\mathbf{t}})_{G^s_{\mathbf{t}}((a,\,b),\,\varGamma)} + (u^{\mathbf{n}},\, v^{\mathbf{n}})_{G^s_{\mathbf{n},\mathrm{av}}((a,\,b),\,\varGamma)}$.

Remark 2.2. The notation $T\Gamma$, in the definition of $G^s_{\mathbf{t}}((a,\,b),\,\Gamma)$, stands for the tangent bundle of Γ; the notation underlines that, at (almost) every instant of time, the value $u(t)$ of an element $u \in G^s_{\mathbf{t}}((a,\,b),\,\Gamma)$ is a vector function tangent to Γ, that is, a vector field in Γ.

Remark 2.3. Notice that the integral $\int_\Gamma \varkappa \,\mathrm{d}\Gamma = 0$ is well-defined, in the sense of distributions, for $\varkappa \in H^r(\Gamma,\,\mathbb{R})$ and all $r \in \mathbb{R}$: for $r \geq 0$, we have $\varkappa \in L^2(\Gamma,\,\mathbb{R})$ and the integral is well-defined; on the other hand for $r < 0$, we have that $H^r(\Gamma,\,\mathbb{R})$ coincides with the dual space of $H^{-r}(\Gamma,\,\mathbb{R})$ (because $\partial\Gamma = \emptyset$), then since the constant function $1_\Gamma(x) := 1,\ x \in \Gamma$, is in $H^{-r}(\Gamma,\,\mathbb{R})$ (because $\int_\Gamma \mathrm{d}\Gamma$ is bounded), the integral $\int_\Gamma \varkappa \,\mathrm{d}\Gamma := \langle \varkappa,\, 1_\Gamma\rangle_{H^r(\Gamma,\,\mathbb{R}),H^{-r}(\Gamma,\,\mathbb{R})}$ is well-defined (considering, as usual, $L^2(\Gamma,\,\mathbb{R})$ as a pivot space).

For technical reasons we relax a little the trace spaces: we define the superspace $G^s((a,\,b),\,\Gamma)$ of $G^s_{\mathrm{av}}((a,\,b),\,\Gamma)$ by just omitting the average constraint:

$$G^s((a,\,b),\,\Gamma) := G^s_{\mathbf{t}}((a,\,b),\,\Gamma) \oplus G^s_{\mathbf{n}}((a,\,b),\,\Gamma)\mathbf{n} \tag{2.3}$$

with $G^s_{\mathbf{n}}((a,\,b),\,\Gamma) := L^2((a,\,b),\,H^{s-\frac{1}{2}}(\Gamma,\,\mathbb{R})) \cap H^{r_{\mathbf{n},1}(s)}((a,\,b),\,H^{r_{\mathbf{n},2}(s)}(\Gamma,\,\mathbb{R}))$. The space $G^s((a,\,b),\,\Gamma)$ is endowed with the scalar product $(u,\,v)_{G^s((a,\,b),\,\Gamma)} = (u^{\mathbf{t}} + u^{\mathbf{n}}\mathbf{n},\, v^{\mathbf{t}} + v^{\mathbf{n}}\mathbf{n})_{G^s_{\mathrm{av}}((a,\,b),\,\Gamma)} := (u^{\mathbf{t}},\, v^{\mathbf{t}})_{G^s_{\mathbf{t}}((a,\,b),\,\Gamma)} + (u^{\mathbf{n}},\, v^{\mathbf{n}})_{G^s_{\mathbf{n}}((a,\,b),\,\Gamma)}$.

Proposition 2.4. *We have that* $G^s((a,\,b),\,\Gamma) = G^s_{\mathrm{av}}((a,\,b),\,\Gamma) \oplus H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\mathbf{n}$. *Moreover, for* $\pi_u := \frac{1}{\int_\Gamma \mathrm{d}\Gamma}\int_\Gamma u\cdot\mathbf{n}\,\mathrm{d}\Gamma$, *the projections*

$$\begin{aligned}\pi^s : G^s((a,\,b),\,\Gamma) &\to H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\mathbf{n} \\ u &\mapsto \pi_u\mathbf{n}\end{aligned} \quad \textit{and} \quad \begin{aligned}1-\pi^s : G^s((a,\,b),\,\Gamma) &\to G^s_{\mathrm{av}}((a,\,b),\,\Gamma) \\ u &\mapsto u_{\mathrm{av}} := u - \pi^s u\end{aligned}$$

are continuous.

From ([34], Chap. 1, Prop. 2.3), there exists a unique vector function $\Theta \in H^2(\Omega,\,\mathbb{R}^3)$ solving the Stokes system

$$-\nu\Delta\Theta + \nabla p = 0,\ \operatorname{div}\Theta = \frac{\int_\Gamma \mathrm{d}\Gamma}{\int_\Omega \mathrm{d}\Omega} \text{ in } \Omega,\ \text{ and } \Theta\big|_\Gamma = \mathbf{n} \text{ on } \Gamma.$$

Now, we can extend the extension E_s above, defined in $G^s_{\mathrm{av}}((a,\,b),\,\Gamma)$, to $G^s((a,\,b),\,\Gamma)$:

Proposition 2.5. *Writing each* $u \in G^s((a,\,b),\,\Gamma)$ *as* $u = u_{\mathrm{av}} + \pi_u\mathbf{n}$, *we define*

$$\begin{aligned}E^e_s : G^s((a,\,b),\,\Gamma) &\to W((a,\,b),\,H^s_{\mathrm{div}}(\Omega,\,\mathbb{R}^3),\,H^{s-2}(\Omega,\,\mathbb{R}^3)) \oplus H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\Theta \\ u &\mapsto E_s u_{\mathrm{av}} + \pi_u\Theta\end{aligned}$$

and we endow the space $H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\Theta$ *with the scalar product*

$$(\varkappa_1\Theta,\,\varkappa_2\Theta)_{H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\Theta} := (\varkappa_1,\,\varkappa_2)_{H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})}.$$

Then, E^e_s *extends* E_s *and is linear and continuous. Moreover, the trace mapping* $v \mapsto v|_\Gamma$ *from the space* $W((a,\,b),\,H^s_{\mathrm{div}}(\Omega,\,\mathbb{R}^3),\,H^{s-2}(\Omega,\,\mathbb{R}^3)) \oplus H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\Theta$ *onto* $G^s((a,\,b),\,\Gamma))$ *is also linear and continuous.*

The proofs of Propositions 2.4 and 2.5 will be given in the Appendix, Section A.3.

2.2. The (illustrating) control space

Let us write $L^2(\Omega,\mathbb{R}^3) = H \oplus H^\perp$, where $H^\perp = \{\nabla\xi \mid \xi \in H^1(\Omega,\,\mathbb{R})\}$ denotes the orthogonal complement of H in $L^2(\Omega,\mathbb{R}^3)$, and denote by Π the orthogonal projection $\Pi : L^2(\Omega,\mathbb{R}^3) \to H$ in $L^2(\Omega,\mathbb{R}^3)$ onto H. For each positive integer N, we now define the N-dimensional space $H_N \subset H$ as follows: let $\{e_i \mid i \in \mathbb{N}_0\}$ be the orthonormal basis in H formed by the eigenfunctions of the Stokes operator L, whose domain is defined by (2.1), and let $0 < \alpha_1 \le \alpha_2 \le \dots$ be the corresponding eigenvalues: $Le_i = \alpha_i e_i$, then put

$$H_N := \operatorname{span}\{e_i \mid i \le N\} \subset \mathrm{D}(L) \subset H, \tag{2.4}$$

and denote by Π_N the orthogonal projection $\Pi_N : H \to H_N$ in H onto H_N.

Let $\mathcal{O} \subseteq \Gamma$ be a connected open subset of the boundary Γ, localized on one side of its boundary. We suppose that $\mathcal{O}$ is a C^∞-smooth manifold, either boundaryless or with C^∞-smooth boundary $\partial\mathcal{O}$. Let $\{\pi_i \mid i \in \mathbb{N}_0\}$ be an orthonormal basis in $L^2(\mathcal{O}, \mathbb{R})$ formed by the eigenfunctions of the Laplace–de Rham (or Laplace–Beltrami) operator $\Delta_{\mathcal{O}}$ on the smooth manifold $\mathcal{O}$, under Dirichlet boundary conditions, $\pi_i(p) = 0$ for all $p \in \partial\mathcal{O}$. Analogously let $\{\tau_i \mid i \in \mathbb{N}_0\}$ be an orthonormal basis in $L^2(\mathcal{O}, T\mathcal{O})$ formed by the vector fields that are eigenfunctions of $\Delta_{\mathcal{O}}$ on $T\mathcal{O}$, also under Dirichlet boundary conditions in the case $\partial\mathcal{O} \ne \emptyset$, $\tau_i(p) = 0 \in T_p\Gamma$ for all $p \in \partial\mathcal{O}$. It is known that π_i and τ_i $(i \in \mathbb{N}_0)$ are smooth. Let $0 \le \beta_1 \le \beta_2 \le \dots$, and $0 \le \gamma_1 \le \gamma_2 \le \dots$ be the eigenvalues associated with the systems $\{\pi_i \mid i \in \mathbb{N}_0\}$ and $\{\tau_i \mid i \in \mathbb{N}_0\}$, respectively.

We may write $L^2(\mathcal{O}, \mathbb{R}^3)$ as an orthogonal sum $L^2(\mathcal{O}, \mathbb{R}^3) = L^2(\mathcal{O}, \mathbb{R})\mathbf{n} \oplus L^2(\mathcal{O}, T\mathcal{O})$. Notice that $\{\pi_i\mathbf{n} \mid i \in \mathbb{N}_0\}$ is an orthonormal basis for $L^2(\mathcal{O}, \mathbb{R})\mathbf{n} = \{f\mathbf{n} \mid f \in L^2(\mathcal{O}, \mathbb{R})\}$, and the system $\{\pi_i\mathbf{n} \mid i \in \mathbb{N}_0\} \cup \{\tau_i \mid i \in \mathbb{N}_0\}$ is an orthonormal basis in the space $L^2(\mathcal{O}, \mathbb{R}^3)$.

Define, for each $M \in \mathbb{N}_0$, the space

$$L^2_M(\mathcal{O}, \mathbb{R}^3) := \operatorname{span}\{\pi_i\mathbf{n},\, \tau_i \mid i \in \mathbb{N}_0,\, i \le M\} \tag{2.5}$$

and, denote by $P_M^{\mathcal{O}}$ the orthogonal projection $P_M^{\mathcal{O}} : L^2(\mathcal{O}, \mathbb{R}^3) \to L^2_M(\mathcal{O}, \mathbb{R}^3)$ in $L^2(\mathcal{O}, \mathbb{R}^3)$ onto $L^2_M(\mathcal{O}, \mathbb{R}^3)$.

We suppose we are able to apply a control through a subset $\Gamma_\mathrm{c} \subseteq \overline{\Gamma_\mathrm{c}} \subseteq \mathcal{O} \subseteq \Gamma$, where $\overline{\Gamma_\mathrm{c}} = \operatorname{supp}(\chi)$ is the support of a function $\chi \in C^\infty(\Gamma, \mathbb{R})$. Further let $\epsilon > 0$ and let $\vartheta \in C^2(\Gamma, \mathbb{R})$ be a function such that for all $x \in \Gamma_\mathrm{c}$, $\vartheta(x) \ge \varepsilon$ and with $\operatorname{supp}(\vartheta) \subseteq \overline{\mathcal{O}}$. For an illustration purpose, we will give particular attention to the case where the boundary control ζ is in the space

$$\begin{aligned}\mathcal{E}^1_M &:= \chi\mathbb{E}_0^{\mathcal{O}} P^{\mathcal{O}}_{\chi^\perp} P^{\mathcal{O}}_M \vartheta G^1((a,\,b),\,\Gamma)|_{\mathcal{O}}\\ &:= \{\zeta \mid \zeta(t) = \chi\mathbb{E}_0^{\mathcal{O}} P^{\mathcal{O}}_{\chi^\perp} P^{\mathcal{O}}_M(\vartheta\eta(t)|_{\mathcal{O}}) \text{ and } \eta \in G^1((a,\,b),\,\Gamma)\}\end{aligned} \tag{2.6}$$

where $\mathbb{E}_0^{\mathcal{O}} : L^2(\mathcal{O}, \mathbb{R}) \to L^2(\Gamma, \mathbb{R})$ stands for the extension by zero outside $\mathcal{O}$, and $P^{\mathcal{O}}_{\chi^\perp} : L^2(\mathcal{O}, \mathbb{R}^3) \to \{f \in L^2(\mathcal{O}, \mathbb{R}^3) \mid (f,\, \chi\mathbf{n})_{L^2(\mathcal{O},\mathbb{R}^3)} = 0\}$ is the orthogonal projection in $L^2(\mathcal{O}, \mathbb{R}^3)$ onto $\{\chi\mathbf{n}|_{\mathcal{O}}\}^\perp$. In other words,

$$\mathbb{E}_0^{\mathcal{O}}\xi(x) := \begin{cases}\xi(x) & \text{if } x \in \mathcal{O}\\ 0 & \text{if } x \in \Gamma \setminus \overline{\mathcal{O}}\end{cases} \qquad \text{and} \qquad P^{\mathcal{O}}_{\chi^\perp} v := v - \frac{(v,\, \chi\mathbf{n})_{L^2(\mathcal{O},\,\mathbb{R}^3)}}{\int_{\mathcal{O}} \chi^2 \,\mathrm{d}\mathcal{O}} \chi\mathbf{n}|_{\mathcal{O}}\,.$$

In particular the controls take their values $\zeta(t)$, for a.e. $t \in (a,\, b)$, in the finite-dimensional space spanned by $\{\chi\mathbb{E}_0^{\mathcal{O}} P^{\mathcal{O}}_{\chi^\perp}\pi_i\mathbf{n},\, \chi\mathbb{E}_0^{\mathcal{O}}\tau_i \mid i \in \mathbb{N}_0, i \le M\}$.

Remark 2.6. Notice that the function $\zeta(t) = \chi\mathbb{E}_0^{\mathcal{O}} P^{\mathcal{O}}_{\chi^\perp} P^{\mathcal{O}}_M(\vartheta\eta(t)|_{\mathcal{O}})$ satisfies the zero-average compatibility condition: $\int_\Gamma \zeta(t) \cdot \mathbf{n}\,\mathrm{d}\Gamma = \int_{\mathcal{O}} P^{\mathcal{O}}_{\chi^\perp} P^{\mathcal{O}}_M(\vartheta\eta(t)|_{\mathcal{O}}) \cdot \chi\mathbf{n}|_{\mathcal{O}}\,\mathrm{d}\mathcal{O} = 0$. The function χ guarantees that the controls are supported in $\overline{\Gamma_\mathrm{c}}$; the function ϑ is needed because we will need suitable continuity properties (*cf.* Props. 2.17 and 5.1, needed in the proofs of Thms. 5.2 and 5.3, respectively). Further, as we said, we propose the space (2.6) mainly as an example guideline; the arguments that will follow may work for other (admissible) control spaces (*cf.* Sect. 5 where we consider a variation of this control space).

2.3. The addressed problem

Consider the following time-forward Oseen–Stokes system, in $(a,\, b) \times \Omega$,

$$\begin{aligned}\partial_t v + \mathcal{B}(\hat{u})v - \nu\Delta v + \nabla p_v + g = 0, &\qquad \operatorname{div} v = 0,\\ v|_\Gamma = \zeta, &\qquad v(a) = v_0,\end{aligned} \tag{2.7}$$

where $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{wk}}$, $g \in L^2((a,\,b),\,H^{-1}(\Omega,\,\mathbb{R}^3))$, $\zeta \in G^1_{\mathrm{av}}((a,\,b),\,\Gamma)$ is a control, $v_0 \in L^2_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)$, and $\mathcal{B}(\hat{u}) : v \mapsto \langle \hat{u}\cdot\nabla\rangle v + \langle v\cdot\nabla\rangle\hat{u}$.

We will start by the derivation of some observability inequalities concerning the "adjoint" Oseen–Stokes time-backward system, in $(a,\,b)\times\Omega$,

$$\begin{aligned} -\partial_t q + \mathcal{B}^*(\hat{u})q - \nu\Delta q + \nabla p_q + f = 0, &\qquad \operatorname{div} q = 0,\\ q\big|_\Gamma = 0, &\qquad q(b) = q_1 \in H, \end{aligned} \tag{2.8}$$

where $f \in L^2((a,\,b),\,H^{-1}(\Omega,\,\mathbb{R}^3))$ and $\mathcal{B}^*(\hat{u})$ is the formal adjoint to $\mathcal{B}(\hat{u})$, that is,

$$(\mathcal{B}^*(\hat{u})q,\,v)_{L^2(\Omega,\,\mathbb{R}^3)} := \langle q,\,\mathcal{B}(\hat{u})v\rangle_{H^1_0(\Omega,\,\mathbb{R}^3),\,H^{-1}(\Omega,\,\mathbb{R}^3)},\quad q\in V,\, v\in H^1_{\mathrm{div}}(\Omega,\,\mathbb{R}^3). \tag{2.9}$$

Then, two of the derived observability inequalities will be used to obtain two new controllability results to the Oseen–Stokes system (2.7), where (a subspace of) $\mathcal{E}^1_M$ is taken as the space of the controls.

Remark 2.7. Notice that for $D_s q := \left(\nabla q + (\nabla q)^\top\right)$, where $A^\top$ denotes the transpose matrix of A, we have $\mathcal{B}^*(\hat{u})q = \langle \hat{u}\cdot D_s\rangle q$, with $w = (w_1,\,w_2,\,w_3) := \langle \hat{u}\cdot D_s\rangle q$ given by $w_j = \sum_{i=1}^3 \hat{u}_i(\partial_{x_i} q_j + \partial_{x_j} q_i)$ for all $j\in\{1,\,2,\,3\}$. In particular we have $(\langle u\cdot D_s\rangle q)\cdot v = (\langle v\cdot D_s\rangle q)\cdot u$, for any given pair of vectors $u,\,v$ in $\mathbb{R}^3$.

2.4. Existence and uniqueness of weak and strong solutions

Here we present some remarks concerning the solutions of the considered systems (2.7) and (2.8). Among the spaces $G^s_{\mathrm{av}}((a,\,b),\,\Gamma)$, the most interesting for us will be the ones corresponding to $s\in\{1,\,2\}$, that will be related to so-called weak and strong solutions. Recall the extensions E_s in Section 2.1.

Definition 2.8. Given $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{wk}}$, $v_0 \in L^2_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)$, $g \in L^2((a,\,b),\,H^{-1}(\Omega,\,\mathbb{R}^3))$, and $\zeta \in G^1_{\mathrm{av}}((a,\,b),\,\Gamma)$; we say that v, in the space $W((a,\,b),\,H^1_{\mathrm{div}}(\Omega,\,\mathbb{R}^3),\,H^{-1}(\Omega,\,\mathbb{R}^3))$, is a weak solution for system (2.7), if $v - E_1\zeta \in L^2((a,\,b),\,V,\,V')$ is a weak solution for the system

$$\begin{aligned} \partial_t y + \mathcal{B}(\hat{u})y - \nu\Delta y + \nabla p_y + f = 0, &\qquad \operatorname{div} y = 0,\\ y\big|_\Gamma = 0, &\qquad y(a) = y_0 \end{aligned} \tag{2.10}$$

with $f = g + \partial_t E_1\zeta + \mathcal{B}(\hat{u})E_1\zeta - \nu\Delta E_1\zeta$, and $y_0 = v_0 - E_1\zeta(a)\in H$. Here weak solution for (2.10) is understood in the classical sense (*cf.* [22, 32, 34]).

Definition 2.9. Given $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{st}}$, $v_0 \in H^1_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)$, $g \in L^2((a,\,b),\,L^2(\Omega,\,\mathbb{R}^3))$, and $\zeta \in G^2_{\mathrm{av}}((a,\,b),\,\Gamma)$; we say that v, in the space $W((a,\,b),\,H^2_{\mathrm{div}}(\Omega,\,\mathbb{R}^3),\,L^2(\Omega,\,\mathbb{R}^3))$, is a strong solution for system (2.7), if $v - E_2\zeta \in L^2((a,\,b),\,\mathrm{D}(L),\,H)$ is a strong solution for system (2.10) with $f = g + \partial_t E_2\zeta + \mathcal{B}(\hat{u})E_2\zeta - \nu\Delta E_2\zeta$, and $y_0 = v_0 - E_2\zeta(a)\in V$. Again, strong solution for (2.10) is understood in the classical sense (*cf.* [32], Sect. 2.4).

Remark 2.10. The existence and uniqueness of a weak solution in $W((a,\,b),\,V,\,V')$ for (2.10), can be proved by standard arguments as in [34] taking into account that, formally

$$\left\langle\, \mathcal{B}(\hat{u})y,\,w\,\right\rangle_{H^{-1}(\Omega,\,\mathbb{R}^3),H^1_0(\Omega,\,\mathbb{R}^3)} = -\sum_{i,\,j=1}^3\int_\Omega \hat{u}_i(\partial_{x_i}w_j)y_j\,\mathrm{d}x - \sum_{i,\,j=1}^3\int_\Omega y_i(\partial_{x_i}w_j)\hat{u}_j\,\mathrm{d}x,$$

which leads to the estimate $|\mathcal{B}(\hat{u})y|_{H^{-1}(\Omega,\,\mathbb{R}^3)} \le C|\hat{u}|_{L^\infty_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)}|y|_{L^2_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)}$ (*cf.* [27], Rem. 3.1). For the existence and uniqueness of a strong solution for (2.10) we can use, in addition,

$$|\mathcal{B}(\hat{u})y|_{L^2(\Omega,\,\mathbb{R}^3)} \le C_1(|\hat{u}|_{L^\infty_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)}|\nabla y|_{L^2(\Omega,\,\mathbb{R}^9)} + |\nabla\hat{u}|_{L^3(\Omega,\,\mathbb{R}^9)}|y|_{L^6_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)}) \le C_2|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}|y|_{H^1_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)}.$$

In the case our control takes its values in the space $\mathcal{E}_M^1$, a natural question is: what are the admissible initial vector fields v_0 for this type of controls, if we want to guarantee the existence of a weak solution? The answer is not difficult if we give it in a general way (*cf.* [27], Sect. 3.1): let $\mathcal{Z}$ be a Hilbert space, and $K_1 : \mathcal{Z} \to G^1_{\mathrm{av}}((a,\, b),\, \Gamma)$ a continuous linear mapping; then the set of admissible weak initial conditions for system (2.7), with $\zeta \in K_1\mathcal{Z}$, is given by $\mathcal{A}_{K_1} = H + \mathcal{H}_{K_1}$ where $\mathcal{H}_{K_1} := E_1K_1\mathcal{Z}(a) = \{\gamma(a) \mid \gamma = E_1K_1\eta \text{ and } \eta \in \mathcal{Z}\}$. Moreover $\mathcal{H}_{K_1}$ and $\mathcal{A}_{K_1}$ are Hilbert spaces, with associated range norms

$$|u|_{\mathcal{H}_{K_1}} := \inf\{|\eta|_{\mathcal{Z}} \mid u = E_1K_1\eta(a),\ \eta \in \mathcal{Z}\}\,,$$
$$|u|_{\mathcal{A}_{K_1}} := \inf\{|(w,\, z)|_{H\times\mathcal{H}_{K_1}} \mid u = w + z \text{ and } (w,\, z) \in H \times \mathcal{H}_{K_1}\},$$

and there are constants $C_1,\, C_2,\, C_3 > 0$ such that

$$|u|_{L^2_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)} \le C_1|u|_{\mathcal{H}_{K_1}}, \quad \text{for all } u \in \mathcal{H}_{K_1}; \tag{2.11a}$$
$$|u|_{L^2_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)} \le C_2|u|_{\mathcal{A}_{K_1}}, \quad \text{for all } u \in \mathcal{A}_{K_1}; \tag{2.11b}$$
$$|u|_{L^2_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)} \ge C_3|u|_{\mathcal{A}_{K_1}}, \quad \text{for all } u \in H. \tag{2.11c}$$

From ([27], Sect. 3.1), we also have the following existence result:

Theorem 2.11. *Given $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{wk}}$, $g \in L^2((a,\, b),\, H^{-1}(\Omega,\, \mathbb{R}^3))$, a Hilbert space $\mathcal{Z}$, a continuous linear mapping $K_1 : \mathcal{Z} \to G^1_{\mathrm{av}}((a,\, b),\, \Gamma)$, $v_0 \in \mathcal{A}_{K_1}$ and $\eta \in \mathcal{Z}$, with $v_0 - E_1K_1\eta(a) \in H$, then there exists a weak solution v in the space $W((a,\, b),\, H^1_{\mathrm{div}}(\Omega,\, \mathbb{R}^3),\, H^{-1}(\Omega,\, \mathbb{R}^3))$ for system (2.7), with $\zeta = K_1\eta$. Moreover v is unique and depends continuously on the given data $(v_0,\, g,\, \eta)$:*

$$|v|^2_{W((a,\,b),\,H^1_{\mathrm{div}}(\Omega,\,\mathbb{R}^3),\,H^{-1}(\Omega,\,\mathbb{R}^3))} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}\left(|v_0|^2_{L^2_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)} + |g|^2_{L^2((a,\,b),\,H^{-1}(\Omega,\,\mathbb{R}^3))} + |\eta|^2_{\mathcal{Z}}\right).$$

Following the same idea in Section 3.1 of [27], we can also prove the analogous results for strong solutions, where now, we consider a continuous linear mapping $K_2 : \mathcal{Z} \to G^2_{\mathrm{av}}((a,\, b),\, \Gamma)$, the set of admissible strong initial conditions is $\mathcal{A}_{K_2} := V + \mathcal{H}_{K_2}$ with $\mathcal{H}_{K_2} := E_2K_2\mathcal{Z}(a)$, and

$$|u|_{\mathcal{H}_{K_2}} := \inf\{|\eta|_{\mathcal{Z}} \mid u = E_2K_2\eta(a),\ \eta \in \mathcal{Z}\}\,,$$
$$|u|_{\mathcal{A}_{K_2}} := \inf\{|(w,\, z)|_{V\times\mathcal{H}_{K_2}} \mid u = w + z \text{ and } (w,\, z) \in V \times \mathcal{H}_{K_2}\},$$

and there are constants $C_1,\, C_2,\, C_3 > 0$ such that

$$|u|_{H^1_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)} \le C_1|u|_{\mathcal{H}_{K_2}}, \quad \text{for all } u \in \mathcal{H}_{K_2}; \tag{2.12a}$$
$$|u|_{H^1_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)} \le C_2|u|_{\mathcal{A}_{K_2}}, \quad \text{for all } u \in \mathcal{A}_{K_2}; \tag{2.12b}$$
$$|u|_{H^1_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)} \ge C_3|u|_{\mathcal{A}_{K_2}}, \quad \text{for all } u \in V. \tag{2.12c}$$

Theorem 2.12. *Given $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{st}}$, $g \in L^2((a,\, b),\, L^2(\Omega,\, \mathbb{R}^3))$, a Hilbert space $\mathcal{Z}$, a continuous linear mapping $K_2 : \mathcal{Z} \to G^2_{\mathrm{av}}((a,\, b),\, \Gamma)$, $v_0 \in \mathcal{A}_{K_2}$ and $\eta \in \mathcal{Z}$, with $v_0 - E_2K_2\eta(a) \in V$, then there exists a strong solution v in the space $W((a,\, b),\, H^2_{\mathrm{div}}(\Omega,\, \mathbb{R}^3),\, L^2(\Omega,\, \mathbb{R}^3))$ for system (2.7), with $\zeta = K_2\eta$. Moreover v is unique and depends continuously on the given data $(v_0,\, g,\, \eta)$:*

$$|v|^2_{W((a,\,b),\,H^2_{\mathrm{div}}(\Omega,\,\mathbb{R}^3),\,L^2(\Omega,\,\mathbb{R}^3))} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}\left(|v_0|^2_{H^1_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)} + |g|^2_{L^2((a,\,b),\,L^2(\Omega,\,\mathbb{R}^3))} + |\eta|^2_{\mathcal{Z}}\right).$$

Analogously, weak and strong solutions for system (2.8) can be defined in the classical sense, just reversing time. We have the following results.

