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INTERNAL CONTROLLABILITY OF THE KORTEWEG–DE VRIES EQUATION ON A BOUNDED DOMAIN

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Abstract. This paper is concerned with the control properties of the Korteweg–de Vries (KdV) equation posed on a bounded interval $(0, L)$ with a distributed control. When the control region is an arbitrary open subdomain (l_1, l_2), we prove the null controllability of the KdV equation by means of a new Carleman inequality. As a consequence, we obtain a regional controllability result, which roughly tells us that any target function arbitrarily chosen on $(0, l_1)$ and null on (l_2, L) is reachable. Finally, when the control region is a neighborhood of the right endpoint, an exact controllability result in a weighted L^2-space is also established.

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1. Introduction

The Korteweg–de Vries (KdV) equation can be written

$$u_t + u_{xxx} + u_x + uu_x = 0,$$

where $u = u(t, x)$ is a real-valued function of two real variables t and x, and $u_t = \partial u/\partial t$, *etc.* The equation was first derived by Boussinesq [3] and Korteweg–de Vries [13] as a model for the propagation of water waves along a channel. The equation furnishes also a very useful approximation model in nonlinear studies whenever one wishes to include and balance a weak nonlinearity and weak dispersive effects. In particular, the equation is now commonly accepted as a mathematical model for the unidirectional propagation of small amplitude long waves in nonlinear dispersive systems.

The KdV equation has been intensively studied from various aspects of mathematics, including the well-posedness, the existence and stability of solitary waves, the integrability, the long-time behavior, *etc.* (see *e.g.* [12, 18]). The practical use of the KdV equation does not always involve the pure initial value problem.

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In numerical studies, one is often interested in using a finite interval (instead of the whole line) with three boundary conditions.

Here, we shall be concerned with the control properties of KdV, the control acting through a forcing term f incorporated in the equation:

$$u_t + u_x + u_{xxx} + uu_x = f, \quad t \in [0,T],\ x \in [0,L], \tag{1.1}$$

together with some boundary conditions. Our main purpose is to see whether one can force the solutions of (1.1) to have certain desired properties by choosing an appropriate control input f. The focus here is on the *controllability* issue:

Given an initial state u_0 and a terminal state u_1 in a certain space, can one find an appropriate control input f so that equation (1.1) *admits a solution u which equals u_0 at time $t=0$ and u_1 at time $t=T$?*

If one can always find a control input f to guide the system described by (1.1) from any given initial state u_0 to any given terminal state u_1, then the system (1.1) is said to be *exactly controllable*. If the system can be driven, by means of a control f, from any state to the origin (*i.e.* $u_1 \equiv 0$), then one says that system (1.1) is *null controllable*.

The study of the controllability and stabilization of the KdV equation started with the works of Russell and Zhang [25] for a system with periodic boundary conditions and an internal control. Since then, both the controllability and the stabilization have been intensively studied. (We refer the reader to [24] for a survey of the results up to 2009.) In particular, the exact boundary controllability of KdV on a finite domain was investigated in *e.g.* [4–6, 8, 9, 20, 22, 28]. Most of those works were concerned with the following system

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & \text{in } (0,T)\times(0,L), \\ u(t,0) = g_1(t),\ u(t,L) = g_2(t),\ u_x(t,L) = g_3(t) & \text{in } (0,T) \end{cases} \tag{1.2}$$

in which the boundary data g_1, g_2, g_3 can be chosen as control inputs. System (1.2) was first studied by Rosier [20] considering only the control input g_3 (*i.e.* $g_1 = g_2 = 0$). It was shown in [20] that the exact controllability of the linearized system holds in $L^2(0,L)$ if, and only if, L does not belong to the following countable set of *critical lengths*

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2} \,:\, k,\ l \in \mathbb{N}^* \right\}. \tag{1.3}$$

The analysis developed in [20] shows that when the linearized system is controllable, the same is true for the nonlinear one. Note that the converse is false, as it was proved in [4–6] that the (nonlinear) KdV equation is controllable even when L is a critical length. The existence of a discrete set of critical lengths for which the exact controllability of the linearized equation fails was also noticed by Glass and Guerrero in [9] when g_2 is taken as control input (*i.e.* $g_1 = g_3 = 0$). Finally, it is worth mentioning the result by Rosier [22] and Glass and Guerrero [8] for which g_1 is taken as control input (*i.e.* $g_2 = g_3 = 0$). They proved that system (1.2) is then null controllable, but not exactly controllable, because of the strong smoothing effect.

As already noticed in [22, 24], system (1.2) with only the left control input g_1 active behaves like a parabolic system and is only null controllable. On the other hand, if one of the right controls g_2 or g_3 is active, then the system behaves like a hyperbolic system and is exactly controllable. The fact that we have so different control properties according to the place where the control is active is related to the propagation to the left of the high wavenumber exponential solutions of the linearized equation (see [22]).

By contrast, the mathematical theory pertaining to the study of the internal controllability in a bounded domain is considerably less advanced. As far as we know, the null controllability problem for system (1.1) was only addressed in [8] when the control acts in a neighborhood of the left endpoint. On the other hand, the exact controllability results in [14, 25] were obtained on a periodic domain.

The aim of this paper is to address the controllability issue for the KdV equation on a bounded domain with a distributed control. Our first main result is a null controllability result valid for any localization of the control region. Actually, a controllability to the trajectories is established.

Theorem 1.1. *Let* $\omega = (l_1, l_2)$ *with* $0 < l_1 < l_2 < L$, *and let* $T > 0$. *For* $\bar{u}_0 \in L^2(0,L)$, *let* $\bar{u} \in C^0([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$ *denote the solution of*

$$\begin{cases} \bar{u}_t + \bar{u}_x + \bar{u}\,\bar{u}_x + \bar{u}_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ \bar{u}(t,0) = \bar{u}(t,L) = \bar{u}_x(t,L) = 0 & \text{in } (0,T), \\ \bar{u}(0,x) = \bar{u}_0(x) & \text{in } (0,L). \end{cases} \tag{1.4}$$

Then there exists $\delta > 0$ *such that for any* $u_0 \in L^2(0,L)$ *satisfying* $\|u_0 - \bar{u}_0\|_{L^2(0,L)} \leq \delta$, *there exists* $f \in L^2((0,T)\times\omega)$ *such that the solution* $u \in C^0([0,T]; L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ *of*

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 1_\omega f(t,x) & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L), \end{cases} \tag{1.5}$$

satisfies $u(T,\cdot) = \bar{u}(T,\cdot)$ *in* $(0,L)$.

The null controllability is first established for a linearized system

$$\begin{cases} u_t + (\xi u)_x + u_{xxx} = 1_\omega f & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L), \end{cases} \tag{1.6}$$

by following the classical duality approach (see [7, 15]), which reduces the null controllability of (1.6) to an observability inequality for the solutions of the adjoint system. To prove the observability inequality, we derive a new Carleman estimate with an internal observation in $(0,T)\times(l_1,l_2)$ and use some interpolation arguments inspired by those in [8], where the authors derived a similar result when the control acts on a neighborhood on the left endpoint (that is, $l_1 = 0$). The null controllability is extended to the nonlinear system by applying Kakutani fixed-point theorem.

The second problem we address is related to the exact internal controllability of system (1.1). As far as we know, the same problem was studied only in [14, 25] in a periodic domain $\mathbb{T}$ with a distributed control of the form

$$f(t,x) = (Gh)(t,x) := g(x)\left(h(t,x) - \int_{\mathbb{T}} g(y)h(t,y)\mathrm{d}y\right),$$

where $g \in C^\infty(\mathbb{T})$ was such that $\{x \in \mathbb{T};\ g(x) > 0\} = \omega$ and $\int_{\mathbb{T}} g(x)\mathrm{d}x = 1$, and the function h was considered as a new control input. Here, we shall consider the system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = f & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L). \end{cases} \tag{1.7}$$

As the smoothing effect is different from those in a periodic domain, the results in this paper turn out to be very different from those in [14, 25]. First, for a controllability result in $L^2(0,L)$, the control f has to be taken in the space $L^2(0,T,H^{-1}(0,L))$. Actually, with any control $f \in L^2(0,T,L^2(0,L))$, the solution of (1.7) starting from $u_0 = 0$ at $t = 0$ would remain in $H_0^1(0,L)$ (see [8]). On the other hand, as for the boundary control, the localization of the distributed control plays a role in the results.

When the control acts in a neighborhood of $x = L$, we obtain the exact controllability in the weighted Sobolev space $L^2_{\frac{1}{L-x}\mathrm{d}x}$ defined as

$$L^2_{\frac{1}{L-x}\mathrm{d}x} := \left\{ u \in L^1_{\mathrm{loc}}(0,L);\ \int_0^L \frac{|u(x)|^2}{L-x}\mathrm{d}x < \infty \right\}.$$

More precisely, we shall obtain the following result:

Theorem 1.2. *Let $T>0$, $\omega=(l_1,l_2)=(L-\nu,L)$ where $0<\nu<L$. Then, there exists $\delta>0$ such that for any u_0, $u_1\in L^2_{\frac{1}{L-x}\mathrm{d}x}$ with*

$$\|u_0\|_{L^2_{\frac{1}{L-x}\mathrm{d}x}}\le\delta \quad \text{and} \quad \|u_1\|_{L^2_{\frac{1}{L-x}\mathrm{d}x}}\le\delta,$$

one can find a control input $f\in L^2(0,T;H^{-1}(0,L))$ with supp$(f)\subset(0,T)\times\omega$ such that the solution $u\in C^0([0,T],L^2(0,L))\cap L^2(0,T,H^1(0,L))$ of (1.7) satisfies $u(T,.)=u_1$ in $(0,L)$ and $u\in C^0([0,T],L^2_{\frac{1}{L-x}\mathrm{d}x})$. Furthermore, $f\in L^2_{(T-t)\mathrm{d}t}(0,T,L^2(0,L))$.

In the above result, we used the notation:

$$L^2_{(T-t)\mathrm{d}t}(0,T,L^2(0,L)):=\left\{f\in L^1_{loc}(0,T,L^2(0,L));\ \int_0^T||f(t,\cdot)||^2_{L^2(0,L)}(T-t)\mathrm{d}t<\infty\right\}.$$

Actually, we shall have to investigate the well-posedness of the linearization of (1.7) in the space $L^2_{\frac{1}{L-x}\mathrm{d}x}$ and the well-posedness of the (backward) adjoint system in the "dual space" $L^2_{(L-x)\mathrm{d}x}$. To do this, we shall follow some ideas borrowed from [10], where the well-posedness was investigated in the weighted space $L^2_{\frac{x}{L-x}\mathrm{d}x}$. The needed observability inequality is obtained by the standard compactness-uniqueness argument, some estimate obtained by the multiplier method in [20] (this estimate gives at once the global Kato smoothing effect and some energy estimate in $L^2_{x\mathrm{d}x}(0,L)$, which explains in part the choice of the spaces in Thm. 1.2), and some unique continuation property. The exact controllability is extended to the nonlinear system by using the contraction mapping principle.

When the control is acting far from the endpoint $x=L$, *i.e.* in some interval $\omega=(l_1,l_2)$ with $0<l_1<l_2<L$, then there is no chance to control exactly the state function on (l_2,L) (see *e.g.* [22]). However, it is possible to control the state function on $(0,l_1)$, so that a "regional controllability" can be established:

Theorem 1.3. *Let $T>0$ and $\omega=(l_1,l_2)$ with $0<l_1<l_2<L$. Pick any number $l_1'\in(l_1,l_2)$. Then there exists a number $\delta>0$ such that for any $u_0,u_1\in L^2(0,L)$ satisfying*

$$||u_0||_{L^2(0,L)}\le\delta,\qquad ||u_1||_{L^2(0,L)}\le\delta,$$

one can find a control $f\in L^2(0,T,H^{-1}(0,L))$ with supp$(f)\subset(0,T)\times\omega$ such that the solution $u\in C^0([0,T],L^2(0,L))\cap L^2(0,T,H^1(0,L))$ of (1.7) satisfies

$$u(T,x)=\begin{cases}u_1(x) & \text{if } x\in(0,l_1');\\ 0 & \text{if } x\in(l_2,L).\end{cases}\tag{1.8}$$

The proof of Theorem 1.3 combines Theorem 1.1, a boundary controllability result from [20], and the use of a cutt-off function. The issue whether u may also be controlled in the interval (l_1',l_2) is open. Note that, as for the boundary control, the internal control gives a control of hyperbolic type in the left direction and a control of parabolic type in the right direction.

The paper is outlined as follows. In Section 2, we review some linear estimates from [8, 20] that will be used thereafter. Section 3 is devoted to the proof of Theorems 1.1 and 1.3. It contains the proof of a new Carleman estimate for the KdV equation with some internal observation (Prop. 3.1). In Section 4 we prove the well-posedness of KdV in the weighted spaces $L^2_{x\mathrm{d}x}$ and $L^2_{\frac{1}{L-x}\mathrm{d}x}$ by using semigroup theory, and derive Theorem 1.2.

2. Linear estimates

We review a series of estimates for the system

$$\begin{cases}u_t+(\xi u)_x+u_{xxx}=f(t,x) & \text{in } (0,T)\times(0,L),\\ u(t,0)=u(t,L)=u_x(t,L)=0 & \text{in } (0,T),\\ u(0,x)=u_0(x) & \text{in } (0,L)\end{cases}\tag{2.1}$$

and its adjoint system. Here $f = f(t, x)$ is a function which stands for the control of the system, and $\xi = \xi(t, x)$ is a given function.

2.1. The linearized KdV equation

It was noticed in [20] that the operator $A = -\frac{\partial^3}{\partial x^3} - \frac{\partial}{\partial x}$ with domain

$$\mathcal{D}(A) = \left\{w \in H^3(0, L);\ w(0) = w(L) = w_x(L) = 0\right\} \subseteq L^2(0, L)$$

is the infinitesimal generator of a strongly continuous semigroup of contractions in $L^2(0, L)$. More precisely, the following result was established in [20].

Proposition 2.1. *Let $u_0 \in L^2(0, L)$, $\xi \equiv 1$ and $f \equiv 0$. There exists a unique (mild) solution u of (2.1) with*

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T, H_0^1(0, L)). \tag{2.2}$$

Moreover, there exist positive constants c_1 and c_2 such that for all $u_0 \in L^2(0, L)$

$$\|u\|_{L^2(0,T;H^1(0,L))} + \|u_x(.,0)\|_{L^2(0,T)} \leq c_1 \|u_0\|_{L^2(0,L)}\,, \tag{2.3}$$

$$\|u_0\|^2_{L^2(0,L)} \leq \frac{1}{T} \|u\|^2_{L^2(0,T;L^2(0,L))} + c_2 \|u_x(.,0)\|^2_{L^2(0,T)}\,. \tag{2.4}$$

If in addition $u_0 \in D(A)$, then (2.1) has a unique (classical) solution u in the class

$$u \in C([0, T]; D(A)) \cap C^1([0, T]; L^2(0, L)). \tag{2.5}$$

2.2. The modified KdV equation

We introduce a system related to the adjoint system to (2.1), namely

$$\begin{cases} -v_t - \xi v_x - v_{xxx} = f & \text{in } (0, T) \times (0, L), \\ v(t, 0) = v(t, L) = v_x(t, 0) = 0 & \text{in } (0, T), \\ v(T, x) = 0 & \text{in } (0, L), \end{cases} \tag{2.6}$$

for which we review some estimates borrowed from [8].

