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SHARP INTERFACE CONTROL IN A PENROSE−FIFE MODEL

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Abstract. In this paper we study a singular control problem for a system of PDEs describing a phase-field model of Penrose−Fife type. The main novelty of this contribution consists in the idea of forcing a sharp interface separation between the states of the system by using heat sources distributed in the domain and at the boundary. We approximate the singular cost functional with a regular one, which is based on the Legendre−Fenchel relations. Then, we obtain a regularized control problem for which we compute the first order optimality conditions using an adapted penalization technique. The proof of some convergence results and the passage to the limit in these optimality conditions lead to the characterization of the desired optimal controller.

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1. Introduction

We are concerned with a control problem of a system governed by the Penrose−Fife phase transition model. Using the distributed heat source and the boundary heat source as controllers we aim at forcing a sharp interface separation between the states of the system, while keeping its temperature at a certain average level θ_f.

The phase-field model considered here has been proposed by Penrose and Fife in [20] and [21] as a thermodynamically consistent model for the description of the kinetics of phase transition and phase separation processes in binary materials. It is a PDE system coupling a singular heat equation (as seen in (1.1) below) for the absolute temperature θ with a nonlinear equation which describes the evolution of the phase variable φ (see (1.2)), which represents the local fraction of one of the two components. These equations are accompanied by initial data for θ and φ (*cf.* (1.5) and (1.6)) and by boundary conditions, considered here of Robin type for θ (*cf.* (1.3)) and of homogeneous Neumann type for φ (*cf.* (1.4)), according to physical considerations. As far as the Penrose−Fife model is concerned, a vast literature is devoted to the well-posedness (*cf.*, *e.g.* [3, 6, 11, 14, 15, 17, 24]) and to

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the long-time behavior of solutions both in term of attractors (*cf.*, *e.g.* [12,22,23]) and of convergence of single trajectories to stationary states (*cf.*, *e.g.*, [7,9]), while the associated control problem is less studied in the literature.

A control problem was introduced first in [25] for a Penrose−Fife type model with Robin-type boundary conditions for the temperature and a heat flux proportional to the gradient of the inverse absolute temperature. The first order optimality conditions were derived without imposing any local constraint on the state and only in case of a double-well potential in the phase equation. Later on the study has been refined in [10], where the authors succeeded in removing such restrictions on the problem and treating the case with state constraints.

Let us finally quote the paper [8] where a phase transition system was controlled by means of the heat supply in order to be guided into a certain state with a solid (or liquid) part in a prescribed subset of the space domain, by maintaining it in this state during a period of time. The system was controlled to form a diffusive boundary between the solid and liquid states.

Coming back to our problem, we assume here that the phase transition takes place in the interval $(0,T)$, with T finite, and that the system occupies an open bounded domain Ω of $\mathbb{R}^3$, having the boundary Γ sufficiently smooth. The Penrose−Fife system we are interested in reads (see [3,20,22])

$$\theta_t - \Delta\beta(\theta) + \varphi_t = u, \quad \text{in } Q := (0,T)\times\Omega, \tag{1.1}$$

$$\varphi_t - \Delta\varphi + (\varphi^3 - \varphi) = \frac{1}{\theta_c} - \frac{1}{\theta}, \quad \text{in } Q, \tag{1.2}$$

$$-\frac{\partial\beta(\theta)}{\partial\nu} = \alpha(x)(\beta(\theta) - v), \quad \text{on } \Sigma := (0,T)\times\Gamma, \tag{1.3}$$

$$\frac{\partial\varphi}{\partial\nu} = 0, \quad \text{on } \Sigma, \tag{1.4}$$

$$\theta|_{t=0} = \theta_0, \quad \text{in } \Omega, \tag{1.5}$$

$$\varphi|_{t=0} = \varphi_0, \quad \text{in } \Omega, \tag{1.6}$$

where $\beta \in C^1(0,\infty)$ and $\beta(r)$ behaves like $-c_1/r$ closed to 0 and like $c_2 r$ in a neighborhood of $+\infty$, for some constants c_1 and c_2. Then, for the sake of simplicity we can assume that

$$\beta(r) = -\frac{1}{r} + r. \tag{1.7}$$

Next, we let

$$\alpha \in H^1(\Gamma)\cap L^\infty(\Gamma), \quad 0 < \alpha_m \le \alpha(x) \le \alpha_M \quad \text{a.e. } x\in\Gamma, \tag{1.8}$$

with α_m, α_M constants. The constant θ_c is the transition temperature, u is the distributed heat source and v is the boundary heat source.