Theorem 2.13. *Given $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{wk}}$, $f \in L^2((a,\, b),\, H^{-1}(\Omega,\, \mathbb{R}^3))$, and $q_1 \in H$, then there exists a weak solution $q \in W((a,\, b),\, V,\, V')$ for system (2.8). Moreover, q is unique and depends continuously on the given data $(q_1,\, f)$, that is, $|q|^2_{W((a,\,b),\,V,\,V')} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}\left(|q_1|^2_H + |f|^2_{L^2((a,\,b),\,H^{-1}(\Omega,\,\mathbb{R}^3))}\right)$.*

Theorem 2.14. *Given $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{wk}}$, $f \in L^2((a,\,b),\,L^2(\varOmega,\,\mathbb{R}^3))$, and $q_1 \in V$, then there exists a strong solution $q \in W((a,\,b),\,\mathrm{D}(L),\,H)$ for system (2.8). Moreover, q is unique and depends continuously on the given data $(q_1,\,f)$, that is, $|q|^2_{W((a,\,b),\,\mathrm{D}(L),\,H)} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}\left(|q_1|^2_V + |f|^2_{L^2((a,\,b),\,L^2(\varOmega,\,\mathbb{R}^3))}\right)$.*

Remark 2.15. Notice that in [27] we find the Lebesgue-like notation $L^\infty((a,\,b)\times\varOmega,\,\mathbb{R}^3)$ instead of the Bochner-like notation $L^\infty((a,\,b),\,L^\infty(\varOmega,\,\mathbb{R}^3))$. Here we use the latter because it will be more convenient below. To see that the spaces coincide, first we observe that the inclusion $L^1((a,\,b)\times\varOmega,\,\mathbb{R}^3) \subseteq L^1((a,\,b),\,L^1(\varOmega,\,\mathbb{R}^3))$ follows from Fubini's Theorem (see *e.g.* [11], Sect. III.11, Thm. 9); and $L^1((a,\,b)\times\varOmega,\,\mathbb{R}^3) \supseteq L^1((a,\,b),\,L^1(\varOmega,\,\mathbb{R}^3))$ can be derived from Theorem 17 in Section III.11 of [11] (recalling that functions in $L^1((a,\,b)\times\varOmega,\,\mathbb{R}^3)$ are defined up to sets of measure zero). Then, we can write $L^\infty((a,\,b)\times\varOmega,\,\mathbb{R}^3) = L^1((a,\,b)\times\varOmega,\,\mathbb{R}^3)' = L^1((a,\,b),\,L^1(\varOmega,\,\mathbb{R}^3))' = L^\infty((a,\,b),\,L^\infty(\varOmega,\,\mathbb{R}^3))$.

Remark 2.16. Notice that, although we have taken the reference solution $\hat{u}$ in the spaces (2.2), for the previous results concerning the existence, uniqueness and continuity of the solutions we do not need the condition on the time-derivative: $\partial_t\hat{u} \in L^2((a,\,b),\,L^\sigma(\varOmega,\,\mathbb{R}^3))$. This condition is, essentially, only needed for the observability and controllability results that follow.

Notice that, $P^{\mathcal{O}}_M\vartheta L^2(\mathcal{O},\,\mathbb{R}^3)$ is a subset of $C^2(\mathcal{O},\,\mathbb{R}^3)$; then it follows that $\left(\chi\mathbb{E}^{\mathcal{O}}_0 P^{\mathcal{O}}_{\chi^\perp} P^{\mathcal{O}}_M\vartheta L^2(\mathcal{O},\,\mathbb{R}^3)\right)|_{\varGamma_\mathrm{c}}$ is a subset of $H^2_0(\varGamma_\mathrm{c},\,\mathbb{R}^3)$, and $\chi\mathbb{E}^{\mathcal{O}}_0 P^{\mathcal{O}}_{\chi^\perp} P^{\mathcal{O}}_M\vartheta L^2(\mathcal{O},\,\mathbb{R}^3) = \mathbb{E}^{\varGamma_\mathrm{c}}_0(\chi\mathbb{E}^{\mathcal{O}}_0 P^{\mathcal{O}}_{\chi^\perp} P^{\mathcal{O}}_M\vartheta L^2(\mathcal{O},\,\mathbb{R}^3))|_{\varGamma_\mathrm{c}}$ is a subset of $H^2(\varGamma,\,\mathbb{R}^3)$; here $\mathbb{E}^{\varGamma_\mathrm{c}}_0 : H^2_0(\varGamma_\mathrm{c},\,\mathbb{R}^3) \to H^2(\varGamma,\,\mathbb{R}^3)$ stands for the extension by 0 outside $\varGamma_\mathrm{c}$.

Proposition 2.17. *The operator $\eta \mapsto K_i\eta = K^{\mathcal{O}}\eta := \chi\mathbb{E}^{\mathcal{O}}_0 P^{\mathcal{O}}_{\chi^\perp} P^{\mathcal{O}}_M(\vartheta\eta|_{\mathcal{O}})$ maps the space $G^i((a,\,b),\,\varGamma)$ into $G^i_{\mathrm{av}}((a,\,b),\,\varGamma)$, and is linear and continuous, for $i \in \{1,\,2\}$.*

We give the proof in the Appendix, Section A.5. We can find the set of admissible weak conditions $\mathcal{A}_{K_i}$ as above, and apply Theorem 2.11 to guarantee the existence of weak solutions, in the case the control functions are in $\mathcal{E}^1_M$. Analogously, we can derive the existence of strong solutions for controls in the space

$$\mathcal{E}^2_M := \chi\mathbb{E}^{\mathcal{O}}_0 P^{\mathcal{O}}_{\chi^\perp} P^{\mathcal{O}}_M\vartheta G^2((a,\,b),\,\varGamma)|_{\mathcal{O}} \subset \mathcal{E}^1_M. \tag{2.13}$$

3. Localized observability inequalities

In this Section the main goal is to derive some observability inequalities that are somehow appropriate to deal with boundary controls problems with controls supported in a given subset of the boundary. The starting point will be a result on null boundary controllability we find in Section 4 of [27]. Let us denote by $G^1_{\mathrm{av,c}}((a,\,b),\,\varGamma) \subseteq G^1_{\mathrm{av}}((a,\,b),\,\varGamma)$ the space of the traces at $\varGamma$ of the functions in $W((a,\,b),\,V_\mathrm{c},\,H^{-1}(\varOmega,\,\mathbb{R}^3))$. We endow $G^1_{\mathrm{av,c}}((a,\,b),\,\varGamma)$ with the norm inherited from $G^1_{\mathrm{av}}((a,\,b),\,\varGamma)$. We observe that

$$G^1_{\mathrm{av,c}}((a,\,b),\,\varGamma) = \{u \in G^1_{\mathrm{av}}((a,\,b),\,\varGamma) \mid u(t) = 0 \text{ in } \varGamma\setminus\overline{\varGamma_\mathrm{c}} \text{ for a.e. } t \in (a,\,b)\}$$

and that $G^1_{\mathrm{av,c}}((a,\,b),\,\varGamma)$ is a closed subspace of $G^1_{\mathrm{av}}((a,\,b),\,\varGamma)$. Notice that if $(u^n)_{n\in\mathbb{N}}$ is a sequence on $G^1_{\mathrm{av,c}}((a,\,b),\,\varGamma)$ and $u^n \to u^\infty$ in $G^1_{\mathrm{av}}((a,\,b),\,\varGamma)$, then in particular $u^n \to u^\infty$ in $L^2((a,\,b),\,L^2(\varGamma,\,\mathbb{R}))$, and so $u^n|_{\varGamma\setminus\overline{\varGamma_\mathrm{c}}}$ converges to $u^\infty|_{\varGamma\setminus\overline{\varGamma_\mathrm{c}}}$ in $L^2((a,\,b),\,L^2(\varGamma\setminus\overline{\varGamma_\mathrm{c}},\,\mathbb{R}))$. Necessarily, $u^\infty|_{\varGamma\setminus\overline{\varGamma_\mathrm{c}}} = 0$. It follows that $G^1_{\mathrm{av,c}}((a,\,b),\,\varGamma)$ is complete and, we can consider the system (2.7) with $\zeta \in G^1_{\mathrm{av,c}}((a,\,b),\,\varGamma)$ and $v_0 \in \mathcal{A}_{K_1}$, where $\mathcal{A}_{K_1} = H + \mathcal{H}_{K_1}$ is the space of admissible weak initial conditions for that system, with $\mathcal{Z} = G^1_{\mathrm{av,c}}((a,\,b),\,\varGamma)$, $K_1 : G^1_{\mathrm{av,c}}((a,\,b),\,\varGamma) \to G^1_{\mathrm{av}}((a,\,b),\,\varGamma)$ the inclusion mapping $\eta \mapsto \eta$, and $\mathcal{H}_{K_1} := \{E_1\zeta(a) \mid \zeta \in G^1_{\mathrm{av,c}}((a,\,b),\,\varGamma)\}$. $\mathcal{H}_{K_1}$ and $\mathcal{A}_{K_1}$ are supposed to be endowed with the respective range scalar products, and range norms. From Section 4 of [27], since $\overline{\varGamma_\mathrm{c}} = \mathrm{supp}(\chi)$ is the support of a function $\chi \in C^\infty(\varGamma,\,\mathbb{R})$, we have the following null controllability property:

Lemma 3.1. *Given $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{wk}}$ and $v_0 \in \mathcal{A}_{K_1}$, there exists a control $\zeta = \zeta(v_0) \in G^1_{\mathrm{av,c}}((a,\,b),\,\varGamma)$ such that, for the corresponding solution v to system (2.7) with $g = 0$, we have $v(b) = 0$. Moreover the control may be chosen so that the mapping $v_0 \mapsto \zeta(v_0)$ is linear and continuous: $|\zeta(v_0)|_{G^1_{\mathrm{av,c}}((a,\,b),\,\varGamma)} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}|v_0|_{\mathcal{A}_{K_1}}$.*

3.1. Some simple observability inequalities

Consider the system (2.8) with data

$$\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{wk}}, \quad f \in L^2((a,\,b),\,H), \quad \text{and } q_1 \in V. \tag{3.1}$$

By Theorem 2.14 there exists a strong solution $q \in W((a,\,b),\,\mathrm{D}(L),\,H)$ for that system. Consider also, the weak solution v of (2.7) with $g = 0$ and $\zeta = \zeta(v_0)$, the control given by Lemma 3.1. We find

$$
\begin{aligned}
&(q_1,\,v(b))_{L^2(\Omega,\,\mathbb{R}^3)} - (q(a),\,v_0)_{L^2(\Omega,\,\mathbb{R}^3)} = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(q,\,v)_{L^2(\Omega,\,\mathbb{R}^3)}(\tau)\,\mathrm{d}\tau\\
&= \int_a^b \Big((\mathcal{B}^*(\hat{u})q - \nu\Delta q + \nabla p_q + f,\,v)_{L^2(\Omega,\,\mathbb{R}^3)} + \langle q,\,-\mathcal{B}(\hat{u})v + \nu\Delta v - \nabla p_v\rangle_{H^1_0(\Omega,\,\mathbb{R}^3),\,H^{-1}(\Omega,\,\mathbb{R}^3)}\Big)\,\mathrm{d}\tau\\
&= \int_a^b (f,\,v)_{L^2(\Omega,\,\mathbb{R}^3)}\,\mathrm{d}\tau + \int_a^b (-\nu\langle \mathbf{n}\cdot\nabla\rangle q + p_q\mathbf{n},\,v)_{L^2(\Gamma,\,\mathbb{R}^3)}\,\mathrm{d}\tau,
\end{aligned}
$$

from which, we obtain

$$-(q(a),\,v_0)_{L^2(\Omega,\,\mathbb{R}^3)} = \int_a^b (f,\,v)_{L^2(\Omega,\,\mathbb{R}^3)}\,\mathrm{d}\tau + \int_a^b (-\nu\langle \mathbf{n}\cdot\nabla\rangle q + p_q\mathbf{n},\,v)_{L^2(\Gamma,\,\mathbb{R}^3)}\,\mathrm{d}\tau. \tag{3.2}$$

Now, considering $L^2((a,\,b),\,L^2(\Gamma,\,\mathbb{R}^3))$ as a pivot space at the boundary, let $G^1((a,\,b),\,\Gamma)'$ be the dual space of $G^1((a,\,b),\,\Gamma) \subset L^2((a,\,b),\,L^2(\Gamma,\,\mathbb{R}^3))$. We have the dense inclusions $G^1((a,\,b),\,\Gamma) \subset L^2((a,\,b),\,L^2(\Gamma,\,\mathbb{R}^3)) \subset G^1((a,\,b),\,\Gamma)'$. On the other hand, since $v|_\Gamma = \zeta(v_0) \in G^1_{\mathrm{av,c}}((a,\,b),\,\Gamma)$, and the norm of $G^1_{\mathrm{av,c}}((a,\,b),\,\Gamma)$ is the one inherited from $G^1((a,\,b),\,\Gamma)$, we have that $\left|(q(a),\,v_0)_{L^2(\Omega,\,\mathbb{R}^3)}\right|_{\mathbb{R}}$ is bounded by

$$C_1\left(|f|_{L^2((a,\,b),\,H)} + \left|\mathcal{I}|_{\Gamma_\mathrm{c}}\,(p_q\mathbf{n} - \nu\langle\mathbf{n}\cdot\nabla\rangle q)\right|_{G^1((a,\,b),\,\Gamma)'}\right)\left(|v|_{L^2((a,\,b),\,L^2(\Omega,\,\mathbb{R}^3))} + |\zeta(v_0)|_{G^1_{\mathrm{av,c}}((a,\,b),\,\Gamma)}\right),$$

where $\mathcal{I}|_{\Gamma_\mathrm{c}}$ the indicator operator: $\mathcal{I}|_{\Gamma_\mathrm{c}} f(t,\,x) := \begin{cases} f(t,\,x) & \text{if } x \in \Gamma_\mathrm{c}\\ 0 & \text{if } x \in \Gamma\setminus\overline{\Gamma_\mathrm{c}}\end{cases}$, mapping $L^2((a,\,b),\,L^2(\Gamma,\,\mathbb{R}^3))$ into itself. Thus, from the continuity of $v_0 \mapsto \zeta(v_0)$, together with (2.11b) and Theorem 2.11, we can find that

$$|v|_{L^2((a,\,b),\,L^2(\Omega,\,\mathbb{R}^3))} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}|v_0|_{\mathcal{A}_{K_1}} \tag{3.3}$$

and, from (2.11c), we arrive to the boundary observability inequality

$$|q(a)|_H^2 \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}\left(|f|^2_{L^2((a,\,b),\,H)} + \left|\mathcal{I}|_{\Gamma_\mathrm{c}}\,(p_q\mathbf{n} - \nu\langle\mathbf{n}\cdot\nabla\rangle q)\right|^2_{G^1((a,\,b),\,\Gamma)'}\right) \tag{3.4}$$

for the solution q of system (2.8), with $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{wk}}$, and "the" corresponding pressure function p_q.

Now, let $G^2_{\mathrm{av,c}}((a,\,b),\,\Gamma) = \{\gamma|_\Gamma \mid \gamma \in W((a,\,b),\,V_c \cap H^2(\Omega,\,\mathbb{R}^3),\,L^2(\Omega,\,\mathbb{R}^3))\} = \{u \in G^2_{\mathrm{av}}((a,\,b),\,\Gamma) \mid u(t) = 0 \text{ in } \Gamma\setminus\overline{\Gamma_\mathrm{c}},\ \text{for a.e. } t \in (a,\,b)\}$.

Lemma 3.2. *Given $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{st}}$ and $v_0 \in \mathcal{A}_{K_2}$, there exists a control $\zeta = \zeta(v_0) \in G^2_{\mathrm{av,c}}((a,\,b),\,\Gamma)$ such that, for the corresponding solution v to system (2.7) with $g = 0$, we have $v(b) = 0$. Moreover the control may be chosen so that the mapping $v_0 \mapsto \zeta(v_0)$ is linear and continuous: $|\zeta(v_0)|_{G^2_{\mathrm{av,c}}((a,\,b),\,\Gamma)} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}|v_0|_{\mathcal{A}_{K_2}}$.*

Proof. We follow the proof of Theorem 4.1 in [27]. Write $v_0 = v_{0V} + v_{0\mathcal{H}}$ where $(v_{0V},\,v_{0\mathcal{H}}) \in V \times \mathcal{H}_{K_2}$. Let $\gamma_{v_{0\mathcal{H}}} \in W((a,\,b),\,V_c\cap H^2(\Omega,\,\mathbb{R}^3),\,L^2(\Omega,\,\mathbb{R}^3))$ be defined by $\gamma_{v_{0\mathcal{H}}} \in \{\gamma \in W((a,\,b),\,V_c\cap H^2(\Omega,\,\mathbb{R}^3),\,L^2(\Omega,\,\mathbb{R}^3)) \mid \gamma(a) = 0\}^\perp$ and $\gamma_{v_{0\mathcal{H}}}(a) = v_{0\mathcal{H}}$. We have that the mapping $v_{0\mathcal{H}} \mapsto \gamma_{v_{0\mathcal{H}}}$ is linear and continuous. Now, put $l = \frac{b-a}{2}$ and let ξ be a smooth real function, defined in $[a,\,a+l]$, taking the value 1 in a neighborhood of $t = a$, and vanishing in a neighborhood of $t = a + l$. On the interval of time $(a,\, a + l)$ we apply the control $\xi\gamma_{v_{0\mathcal{H}}}|_\Gamma$; in this way we arrive to a point $v(a + l) \in V$ at time $t = a + l$. Moreover $|v(a + l)|_V \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,a+l)|\mathrm{st}}}\right]}|v_0|_{\mathcal{A}_{K_2}}$. Then, in the interval of time $(a + l,\, b)$ proceeding as in the proof of Theorem 4.1 in [27], we can conclude that there exists a control γ_1 driving the system to zero at time $t = b$; moreover $\gamma_1 = \bar{v}|_\Gamma$ is the restriction to Γ of a suitable $\bar{v} \in W((a+l,\, b),\, V_c \cap H^2(\Omega,\, \mathbb{R}^3),\, L^2(\Omega,\, \mathbb{R}^3))$, and the mapping $v(a+l) \mapsto \gamma_1$ is linear and continuous, $|\gamma_1|_{G^2_{\mathrm{av,c}}((a+l,\,b),\,\Gamma)} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a+l,\,b)|\mathrm{st}}}\right]}|v(a + l)|_V$.

Therefore the concatenation $\zeta(v_0) := \gamma_1 \circ \xi\gamma_{v_{0\mathcal{H}}}|_\Gamma$, of the controls $\xi\gamma_{v_{0\mathcal{H}}}|_\Gamma$ and γ_1 drives the system from v_0, at time $t = a$, to 0 at time $t = b$. Moreover $v_0 \mapsto \zeta(v_0)$ is linear and continuous, $|\zeta(v_0)|_{G^2_{\mathrm{av,c}}((a,\,b),\,\Gamma)} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}|v_0|_{\mathcal{A}_{K_2}}$ □

Remark 3.3. Notice that in the previous proof if $v_0 \in V$, then the first control $\xi\gamma_{v_{0\mathcal{H}}}|_\Gamma$ vanishes and, by standard arguments taking into account the smoothing property of the system (*cf.* [5], Eq. (2.6)), we can conclude that $|v(a + l)|_V \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,a+l)|\mathrm{st}}}\right]}|v_0|_H$. Thus we obtain that the concatenation satisfies $|\zeta(v_0)|_{G^2_{\mathrm{av,c}}((a,\,b),\,\Gamma)} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}|v_0|_H$.

Now, from (3.2), we can also obtain that $\left|(q(a),\, v_0)_{L^2(\Omega,\,\mathbb{R}^3)}\right|_{\mathbb{R}}$ is bounded by

$$C_1\left(|f|_{L^2((a,\,b),\,H)} + \left|\mathcal{I}\right|_{\Gamma_\mathrm{c}}(p_q\mathbf{n} - \nu\langle\mathbf{n}\cdot\nabla\rangle q)\big|_{G^2((a,\,b),\,\Gamma)'}\right)\left(|v|_{L^2((a,\,b),\,L^2(\Omega,\,\mathbb{R}^3))} + |\zeta(v_0)|_{G^2_{\mathrm{av,c}}((a,\,b),\,\Gamma)}\right)$$

and, using (3.3), (2.11c), and Remark 3.3, we derive that for all $v_0 \in V$ we have

$$|(q(a),\, v_0)_H|_{\mathbb{R}} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}\left(|f|^2_{L^2((a,\,b),\,H)} + \left|\mathcal{I}\right|_{\Gamma_\mathrm{c}}(p_q\mathbf{n} - \nu\langle\mathbf{n}\cdot\nabla\rangle q)\big|^2_{G^2((a,\,b),\,\Gamma)'}\right)|v_0|_H.$$

Then from the density of the inclusion $V \subset H$, it follows that

$$|q(a)|^2_H \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}\left(|f|^2_{L^2((a,\,b),\,H)} + \left|\mathcal{I}\right|_{\Gamma_\mathrm{c}}(p_q\mathbf{n} - \nu\langle\mathbf{n}\cdot\nabla\rangle q)\big|^2_{G^2((a,\,b),\,\Gamma)'}\right) \tag{3.5}$$

for the solution q of system (2.8), with $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{st}}$, and "the" corresponding pressure function p_q.

Remark 3.4. In inequalities (3.4) and (3.5), we suppose we have fixed a well defined choice of p_q, that is known to be unique up to an additive constant. The constants $\overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}$ and $\overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}$ in those inequalities depend also on Γ_c and on the length of $(a,\, b)$ but, they can be taken independent of the choice of p_q, because (3.2) holds independently of the choice of p_q. Indeed if we replace p_q by $\tilde{p} = p_q + c$, with $c \in \mathbb{R}$, then from $\operatorname{div} v = 0$ we derive that $(\tilde{p}\mathbf{n},\, v)_{L^2(\Gamma,\,\mathbb{R}^3)} = (\nabla\tilde{p},\, v)_{L^2(\Omega,\,\mathbb{R}^3)} = (\nabla p_q,\, v)_{L^2(\Omega,\,\mathbb{R}^3)} = (p_q\mathbf{n},\, v)_{L^2(\Gamma,\,\mathbb{R}^3)}$.

3.2. Choice of the pressure function

Often the pressure function $p = p_q$ is chosen to have zero average in Ω but, in the study of specific problems, as we will see later in Section 5, it may be convenient to set another choice.

Definition 3.5. We say that the linear mapping $p \mapsto c_\varsigma p := p - \frac{\varsigma p}{\varsigma 1_\Omega}$ is an appropriate choice of the pressure function if $\varsigma : H^1(\Omega,\, \mathbb{R}) \to \mathbb{R}$ is a continuous linear function, with $\varsigma 1_\Omega \ne 0$. Here 1_Ω stands for the function $1_\Omega(x) := 1$, for all $x \in \Omega$.

Remark 3.6. Notice that if ς defines an appropriate choice, then $p - c_\varsigma p = \frac{\varsigma p}{\varsigma 1_\Omega}$ is a constant function, $c_\varsigma 1_\Omega = 0$, $\varsigma c_\varsigma p = 0$, and $c_\varsigma c_\varsigma p = c_\varsigma p$. Moreover $|\varsigma \cdot|_{\mathbb{R}}$ is a seminorm in $H^1(\Omega,\, \mathbb{R})$ and, since $\varsigma 1_\Omega \ne 0$, the norms $|\cdot|_{H^1(\Omega,\,\mathbb{R})}$ and $|\nabla \cdot|_{L^2(\Omega,\,\mathbb{R}^3)} + |\varsigma \cdot|_{\mathbb{R}}$ are equivalent in $H^1(\Omega,\, \mathbb{R})$ (see *e.g.* [33], Sect. II.1.4). With the above terminology, the "usual" choice of zero-averaged p in Ω corresponds to $\varsigma = \varsigma_\Omega$, $p = c_{\varsigma_\Omega}p$, with $\varsigma_\Omega p := \int_\Omega p\,\mathrm{d}\Omega$.

Given an appropriate choice c_ς, the pressure function p in (2.8) may be supposed, or chosen, to satisfy $\varsigma p = 0$. For strong solutions, we know that the choice $p = c_{\varsigma_\Omega} p$ is in $L^2((a,\, b),\, H^1(\Omega,\, \mathbb{R}))$. Now, if c_{ς_1} and c_{ς_2} are two appropriate choices, then we have that

$$c_{\varsigma_1} c_{\varsigma_2} p = c_{\varsigma_2} p - \tfrac{\varsigma_1 c_{\varsigma_2} p}{\varsigma_1 1_\Omega} = p - \tfrac{\varsigma_2 p}{\varsigma_2 1_\Omega} - \tfrac{\varsigma_1 \left(p - \frac{\varsigma_2 p}{\varsigma_2 1_\Omega}\right)}{\varsigma_1 1_\Omega} = c_{\varsigma_1} p - \tfrac{\varsigma_2 p}{\varsigma_2 1_\Omega} + \tfrac{\varsigma_1 \frac{\varsigma_2 p}{\varsigma_2 1_\Omega}}{\varsigma_1 1_\Omega} = c_{\varsigma_1} p.$$

That is, $c_{\varsigma_1} c_{\varsigma_2} p$ coincides with the appropriate choice $c_{\varsigma_1} p$, which means that we may choose p having zero average on Γ_c, which corresponds to $\varsigma = \varsigma_\mathrm{c}$, $p = c_{\varsigma_\mathrm{c}} p = c_{\varsigma_\mathrm{c}} c_{\varsigma_\Omega} p$, with $\varsigma_\mathrm{c} p := \int_{\Gamma_\mathrm{c}} p \,\mathrm{d}\Gamma_\mathrm{c}$.

Remark 3.7. Here we will consider only solutions of system (2.8) with data (3.1); since these solutions are strong we can guarantee (choosing *a priori*, *e.g.*, $p = c_{\varsigma_\Omega} p$) that the corresponding pressure function p is in $H^1(\Omega,\, \mathbb{R})$; this is why we have defined "appropriate choice" ($p = c_\varsigma p = c_\varsigma c_{\varsigma_\Omega} p$) for this regularity. Of course for weak regularity, *i.e.*, to the case $p \in L^2(\Omega,\, \mathbb{R})$, and not necessarily in $H^1(\Omega,\, \mathbb{R})$, we should consider continuous linear functions $\varsigma : L^2(\Omega,\, \mathbb{R}) \to \mathbb{R}$, with $\varsigma 1_\Omega \neq 0$.

3.3. Smoother observability inequalities

We see that the boundary term in inequalities (3.4) and (3.5) vanishes if the "observed" trace $p_q \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q$ vanishes in Γ_c; in this sense we may understand those inequalities as inequalities localized on Γ_c. However, the indicator operator $\mathcal{I}|_{\Gamma_\mathrm{c}}$ would, roughly speaking, suit the case in which we take controls like $\zeta = \mathcal{I}|_{\Gamma_\mathrm{c}} \eta$ in system (2.7), and it would destroy all regularity of η we may be interested to (or need to) preserve for ζ across the boundary of Γ_c (*cf.* Prop. 2.17, where the operator $\eta \mapsto K^{\mathcal{O}} \eta$ returns us a control with enough regularity to guarantee the existence of a weak solution for (2.7)).

Here we present a class of observability inequalities localized on open subsets of $(a,\, b) \times \Gamma$. In particular we will see that $\mathcal{I}|_{\Gamma_\mathrm{c}}$ can be replaced by a general smoother operator. We start by some straightforward corollaries of Lemmas 3.1 and 3.2.