2.2.1. Energy Estimates

We introduce the following spaces

$$\begin{aligned} &X_0 := L^2(0, T; H^{-2}(0, L)), \quad X_1 := L^2(0, T; H_0^2(0, L)), \\ &\tilde{X}_0 := L^1(0, T; H^{-1}(0, L)), \quad \tilde{X}_1 := L^1(0, T; (H^3 \cap H_0^2)(0, L)), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &Y_0 := L^2((0, T) \times (0, L)) \cap C^0([0, T]\,; H^{-1}(0, L)), \\ &Y_1 := L^2(0, T; H^4(0, L)) \cap C^0([0, T]\,; H^3(0, L)). \end{aligned} \tag{2.8}$$

The spaces $X_0, X_1, \tilde{X}_0, \tilde{X}_1, Y_0$, and Y_1 are equipped with their natural norms. For instance, the spaces Y_0 and Y_1 are equipped with the norms

$$\|w\|_{Y_0} := \|w\|_{L^2((0,T)\times(0,L))} + \|w\|_{L^\infty(0,T;H^{-1}(0,L))}$$

and

$$\|w\|_{Y_1} := \|w\|_{L^2(0,T;H^4(0,L))} + \|w\|_{L^\infty(0,T;H^3(0,L))}\,.$$

For $\theta \in [0,1]$, we define the complex interpolation spaces (see [2] and [16])

$$X_\theta = (X_0, X_1)_{[\theta]},\ \tilde{X}_\theta = (\tilde{X}_0, \tilde{X}_1)_{[\theta]} \text{ and } Y_\theta = (Y_0, Y_1)_{[\theta]}.$$

Then,

$$X_{1/4} = L^2(0,T;H^{-1}(0,L)), \quad \tilde{X}_{1/4} = L^1(0,T;L^2(0,L)) \tag{2.9}$$

and

$$Y_{1/4} = L^2(0,T;H^1(0,L)) \cap C^0([0,T];L^2(0,L)). \tag{2.10}$$

Furthermore,

$$X_{1/2} = L^2((0,T)\times(0,L)), \quad \tilde{X}_{1/2} = L^1(0,T;H_0^1(0,L)) \tag{2.11}$$

and

$$Y_{1/2} = L^2(0,T;H^2(0,L)) \cap C^0([0,T];H^1(0,L)). \tag{2.12}$$

Proposition 2.2 ([8], Sect. 2.2.2)**.** *Let $\xi \in Y_{\frac14}$ and $f \in X_{\frac14} \cup \tilde{X}_{\frac14} = L^2(0,T;H^{-1}(0,L)) \cup L^1(0,T;L^2(0,L))$. Then the solution v of (2.6) belongs to $Y_{\frac14}$, and there exists some constant $C = C(||\xi||_{Y_{\frac14}}) > 0$ such that*

$$\|v\|_{L^\infty(0,T,L^2(0,L))} + \|v\|_{L^2(0,T;H^1(0,L))} + \|v_x(\cdot,L)\|_{L^2(0,T)} \le C\left(\|\xi\|_{Y_{1/4}}\right)\|f\|_{L^2(0,T;H^{-1}(0,L))} \tag{2.13}$$

and

$$\|v\|_{L^\infty(0,T,L^2(0,L))} + \|v\|_{L^2(0,T;H^1(0,L))} + \|v_x(\cdot,L)\|_{L^2(0,T)} \le C\left(\|\xi\|_{Y_{1/4}}\right)\|f\|_{L^1(0,T;L^2(0,L))}. \tag{2.14}$$

More can be said when $\xi \equiv 0$. Consider the following system

$$\begin{cases} -v_t - v_{xxx} = g & \text{in } (0,T)\times(0,L), \\ v(t,0) = v(t,L) = v_x(t,0) = 0 & \text{in } (0,T), \\ v(T,x) = 0 & \text{in } (0,L). \end{cases} \tag{2.15}$$

Proposition 2.3 ([8], Sect. 2.3.1)**.** *If $g \in X_1 \cup \tilde{X}_1$, then $v \in Y_1$, and there exists some constant $C > 0$ such that*

$$\|v\|_{Y_1} + \|v_x(\cdot,L)\|_{H^1(0,T)} \le C\|g\|_{X_1} \tag{2.16}$$

and

$$\|v\|_{Y_1} + \|v_x(\cdot,L)\|_{H^1(0,T)} \le C\|g\|_{\tilde{X}_1}. \tag{2.17}$$

Proposition 2.4 ([8], Sect. 2.3.2)**.** *If $g \in X_{1/2} \cup \tilde{X}_{1/2}$, then $v \in Y_{1/2}$, and there exists some constant $C > 0$ such that*

$$\|v\|_{Y_{1/2}} + \|v_x(\cdot,L)\|_{H^{1/3}(0,T)} + \|v_{xx}(\cdot,0)\|_{L^2(0,T)} + \|v_{xx}(\cdot,L)\|_{L^2(0,T)} \le C\|g\|_{X_{1/2}} \tag{2.18}$$

and

$$\|v\|_{Y_{1/2}} + \|v_x(\cdot,L)\|_{H^{1/3}(0,T)} + \|v_{xx}(\cdot,0)\|_{L^2(0,T)} + \|v_{xx}(\cdot,L)\|_{L^2(0,T)} \le C\|g\|_{\tilde{X}_{1/2}}. \tag{2.19}$$

3. Null controllability results

This section is devoted to the proof of Theorems 1.1 and 1.3.

3.1. Null controllability of a linearized equation

We first consider the system

$$\begin{cases} u_t + (\xi u)_x + u_{xxx} = 1_\omega f(t,x) & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L), \end{cases} \tag{3.1}$$

where $\xi = \xi(t,x)$ is a given function in $Y_{\frac{1}{4}}$, and $\omega = (l_1, l_2) \subset (0,L)$. Our aim is to prove the null controllability of (3.1). To this end, we shall establish an observability inequality for the corresponding adjoint system

$$\begin{cases} -v_t - \xi(t,x)v_x - v_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ v(t,0) = v(t,L) = v_x(t,0) = 0 & \text{in } (0,T), \\ v(T,x) = v_T(x) & \text{in } (0,L) \end{cases} \tag{3.2}$$

by using some Carleman inequality.

3.1.1. Carleman inequality with internal observation

Assume that $\omega = (l_1, l_2)$ with

$$0 < l_1 < l_2 < L.$$

Pick any function $\psi \in C^3([0,L])$ with

$$\psi > 0 \text{ in } [0,L]; \tag{3.3}$$

$$|\psi'| > 0,\ \ \psi'' < 0,\ \ \text{and } \psi'\psi''' < 0 \text{ in } [0,L]\setminus\omega; \tag{3.4}$$

$$\psi'(0) < 0 \text{ and } \psi'(L) > 0; \tag{3.5}$$

$$\min_{x\in[l_1,l_2]} \psi(x) = \psi(l_3) < \max_{x\in[l_1,l_2]} \psi(x) = \psi(l_1) = \psi(l_2), \qquad \max_{x\in[0,L]} \psi(x) = \psi(0) = \psi(L) \tag{3.6}$$

$$\psi(0) < \frac{4}{3}\psi(l_3), \tag{3.7}$$

for some $l_3 \in (l_1, l_2)$. A convenient function ψ is defined on $[0,L]\setminus\omega$ as

$$\psi(x) = \begin{cases} \varepsilon x^3 - x^2 - x + c_1 & \text{if } x \in [0,l_1], \\ -\varepsilon x^3 + ax + c_2 & \text{if } x \in [l_2, L] \end{cases}$$

with $\varepsilon, a, c_1, c_2 > 0$ conveniently chosen. Note first that $\psi(l_1) = \psi(l_2)$ and $\psi(0) = \psi(L)$ if, and only if,

$$a = (L-l_2)^{-1}(l_1^2 + l_1 - \varepsilon l_2^3 - \varepsilon l_1^3 + \varepsilon L^3), \qquad c_1 = c_2 - \varepsilon L^3 + aL.$$

Then $a > 0$, $c_1 - c_2 > 0$ and the conditions (3.4) and (3.5) hold provided that $0 < \varepsilon \ll 1$. The conditions (3.3) and (3.7) hold for $c_2 \gg 1$. Finally, the condition (3.6) is easy to satisfy.

Set

$$\varphi(t,x) = \frac{\psi(x)}{t(T-t)}\cdot \tag{3.8}$$

For $f \in L^2(0,T;L^2(0,L))$ and $q_0 \in L^2(0,L)$, let q denote the solution of the system

$$q_t + q_{xxx} = f, \quad t\in(0,T),\ x\in(0,L), \tag{3.9}$$

$$q(t,0) = q(t,L) = q_x(t,L) = 0, \quad t \in (0,T), \tag{3.10}$$

$$q(0,x) = q_0(x), \quad x\in(0,L). \tag{3.11}$$

Then the following Carleman inequality holds.

Proposition 3.1. *Pick any $T > 0$. There exist two constants $C > 0$ and $s_0 > 0$ such that any $f \in L^2(0,T;L^2(0,L))$, any $q_0 \in L^2(0,L)$ and any $s \geq s_0$, the solution q of (3.9)–(3.11) fulfills*

$$\int_0^T\int_0^L \left[s\varphi|q_{xx}|^2+(s\varphi)^3|q_x|^2+(s\varphi)^5|q|^2\right]\mathrm{e}^{-2s\varphi}\mathrm{d}x\mathrm{d}t+\int_0^T\left[\left(s\varphi|q_{xx}|^2+(s\varphi)^3|q_x|^2\right)\mathrm{e}^{-2s\varphi}\right]_{|x=0}+\left[s\varphi|q_{xx}|^2\mathrm{e}^{-2s\varphi}\right]_{|x=L}\mathrm{d}t$$
$$\leq C\left(\int_0^T\int_0^L|f|^2\mathrm{e}^{-2s\varphi}\mathrm{d}x\mathrm{d}t+\int_0^T\int_\omega\left[s\varphi|q_{xx}|^2+(s\varphi)^3|q_x|^2+(s\varphi)^5|q|^2\right]\mathrm{e}^{-2s\varphi}\mathrm{d}x\mathrm{d}t\right) \quad (3.12)$$

Actually, we shall need a Carleman estimate for (3.2) with the potential $\xi\in Y_{\frac14}$. Let

$$\tilde\varphi(t,x)=\varphi(t,L-x).$$

Corollary 3.2. *Let $\xi\in Y_{\frac14}$. Then there exist some positive constants $\tilde s_0=\tilde s_0(T,||\xi||_{Y_{\frac14}})$ and $C=C(T,||\xi||_{Y_{\frac14}})$ such that for all $s\geq\tilde s_0$ and all $v_T\in L^2(0,L)$, the solution v of (3.2) fulfills*

$$\int_0^T\int_0^L[s\tilde\varphi|v_{xx}|^2+(s\tilde\varphi)^3|v_x|^2+(s\tilde\varphi)^5|v|^2]\mathrm{e}^{-2s\tilde\varphi}\mathrm{d}x\mathrm{d}t$$
$$\leq C\int_0^T\int_\omega[s\tilde\varphi|v_{xx}|^2+(s\tilde\varphi)^3|v_x|^2+(s\tilde\varphi)^5|v|^2]\mathrm{e}^{-2s\tilde\varphi}\mathrm{d}x\mathrm{d}t. \quad (3.13)$$

Proof of Proposition 3.1. We first assume that $q_0\in D(A)$ and that $f\in C([0,T];D(A))$, so that $q\in C([0,T];D(A))\cap C^1([0,T];L^2(0,L))$. This will be sufficient to legitimate the following computations. The general case ($q_0\in L^2(0,L)$ and $f\in L^2(0,T;L^2(0,L))$) follows by density. Indeed, if we set

$$p(t,x):=\sqrt{\varphi(t,l_3)}\mathrm{e}^{-s\varphi(t,l_3)}q(t,x)$$

then p solves (3.9)–(3.11) with q_0 replaced by 0, and f replaced by

$$\tilde f=\sqrt{\varphi(t,l_3)}\mathrm{e}^{-s\varphi(t,l_3)}f+\left(\frac12\varphi_t(t,l_3)\varphi^{-\frac12}(t,l_3)-s\varphi_t(t,l_3)\sqrt{\varphi(t,l_3)}\right)\mathrm{e}^{-s\varphi(t,l_3)}q,$$

so that (with different constants C)

$$\int_0^T\int_0^L\varphi|q_{xx}|^2\mathrm{e}^{-2s\varphi}\mathrm{d}x\mathrm{d}t\leq C||p||^2_{L^2(0,T,H^2(0,L))}\leq C||\tilde f||^2_{L^2(0,T,L^2(0,L))}\leq C\left(||f||^2_{L^2(0,T,L^2(0,L))}+||q_0||^2_{L^2(0,L)}\right).$$

Since

$$||q||^2_{L^2(0,T,H^1(0,L))}\leq C\left(||f||^2_{L^2(0,T,L^2(0,L))}+||q_0||^2_{L^2(0,L)}\right)$$

we conclude that we can pass to the limit in each term in (3.12), if we take a sequence $\{(q_0^n,f^n)\}_{n\geq0}$ in $\mathcal{D}(A)\times C([0,T],\mathcal{D}(A))$ such that $q_0^n\to q_0$ in $L^2(0,L)$ and $f^n\to f$ in $L^2(0,T,L^2(0,L))$.

Assume from now on that $q_0\in\mathcal{D}(A)$ and that $f\in C([0,T];\mathcal{D}(A))$. Let q denote the solution of (3.9)–(3.11), and let $u=\mathrm{e}^{-s\varphi}q$, $w=\mathrm{e}^{-s\varphi}L(\mathrm{e}^{s\varphi}u)$, where

$$L=\partial_t+\partial_x^3. \quad (3.14)$$

Straightforward computations show that

$$w=Mu:=u_t+u_{xxx}+3s\varphi_xu_{xx}+\left(3s^2\varphi_x^2+3s\varphi_{xx}\right)u_x+\left(s^3\varphi_x^3+3s^2\varphi_x\varphi_{xx}+s\left(\varphi_t+\varphi_{xxx}\right)\right)u. \quad (3.15)$$

Let M_1 and M_2 denote the formal self-adjoint and skew-adjoint parts of the operator M. We readily obtain that

$$M_1 u := 3s\left(\varphi_x u_{xx} + \varphi_{xx} u_x\right) + \left[s\left(\varphi_t + \varphi_{xxx}\right) + s^3\varphi_x^3\right] u, \tag{3.16}$$

$$M_2 u := u_t + u_{xxx} + 3s^2\left(\varphi_x^2 u_x + \varphi_x\varphi_{xx} u\right). \tag{3.17}$$

On the other hand

$$||w||^2 = ||M_1 u||^2 + ||M_2 u||^2 + 2\left(M_1 u, M_2 u\right) \tag{3.18}$$

where $(u,v) = \int_0^T\!\!\int_0^L uv\mathrm{d}x\mathrm{d}t$ and $||w||^2 = (w,w)$. From now on, for the sake of simplicity, we write $\iint u$ (resp. $\int u\big|_0^L$) instead of $\int_0^T\!\!\int_0^L u(t,x)\mathrm{d}x\mathrm{d}t$ (resp. $\int_0^T u(t,x)\big|_{x=0}^L \mathrm{d}t$). The proof of the Carleman inequality follows the same pattern as in [17, 23]. The first step provides an exact computation of the scalar product (M_1u, M_2u), whereas the second step gives the estimates obtained thanks to the pseudoconvexity conditions (3.3)–(3.7).