Note that the heat flux law (1.7) is a common choice in several types of phase-transition and phase-separation models both in liquids and in crystalline solids (*cf.*, *e.g.*, [3,20,22] where similar growth conditions are postulated).

We denote the signum multivalued function (translated by θ_c) by

$$H(r) = \begin{cases} 1, & r > \theta_c \\ [-1,1], & r = \theta_c \\ -1, & r < \theta_c \end{cases} \tag{1.9}$$

and this will be useful to set the third control variable in our problem. Indeed, let us define the cost functional as

$$J(u,v,\eta) = \lambda_1 \int_Q (\theta - \theta_f)^2 \,\mathrm{d}x\,\mathrm{d}t + \lambda_2 \int_Q (\varphi - \eta)^2 \,\mathrm{d}x\,\mathrm{d}t \tag{1.10}$$

and introduce the control problem:

$$\text{Minimize } J(u,v,\eta) \text{ for all } (u,v,\eta) \in K_1 \times K_2 \times K_3, \tag{P}$$

subject to (1.1)–(1.6), where

$$K_1 = \{u \in L^\infty(Q) : \ u_m \le u(t,x) \le u_M \ \text{ a.e. } (t,x) \in Q\}, \tag{1.11}$$
$$K_2 = \{v \in L^\infty(\Sigma) : \ v_m \le v(t,x) \le v_M \ \text{ a.e. } (t,x) \in \Sigma\}, \tag{1.12}$$
$$K_3 = \{\eta \in L^\infty(Q) : \ \eta(t,x) \in H(\theta(t,x)) \ \text{ a.e. } (t,x) \in Q\}, \tag{1.13}$$

and u_m, u_M, v_m, v_M are fixed real values. The positive constants λ_1, λ_2 are used to give more importance to one term or the other in (P).

With a general approach, we can consider that

$$\theta_f \text{ is a function of } t \text{ and } x, \text{ and } \theta_f \in L^2(Q). \tag{1.14}$$

All the results in this paper hold under this condition. If by the control problem one intends to preserve the system separated in two phases by the sharp interface, it should be added that θ_f must belong to a neighboorhood of θ_c, *i.e.*, $\|\theta_f - \theta_c\|_{L^2(Q)} \le \delta$, with δ rather small.

The problem (P) is introduced in order to enforce the formation of a sharp interface between the two phases by the constraint $\eta \in K_3$. As far as we know such a control problem has not been previously studied.

Let us note, however, that the well-posedness of an initial-boundary value Stefan-type problem with phase relaxation or with standard interphase equilibrium conditions (*cf.* (1.13)), where the heat flux is proportional to the gradient of the inverse absolute temperature, was studied in [4, 5], for Robin-type boundary conditions. It was shown in these contributions that the Stefan problems with singular heat flux are the natural limiting cases of a thermodynamically consistent model of Penrose−Fife type.

The layout of this paper is as follows. In Section 2, Theorem 2.2, we review the existence results for the state system and provide new results concerning the supplementary regularity of the state which will be necessary in the computation of the optimality conditions. Then, we prove in Theorem 2.3 the existence of at least one solution to problem (P), represented by an optimal triplet of controllers (u, v, η) and the corresponding pair of states (θ, φ).

Due to the singularity induced by the graph representing the sharp interface, the conditions of optimality cannot be deduced directly for (P). In order to avoid working with the graph $H(\theta)$, in Section 3 we introduce an approximating problem (P_ε) in which the constraint $\eta \in H(\theta)$ is replaced by an equivalent relation based on the Legendre−Fenchel relations between a proper convex lower semicontinuous (l.c.s.) function j and its conjugate, j^*. In this case j is the potential of H. This approximating problem has at least one solution (see Prop. 3.1) which is the appropriate approximation of a solution to (P). This last assertion relies on the convergence result of (P_ε) to (P) given in Theorem 3.2. In Section 4, we rigorously examine the question concerning the computation of the optimality conditions. A second approximation is represented by a penalized minimization problem $(P_{\varepsilon,\sigma})$ in which j is replaced by its Moreau−Yosida regularization. The optimality conditions for $(P_{\varepsilon,\sigma})$ are provided by explicit expressions in Proposition 4.5. Some estimates and the proof of the strong convergence (as $\sigma \to 0$) of the controllers in $(P_{\varepsilon,\sigma})$ allow the passage to the limit as $\sigma \to 0$ in order to recover the form of the controllers in problem (P_ε): this is performed in Theorem 4.6. Recalling Theorem 2.3, the optimal controller in (P) is obtained as the limit of a sequence of optimal controllers in (P_ε), on the basis of the convergence of (P_ε) to (P).