Corollary 3.8. *Given $\hat{u} \in \mathcal{W}^{(a,\, b)|\mathrm{wk}}$, $v_0 \in H$, and $\emptyset \neq (c,\, d) \subseteq (a,\, b)$, there exists a control $\zeta = \zeta(v_0) \in G^1_{\mathrm{av,c}}((a,\, b),\, \Gamma)$ such that, for the corresponding solution v to system (2.7) with $g = 0$, we have $v(b) = 0$. Moreover the support of the control is contained in $[c,\, d] \times \overline{\Gamma_\mathrm{c}}$, and the mapping $v_0 \mapsto \zeta(v_0)$ is linear and continuous: $|\zeta(v_0)|_{G^1_{\mathrm{av,c}}((a,\, b),\, \Gamma)} \leq \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\, b)|\mathrm{wk}}}\right]} |v_0|_H$.*

Proof. If $a < c$ we apply zero boundary control for time $t \in (a,\, c)$. Then we apply the control given in Lemma 3.1 (with $(c,\, d)$ in the role of $(a,\, b)$) driving the system to 0 at time $t = d$. Finally, if $d < b$ we apply zero control for time $t \in (d,\, b)$. Now using Theorem 2.11, it is straightforward to check that the proposed concatenated control satisfy the required properties. □

Corollary 3.9. *Given $\hat{u} \in \mathcal{W}^{(a,\, b)|\mathrm{st}}$, $v_0 \in V$, and $\emptyset \neq (c,\, d) \subseteq (a,\, b)$, there exists a control $\zeta = \zeta(v_0) \in G^2_{\mathrm{av,c}}((a,\, b),\, \Gamma)$ such that, for the corresponding solution v to system (2.7) with $g = 0$, we have $v(b) = 0$. Moreover the support of the control is contained in $[c,\, d] \times \overline{\Gamma_\mathrm{c}}$, and the mapping $v_0 \mapsto \zeta(v_0)$ is linear and continuous: $|\zeta(v_0)|_{G^2_{\mathrm{av,c}}((a,\, b),\, \Gamma)} \leq \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\, b)|\mathrm{st}}}\right]} |v_0|_V$.*

Proof. The proof is similar to that of Corollary 3.8; we have just to take the control given in Lemma 3.2 in the interval $(c,\, d)$, and use Theorem 2.12 instead. □

Now, proceeding as in Section 3.1, using (3.2) and the controls given by Corollaries 3.8 and 3.9, we can arrive to the following observability inequalities for the solution q of system (2.8) and the corresponding pressure function p_q:

$$|q(a)|_H^2 \leq \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\, b)|\mathrm{wk}}}\right]} \left(|f|^2_{L^2((a,\, b),\, H)} + \left| \widetilde{\psi}(p_q \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q) \right|^2_{G^1((a,\, b),\, \Gamma)'} \right), \qquad \text{if } \hat{u} \in \mathcal{W}^{(a,\, b)|\mathrm{wk}}; \tag{3.6}$$

$$|q(a)|_H^2 \leq \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\, b)|\mathrm{st}}}\right]} \left(|f|^2_{L^2((a,\, b),\, H)} + \left| \widetilde{\psi}(p_q \mathbf{n} - \nu \langle \mathbf{n} \cdot \nabla \rangle q) \right|^2_{G^2((a,\, b),\, \Gamma)'} \right), \qquad \text{if } \hat{u} \in \mathcal{W}^{(a,\, b)|\mathrm{st}}; \tag{3.7}$$

where $\widetilde{\psi} \in L^\infty((a,\,b),\,L^\infty(\Gamma,\,\mathbb{R}))$ is any function taking the value 1 in $(c,\,d)\times\Gamma_\mathrm{c}$ (recall that the support of the control is contained in $[c,\,d]\times\overline{\Gamma_\mathrm{c}}$).

Next we relax a little the observability inequalities (3.6) and (3.7). We will need the following auxiliary result, whose proof is given in the Appendix, Section A.4.

Proposition 3.10. *Given $u \in G^i((a,\,b),\,\Gamma)$ and $\psi \in C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))$, then $\psi u \in G^i((a,\,b),\,\Gamma)$ and we have $|\psi u|_{G^i((a,\,b),\,\Gamma)} \le C|\psi|_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}|u|_{G^i((a,\,b),\,\Gamma)}$, for $i\in\{1,\,2\}$.*

Now, let ϕ be a function satisfying

$$\begin{aligned}&\phi\in L^\infty((a,\,b),\,L^\infty(\Gamma,\,\mathbb{R})),\ \text{and for some } (t_0,\,x_0)\in[a,\,b]\times\Gamma\\&\phi(t_0,\,x_0)\neq 0 \text{ and } \phi\in C^1([t_0-\delta,\,t_0+\delta]\cap[a,\,b],\,C^2(\overline{\mathcal{N}_{x_0}},\,\mathbb{R})),\end{aligned}\tag{3.8}$$

for some $\delta>0$ and some neighborhood $\mathcal{N}_{x_0}\subseteq\Gamma$ of x_0.

Theorem 3.11. *Let ϕ satisfy (3.8), and let $(q,\,p_q)$ solve system (2.8), for a fixed appropriate choice of the pressure function p_q. Then,*

$$|q(a)|_H^2 \le \overline{C}_{\left[|\hat u|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}\Big(|f|^2_{L^2((a,\,b),\,H)}+|\phi(p_q\mathbf{n}-\nu\langle\mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\Gamma)'}\Big),\qquad \textit{if } \hat u\in\mathcal{W}^{(a,\,b)|\mathrm{wk}};\tag{3.9}$$

$$|q(a)|_H^2 \le \overline{C}_{\left[|\hat u|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}\Big(|f|^2_{L^2((a,\,b),\,H)}+|\phi(p_q\mathbf{n}-\nu\langle\mathbf{n}\cdot\nabla\rangle q)|^2_{G^2((a,\,b),\,\Gamma)'}\Big),\qquad \textit{if } \hat u\in\mathcal{W}^{(a,\,b)|\mathrm{st}};\tag{3.10}$$

where now the constants $\overline{C}_{\left[|\hat u|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}$ and $\overline{C}_{\left[|\hat u|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}$ depend also on ϕ.

Proof. We prove (3.9); the proof of (3.10) is completely analogous. First of all, for any $h\in G^1((a,\,b),\,\Gamma)'$ and $\psi\in C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))$, from the definitions

$$|\psi h|_{G^1((a,\,b),\,\Gamma)'} := \sup_{\substack{v\in G^1((a,\,b),\,\Gamma)\\ |v|_{G^1((a,\,b),\,\Gamma)}=1}} \langle\psi h,\,v\rangle_{G^1((a,\,b),\,\Gamma)',G^1((a,\,b),\,\Gamma)},$$

$$\langle\psi h,\,v\rangle_{G^1((a,\,b),\,\Gamma)',G^1((a,\,b),\,\Gamma)} := \langle h,\,\psi v\rangle_{G^1((a,\,b),\,\Gamma)',G^1((a,\,b),\,\Gamma)},$$

and from Proposition 3.10, we obtain

$$|\psi h|_{G^1((a,\,b),\,\Gamma)'} \le C|\psi|_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}|h|_{G^1((a,\,b),\,\Gamma)'}.\tag{3.11}$$

Next, since $\phi(t_0,\,x_0)\neq 0$, and ϕ is regular enough in $\mathcal{N}^\times := [t_0-\delta,\,t_0+\delta]\cap[a,\,b]\times\overline{\mathcal{N}_{x_0}}$, we can set two open subsets $(c,\,d)\times\mathcal{O}_\phi$ and $(c^1,\,d^1)\times\mathcal{O}_\phi^1$ such that

$$\begin{cases}\overline{\mathcal{O}_\phi}=\mathrm{supp}(\tilde\chi_\phi) \text{ and } \overline{\mathcal{O}^1_\phi}=\mathrm{supp}(\tilde\chi^1_\phi),\ \text{for smooth functions } \tilde\chi_\phi \text{ and } \tilde\chi^1_\phi;\\ (c,\,d)\times\mathcal{O}_\phi\subset[c,\,d]\times\overline{\mathcal{O}_\phi}\subset(c^1,\,d^1)\times\mathcal{O}^1_\phi\subset[c^1,\,d^1]\times\overline{\mathcal{O}^1_\phi}\subset\mathcal{N}^\times;\\ \Big|\phi([c^1,\,d^1]\times\overline{\mathcal{O}^1_\phi})\Big|_{\mathbb{R}}\subseteq[\varepsilon,\,+\infty),\ \text{with } \varepsilon>0.\end{cases}$$

Now, let $\gamma\in C^\infty([a,\,b]\times\Gamma,\,\mathbb{R})$ be a smooth function such that $\gamma=1$ in $[c,\,d]\times\overline{\mathcal{O}_\phi}$ and $\gamma=0$ in $[a,\,b]\times\Gamma\setminus(c^1,\,d^1)\times\mathcal{O}^1_\phi$. Thus $\phi^{-1}\gamma(t,\,x) := \begin{cases}\frac{\gamma(t,\,x)}{\phi(t,\,x)} & \text{if } \phi(t,\,x)\neq 0\\ 0 & \text{if } \phi(t,\,x)=0\end{cases}$ is a differentiable mapping, that is, $\phi^{-1}\gamma\in C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))$.

Consider also the subspace $G^1_\phi((a,\,b),\,\Gamma) := \big\{v\in G^1((a,\,b),\,\Gamma)\ \big|\ v(t)=0 \text{ in } \Gamma\setminus\overline{\mathcal{O}_\phi} \text{ for a.e. } t\in(a,\,b)\big\}$. From inequality (3.6), with $\mathcal{O}_\phi$ in the role of Γ_c, we obtain

$$|q(a)|_H^2 \le \overline{C}_{\left[|\hat u|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}\Big(|f|^2_{L^2((a,\,b),\,H)}+|\gamma(p_q\mathbf{n}-\nu\langle\mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\Gamma)'}\Big),$$

and, since $\gamma = \phi^{-1}\gamma\phi$, from (3.11) it follows that

$$|\gamma(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\Gamma)'} \le C_1|\phi^{-1}\gamma|_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}\,|\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\Gamma)'}\,.$$

Therefore, $|q(a)|^2_H \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}(\phi)\Big(|f|^2_{L^2((a,\,b),\,H)} + |\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\Gamma)'}\Big)$, that is, (3.9) holds. □

Remark 3.12. We notice that (3.9) is an observability inequality localized on $\mathrm{supp}(\phi)$. In many applications, taking $\phi \in C^1((a,\,b),\,C^2(\Gamma,\,\mathbb{R}))$ instead of (3.8) should be sufficient and sometimes necessary (see the discussion in the beginning of this Sect. 3.3). We take ϕ satisfying (3.8) in (3.9) because it does not bring any real additional difficulties to the proof.

4. Truncated observability inequalities

In the case of finite-dimensional controls, we need suitably truncated observability inequalities, that is, we need to focus the observation on a suitable finite-dimensional space, closely related to the control space. Inspired by the work in [5] for the case of internal controls, we show below that if $f = 0$ and $q(b)$ is finite-dimensional, then the "observed space" can be truncated, and we still have a boundary observability inequality.

4.1. Auxiliary results

Lemma 4.1. *Let X and Y be two Banach spaces, and let $L : X \to Y$ be a linear continuous mapping. If $(x^n)_{n\in\mathbb{N}}$ is a sequence in X such that $x^n \rightharpoonup x$ in X, then $Lx^n \rightharpoonup Lx$ in Y.*

Proof. Given $f \in Y'$, the composition $f\circ L$ is in X', which implies $\langle f,\,Lx^n\rangle_{Y',\,Y} =: \langle f\circ L,\,x^n\rangle_{X',\,X} \to \langle f\circ L,\,x\rangle_{X',\,X} := \langle f,\,Lx\rangle_{Y',\,Y}$. □

Now, recall the space $H_N \subset H$, defined in (2.4), spanned by the first N eigenfunctions of the Stokes operator. We have the following:

Lemma 4.2. *Let $\phi \in C^1((a,\,b),\,C^2(\Gamma,\,\mathbb{R}))$ be non-identically zero, and let $(q,\,p_q)$ solve system (2.8), with $f = 0$ and $q_1 \in H_N$, for a fixed appropriate choice c_ς for the pressure function p_q,* i.e., *$\varsigma p_q = 0$. Then,*

$$|\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{L^2((a,\,b),\,H^{\frac12}(\Gamma,\mathbb{R}^3))} \le \overline{C}_{\left[N,\,|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}|\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\Gamma)'},\quad \textit{if } \hat{u}\in\mathcal{W}^{(a,\,b)|\mathrm{wk}};\tag{4.1}$$

$$|\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{L^2((a,\,b),\,H^{\frac12}(\Gamma,\mathbb{R}^3))} \le \overline{C}_{\left[N,\,|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}|\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{G^2((a,\,b),\,\Gamma)'},\quad \textit{if } \hat{u}\in\mathcal{W}^{(a,\,b)|\mathrm{st}};\tag{4.2}$$

where the constants $\overline{C}_{\left[N,\,|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}$ and $\overline{C}_{\left[N,\,|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}$ depend only on N, Ω, ϕ, on the length of $(a,\,b)$, and on the respective norm of $\hat{u}$.

Proof. We prove (4.1). The proof of (4.2) is completely analogous. We argue by contradiction. Suppose that there exists a sequence of pairs $\big((q_1^n,\,\hat{u}^n)\big)_{n\in\mathbb{N}}$ in $H_N\times\mathcal{W}^{(a,\,b)|\mathrm{wk}}$ with $(|\hat{u}^n|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}})_{n\in\mathbb{N}}$ bounded, such that the solution $(q^n,\,p_{q^n})$ of the system

$$\begin{aligned} -\partial_t q^n + \mathcal{B}^*(\hat{u}^n)q^n - \nu\Delta q^n + \nabla p_{q^n} &= 0, \qquad \operatorname{div} q^n = 0,\\ q^n\big|_\Gamma &= 0, \qquad q^n(b) = q_1^n \in H_N \end{aligned}\tag{4.3}$$

satisfies the inequality

$$|\phi(p_{q^n}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^n)|^2_{L^2((a,\,b),\,H^{\frac12}(\Gamma,\mathbb{R}^3))} > n|\phi(p_{q^n}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^n)|^2_{G^1((a,\,b),\,\Gamma)'},\tag{4.4}$$

where the pressure functions p_{q^n} are supposed to agree with the fixed choice, *i.e.*, $\varsigma p_{q^n} = 0$, for all $n\in\mathbb{N}$. Notice that $q_1^n = 0$ implies that $q^n = 0$ and that p_{q^n} is a constant function, and from $\varsigma p_{q^n} = 0$, we obtain that $p_{q^n} = 0$;

in this case (4.4) is not satisfied, *i.e.*, necessarily $q_1^n \neq 0$, for all $n \in \mathbb{N}$. On the other hand, since the mapping sending $q(b)$ to the corresponding solution $(q,\, p_q)$ is linear, there is no loss of generality in assuming that $|q_1^n|_V = 1$. The boundedness of $(|\hat{u}^n|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}})_{n\in\mathbb{N}}$ implies that $(\hat{u}^n)_{n\in\mathbb{N}}$ and $(\partial_t \hat{u}^n)_{n\in\mathbb{N}}$ are bounded sequences in $L^\infty((a,\,b),\, L^\infty_{\mathrm{div}}(\Omega,\,\mathbb{R}^3))$ and $L^2((a,\,b),\, L^\sigma(\Omega,\,\mathbb{R}^3))$, respectively. It follows, from Theorem 2.14, that the sequences $(q^n)_{n\in\mathbb{N}}$ and $(\partial_t q^n)_{n\in\mathbb{N}}$ are bounded in $L^2((a,\,b),\, \mathrm{D}(L))$ and $L^2((a,\,b),\, H)$, respectively. Since, by the Kakutani's Theorem (see *e.g.* [8], Chap. V, Thm. 4.2), a ball in a reflexive Banach space is weakly compact and, by the Alaoglu's Theorem (see *e.g.* [8], Chap. V, Thm. 3.1), a ball in $L^\infty((a,\,b),\, L^\infty_{\mathrm{div}}(\Omega,\,\mathbb{R}^3))$ is compact in the weak-$*$ topology, there exist a subsequence of $(q_1^n,\, q^n,\, \hat{u}^n)$ (for which we preserve the notation), a V-unit vector $q_1^\infty \in H_N$, $q^\infty \in W((a,\,b), \mathrm{D}(L), H)$, and $\hat{u}^\infty \in \mathcal{W}^{(a,\,b)|\mathrm{wk}}$ such that

$$\begin{array}{rll} q_1^n \to & q_1^\infty & \text{in } H_N;\\ q^n \rightharpoonup & q^\infty & \text{in } L^2((a,\,b),\, \mathrm{D}(L));\\ \partial_t q^n \rightharpoonup & \partial_t q^\infty & \text{in } L^2((a,\,b),\, H);\\ \hat{u}^n \rightharpoonup_* & \hat{u}^\infty & \text{in } L^\infty((a,\,b),\, L^\infty_{\mathrm{div}}(\Omega,\,\mathbb{R}^3));\\ \partial_t \hat{u}^n \rightharpoonup & \partial_t \hat{u}^\infty & \text{in } L^2((a,\,b),\, L^\sigma(\Omega,\,\mathbb{R}^3)). \end{array}$$

Since $W((a,\,b),\, H^2(\Omega,\,\mathbb{R}^3),\, L^2(\Omega,\,\mathbb{R}^3)) \subset L^2((a,\,b),\, H^1(\Omega,\,\mathbb{R}^3))$ is a compact inclusion (see *e.g.* [34], Chap. 3, Thm. 2.1), we can suppose (taking again a subsequence) that

$$q^n \to q^\infty \text{ in } L^2((a,\,b),\, V).$$

Now, $|\nabla p_{q^n}|^2_{L^2(\Omega,\,\mathbb{R}^3)} = (\nabla p_{q^n},\, \partial_t q^n - \mathcal{B}^*(\hat{u}^n)q^n + \nu\Delta q^n)_{L^2(\Omega,\,\mathbb{R}^3)} = (\nabla p_{q^n},\, -\mathcal{B}^*(\hat{u}^n)q^n + \nu\Delta q^n)_{L^2(\Omega,\,\mathbb{R}^3)}$, implies that $|\nabla p_{q^n}|^2_{L^2(\Omega,\,\mathbb{R}^3)} \le \overline{C}_{\left[|\hat{u}^n|_{L^\infty((a,\,b),\, L^\infty_{\mathrm{div}}(\Omega,\,\mathbb{R}^3))}\right]} |\nabla p_{q^n}|_{L^2(\Omega,\,\mathbb{R}^3)} |q^n|_{H^2_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)}$. Necessarily, $|\nabla p_{q^n}|_{L^2(\Omega,\,\mathbb{R}^3)} \le \overline{C}_{\left[|\hat{u}^n|_{L^\infty((a,\,b),\, L^\infty_{\mathrm{div}}(\Omega,\,\mathbb{R}^3))}\right]} |q^n|_{H^2_{\mathrm{div}}(\Omega,\,\mathbb{R}^3)}$, and we can conclude that $(\nabla p_{q^n})_{n\in\mathbb{N}}$ is bounded in $L^2((a,\,b),\, L^2(\Omega,\,\mathbb{R}^3))$. Furthermore since $\varsigma p_{q^n} = 0$ we obtain that $(p_{q^n})_{n\in\mathbb{N}}$ is bounded in $L^2((a,\,b),\, H^1(\Omega,\,\mathbb{R}))$ (see Rem. 3.6). Thus, there exists $p^\infty \in L^2((a,\,b),\, H^1(\Omega,\,\mathbb{R}))$ such that (taking a subsequence and using Lem. 4.1)

$$\begin{array}{rll} p_{q^n} \rightharpoonup p^\infty & \text{in } L^2((a,\,b),\, L^2(\Omega,\,\mathbb{R}))\\ \nabla p_{q^n} \rightharpoonup \nabla p^\infty & \text{in } L^2((a,\,b),\, L^2(\Omega,\,\mathbb{R}^3)) \end{array}, \quad \text{and} \quad p^\infty = c_\varsigma p^\infty \in L^2((a,\,b),\, H^1(\Omega,\,\mathbb{R})). \tag{4.5}$$

Next step is to pass to the limit in (4.3), for that we will need some preliminary computations. Let us rewrite

$$\mathcal{B}^*(\hat{u}^n)q^n - \mathcal{B}^*(\hat{u}^\infty)q^\infty = \mathcal{B}^*(\hat{u}^n)(q^n - q^\infty) + \mathcal{B}^*(\hat{u}^n - \hat{u}^\infty)q^\infty$$

and let us be given $v \in L^2((a,\,b),\, L^2(\Omega,\,\mathbb{R}^3))$; we find

$$|(\mathcal{B}^*(\hat{u}^n)(q^n - q^\infty),\, v)_{L^2((a,\,b),\, L^2(\Omega,\,\mathbb{R}^3))}|_{\mathbb{R}} \le C|\hat{u}^n|_{L^\infty((a,\,b),\, L^\infty_{\mathrm{div}}(\Omega,\,\mathbb{R}^3))}|q^n - q^\infty|_{L^2((a,\,b),\, V)}|v|_{L^2((a,\,b),\, L^2(\Omega,\,\mathbb{R}^3))} \to 0$$

and, from Remark 2.7, we also obtain

$$\begin{aligned} (\mathcal{B}^*(\hat{u}^n - \hat{u}^\infty)q^\infty,\, v)_{L^2((a,\,b),\, L^2(\Omega,\,\mathbb{R}^3))} &= \int_a^b \left(\int_\Omega (\langle (\hat{u}^n - \hat{u}^\infty) \cdot D_s \rangle q^\infty) \cdot v \,\mathrm{d}\Omega \right) \mathrm{d}t \\ = \int_a^b \left(\int_\Omega (\langle v \cdot D_s \rangle q^\infty) \cdot (\hat{u}^n - \hat{u}^\infty) \,\mathrm{d}\Omega \right) \mathrm{d}t &= \langle \hat{u}^n - \hat{u}^\infty,\, \langle v \cdot D_s \rangle q^\infty \rangle_{L^1((a,\,b),\, L^1(\Omega,\,\mathbb{R}^3))',\, L^1((a,\,b),\, L^1(\Omega,\,\mathbb{R}^3))} \to 0 \end{aligned}$$

and can conclude that

$$\mathcal{B}^*(\hat{u}^n)q^n \rightharpoonup \mathcal{B}^*(\hat{u}^\infty)q^\infty \text{ in } L^2((a,\,b),\, L^2(\Omega,\,\mathbb{R}^3)). \tag{4.6}$$

From Lemma 4.1 it also follows that $\Delta q^n \rightharpoonup \Delta q^\infty$ in $L^2((a,\,b),\, L^2(\Omega,\,\mathbb{R}^3))$, and then we can pass to the weak limit in (4.3) and obtain (for $p_{q^\infty} := p^\infty$)

$$\begin{array}{rl} -\partial_t q^\infty + \mathcal{B}^*(\hat{u}^\infty)q^\infty - \nu\Delta q^\infty + \nabla p_{q^\infty} = 0, & \operatorname{div} q^\infty = 0,\\ q^\infty|_\Gamma = 0, & q^\infty(b) = q_1^\infty \in H_N. \end{array} \tag{4.7}$$

Accordingly to Propositions 2.5 and 3.10, given $v \in G^1((a,\,b),\,\Gamma)$, we can extend $\phi v \in G^1((a,\,b),\,\Gamma)$ to $E_1^e\phi v \in W((a,\,b),\,H^1_{\mathrm{div}}(\Omega,\,\mathbb{R}^3),\,H^{-1}(\Omega,\,\mathbb{R}^3)) \oplus H^{\frac{3}{4}}((a,\,b),\,\mathbb{R})\Theta$, with $\phi v = E_1^e\phi v|_\Gamma$; hence we can derive

$$\begin{aligned}&\langle \phi(p_{q^n}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^n),\, v\rangle_{G^1((a,\,b),\,\Gamma)',G^1((a,\,b),\,\Gamma)} = \langle p_{q^n}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^n,\, \phi v\rangle_{G^1((a,\,b),\,\Gamma)',G^1((a,\,b),\,\Gamma)}\\ &= (\nabla p_{q^n} - \nu\Delta q^n,\, E_1^e\phi v)_{L^2((a,\,b),\,L^2(\Omega,\,\mathbb{R}^3))} + (p_{q^n},\, \mathrm{div}(E_1^e\phi v))_{L^2((a,\,b),\,L^2(\Omega,\,\mathbb{R}))}\\ &\quad + (-\nu\nabla q^n,\, \nabla(E_1^e\phi v))_{L^2((a,\,b),\,L^2(\Omega,\,\mathbb{R}^9))}\end{aligned}$$

and, taking the limit, we obtain

$$\begin{aligned}&\langle \phi(p_{q^n}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^n),\, v\rangle_{G^1((a,\,b),\,\Gamma)',G^1((a,\,b),\,\Gamma)}\\ &\to (\nabla p_{q^\infty} - \nu\Delta q^\infty,\, E_1^e\phi v)_{L^2((a,\,b),\,L^2(\Omega,\,\mathbb{R}^3))} + (p_{q^\infty},\, \mathrm{div}(E_1^e\phi v))_{L^2((a,\,b),\,L^2(\Omega,\,\mathbb{R}))}\\ &\quad + (-\nu\nabla q^\infty,\, \nabla(E_1^e\phi v))_{L^2((a,\,b),\,L^2(\Omega,\,\mathbb{R}^9))} = \langle \phi(p_{q^\infty}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^\infty),\, v\rangle_{G^1((a,\,b),\,\Gamma)',G^1((a,\,b),\,\Gamma)},\end{aligned}$$

that is, $\phi(p_{q^n}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^n) \rightharpoonup_* \phi(p_{q^\infty}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^\infty)$ in $G^1((a,\,b),\,\Gamma)'$. In particular we have $|\phi(p_{q^\infty}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^\infty)|_{G^1((a,\,b),\,\Gamma)'} \le \liminf_{n\to+\infty} |\phi(p_{q^n}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^n)|_{G^1((a,\,b),\,\Gamma)'}$. Therefore, from (4.4), we can conclude that $|\phi(p_{q^\infty}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^\infty)|^2_{G^1((a,\,b),\,\Gamma)'} \le \liminf_{n\to+\infty} \frac{1}{n}|\phi(p_{q^n}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^n)|_{L^2((a,\,b),\,H^{\frac{1}{2}}(\Omega,\,\mathbb{R}^3))}$, which implies

$$|\phi(p_{q^\infty}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^\infty)|^2_{G^1((a,\,b),\,\Gamma)'} = 0,$$

because $\phi(p_{q^n}\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q^n)$ is bounded in $L^2((a,\,b),\,H^{\frac{1}{2}}(\Gamma,\,\mathbb{R}^3))$.

Applying now the observability inequality (3.9) to system (4.7) considered on the interval $(a+r, b)$ with $0 \le r < b-a$, we conclude that $q^\infty(t) = 0$ for $a \le t < b$. Since $q^\infty \in C([a,\,b], V)$, we obtain $q_1^\infty = q^\infty(b) = 0$. This contradicts the fact that $q_1^\infty \in H_N$ is a V-unit vector. The contradiction proves that (4.1) holds. □

4.2. Truncation in space variable

Let $\mathcal{O} \subseteq \Gamma = \partial\Omega$ be an open connected smooth submanifold, and recall the orthogonal projection $P_M^{\mathcal{O}} : L^2(\mathcal{O},\,\mathbb{R}^3) \to L^2_M(\mathcal{O},\,\mathbb{R}^3)$, and extension $\mathbb{E}_0^{\mathcal{O}}$, defined in Section 2.2.