Step 1. Exact computation of the scalar product in (3.18).
Write

$$2(M_1u, M_2u) = 2\iint [s(\varphi_t + \varphi_{xxx}) + s^3\varphi_x^3]uM_2u + 2\iint 3s(\varphi_x u_{xx} + \varphi_{xx}u_x)M_2u =: I_1 + I_2.$$

Let

$$\alpha := s(\varphi_t + \varphi_{xxx}) + s^3\varphi_x^3. \tag{3.19}$$

Using (3.17), we decompose I_1 into

$$I_1 = \iint 2\alpha u u_t + \iint 2\alpha u u_{xxx} + 3s^2\iint 2\alpha u(\varphi_x^2 u_x + \varphi_x\varphi_{xx}u).$$

Integrating by parts with respect to t or x, noticing that $u_{|x=0} = u_{|x=L} = u_{x|x=L} = 0$, and that $u_{|t=0} = u_{|t=T} = 0$ by (3.3), we obtain that

$$\begin{aligned} I_1 &= -\iint \alpha_t u^2 + (3\iint \alpha_x u_x^2 - \iint \alpha_{xxx}u^2 - \int \alpha u_x^2\big|_0^L) - 3s^2\iint \varphi_x^2\alpha_x u^2 \\ &= -\iint (\alpha_t + \alpha_{xxx} + 3s^2\varphi_x^2\alpha_x)u^2 + 3\iint \alpha_x u_x^2 - \int \alpha u_x^2\big|_0^L. \end{aligned} \tag{3.20}$$

Next, we compute

$$I_2 = 2\iint 3s(\varphi_x u_{xx} + \varphi_{xx}u_x)(u_t + u_{xxx} + 3s^2(\varphi_x^2 u_x + \varphi_x\varphi_{xx}u)).$$

Performing integrations by parts, we obtain successively

$$\begin{aligned} &2\iint (\varphi_x u_{xx} + \varphi_{xx}u_x)u_t = \iint \varphi_{xt}u_x^2, \\ &2\iint (\varphi_x u_{xx} + \varphi_{xx}u_x)u_{xxx} = -3\iint \varphi_{xx}u_{xx}^2 + \iint \varphi_{4x}u_x^2 + \int \left(\varphi_x u_{xx}^2 - \varphi_{3x}u_x^2 + 2\varphi_{xx}u_{xx}u_x\right)\big|_0^L, \end{aligned}$$

and

$$2\iint (\varphi_x u_{xx} + \varphi_{xx}u_x)\left(\varphi_x^2 u_x + \varphi_x\varphi_{xx}u\right) = -3\iint \varphi_x^2\varphi_{xx}u_x^2 + \iint \left[\left(\varphi_x^2\varphi_{xx}\right)_{xx} - \left(\varphi_x\varphi_{xx}^2\right)_x\right]u^2 + \int \varphi_x^3 u_x^2\big|_0^L.$$

Thus

$$\begin{aligned} I_2 = &-9s\iint \varphi_{xx}u_{xx}^2 + \iint \left[-27s^3\varphi_x^2\varphi_{xx} + 3s(\varphi_{xt} + \varphi_{4x})\right]u_x^2 \\ &+ \iint 9s^3\left[\left(\varphi_x^2\varphi_{xx}\right)_{xx} - \left(\varphi_x\varphi_{xx}^2\right)_x\right]u^2 + \int \left[3s\left(\varphi_x u_{xx}^2 - \varphi_{3x}u_x^2 + 2\varphi_{xx}u_x u_{xx}\right) + 9s^3\varphi_x^3 u_x^2\right]\big|_0^L \end{aligned} \tag{3.21}$$

Gathering together (3.20) and (3.21), we infer that

$$\begin{aligned}2(M_1u, M_2u) = &\iint \left[-\left(\alpha_t + \alpha_{xxx} + 3s^2\varphi_x^2\alpha_x\right) + 9s^3\left(\left(\varphi_x^2\varphi_{xx}\right)_{xx} - \left(\varphi_x\varphi_{xx}^2\right)_x\right)\right]u^2\\ &+ \iint \left[3\alpha_x - 27s^3\varphi_x^2\varphi_{xx} + 3s\left(\varphi_{xt} + \varphi_{4x}\right)\right]u_x^2 - 9s\iint \varphi_{xx}u_{xx}^2\\ &+ \int \left[3s\varphi_x u_{xx}^2 + \left(9s^3\varphi_x^3 - 3s\varphi_{xxx} - \alpha\right)u_x^2 + 2\varphi_{xx}u_xu_{xx}\right]\Big|_0^L \end{aligned} \tag{3.22}$$

Step 2. Estimation of each term in (3.22).
The estimates are given in a series of claims.

Claim 3.3. There exist some constants $s_1 > 0$ and $C_1 > 1$ such that for all $s \geq s_1$, we have

$$\iint \left[-\left(\alpha_t + \alpha_{xxx} + 3s^2\varphi_x^2\alpha_x\right) + 9s^3\left(\left(\varphi_x^2\varphi_{xx}\right)_{xx} - \left(\varphi_x\varphi_{xx}^2\right)_x\right)\right]u^2 \geq C_1^{-1}\iint (s\varphi)^5u^2 - C_1\int_0^T\int_\omega (s\varphi)^5u^2.$$

From (3.19), we see that the term in s^5 in the brackets reads

$$-3s^5\varphi_x^2\left(\varphi_x^3\right)_x = -9s^5\varphi_x^4\varphi_{xx} = -9s^5\frac{(\psi')^4\psi''}{t^5(T-t)^5}.$$

We infer from (3.4) that for some $\kappa_1 > 0$ and all $s > 0$

$$-9s^5\varphi_x^4\varphi_{xx} \geq \kappa_1(s\varphi)^5 \qquad (t,x) \in (0,T)\times([0,L]\setminus\omega).$$

On the other hand, we have for some $\kappa_2 > 0$ and all $s > 0$

$$\begin{aligned}|\alpha_t| + |\alpha_{xxx}| + \left|9s^3\left(\left(\varphi_x^2\varphi_{xx}\right)_{xx} - \left(\varphi_x\varphi_{xx}^2\right)_x\right)\right| &\leq \kappa_2 s^3\varphi^4 \quad (t,x) \in (0,T)\times(0,L),\\ \left|3s^2\varphi_x^2\alpha_x\right| &\leq \kappa_2(s\varphi)^5 \quad (t,x) \in (0,T)\times\omega.\end{aligned}$$

Claim 3.3 follows then for all $s > s_1$ with s_1 large enough and some $C_1 > 1$.

Claim 3.4. There exist some constants $s_2 > 0$ and $C_2 > 1$ such that for all $s \geq s_2$, we have

$$\iint \left[3\alpha_x - 27s^3\varphi_x^2\varphi_{xx} + 3s(\varphi_{xt} + \varphi_{4x})\right]u_x^2 \geq C_2^{-1}\iint (s\varphi)^3u_x^2 - C_2\int_0^T\int_\omega (s\varphi)^3u_x^2. \tag{3.23}$$

Indeed, the term in s^3 in the brackets is found to be

$$-18s^3\varphi_x^2\varphi_{xx} \geq \kappa_3(s\varphi)^3 \quad (t,x) \in (0,T)\times([0,L]\setminus\omega)$$

for some $\kappa_3 > 0$ and all $s > 0$, by (3.4). On the other hand, we have for some $\kappa_4 > 0$ and all $s > 0$

$$\begin{aligned}|6s(\varphi_{tx} + \varphi_{4x})| &\leq \kappa_4 s\varphi^2 \quad (t,x) \in (0,T)\times(0,L),\\ |18s^3\varphi_x^2\varphi_{xx}| &\leq \kappa_4(s\varphi)^3 \quad (t,x) \in (0,T)\times\omega.\end{aligned}$$

Claim 3.4 follows for all $s \geq s_2$ with s_2 large enough and some $C_2 > 1$.

Claim 3.5. There exist some constants $s_3 > 0$ and $C_3 > 1$ such that for all $s \geq s_3$, we have

$$-9s\iint \varphi_{xx}u_{xx}^2 \geq C_3^{-1}\iint s\varphi u_{xx}^2 - C_3\int_0^T\int_\omega s\varphi u_{xx}^2. \tag{3.24}$$

Claim 3.5 is clear, for $\psi'' < 0$ on $[0,L]\setminus\omega$.

Claim 3.6. There exist some constants $s_4 > 0$ and $C_4 > 1$ such that for all $s \geq s_4$, we have

$$\int \left[3s\varphi_x u_{xx}^2 + \left(9s^3\varphi_x^3 - 3s\varphi_{xxx} - \alpha\right)u_x^2 + 2\varphi_{xx}u_x u_{xx}\right]\Big|_0^L \geq C_4^{-1}\int_0^T \left[\left(s\varphi u_{xx}^2\right)_{|x=0} + \left(s\varphi u_{xx}^2\right)_{|x=L} + \left(s^3\varphi^3 u_x^2\right)_{|x=0}\right]\mathrm{d}t.$$

Since $u_{x|x=L} = 0$ and

$$\left[\left(9s^3\varphi_x^3 - 3s\varphi_{xxx} - \alpha\right)u_x^2\right]_{|x=0} = \left[\left(8s^3\varphi_x^3 - s\left(\varphi_t + 4\varphi_{xxx}\right)\right)u_x^2\right]_{|x=0},$$

we obtain with (3.5) for $s \geq s_4$ with s_4 large enough,

$$\left[\left(9s^3\varphi_x^3 - 3s\varphi_{xxx} - \alpha\right)u_x^2\right]\Big|_0^L \geq \kappa_5\left[(s\varphi)^3 u_x^2\right]_{|x=0}$$

and

$$3s\varphi_x u_{xx}^2|_0^L \geq \kappa_6\left(\left[s\varphi u_{xx}^2\right]_{|x=0} + \left[s\varphi u_{xx}^2\right]_{|x=L}\right)$$

for some constant $\kappa_5, \kappa_6 > 0$. Finally

$$|[2s\varphi_{xx}u_x u_{xx}]_{x=0}| \leq \frac{\kappa_6}{2}\left[s\varphi u_{xx}^2\right]_{|x=0} + \kappa_7\left[s\varphi u_x^2\right]_{|x=0}$$

for some constant $\kappa_7 > 0$. Since $s\varphi(t,0) \ll (s\varphi)^3(t,0)$ for $s \gg 1$, Claim 3.6 follows.

We infer from Claims 3.3, 3.4, 3.5, and 3.6 that for some positive constants s_0, C and all $s \geq s_0$

$$\iint\left[(s\varphi)^5|u|^2 + (s\varphi)^3|u_x|^2 + s\varphi|u_{xx}|^2\right] + \int_0^T\left[\left(s\varphi u_{xx}^2\right)_{|x=0} + \left(s\varphi u_{xx}^2\right)_{|x=L} + \left(s^3\varphi^3 u_x^2\right)_{|x=0}\right]\mathrm{d}t \leq C\left(\iint|w|^2 + \int_0^T\int_\omega\left[(s\varphi)^5|u|^2 + (s\varphi)^3|u_x|^2 + s\varphi|u_{xx}|^2\right]\right). \quad (3.25)$$

Replacing u by $\mathrm{e}^{-s\varphi}q$ yields (3.12).

□

Proof of Corollary 3.2. Note first that for $\xi \in Y_{\frac{1}{4}}$ and $v_T \in L^2(0,L)$, one can prove that (3.2) has a unique solution $v \in Y_{\frac{1}{4}}$, by using the contraction mapping principle for the integral equation. Corollary 3.2 follows from Proposition 3.1 by taking $q_0(x) = v_T(L-x)$, $q(t,x) = v(T-t, L-x)$, and $f(t,x) = -\xi(T-t, L-x)q_x(t,x)$, assuming first that $\xi \in Y_{\frac{1}{4}} \cap L^\infty(Q)$ (so that $f \in L^2(Q)$). Indeed, with $u = \mathrm{e}^{-s\varphi}q$,

$$w = \mathrm{e}^{-s\varphi}L(\mathrm{e}^{s\varphi}u) = -\xi(T-t, L-x)(u_x + s\varphi_x u),$$

so that

$$\begin{aligned}\iint|w|^2\mathrm{d}x\mathrm{d}t &\leq C\int_0^T\int_0^L|\xi(T-t,L-x)|^2\left(|u_x|^2 + |s\varphi_x u|^2\right)\mathrm{d}x\mathrm{d}t\\ &\leq C\int_0^T||\xi(T-t)||^2_{L^2(0,L)}\left(||u_x||^2_{L^\infty(0,L)} + ||s\varphi_x u||^2_{L^\infty(0,L)}\right)\mathrm{d}t\\ &\leq C||\xi||^2_{L^\infty(0,T,L^2(0,L))}\int_0^T\int_0^L\left[u_x^2 + u_{xx}^2 + \frac{s^2}{t^2(T-t)^2}\left(u^2 + u_x^2\right)\right]\mathrm{d}x.\end{aligned} \quad (3.26)$$

Combining (3.25) with (3.26), picking $s \gg 1$, and replacing again u by $\mathrm{e}^{-s\varphi}v(T-t, L-x)$ yields (3.13). The result for $\xi \in Y_{\frac{1}{4}}$ follows by density. □

3.1.2. Internal observation

We go back to the adjoint system (3.2). Our next goal is to remove the terms v_{xx} and v_x from the r.h.s. of (3.13). In addition to the weight $\tilde{\varphi}(t,x) = \frac{1}{t(T-t)}\psi(L-x)$, we introduce the functions

$$\hat{\varphi}(t) = \frac{1}{t(T-t)} \max_{x\in[0,L]} \psi(x) = \frac{\psi(0)}{t(T-t)} \text{ and } \check{\varphi}(t) = \frac{1}{t(T-t)} \min_{x\in[0,L]} \psi(x) = \frac{\psi(l_3)}{t(T-t)}, \tag{3.27}$$

where we used (3.6). By (3.7), we have

$$\hat{\varphi}(t) < \frac{4}{3}\check{\varphi}(t), \quad t \in (0,T). \tag{3.28}$$

Lemma 3.7. *Let $0 < l_1 < l_2 < L$, $\xi \in Y_{\frac{1}{4}}$, and $\tilde{s}_0$ be as in Corollary 3.2. Then there exists a constant $C = C(T, ||\xi||_{Y_{\frac{1}{4}}}) > 0$ such that for any $s \geq \tilde{s}_0$ and any $v_T \in L^2(0,L)$, the solution v of (3.2) satisfies*

$$\int_Q \left\{(s\check{\varphi})^5|v|^2 + (s\check{\varphi})^3|v_x|^2 + s\check{\varphi}|v_{xx}|^2\right\} \mathrm{e}^{-2s\hat{\varphi}}\mathrm{d}x\mathrm{d}t \leq C_1 s^{10}\int_0^T \mathrm{e}^{s(6\hat{\varphi}-8\check{\varphi})}\check{\varphi}^{31} \left\|v(t,\cdot)\right\|^2_{L^2(\omega)} \mathrm{d}t, \tag{3.29}$$

where $Q = (0,T) \times (0,L)$ and $\omega = (l_1, l_2) \subset (0,L)$.

Proof. We follow the same approach as in [8]. From (3.13) and (3.27)–(3.28), we first obtain

$$\int_Q \left\{s^5\check{\varphi}^5|v|^2 + s^3\check{\varphi}^3|v_x|^2 + s\check{\varphi}|v_{xx}|^2\right\} \mathrm{e}^{-2s\hat{\varphi}}\mathrm{d}x\mathrm{d}t$$
$$\leq C\int_0^T\int_\omega \left\{s^5\check{\varphi}^5\left|v\right|^2 + s^3\check{\varphi}^3|v_x|^2 + s\check{\varphi}|v_{xx}|^2\right\} \mathrm{e}^{-2s\check{\varphi}}\mathrm{d}x\mathrm{d}t =: C(I_0 + I_1 + I_2). \tag{3.30}$$

Since $\check{\varphi}$ and $\hat{\varphi}$ do not depend on x, we clearly have that

$$I_1 \leq s^3 \int_0^T \check{\varphi}^3\mathrm{e}^{-2s\check{\varphi}} \left\|v(t,\cdot)\right\|^2_{H^1(\omega)} \mathrm{d}t \tag{3.31}$$

and

$$I_2 \leq s \int_0^T \check{\varphi}\mathrm{e}^{-2s\check{\varphi}} \left\|v(t,\cdot)\right\|^2_{H^2(\omega)} \mathrm{d}t. \tag{3.32}$$

Using interpolation in the Sobolev spaces $H^s(\omega)$ $(s \geq 0)$, we obtain for some positive constants K_1, K_2

$$\left\|v(t,\cdot)\right\|_{H^1(\omega)} \leq K_1 \left\|v(t,\cdot)\right\|^{3/8}_{H^{8/3}(\omega)} \left\|v(t,\cdot)\right\|^{5/8}_{L^2(\omega)} \tag{3.33}$$

and

$$\left\|v(t,\cdot)\right\|_{H^2(\omega)} \leq K_2 \left\|v(t,\cdot)\right\|^{3/4}_{H^{8/3}(\omega)} \left\|v(t,\cdot)\right\|^{1/4}_{L^2(\omega)}. \tag{3.34}$$

Replacing (3.33) and (3.34) in (3.31) and (3.32), respectively, yields

$$I_1 \leq Cs^3 \int_0^T \check{\varphi}^3\mathrm{e}^{-2s\check{\varphi}} \left\|v(t,\cdot)\right\|^{3/4}_{H^{8/3}(\omega)} \left\|v(t,\cdot)\right\|^{5/4}_{L^2(\omega)} \mathrm{d}t \tag{3.35}$$

and

$$I_2 \leq Cs \int_0^T \check{\varphi}\mathrm{e}^{-2s\check{\varphi}} \left\|v(t,\cdot)\right\|^{3/2}_{H^{8/3}(\omega)} \left\|v(t,\cdot)\right\|^{1/2}_{L^2(\omega)} \mathrm{d}t. \tag{3.36}$$