2. Existence in the state system and control problem

We denote by V the Sobolev space $H^1(\Omega)$ endowed with the standard scalar product

$$\|\psi\|_V = \left(\int_\Omega |\nabla\psi(x)|^2\,\mathrm{d}x + \int_\Omega |\psi(x)|^2\,\mathrm{d}x\right)^{1/2}. \tag{2.1}$$

We identify $L^2(\Omega)$ with its dual space, in order that $V \subset L^2(\Omega) \subset V'$ with dense and compact embeddings. We recall that if α satisfies (1.8), then the norm

$$|||\,\psi\,||| = \int_\Omega |\nabla\psi(x)|^2\,\mathrm{d}x + \int_\Gamma \alpha(x)\,|\psi(x)|^2\,\mathrm{d}s \tag{2.2}$$

is equivalent to $\|\psi\|_V$, due to the inequality

$$\|\psi\|_V^2 \le C_P\left(\int_\Omega |\nabla\psi(x)|^2\,\mathrm{d}x + \int_\Gamma |\psi(x)|^2\,\mathrm{d}s\right),\ \forall\psi\in V, \tag{2.3}$$

(see [18], p. 20), with C_P depending on Ω. For simplicity, in the following let us not indicate the arguments of functions in the integrals.

Definition 2.1. Let

$$\begin{aligned}\theta_0 \in L^2(\Omega),\quad \theta_0>0\ \text{ a.e. in } \Omega,\quad \ln\theta_0\in L^1(\Omega),\quad \varphi_0\in H^1(\Omega),\\ u\in L^2(Q),\quad v\in L^2(\Sigma),\quad \alpha \text{ satisfies (1.8)}.\end{aligned} \tag{2.4}$$

We call a *solution* to (1.1)–(1.6) a pair (θ,φ) such that

$$\theta\in L^2(0,T;V)\cap C([0,T];L^2(\Omega))\cap W^{1,2}([0,T];V'),\quad \beta(\theta),\ \frac{1}{\theta}\in L^2(0,T;V), \tag{2.5}$$

$$\varphi\in L^2(0,T;H^2(\Omega))\cap C([0,T];V)\cap W^{1,2}([0,T];L^2(\Omega)), \tag{2.6}$$

which satisfies (1.1)–(1.4) in the form

$$\begin{aligned}\int_0^T \left\langle\frac{\mathrm{d}\theta}{\mathrm{d}t}(t),\psi_1(t)\right\rangle_{V',V}\mathrm{d}t + \int_Q \nabla\beta(\theta)\cdot\nabla\psi_1\,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{\mathrm{d}\varphi}{\mathrm{d}t}\psi_1\,\mathrm{d}x\,\mathrm{d}t\\ + \int_\Sigma \alpha\beta(\theta)\psi_1\,\mathrm{d}s\,\mathrm{d}t = \int_Q u\psi_1\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma \alpha v\psi_1\,\mathrm{d}s\,\mathrm{d}t,\end{aligned} \tag{2.7}$$

$$\begin{aligned}\int_Q \frac{\mathrm{d}\varphi}{\mathrm{d}t}\psi_2\,\mathrm{d}x\,\mathrm{d}t + \int_Q \nabla\varphi\cdot\nabla\psi_2\,\mathrm{d}x\,\mathrm{d}t + \int_Q (\varphi^3-\varphi)\psi_2\,\mathrm{d}x\,\mathrm{d}t\\ = \int_Q\left(\frac{1}{\theta_c}-\frac{1}{\theta}\right)\psi_2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{2.8}$$

for any $\psi_1,\ \psi_2\in L^2(0,T;V)$, and such that the initial conditions (1.5)–(1.6) hold.

The next statement collects a number of properties of solutions to (1.1)–(1.6).