Theorem 4.3. *Let $N \in \mathbb{N}_0$ and let $(q,\,p_q)$ solve system (2.8) with $q_1 \in H_N$ and $f = 0$. Fix also an appropriate choice for the pressure function p_q. Let us be given also two differentiable functions ϕ and $\tilde{\phi}$ in $C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))$ with nonempty support, and $\varepsilon > 0$ such that* $\mathrm{supp}(\phi) \subseteq [a,\,b] \times \overline{\mathcal{O}}$ *and $|\tilde{\phi}(t,\,x)|_{\mathbb{R}} \ge \varepsilon$ for all $(t,\,x) \in \mathrm{supp}(\phi)$. Then, if $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{wk}}$ there exists a positive integer $M = \overline{C}_{\left[N,\,|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}$ such that*

$$|q(a)|^2_H \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]} \left|\tilde{\phi}\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})\right|^2_{G^1((a,\,b),\,\Gamma)'} \tag{4.8}$$

and, if $\hat{u} \in \mathcal{W}^{(a,\,b)|\mathrm{st}}$ there exists a positive integer $M = \overline{C}_{\left[N,\,|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}$ such that

$$|q(a)|^2_H \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]} \left|\tilde{\phi}\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})\right|^2_{G^2((a,\,b),\,\Gamma)'}, \tag{4.9}$$

where the constants $\overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{wk}}}\right]}$ and $\overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a,\,b)|\mathrm{st}}}\right]}$ depend only on Ω, $\mathcal{O}$, ϕ, $\tilde{\phi}$, $b-a$, and on the respective norm of $\hat{u}$.

Proof. Again we prove (4.8), the proof of (4.9) is completely analogous. Consider the Laplace–de Rham operator, defined by:

$$\begin{aligned}\Delta_{\mathcal{O}} : H^2(\mathcal{O},\,\mathbb{R}^3) \cap H^1_0(\mathcal{O},\,\mathbb{R}^3) &\to L^2(\mathcal{O},\,\mathbb{R}^3)\\ u = (u\cdot\mathbf{n})\mathbf{n} + u_{\mathbf{t}} &\mapsto (\Delta_{\mathcal{O}}(u\cdot\mathbf{n}))\mathbf{n} + \Delta_{\mathcal{O}}u_{\mathbf{t}}\end{aligned} \tag{4.10}$$

mapping $H^2(\mathcal{O},\,\mathbb{R}^3) \cap H^1_0(\mathcal{O},\,\mathbb{R}^3)$ onto $L^2(\mathcal{O},\,\mathbb{R}^3)$; see Section 2.2.

Denote by $\mathrm{D}(\varDelta_{\mathcal{O}}^s) := \{u \in L^2(\mathcal{O},\, \mathbb{R}^3) \mid \varDelta_{\mathcal{O}}^s u \in L^2(\mathcal{O},\, \mathbb{R}^3)\}$, the domain of its fractional power $\varDelta_{\mathcal{O}}^s$, $s \in [0,\, 1]$. Notice that, for $u = \sum_{i\in\mathbb{N}_0} u_{\mathbf{n}}^i \pi_i \mathbf{n} + \sum_{i\in\mathbb{N}_0} u_{\mathbf{t}}^i \tau_i$, we may write $\varDelta_{\mathcal{O}}^s u = \sum_{i\in\mathbb{N}_0} u_{\mathbf{n}}^i \beta_i^s \pi_i \mathbf{n} + \sum_{i\in\mathbb{N}_0} u_{\mathbf{t}}^i \gamma_i^s \tau_i$, where β_i and γ_i are the eigenvalues associated with π_i and τ_i, respectively. Moreover we can endow $\mathrm{D}(\varDelta_{\mathcal{O}}^s)$ with the scalar product $(u,\, v)_{\mathrm{D}(\varDelta_{\mathcal{O}}^s)} := (u,\, v)_{L^2(\mathcal{O},\,\mathbb{R}^3)} + (\varDelta_{\mathcal{O}}^s u,\, \varDelta_{\mathcal{O}}^s v)_{L^2(\mathcal{O},\,\mathbb{R}^3)}$. Notice that the system $\{\pi_i \mathbf{n},\, \tau_i \mid i \in \mathbb{N}_0\}$ is orthogonal in $\mathrm{D}(\varDelta_{\mathcal{O}}^s)$, for all $s \in [0,\, 1]$. We find

$$\begin{aligned} |\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a,\,b),\,\varGamma)'} &\le C|\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{L^2((a,\,b),\,L^2(\varGamma,\,\mathbb{R}^3))} \\ &= C|(1 - P_M^{\mathcal{O}})(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{L^2((a,\,b),\,L^2(\mathcal{O},\,\mathbb{R}^3))}. \end{aligned} \tag{4.11}$$

On the other side, let $0 \le k \le 2$. Since the mapping $f \mapsto \phi f|_{\mathcal{O}}$ is in the intersection

$$\mathcal{L}\left(L^2((a,\,b),\,L^2(\varGamma,\,\mathbb{R}^3)) \to L^2((a,\,b),\,L^2(\mathcal{O},\,\mathbb{R}^3))\right) \,\cap\, \mathcal{L}\left(L^2((a,\,b),\,H^2(\varGamma,\,\mathbb{R}^3)) \to L^2((a,\,b),\,H_0^2(\mathcal{O},\,\mathbb{R}^3))\right),$$

by an interpolation argument, we can conclude (*e.g.*, using Thm. A.4 and Lem. A.6) that it also maps $L^2((a,\,b),\,H^k(\varGamma,\,\mathbb{R}^3)) = [L^2((a,\,b),\,H^2(\varGamma,\,\mathbb{R}^3)),\,L^2((a,\,b),\,L^2(\varGamma,\,\mathbb{R}^3))]_{1-\frac{k}{2}}$ continuously into

$$\begin{aligned} &[L^2((a,\,b),\,H_0^2(\mathcal{O},\,\mathbb{R}^3)),\,L^2((a,\,b),\,L^2(\mathcal{O},\,\mathbb{R}^3))]_{1-\frac{k}{2}} = L^2\left((a,\,b),\,[H_0^2(\mathcal{O},\,\mathbb{R}^3),\,L^2(\mathcal{O},\,\mathbb{R}^3)]_{1-\frac{k}{2}}\right) \\ &\subseteq L^2\left((a,\,b),\,[\mathrm{D}\left(\varDelta_{\mathcal{O}}^1\right),\,\mathrm{D}\left(\varDelta_{\mathcal{O}}^0\right)]_{1-\frac{k}{2}}\right) = L^2\left((a,\,b),\,\mathrm{D}\left(\varDelta_{\mathcal{O}}^{\frac{k}{2}}\right)\right). \end{aligned}$$

Then, from (4.11), we can write in particular

$$\begin{aligned} |\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a,\,b),\,\varGamma)'} &\le \tfrac{C_1}{1+\theta_M^{\frac{1}{3}}}|(1 - P_M^{\mathcal{O}})(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{L^2\left((a,\,b),\,\mathrm{D}\left(\varDelta_{\mathcal{O}}^{\frac{1}{6}}\right)\right)} \\ &\le C_1\theta_M^{-\frac{1}{3}}|\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}}|^2_{L^2\left((a,\,b),\,\mathrm{D}\left(\varDelta_{\mathcal{O}}^{\frac{1}{6}}\right)\right)} \end{aligned}$$

where $\theta_M = \min\{\beta_i,\,\gamma_i \mid i > M\}$. Now, from $[H_0^2(\mathcal{O},\,\mathbb{R}),\,L^2(\mathcal{O},\,\mathbb{R})]_{1-\frac{1}{6}} \subseteq \mathrm{D}\left(\varDelta_{\mathcal{O}}^{\frac{1}{6}}\right) \subseteq [H^2(\mathcal{O},\,\mathbb{R}),\,L^2(\mathcal{O},\,\mathbb{R})]_{1-\frac{1}{6}}$, we can conclude (*cf.* [23], Chap. 1, Thms. 9.6 and 11.6) that $\mathrm{D}\left(\varDelta_{\mathcal{O}}^{\frac{1}{6}}\right) = H^{\frac{1}{3}}(\mathcal{O},\,\mathbb{R}^3)$, with equivalent norms. Thus we can write $|\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a,\,b),\,\varGamma)'} \le C_2\theta_M^{-\frac{1}{3}}|\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}}|^2_{L^2((a,\,b),\,H^{\frac{1}{3}}(\mathcal{O},\,\mathbb{R}^3))}$ and, from the continuity of the restriction to $\mathcal{O}$, from $H^1(\varGamma,\,\mathbb{R}^3)$ onto $H^1(\mathcal{O},\,\mathbb{R}^3)$ and from $L^2(\varGamma,\,\mathbb{R}^3)$ onto $L^2(\mathcal{O},\,\mathbb{R}^3)$, again by an interpolation argument, we conclude that it is also continuous from $H^{\frac{1}{3}}(\varGamma,\,\mathbb{R}^3)$ onto $H^{\frac{1}{3}}(\mathcal{O},\,\mathbb{R}^3)$ (*cf.* Sect. A.6). Therefore

$$|\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a,\,b),\,\varGamma)'} \le C_2\theta_M^{-\frac{1}{3}}|\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{L^2((a,\,b),\,H^{\frac{1}{3}}(\varGamma,\,\mathbb{R}^3))}$$

and, using the inequality (4.1) in Lemma 4.2, we arrive to

$$|\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\varGamma)'} \le \theta_M^{-\frac{1}{3}}\overline{C}_{\left[N,\,|\hat{u}|_{\mathcal{W}(a,\,b)|\mathrm{wk}}\right]}|\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\varGamma)'}. \tag{4.12}$$

Now let $\xi \in C^\infty([a,\,b],\,C^\infty(\varGamma,\,\mathbb{R}))$ be a nonnegative function taking the value 1 if $|\tilde{\phi}|_{\mathbb{R}} \ge \varepsilon$ and vanishing if $|\tilde{\phi}|_{\mathbb{R}} \le \frac{\varepsilon}{2}$. In particular $\xi\phi = \phi$ and $\tilde{\phi}^{-1}\xi = \frac{\xi}{\tilde{\phi}} \in C^1([a,\,b],\,C^2(\varGamma,\,\mathbb{R}))$. Hence

$$\begin{aligned} &|\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\varGamma)'} = |\tilde{\phi}\tilde{\phi}^{-1}\xi\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\varGamma)'} \\ &= |\tilde{\phi}\tilde{\phi}^{-1}\xi\mathbb{E}_0^{\mathcal{O}}(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a,\,b),\,\varGamma)'} \\ &\le 2C_3|\tilde{\phi}^{-1}\xi|^2_{C^1([a,\,b],\,C^2(\varGamma,\,\mathbb{R})}|\tilde{\phi}\mathbb{E}_0^{\mathcal{O}}P_M^{\mathcal{O}}(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a,\,b),\,\varGamma)'} \\ &\quad + 2C_3|\xi|^2_{C^1([a,\,b],\,C^2(\varGamma,\,\mathbb{R})}|\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a,\,b),\,\varGamma)'} \\ &\le C_4|\tilde{\phi}\mathbb{E}_0^{\mathcal{O}}P_M^{\mathcal{O}}(\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a,\,b),\,\varGamma)'} + \theta_M^{-\frac{1}{3}}C_5|\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\varGamma)'} \end{aligned}$$

and, choosing the integer M so large that $\theta_M^{-\frac{1}{3}} C_5 = \theta_M^{-\frac{1}{3}} \overline{C}_{\left[N,\, |\hat{u}|_{\mathcal{W}(a,\, b)|\mathrm{wk}}\right]} |\xi|^2_{C^1([a,\, b],\, C^2(\Gamma,\, \mathbb{R})} \le \frac{1}{2}$, we obtain $|\phi(p_q \mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\, b),\, \Gamma)'} \le 2C_4 |\tilde{\phi} \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}} (\phi(p_q\mathbf{n} - \nu\langle \mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a,\, b),\, \Gamma)'}$. Combining this with (3.9) (with $f = 0$), we arrive to the required inequality (4.8). □

Remark 4.4. Notice that the integer M in Theorem 4.3, depends on N but, the constants in the observability inequalities (4.8) and (4.9) do not. We can, of course, take $\tilde{\phi} = 1$ identically; however, as we will see in the example in Section 5, it is useful to consider the more general case.

4.3. Further truncation in time variable

In [29], an observability inequality truncated in both space and time variables was used to derive suitable results for the stochastic Navier–Stokes equations perturbed by an internal random force localized in a subset of the domain Ω. Inspired by these results, here we show that we can also truncate the observability inequality in time variable. We will need the following proposition, whose proof is given in the Appendix, Section A.7.

Proposition 4.5. *The inclusion $G^1((a,\, b),\, \Gamma) \subseteq H^{\frac{1}{4}}\left((a,\, b),\, H^{-\frac{1}{4}}\left(\Gamma,\, \mathbb{R}^3\right)\right)$ holds and is continuous.*

Let us consider the Laplace–de Rham operator in $(a,\, b)$ with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} \Delta_t : H^2((a,\, b),\, \mathbb{R}) \cap H_0^1((a,\, b),\, \mathbb{R}) &\to L^2((a,\, b),\, \mathbb{R}) \\ f &\mapsto -\partial_t \partial_t f. \end{aligned}$$

It is well known that the orthonormal system of eigenfuntions, and corresponding eigenvalues, are given by $\left\{\bar{s}_n := (\frac{2}{b-a})^{\frac{1}{2}} \sin(n\pi(\frac{x-a}{b-a})) \mid n \in \mathbb{N}_0\right\}$, and $\left\{\lambda_n = (\frac{n\pi}{b-a})^2 \mid n \in \mathbb{N}_0\right\}$; $\Delta_t \bar{s}_n = \lambda_n \bar{s}_n$. Next, given a Hilbert space X, we define the following mapping P_M^t in $L^2((a,\, b),\, X)$

$$P_M^t f(t) := \sum_{n=1}^{M} \left(\int_a^b \bar{s}_n(\tau) f(\tau)\, \mathrm{d}\tau\right) \bar{s}_n.$$

Proposition 4.6. *The mapping P_M^t is an orthogonal projection both in $L^2((a,\, b),\, X)$ and in $H_0^1((a,\, b),\, X)$, onto $P_M^t L^2((a,\, b),\, X) = \sum_{n=1}^{M} \bar{s}_n X$. Moreover we may write $f = \sum_{n\in\mathbb{N}_0} \left(\int_a^b \bar{s}_n(\tau) f(\tau)\, \mathrm{d}\tau\right) \bar{s}_n = \lim_{M\to+\infty} P_M^t f$, $|f|^2_{L^2((a,\, b),\, X)} = \sum_{n\in\mathbb{N}_0} \left|\int_a^b \bar{s}_n(\tau) f(\tau)\, \mathrm{d}\tau\right|^2_X$, and $|f|^2_{H_0^1((a,\, b),\, X)} = \sum_{n\in\mathbb{N}_0} (1 + \lambda_n) \left|\int_a^b \bar{s}_n(\tau) f(\tau)\, \mathrm{d}\tau\right|^2_X$.*

The proof of this proposition is straightforward, though nontrivial; for the sake of completeness we present it in the Appendix, Section A.8.

Now, for simplicity, given a finite orthogonal sequence $\mathcal{S} = \{v_i \mid i = 1,\, 2,\, \ldots,\, k\} \subseteq X$ in the Hilbert space X, let $\mathcal{F} = \operatorname{span} \mathcal{S}$ and define the operator $\Delta_{t,\, \mathcal{F}} : H^2((a,\, b),\, \mathcal{F}) \cap H_0^1((a,\, b),\, \mathcal{F}) \to L^2((a,\, b),\, \mathcal{F})$, sending $f(t) = \sum_{i=1}^k f_i(t) v_i$ to $\sum_{i=1}^k \Delta_t f_i(t) v_i$. It turns out that $\Delta_{t,\mathcal{F}}^s f(t) = \sum_{i=1}^k (\Delta_t^s f_i(t)) v_i$, and

$$|f|^2_{\mathrm{D}\left(\Delta_{t,\mathcal{F}}^s\right)} = \sum_{i=1}^{k} |f_i|^2_{\mathrm{D}(\Delta_t^s)} |v_i|^2_X; \text{ for } s \in [0,\, 1]. \tag{4.13}$$

Theorem 4.7. *Let $N \in \mathbb{N}_0$ and let $(q,\, p_q)$ solve system (2.8) with $q_1 \in H_N$ and $f = 0$, for an appropriate choice for the pressure function p_q. Let us be given also two differentiable functions $\phi,\, \tilde{\phi} \in C^1([a,\, b],\, C^2(\Gamma,\, \mathbb{R}))$, with nonempty support, an open connected smooth submanifold $\mathcal{O} \subseteq \Gamma$, and $\varepsilon > 0$ such that* $\operatorname{supp}(\phi) \subseteq [a,\, b] \times \overline{\mathcal{O}}$ *and*

$|\tilde{\phi}(t, x)|_{\mathbb{R}} \geq \varepsilon$ *for all* $(t, x) \in \operatorname{supp}(\phi)$. *Then, if* $\hat{u} \in \mathcal{W}^{(a, b)|\mathrm{wk}}$ *there exists a positive integer* $M = \overline{C}_{\left[N, |\hat{u}|_{\mathcal{W}^{(a, b)|\mathrm{wk}}}\right]}$ *such that*

$$|q(a)|_H^2 \leq \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a, b)|\mathrm{wk}}}\right]} \left|\tilde{\phi} P_M^t \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})\right|^2_{G^1((a, b),\, \Gamma)'} \tag{4.14}$$

and, if $\hat{u} \in \mathcal{W}^{(a, b)|\mathrm{st}}$ *there exists a positive integer* $M = \overline{C}_{\left[N, |\hat{u}|_{\mathcal{W}^{(a, b)|\mathrm{st}}}\right]}$ *such that*

$$|q(a)|_H^2 \leq \overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a, b)|\mathrm{st}}}\right]} \left|\tilde{\phi} P_M^t \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})\right|^2_{G^2((a, b),\, \Gamma)'}, \tag{4.15}$$

where the constants $\overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a, b)|\mathrm{wk}}}\right]}$ *and* $\overline{C}_{\left[|\hat{u}|_{\mathcal{W}^{(a, b)|\mathrm{st}}}\right]}$ *depend only on* Ω, $\mathcal{O}$, ϕ, $\tilde{\phi}$, $b - a$, *and on the respective norm of* $\hat{u}$.

Proof. Again we prove (4.14), the proof of (4.15) is completely analogous (*e.g.*, starting by using the continuity of the inclusion $G^1(a, b), \Gamma)' \subset G^2(a, b), \Gamma)')$. From Proposition 4.5, we can derive

$$\begin{aligned}
&|(1 - P_M^t)\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a, b),\, \Gamma)'}\\
&\leq C|(1 - P_M^t)\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})|^2_{H^{-\frac14}((a, b),\, H^{\frac14}(\Gamma, \mathbb{R}^3))}
\end{aligned}$$

and, from the continuity of the extension by zero outside $\mathcal{O}$, from $H^s(\mathcal{O}, \mathbb{R}^3)$ into $H^s(\Gamma, \mathbb{R}^3)$ for $0 \leq s < \frac12$ (*cf.* [23], Chap. 1, Sect. 11.3), we can write

$$\begin{aligned}
&|(1 - P_M^t)\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a, b),\, \Gamma)'}\\
&\leq C|(1 - P_M^t) P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})|^2_{H^{-\frac14}((a, b),\, H^{\frac14}(\mathcal{O}, \mathbb{R}^3))},
\end{aligned}$$

Now, set $\mathcal{F} := L^2_M(\mathcal{O}, \mathbb{R}^3) = P_M^{\mathcal{O}} L^2(\mathcal{O}, \mathbb{R}^3)$. By an analogous argument as in the proof of Theorem 4.3 we can prove that $H^{\frac14}(\mathcal{O}, \mathbb{R}^3) = \mathrm{D}\left(\Delta_{\mathcal{O}}^{\frac18}\right)$, $H^{\frac14}((a, b), \mathcal{F}) = \mathrm{D}(\Delta_{t,\mathcal{F}}^{\frac18})$, and $H^{-\frac14}((a, b), \mathcal{F}) = \mathrm{D}(\Delta_{t,\mathcal{F}}^{-\frac18})$, with equivalent norms. Thus, using (4.13), we can derive

$$\begin{aligned}
&|(1 - P_M^t)\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a, b),\, \Gamma)'} \leq C_1|(1 - P_M^t) P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})|^2_{\mathrm{D}\left(\Delta_{t,\mathcal{F}_{\frac18}}^{-\frac18}\right)}\\
&= C_1|\Delta_{t,\mathcal{F}_{\frac18}}^{-\frac18}(1 - P_M^t) P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})|^2_{\mathrm{D}\left(\Delta_{t,\mathcal{F}_{\frac18}}^{0}\right)} \leq C_1 \Theta_{t,M}^{-\frac14}|P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})|^2_{\mathrm{D}\left(\Delta_{t,\mathcal{F}_{\frac18}}^{0}\right)}\\
&\leq C_1 \Theta_{t,M}^{-\frac14}|\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}}|^2_{L^2\left((a, b),\, \mathrm{D}\left(\Delta_{\mathcal{O}}^{\frac18}\right)\right)}
\end{aligned}$$

where $\theta_{t,M} = \min\{\lambda_i | i \rangle M\} = \left(\frac{(M+1)\pi}{b-a}\right)^2$, and $\mathcal{F}_{\frac18}$ means that $\mathcal{F}$ is endowed with the $\mathrm{D}\left(\Delta_{\mathcal{O}}^{\frac18}\right)$-norm. Proceeding as in the proof of Theorem 4.3, and using (4.1), we obtain

$$\begin{aligned}
&|(1 - P_M^t)\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a, b),\, \Gamma)'} \leq C_2 \Theta_{t,M}^{-\frac14}|\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|^2_{L^2((a, b),\, H^{\frac14}(\Gamma, \mathbb{R}^3))}\\
&\leq \Theta_{t,M}^{-\frac14} \overline{C}_{\left[N, |\hat{u}|_{\mathcal{W}^{(a, b)|\mathrm{wk}}}\right]}|\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|^2_{G^1((a, b),\, \Gamma)'}.
\end{aligned} \tag{4.16}$$

Next, again as in the proof of Theorem 4.3, we set $\xi \in C^\infty([a, b], C^\infty(\Gamma, \mathbb{R}))$ be a nonnegative function taking the value 1 if $|\tilde{\phi}|_{\mathbb{R}} \geq \varepsilon$ and vanishing if $|\tilde{\phi}|_{\mathbb{R}} \leq \frac{\varepsilon}{2}$. Writing

$$\begin{aligned}
&\tilde{\phi}\tilde{\phi}^{-1}\xi\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q) = \tilde{\phi}\tilde{\phi}^{-1}\xi\mathbb{E}_0^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})\\
&= \tilde{\phi}\tilde{\phi}^{-1}\xi P_M^t \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}}) + \tilde{\phi}\tilde{\phi}^{-1}\xi(1 - P_M^t)\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})\\
&\quad + \tilde{\phi}\tilde{\phi}^{-1}\xi\mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\phi(p_q \mathbf{n} - \nu\langle \mathbf{n} \cdot \nabla\rangle q)|_{\mathcal{O}})
\end{aligned}$$

and using (4.12), (4.16), and $\tilde{\phi}\tilde{\phi}^{-1}\xi\phi = \phi$, we find

$$\begin{aligned}|\phi(p_q\mathbf{n} - \nu\langle\mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\Gamma)'} &\le C_3|\tilde{\phi}^{-1}\xi|^2_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}|\tilde{\phi}P^t_M\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_M(\phi(p_q\mathbf{n} - \nu\langle\mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a,\,b),\,\Gamma)'}\\ &+ \left(\Theta_{t,M}^{-\frac{1}{4}} + \Theta_M^{-\frac{1}{3}}\right)|\xi|^2_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}\overline{C}_{\left[N,\,|\hat{u}|_{\mathcal{W}(a,\,b)|\mathrm{wk}}\right]}|\phi(p_q\mathbf{n} - \nu\langle\mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\Gamma)'}\end{aligned}$$

and, choosing $M \in \mathbb{N}_0$ so large that $\left(\Theta_M^{-\frac{1}{3}} + \Theta_{t,M}^{-\frac{1}{4}}\right)|\xi|^2_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}\overline{C}_{\left[N,\,|\hat{u}|_{\mathcal{W}(a,\,b)|\mathrm{wk}}\right]} \le \frac{1}{2}$, we obtain $|\phi(p_q\mathbf{n} - \nu\langle\mathbf{n}\cdot\nabla\rangle q)|^2_{G^1((a,\,b),\,\Gamma)'} \le C_{4(\phi,\tilde{\phi})}|\tilde{\phi}P^t_M\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_M(\phi(p_q\mathbf{n} - \nu\langle\mathbf{n}\cdot\nabla\rangle q)|_{\mathcal{O}})|^2_{G^1((a,\,b),\,\Gamma)'}$. Combining this with (3.9) (with $f = 0$), we arrive to the required inequality (4.14). □

5. Examples of application

Recall the spaces $\mathcal{E}^i_M = \chi\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_{\chi^\perp}P^{\mathcal{O}}_M\vartheta G^i((a,\,b),\,\Gamma)|_{\mathcal{O}}$, $i \in \{1,\,2\}$ defined in (2.6) and in (2.13). Here we use the truncated observability inequalities (4.9) and (4.15) to derive two controllability results for the Oseen–Stokes system (2.7), where the control ζ is taken in (a subspace of) $\mathcal{E}^2_M$.

It turns out that, while inequality (4.9) is appropriate to deal with the control space $\mathcal{E}^2_M$, inequality (4.15) is (taking $\phi = \varphi\chi$ and $\tilde{\phi} = \tilde{\varphi}\vartheta$, for suitable functions φ and $\tilde{\varphi}$) appropriate to deal with controls in

$$\mathcal{G}_M := \varphi\chi\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_{\chi^\perp}P^{\mathcal{O}}_MP^t_M\tilde{\varphi}\vartheta G^2((a,\,b),\,\Gamma)|_{\mathcal{O}} := \{\zeta \mid \zeta = \varphi\chi\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_{\chi^\perp}P^{\mathcal{O}}_MP^t_M(\tilde{\varphi}\vartheta\eta|_{\mathcal{O}}) \text{ and } \eta \in G^2((a,\,b),\,\Gamma)\}.$$

Let $\varphi \in C^1((a,\,b),\,\mathbb{R})$ be such that $\mathrm{supp}(\varphi) \neq \emptyset$ and $\varphi(t)$ vanishes in a neighborhood of $\{a,\,b\}$, say $\varphi(t) = 0$ for some $0 < \delta < \frac{b-a}{2}$ and all $t \in [a,\,a+\delta] \cup [b-\delta,\,b]$. Also, let $\tilde{\varphi} \in C^1([a,\,b],\,\mathbb{R}) \cap H^1_0((a,\,b),\,\mathbb{R})$ be a function such that $|\tilde{\varphi}(t)|_{\mathbb{R}} \ge \varepsilon > 0$ for all $t \in \mathrm{supp}(\phi)$. Consider the operator

$$\eta \mapsto K^{\mathcal{O}}_t\eta := \varphi\chi\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_{\chi^\perp}P^{\mathcal{O}}_MP^t_M(\tilde{\varphi}\vartheta\eta|_{\mathcal{O}}). \tag{5.1}$$

Proposition 5.1. *$K^{\mathcal{O}}_t \in \mathcal{L}(G^i((a,\,b),\,\Gamma) \to G^i_{\mathrm{av}}((a,\,b),\,\Gamma))$, for $i \in \{1,\,2\}$.*

The Proof of Proposition 5.1 will be given in the Appendix, Section A.9.

Next, we recall also the space H_N and the orthogonal projection $\mathit{\Pi}_N : H \to H_N$; see Section 2.2.