Next, an application of Young inequality in (3.35) and (3.36) gives

$$I_1 \le Cs^3 \int_0^T \check{\varphi}^3 e^{-2s\check{\varphi}} e^{-\frac{3}{4}s\hat{\varphi}} e^{\frac{3}{4}s\hat{\varphi}} \check{\varphi}^{-\frac{27}{8}} \check{\varphi}^{\frac{27}{8}} \|v(t,\cdot)\|^{3/4}_{H^{8/3}(\omega)} \|v(t,\cdot)\|^{5/4}_{L^2(\omega)} \,dt$$
$$\le C_\epsilon s^6 \int_0^T e^{s\left(\frac{6}{5}\hat{\varphi}-\frac{16}{5}\check{\varphi}\right)} \check{\varphi}^{51/5} \|v(t,\cdot)\|^2_{L^2(\omega)} \,dt + \epsilon s^{-2} \int_0^T e^{-2s\hat{\varphi}} \check{\varphi}^{-9} \|v(t,\cdot)\|^2_{H^{8/3}(\omega)} \,dt \tag{3.37}$$

and

$$I_2 \le Cs \int_0^T e^{-2s\check{\varphi}} e^{-\frac{3}{2}s\hat{\varphi}} e^{\frac{3}{2}s\hat{\varphi}} \check{\varphi}^{-\frac{27}{4}} \check{\varphi}^{\frac{31}{4}} \|v(t,\cdot)\|^{3/2}_{H^{8/3}(\omega)} \|v(t,\cdot)\|^{1/2}_{L^2(\omega)} \,dt$$
$$\le C_\epsilon s^{10} \int_0^T e^{s(6\hat{\varphi}-8\check{\varphi})} \check{\varphi}^{31} \|v(t,\cdot)\|^2_{L^2(\omega)} \,dt + \epsilon s^{-2} \int_0^T e^{-2s\hat{\varphi}} \check{\varphi}^{-9} \|v(t,\cdot)\|^2_{H^{8/3}(\omega)} \,dt, \tag{3.38}$$

for any $\epsilon > 0$. Note that

$$I_0 + s^6 \int_0^T e^{s(\frac{6}{5}\hat{\varphi}-\frac{16}{5}\check{\varphi})} \check{\varphi}^{51/5} \|v(t,\cdot)\|^2_{L^2(\omega)} \,dt \le Cs^{10} \int_0^T e^{s(6\hat{\varphi}-8\check{\varphi})} \check{\varphi}^{31} \|v(t,\cdot)\|^2_{L^2(\omega)} \,dt. \tag{3.39}$$

Gathering together (3.30) and (3.37)–(3.39), we obtain

$$\int_Q \left\{ s^5\check{\varphi}^5|v|^2 + s^3\check{\varphi}^3|v_x|^2 + s\check{\varphi}|v_{xx}|^2 \right\} e^{-2s\hat{\varphi}} dxdt$$
$$\le Cs^{10} \int_0^T e^{s(6\hat{\varphi}-8\check{\varphi})} \check{\varphi}^{31} \|v(t,\cdot)\|^2_{L^2(\omega)} \,dt + 2\epsilon s^{-2} \int_0^T e^{-2s\hat{\varphi}} \check{\varphi}^{-9} \|v(t,\cdot)\|^2_{H^{8/3}(\omega)} \,dt. \tag{3.40}$$

It remains to estimate the integral term

$$\int_0^T e^{-2s\hat{\varphi}} \check{\varphi}^{-9} \|v(t,\cdot)\|^2_{H^{8/3}(\omega)} \,dt.$$

This is done by a bootstrap argument based on the smoothing effect of the KdV equation.

Let $v_1(t,x) := \theta_1(t)v(t,x)$ with

$$\theta_1(t) = \exp(-s\hat{\varphi})\check{\varphi}^{-\frac{1}{2}}.$$

Then v_1 satisfies the system

$$\begin{cases} -v_{1t} - v_{1xxx} = f_1 := \xi\theta_1 v_x - \theta_{1t} v & \text{in } (0,T)\times(0,L), \\ v_1(t,0) = v_1(t,L) = v_{1x}(t,0) = 0 & \text{in } (0,T), \\ v_1(T,x) = 0 & \text{in } (0,L). \end{cases} \tag{3.41}$$

Now, observe that, since $v_x(t,0) = 0$, $\xi \in L^\infty(0,T,L^2(0,L))$ and $|\theta_{1t}| \le Cs\check{\varphi}^{\frac{3}{2}} \exp(-s\hat{\varphi})$, we have

$$\|f_1\|^2_{L^2((0,T)\times(0,L))} \le C||\xi||^2_{L^\infty(0,T,L^2(0,L))} \int_0^T e^{-2s\hat{\varphi}} ||v_x||^2_{L^\infty(0,L)} dt + C \int_Q e^{-2s\hat{\varphi}} s^2\check{\varphi}^3|v|^2 dxdt$$
$$\le C \int_Q \left\{ s^2\check{\varphi}^3|v|^2 + s|v_x|^2 + s^{-1}|v_{xx}|^2 \right\} e^{-2s\hat{\varphi}} dxdt \tag{3.42}$$

for some constant $C > 0$ and all $s \ge s_0$. Moreover, by Proposition 2.4, $v_1 \in Y_{1/2}$. Then, interpolating between $L^2(0,T;H^2(0,L))$ and $L^\infty(0,T;H^1(0,L))$, we infer that $v_1 \in L^4(0,T;H^{3/2}(0,L))$ and

$$\|v_1\|_{L^4(0,T;H^{3/2}(0,L))} \le C \|f_1\|_{L^2((0,T)\times(0,L))}. \tag{3.43}$$

Let $v_2(t,x) := \theta_2(t)v(t,x)$ with
$$\theta_2 = \exp(-s\hat{\varphi})\check{\varphi}^{-\frac{5}{2}}.$$
Then v_2 satisfies system (3.41) with f_1 replaced by
$$f_2 := \xi\theta_2\theta_1^{-1}v_{1x} - \theta_{2t}\theta_1^{-1}v_1.$$

Observe that
$$\left|\theta_2\theta_1^{-1}\right| + \left|\theta_{2t}\theta_1^{-1}\right| \le Cs.$$
On the other hand, since $\xi \in L^4(0,T;H^{\frac{1}{2}}(0,L))$ and $v_{1x} \in L^4(0,T;H^{\frac{1}{2}}(0,L))$ by (3.43), we infer that $\xi v_{1x} \in L^2(0,T;H^{1/3}(0,L))$. Indeed, the product of two functions in $H^{\frac{1}{2}}(0,L)$ belongs to $H^s(0,L)$ for any $s<1/2$, and in particular to $H^{\frac{1}{3}}(0,L)$. (This fact is proved for $H^{\frac{1}{2}}(\mathbb{R})$ in Theorem 8.3.1 of [11], and a similar result can be deduced for $H^{\frac{1}{2}}(0,L)$ by using a classical extension argument.) Thus, we obtain
$$\|f_2\|_{L^2(0,T;H^{1/3}(0,L))} \le Cs\,\|v_1\|_{L^4(0,T;H^{3/2}(0,L))}. \tag{3.44}$$

Interpolating between (2.16) and (2.18), we have that $v_2 \in L^2(0,T;H^{7/3}(0,L)) \cap L^\infty(0,T;H^{4/3}(0,L))$ with
$$\|v_2\|_{L^2(0,T;H^{7/3}(0,L))\cap L^\infty(0,T;H^{4/3}(0,L))} \le C\,\|f_2\|_{L^2(0,T;H^{1/3}(0,L))}. \tag{3.45}$$

Finally, let $v_3 := \theta_3(t)v(t,x)$ with
$$\theta_3(t) = \exp(-s\hat{\varphi})\check{\varphi}^{-\frac{9}{2}}.$$
Then v_3 satisfies system (3.41) with f_1 replaced by
$$f_3 := \xi\theta_3\theta_2^{-1}v_{2x} - \theta_{3t}\theta_2^{-1}v_2.$$
Again
$$\left|\theta_3\theta_2^{-1}\right| + \left|\theta_{3t}\theta_2^{-1}\right| \le Cs.$$
Interpolating again between (2.16) and (2.18), we have that
$$\|v_3\|_{L^2(0,T;H^{8/3}(0,L))\cap L^\infty(0,T;H^{5/3}(0,L))} \le C\,\|f_3\|_{L^2(0,T;H^{2/3}(0,L))}. \tag{3.46}$$
Since $\xi \in Y_{\frac{1}{4}}$, we have that $\xi \in L^3(0,T;H^{\frac{2}{3}}(0,L))$. On the other hand, by (3.45),
$$v_{2x} \in L^2\left(0,T;H^{4/3}(0,L)\right) \cap L^\infty\left(0,T;H^{1/3}(0,L)\right).$$
It follows that $v_{2x} \in L^6(0,T,H^{\frac{2}{3}}(0,L))$. Since $H^{\frac{2}{3}}(0,L)$ is an algebra, we conclude that $\xi v_{2x} \in L^2(0,T,H^{\frac{2}{3}}(0,L))$. Therefore
$$\|f_3\|_{L^2(0,T;H^{2/3}(0,L))} \le Cs\,\|v_2\|_{L^2(0,T;H^{7/3}(0,L))\cap L^\infty(0,T;H^{4/3}(0,L))}. \tag{3.47}$$
Thus we infer from (3.42)–(3.47) that for some constants $C_1, C_2 > 0$ and all $s \ge s_0$
$$\begin{aligned}\|v_3\|^2_{L^2(0,T;H^{8/3}(0,L))} &\le C_1 s^4 ||f_1||^2_{L^2((0,T)\times(0,L))}\\ &\le C_2\int_Q \left\{s^6\check{\varphi}^3|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2\right\} \mathrm{e}^{-2s\hat{\varphi}}\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.48}$$

Hence, replacing $v_3 = \exp(-s\hat{\varphi})\check{\varphi}^{-\frac{9}{2}}v$ in (3.48) yields for some constant $C_3 > 0$
$$\int_0^T \mathrm{e}^{-2s\hat{\varphi}}\check{\varphi}^{-9}\,\|v(t,\cdot)\|^2_{H^{8/3}(\omega)}\,\mathrm{d}t \le C_3 s^2\int_Q \left\{(s\check{\varphi})^5|v|^2 + (s\check{\varphi})^3|v_x|^2 + s\check{\varphi}|v_{xx}|^2\right\}\mathrm{e}^{-2s\hat{\varphi}}\mathrm{d}x\mathrm{d}t. \tag{3.49}$$

Then, picking $\epsilon = 1/(4C_3)$ in (3.40) results in

$$\int_Q s\check{\varphi}\mathrm{e}^{-2s\hat{\varphi}}\left\{s^4\check{\varphi}^4|v|^2 + s^2\check{\varphi}^2|v_x|^2 + |v_{xx}|^2\right\}\mathrm{d}x\mathrm{d}t \le C_4 s^{10}\int_0^T \mathrm{e}^{s(6\hat{\varphi}-8\check{\varphi})}\check{\varphi}^{31}\,\|v(t,\cdot)\|^2_{L^2(\omega)}\,\mathrm{d}t$$

for all $s \ge \tilde{s}_0$ and some positive constant $C_4 = C_4(T, ||\xi||_{Y_{\frac{1}{4}}})$. □

We are in a position to prove the null controllability of system (3.1).

Theorem 3.8. *Let $T > 0$. Then for any $\xi \in Y_{1/4}$ and any $u_0 \in L^2(0, L)$, one can find a control $f \in L^2((0, T) \times \omega)$ such that the solution u of (3.1) fulfills $u(T, \cdot) = 0$.*

Proof. Scaling in (3.2) by v and $(L - x)v$, integrating over $(0, L)$ and adding the two resulting equations, we obtain

$$-\frac{1}{2}\frac{d}{\mathrm{d}t}\int_0^L (1 + L - x)v^2\mathrm{d}x + \frac{1}{2}v_x^2(L, t) + \frac{3}{2}\int_0^L v_x^2\mathrm{d}x = \int_0^L (1 + L - x)\xi v v_x\mathrm{d}x. \tag{3.50}$$

We estimate the term in the r.h.s. of (3.50) as

$$\begin{aligned}\left|\int_0^L (1 + L - x)\xi v v_x\mathrm{d}x\right| &\le ||(1 + L - x)\xi||_{L^\infty(0,L)}||v_x||_{L^2(0,L)}||v||_{L^2(0,L)}\\ &\le \frac{1}{2}||v_x||^2_{L^2(0,L)} + \frac{1}{2}(1 + L)^2||\xi||^2_{L^\infty(0,L)}||v||^2_{L^2(0,L)}\\ &\le \frac{1}{2}||v_x||^2_{L^2(0,L)} + C(L)||\xi||^2_{H^1(0,L)}||v||^2_{L^2(0,L)}\end{aligned} \tag{3.51}$$

where $C(L) > 0$. Combining (3.50)–(3.51) and using Gronwall lemma, we obtain

$$\max_{t\in[0,T]}||v(t)||^2_{L^2(0,L)} + ||v_x||^2_{L^2(0,T,L^2(0,L))} \le \widehat{C}(L, ||\xi||_{L^2(0,T,H^1(0,L))})||v_T||^2_{L^2(0,L)} \tag{3.52}$$

for some constant $\widehat{C}(L, ||\xi||_{L^2(0,T,H^1(0,L))}) > 0$ which is nondecreasing in its second variable. Replacing $v(t)$ by $v(0)$ and v_T by $v(\tau)$ for $T/3 < \tau < 2T/3$ in (3.52), and integrating over $\tau \in (T/3, 2T/3)$, we obtain that

$$||v(0)||^2_{L^2(0,L)} \le \frac{3}{T}\widehat{C}(L, ||\xi||_{L^2(0,T,H^1(0,L))})\int_{\frac{T}{3}}^{\frac{2T}{3}} ||v(\tau)||^2_{L^2(0,L)}\mathrm{d}\tau. \tag{3.53}$$

Combining (3.53) with Lemma 3.7 for a fixed value of $s \ge \tilde{s}_0$, we derive the following observability inequality

$$\int_0^L |v(0, x)|^2\,\mathrm{d}x \le C_*\int_0^T \|v(t, \cdot)\|^2_{L^2(\omega)}\,\mathrm{d}t \tag{3.54}$$

where $C_* = C_*(T, L, ||\xi||_{Y_{1/4}}) > 0$ is nondecreasing in its last variable. Using (3.54), we can deduce the existence of a function $f \in L^2((0, T) \times \omega)$ as in Theorem 3.8 proceeding as follows.

On $L^2(0, L)$, we define the norm

$$\|v_T\|_B := \|v\|_{L^2((0,T)\times\omega)}\,,$$

where v is the solution of (3.2) associated with v_T. The fact that $||\cdot||_B$ is a norm comes from (3.54) applied on (t, T) for $0 < t < T$.

Let B denote the completion of $L^2(0, L)$ with respect to the above norm. We define a functional J on B by

$$J(v_T) := \frac{1}{2}\|v_T\|^2_B + \int_0^L v(0, x)u_0(x)\mathrm{d}x.$$

From (3.54) we infer that J is well defined and continuous on B. As it is strictly convex and coercive, it admits a unique minimum v_T^*, characterized by the Euler-Lagrange equation

$$\int_0^T\int_\omega v^* w \mathrm{d}x\mathrm{d}t + \int_0^L w(0,x)u_0(x)\mathrm{d}x = 0, \qquad \forall w_T \in B, \tag{3.55}$$

where w (resp. v^*) denotes the solution of (3.2) associated with $w_T \in B$ (resp. $v_T^* \in B$). Define $f \in L^2((0,T)\times\omega)$ by

$$f := 1_\omega v^*, \tag{3.56}$$

and let u denote the solution of (3.1) associated with u_0 and f. Multiplying (3.1) by $w(t,x)$ and integrating by parts, we obtain for all $w_T \in L^2(0,L)$

$$\int_0^L u(T,x)w_T\mathrm{d}x = \int_0^L u_0(x)w(0,x)\mathrm{d}x + \int_0^T\int_\omega v^* w\mathrm{d}x\mathrm{d}t = 0, \tag{3.57}$$

where the second equality follows from (3.55). Therefore $u(T,\cdot) = 0$. Finally, letting $w_T = v_T^*$ in (3.55) and using (3.54), we obtain

$$\int_0^T\int_\omega |f|^2\mathrm{d}x\mathrm{d}t \le C_* \int_0^L |u_0(x)|^2\mathrm{d}x. \tag{3.58}$$

□

3.2. Null controllability of the nonlinear equation

In this section we prove Theorem 1.1. This is done by using a fixed-point argument.