Theorem 2.2. *Let assumptions (2.4) hold. Then (1.1)–(1.6) has a unique solution, fulfiling*

$$\theta>0\ \text{ a.e. in } Q,\quad \ln\theta\in L^\infty(0,T;L^1(\Omega))$$

and satisfying the estimates

$$\|\theta\|_{L^2(0,T;V)} + \|\theta\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta\|_{W^{1,2}([0,T];V')} + \left\|\frac{1}{\theta}\right\|_{L^2(0,T;V)} \le C, \tag{2.9}$$

$$\|\varphi\|_{L^2(0,T;H^2(\Omega))} + \|\varphi\|_{L^\infty(0,T;V)} + \|\varphi\|_{W^{1,2}([0,T];L^2(\Omega))} \le C. \tag{2.10}$$

Moreover, let us set $\bar\theta := \theta_1 - \theta_2$, $\bar\varphi := \varphi_1 - \varphi_2$, $\bar u := u_1 - u_2$, $\bar v := v_1 - v_2$, *where* (θ_1, φ_1), *and* (θ_2, φ_2) *are the solutions of* (1.1)–(1.6) *corresponding respectively to the data* u_1, v_1 *and* u_2, v_2, *to the same initial data* θ_0, φ_0 *and to the same coefficient* α*; then, we have the following continuous dependence estimate of the solution with respect to the data:*

$$\|\bar\theta\|^2_{L^2(Q)} + \|\bar\varphi\|^2_{C([0,T];L^2(\Omega))} + \|\bar\varphi\|^2_{L^2(0,T;V)} \le C\left(\|\bar u\|_{L^2(Q)} + \|\bar v\|^2_{L^2(\Sigma)}\right), \tag{2.11}$$

with the positive constant C *depends only on the problem parameters, but not on* u_i, v_i, $1 = 1, 2$. *Next, we list some regularity properties of the solution: if, in addition to* (2.4)*, we suppose that*

$$\varphi_0 \in H^2(\Omega), \quad \frac{\partial \varphi_0}{\partial \nu} = 0 \text{ on } \Gamma, \tag{2.12}$$

then we have

$$\varphi \in L^\infty(Q) \cap L^\infty(0,T;H^2(\Omega)) \cap W^{1,2}([0,T];V) \tag{2.13}$$

and

$$\|\varphi\|_{L^\infty(Q)} + \|\varphi\|_{L^\infty(0,T;H^2(\Omega))} + \|\varphi\|_{W^{1,2}([0,T];V)} \le C; \tag{2.14}$$

further, if, in addition to (2.4)*, there hold*

$$\theta_0,\ \frac{1}{\theta_0} \in L^\infty(\Omega), \quad u \in L^2(0,T;L^6(\Omega)), \quad v \in L^\infty(\Sigma) \ \text{and} \quad v \le v_M \quad \text{a.e. in } \Sigma, \tag{2.15}$$

then we have

$$\theta,\ \frac{1}{\theta} \in L^\infty(Q) \tag{2.16}$$

with

$$\|\theta\|_{L^\infty(Q)} + \left\|\frac{1}{\theta}\right\|_{L^\infty(Q)} \le C, \tag{2.17}$$

where C *denotes several positive constants depending only on the problem parameters.*

Proof. The proof of existence of solutions to (1.1)–(1.6) follows from an adaptation of ([3], Thm. 2.3) to the case of α non constant in (1.3). The uniqueness of solutions has been proved in ([6], Thm. 1) and it has been then generalized to the case of less regular data (satisfying exactly assumptions (2.4)) in ([22], Thm. 3.5), where also a continuous dependence result of the solution with respect to the data has been shown. We also refer to the above-mentioned papers for the proof of estimates (2.9)–(2.10). In what follows the positive constants C, which may also differ from line to line, will depend only on the problem parameters.

Proof of estimate (2.11). Following the lines of ([6], Thm. 1) and ([22], Thm. 3.5), we write (1.1) firstly for (θ_1, φ_1) and then for (θ_2, φ_2), (θ_1, φ_1), and (θ_2, φ_2) being two solutions to (1.1)–(1.6) corresponding to the data u_1, v_1 and u_2, v_2, respectively, to the same initial data θ_0, φ_0, and to the same coefficient α. Taking the difference, integrating with respect to time, choosing as test function $\psi_1 = \beta(\theta_1) - \beta(\theta_2)$, and using the monotonicity properties of the function $\theta \mapsto -1/\theta$, we find out that