Theorem 5.2. *For each $N \in \mathbb{N}$ there exists an integer $M = \overline{C}_{\left[N,\,|\hat{u}|_{\mathcal{W}(a,\,b)|\mathrm{st}}\right]} \in \mathbb{N}_0$ such that, for every $v_0 \in H$, we can find $\eta = \eta(v_0) \in G^2((a,\,b),\,\Gamma)$, depending linearly on v_0, such that the boundary control $\zeta = \varphi\chi\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_{\chi^\perp}P^{\mathcal{O}}_M(\vartheta\eta|_{\mathcal{O}})$ drives the system (2.7), with $g = 0$, to a vector $v(b) \in V$ such that $\mathit{\Pi}_Nv(b) = 0$. Moreover, there exists a constant $\overline{C}_{\left[|\hat{u}|_{\mathcal{W}(a,\,b)|\mathrm{st}}\right]}$, depending on $|\hat{u}|_{\mathcal{W}(a,\,b)|\mathrm{st}}$, φ, and $b-a$ but, not on the pair $(N,\,v_0)$, such that $|\eta|^2_{G^2((a,\,b),\,\Gamma)} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}(a,\,b)|\mathrm{st}}\right]}|v_0|^2_H$.*

Theorem 5.3. *For each $N \in \mathbb{N}$ there exists an integer $M = \overline{C}_{\left[N,\,|\hat{u}|_{\mathcal{W}(a,\,b)|\mathrm{st}}\right]} \in \mathbb{N}_0$ such that, for every $v_0 \in H$, we can find $\eta = \eta(v_0) \in G^2((a,\,b),\,\Gamma)$, depending linearly on v_0, such that the boundary control $\zeta = \varphi\chi\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_{\chi^\perp}P^{\mathcal{O}}_MP^t_M(\tilde{\varphi}\vartheta\eta|_{\mathcal{O}})$ drives the system (2.7), with $g = 0$, to a vector $v(b) \in V$ such that $\mathit{\Pi}_Nv(b) = 0$. Moreover, there exists a constant $\overline{C}_{\left[|\hat{u}|_{\mathcal{W}(a,\,b)|\mathrm{st}}\right]}$, depending on $|\hat{u}|_{\mathcal{W}(a,\,b)|\mathrm{st}}$, φ, $\tilde{\varphi}$, and $b-a$ but, not on the pair $(N,\,v_0)$, such that*

$$|\eta|^2_{G^2((a,\,b),\,\Gamma)} \le \overline{C}_{\left[|\hat{u}|_{\mathcal{W}(a,\,b)|\mathrm{st}}\right]}|v_0|^2_H. \tag{5.2}$$

The proofs of Theorems 5.2 and 5.3 are completely analogous. So we will prove only Theorem 5.3 where we shall use the observability inequality (4.15); to prove Theorem 5.2 we can use (4.9) instead. We start by recalling the following:

Lemma 5.4. *Let Z be a Banach space such that $Z = X \oplus Y$, where X and Y are closed subspaces of Z. Then we can rewrite, in a unique way, each $z \in Z$ as $z = z_X + z_Y$ with $(z_X,\, z_Y) \in X \times Y$, and the projections $z \mapsto z_X$, and $z \mapsto z_Y$ are continuous. Moreover the norms $|\cdot|_Z$ and $|\cdot|_{X\oplus Y}$ are equivalent in Z.*

Proof. Consider the graph $G_X = \{(z,\, w) \in Z \times Z \mid z \in Z \text{ and } w = z_X\}$ of the projection onto X. It is straightforward to prove that G_X is closed, then by the Closed Graph Theorem (see *e.g.* [8], Sect. III.12) it follows the continuity of the projection $z \mapsto z_X$, and also that of the projection $z \mapsto z_Y = z - z_X$. Finally, from $|z|_Z^2 \le (|z_X|_Z + |z_Y|_Z)^2 \le 2(|z_X|_Z^2 + |z_Y|_Z^2) = 2|z|_{X\oplus Y}^2 \le C|z|_Z^2$, we have the equivalence of the norms. □

Lemma 5.5. *If $q \in \mathrm{D}(L)$, then $\langle \mathbf{n} \cdot \nabla \rangle q$ is tangent to Γ.*

Proof. Since $q|_\Gamma = 0$ we have that $\nabla q_j = \alpha_j \mathbf{n}$ on Γ, for a suitable function α_j and for each $j \in \{1,\, 2,\, 3\}$. Then we can derive that $\partial_{x_i} q_j = \alpha_j \mathbf{n}_i$ and $(\langle \mathbf{n}\cdot\nabla\rangle q)\cdot \mathbf{n} = \sum_{j=1}^3 \left(\sum_{i=1}^3 \mathbf{n}_i \partial_{x_i} q_j\right)\mathbf{n}_j = \sum_{j=1}^3 \left(\sum_{i=1}^3 \mathbf{n}_i^2 \alpha_j\right)\mathbf{n}_j = \sum_{j=1}^3 \alpha_j \mathbf{n}_j$, on the boundary Γ. On the other hand, from $0 = \operatorname{div} q = \sum_{j=1}^3 \partial_{x_j} q_j$, we obtain that $0 = (\operatorname{div} q)|_\Gamma = \sum_{j=1}^3 \alpha_j \mathbf{n}_j$. Therefore, we have $(\langle \mathbf{n}\cdot\nabla\rangle q)\cdot \mathbf{n} = 0$ on Γ. □

Proof of Theorem 5.3. We shall follow the idea in the proof of Lemma 3.2 in [5]. First, we extend the orthogonal projection $\Pi : L^2(\Omega,\, \mathbb{R}^3) \to H$ to a projection mapping $\Pi : H^{-1}(\Omega,\, \mathbb{R}^3) \to V'$ by setting $\langle \Pi f,\, u\rangle_{V',V} := \langle f,\, u\rangle_{H^{-1}(\Omega,\mathbb{R}^3),H_0^1(\Omega,\mathbb{R}^3)}$. Recall that we can write $H^{-1}(\Omega,\, \mathbb{R}^3) = V' \oplus \{\nabla p \mid p \in L^2(\Omega,\, \mathbb{R})\}$ (see [34], Chap. 1, Sect. 1.4, Prop. 1.1 and Rem. 1.9). Observe that, given $p \in L^2(\Omega,\, \mathbb{R})$, we have $\langle \Pi \nabla p,\, u\rangle_{V',V} = \langle \nabla p,\, u\rangle_{H^{-1}(\Omega,\mathbb{R}^3),H_0^1(\Omega,\mathbb{R}^3)} = 0$, that is, $\Pi\nabla p = 0$; in other words, Π coincides with the projection in $V' \oplus \{\nabla p \mid p \in L^2(\Omega,\, \mathbb{R})\}$ onto the component V'. In particular, from Lemma 5.4, Π is continuous.

Then, we fix $\epsilon > 0$ and consider the following minimization problem:

Problem 5.6. Given $M, N \in \mathbb{N}$ and $v_0 \in H$, find the minimum of the quadratic functional

$$J_\epsilon(v,\, \eta) := |\eta|^2_{G^2((a,\,b),\,\Gamma)} + \tfrac{1}{\epsilon}|\Pi_N v(b)|_H^2,$$

subject to the constraint $F(v,\, \eta) = (0,\, 0,\, 0)$, in the space

$$\mathcal{X} := W_H((a,\, b),\, H^1_{\mathrm{div}}(\Omega,\, \mathbb{R}^3),\, H^{-1}(\Omega,\, \mathbb{R}^3)) \times G^2((a,\, b),\, \Gamma),$$

where F is defined as $\begin{cases} F : \mathcal{X} \to \mathcal{Y} := H \times L^2((a,\, b),\, V') \times G^1_{\mathrm{av},\,H}((a,\, b),\, \Gamma) \\ (v,\, \eta) \mapsto \left(v(a) - v_0,\, \Pi\left(v_t - \nu\Delta v + \mathcal{B}(\hat u)v\right),\, v|_\Gamma - K_t^{\mathcal{O}}\eta\right)\end{cases}$ with

$$\begin{aligned} W_H((a,\, b),\, H^1_{\mathrm{div}}(\Omega,\, \mathbb{R}^3),\, H^{-1}(\Omega,\, \mathbb{R}^3)) &:= \{u \in W((a,\, b),\, H^1_{\mathrm{div}}(\Omega,\, \mathbb{R}^3),\, H^{-1}(\Omega,\, \mathbb{R}^3)) \mid u(a) \in H\};\\ G^1_{\mathrm{av},\,H}((a,\, b),\, \Gamma) &:= \{u|_\Gamma \mid u \in W_H((a,\, b),\, H^1_{\mathrm{div}}(\Omega,\, \mathbb{R}^3),\, H^{-1}(\Omega,\, \mathbb{R}^3))\}.\end{aligned}$$

Since the constraint can be rewritten as $A(v,\, \eta) = (v_0,\, 0,\, 0)$ where A is the linear mapping $A(v,\, \eta) := F(v,\, \eta) + (v_0,\, 0,\, 0)$, we have that Problem 5.6 has a unique minimizer $(\bar v_\epsilon, \bar\eta_\epsilon)$, which linearly depends on $v_0 \in H$ (*e.g.*, using Lem. A.14 and Rem. A.15 in the Appendix, Sect. A.10; together with Prop. 5.1 and Thm. 2.11). By the Karush–Kuhn–Tucker Theorem (*e.g.*, see [5], Sect. A.1), it follows that there is a Lagrange multiplier $(\mu^\epsilon,\, q^\epsilon,\, \rho^\epsilon) \in \mathcal{Y}' = H \times L^2((a,\, b),\, V) \times G^1_{\mathrm{av},\,H}((a,\, b),\, \Gamma)'$ such that

$$\mathrm{d}J_\epsilon(\bar v^\epsilon,\, \bar\eta^\epsilon) + (\mu^\epsilon,\, q^\epsilon,\, \rho^\epsilon) \circ \mathrm{d}F(\bar v^\epsilon,\, \bar\eta^\epsilon) = 0,$$

where the symbol "∘" stands for the composition of two linear operators. It follows that, for all $(z,\, \xi) \in \mathcal{X}$,

$$\begin{aligned} 0 &= 2\tfrac{1}{\epsilon}(\Pi_N \bar v^\epsilon(b),\, z(b))_H + (\mu^\epsilon,\, z(a))_H + \int_a^b \langle z_t + \mathcal{B}(\hat u)z - \nu\Delta z,\, q^\epsilon\rangle_{H^{-1}(\Omega,\mathbb{R}^3),H_0^1(\Omega,\mathbb{R}^3)}\,\mathrm{d}t\\ &\quad + (\rho^\epsilon,\, z|_\Gamma)_{G^1_{\mathrm{av},\,H}((a,b),\,\Gamma)',\,G^1_{\mathrm{av},\,H}((a,b),\,\Gamma)}, \end{aligned} \tag{5.3}$$

$$0 = 2(\bar\eta^\epsilon,\, \xi)_{G^2((a,b),\,\Gamma)} + (\rho^\epsilon,\, -K_t^{\mathcal{O}}\xi)_{G^1_{\mathrm{av},\,H}((a,b),\,\Gamma)',\,G^1_{\mathrm{av},\,H}((a,b),\,\Gamma)}. \tag{5.4}$$

Letting z run over $W((a,\,b),\,V,\,V')$ (*e.g.*, proceeding as in the proof of Lem. 3.2 in [5]) we can verify that relation (5.3) implies that q^ϵ solves system (2.8) with $f=0$, $q^\epsilon(b)=-\frac{2}{\epsilon}\Pi_N\bar v^\epsilon(b)$, and a suitable pressure function p_{q^ϵ}. Further, $q^\epsilon(a)=\mu^\epsilon$.

Next, we let z run over $W_H((a,\,b),\,H^1_{\rm div}(\Omega,\,\mathbb{R}^3),\,H^{-1}(\Omega,\,\mathbb{R}^3))$ in (5.3); we can derive that $\rho^\epsilon=p_{q^\epsilon}\mathbf{n}-\langle\mathbf{n}\cdot\nabla\rangle q^\epsilon$ and, in particular, we have that $\rho^\epsilon\in L^2((a,\,b),\,L^2(\Gamma,\,\mathbb{R}^3))$. Therefore, we can obtain

$$\begin{aligned}(\rho^\epsilon,\,K^{\mathcal{O}}_t\xi)_{G^1_{\mathrm{av},\,H}((a,\,b),\,\Gamma)',\,G^1_{\mathrm{av},\,H}((a,\,b),\,\Gamma)}&=(\rho^\epsilon,\,\varphi\chi\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_{\chi^\perp}P^{\mathcal{O}}_MP^t_M(\tilde\varphi\vartheta\xi|_{\mathcal{O}}))_{L^2((a,\,b),\,L^2(\Gamma,\mathbb{R}^3))}\\&=(\tilde\varphi\vartheta P^t_M\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_MP^{\mathcal{O}}_{\chi^\perp}(\varphi\chi\rho^\epsilon|_{\mathcal{O}}),\,\xi)_{L^2((a,\,b),\,L^2(\Gamma,\mathbb{R}^3))}\end{aligned}$$

and, from (5.4), it follows that necessarily $2\mathcal{A}\bar\eta^\epsilon=\tilde\varphi\vartheta P^t_M\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_MP^{\mathcal{O}}_{\chi^\perp}(\varphi\chi\rho^\epsilon|_{\mathcal{O}})$, where $\mathcal{A}$ is the natural isomorphism

$$\langle\mathcal{A}u,\,v\rangle_{G^2((a,\,b),\,\Gamma)',\,G^2((a,\,b),\,\Gamma)}:=(u,\,v)_{G^2((a,\,b),\,\Gamma)}\tag{5.5}$$

from $G^2((a,\,b),\,\Gamma)$ onto $G^2((a,\,b),\,\Gamma)'$. Notice that the mapping $v\mapsto(u,\,v)_{G^2((a,\,b),\,\Gamma)}$ is in $G^2((a,\,b),\,\Gamma)'$, and (5.5) just says that we denote this mapping by $\mathcal{A}u$. That $\mathcal{A}$ is, indeed, bijective follows from the Lax–Milgram Lemma (*cf.* [34], Chap. 1, Thm. 2.2; [24], Chap. 1, Sect. 3.1).

Therefore, we obtain

$$2\mathcal{A}\bar\eta^\epsilon=\tilde\varphi\vartheta P^t_M\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_MP^{\mathcal{O}}_{\chi^\perp}(\varphi\chi(p_{q^\epsilon}\mathbf{n}-\langle\mathbf{n}\cdot\nabla\rangle q^\epsilon)|_{\mathcal{O}}).\tag{5.6}$$

Combining the above identities, we can arrive to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(q^\epsilon,\,\bar v^\epsilon)_{L^2(\Omega,\mathbb{R}^3)}&=(q^\epsilon_t,\,\bar v^\epsilon)_{L^2(\Omega,\mathbb{R}^3)}+\langle q^\epsilon,\,\bar v^\epsilon_t\rangle_{H^1_0(\Omega,\mathbb{R}^3),H^{-1}(\Omega,\mathbb{R}^3)}\\&=(p_{q^\epsilon}\mathbf{n}-\langle\mathbf{n}\cdot\nabla\rangle q^\epsilon,\,\bar v^\epsilon|_\Gamma)_{L^2(\Gamma,\mathbb{R}^3)}=(p_{q^\epsilon}\mathbf{n}-\langle\mathbf{n}\cdot\nabla\rangle q^\epsilon,\,K^{\mathcal{O}}_t\bar\eta^\epsilon)_{L^2(\Gamma,\mathbb{R}^3)}\\&=\left(\tilde\varphi\vartheta P^t_M\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_MP^{\mathcal{O}}_{\chi^\perp}(\varphi\chi(p_{q^\epsilon}\mathbf{n}-\langle\mathbf{n}\cdot\nabla\rangle q^\epsilon)|_{\mathcal{O}}),\,\bar\eta^\epsilon\right)_{L^2(\Gamma,\mathbb{R}^3)}\end{aligned}$$

and, integrating in time over the interval $(a,\,b)$,

$$\begin{aligned}(q^\epsilon(b),\,\bar v^\epsilon(b))_H-(q^\epsilon(a),\,\bar v^\epsilon(a))_H&=\left(\tilde\varphi\vartheta P^t_M\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_MP^{\mathcal{O}}_{\chi^\perp}(\varphi\chi(p_{q^\epsilon}\mathbf{n}-\langle\mathbf{n}\cdot\nabla\rangle q^\epsilon)|_{\mathcal{O}}),\,\bar\eta^\epsilon\right)_{L^2((a,\,b),\,L^2(\Gamma,\mathbb{R}^3))}\\&=(2\mathcal{A}\bar\eta^\epsilon,\,\bar\eta^\epsilon)_{L^2((a,\,b),\,L^2(\Gamma,\mathbb{R}^3))}=2|\bar\eta^\epsilon|^2_{G^2((a,\,b),\,\Gamma)};\end{aligned}$$

so, from $q^\epsilon(b)=-\frac{2}{\epsilon}\Pi_N\bar v^\epsilon(b)$, we obtain

$$2|\bar\eta^\epsilon|^2_{G^2((a,\,b),\,\Gamma)}+\tfrac{2}{\epsilon}|\Pi_N\bar v^\epsilon(b)|^2_H=-(q^\epsilon(a),\,\bar v^\epsilon(a))_H.\tag{5.7}$$

We wish to use the truncated observability inequality (4.15) to estimate the right-hand side of (5.7); to this end, it will be convenient to choose the pressure function p_{q^ϵ} in a suitable way (*cf.* Sect. 3.2). We choose p_{q^ϵ} such that $\varsigma(p_{q^\epsilon}):=\int_\Gamma\chi^2p_{q^\epsilon}\,\mathrm{d}\Gamma=0$. Then, using also Lemma 5.5, we observe that $P^{\mathcal{O}}_{\chi^\perp}(\varphi\chi(p_{q^\epsilon}\mathbf{n}-\langle\mathbf{n}\cdot\nabla\rangle q^\epsilon)|_{\mathcal{O}})=\varphi P^{\mathcal{O}}_{\chi^\perp}(\chi p_{q^\epsilon}\mathbf{n}|_{\mathcal{O}})-\varphi\chi\langle\mathbf{n}\cdot\nabla\rangle q^\epsilon|_{\mathcal{O}}=\varphi\chi(p_{q^\epsilon}\mathbf{n}-\langle\mathbf{n}\cdot\nabla\rangle q^\epsilon)|_{\mathcal{O}}$, and by the observability inequality (4.15), with $\phi(t,\,x)=\varphi(t)\chi(x)$ and $\tilde\phi(t,\,x)=\tilde\varphi(t)\vartheta(x)$ for $(t,\,x)\in[a,\,b]\times\Gamma$, there exists an integer M such that

$$\begin{aligned}|q^\epsilon(a)|^2_H&\le\overline{C}_{\left[|\hat u|_{\mathcal{W}(a,\,b)|\mathrm{st}}\right]}|\tilde\varphi\vartheta P^t_M\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_M(\varphi\chi(p_{q^\epsilon}\mathbf{n}-\nu\langle\mathbf{n}\cdot\nabla\rangle q^\epsilon)|_{\mathcal{O}})|^2_{G^2((a,\,b),\,\Gamma)'}\\&=\overline{C}_{\left[|\hat u|_{\mathcal{W}(a,\,b)|\mathrm{st}}\right]}|2\mathcal{A}\bar\eta^\epsilon|^2_{G^2((a,\,b),\,\Gamma)'}.\end{aligned}\tag{5.8}$$

Further, from (5.7), for every $\alpha>0$ we can write

$$4|\bar\eta^\epsilon|^2_{G^2((a,\,b),\,\Gamma)}+\tfrac{4}{\epsilon}|\Pi_N\bar v^\epsilon(b)|^2_H\le\alpha|q^\epsilon(a)|^2_H+\tfrac{1}{\alpha}|\bar v^\epsilon(a)|^2_H\le4\alpha\overline{C}_{\left[|\hat u|_{\mathcal{W}(a,\,b)|\mathrm{st}}\right]}|\mathcal{A}\bar\eta^\epsilon|^2_{G^2((a,\,b),\,\Gamma)'}+\tfrac{1}{\alpha}|\bar v^\epsilon(a)|^2_H$$

and, setting $\alpha=\left(2\overline{C}_{\left[|\hat u|_{\mathcal{W}(a,\,b)|\mathrm{st}}\right]}\right)^{-1}$, we obtain

$$|\bar\eta^\epsilon|^2_{G^2((a,\,b),\,\Gamma)}+\tfrac{2}{\epsilon}|\Pi_N\bar v^\epsilon(b)|^2_H\le\overline{C}_{\left[|\hat u|_{\mathcal{W}(a,\,b)|\mathrm{st}}\right]}|v_0|^2_H.\tag{5.9}$$

In particular, the family $\{\bar\eta^\epsilon \mid \epsilon > 0\}$ is bounded in $G^2((a,\,b),\,\Gamma)$, from which it follows the boundedness of the family $\{\bar v^\epsilon \mid \epsilon > 0\}$ in $W_H((a,\,b),\,H^1_{\rm div}(\Omega,\,\mathbb{R}^3),\,H^{-1}(\Omega,\,\mathbb{R}^3))$. Indeed, we notice that the constraint $F(\bar v^\epsilon,\,\bar\eta^\epsilon) = (0,\,0,\,0)$ means that the triple $(v,\,g,\,\zeta) = (\bar v^\epsilon,\,0,\,K^{\mathcal{O}}_t\bar\eta^\epsilon)$ solves (2.7), with $v(a) = v_0$ and then the boundedness follows from Proposition 5.1 and Theorem 2.11.

Thus, we can find a decreasing sequence $\epsilon_n \searrow 0$ such that $\eta^{\epsilon_n} \rightharpoonup \eta^0$ in $G^2((a,\,b),\,\Gamma)$ and $\bar v^{\epsilon_n} \rightharpoonup v^0$ in $W_H((a,\,b),\,H^1_{\rm div}(\Omega,\,\mathbb{R}^3),\,H^{-1}(\Omega,\,\mathbb{R}^3))$. From this, it follows (*e.g.*, from Lemma 4.1) that $\bar v^{\epsilon_n} \rightharpoonup v^0$ in $L^2((a,\,b),\,H^1_{\rm div}(\Omega,\,\mathbb{R}^3))$, $\bar v^{\epsilon_n}_t \rightharpoonup v^0_t$ in $L^2((a,\,b),\,H^{-1}(\Omega,\,\mathbb{R}^3))$, and $\bar v^{\epsilon_n}(b) \rightharpoonup v^0(b)$ in H. A standard limiting argument shows that also $(v,\,g,\,\zeta) = (v^0,\,0,\,K^{\mathcal{O}}_t\eta^0)$ solves (2.7), with $v(a) = v_0$. Furthermore, from (5.9), we have $2|\Pi_N\bar v^\epsilon(b)|^2_H \le \epsilon\overline{C}_{\left[|\hat u|_{\mathcal{W}(a,\,b)|\rm st}\right]}|v_0|^2_H \to 0$ as $\epsilon \to 0$. Now, the fact that $\Pi_N\bar v^{\epsilon_n}(b) \rightharpoonup \Pi_N v^0(b)$ in H, implies that $|\Pi_N v^0(b)|_H \le \liminf_{n\to+\infty}|\Pi_N\bar v^{\epsilon_n}(b)|_H = 0$. Analogously, it follows from (5.9) that $\eta = \eta^0$ satisfies (5.2): $|\eta^0|^2_{G^2((a,\,b),\,\Gamma)} \le \liminf_{n\to+\infty}|\bar\eta^{\epsilon_n}|^2_{G^2((a,\,b),\,\Gamma)} \le \overline{C}_{\left[|\hat u|_{\mathcal{W}(a,\,b)|\rm st}\right]}|v_0|^2_H$.

It remains to show that the control η may be chosen depending linearly on v_0. For that we follow the idea in the proof of Lemma 3.5 in [5]: let $N \in \mathbb{N}_0$ and let M be the integer in (5.8); consider the following variation of Problem 5.6.

Problem 5.7. Given $v_0 \in H$, find the minimum of the quadratic functional $J_\infty(v,\,\eta) := |\eta|^2_{G^2((a,\,b),\,\Gamma)}$ subject to the constraint $\widetilde{A}(v,\,\eta) = (v_0,\,0,\,0,\,0)$, in the space $\mathcal{X}$, where $\begin{cases}\widetilde{A} : \mathcal{X} \to \widetilde{\mathcal{Y}} := \widetilde{A}\mathcal{X} \subseteq \mathcal{Y}\times H_N\\ (v,\,\eta) \mapsto \big(A(v,\,\eta),\,\Pi_N v(b)\big)\end{cases}$.

It follows (using again Lemma A.14 and Remark A.15) that Problem 5.7 has an unique minimizer $(\bar v,\,\bar\eta)(v_0)$ depending linearly on v_0. Notice that necessarily $|\bar\eta|^2_{G^2((a,\,b),\,\Gamma)} \le |\eta^0|^2_{G^2((a,\,b),\,\Gamma)} \le \overline{C}_{\left[|\hat u|_{\mathcal{W}(a,\,b)|\rm st}\right]}|v_0|^2_H$. □

Constancy of the control. Notice that the control $\zeta = K^{\mathcal{O}}_t\eta = K^{\mathcal{O}}_t\bar\eta$, given in Theorem 5.3 (and minimizing Problem 5.7), can be "realized" by an element $\kappa \in \mathbb{R}^{2M^2}$:

$$K^{\mathcal{O}}_t\eta(t,\,x) = \breve{K}\kappa = \sum_{i,j=1}^{M}\kappa_{i,j}\varphi(t)\bar s_i(t)\chi(x)\mathbb{E}^{\mathcal{O}}_0 P^{\mathcal{O}}_{\chi^\perp}(\pi_j\mathbf{n})|_x + \sum_{i,j=1}^{M}\kappa_{i,M+j}\varphi(t)\bar s_i(t)\chi(x)\mathbb{E}^{\mathcal{O}}_0\tau_j|_x\,,$$

where $\bar s_i$ (see Sect. 4.3), π_j, and τ_j (see Sect. 2.2) are eigenfunctions and eigenvector fields of the Dirichlet Laplacean operator in $(a,\,b)$ and in $\mathcal{O}$. Notice that if $\breve{K}$ has a nontrivial kernel $\mathcal{N}(\breve{K}) = \{\kappa \in \mathbb{R}^{2M^2} \mid \breve{K}\kappa = 0\}$, then κ is not unique but, for given $\kappa \in \mathbb{R}^{2M^2}$ we can set the unique $\breve\kappa \in \mathbb{R}^{2M^2}$ solving

$$\breve{K}\breve\kappa = \breve{K}\kappa \quad\text{and}\quad \breve\kappa \in \mathcal{N}(\breve{K})^\perp$$

where $\mathcal{N}(\breve{K})^\perp$ stands for the orthogonal complement, in $\mathbb{R}^{2M^2}$, of the kernel $\mathcal{N}(\breve{K})$. In this way $|\breve\kappa|_{\mathbb{R}^{2M^2}}$ and $|\breve{K}\breve\kappa|_{G^2_{\rm av}((a,\,b),\,\Gamma)}$ are two norms in the finite-dimensional space $\mathcal{N}(\breve{K})^\perp$; it follows that for $\breve\kappa(v_0) \in \mathcal{N}(\breve{K})^\perp$ with $\breve{K}\breve\kappa(v_0) = K^{\mathcal{O}}_t\bar\eta(v_0)$, we have $|\breve\kappa(v_0)|^2_{\mathbb{R}^{2M^2}} \le \overline{C}_{[M]}|K^{\mathcal{O}}_t\bar\eta(v_0)|^2_{G^2_{\rm av}((a,\,b),\,\Gamma)} \le \overline{C}_{\left[N,\,|\hat u|_{\mathcal{W}(a,\,b)|\rm st}\right]}|v_0|^2_H$.