3.2.1. Proof of Theorem 1.1

Consider u and $\bar{u}$ fulfilling system (1.5) and (1.4), respectively. Then $q = u - \bar{u}$ satisfies

$$\begin{cases} q_t + q_x + (\frac{q^2}{2} + \bar{u}q)_x + q_{xxx} = 1_\omega f(t,x) & \text{in } (0,T)\times(0,L),\\ q(t,0) = q(t,L) = q_x(t,L) = 0 & \text{in } (0,T),\\ q(0,x) = q_0(x) := u_0(x) - \bar{u}_0(x) & \text{in } (0,L). \end{cases} \tag{3.59}$$

The objective is to find f such that the solution q of (3.59) satisfies

$$q(T,\cdot) = 0.$$

Given $\xi \in Y_{\frac{1}{4}}$ and $q_0 := u_0 - \bar{u}_0 \in L^2(0,L)$, we consider the control problem

$$q_t + q_x + (\xi q)_x + q_{xxx} = 1_\omega f(t,x) \ \text{ in } (0,T)\times(0,L), \tag{3.60}$$

$$q(t,0) = q(t,L) = q_x(t,L) = 0 \ \text{ in } (0,T), \tag{3.61}$$

$$q(0,x) = q_0(x) \ \text{ in } (0,L). \tag{3.62}$$

Proceeding as in the proof of Theorem 3.8, we can establish the following estimate

$$||q||^2_{L^\infty(0,T,L^2(0,L))} + ||q_x||^2_{L^2(0,T,L^2(0,L))} \le \tilde{C}(T,L,||\xi||_{Y_{1/4}})\left(||q_0||^2_{L^2(0,L)} + ||f||^2_{L^2((0,T)\times\omega)}\right) \tag{3.63}$$

We introduce the space

$$E := C^0([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L)) \cap H^1(0,T;H^{-2}(0,L))$$

endowed with its natural norm

$$\|z\|_E := ||z||_{Y_{1/4}} + ||z||_{H^1(0,T,H^{-2}(0,L))}.$$

We consider in $L^2((0,T)\times(0,L))$ the following set

$$B := \{z \in E;\ \|z\|_E \leq 1\}.$$

B is compact in $L^2((0,T)\times(0,L))$, by Aubin–Lions's lemma. We will limit ourselves to controls f fulfilling the condition

$$||f||^2_{L^2((0,T)\times\omega)} \leq C_*||q_0||^2_{L^2(0,L)} \tag{3.64}$$

where $C_* := C_*(T,L,||\bar{u}||_{Y_{1/4}}+\frac{1}{2})$. We associate with any $z \in B$ the set

$$P(z) := \big\{q \in B;\ \ \exists f \in L^2((0,T)\times\omega) \text{ such that } f \text{ satisfies (3.64) and } q \text{ solves (3.60)–(3.62) with } \xi = \bar{u} + \tfrac{z}{2} \text{ and } q(T,\cdot) = 0\big\}.$$

By Theorem 3.8, (3.58) and (3.63), we see that if $\|q_0\|_{L^2(0,L)}$ is sufficiently small, then $P(z)$ is nonempty for all $z \in B$. We shall use the following version of Kakutani fixed point theorem (see *e.g.* [27], Thm. 9.B):

Theorem 3.9. *Let F be a locally convex space, let $B \subset F$ and let $P : B \longrightarrow 2^B$. Assume that*

(1) *B is a nonempty, compact, convex set;*
(2) *$P(z)$ is a nonempty, closed, convex set for all $z \in B$;*
(3) *The set-valued map $P : B \longrightarrow 2^B$ is upper-semicontinuous;* i.e., *for every closed subset A of F, $P^{-1}(A) = \{z \in B;\ P(z) \cap A \neq \varnothing\}$ is closed.*

Then P has a fixed point, i.e., *there exists $z \in B$ such that $z \in P(z)$.*

Let us check that Theorem 3.9 can be applied to P and

$$F = L^2((0,T)\times(0,L)).$$

The convexity of B and $P(z)$ for all $z \in B$ is clear. Thus (1) is satisfied. For (2), it remains to check that $P(z)$ is closed in F for all $z \in B$. Pick any $z \in B$ and a sequence $\{q^k\}_{k\in\mathbb{N}}$ in $P(z)$ which converges in F towards some function $q \in B$. For each k, we can pick some control function $f^k \in L^2((0,T)\times\omega)$ fulfilling (3.64) such that (3.60)–(3.62) are satisfied with $\xi = \bar{u} + \frac{z}{2}$ and $q^k(T,\cdot) = 0$. Extracting subsequences if needed, we may assume that as $k \to \infty$

$$f^k \to f \ \text{ in } L^2((0,T)\times\omega) \text{ weakly,} \tag{3.65}$$

$$q^k \to q \ \text{ in } L^2(0,T;H^1(0,L)) \cap H^1(0,T;H^{-2}(0,L)) \text{ weakly,} \tag{3.66}$$

By (3.66), the boundedness of $||q^k||_{L^\infty(0,T,L^2(0,L))}$ and Aubin–Lions's lemma, $\{q^k\}_{k\in\mathbb{N}}$ is relatively compact in $C^0([0,T],H^{-1}(0,L))$. Extracting a subsequence if needed, we may assume that

$$q^k \to q \text{ strongly in } C^0([0,T],H^{-1}(0,L)).$$

In particular, $q(0,x) = q_0(x)$ and $q(T,x) = 0$. On the other hand, we infer from (3.66) that

$$\xi q^k \to \xi q \text{ in } L^2((0,T)\times(0,L)) \text{ weakly.}$$

Therefore, $(\xi q^k)_x \to (\xi q)_x$ in $\mathcal{D}'((0,T)\times(0,L))$. Finally, it is clear that

$$||f||^2_{L^2((0,T)\times\omega)} \leq C_*||q_0||^2_{L^2(0,L)}$$

and that q satisfies (3.60) with $\xi = \bar{u} + \frac{z}{2}$ and $q(T,\cdot) = 0$. Thus $q \in P(z)$ and $P(z)$ is closed. Now, let us check (3). To prove that P is upper-semicontinuous, consider any closed subset A of F and any sequence $\{z^k\}_{k\in\mathbb{N}}$ in B such that

$$z^k \in P^{-1}(A), \quad \forall k \geq 0, \tag{3.67}$$

and

$$z^k \to z \quad \text{in} \quad F \tag{3.68}$$

for some $z \in B$. We aim to prove that $z \in P^{-1}(A)$. By (3.67), we can pick a sequence $\left\{q^k\right\}_{k\in\mathbb{N}}$ in B with $q^k \in P(z^k) \cap A$ for all k, and a sequence $\left\{f^k\right\}_{k\in\mathbb{N}}$ in $L^2((0,T)\times\omega)$ such that

$$\begin{cases} q_t^k + q_x^k + \left(\left(\bar{u} + \dfrac{z^k}{2}\right) q^k\right)_x + q_{xxx}^k = 1_\omega f^k(t,x) & \text{in } (0,T)\times(0,L),\\ q^k(t,0) = q^k(t,L) = q_x^k(t,L) = 0 & \text{in } (0,T),\\ q^k(0,x) = q_0(x) & \text{in } (0,L), \end{cases} \tag{3.69}$$

$$q^k(T,x) = 0, \qquad \text{in } (0,L), \tag{3.70}$$

and

$$\left\|f^k\right\|^2_{L^2((0,T)\times\omega)} \le C_* \left\|q_0\right\|^2_{L^2(0,L)}. \tag{3.71}$$

From (3.71) and the fact that z^k, $q^k \in B$, extracting subsequences if needed, we may assume that as $k \to \infty$,

$$\begin{aligned} &f^k \to f \quad \text{in } L^2((0,T)\times\omega) \text{ weakly},\\ &q^k \to q \quad \text{in } L^2(0,T;H^1(0,L)) \cap H^1(0,T;H^{-2}(0,L)) \text{ weakly},\\ &q^k \to q \quad \text{in } C^0([0,T],H^{-1}(0,L)) \text{ strongly},\\ &q^k \to q \quad \text{in } F \text{ strongly},\\ &z^k \to z \quad \text{in } F \text{ strongly}, \end{aligned}$$

where $f \in L^2((0,T)\times\omega)$ and $q \in B$. Again, $q(0,x) = q_0(x)$ and $q(T,x) = 0$. We also see that (3.61) and (3.64) are satisfied. It remains to check that

$$q_t + q_x + \left(\left(\bar{u} + \frac{z}{2}\right) q\right)_x + q_{xxx} = 1_\omega f(t,x). \tag{3.72}$$

Observe that the only nontrivial convergence in (3.69) is that of the nonlinear term $(z^k q^k)_x$. Note first that

$$||z^k q^k||_{L^2(0,T,L^2(0,L))} \le ||z^k||_{L^\infty(0,T,L^2(0,L))} ||q^k||_{L^2(0,T,L^\infty(0,L))} \le C,$$

so that, extracting a subsequence, one can assume that $z^k q^k \to f$ weakly in $L^2((0,T)\times(0,L))$. To prove that $f = zq$, it is sufficient to observe that for any $\varphi \in \mathcal{D}(Q)$,

$$\int_0^T\!\!\int_0^L z^k q^k \varphi \,\mathrm{d}x\mathrm{d}t \to \int_0^T\!\!\int_0^L zq\varphi \,\mathrm{d}x\mathrm{d}t,$$

for $z^k \to z$ and $q^k\varphi \to q\varphi$ in F. Thus

$$z^k q^k \to zq \ \text{ in } L^2((0,T)\times(0,L)) \text{ weakly}.$$

It follows that $(z^k q^k)_x \to (zq)_x$ in $\mathcal{D}'((0,T)\times(0,L))$. Therefore, (3.72) holds and $q \in P(z)$. On the other hand, $q \in A$, since $q^k \to q$ in F and A is closed. We conclude that $z \in P^{-1}(A)$, and hence $P^{-1}(A)$ is closed.

It follows from Theorem 3.9 that there exists $q \in B$ with $q \in P(q)$; that is, we have found a control $f \in L^2((0,T)\times\omega)$ such that the solution of (3.59) satisfies $q(T,\cdot) = 0$ in $(0,L)$. The proof of Theorem 1.1 is complete.

With Theorem 1.1 at hand, one can prove Theorem 1.3 about the regional controllability.

3.3. Proof of Theorem 1.3

By Theorem 1.1, if δ is small enough one can find a control input $f \in L^2(0,T/2,L^2(0,L))$ with $\text{supp}(f) \subset (0,T)\times\omega$ such that the solution of (1.7) satisfies $u(T/2,.)\equiv 0$ in $(0,L)$. Pick any number $l_2' \in (l_1',l_2)$ with $l_2' \notin \mathcal{N}$. (This is possible, the set $\mathcal{N}$ being discrete.) By ([20], Thm. 1.3), if δ is small enough one can pick a function $h\in L^2(T/2,T)$ such that the solution $y\in C^0([T/2,T],L^2(0,l_2'))\cap L^2(T/2,T,H^1(0,l_2'))$ of the system

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{in } (T/2,T)\times(0,l_2'),\\ y(t,0)=y(t,l_2')=0, \quad y_x(t,l_2')=h(t) & \text{in } (T/2,T),\\ y(T/2,x)=0 & \text{in } (0,l_2') \end{cases}$$

satisfies $y(T,x)=u_1(x)$ for $0<x<l_2'$. We pick a function $\mu\in C^\infty([0,L])$ such that

$$\mu(x)=\begin{cases} 1 & \text{if } x<l_1',\\ 0 & \text{if } x>\frac{l_1'+l_2'}{2} \end{cases}$$

and set for $T/2<t\le T$

$$u(t,x)=\begin{cases} \mu(x)y(t,x) & \text{if } x<l_2',\\ 0 & \text{if } x>l_2'. \end{cases}$$

Note that, for $T/2<t<T$, $u_t+u_{xxx}+u_x+uu_x=f$ with

$$f=\mu(\mu-1)yy_x+(\mu_{xxx}y+3\mu_{xx}y_x+3\mu_x y_{xx}+\mu_x y)+\mu\mu_x y^2.$$

Since $||y||^4_{L^4(0,T,L^4(0,l_2'))}\le C||y||^2_{L^\infty(0,T,L^2(0,L))}||y||^2_{L^2(0,T,H^1(0,L))}$, it is clear that $f\in L^2(0,T,H^{-1}(0,L))$ with $\text{supp}(f)\subset(0,T)\times(l_1,l_2)$. Furthermore, $u\in C([0,T],L^2(0,L))\cap L^2(0,T,H^1(0,L))$ solves (1.7) and satisfies (1.8).

4. Exact controllability results

Pick any function $\rho\in C^\infty(0,L)$ with

$$\rho(x)=\begin{cases} 0 & \text{if } 0<x<L-\nu,\\ 1 & \text{if } L-\frac{\nu}{2}<x<L, \end{cases} \tag{4.1}$$

for some $\nu\in(0,L)$.

This section is devoted to the investigation of the exact controllability of the system

$$\begin{cases} u_t+u_x+uu_x+u_{xxx}=f=(\rho(x)h)_x & \text{in } (0,T)\times(0,L),\\ u(t,0)=u(t,L)=u_x(t,L)=0 & \text{in } (0,T),\\ u(0,x)=u_0(x) & \text{in } (0,L). \end{cases} \tag{4.2}$$

More precisely, we aim to find a control input $h\in L^2(0,T;L^2(0,L))$ (actually, with $(\rho(x)h(t,x))_x$ in some space of functions) to guide the system described by (4.2) in the time interval $[0,T]$ from any (small) given initial state u_0 in $L^2_{\frac{1}{L-x}\mathrm{d}x}$ to any (small) given terminal state u_T in the same space. We first consider the linearized system, and next proceed to the nonlinear one. The results involve some weighted Sobolev spaces.

4.1. The linear system

For any measurable function $w:(0,L)\to(0,+\infty)$ (not necessarily in $L^1(0,L)$), we introduce the weighted L^2-space

$$L^2_{w(x)\mathrm{d}x}=\{u\in L^1_{loc}(0,L);\ \int_0^L u(x)^2 w(x)\mathrm{d}x<\infty\}.$$

It is a Hilbert space when endowed with the scalar product

$$(u,v)_{L^2_{w(x)\mathrm{d}x}} = \int_0^L u(x)v(x)w(x)\mathrm{d}x.$$

We first prove the well-posedness of the linear system associated with (4.2), namely

$$\begin{cases} u_t + u_x + u_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L), \end{cases} \tag{4.3}$$

in both the spaces $L^2_{x\mathrm{d}x}$ and $L^2_{\frac{1}{L-x}\mathrm{d}x}$, following [10] where the well-posedness was established in $L^2_{\frac{x}{L-x}\mathrm{d}x}$. We need the following result.