6. Final remarks

6.1. On further plausible consequences

Departing from a theorem analogous to Theorem 5.2, in [5] it was proven the internal feedback stabilization to a nonstationary solution for the Navier–Stokes equations. We can conjecture that the analogous result holds in the boundary control case. Of course, there are details that must be checked that we prefer to address in a future paper; here, we confine the illustration of applications of the observability inequalities to the examples in Theorems 5.2 and 5.3.

For applications it is also important to have an estimate, as sharp as possible, for the dimension M of the feedback stabilizing controller. This study has been started, for the simpler case of the viscous 1D Burgers equation with internal controls, in [21] where results of some numerical simulations can also be found.

Also, Theorem 4.7 is inspired by the work in [29] concerning the randomly forced Navier–Stokes equation with space-time internal localized noise. From a localized internal observability inequality, analogous to the boundary inequalities in Theorem 4.7, and using appropriate controls, it was proven in [29] that the Markov process generated by the restriction of solutions to the instants of time proportional to the period possesses a unique stationary distribution, which is exponentially mixing. Then we can also conjecture that the analogous result holds with space-time boundary localized noise, as a consequence of the observability inequalities in Theorem 4.7. Again, there are details to be checked.

6.2. On some of the regularity assumptions

In Section 2.2 we suppose we are able to apply a control through a subset $\Gamma_{\mathrm{c}} \subseteq \Gamma$, that is the support of a given function $\chi \in C^\infty(\Gamma, \mathbb{R})$. Then, we (must) choose an open superset $\mathcal{O} \supseteq \overline{\Gamma_{\mathrm{c}}}$ for which we have/know the existence of the systems of eigenfunctions and eigenvector fields of the Laplace–de Rham operator. This freedom to choose an auxiliary superset on the support of the controls can be important for applications. Moreover, the asked smoothness of $\partial\mathcal{O}$ may be not necessary; for example if $\overline{\Gamma_{\mathrm{c}}} \subset R \subset \Gamma$ where R is an open flat rectangle, we can find the corresponding systems of smooth eigenfunctions and eigenvector fields. Indeed, identifying $R \sim [0,\, s] \times [0,\, r]$, we find the system of eigenfunctions $\mathcal{F} = \left\{ \frac{2}{(sr)^{\frac{1}{2}}} \sin(\frac{n_1 \pi z_1}{s}) \sin(\frac{n_2 \pi z_2}{r}) \;\middle|\; n = (n_1,\, n_2) \in \mathbb{N}_0^2 \right\}$, and the system of eigenvector fields $(\mathcal{F},\, 0) \cup (0,\, \mathcal{F})$, with $(z_1,\, z_2) \in [0,\, s] \times [0,\, r]$ being "the" coordinates in R.

In Section 2, we suppose C^∞ regularity for the boundary $\partial\Omega$ because we use some results that have been derived for C^∞-smooth Riemannian manifolds, namely results from [1,14,28,31]. The derivation of the necessary results for less regular boundaries is out of the scope of this work. Anyway, concerning the control space in Section 5, the C^∞-regularity is only needed for the auxiliary subset $\mathcal{O} \subseteq \Gamma$ containing the support $\overline{\Gamma_{\mathrm{c}}}$ of the admissible boundary controls; away from $\overline{\mathcal{O}} \subseteq \Gamma$ the C^4-regularity is sufficient (to use, in Sect. 3, the results from [27]).

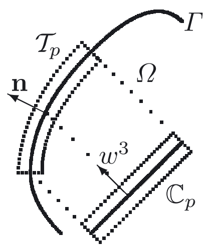
Appendix

A.1. Laplace–de Rham operator

Familiarity with basic tools from differential geometry is assumed. We refer to [7, 9, 20, 35]. Below, the nonfamiliar reader may suppose for simplicity that $\mathcal{O}$ is flat, say $\mathcal{O} \subset \Gamma \cap \{(x_1,\, x_2,\, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}$.

Let $\Omega \subset \mathbb{R}^3$ be a connected bounded domain of class C^∞ located locally on one side of its boundary $\Gamma = \partial\Omega$. More precisely, we suppose that each point $p \in \Gamma$ has a tubular neighborhood $\mathcal{T}_p \subset \mathbb{R}^3$ that is diffeomorphic to a cylinder $\mathbb{C}_p := \{(w^1,\, w^2,\, w^3) \in \mathbb{R}^3 \mid (w^1)^2 + (w^2)^2 < \rho_p \text{ and } |w^3|_{\mathbb{R}} < \varepsilon_p\}$, for suitable $\rho_p,\, \varepsilon_p > 0$. That is, we suppose that there is a bijective mapping Φ_p as follows:

- $\begin{cases} \Phi_p : \mathbb{C}_p \to \mathcal{T}_p \\ (w^1,\, w^2,\, w^3) \mapsto \left(w^1,\, w^2,\, \Phi_p^0(w^1,\, w^2)\right) + w^3 \mathbf{n}_{\Phi_p^0(w^1,\, w^2)} \end{cases}$;
- both Φ_p, its inverse $\Phi_p^{-1} : \mathcal{T}_p \to \mathbb{C}_p$, and Φ_p^0 are of class C^∞;
- $\mathbf{n}_{\Phi_p^0(w^1,\, w^2)}$ is the unit outward normal vector to Γ at $(w^1,\, w^2,\, \Phi_p^0(w^1,\, w^2)) \in \Gamma$;
- $\begin{cases} \Phi_p(\mathbb{C}_p^0) = \mathcal{T}_p \cap \Gamma, \\ \Phi_p(\mathbb{C}_p^-) = \mathcal{T}_p \cap \Omega \end{cases}$ where $\begin{cases} \mathbb{C}_p^0 := \{(w^1,\, w^2,\, w^3) \in \mathbb{C}_p \mid w^3 = 0\} \\ \mathbb{C}_p^- := \{(w^1,\, w^2,\, w^3) \in \mathbb{C}_p \mid w^3 < 0\} \end{cases}$



We recall (*cf.* [27], Rem. A.1) that we may see the open subset $\mathcal{T}_p \subset \mathbb{R}^3$ with its induced Euclidean metric as the Riemannian manifold $(\mathbb{C}_p,\, g)$ with the following metric tensor $g = g_{ij} \mathrm{d}w^i \otimes \mathrm{d}w^j$.

$$g = \left(1 + \left(\frac{\partial \Phi_p^0}{\partial w^1}\right)^2\right) \mathrm{d}w^1 \otimes \mathrm{d}w^1 + \frac{\partial \Phi_p^0}{\partial w^1} \frac{\partial \Phi_p^0}{\partial w^2} \left(\mathrm{d}w^1 \otimes \mathrm{d}w^2 + \mathrm{d}w^2 \otimes \mathrm{d}w^1\right) + \left(1 + \left(\frac{\partial \Phi_p^0}{\partial w^2}\right)^2\right) \mathrm{d}w^2 \otimes \mathrm{d}w^2 + \mathrm{d}w^3 \otimes \mathrm{d}w^3.$$

The Euclidean volume element in $\mathcal{T}_p$ may then be written as $\mathrm{d}\mathbb{C}_p = \left(1 + \left(\frac{\partial \Phi_p^0}{\partial w^1}\right)^2 + \left(\frac{\partial \Phi_p^0}{\partial w^2}\right)^2\right)^{\frac{1}{2}} \mathrm{d}w^1 \wedge \mathrm{d}w^2 \wedge \mathrm{d}w^3.$

Let $\mathcal{O} \subseteq \Gamma$ be a smooth connected two-dimensional manifold, either with or without boundary. The Laplace–de Rham operator $\Delta_{\mathcal{O}}$ on $\mathcal{O}$, is defined locally in $\mathcal{T}_p \cap \mathcal{O}$ by means of compositions of the Hodge star $*$, exterior derivative d, sharp $^\sharp$ and flat $^\flat$ mappings: for a given k-differential form α we put $\Delta_{\mathcal{O}}\alpha := -(*\mathrm{d}*\mathrm{d} + \mathrm{d}*\mathrm{d}*)\alpha$. $\Delta_{\mathcal{O}}$ maps k-forms into k-forms. A function f is a 0-form, and it turns out that for functions we have $\mathrm{d}*f = 0$ so $\Delta_{\mathcal{O}}f = -*\mathrm{d}*\mathrm{d}f$. To compute the Laplacean (Laplace–de Rham) of a vector field $v \in T\mathcal{O}$ we use in addition the sharp $^\sharp$ and flat $^\flat$ mappings:

$$\Delta_{\mathcal{O}}v := (\Delta_{\mathcal{O}}v^\sharp)^\flat. \tag{A.1}$$

We recall that $^\sharp$ maps vector fields into 1-forms, and $^\flat$ maps 1-forms into vector fields: for a vector field $V = \sum_{i=1}^{3} V^i \frac{\partial}{\partial w^i}$ and a 1-form $\alpha = \sum_{i=1}^{3} \alpha_i \mathrm{d}w^i$, we have $V^\sharp := \sum_{i,j=1}^{3} g_{ij}V^i\mathrm{d}w^j$ and $\alpha^\flat := \sum_{i,j=1}^{3} g^{ij}\alpha_i \frac{\partial}{\partial w^j}$, where $[g^{ij}]$ stands for the inverse matrix of $[g_{ij}]$. It turns out that $\sharp$ and $\flat$ are inverse to each other: $(V^\sharp)^\flat = V$ and $(\alpha^\flat)^\sharp = \alpha$. [2]

Eigenfunctions and eigenvector fields

We are interested in functions and vector fields vanishing outside a submanifold $\mathcal{O} \subseteq \Gamma$. Since we need some regularity for those functions and vector fields, two cases must be considered: $\partial\mathcal{O} \neq \emptyset$ and $\partial\mathcal{O} = \emptyset$.

- *The case $\partial\mathcal{O} \neq \emptyset$.* Consider the Laplace–de Rham operator $\Delta_{\mathcal{O}}$:

$$\begin{aligned}\Delta_{\mathcal{O}} : H^2(\mathcal{O},\, Y) \cap H^1_0(\mathcal{O},\, Y) &\to L^2(\mathcal{O},\, Y)\\ u &\mapsto \Delta_{\mathcal{O}}u,\end{aligned}$$

where $H^1_0(\mathcal{O},\, Y)$ is the closure of the space of smooth mappings $C^\infty_c(\mathcal{O},\, Y)$, having a compact support contained in $\mathcal{O}$, in the $H^1(\mathcal{O},\, Y)$-norm.

For the case of functions and 1-forms, respectively $Y = \mathbb{R}$ and $Y = T^*\mathcal{O}$, it follows that $\Delta_{\mathcal{O}}$ is an isomorphism (see *e.g.* [28], Thm. 3.4.10). See also Section 5.1 of [31], for the particular case of functions.

Notice that we consider that the 1-forms satisfy the (homogeneous) Dirichlet boundary conditions $w|_{\partial\mathcal{O}} = 0$, where the restriction has the same meaning as in [28], *i.e.*, $w|_{\partial\mathcal{O}} : \cup_{p\in\partial\mathcal{O}}\, T_p\Gamma \to \mathbb{R}$, $(w|_{\partial\mathcal{O}})|_p\,(v) := w|_p\,(v)$, for any $v \in T_p\Gamma$, $p \in \partial\mathcal{O}$. [3]

For vector fields, *i.e.*, in the case $Y = T\mathcal{O}$, from (A.1) follows that $\Delta_{\mathcal{O}}V = U$ if, and only if, $\Delta_{\mathcal{O}}V^\sharp = U^\sharp$ and, then $\Delta_{\mathcal{O}} : H^2(\mathcal{O},\, Y) \cap H^1_0(\mathcal{O},\, Y) \to L^2(\mathcal{O},\, Y)$ is also a isomorphism in this case. Notice that from well known properties of the Hodge star, wedge product and interior product mappings (*cf.* [31] or [26], Sect. 5.7), we can write $*(\alpha \wedge *\beta) = -\iota_{\alpha^\flat} **\beta = \beta(\alpha^\flat) = g(\beta^\flat,\, \alpha^\flat)$, from which we conclude that $(\alpha,\, \beta)_{L^2(\mathcal{O},\, T^*\mathcal{O})} := \int_{\mathcal{O}} \alpha \wedge *\beta = \int_{\mathcal{O}} *(\alpha \wedge *\beta)\, \mathrm{d}\mathcal{O} = \int_{\mathcal{O}} g(\beta^\flat,\, \alpha^\flat)\, \mathrm{d}\mathcal{O} =: (\alpha^\flat,\, \beta^\flat)_{L^2(\mathcal{O},\, T\mathcal{O})}$.

Moreover, $\Delta_{\mathcal{O}}$ is self-adjoint and have compact inverse. We can deduce the existence of a system of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$, with $\lambda_k \to +\infty$, and corresponding eigenforms $\Delta_{\mathcal{O}}\zeta_k = \lambda_k\zeta_k$, forming a complete orthonormal system $\{\zeta_k \mid k \in \mathbb{N}_0\}$ in $L^2(\mathcal{O},\, Y)$. That the first eigenvalue is nonzero follows from the fact that $\Delta_{\mathcal{O}}w = 0$ and $w|_{\partial\mathcal{O}} = 0$ imply that $0 = (\Delta_{\mathcal{O}}w,\, w)_{L^2(\mathcal{O},\, Y)} = (\mathrm{d}w,\, \mathrm{d}w)_{L^2(\Gamma_{\mathcal{O}},\, Y)} + (\operatorname{div} w,\, \operatorname{div} w)_{L^2(\mathcal{O},\, Y)}$, *i.e.*, $\mathrm{d}w = 0 = \operatorname{div} w$, where $\operatorname{div} w := -*\mathrm{d}*w$, and by [28], Theorem 3.4.4, it follows that $w = 0$, necessarily. In the case of functions, $Y = \mathbb{R}$, we have also that the first eigenfunction does not change sign in $\mathcal{O}$ (see [31], Chap. 5, Prop. 2.4). The eigenforms are C^∞-smooth due to Theorem 3.4.10 of [28].

- *The case $\partial\mathcal{O} = \emptyset$.* In this case $\mathcal{O}$ is a connected component of Γ. In the boundaryless case we still have the existence of a system of eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$, with $\lambda_k \to +\infty$, and corresponding smooth eigenforms ζ_k, with $\Delta_{\mathcal{O}}\zeta_k = \lambda_k\zeta_k$, forming a complete orthonormal system $\{\zeta_k \mid k \in \mathbb{N}_0\}$ in $L^2(\mathcal{O},\, Y)$. The finite-dimensional eigenspace corresponding to the first eigenvalue $\lambda_1 = 0$, is the space of harmonic forms w, defined by $\mathrm{d}w = 0$ if w is a function, and by $\mathrm{d}w = 0$ and $\operatorname{div} w = 0$ if w is a 1-form. For more details see Section 5.8 of [31].

[2] In some works the roles of $\sharp$ and $\flat$ are changed. The Laplace–de Rham operator is defined to have nonnegative eigenvalues; in Euclidean (flat) domains it coincides with the symmetric of the "usual" Laplacean, $\Delta_{\mathcal{O}} = -\Delta$, in $\mathcal{O}$.

[3] As we see, to define $w|_{\partial\mathcal{O}}$, we essentially need $w|_p$ to be well defined in $T_p\Gamma$, for $p \in \partial\mathcal{O}$. Notice also that, for some authors, the terminology "Dirichlet boundary conditions", for 1-forms, stand for different boundary conditions as in Section 5.9 of [31]; the meaning we use here coincides, in the Euclidean case, to say that of all coordinate components of the 1-form w must vanish.

A.2. On Interpolation and fractional Sobolev-Bochner spaces

Here we recall some results on Interpolation, mainly from [23]. Given a Banach space X, the norm of the Sobolev-like space $H^s((a,\, b),\, X)$ can be defined by means of the Fourier transform. First $H^s((a,\, b),\, X)$ can be defined as $H^s((a,\, b),\, X) := \{\tilde{u}|_{(a,\, b)} \mid \tilde{u} \in H^s(\mathbb{R},\, X)\}$; and the Fourier transform, in the (time) variable $t \in (a,\, b)$, of $\tilde{u}$ is defined by $\mathcal{F}_t(\tilde{u})(\tau) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-i\tau r}\tilde{u}(r)\,\mathrm{d}r$. Then, the space $H^s(\mathbb{R},\, X)$ is endowed with the norm

$$|\tilde{u}|_{H^s(\mathbb{R},\, X)} := |(1+|\tau|^2)^{\frac{s}{2}}\mathcal{F}_t(\tilde{u})|_{L^2(\mathbb{R},\, X)}, \tag{A.2}$$

and the space $H^s((a,\, b),\, X)$ with the (quotient) norm

$$|u|_{H^s((a,\, b),\, X)} := \inf_{F_s} \left\{ |F_s u|_{H^s(\mathbb{R},\, X)} \;\middle|\; F_s u|_{(a,\, b)} = u \right\}, \tag{A.3}$$

where F_s runs over $\mathcal{L}(H^s((a,\, b),\, X) \to H^s(\mathbb{R},\, X))$ (*cf.* [23], Chap. 1, Sects. 7.1 and 9.1).

Remark A.1. Let $m \in \mathbb{N}_0$. Then, a continuous extension $\hat{F}_m \colon H^m((a,\, b),\, X) \to H^m(\mathbb{R},\, X)$, $u \mapsto \hat{F}_m u$ can be constructed from a standard procedure; for example, for $l = b-a > 0$, we can define the function $\bar{u}$ as follows: $\begin{cases} \bar{u}(a-t) := \sum_{i=1}^m \lambda_i u\left(a+\frac{t}{i}\right), & \text{if } t \in (0,\, l) \\ \bar{u}(t) := u(t), & \text{if } t \in (a,\, b) \end{cases}$. Similarly we can construct an extension to $(b,\, b+l)$, and we arrive to an extension $\hat{u}$ to $(a-l,\, b+l)$. Now we may multiply by a C^∞-smooth function ϕ supported in $\left[a-\frac{l}{2},\, b+\frac{l}{2}\right]$ and taking the value 1 in $[a,\, b]$. Then, the obtained extension $u \mapsto \hat{F}_m u := \phi\hat{u}$ satisfies

$$\hat{F}_m \in \mathcal{L}(H^j((a,\, b),\, X) \to H^j(\mathbb{R},\, X)), \text{ for all } j \in \mathbb{N},\ j \leq m,$$

if the constants λ_i ($i \in \{1,\, 2,\, \ldots,\, m\}$) solve the system $1 = \sum_{i=1}^m (-1)^{k-1}\frac{\lambda_i}{i^{k-1}}$, $k = 1,\, 2,\, \ldots,\, m$ (*cf.* [15], Sect. 2.1; see also [23], Chap. 1, Sect. 8.1).

Remark A.2. Notice that the extension $\bar{u}$ in Remark A.1 is well defined for functions in $L^2((a,\, b),\, X)$, indeed we can suppose that $u(a+t)$ is defined for all $t \in (0,\, l) \setminus \mathcal{N}$ where $\mathcal{N}$ is a set of measure zero. Then $\bar{u}(a-t)$ is defined for all $t \in (0,\, l) \setminus \cup_{i=1}^m i\mathcal{N}$, that is, $\bar{u}(a-t)$ is well defined for a.e. $t \in (0,\, l)$.

Definition A.3. A pair of Hilbert spaces $(X,\, Y)$ is said an interpolation pair if $X \subseteq Y$, and the inclusion is dense and continuous[4].

Theorem A.4 (Interpolation Theorem)**.** *Let us be given two interpolation pairs $(X,\, Y)$ and $(X_1,\, Y_1)$. If L is a linear and continuous operator both from X into Y and from X_1 into Y_1, then it is also linear and continuous from $[X,\, Y]_\theta$ into $[X_1,\, Y_1]_\theta$, $0 < \theta < 1$. Moreover $|L|_{\mathcal{L}([X,\, Y]_\theta \to [X_1,\, Y_1]_\theta)} \leq C\max\{|L|_{\mathcal{L}(X \to X_1)},\, |L|_{\mathcal{L}(Y \to Y_1)}\}$.*

This theorem can be found in [23], Chapter 1, Theorem 5.1. The estimate follows from the last equation in the proof of the theorem and from Remark 4.2 in the same chapter.

Remark A.5. From Chapter 1, Theorem 4.2 of [23], for $0 < \theta < 1$, we have the following trace characterization: $[X,\, Y]_\theta = \{f(0) \mid \mathcal{F}_t f \in L^2(\mathbb{R},\, X) \text{ and } |\tau|^{(2\theta)^{-1}}\mathcal{F}_t f \in L^2(\mathbb{R},\, Y)\}$. This characterization is used in [18] as definition of the interpolation space.

Lemma A.6. *Let $s_1 \geq s_2$ and let $(X,\, Y)$ be an interpolation pair. It follows that $(H^{s_1}(I,\, X),\, H^{s_2}(I,\, Y))$ is also an interpolation pair and that $[H^{s_1}(I,\, X),\, H^{s_2}(I,\, Y)]_\theta = H^{(1-\theta)s_1+\theta s_2}(I,\, [X,\, Y]_\theta)$.*

[4] In the Literature we can find more general definitions for interpolation pair.

Proof. Though a bit long, the proof of the fact that $(H^{s_1}(I,\,X),\,H^{s_2}(I,\,Y))$ is an interpolation pair is straightforward; we skip it. To prove the interpolation identity we can follow ([23], Chap. 1, Sect. 2.1): first we notice that we can identify X with the domain of a suitable auto-adjoint, positive and unbounded operator $\Lambda : X \to Y$; then we can make use of the operator $\mathcal{F}_t u \mapsto \widehat{\Lambda}\mathcal{F}_t u := (1+|\tau|^2)^{s_1-s_2}\Lambda\mathcal{F}_t u$, defined in $H^{s_2}(I,\,Y)$, to prove the identity $[H^{s_1}(\mathbb{R},\,X),\,H^{s_2}(\mathbb{R},\,Y)]_\theta = H^{(1-\theta)s_1+\theta s_2}(\mathbb{R},\,[X,\,Y]_\theta)$; the analogous identity for a given interval I will follow by a restriction and interpolation argument. Notice that we can identify $[H^{s_1}(\mathbb{R},\,X),\,H^{s_2}(\mathbb{R},\,Y)]_\theta$ with $\mathcal{F}_t^{-1}\mathrm{D}(\widehat{\Lambda}^{1-\theta})$, and from $\widehat{\Lambda}^{1-\theta} = (1+|\tau|^2)^{(s_1-s_2)(1-\theta)}\Lambda^{1-\theta}$ we have that $\mathrm{D}(\widehat{\Lambda}^{1-\theta}) = \{\mathcal{F}_t u \in L^2(\mathbb{R},\,Y) \mid (1+|\tau|^2)^{s2+(1-\theta)(s_1-s_2)}\Lambda^{1-\theta}\mathcal{F}_t u \in L^2(\mathbb{R},\,Y)\}$, from which it follows that $\mathcal{F}_t^{-1}\mathrm{D}(\widehat{\Lambda}^{1-\theta}) = H^{(1-\theta)s_1+\theta s_2}(\mathbb{R},\,\mathrm{D}(\Lambda^{1-\theta}))$. Finally, from the identity $\mathrm{D}(\Lambda^{1-\theta}) = [X,\,Y]_\theta$, we can conclude that $[H^{s_1}(\mathbb{R},\,X),\,H^{s_2}(\mathbb{R},\,Y)]_\theta = H^{(1-\theta)s_1+\theta s_2}(\mathbb{R},\,[X,\,Y]_\theta)$. □

Remark A.7. The identity $[H^{s_1}(I,\,X),\,H^{s_2}(I,\,Y)]_\theta = H^{(1-\theta)s_1+\theta s_2}(I,\,[X,\,Y]_\theta)$ can be found in [23], Chapter 1, Section 9.4. However, to be coherent with our definition of interpolation space –and with the setting in ([23], Chap. 1, Sect. 2.1)– we need to impose the condition $s_1 \ge s_2$. Just to give an idea, set $I = (0,\,1)$; then for any $\bar{x} \in X \subseteq Y$, we have that the function $t \mapsto t^{\frac{1}{2}}\bar{x}$ is in $H^0(I,\,X) \setminus H^1(I,\,Y)$. That is, $(H^0(I,\,X),\,H^1(I,\,Y))$ is not an interpolation pair.

A.3. Proofs of Propositions 2.4 and 2.5

Lemma A.8. *Given $v \in G^s_{\mathbf{n}}((a,\,b),\,\Gamma)$, we have that $\int_\Gamma v\,\mathrm{d}\Gamma \in H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})$. Moreover, the mapping $I_\Gamma : v \mapsto \int_\Gamma v\,\mathrm{d}\Gamma$ is in $\mathcal{L}(G^s_{\mathbf{n}}((a,\,b),\,\Gamma) \to H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R}))$.*

Proof. For a given $u \in G^s_{\mathbf{n}}(\mathbb{R},\,\Gamma)$, we find that

$$\begin{aligned}
|I_\Gamma u|^2_{H^{r_{\mathbf{n},1}(s)}(\mathbb{R},\,\mathbb{R})} &= \int_{\mathbb{R}} (1+|\tau|^2)^{r_{\mathbf{n},1}(s)}|\mathcal{F}_t I_\Gamma u(\tau)|^2_{\mathbb{R}}\,\mathrm{d}\tau = \int_{\mathbb{R}} (1+|\tau|^2)^{r_{\mathbf{n},1}(s)}\left|\int_\Gamma \mathcal{F}_t u(\tau,\,x)\,\mathrm{d}\Gamma\right|^2_{\mathbb{R}}\mathrm{d}\tau \\
&= \int_{\mathbb{R}} (1+|\tau|^2)^{r_{\mathbf{n},1}(s)}\left|\langle \mathcal{F}_t u(\tau,\,x),\,1_\Gamma\rangle_{H^{r_{\mathbf{n},2}(s)}(\Gamma,\,\mathbb{R}),\,H^{-r_{\mathbf{n},2}(s)}(\Gamma,\,\mathbb{R})}\right|^2_{\mathbb{R}}\mathrm{d}\tau \\
&\le \int_{\mathbb{R}} (1+|\tau|^2)^{r_{\mathbf{n},1}(s)}|\mathcal{F}_t u(\tau,\,x)|^2_{H^{r_{\mathbf{n},2}(s)}(\Gamma,\,\mathbb{R})}|1_\Gamma|^2_{H^{-r_{\mathbf{n},2}(s)}(\Gamma,\,\mathbb{R})}\,\mathrm{d}\tau \\
&= |1_\Gamma|^2_{H^{-r_{\mathbf{n},2}(s)}(\Gamma,\,\mathbb{R})}|u|^2_{H^{r_{\mathbf{n},1}(s)}(\mathbb{R},\,H^{r_{\mathbf{n},2}(s)}(\Gamma,\,\mathbb{R}))} \le |1_\Gamma|^2_{H^{-r_{\mathbf{n},2}(s)}(\Gamma,\,\mathbb{R})}|u|^2_{G^s_{\mathbf{n}}(\mathbb{R},\,\Gamma)}.
\end{aligned}$$

Now, we observe that $G^s_{\mathbf{n}}((a,\,b),\,\Gamma) = G^s_{\mathbf{n}}(\mathbb{R},\,\Gamma)|_{(a,\,b)}$, and $\int_\Gamma u|_{(a,\,b)}\,\mathrm{d}\Gamma = \left(\int_\Gamma u\,\mathrm{d}\Gamma\right)|_{(a,\,b)}$, for each $u \in G^s_{\mathbf{n}}(\mathbb{R},\,\Gamma)$; therefore, $\int_\Gamma G^s_{\mathbf{n}}((a,\,b),\,\Gamma)\,\mathrm{d}\Gamma = \int_\Gamma G^s_{\mathbf{n}}(\mathbb{R},\,\Gamma)|_{(a,\,b)}\,\mathrm{d}\Gamma = \left(\int_\Gamma G^s_{\mathbf{n}}(\mathbb{R},\,\Gamma)\,\mathrm{d}\Gamma\right)|_{(a,\,b)} \subseteq H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})$. The linearity of the mapping I_Γ is clear. Now, we observe that the extension $\hat{F}_1$ constructed in Remark A.1, satisfies $\hat{F}_1 \in \mathcal{L}(H^\rho((a,\,b),\,X) \to H^\rho(\mathbb{R},\,X))$ for all $0 \le \rho \le 1$, and any Banach space X (*e.g.*, we can use a suitable interpolation argument, with Thm. A.4 and Lem. A.6). Then, since $0 \le r_{\mathbf{n},1}(s) \le 1$, we can write $|I_\Gamma v|^2_{H^{r_{\mathbf{n},1}(s)}((a,b),\mathbb{R})} = \inf_F |I_\Gamma F v|^2_{H^{r_{\mathbf{n},1}(s)}(\mathbb{R},\,\mathbb{R})} \le |1_\Gamma|^2_{H^{-r_{\mathbf{n},2}(s)}(\Gamma,\,\mathbb{R})}|\hat{F}_1 v|^2_{G^s_{\mathbf{n}}(\mathbb{R},\,\Gamma)} \le C|v|^2_{G^s_{\mathbf{n}}((a,\,b),\,\Gamma)}$, where F runs over $\mathcal{L}(H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R}),\,H^{r_{\mathbf{n},1}(s)}(\mathbb{R},\,\mathbb{R}))$. □

Proof of Proposition 2.4. Given $u \in G^s_{\mathbf{n}}((a,\,b),\,\Gamma)$ we write $u = (u - \varkappa_u) + \varkappa_u$, with $\varkappa_u = \frac{1}{\int_\Gamma \mathrm{d}\Gamma}\int_\Gamma u\,\mathrm{d}\Gamma$. By Lemma A.8, we have that $\varkappa_u \in H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})$; we also observe that $\varkappa_u$ is independent of $x \in \Gamma$, $\varkappa_u(t,\,x) = \varkappa_u(t)$ for all $(t,\,x) \in (a,\,b) \times \Gamma$, thus

$$\begin{aligned}
&|\varkappa_u|^2_{H^{r_1}((a,\,b),\,H^{r_2}(\Gamma,\,\mathbb{R}))} = \inf_{\bar{E}} \int_{\mathbb{R}} (1+|\tau|^2)^{r_1}|\mathcal{F}_t(\bar{E}\varkappa_u)(\tau,\,x)|^2_{H^{r_2}(\Gamma,\,\mathbb{R})}\,\mathrm{d}\tau \\
&\le \inf_E \int_{\mathbb{R}} (1+|\tau|^2)^{r_1}|\mathcal{F}_t(E\varkappa_u)(\tau)|^2_{H^{r_2}(\Gamma,\,\mathbb{R})}\,\mathrm{d}\tau = \inf_E \int_{\mathbb{R}} (1+|\tau|^2)^{r_1}|\mathcal{F}_t(E\varkappa_u)(\tau)|^2_{\mathbb{R}}|1_\Gamma|^2_{H^{r_2}(\Gamma,\,\mathbb{R})}\,\mathrm{d}\tau \\
&= |1_\Gamma|^2_{H^{r_2}(\Gamma,\,\mathbb{R})}|\varkappa_u|^2_{H^{r_1}((a,\,b),\,\mathbb{R})},
\end{aligned}$$

where $\bar{E}$ runs over all continuous extensions $H^{r_1}((a,\,b),\,H^{r_2}(\Gamma,\,\mathbb{R})) \to H^{r_1}(\mathbb{R},\,H^{r_2}(\Gamma,\,\mathbb{R}))$, and E runs over all continuous extensions $H^{r_1}((a,\,b),\,\mathbb{R}) \to H^{r_1}(\mathbb{R},\,\mathbb{R})$. Now, setting $(r_1,\,r_2) = (0,\,s-\frac{1}{2})$, from $0 < r_{\mathbf{n},1}(s)$ we can conclude that $\varkappa_u \in L^2((a,\,b),\,H^{s-\frac{1}{2}}(\Gamma,\,\mathbb{R}))$, and setting $(r_1,\,r_2) = (r_{\mathbf{n},1}(s),\,r_{\mathbf{n},2}(s))$ we obtain $\varkappa_u \in H^{r_{\mathbf{n},1}(s)}((a,\,b),\,H^{r_{\mathbf{n},2}(s)}(\Gamma,\,\mathbb{R}))$; it follows that $\varkappa_u \in G^s_{\mathbf{n}}((a,\,b),\,\Gamma)$ and $u - \varkappa_u \in G^s_{\mathbf{n},\,\mathrm{av}}((a,\,b),\,\Gamma)$. Therefore, we have that $G^s((a,\,b),\,\Gamma) = G^s_{\mathrm{av}}((a,\,b),\,\Gamma) \oplus H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\mathbf{n}$. Notice that $G^s_{\mathrm{av}}((a,\,b),\,\Gamma) \cap H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\mathbf{n} = \{0\}$, because for each v in the intersection, we have $v = \varkappa\mathbf{n}$ with $\varkappa \in H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})$, and $0 = \int_\Gamma v \cdot \mathbf{n}\,\mathrm{d}\Gamma$, which implies $0 = \varkappa \int_\Gamma \mathrm{d}\Gamma$, *i.e.*, $\varkappa = 0$. From the proof of Lemma A.8, we can also conclude the continuity of the projection $u \mapsto \varkappa_u$ in $G^s_{\mathbf{n}}((a,\,b),\,\Gamma)$. Thus, since $v \mapsto v \cdot \mathbf{n}$ is continuous from $G^s((a,\,b),\,\Gamma)$ onto $G^s_{\mathbf{n}}((a,\,b),\,\Gamma)$, it follows also the continuity of $v \mapsto \pi_v \mathbf{n}$ from $G^s((a,\,b),\,\Gamma)$ onto itself, where $\pi_v := \frac{1}{\int_\Gamma \mathrm{d}\Gamma} \int_\Gamma v \cdot \mathbf{n}\,\mathrm{d}\Gamma$. □

Proof of Proposition 2.5. Clearly E^e_s extends E_s, that is, $E^e_s u = E_s u$ for all $u \in G^s_{\mathrm{av}}((a,\,b),\,\Gamma)$; the linearity also follows straightforwardly. Notice that $W((a,\,b),\,H^s_{\mathrm{div}}(\Omega,\,\mathbb{R}^3),\,H^{s-2}(\Omega,\,\mathbb{R}^3)) \cap H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\Theta = \{0\}$, because for each w in the intersection, we have $w = \varkappa\Theta$, with $\varkappa \in H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})$ and $\operatorname{div} w = 0$, from which we obtain $0 = \operatorname{div}(\varkappa\Theta) = \varkappa\left(\frac{\int_\Gamma \mathrm{d}\Gamma}{\int_\Omega \mathrm{d}\Omega}\right)$, *i.e.*, $\varkappa = 0$. Next, for simplicity we denote $S^s := W((a,\,b),\,H^s_{\mathrm{div}}(\Omega,\,\mathbb{R}^3),\,H^{s-2}(\Omega,\,\mathbb{R}^3)) \oplus H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\Theta$, and we find

$$\begin{aligned}|E^e_s u|_{S^s} &\le |E_s u_{\mathrm{av}}|_{S^s} + |\pi_u\Theta|_{S^s} = |E_s u_{\mathrm{av}}|_{W((a,\,b),\,H^s_{\mathrm{div}}(\Omega,\,\mathbb{R}^3),\,H^{s-2}(\Omega,\,\mathbb{R}^3))} + |\pi_u\Theta|_{H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\Theta}\\ &\le C|u_{\mathrm{av}}|_{G^s_{\mathrm{av}}((a,\,b),\,\Gamma)} + |\pi_u|_{H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})}\end{aligned}$$

and, from Proposition 2.4, we obtain $|E^e_s u|_{S^s} \le C_1 |u|_{G^s((a,\,b),\,\Gamma)}$.

It remains to prove the continuity of the trace. We know that the trace mapping is continuous from $W((a,\,b),\,H^s_{\mathrm{div}}(\Omega,\,\mathbb{R}^3),\,H^{s-2}(\Omega,\,\mathbb{R}^3))$ onto $G^s_{\mathrm{av}}((a,\,b),\,\Gamma) \subset G^s((a,\,b),\,\Gamma)$ and, on the other hand, we also have that $|\varkappa\Theta|_\Gamma|_{G^s((a,\,b),\,\Gamma)} = |\varkappa\mathbf{n}|_{G^s((a,\,b),\,\Gamma)} = |\varkappa|_{G^s_{\mathbf{n}}((a,\,b),\,\Gamma)} \le C|\varkappa|_{H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})} = C|\varkappa\Theta|_{H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\Theta}$. Therefore the continuity of the trace $v \mapsto v|_\Gamma = v_{\mathrm{div}}|_\Gamma + \varkappa\Theta|_\Gamma$ follows from the continuity of the projections $v \mapsto v_{\mathrm{div}} \in W((a,\,b),\,H^s_{\mathrm{div}}(\Omega,\,\mathbb{R}^3),\,H^{s-2}(\Omega,\,\mathbb{R}^3))$ and $v \mapsto \varkappa\Theta \in H^{r_{\mathbf{n},1}(s)}((a,\,b),\,\mathbb{R})\Theta$ (*cf.* Lem. 5.4). □

A.4. Proof of Proposition 3.10

The proof will follow from a reiteration-like argument. We start with the following auxiliary result:

Lemma A.9. *Let $v \in H^k((a,\,b),\,H^j(\Gamma,\,Z))$, with $\{k,\,j\} \in \{0,\,1\} \times \{-2,\,-1,\,0,\,1,\,2\}$, where Z is either $\mathbb{R}$ or $T\Gamma$. Let also $\psi \in C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))$. Then $\Psi : v \mapsto \psi v$ maps $H^k((a,\,b),\,H^j(\Gamma,\,Z))$ into itself, and we have the estimate $|\psi v|_{H^k((a,\,b),\,H^j(\Gamma,\,Z))} \le C|\psi|_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}|v|_{H^k((a,\,b),\,H^j(\Gamma,\,Z))}$.*

Proof. If j is nonnegative, it is straightforward to check that $\psi v \in H^k((a,\,b),\,H^j(\Gamma,\,Z))$ and to find the estimate $|\psi v|_{H^k((a,\,b),\,H^j(\Gamma,\,Z))} \le C|\psi|_{C^k([a,\,b],\,C^j(\Gamma,\,\mathbb{R}))}|v|_{H^k((a,\,b),\,H^j(\Gamma,\,Z))}$. In the case $j < 0$, we recall that, for each $t \in (a,\,b)$ and each $\xi \in C^{-j}(\Gamma,\,\mathbb{R})$, we have $\langle \xi v(t),\,\cdot\rangle_{H^j(\Gamma,\,Z),\,H^{-j}(\Gamma,\,Z)} := \langle v(t),\,\xi\cdot\rangle_{H^j(\Gamma,\,Z),\,H^{-j}(\Gamma,\,Z)}$ and that, for $k \in \{0,\,1\}$, we may write $|\psi v|_{H^k((a,\,b),\,H^j(\Gamma,\,Z))} = |\psi v|_{L^2((a,\,b),\,H^j(\Gamma,\,Z))} + |k\partial_t(\psi v)|_{L^2((a,\,b),\,H^j(\Gamma,\,Z))}$. Then, we obtain $|\psi v|_{H^k((a,\,b),\,H^j(\Gamma,\,Z))} \le C|\psi|_{C^k([a,\,b],\,C^{-j}(\Gamma,\,\mathbb{R}))}|v|_{L^2((a,\,b),\,H^j(\Gamma,\,Z))} + C|\psi|_{C^0([a,\,b],\,C^{-j}(\Gamma,\,\mathbb{R}))}|k\partial_t v|_{L^2((a,\,b),\,H^j(\Gamma,\,Z))}$, from which we derive $|\psi v|_{H^k((a,\,b),\,H^j(\Gamma,\,Z))} \le C|\psi|_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}|v|_{H^k((a,\,b),\,H^j(\Gamma,\,Z))}$. □

Now, from Lemma A.6, we observe that

$$\begin{aligned}L^2((a,\,b),\,H^{\frac{3}{2}}(\Gamma,\,Z)) &= [L^2((a,\,b),\,H^2(\Gamma,\,Z)),\,L^2((a,\,b),\,L^2(\Gamma,\,Z))]_{\frac{1}{4}},\\ H^1((a,\,b),\,H^{-\frac{1}{2}}(\Gamma,\,\mathbb{R})) &= [H^1((a,\,b),\,L^2(\Gamma,\,\mathbb{R})),\,H^1((a,\,b),\,H^{-1}(\Gamma,\,\mathbb{R}))]_{\frac{1}{2}},\\ H^{\frac{3}{4}}((a,\,b),\,L^2(\Gamma,\,T\Gamma)) &= [H^1((a,\,b),\,L^2(\Gamma,\,T\Gamma)),\,L^2((a,\,b),\,L^2(\Gamma,\,T\Gamma))]_{\frac{1}{4}},\end{aligned}$$

from which, using the Interpolation Theorem and Lemma A.9, we can conclude that Proposition 3.10 holds for $i = 2$. Indeed, we obtain that $\Psi : v \mapsto \psi v$ maps $L^2((a,\,b),\,H^{\frac{3}{2}}(\Gamma,\,Z))$ into itself, and that, for a suitable constant

$C > 0$, $|\Psi|_{\mathcal{L}(L^2((a,\,b),\,H^{\frac{3}{2}}(\Gamma,\,Z))\to L^2((a,\,b),\,H^{\frac{3}{2}}(\Gamma,\,Z)))} \le C \max_{S\in\{L^2((a,\,b),\,H^2(\Gamma,\,Z)),\,L^2((a,\,b),\,L^2(\Gamma,\,Z))\}} \left\{|\Psi|_{\mathcal{L}(S\to S)}\right\}$, that is, $|\psi v|_{L^2((a,\,b),\,H^{\frac{3}{2}}(\Gamma,\,Z))} \le C_1|\psi|_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}|v|_{L^2((a,\,b),\,H^{\frac{3}{2}}(\Gamma,\,Z))}$. By a similar reasoning we can obtain similar estimates $|\psi v|_S \le C|\psi|_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}|v|_S$, where S is either the space $H^1((a,\,b),\,H^{-\frac{1}{2}}(\Gamma,\,\mathbb{R}))$ or the space $H^{\frac{3}{4}}((a,\,b),\,L^2(\Gamma,\,T\Gamma))$. These estimates imply that

$$|\psi v|_{G^2((a,\,b),\,\Gamma)} \le C|\psi|_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}|v|_{G^2((a,\,b),\,\Gamma)}. \tag{A.4}$$

Analogously, we can derive that

$$\begin{aligned}
L^2((a,\,b),\,H^{\frac{1}{2}}(\Gamma,\,Z)) &= [L^2((a,\,b),\,H^1(\Gamma,\,Z)),\,L^2((a,\,b),\,L^2(\Gamma,\,Z))]_{\frac{1}{2}},\\
H^{\frac{3}{4}}((a,\,b),\,H^{-1}(\Gamma,\,\mathbb{R})) &= [H^1((a,\,b),\,H^{-1}(\Gamma,\,\mathbb{R})),\,L^2((a,\,b),\,H^{-1}(\Gamma,\,\mathbb{R}))]_{\frac{1}{4}},\\
H^{\frac{1}{2}}((a,\,b),\,H^{-\frac{1}{2}}(\Gamma,\,T\Gamma)) &= [H^1((a,\,b),\,L^2(\Gamma,\,T\Gamma)),\,L^2((a,\,b),\,H^{-1}(\Gamma,\,T\Gamma))]_{\frac{1}{2}}
\end{aligned}$$

and, arguing as above, we can conclude that $|\psi v|_S \le C|\psi|_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}|v|_S$, where S stands is either for $L^2((a,\,b),\,H^{\frac{1}{2}}(\Gamma,\,Z))$, for $H^{\frac{3}{4}}((a,\,b),\,H^{-1}(\Gamma,\,\mathbb{R}))$, or for $H^{\frac{1}{2}}((a,\,b),\,H^{-\frac{1}{2}}(\Gamma,\,T\Gamma))$; which allow us to conclude that

$$|\psi v|_{G^1((a,\,b),\,\Gamma)} \le C|\psi|_{C^1([a,\,b],\,C^2(\Gamma,\,\mathbb{R}))}|v|_{G^1((a,\,b),\,\Gamma)}. \tag{A.5}$$

From (A.4) and (A.5), it follows the statement of Proposition 3.10. □

A.5. Proof of Proposition 2.17

Lemma A.10. *Let $r \ge 0$ and $0 \le s \le 2$, then $K^{\mathcal{O}} : \eta \mapsto K^{\mathcal{O}}\eta := \chi E_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}}(\vartheta\eta|_{\mathcal{O}})$ maps $H^r(\mathbb{R},\,H^s(\Gamma,\,\mathbb{R}^3))$ into itself, linearly and continuously.*

Proof. We find

$$\begin{aligned}
|K^{\mathcal{O}}\eta|^2_{H^r(\mathbb{R},\,H^s(\Gamma,\,\mathbb{R}^3))} &= \int_{\mathbb{R}} (1+|\tau|^2)^r |\mathcal{F}_t K^{\mathcal{O}}\eta(\tau,\,\cdot)|^2_{H^s(\Gamma,\,\mathbb{R}^3)}\,\mathrm{d}\tau\\
&= \int_{\mathbb{R}} (1+|\tau|^2)^r |\mathcal{F}_t \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}}(\vartheta\eta|_{\mathcal{O}}(\tau,\,\cdot))|^2_{H^s(\Gamma,\,\mathbb{R}^3)}\,\mathrm{d}\tau.
\end{aligned} \tag{A.6}$$

Using analogous arguments as in Section A.4 we can derive that

$$\begin{aligned}
&|\mathcal{F}_t \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}}(\vartheta\eta|_{\mathcal{O}}(\tau,\,\cdot))|^2_{H^s(\Gamma,\,\mathbb{R}^3)} = |\chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}}(\vartheta\mathcal{F}_t\eta|_{\mathcal{O}}(\tau,\,\cdot))|^2_{H^s(\Gamma,\,\mathbb{R}^3)}\\
&= |\mathbb{E}_0^{\mathcal{O}} \chi|_{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}}(\vartheta\mathcal{F}_t\eta|_{\mathcal{O}}(\tau,\,\cdot))|^2_{H^s(\Gamma,\,\mathbb{R}^3)} \le \left|\chi|_{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}}(\vartheta\mathcal{F}_t\eta|_{\mathcal{O}}(\tau,\,\cdot))\right|^2_{H^s(\mathcal{O},\,\mathbb{R}^3)}\\
&\le 2|\chi|^2_{C^2(\Gamma,\,\mathbb{R})}\left(|P_M^{\mathcal{O}}(\vartheta\mathcal{F}_t\eta|_{\mathcal{O}}(\tau,\,\cdot))|^2_{H^s(\mathcal{O},\,\mathbb{R}^3)} + \left|\frac{(P_M^{\mathcal{O}}(\vartheta\mathcal{F}_t\eta|_{\mathcal{O}}(\tau,\,\cdot)),\,\chi\mathbf{n})_{L^2(\mathcal{O},\,\mathbb{R}^3)}}{\int_{\mathcal{O}}\chi^2\mathrm{d}\mathcal{O}}\chi\mathbf{n}\right|^2_{H^s(\mathcal{O},\,\mathbb{R}^3)}\right)\\
&\le C\left(|P_M^{\mathcal{O}}(\vartheta\mathcal{F}_t\eta|_{\mathcal{O}}(\tau,\,\cdot))|^2_{H^s(\mathcal{O},\,\mathbb{R}^3)} + |P_M^{\mathcal{O}}(\vartheta\mathcal{F}_t\eta|_{\mathcal{O}}(\tau,\,\cdot))|^2_{L^2(\mathcal{O},\,\mathbb{R}^3)}\right) \le C_1|P_M^{\mathcal{O}}(\vartheta\mathcal{F}_t\eta|_{\mathcal{O}}(\tau,\,\cdot))|^2_{H^s(\mathcal{O},\,\mathbb{R}^3)}.
\end{aligned}$$

Further using some interpolation arguments, we have that $\mathrm{D}\left(\Delta_{\mathcal{O}}^{\frac{s}{2}}\right) \subseteq H^s(\mathcal{O},\,\mathbb{R}^3)$ is a continuous inclusion, where $\mathrm{D}\left(\Delta_{\mathcal{O}}^{\frac{s}{2}}\right)$ is the fractional domain of the (Dirichlet) Laplacean, defined in (4.10). In particular, using the orthogonality of the eigenfunctions π_i and eigenvector fields τ_i in $\mathrm{D}\left(\Delta_{\mathcal{O}}^{\frac{s}{2}}\right)$, we can write

$$|\mathcal{F}_t \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}}(\vartheta\eta|_{\mathcal{O}}(\tau,\,\cdot))|^2_{H^s(\Gamma,\,\mathbb{R}^3)} \le C_2|P_M^{\mathcal{O}}(\vartheta\mathcal{F}_t\eta|_{\mathcal{O}}(\tau,\,\cdot))|^2_{\mathrm{D}\left(\Delta_{\mathcal{O}}^{\frac{s}{2}}\right)} \le C_2|\vartheta\mathcal{F}_t\eta|_{\mathcal{O}}(\tau,\,\cdot))|^2_{\mathrm{D}\left(\Delta_{\mathcal{O}}^{\frac{s}{2}}\right)}.$$

Again, by interpolation arguments, we can also derive that $z \mapsto \vartheta z|_{\mathcal{O}}$ maps $H^s(\Gamma,\, \mathbb{R}^3)$ into $\mathrm{D}\left(\Delta_{\mathcal{O}}^{\frac{s}{2}}\right)$ continuously, from which it follows $|\mathcal{F}_t \chi \mathbb{E}_0^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}} P_M^{\mathcal{O}} (\vartheta\eta|_{\mathcal{O}}(\tau,\,\cdot))|^2_{H^s(\Gamma,\,\mathbb{R}^3)} \leq C_3 |\mathcal{F}_t \eta(\tau,\,\cdot))|^2_{H^s(\Gamma,\,\mathbb{R}^3)}$ and, from (A.6), we can arrive to $|K^{\mathcal{O}}\eta|^2_{H^r(\mathbb{R},\, H^s(\Gamma,\,\mathbb{R}^3))} \leq C_3 |\eta|^2_{H^r(\mathbb{R},\, H^s(\Gamma,\,\mathbb{R}^3))}$. □

Lemma A.11. *If $r \geq 0$ and $-1 \leq s < 0$, there is a constant $C > 0$ such that, for any $\eta \in H^r(\mathbb{R},\, L^2(\Gamma,\, \mathbb{R}^3))$, we have $|K^{\mathcal{O}}\eta|^2_{H^r(\mathbb{R},\, H^s(\Gamma,\,\mathbb{R}^3))} \leq C|\eta|^2_{H^r(\mathbb{R},\, H^s(\Gamma,\,\mathbb{R}^3))}$.*

Proof. We just notice that, for $\xi \in H^{-s}(\Gamma,\, \mathbb{R}^3)$,

$$\begin{aligned} \langle \mathcal{F}_t K^{\mathcal{O}}\eta,\, \xi\rangle_{H^s(\Gamma,\,\mathbb{R}^3),\, H^{-s}(\Gamma,\,\mathbb{R}^3)} &= (\mathcal{F}_t K^{\mathcal{O}}\eta,\, \xi)_{L^2(\Gamma,\,\mathbb{R}^3)} = (\mathcal{F}_t\eta,\, \vartheta\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}}(\chi\xi|_{\mathcal{O}}))_{L^2(\Gamma,\,\mathbb{R}^3)} \\ &= \langle \mathcal{F}_t\eta,\, \vartheta\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}}(\chi\xi|_{\mathcal{O}})\rangle_{H^s(\Gamma,\,\mathbb{R}^3),\, H^{-s}(\Gamma,\,\mathbb{R}^3)} \end{aligned}$$

and, using an argument similar to that in the proof of Lemma A.10, we have $|\vartheta\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}}(\chi\xi|_{\mathcal{O}})|^2_{H^{-s}(\Gamma,\,\mathbb{R}^3)} \leq C|P_M^{\mathcal{O}} P_{\chi^\perp}^{\mathcal{O}}(\chi\xi|_{\mathcal{O}})|^2_{H^{-s}(\mathcal{O},\,\mathbb{R}^3)} \leq C_1 |P_M^{\mathcal{O}}\left(\chi\xi|_{\mathcal{O}} - (\frac{(\chi\xi|_{\mathcal{O}},\, \chi\mathbf{n})_{L^2(\mathcal{O},\,\mathbb{R}^3)}}{\int_{\mathcal{O}} \chi^2 \mathrm{d}\mathcal{O}})\chi\mathbf{n}\right)|^2_{\mathrm{D}(\Delta_{\mathcal{O}}^{\frac{s}{2}})} \leq C_2|\xi|^2_{H^{-s}(\Gamma,\,\mathbb{R}^3)}$. It follows that $|\mathcal{F}_t K^{\mathcal{O}}\eta|^2_{H^s(\Gamma,\,\mathbb{R}^3)} \leq C_2|\mathcal{F}_t\eta|^2_{H^s(\Gamma,\,\mathbb{R}^3)}$, and $|K^{\mathcal{O}}\eta|^2_{H^r(\mathbb{R},\, H^s(\Gamma,\,\mathbb{R}^3))} \leq C_2|\eta|^2_{H^r(\mathbb{R},\, H^s(\Gamma,\,\mathbb{R}^3))}$. □

Next, we show that from Lemmas A.10 and A.11 we can derive that $K^{\mathcal{O}} \in \mathcal{L}(G^i(\mathbb{R},\, \Gamma) \to G^i_{\mathrm{av}}(\mathbb{R},\, \Gamma))$, $i \in \{1,\, 2\}$. Indeed, for given $\eta \in G^i(\mathbb{R},\, \Gamma)$, we find

$$\begin{aligned} |K^{\mathcal{O}}\eta|^2_{G^i_{\mathrm{av}}(\mathbb{R},\,\Gamma)} &= |K^{\mathcal{O}}\eta|^2_{L^2(\mathbb{R},\, H^{i-\frac{1}{2}}(\Gamma,\,\mathbb{R}^3))} + |\mathbf{n}\cdot K^{\mathcal{O}}\eta|^2_{H^{r_{\mathbf{n},1}(i)}(\mathbb{R},\, H^{r_{\mathbf{n},2}(i)}_{\mathrm{av}}(\Gamma,\,\mathbb{R}))} \\ &\quad + |K^{\mathcal{O}}\eta - (\mathbf{n}\cdot K^{\mathcal{O}}\eta)\mathbf{n}|^2_{H^{r_{\mathbf{t},1}(i)}(\mathbb{R},\, H^{r_{\mathbf{t},2}(i)}(\Gamma,\, T\Gamma))}; \end{aligned} \tag{A.7}$$

and from the identities

$$\begin{aligned} \mathbf{n}\cdot K^{\mathcal{O}}\eta &= \mathbf{n}\cdot\chi\mathbb{E}_0^{\mathcal{O}} \sum_{i=1}^{M} (\pi_i\mathbf{n},\, \vartheta\eta|_{\mathcal{O}})_{L^2(\mathcal{O},\,\mathbb{R}^3)} P_{\chi^\perp}^{\mathcal{O}}(\pi_i\mathbf{n}) \\ &= \mathbf{n}\cdot\chi\mathbb{E}_0^{\mathcal{O}} \sum_{i=1}^{M} (\pi_i\mathbf{n},\, \vartheta(\mathbf{n}\cdot\eta)\mathbf{n}|_{\mathcal{O}})_{L^2(\mathcal{O},\,\mathbb{R}^3)} P_{\chi^\perp}^{\mathcal{O}}(\pi_i\mathbf{n}) = \mathbf{n}\cdot K^{\mathcal{O}}((\mathbf{n}\cdot\eta)\mathbf{n}); \end{aligned} \tag{A.8}$$

$$\begin{aligned} K^{\mathcal{O}}\eta - (\mathbf{n}\cdot K^{\mathcal{O}}\eta)\mathbf{n} &= K^{\mathcal{O}}(\eta - (\mathbf{n}\cdot\eta)\mathbf{n} + (\mathbf{n}\cdot\eta)\mathbf{n}) - (\mathbf{n}\cdot K^{\mathcal{O}}((\mathbf{n}\cdot\eta)\mathbf{n}))\mathbf{n} \\ &= K^{\mathcal{O}}(\eta - (\mathbf{n}\cdot\eta)\mathbf{n}) + K^{\mathcal{O}}((\mathbf{n}\cdot\eta)\mathbf{n}) - (\mathbf{n}\cdot K^{\mathcal{O}}((\mathbf{n}\cdot\eta)\mathbf{n}))\mathbf{n} = K^{\mathcal{O}}(\eta - (\mathbf{n}\cdot\eta)\mathbf{n}); \end{aligned} \tag{A.9}$$

it follows that $|K^{\mathcal{O}}\eta|^2_{G^i_{\mathrm{av}}(\mathbb{R},\,\Gamma)} \leq |K^{\mathcal{O}}\eta|^2_{L^2(\mathbb{R},\, H^{i-\frac{1}{2}}(\Gamma,\,\mathbb{R}^3))} + C|\mathbf{n}|^2_{C^1(\Gamma,\,\mathbb{R}^3)}|K^{\mathcal{O}}((\mathbf{n}\cdot\eta)\mathbf{n})|^2_{H^{r_{\mathbf{n},1}(i)}(\mathbb{R},\, H^{r_{\mathbf{n},2}(i)}(\Gamma,\,\mathbb{R}^3))} + |K^{\mathcal{O}}(\eta - (\mathbf{n}\cdot\eta)\mathbf{n})|^2_{H^{r_{\mathbf{t},1}(i)}(\mathbb{R},\, H^{r_{\mathbf{t},2}(i)}(\Gamma,\, T\Gamma))}$. Using appropriately Lemmas A.10 and A.11 we obtain

$$\begin{aligned} |K^{\mathcal{O}}\eta|^2_{G^i_{\mathrm{av}}(\mathbb{R},\,\Gamma)} &\leq C_1|\eta|^2_{L^2(\mathbb{R},\, H^{i-\frac{1}{2}}(\Gamma,\,\mathbb{R}^3))} + C_1|(\mathbf{n}\cdot\eta)\mathbf{n}|^2_{H^{r_{\mathbf{n},1}(i)}(\mathbb{R},\, H^{r_{\mathbf{n},2}(i)}(\Gamma,\,\mathbb{R}^3))} \\ &\quad + C_1|\eta - (\mathbf{n}\cdot\eta)\mathbf{n}|^2_{H^{r_{\mathbf{t},1}(i)}(\mathbb{R},\, H^{r_{\mathbf{t},2}(i)}(\Gamma,\, T\Gamma))} \leq C_2|\eta|^2_{G^i(\mathbb{R},\,\Gamma)} \end{aligned}$$

and we can conclude that Proposition 2.17 holds with $\mathbb{R}$ in the role of $(a,\, b)$.