Theorem 4.1 (see [10])**.** *Let $W \subset V \subset H$ be three Hilbert spaces with continuous and dense embeddings. Let $a(v,w)$ be a bilinear form defined on $V \times W$ that satisfies the following properties:*
(i) (Continuity)

$$a(v,w) \le M||v||_V||w||_W, \quad \forall v \in V, \ \forall w \in W. \tag{4.4}$$

(ii) (Coercivity)

$$a(w,w) \ge m||w||_V^2, \quad \forall w \in W. \tag{4.5}$$

Then for all $f \in V'$ (the dual space of V), there exists $v \in V$ such that

$$a(v,w) = f(w) \qquad \forall w \in W. \tag{4.6}$$

Assume that, in addition to (i) *and* (ii)*, $a(v,w)$ satisfies:*
(iii) (Regularity) *for all $g \in H$, any solution $v \in V$ of*

$$a(v,w) = (g,w)_H \qquad \forall w \in W, \tag{4.7}$$

belongs to W. Then equation (4.7) has a unique solution $v = v(g) \in W$. Let $D(A) := \{v(g);\ g \in H\} \subset W \subset H$ and set $Av := -g$ for $v \in D(A)$. (Note that there is a unique $g \in H$ satisfying (4.7).) Then A is a maximal dissipative operator, and hence it generates a continuous semigroup of contractions $(\mathrm{e}^{tA})_{t\ge 0}$ in H.

4.2. Well-posedness in $L^2_{x\mathrm{d}x}$

Theorem 4.2. *Let $A_1 u = -u_{xxx} - u_x$ with domain*

$$D(A_1) = \left\{u \in H^2(0,L) \cap H_0^1(0,L);\ u_{xxx} \in L^2_{x\mathrm{d}x},\ u_x(L) = 0\right\} \subset L^2_{x\mathrm{d}x}.$$

Then A_1 generates a strongly continuous semigroup in $L^2_{x\mathrm{d}x}$.

Proof. Let

$$H = L^2_{x\mathrm{d}x}, \quad V = H_0^1(0,L), \quad W = \left\{w \in H_0^1(0,L),\ w_{xx} \in L^2_{x^2\mathrm{d}x}\right\},$$

be endowed with the respective norms

$$||u||_H := ||\sqrt{x}u||_{L^2(0,L)}, \quad ||v||_V := ||v_x||_{L^2(0,L)}, \quad ||w||_W := ||xw_{xx}||_{L^2(0,L)}.$$

Clearly, $V \subset H$ with a continuous (dense) embedding between two Hilbert spaces. On the other hand, we claim that

$$||w_x||_{L^2} \le C||xw_{xx}||_{L^2} \qquad \forall w \in W. \tag{4.8}$$

Indeed, we note first that we have for $w \in \mathcal{T} := C^\infty([0,L]) \cap H_0^1(0,L)$ and $p \in \mathbb{R}$

$$0 \le \int_0^L (xw_{xx} + pw_x)^2 \mathrm{d}x = \int_0^L (x^2 w_{xx}^2 + 2pxw_x w_{xx} + p^2 w_x^2)\mathrm{d}x = \int_0^L x^2 w_{xx}^2 \mathrm{d}x + \left(p^2 - p\right)\int_0^L w_x^2 \mathrm{d}x + pLw_x^2(L).$$

Taking $p = 1/2$ results in

$$\int_0^L w_x^2 \mathrm{d}x \le 4\int_0^L x^2 w_{xx}^2 \mathrm{d}x + 2L|w_x(L)|^2. \tag{4.9}$$

The estimate (4.9) is also true for any $w \in W$, since $\mathcal{T}$ is dense in W. Let us prove (4.8) by contradiction. If (4.8) is false, then there exists a sequence $\{w^n\}_{n\ge 0}$ in W such that

$$1 = ||w_x^n||_{L^2} \ge n||xw_{xx}^n||_{L^2} \qquad \forall n \ge 0.$$

Extracting subsequences, we may assume that

$$\begin{aligned} w^n \to w &\quad \text{in } H_0^1(0,L) \text{ weakly} \\ xw_{xx}^n \to 0 &\quad \text{in } L^2(0,L) \text{ strongly} \end{aligned}$$

and hence $xw_{xx} = 0$, which gives $w(x) = c_1 x + c_2$. Since $w \in H_0^1(0,L)$, we infer that $w \equiv 0$. Since w^n is bounded in $H^2(L/2,L)$, extracting subsequences we may also assume that $w_x^n(L)$ converges in $\mathbb{R}$. We infer then from (4.9) that w^n is a Cauchy sequence in $H_0^1(0,L)$, so that

$$w^n \to w \quad \text{in } H_0^1(0,L) \text{ strongly},$$

and hence $||w_x||_{L^2} = \lim_{n\to\infty} ||w_x^n||_{L^2} = 1$. This contradicts the fact that $w \equiv 0$. The proof of (4.8) is achieved.

Thus $||\cdot||_W$ is a norm in W, which is clearly a Hilbert space, and $W \subset V$ with continuous (dense) embedding. Let

$$a(v,w) = \int_0^L v_x[(xw)_{xx} + xw]\mathrm{d}x, \qquad v \in V,\ w \in W.$$

Let us check that (i), (ii), and (iii) in Theorem 4.1 hold. For $v \in V$ and $w \in W$,

$$\begin{aligned} |a(v,w)| &\le ||v_x||_{L^2}||xw_{xx} + 2w_x + xw||_{L^2} \\ &\le ||v_x||_{L^2}\left(\|xw_{xx}||_{L^2} + C\|\, w_x||_{L^2}\right) \\ &\le C||v||_V||w||_W \end{aligned}$$

where we used Poincaré inequality and (4.8). This proves that the bilinear form a is well defined and continuous on $V \times W$. For (ii), we first pick any $w \in \mathcal{T}$ to obtain

$$\begin{aligned} a(w,w) &= \int_0^L w_x(xw_{xx} + 2w_x + xw)\mathrm{d}x \\ &= \frac{3}{2}\int_0^L w_x^2 \mathrm{d}x + \left[x\frac{w_x^2}{2}\right]|_0^L - \frac{1}{2}\int_0^L w^2 \mathrm{d}x \\ &\ge \frac{3}{2}\int_0^L w_x^2 \mathrm{d}x - \frac{1}{2}\int_0^L w^2 \mathrm{d}x. \end{aligned}$$

By Poincaré inequality

$$\int_0^L w^2(x)\mathrm{d}x \le \left(\frac{L}{\pi}\right)^2 \int_0^L w_x^2(x)\mathrm{d}x,$$

and hence

$$a(w,w) \geq \left(\frac{3}{2} - \frac{L^2}{2\pi^2}\right) \int_0^L w_x^2 \mathrm{d}x.$$

This shows the coercivity when $L < \pi\sqrt{3}$. When $L \geq \pi\sqrt{3}$, we have to consider, instead of a, the bilinear form $a_\lambda(v,w) := a(v,w) + \lambda(v,w)_H$ for $\lambda \gg 1$. Indeed, we have by Cauchy–Schwarz inequality and Hardy inequality

$$\begin{aligned} ||w||_{L^2}^2 &\leq ||x^{\frac{1}{2}}w||_{L^2}||x^{-\frac{1}{2}}w||_{L^2} \\ &\leq \sqrt{L}||w||_H||x^{-1}w||_{L^2} \\ &\leq \varepsilon||w_x||_{L^2}^2 + C_\varepsilon||w||_H^2 \end{aligned}$$

and hence

$$a_\lambda(w,w) \geq \left(\frac{3}{2} - \frac{\varepsilon}{2}\right)||w||_V^2 + \left(\lambda - \frac{C_\varepsilon}{2}\right)||w||_H^2.$$

Therefore, if $\varepsilon < 3$ and $\lambda > C_\varepsilon/2$, then a_λ is a continuous bilinear form which is coercive.

Let us have a look at the regularity issue. For given $g \in H$, let $v \in V$ be such that

$$a_\lambda(v,w) = (g,w)_H \qquad \forall w \in W,$$

i.e.

$$\int_0^L v_x((xw)_{xx} + xw)\mathrm{d}x + \lambda\int_0^L v(x)w(x)x\mathrm{d}x = \int_0^L g(x)w(x)x\mathrm{d}x. \tag{4.10}$$

Picking any $w \in \mathcal{D}(0,L)$ results in

$$\langle x(v_{xxx} + v_x + \lambda v), w\rangle_{\mathcal{D}',\mathcal{D}} = \langle xg, w\rangle_{\mathcal{D}',\mathcal{D}} \qquad \forall w \in \mathcal{D}(0,L), \tag{4.11}$$

and hence

$$v_{xxx} + v_x + \lambda v = g \qquad \text{in } \mathcal{D}'(0,L). \tag{4.12}$$

Since $v \in H_0^1(0,L)$ and $g \in L^2_{x\mathrm{d}x}$, we have that $v \in H^3(\varepsilon,L)$ for all $\varepsilon \in (0,L)$ and $v_{xxx} \in L^2_{x\mathrm{d}x}$. Picking any $w \in \mathcal{T}$ and $\varepsilon \in (0,L)$, and scaling in (4.12) by xw yields

$$\int_\varepsilon^L v_x((xw)_{xx} + xw)\mathrm{d}x + [v_{xx}(xw) - v_x(xw)_x]|_\varepsilon^L = \int_\varepsilon^L (g - \lambda v)xw\mathrm{d}x.$$

Letting $\varepsilon \to 0$ and comparing with (4.10), we obtain

$$-Lv_x(L)w_x(L) = \lim_{\varepsilon\to 0}\big(\varepsilon v_{xx}(\varepsilon)w(\varepsilon) - v_x(\varepsilon)(w(\varepsilon) + \varepsilon w_x(\varepsilon))\big). \tag{4.13}$$

Since $v_{xxx} \in L^2_{x\mathrm{d}x}$, we obtain successively for some constant $C > 0$ and all $\varepsilon \in (0,L)$

$$|v_{xx}(\varepsilon) - v_{xx}(L)| \leq \left(\int_\varepsilon^L x|v_{xxx}|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\varepsilon^L x^{-1}\mathrm{d}x\right)^{\frac{1}{2}} \leq C|\log\varepsilon| \tag{4.14}$$

$$|v_x(\varepsilon)| \leq C. \tag{4.15}$$

We infer from (4.14) that $v \in H^2(0,L)$, and hence $v \in W$. Furthermore, letting $\varepsilon \to 0$ in (4.13) and using (4.14)–(4.15) yields $v_x(L) = 0$, since $w_x(L)$ was arbitrary. We conclude that $v \in \mathcal{D}(A_1)$. Conversely, it is clear that the operator $A_1 - \lambda$ maps $\mathcal{D}(A_1)$ into H, and actually onto H from the above computations. Hence $A_1 - \lambda$ generates a strongly semigroup of contractions in H. □

4.3. Well-posedness in $L^2_{(L-x)^{-1}\mathrm{d}x}$

Theorem 4.3. *Let $A_2 u = -u_{xxx} - u_x$ with domain*

$$\mathcal{D}(A_2) = \{u \in H^3(0,L) \cap H^1_0(0,L);\ u_{xxx} \in L^2_{\frac{1}{L-x}\mathrm{d}x} \text{ and } u_x(L) = 0\} \subset L^2_{\frac{1}{L-x}\mathrm{d}x}.$$

Then A_2 generates a strongly continuous semigroup in $L^2_{\frac{1}{L-x}\mathrm{d}x}$.

Proof. We will use Hille–Yosida theorem, and (partially) Theorem 4.1. Let

$$H = L^2_{\frac{1}{L-x}\mathrm{d}x}, \quad V = \left\{u \in H^1_0(0,L),\ u_x \in L^2_{\frac{1}{(L-x)^2}\mathrm{d}x}\right\}, \quad W = H^2_0(0,L), \tag{4.16}$$

be endowed respectively with the norms

$$||u||_H = ||(L-x)^{-\frac{1}{2}} u||_{L^2}, \quad ||u||_V = \left\|(L-x)^{-1} u_x\right\|_{L^2}, \quad ||u||_W = ||u_{xx}||_{L^2}. \tag{4.17}$$

From [10], we know that V endowed with $||\cdot||_V$ is a Hilbert space, and that

$$||(L-x)^{-2} u||_{L^2} \le \frac{2}{3} ||(L-x)^{-1} u_x||_{L^2} \qquad \forall u \in V, \tag{4.18}$$

and hence

$$||u||_H \le \left(\int_0^L \frac{L^3}{(L-x)^4} u^2(x)\mathrm{d}x\right)^{\frac{1}{2}} \le \frac{2}{3} L^{\frac{3}{2}} ||u||_V \qquad \forall u \in V. \tag{4.19}$$

Thus $V \subset H$ with continuous embedding. From Poincaré inequality, we have that $||\cdot||_W$ is a norm on W equivalent to the H^2-norm. On the other hand, from Hardy inequality

$$\int_0^L \frac{v^2}{(L-x)^2}\mathrm{d}x \le C \int_0^L v_x^2 \mathrm{d}x \quad \forall v \in H^1(0,L) \text{ with } v(L) = 0, \tag{4.20}$$

we have that

$$||v||_V \le C||v||_W \qquad \forall v \in W. \tag{4.21}$$

Thus $W \subset V$ with continuous embedding. It is easily seen that $\mathcal{D}(0,L)$ is dense in H, V, and W. Let

$$a(v,w) = \int_0^L \left[v_x \left(\frac{w}{L-x}\right)_{xx} + v_x \frac{w}{L-x}\right]\mathrm{d}x \qquad (v,w) \in V \times W.$$

Then

$$\begin{aligned}
|a(v,w)| &\le \left|\int_0^L v_x \left(\frac{w_{xx}}{L-x} + 2\frac{w_x}{(L-x)^2} + 2\frac{w}{(L-x)^3} + \frac{w}{L-x}\right)\mathrm{d}x\right| \\
&\le ||w_{xx}||_{L^2} \left\|\frac{v_x}{L-x}\right\|_{L^2} + 2\left\|\frac{w_x}{L-x}\right\|_{L^2}\left\|\frac{v_x}{L-x}\right\|_{L^2} + \left\|\frac{v_x}{L-x}\right\|_{L^2}\left(2\left\|\frac{w}{(L-x)^2}\right\|_{L^2} + ||w||_{L^2}\right) \\
&\le C||v||_V ||w||_W
\end{aligned}$$

by (4.18), (4.19), and (4.21). This shows that a is well defined and continuous. Let us look at the coercivity of a. Pick any $w \in \mathcal{D}(0,L)$. Then

$$\begin{aligned}
a(w,w) &= \int_0^L w_x \left(\frac{w_{xx}}{L-x} + 2\frac{w_x}{(L-x)^2} + 2\frac{w}{(L-x)^3} + \frac{w}{L-x}\right)\mathrm{d}x \\
&= \frac{3}{2}\int_0^L \frac{w_x^2}{(L-x)^2}\mathrm{d}x - 3\int_0^L \frac{w^2}{(L-x)^4}\mathrm{d}x - \frac{1}{2}\int_0^L \frac{w^2}{(L-x)^2}\mathrm{d}x \\
&\ge \frac{1}{6}\int_0^L \frac{w_x^2}{(L-x)^2}\mathrm{d}x - \frac{1}{2}\int_0^L \frac{w^2}{(L-x)^2}\mathrm{d}x
\end{aligned}$$

where we used (4.18) for the last line. Note that, using Cauchy–Schwarz inequality and (4.18), we have that

$$\begin{aligned}\left\|\frac{w}{L-x}\right\|_{L^2}^2 &\le \left\|(L-x)^{-\frac{1}{2}}w\right\|_{L^2}\left\|(L-x)^{-\frac{3}{2}}w\right\|_{L^2}\\ &\le \frac{2\sqrt{L}}{3}||w||_H||w||_V\\ &\le \varepsilon||w||_V^2+\frac{L}{9\varepsilon}||w||_H^2.\end{aligned} \tag{4.22}$$

If we pick $\varepsilon\in(0,1/3)$, we infer that for all $w\in\mathcal{D}(0,L)$

$$a(w,w)+\frac{L}{18\varepsilon}||w||_H^2\ge\left(\frac{1}{6}-\frac{\varepsilon}{2}\right)||w||_V^2\ge C||w||_V^2. \tag{4.23}$$

The result is also true for any $w\in W$, by density. This shows that the continuous bilinear form

$$a_\lambda(v,w)=a(v,w)+\lambda(v,w)_H$$

is coercive for $\lambda>L/6$. Let $g\in H$ be given. By Theorem 4.1, there is at least one solution $v\in V$ of

$$a_\lambda(v,w)=(g,w)_H\qquad\forall w\in W. \tag{4.24}$$

Pick such a solution $v\in V$, and let us prove that $v\in\mathcal{D}(A_2)$. Picking any $w\in\mathcal{D}(0,L)$ in (4.24) yields

$$v_{xxx}+v_x+\lambda v=g\qquad\text{in }\mathcal{D}'(0,L). \tag{4.25}$$

As $g\in L^2(0,L)$ and $v\in H^1(0,L)$, we have that $v_{xxx}\in L^2(0,L)$, and $v\in H^3(0,L)$. Pick finally w of the form $w(x)=x^2(L-x)^2\overline{w}(x)$, where $\overline{w}\in C^\infty([0,L])$ is arbitrary chosen. Note that $w\in W$ and that $w/(L-x)\in H_0^1(0,L)\cap C^\infty([0,L])$. Multiplying in (4.25) by $w/(L-x)$ and integrating over $(0,L)$, we obtain after comparing with (4.24)

$$0=-v_x\left(\frac{w}{L-x}\right)_x|_0^L=-v_x\left((2xL-3x^2)\overline{w}+x^2(L-x)\overline{w}_x\right)|_0^L=v_x(L)L^2\overline{w}(L).$$

As $\overline{w}(L)$ can be chosen arbitrarily, we conclude that $v_x(L)=0$. Using (4.20) twice, we infer that $v_x+\lambda v\in H$, and hence $v_{xxx}=g-(v_x+\lambda v)\in H$. Therefore $v\in\mathcal{D}(A_2)$. Thus, for $\lambda>L/6$ we have that $A_2-\lambda:\mathcal{D}(A_2)\to H$ is onto. Let us check that $A_2-\lambda$ is also dissipative in H. Pick any $w\in\mathcal{D}(A_2)$. Then we obtain after some integrations by parts that

$$(A_2w,w)_H=-\frac{3}{2}\int_0^L\frac{w_x^2}{(L-x)^2}\mathrm{d}x+3\int_0^L\frac{w^2}{(L-x)^4}\mathrm{d}x+\frac{1}{2}\int_0^L\frac{w^2}{(L-x)^2}\mathrm{d}x-\frac{w_x^2(0)}{2L}$$

and

$$(A_2w-\lambda w,w)_H\le-(\frac{1}{6}-\frac{\varepsilon}{2})||w||_V^2-\frac{w_x^2(0)}{2L}\le0$$

for $\varepsilon<1/3$ and $\lambda=L/(18\varepsilon)$. We conclude that $A_2-\lambda$ is maximal dissipative for $\lambda>L/6$, and thus it generates a strongly continuous semigroup of contractions in H by Hille–Yosida theorem. □

A global Kato smoothing effect as in [10, 20] can as well be derived.

Proposition 4.4. *Let H and V be as in (4.16)–(4.17), and let $T>0$ be given. Then there exists some constant $C=C(L,T)$ such that for any $u_0\in H$, the solution $u(t)=\mathrm{e}^{tA_2}u_0$ of (4.3) satisfies*

$$||u||_{L^\infty(0,T,H)}+||u||_{L^2(0,T,V)}\le C||u_0||_H. \tag{4.26}$$

Proof. We proceed as in [10]. First, we notice that $\mathcal{D}(A_2)$ is dense in H, so that it is sufficient to prove the result when $u_0 \in \mathcal{D}(A_2)$. Note that the estimate $||u||_{L^\infty(0,T,H)} \leq C||u_0||_H$ is a consequence of classical semigroup theory. Assume $u_0 \in \mathcal{D}(A_2)$, so that $u_t = A_2 u$ in the classical sense. Taking the inner product in H with u yields

$$(u_t, u)_H = -a(u,u) \leq -C||u||_V^2 + \frac{L}{18\varepsilon}||u||_H^2$$

where we used (4.23). An integration over $(0,T)$ completes the proof of the estimate of $||u||_{L^2(0,T,V)}$. □

4.4. Non-homogeneous system

In this section we consider the nonhomogeneous system

$$u_t + u_x + u_{xxx} = f(t,x) \ \text{ in } (0,T)\times(0,L), \tag{4.27}$$

$$u(t,0) = u(t,L) = u_x(t,L) = 0 \ \text{ in } (0,T), \tag{4.28}$$

$$u(0,x) = u_0 \ \text{ in } (0,L). \tag{4.29}$$

We need to prove the existence of a solution $u \in C([0,T], L^2_{x\mathrm{d}x}) \cap L^2(0,T,H^1(0,L))$ when solely $f \in L^2(0,T,H^{-1}(0,L))$.

Proposition 4.5. *Let $u_0 \in L^2_{x\mathrm{d}x}$ and $f \in L^2(0,T;H^{-1}(0,L))$. Then there exists a unique solution $u \in C([0,T], L^2_{x\mathrm{d}x}) \cap L^2(0,T,H^1(0,L))$ to (4.27)–(4.29). Furthermore, there is some constant $C > 0$ such that*

$$||u||_{L^\infty(0,T,L^2_{x\mathrm{d}x})} + ||u||_{L^2(0,T,H^1(0,L))} \leq C\big(||u_0||_{L^2_{x\mathrm{d}x}} + ||f||_{L^2(0,T,H^{-1}(0,L))}\big). \tag{4.30}$$

Proof. Assume first that $u_0 \in \mathcal{D}(A_1)$ and $f \in C^0([0,T],\mathcal{D}(A_1))$ to legitimate the following computations. Multiplying each term in (4.27) by xu and integrating over $(0,\tau)\times(0,L)$ where $0<\tau<T$ yields

$$\frac{1}{2}\int_0^L x|u(\tau,x)|^2\mathrm{d}x - \frac{1}{2}\int_0^L x|u_0(x)|^2\mathrm{d}x + \frac{3}{2}\int_0^\tau\int_0^L |u_x|^2\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^\tau\int_0^L |u|^2\mathrm{d}x\mathrm{d}t = \int_0^\tau\int_0^L xuf\mathrm{d}x\mathrm{d}t. \tag{4.31}$$

$\langle .,.\rangle_{H^{-1},H^1_0}$ denoting the duality pairing between $H^{-1}(0,L)$ and $H^1_0(0,L)$, we have that for all $\varepsilon > 0$

$$\int_0^\tau\int_0^L xuf\mathrm{d}x\mathrm{d}t = \int_0^\tau \langle f, xu\rangle_{H^{-1},H^1_0} \leq \frac{\varepsilon}{2}\int_0^\tau\int_0^L u_x^2\mathrm{d}x\mathrm{d}t + C_\varepsilon\int_0^\tau ||f||_{H^{-1}}^2\mathrm{d}t. \tag{4.32}$$

For $0<\varepsilon<L^2$, the last term in the l.h.s. of (4.31) is decomposed as

$$\frac{1}{2}\int_0^\tau\int_0^L |u|^2\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_0^\tau\int_0^{\sqrt{\varepsilon}} |u|^2\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^\tau\int_{\sqrt{\varepsilon}}^L |u|^2\mathrm{d}x\mathrm{d}t =: I_1 + I_2.$$

We claim that

$$I_1 \leq \frac{\varepsilon}{2}\int_0^\tau\int_0^L |u_x|^2\mathrm{d}x\mathrm{d}t, \tag{4.33}$$

$$I_2 \leq \frac{1}{2\sqrt{\varepsilon}}\int_0^\tau\int_0^L x|u|^2\mathrm{d}x\mathrm{d}t. \tag{4.34}$$

For (4.33), since $u(0,t)=0$ we have that for $(t,x)\in(0,T)\times(0,\sqrt{\varepsilon})$

$$|u(x,t)| \leq \int_0^{\sqrt{\varepsilon}} |u_x|\mathrm{d}x \leq \varepsilon^{\frac{1}{4}}\left(\int_0^{\sqrt{\varepsilon}}|u_x|^2\mathrm{d}x\right)^{\frac{1}{2}}$$

and hence

$$\int_0^{\sqrt{\varepsilon}} |u|^2 \mathrm{d}x \le \varepsilon \int_0^{\sqrt{\varepsilon}} |u_x|^2 \mathrm{d}x$$

which gives (4.33) after integrating over $t \in (0, \tau)$. (4.34) is obvious.

Gathering together (4.31)–(4.34), we obtain for $0 < \varepsilon < L^2$

$$\frac{1}{2}\int_0^L x|u(\tau,x)|^2 \mathrm{d}x + \left(\frac{3}{2} - \varepsilon\right) \int_0^\tau\!\!\int_0^L |u_x|^2 \mathrm{d}x\mathrm{d}t \le \frac{1}{2}\int_0^L x|u_0(x)|^2 \mathrm{d}x + \frac{1}{2\sqrt{\varepsilon}} \int_0^\tau\!\!\int_0^L x|u|^2 \mathrm{d}x\mathrm{d}t + C_\varepsilon \int_0^\tau ||f||^2_{H^{-1}} \mathrm{d}t.$$

Picking $\varepsilon \in (0, \min(L^2, 3/2))$ and applying Gronwall's lemma, we obtain

$$||u||^2_{L^\infty(0,T,L^2_{x\mathrm{d}x})} + ||u_x||^2_{L^2(0,T,L^2(0,L))} \le C(T)\big(||u_0||^2_{L^2_{x\mathrm{d}x}} + ||f||^2_{L^2(0,T,H^{-1}(0,L))}\big).$$

This gives (4.30) for $u_0 \in D(A_1)$ and $f \in C^0([0,T], D(A_1))$. A density argument allows us to construct a solution $u \in C([0,T], L^2_{x\mathrm{d}x}) \cap L^2(0,T,H^1(0,L))$ of (4.27)–(4.29) satisfying (4.30) for $u_0 \in L^2_{x\mathrm{d}x}$ and $f \in L^2(0,T,H^{-1}(0,L))$. The uniqueness follows from classical semigroup theory. □

Our goal now is to obtain a similar result in the spaces H and V introduced in (4.16)–(4.17). To do that, we limit ourselves to the situation when $f = (\rho(x)h)_x$ with $h \in L^2(0,T,L^2(0,L))$.

Proposition 4.6. *Let $u_0 \in H$ and $h \in L^2(0,T,L^2(0,L))$, and set $f := (\rho(x)h)_x$, where $\rho \in C^\infty([0,L])$ is as in (4.1). Then there exists a unique solution $u \in C([0,T],H) \cap L^2(0,T,V)$ to (4.27)–(4.29). Furthermore, there is some constant $C > 0$ such that*

$$||u||_{L^\infty(0,T,H)} + ||u||_{L^2(0,T,V)} \le C\big(||u_0||_H + ||h||_{L^2(0,T,L^2(0,L))}\big). \tag{4.35}$$

Proof. Assume that $u_0 \in \mathcal{D}(A_2)$ and $h \in C_0^\infty((0,T) \times (0,L))$, so that $f \in C^1([0,T],H)$. Taking the inner product of $u_t - A_2 u - f = 0$ with u in H yields

$$(u_t, u)_H = -a(u,u) + (f,u)_H \le -C||u||^2_V + \frac{L}{18\varepsilon}||u||^2_H + (f,u)_H, \tag{4.36}$$

where we used (4.23) with some $\varepsilon \in (0, 1/3)$. Then

$$\begin{aligned} |(f,u)_H| &= \left| \int_0^L (\rho(x)h)_x \frac{u}{L-x} \mathrm{d}x \right| \\ &= \left| \int_0^L \rho(x)h \left(\frac{u_x}{L-x} + \frac{u}{(L-x)^2} \right) \mathrm{d}x \right| \\ &\le C||h||_{L^2} \left(\left\| \frac{u_x}{L-x} \right\|_{L^2} + \left\| \frac{u}{(L-x)^2} \right\|_{L^2} \right) \\ &\le C||h||_{L^2}||u||_V, \end{aligned}$$

where we used (4.18) in the last line. Thus, we have that

$$|(f,u)_H| \le \frac{C}{2}||u||^2_V + C'||h||^2_{L^2}$$

which, when combined with (4.36), gives after integration over $(0,\tau)$ for $0<\tau<T$

$$||u(\tau)||_H^2 + C\int_0^\tau ||u||_V^2 \mathrm{d}t \le ||u_0||_H^2 + C''\left(\int_0^\tau ||u||_H^2 \mathrm{d}t + \int_0^\tau\int_0^L |h|^2 \mathrm{d}x\mathrm{d}t\right).$$

An application of Gronwall's lemma yields (4.35) for $u_0 \in \mathcal{D}(A_2)$ and $h \in C_0^\infty((0,T)\times(0,L))$. A density argument allows us to construct a solution $u \in C([0,T],H)\cap L^2(0,T,V)$ of (4.27)–(4.29) satisfying (4.35) for $u_0 \in H$ and $h \in L^2(0,T,L^2(0,L))$. The uniqueness follows from classical semigroup theory. □

4.5. Controllability of the linearized system

We turn our attention to the control properties of the linear system

$$u_t + u_{xxx} + u_x = f = (\rho(x)h)_x, \tag{4.37}$$

$$u(t,0) = u(t,L) = u_x(t,L) = 0, \tag{4.38}$$

$$u(0,x) = u_0(x). \tag{4.39}$$

Theorem 4.7. *Let $T>0$, $\nu \in (0,L)$ and $\rho(x)$ as in (4.1). Then there exists a continuous linear operator $\Gamma : L^2_{\frac{1}{L-x}\mathrm{d}x} \to L^2(0,T,L^2(0,L)) \cap L^2_{(T-t)\mathrm{d}t}(0,T,H^1(0,L))$ such that for any $u_0,u_1 \in L^2_{\frac{1}{L-x}\mathrm{d}x}$, the solution u of (4.37)–(4.39) with $h=\Gamma(u_1)$ satisfies $u(T,x)=u_1(x)$ in $(0,L)$.*

Note that the forcing term $f=(\rho(x)h)_x$ is actually a *function* in $L^2_{(T-t)\mathrm{d}t}(0,T,L^2(0,L))$ supported in $(0,T)\times(L-\nu,L)$.

Proof. By using Proposition 4.6, we can assume that $u_0=0$ without loss of generality. We use the Hilbert Uniqueness Method (see *e.g.* [15]). Introduce the adjoint system

$$-v_t - v_{xxx} - v_x = 0, \tag{4.40}$$

$$v(t,0) = v(t,L) = v_x(t,0) = 0, \tag{4.41}$$

$$v(T,x) = v_T(x). \tag{4.42}$$

If $u_0 \equiv 0$, $v_T \in \mathcal{D}(0,L)$, and $h \in \mathcal{D}((0,T)\times(0,L))$, then multiplying in (4.37) by v and integrating over $(0,T)\times(0,L)$ gives

$$\int_0^L u(T,x)v_T(x)\mathrm{d}x = \int_0^T\int_0^L (\rho(x)h)_x v \mathrm{d}x\mathrm{d}t = -\int_0^T\int_0^L \rho(x)hv_x \mathrm{d}x\mathrm{d}t.$$

The usual change of variables $x\to L-x$, $t\to T-t$, combined with Proposition 4.5, gives

$$||v||_{L^\infty\left(0,T,L^2_{(L-x)\mathrm{d}x}\right)} + ||v||_{L^2(0,T,H^1(0,L))} \le C||v_T||_{L^2_{(L-x)\mathrm{d}x}}.$$

By a density argument, we obtain that for all $h \in L^2(0,T,L^2(0,L))$ and all $v_T \in L^2_{(L-x)\mathrm{d}x}$,

$$\langle u(T,.), v_T\rangle_{L^2_{\frac{1}{L-x}\mathrm{d}x}, L^2_{(L-x)\mathrm{d}x}} = -\int_0^T (h,\rho(x)v_x)_{L^2}\mathrm{d}t,$$

where u and v denote the solutions of (4.37)–(4.39) and (4.40)–(4.42), respectively, and $\langle\cdot,\cdot\rangle_{L^2_{\frac{1}{L-x}\mathrm{d}x}, L^2_{(L-x)\mathrm{d}x}}$ denotes the duality pairing between $L^2_{\frac{1}{L-x}\mathrm{d}x}$ and $L^2_{(L-x)\mathrm{d}x}$. We have to prove the following observability inequality

$$||v_T||^2_{L^2_{(L-x)\mathrm{d}x}} \le C\int_0^T\int_0^L |\rho(x)v_x|^2\mathrm{d}x\mathrm{d}t \tag{4.43}$$

or, equivalently, letting $w(t,x) = v(T-t, L-x)$,

$$||w_0||^2_{L^2_{x\mathrm{d}x}} \le C \int_0^T\int_0^L |\rho(L-x)w_x|^2 \mathrm{d}x\mathrm{d}t \tag{4.44}$$

where w solves

$$\begin{cases} w_t + w_{xxx} + w_x = 0, \\ w(t,0) = w(t,L) = w_x(t,L) = 0, \\ w(0,x) = w_0(x). \end{cases} \tag{4.45}$$