Finally, using the idea in Remark A.1 (*cf.* proof of Lem. A.8), we notice that we can construct an extension $F \in \mathcal{L}(G^i((a,\, b),\, \Gamma) \to G^i(\mathbb{R},\, \Gamma)) \cap \mathcal{L}(G^i_{\mathrm{av}}((a,\, b),\, \Gamma) \to G^i_{\mathrm{av}}(\mathbb{R},\, \Gamma))$. Moreover we will have $FK^{\mathcal{O}} = K^{\mathcal{O}}F$, roughly speaking, because $K^{\mathcal{O}}$ does not depend on the time variable, while F depends essentially on the time variable. Then we obtain

$$\begin{aligned} |K^{\mathcal{O}}\eta|^2_{G^i_{\mathrm{av}}((a,\,b),\,\Gamma)} &\leq \inf_{E\in\mathcal{L}(G^i_{\mathrm{av}}((a,\,b),\,\Gamma)\to G^i_{\mathrm{av}}(\mathbb{R},\,\Gamma))} |EK^{\mathcal{O}}\eta|^2_{G^i_{\mathrm{av}}(\mathbb{R},\,\Gamma)} \leq |FK^{\mathcal{O}}\eta|^2_{G^i_{\mathrm{av}}(\mathbb{R},\,\Gamma)} \\ &= |K^{\mathcal{O}}F\eta|^2_{G^i_{\mathrm{av}}(\mathbb{R},\,\Gamma)} \leq C_2|F\eta|^2_{G^i(\mathbb{R},\,\Gamma)} \leq C_3|\eta|^2_{G^i((a,\,b),\,\Gamma)}, \end{aligned}$$

which proves the Proposition 2.17. □

A.6. On the definitions of the Sobolev spaces on the boundary

We can see the space $H^s(\Gamma,\,\mathbb{R})$ $(s>0)$ as the space of traces on Γ of the functions in $H^{s+\frac{1}{2}}(\Omega,\,\mathbb{R})$ (see for example [24], Chap. 2, Sect. 5). Alternatively, say for $s\in[0,\,2]$, we can identify $H^s(\Gamma,\,\mathbb{R})$ with the domain, in $L^2(\Gamma,\,\mathbb{R})$, of the domain of Laplace–de Rham operator Δ_Γ on the manifold Γ, that is,

$$H^s(\Gamma,\,\mathbb{R})=\mathrm{D}\left((1+\Delta_\Gamma)^{\frac{s}{2}}\right)=\left[\mathrm{D}\left((1+\Delta_\Gamma)^1\right),\,\mathrm{D}\left((1+\Delta_\Gamma)^0\right)\right]_{1-\frac{s}{2}};$$

see for example [10]. The space $H^s(\Gamma,\,\mathbb{R})$ can be endowed with the norm

$$|f|_{H^s(\Gamma,\,\mathbb{R})}=\left|(1+\Delta_\Gamma)^{\frac{s}{2}}\right|_{L^2(\Gamma,\,\mathbb{R})};\tag{A.10}$$

and the Sobolev space $H^s(\mathcal{O},\,\mathbb{R})$ can be identified with $H^s(\Gamma,\,\mathbb{R})|_{\mathcal{O}}$, and endowed with the (quotient) norm $|g|_{H^s(\mathcal{O},\,\mathbb{R})}:=\inf_{f|_{\mathcal{O}}=g}|f|_{L^2(\Gamma,\,\mathbb{R})}$; see also the discussion in Section A.2.

We can also define the Sobolev spaces by means of the Levi-Civita connection (covariant derivative), as in [1, 10], or using an atlas of Γ and a partition of unity argument, as in [31].

For compact manifolds, either with or without boundary, all these definitions are essentially equivalent. For the equivalence of the covariant derivative and domains of fractional powers of $(1+\Delta_\Gamma)$ approaches we refer to [10]. For the equivalence of the atlas and domains of fractional powers of $(1+\Delta_\Gamma)$ (*i.e.*, interpolation) approaches we refer to [31], Chapter 4, Section 3.

A.7. Proof of Proposition 4.5

From Section 2.1 we have that $G^1((a,\,b),\,\Gamma)\subseteq L^2((a,\,b),\,H^{\frac{1}{2}}(\Gamma,\,\mathbb{R}^3))\cap H^{\frac{1}{2}}((a,\,b),\,H^{-1}(\Gamma,\,\mathbb{R}^3))$ continuously. Then, Proposition 4.5 will follow from:

Lemma A.12. *Let $r_1,\,r_2\geq 0$ and $s_1,\,s_2\in\mathbb{R}$ be real numbers. Let I be any open interval $I\subseteq\mathbb{R}$ (either bounded or not), then the inclusion $H^{r_1}(I,\,H^{s_1}(\Gamma,\,\mathbb{R}^3))\cap H^{r_2}(I,\,H^{s_2}(\Gamma,\,\mathbb{R}^3))\subseteq H^{\frac{r_1+r_2}{2}}\left(I,\,H^{\frac{s_1+s_2}{2}}(\Gamma,\,\mathbb{R}^3)\right)$ holds and is continuous.*

Proof. We may suppose that Γ is a connected manifold; if that is not the case, then it is a disjoint union of such manifolds and the Sobolev spaces in Γ will be just the Cartesian product of the corresponding spaces in each connected component. We will use the characterization (A.10) (recalling that is is meaningful also for negative values of s; see also the discussion in [23]). Consider the Laplace–de Rham operator (*cf.* (4.10)),

$$\Delta_\Gamma:u=(u\cdot\mathbf{n})\mathbf{n}+u_{\mathbf{t}}\mapsto(\Delta_\Gamma(u\cdot\mathbf{n}))\mathbf{n}+\Delta_\Gamma u_{\mathbf{t}}.$$

Notice that $1+\Delta_\Gamma$, is a symmetric operator, in $L^2(\Gamma,\,\mathbb{R}^3)$, and the same holds for its (fractional) powers, $u=\sum_{i\in\mathbb{N}_0}u^i_{\mathbf{n}}\pi_i\mathbf{n}+\sum_{i\in\mathbb{N}_0}u^i_{\mathbf{t}}\tau_i\mapsto(1+\Delta_\Gamma)^s u=\sum_{i\in\mathbb{N}_0}u^i_{\mathbf{n}}(1+\beta_i)^s\pi_i\mathbf{n}+\sum_{i\in\mathbb{N}_0}u^i_{\mathbf{t}}(1+\gamma_i)^s\tau_i$. where β_i and γ_i are the eigenvalues associated with the smooth eigenfunction π_i and smooth eigenvector field τ_i of Δ_Γ, respectively. For $s\geq 0$ we have the characterization

$$H^s(\Gamma,\,\mathbb{R}^3)=\left\{u\in L^2(\Gamma,\,\mathbb{R}^3)\;\middle|\;\sum_{i\in\mathbb{N}_0}(u^i_{\mathbf{n}})^2(1+\beta_i)^{2s}+(u^i_{\mathbf{t}})^2(1+\gamma_i)^{2s}<+\infty\right\}\tag{A.11}$$

and $H^{-s}(\Gamma,\,\mathbb{R}^3)$ is the completion of $L^2(\Gamma,\,\mathbb{R}^3)$ in the norm $\left(\sum_{i\in\mathbb{N}_0}(u^i_{\mathbf{n}})^2(1+\beta_i)^{-2s}+(u^i_{\mathbf{t}})^2(1+\gamma_i)^{-2s}\right)^{\frac{1}{2}}$. From

$$\begin{aligned}
&|u|^2_{H^{\frac{r_1+r_2}{2}}\left(\mathbb{R},\,H^{\frac{s_1+s_2}{2}}(\Gamma,\mathbb{R}^3)\right)}=\int_{\mathbb{R}}(1+|\tau|^2)^{\frac{r_1+r_2}{2}}|(1+\Delta_\Gamma)^{\frac{s_1+s_2}{4}}\mathcal{F}_t u(\tau,\,x)|^2_{L^2(\Gamma,\mathbb{R}^3)}\,\mathrm{d}\tau\\
&\leq\prod_{i=1}^{2}\left(\int_{\mathbb{R}}(1+|\tau|^2)^{r_i}|(1+\Delta_\Gamma)^{\frac{s_i}{2}}\mathcal{F}_t u(\tau,\,x)|^2_{L^2(\Gamma,\mathbb{R}^3)}\,\mathrm{d}\tau\right)^{\frac{1}{2}}=|u|_{H^{r_1}(\mathbb{R},\,H^{s_1}(\Gamma,\mathbb{R}^3))}|u|_{H^{r_2}(\mathbb{R},\,H^{s_2}(\Gamma,\mathbb{R}^3))}\\
&\leq\tfrac{1}{2}\left(|u|^2_{H^{r_1}(\mathbb{R},\,H^{s_1}(\Gamma,\mathbb{R}^3))}+|u|^2_{H^{r_2}(\mathbb{R},\,H^{s_2}(\Gamma,\mathbb{R}^3))}\right)=\tfrac{1}{2}|u|^2_{H^{r_1}(\mathbb{R},\,H^{s_1}(\Gamma,\mathbb{R}^3))\cap H^{r_2}(\mathbb{R},\,H^{s_2}(\Gamma,\mathbb{R}^3))},
\end{aligned}$$

we conclude that the Lemma holds with $I=\mathbb{R}$. If the interval I is strictly contained in $\mathbb{R}$, we also find

$$\begin{aligned} &H^{\frac{r_1+r_2}{2}}\left(I,\, H^{\frac{s_1+s_2}{2}}(\Gamma,\,\mathbb{R}^3)\right) = H^{\frac{r_1+r_2}{2}}\left(\mathbb{R},\, H^{\frac{s_1+s_2}{2}}(\Gamma,\,\mathbb{R}^3)\right)\Big|_I \\ &\supseteq \left(H^{r_1}(\mathbb{R},\, H^{s_1}(\Gamma,\,\mathbb{R}^3))\cap H^{r_2}(\mathbb{R},\, H^{s_2}(\Gamma,\,\mathbb{R}^3))\right)\big|_I = H^{r_1}(I,\, H^{s_1}(\Gamma,\,\mathbb{R}^3))\cap H^{r_2}(I,\, H^{s_2}(\Gamma,\,\mathbb{R}^3)). \end{aligned}$$

Now (proceeding *e.g.* as in Remark A.1), setting $m\in\mathbb{N}_0$ such that $m\geq\max\{r_1,\, r_2\}$, we can find an extension $\hat{F}_m$ in the intersection

$$\bigcap_{(\rho_1,\,\rho_2)\in\left\{(r_1,\,s_1),\,(r_2,\,s_2),\,\frac{(r_1+r_2,\,s_1+s_2)}{2}\right\}} \mathcal{L}(H^{\rho_1}(I,\, H^{\rho_2}(\Gamma,\,\mathbb{R}^3))\to H^{\rho_1}(\mathbb{R},\, H^{\rho_2}(\Gamma,\,\mathbb{R}^3)))$$

(*cf.* proof of Lem. A.8). Thus,

$$\begin{aligned} &|u|^2_{H^{\frac{r_1+r_2}{2}}\left(I,\,H^{\frac{s_1+s_2}{2}}(\Gamma,\mathbb{R}^3)\right)} := \inf_E |Eu|^2_{H^{\frac{r_1+r_2}{2}}\left(\mathbb{R},\,H^{\frac{s_1+s_2}{2}}(\Gamma,\mathbb{R}^3)\right)} \leq |\hat{F}_m u|^2_{H^{\frac{r_1+r_2}{2}}\left(\mathbb{R},\,H^{\frac{s_1+s_2}{2}}(\Gamma,\mathbb{R}^3)\right)} \\ &\leq \tfrac{1}{2}|\hat{F}_m u|^2_{H^{r_1}(\mathbb{R},\,H^{s_1}(\Gamma,\mathbb{R}^3))\cap H^{r_2}(\mathbb{R},\,H^{s_2}(\Gamma,\mathbb{R}^3))} \leq C|u|^2_{H^{r_1}(I,\,H^{s_1}(\Gamma,\mathbb{R}^3))\cap H^{r_2}(I,\,H^{s_2}(\Gamma,\mathbb{R}^3))}, \end{aligned}$$

with E running over $\mathcal{L}\left(H^{\frac{r_1+r_2}{2}}\left(I,\, H^{\frac{s_1+s_2}{2}}(\Gamma,\,\mathbb{R}^3)\right)\to H^{\frac{r_1+r_2}{2}}\left(\mathbb{R},\, H^{\frac{s_1+s_2}{2}}(\Gamma,\,\mathbb{R}^3)\right)\right)$. □

A.8. Proof of Proposition 4.6

Let $\{x_j \mid j\in\mathbb{N}_0\}$ be an orthonormed basis in X; then $\mathcal{F}:=\{\bar{s}_n x_j \mid (j,\,n)\in\mathbb{N}_0^2\}$ is an orthonormed basis in $L^2((a,\,b),\,X)$: that $\mathcal{F}$ is orthonormed follows from $\int_a^b(\bar{s}_n x_j,\,\bar{s}_m x_i)_X\,\mathrm{d}t = (x_j,\,x_i)_X\int_a^b\bar{s}_n\bar{s}_m\,\mathrm{d}t = \delta_{(j,\,n)}^{(i,\,m)}$, where $\delta_{p_1}^{p_2}$ is the Kronecker delta: $\delta_{p_1}^{p_2}=\begin{cases}1, \text{ if } p_1=p_2\\ 0, \text{ if } p_1\neq p_2\end{cases}$. On the other side given $f\in L^2((a,\,b),\,X)$ orthogonal to all the elements of $\mathcal{F}$, we find that $0=\int_a^b\bar{s}_n(f,\,x_j)_X\,\mathrm{d}t$, and that $\int_a^b|(f,\,x_j)_X|^2_{\mathbb{R}}\,\mathrm{d}t\leq\int_a^b|f|_X^2|x_j|_X^2\,\mathrm{d}t=\int_a^b|f|_X^2\,\mathrm{d}t<+\infty$, which allow us to conclude that for all $j\in\mathbb{N}_0$, $(f(t),\,x_j)_X=0$ for a.e. $t\in(a,\,b)$; which in turn implies that $f(t)=0$ for a.e. $t\in(a,\,b)$, that is, $f=0$. Recall that the union of a sequence of sets with Lebesgue measure zero is still a set of Lebesgue measure zero, see ([30], Chap. 13, Sect. 2) and ([36], Sect. 2.2).

Since $L^2((a,\,b),\,X)$ is complete, and an orthogonal basis is inconditional, we can write $\left(\int_a^b\bar{s}_n(t)f(t)\,\mathrm{d}t\right)\bar{s}_n = \lim_{k\to+\infty}\sum_{j=1}^k\left(\int_a^b(\bar{s}_n(t)x_j,\,f(t))_X\,\mathrm{d}t\right)\bar{s}_n x_j$, and $f=\lim_{M\to+\infty}\sum_{n=1}^M\left(\int_a^b\bar{s}_n(t)f(t)\,\mathrm{d}t\right)\bar{s}_n$. Furthermore,

$$\begin{aligned} |f|^2_{L^2((a,\,b),\,X)} &= \sum_{n,\,j\in\mathbb{N}_0}(f,\,\bar{s}_n x_j)^2_{L^2((a,\,b),\,X)} = \sum_{n\in\mathbb{N}_0}\sum_{j\in\mathbb{N}_0}\left(\int_a^b(\bar{s}_n(t)x_j,\,f(t))_X\,\mathrm{d}t\right)^2 \\ &= \sum_{n\in\mathbb{N}_0}\left|\left(\int_a^b\bar{s}_n(t)f(t)\,\mathrm{d}t\right)\bar{s}_n\right|^2_{L^2((a,\,b),\,X)} = \sum_{n\in\mathbb{N}_0}\left|\int_a^b\bar{s}_n(t)f(t)\,\mathrm{d}t\right|^2_X, \text{ for } f\in L^2((a,\,b),\,X); \\ |f|^2_{H^1_0((a,\,b),\,X)} &= \left|\sum_{n\in\mathbb{N}_0}\left(\int_a^b\bar{s}_n(\tau)f(\tau)\,\mathrm{d}\tau\right)\bar{s}_n\right|^2_{L^2((a,\,b),\,X)} + \left|\sum_{n\in\mathbb{N}_0}\left(\int_a^b\bar{s}_n(\tau)f(\tau)\,\mathrm{d}\tau\right)\partial_t\bar{s}_n\right|^2_{L^2((a,\,b),\,X)} \\ &= \sum_{n\in\mathbb{N}_0}(1+\lambda_n)\left|\int_a^b\bar{s}_n(\tau)f(\tau)\,\mathrm{d}\tau\right|^2_X, \text{ for } f\in H^1_0((a,\,b),\,X). \end{aligned}$$

Notice that we have $(\bar{s}_m,\,\bar{s}_n)_{L^2((a,\,b),\,\mathbb{R})}=\delta_n^m$ and $(\partial_t\bar{s}_m,\,\partial_t\bar{s}_n)_{L^2((a,\,b),\,\mathbb{R})}=(-\partial_t\partial_t\bar{s}_m,\,\bar{s}_n)_{L^2((a,\,b),\,\mathbb{R})}=\delta_n^m\lambda_m$. We can also find

$$|P_M^t f|^2_{L^2((a,\,b),\,X)} = \sum_{n=1}^M\left|\int_a^b\bar{s}_n(t)f(t)\,\mathrm{d}t\right|^2_X; \quad |P_M^t f|^2_{H^1_0((a,\,b),\,X)} = \sum_{n=1}^M(1+\lambda_n)\left|\int_a^b\bar{s}_n(\tau)f(\tau)\,\mathrm{d}\tau\right|^2_X.$$

Now, it is straightforward to check that P_M^t is an orthogonal projection both in $L^2((a, b), X)$ and in $H_0^1((a, b), X)$, onto $\sum_{n=1}^M \bar{s}_n X$. □

A.9. Proof of Proposition 5.1

Setting $\psi = \phi 1_\Gamma$, from Propositions 2.17 and 3.10 we have $\varphi K^{\mathcal{O}} = \psi K^{\mathcal{O}} \in \mathcal{L}(G^i((a, b), \Gamma) \to G^i_{\rm av}((a, b), \Gamma))$, with $i \in \{1, 2\}$. Thus, since $K_t^{\mathcal{O}} = \varphi K^{\mathcal{O}} P_M^t \tilde{\varphi}$, it follows that it remains to show the continuity of the mapping $\eta \mapsto P_M^t(\tilde{\varphi}\eta)$, from $G^i((a, b), \Gamma)$ into itself.

From Section 4.3 we know that $f \mapsto P_M^t f$ is a an orthogonal projection in $L^2((a, b), X)$ and in $H_0^1((a, b), X)$. It follows that $P_M^t \tilde{\varphi} \in \mathcal{L}(H^1((a, b), X) \to H_0^1((a, b), X)) \cap \mathcal{L}(L^2((a, b), X) \to L^2((a, b), X))$. By an interpolation argument it will follow that $P_M^t \tilde{\varphi} \in \mathcal{L}(H^s((a, b), X) \to [H_0^1((a, b), X), L^2((a, b), X)]_{1-s})$, and $[H_0^1((a, b), X), L^2((a, b), X)]_{1-s} \subseteq H^s((a, b), X)$.

Notice that for X we can take $H^r(\Gamma, Z)$ with Z either $\mathbb{R}$ or $T\Gamma$, and with $r \in \mathbb{R}$. Letting the triple (s, r, Z) run over the set $\{(0, i - \frac{1}{2}, T\Gamma), (r_{\mathbf{t},1}(i), r_{\mathbf{t},1}(i), T\Gamma), (0, i - \frac{1}{2}, \mathbb{R}), (r_{\mathbf{n},1}(i), r_{\mathbf{n},1}(i), \mathbb{R})\}$, from $P_M^t \tilde{\varphi} \in \mathcal{L}(H^s((a, b), H^r(\Gamma, Z)) \to H^s((a, b), H^r(\Gamma, Z)))$, we can obtain that $P_M^t \tilde{\varphi} \in \mathcal{L}(G^i((a, b), \Gamma) \to G^i((a, b), \Gamma))$, $i \in \{1, 2\}$. Recall that $r_{\mathbf{t},1}(i)$, $r_{\mathbf{t},2}(i)$, $r_{\mathbf{n},1}(i)$, and $r_{\mathbf{n},2}(i)$ are defined in Section 2.1. □

A.10. Quadratic functionals with linear constraint

We present some remarks on the Theorem A.2 in [5], just to simplify the exposition in Section 5. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces, let $\tilde{J}(x, y)$ be a bounded symmetric bilinear form on $\mathcal{X}$ that is weakly continuous with respect to each of its arguments, and let $A : \mathcal{X} \to \mathcal{Y}$ be a continuous surjective linear operator. Denote $J(x) := \tilde{J}(x, x)$. Given a vector $y \in \mathcal{Y}$, consider the minimization problem

$$J(x) \to \min, \quad Ax = y. \tag{A.12}$$

Definition A.13. We say that $\bar{x} \in \mathcal{X}$ is a *global minimum* for (A.12) if $A\bar{x} = y$ and $J(\bar{x}) \leq J(x)$ for all $x \in A^{-1}(y) := \{x \in \mathcal{X} \mid Ax = y\}$.

Lemma A.14. *Suppose that for each $y \in \mathcal{Y}$ and $c > 0$, we have that J is nonnegative and strictly convex on each affine space $A^{-1}(y)$, and that the set $\mathcal{S} := \{x \in A^{-1}(y) : J(x) \leq c\}$ is weakly compact. Then problem* (A.12) *has a unique global minimum $\bar{x} \in \mathcal{X}$, and the function $L : \mathcal{Y} \to \mathcal{X}$ taking y to $\bar{x}$ is linear.*

Proof. For the uniqueness and linearity we may repeat the respective arguments in the proof of Theorem A.2 in [5], while for the existence we just need to adapt/rewrite the respective arguments in there, as follows: let $(x_n)_{n\in\mathbb{N}}$ be a minimizing sequence $J(x_n) \to \inf_{Ax=y} J(x)$, in $\mathcal{S}$. Since $\mathcal{S}$ is weakly compact we can assume (taking a subsequence) that there exists $\bar{x} \in A^{-1}(y)$ such that $x_n \rightharpoonup \bar{x}$, from which we can derive that necessarily $\bar{x}$ is a global minimum for J. □

Remark A.15. If $\mathcal{X}$ is a reflexive Banach space, it is sufficient to assume the boundedness of the set $\mathcal{S}$ in $\mathcal{X}$, instead of its weak compactness. Indeed, if $\mathcal{S} \subseteq B_r := \{x \in \mathcal{X} \mid |x|_{\mathcal{X}} \leq r\}$, for some $r > 0$, then by the Kakutani's Theorem follows that any sequence $(x_n)_{n\in\mathbb{N}}$, in $\mathcal{S}$, has a subsequence $(x_{\xi(n)})_{n\in\mathbb{N}}$ with $x_{\xi(n)} \rightharpoonup \bar{x}$, for some $\bar{x} \in B_r$. Necessarily we have $y = A\bar{x}$, because if that was not the case then, the Hahn–Banach Theorem would guarantee the existence of $f \in \mathcal{Y}'$ such that $\langle f, y - A\bar{x}\rangle_{\mathcal{Y}',\mathcal{Y}} = 1$ (*cf.* [8], Chap. III, Cor. 6.6) but, by the continuity of A it follows that the composition $f \circ A$ is in $\mathcal{X}'$, and that $\langle f, y\rangle_{\mathcal{Y}',\mathcal{Y}} = \langle f, Ax_{\xi(n)}\rangle_{\mathcal{Y}',\mathcal{Y}} =: \langle f \circ A, x_{\xi(n)}\rangle_{\mathcal{X}',\mathcal{X}} \to \langle f \circ A, \bar{x}\rangle_{\mathcal{X}',\mathcal{X}} := \langle f, A\bar{x}\rangle_{\mathcal{Y}',\mathcal{Y}}$, that is, $\langle f, y - A\bar{x}\rangle_{\mathcal{Y}',\mathcal{Y}} = 0$, which is a contradiction. Finally, by $0 \leq J(x_n - \bar{x}) = J(x_n) - 2\tilde{J}(x_n, \bar{x}) + J(\bar{x})$ and the weak continuity of $\tilde{J}$, we have that $J(\bar{x}) \leq \liminf_{n\to\infty} J(x_n) \leq c$. Thus $\bar{x} \in \mathcal{S}$, and we can conclude that $\mathcal{S}$ is weakly compact.

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