From [20], we know that for any $q \in C^\infty([0,T]\times[0,L])$

$$-\int_0^T\int_0^L (q_t + q_{xxx} + q_x)\frac{w^2}{2}\mathrm{d}x\mathrm{d}t + \int_0^L \left(q\frac{w^2}{2}\right)(T,x)\mathrm{d}x - \int_0^L \left(q\frac{w^2}{2}\right)(0,x)\mathrm{d}x \\ + \frac{3}{2}\int_0^T\int_0^L q_x w_x^2 \mathrm{d}x\mathrm{d}t + \int_0^T \left(q\frac{w_x^2}{2}\right)(t,0)\mathrm{d}t = 0.$$

We pick $q(t,x) = (T-t)b(x)$, where $b \in C^\infty([0,L])$ is nondecreasing and satisfies

$$b(x) = \begin{cases} x & \text{if } \ 0 < x < \nu/4, \\ 1 & \text{if } \ \nu/2 < x < L. \end{cases}$$

with $\nu \in (0,L)$. This yields

$$\begin{aligned} ||w_0||^2_{L^2_{x\mathrm{d}x}} &\le C(L,\nu)\int_0^L b(x)w_0^2(x)\mathrm{d}x \\ &\le C(T,L,\nu)\left(\int_0^T\int_0^{\frac{\nu}{2}} w_x^2\mathrm{d}x\mathrm{d}t + \int_0^T\int_0^L w^2\mathrm{d}x\mathrm{d}t\right). \end{aligned} \tag{4.46}$$

If the estimate

$$||w_0||^2_{L^2_{x\mathrm{d}x}} \le C\int_0^T\int_0^{\frac{\nu}{2}} w_x^2\mathrm{d}x\mathrm{d}t \tag{4.47}$$

fails, then one can find a sequence $\{w_0^n\} \subset L^2_{x\mathrm{d}x}$ such that

$$1 = ||w_0^n||^2_{L^2_{x\mathrm{d}x}} > n\int_0^T\int_0^{\frac{\nu}{2}} |w_x^n|^2\mathrm{d}x\mathrm{d}t, \tag{4.48}$$

where w^n denotes the solution of (4.45) with w_0 replaced by w_0^n. By (4.30) and (4.48), $\{w^n\}$ is bounded in $L^2(0,T,H^1(0,L))$, hence also in $H^1(0,T,H^{-2}(0,L))$ by (4.45). Extracting a subsequence, we have by Aubin–Lions's lemma that w^n converges strongly in $L^2(0,T,L^2(0,L))$. Thus, using (4.46) and (4.48), we see that w_0^n is a Cauchy sequence in $L^2_{x\mathrm{d}x}$, and hence it converges strongly in this space. Let w_0 denote its limit in $L^2_{x\mathrm{d}x}$, and let w denote the corresponding solution of (4.45). Then

$$\begin{aligned} &||w_0||_{L^2_{x\mathrm{d}x}} = 1, \\ &w^n \to w \qquad \text{in } L^2(0,T,H^1(0,L)). \end{aligned}$$

But $w_x^n \to 0$ in $L^2(0,T,L^2(0,\nu/2))$ by (4.48). Thus $w_x \equiv 0$ in $(0,T)\times(0,\nu/2)$, and hence $w(t,x) = g(t)$ (for some function g) in $(0,T)\times(0,\nu/2)$. Since w satisfies (4.45), we infer from $w(t,0) = 0$ that $w \equiv 0$ in $(0,T)\times(0,\nu/2)$, and also in $(0,T)\times(0,L)$ by Holmgren's theorem. This would imply that $w(0,x) = 0$, in contradiction with $||w_0||_{L^2_{x\mathrm{d}x}} = 1$. Therefore (4.47) is proved, and (4.44) follows at once.

We are in a position to apply H.U.M. Let $\Lambda(v_T)=(L-x)^{-1}u(T,.)\in L^2_{(L-x)\mathrm{d}x}$, where u solves (4.37)–(4.39) with $h=-\rho(x)v_x$. Then $\Lambda: L^2_{(L-x)\mathrm{d}x}\to L^2_{(L-x)\mathrm{d}x}$ is clearly continuous. On the other hand, from (4.43)

$$\big(\Lambda(v_T),v_T\big)_{L^2_{(L-x)\mathrm{d}x}}=\langle u(T,.),v_T\rangle_{L^2_{\frac{1}{L-x}\mathrm{d}x},L^2_{(L-x)\mathrm{d}x}}=\int_0^T ||\rho(x)v_x||^2_{L^2}\mathrm{d}t\geq C||v_T||^2_{L^2_{(L-x)\mathrm{d}x}},$$

and it follows that the map $v_T\to\Lambda(v_T)$ is invertible in $L^2_{(L-x)\mathrm{d}x}$.

Define the map $\Gamma:\ L^2_{\frac{1}{L-x}\mathrm{d}x}\to L^2(0,T,L^2(0,L))$ by $\Gamma(u_1)=h:=-\rho(x)v_x$, where v is the solution of (4.40)–(4.42) with $v_T=\Lambda^{-1}((L-x)^{-1}u_1)$. Γ is continuous from $L^2_{\frac{1}{L-x}\mathrm{d}x}$ to $L^2(0,T,L^2(0,L))$, and the solution u of (4.37)–(4.39) with $u_0=0$ and $h=\Gamma(u_1)$ satisfies $u(T,.)=u_1$. To prove that Γ is also continuous from $L^2_{\frac{1}{L-x}\mathrm{d}x}$ into $L^2_{(T-t)\mathrm{d}t}(0,T,H^1(0,L))$, it is sufficient to prove the following estimate

$$\int_0^T ||v(t)||^2_{H^2}(T-t)\mathrm{d}t\leq C||v_T||^2_{L^2_{(L-x)\mathrm{d}x}},$$

for the solutions of (4.40)–(4.42) or, alternatively, the estimate

$$\int_0^T ||w||^2_{H^2}\,t\mathrm{d}t\leq C||w_0||^2_{L^2_{x\mathrm{d}x}} \tag{4.49}$$

for the solutions of (4.45). By Proposition 4.5,

$$\int_0^T ||w||^2_{H^1_0(0,L)}\mathrm{d}t\leq C||w_0||^2_{L^2_{x\mathrm{d}x}}. \tag{4.50}$$

This yields for $w_0\in L^2(0,L)$

$$\int_0^T ||w||^2_{H^1_0(0,L)}\mathrm{d}t\leq C||w_0||^2_{L^2}. \tag{4.51}$$

Assume now that $w_0\in\mathcal{D}(A)$, and let $u_0=Aw_0=-w_{0,xxx}-w_{0,x}$. Denote by w (resp. u) the solution of (4.45) issuing from w_0 (resp. u_0). Then

$$Aw=-w_{xxx}-w_x=u\in L^2(0,T,H^1_0(0,L)),$$

and we infer that $w\in L^2(0,T,H^4(0,L))$. By interpolation, this gives that $w\in L^2(0,T,H^2(0,L))$ if $w_0\in H^1_0(0,L)$, with an estimate of the form

$$\int_0^T ||w||^2_{H^2(0,L)}\mathrm{d}t\leq C||w_0||^2_{H^1_0(0,L)}. \tag{4.52}$$

The different constants C in (4.50)–(4.52) may be taken independent of T for $0<T<T_0$. Thus, using Fubini's theorem, we obtain

$$\int_0^T s||w(s)||^2_{H^2}ds=\int_0^T\left(\int_t^T ||w(s)||^2_{H^2}ds\right)\mathrm{d}t\leq C\int_0^T ||w(t)||^2_{H^1_0(0,L)}\mathrm{d}t\leq C||w_0||^2_{L^2_{x\mathrm{d}x}}.$$

This completes the proof of (4.49) and of Theorem 4.7. □

4.6. Exact controllability of the nonlinear system

Our aim is to prove the local exact controllability in $L^2_{\frac{1}{L-x}\mathrm{d}x}$ of system (4.2). Note that the solutions of (4.2) can be written as

$$u = u_L + u_1 + u_2,$$

where u_L is the solution of (4.3) with initial data $u_0 \in L^2_{\frac{1}{L-x}\mathrm{d}x}$, u_1 is solution of

$$\begin{cases} u_{1,t} + u_{1,x} + u_{1,xxx} = f = (\rho(x)h)_x & \text{in } (0,T)\times(0,L),\\ u_1(t,0) = u_1(t,L) = u_{1,x}(t,L) = 0 & \text{in } (0,T),\\ u_1(0,x) = 0 & \text{in } (0,L) \end{cases} \tag{4.53}$$

with $h = h(t,x) \in L^2(0,T;L^2(0,L))$, and u_2 is solution of

$$\begin{cases} u_{2,t} + u_{2,x} + u_{2,xxx} = g(t,x) & \text{in } (0,T)\times(0,L),\\ u_2(t,0) = u_2(t,L) = u_{2,x}(t,L) = 0 & \text{in } (0,T),\\ u_2(0,x) = 0 & \text{in } (0,L), \end{cases} \tag{4.54}$$

with $g = g(t,x) = -uu_x$.

The following result is concerned with the solutions of the non-homogeneous system (4.54).

Proposition 4.8.
(i) *Let H and V be as in (4.16)–(4.17) If $u, v \in L^2(0,T;V)$, then $uv_x \in L^1(0,T;H)$. Furthermore, the map*

$$(u,v) \in L^2(0,T;V)^2 \to uv_x \in L^1(0,T;H)$$

is continuous and there exists a constant $c > 0$ such that

$$\|uv_x\|_{L^1(0,T;H)} \le c\,\|u\|_{L^2(0,T;V)}\,\|v\|_{L^2(0,T;V)}\,. \tag{4.55}$$

(ii) *For $g \in L^1(0,T;H)$, the mild solution u of (4.54) given by Duhamel formula satisfies*

$$u_2 \in C([0,T]\,;H) \cap L^2(0,T;V) =: \mathcal{G}$$

and we have the estimate

$$||u_2||_{L^\infty(0,T,H)} + ||u_2||_{L^2(0,T,V)} \le C||g||_{L^1(0,T,H)}. \tag{4.56}$$

Proof. For $u, v \in V$, we have

$$||uv_x||_{L^2_{\frac{1}{L-x}\mathrm{d}x}} \le ||u||_{L^\infty}\left\|\frac{v_x}{\sqrt{L-x}}\right\|_{L^2} \le C||u||_V||v||_V.$$

This gives (i). For (ii), we first assume that $g \in C^1([0,T],H)$, so that $u_2 \in C^1([0,T],H) \cap C^0([0,T],\mathcal{D}(A_2))$. Taking the inner product of $u_{2,t} = A_2u_2 + g$ with u_2 in H yields

$$(u_{2,t}, u_2)_H \le -C||u_2||_V^2 + C'||u_2||_H^2 + (g, u_2)_H \tag{4.57}$$

where C, C' denote some positive constants. Integrating over $(0,T)$ and using the classical estimate

$$||u_2||_{L^\infty(0,T,H)} \le C||g||_{L^1(0,T,H)}$$

coming from semigroup theory, we obtain (ii) when $g \in C^1([0,T],H)$. The general case ($g \in L^1(0,T,H)$) follows by density. ∎

Let $\Theta_1(h) := u_1$ and $\Theta_2(g) := u_2$, where u_1 (resp. u_2) denotes the solution of (4.53) (resp. (4.54)). Then $\Theta_1 : L^2(0,T;L^2(0,L)) \to \mathcal{G}$ and $\Theta_2 : L^1(0,T;L^2_{\frac{1}{L-x}\mathrm{d}x}) \to \mathcal{G}$ are well-defined continuous operators, by Propositions 4.6 and 4.8.

Using Proposition 4.8 and the contraction mapping principle, one can prove as in [10, 19, 20] the existence and uniqueness of a solution $u \in \mathcal{G}$ of (4.2) when the initial data u_0 and the forcing term h are small enough. As the proof is similar to those of Theorem 4.9, it will be omitted.

We are in a position to prove the main result of Section 4, namely the (local) exact controllability of system (4.2).

Theorem 4.9. *Let $T > 0$. Then there exists $\delta > 0$ such that for any u_0, $u_1 \in L^2_{\frac{1}{L-x}\mathrm{d}x}$ satisfying $\|u_0\|_{L^2_{\frac{1}{L-x}\mathrm{d}x}} \leq \delta$, $\|u_1\|_{L^2_{\frac{1}{L-x}\mathrm{d}x}} \leq \delta$, one can find a control function $h \in L^2(0,T;L^2(0,L))$ such that the solution $u \in \mathcal{G}$ of (4.2) satisfies $u(T,\cdot) = u_1$ in $(0,L)$.*

As in the linear case, the forcing term $f = (\rho(x)h)_x$ is actually a *function* in $L^2_{(T-t)\mathrm{d}t}(0,T,L^2(0,L))$ supported in $(0,T) \times (L-\nu, L)$.

Proof. To prove this result, we apply the contraction mapping principle, following closely [20]. Let $\mathcal{F}$ denote the nonlinear map

$$\mathcal{F} : L^2(0,T;V) \to \mathcal{G},$$

defined by

$$\mathcal{F}(u) = u_L + \Theta_1 \circ \Gamma(u_T - u_L(T,\cdot) + \Theta_2(uu_x)(T,\cdot)) - \Theta_2(uu_x),$$

where u_L is the solution of (4.3) with initial data $u_0 \in L^2_{\frac{1}{L-x}\mathrm{d}x}$, Θ_1 and Θ_2 are defined as above, and Γ is as in Theorem 4.7.

Remark that if u is a fixed point of $\mathcal{F}$, then u is a solution of (4.2) with the control $h = \Gamma(u_T - u_L(T,\cdot) + \Theta_2(uu_x)(T,\cdot))$, and it satisfies

$$u(T,\cdot) = u_T,$$

as desired. In order to prove the existence of a fixed point of $\mathcal{F}$, we apply the Banach fixed-point theorem to the restriction of $\mathcal{F}$ to some closed ball $\overline{B}(0,R)$ in $L^2(0,T;V)$.

(i) *$\mathcal{F}$ is contractive.* Pick any $u, \tilde{u} \in \overline{B}(0,R)$. Using (4.35) and (4.55)–(4.56), we deduce that for some constant C, independent of u, $\tilde{u}$, and R, we have

$$\|\mathcal{F}(u) - \mathcal{F}(\tilde{u})\|_{L^2(0,T;V)} \leq 2CR\,\|u - \tilde{u}\|_{L^2(0,T;V)}\,. \tag{4.58}$$

Hence, $\mathcal{F}$ is contractive if R satisfies

$$R < \frac{1}{4C}, \tag{4.59}$$

where C is the constant in (4.58).

(ii) *$\mathcal{F}$ maps $\overline{B}(0,R)$ into itself.* Using Proposition 4.4 and the continuity of the operators Γ, Θ_1, and Θ_2, we infer the existence of a constant $C' > 0$ such that for any $u \in \overline{B}(0,R)$, we have

$$\|\mathcal{F}(u)\|_{L^2(0,T;V)} \leq C'\left(\|u_0\|_{L^2_{\frac{1}{L-x}\mathrm{d}x}} + \|u_T\|_{L^2_{\frac{1}{L-x}\mathrm{d}x}} + R^2\right).$$

Thus, taking R satisfying (4.59) and $R < 1/(2C')$ and assuming that $\|u_0\|_{L^2_{\frac{1}{L-x}\mathrm{d}x}}$ and $\|u_T\|_{L^2_{\frac{1}{L-x}\mathrm{d}x}}$ are small enough, we obtain that the operator $\mathcal{F}$ maps $\overline{B}(0,R)$ into itself. Therefore the map $\mathcal{F}$ has a fixed point in $\overline{B}(0,R)$ by the Banach fixed-point Theorem. The proof of Theorem 4.9 is complete. □

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