$$\begin{aligned}
&\int_{Q_t} |\bar{\theta}|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_{\Omega} |\nabla 1 * (\beta(\theta_1) - \beta(\theta_2))\,(t)|^2 \,\mathrm{d}x \\
&\qquad + \int_{Q_t} \bar{\varphi}\,(\beta(\theta_1) - \beta(\theta_2))\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_{\Gamma} \alpha\,|1 * (\beta(\theta_1) - \beta(\theta_2))\,(t)|^2\,\mathrm{d}s \\
&\leq \int_{Q_t} (1 * \bar{u})\,(\beta(\theta_1) - \beta(\theta_2))\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\Sigma_t} \alpha\,(1 * \bar{v})\,(\beta(\theta_1) - \beta(\theta_2))\,\mathrm{d}s\,\mathrm{d}\tau, \qquad (2.18)
\end{aligned}$$

where $1 * b(t) := \int_0^t b(\tau)\,\mathrm{d}\tau,\ t \in (0, T]$. Here we have used the notation of the statement for $\bar{\theta}$, $\bar{\varphi}$, $\bar{u}$, $\bar{v}$. Next, taking the differences of (1.2), testing by $\psi_2 = \bar{\varphi}$, and exploiting the monotonicity of $\varphi \mapsto \varphi^3$, we have that

$$\frac{1}{2}\int_{\Omega} |\bar{\varphi}(t)|^2 \,\mathrm{d}x + \int_{Q_t} |\nabla\bar{\varphi}|^2 \,\mathrm{d}x\,\mathrm{d}\tau \leq \int_{Q_t} |\bar{\varphi}|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \int_{Q_t} \left(-\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)\bar{\varphi}\,\mathrm{d}x\,\mathrm{d}\tau. \qquad (2.19)$$

Now, summing up (2.18) and (2.19), we take advantage of a cancellation of one term due to the special form (1.7) of β. Then, in view of assumption (1.8) on α, we arrive at

$$\begin{aligned}
&\int_{Q_t} |\theta|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \|1 * (\beta(\theta_1) - \beta(\theta_2))\,(t)\|_V^2 + \|\bar{\varphi}(t)\|^2_{L^2(\Omega)} + \int_{Q_t} |\nabla\bar{\varphi}|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq C_1 \left(\int_{Q_t} |\bar{\varphi}\bar{\theta}|\,\mathrm{d}x\,\mathrm{d}\tau + \int_{Q_t} |(1 * \bar{u})\,(\beta(\theta_1) - \beta(\theta_2))\,|\,\mathrm{d}x\,\mathrm{d}\tau \right. \\
&\qquad \left. + \int_{\Sigma_t} |\alpha\,(1 * \bar{v})\,(\beta(\theta_1) - \beta(\theta_2))\,|\,\mathrm{d}s\,\mathrm{d}\tau + \int_{Q_t} |\bar{\varphi}|^2 \,\mathrm{d}x\,\mathrm{d}\tau \right). \qquad (2.20)
\end{aligned}$$

Let us now estimate the integrals on the right hand side of (2.20) as follows. Using Young's inequality and integrating by parts in time, we obtain

$$\begin{aligned}
&\int_{Q_t} |(1 * \bar{u})\,(\beta(\theta_1) - \beta(\theta_2))\,|\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq \int_{\Omega} |1 * \bar{u}|(t)|1 * (\beta(\theta_1) - \beta(\theta_2))\,|(t)\,\mathrm{d}x + \int_{Q_t} |\bar{u}\,(1 * (\beta(\theta_1) - \beta(\theta_2)))\,|\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq \delta_1 \|1 * (\beta(\theta_1) - \beta(\theta_2))\,(t)\|_V^2 + C_{\delta_1}\|1 * \bar{u}(t)\|^2_{L^2(\Omega)} \\
&\qquad + \int_{Q_t} |\bar{u}|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \|1 * (\beta(\theta_1) - \beta(\theta_2))\,\|^2_{L^2(\Omega)}\,\mathrm{d}\tau. \qquad (2.21)
\end{aligned}$$

Thanks to (1.8), we get

$$\begin{aligned}
&\int_{\Sigma_t} |\alpha\,(1 * \bar{v})\,(\beta(\theta_1) - \beta(\theta_2))\,|\,\mathrm{d}s\,\mathrm{d}\tau \\
&\leq \int_{\Gamma} |\alpha||1 * \bar{v}|(t)|1 * (\beta(\theta_1) - \beta(\theta_2))\,|(t)\,\mathrm{d}s + \int_{\Sigma_t} |\alpha\bar{v}\,(1 * (\beta(\theta_1) - \beta(\theta_2)))\,|\,\mathrm{d}s\,\mathrm{d}\tau \\
&\leq \delta_2 \|1 * (\beta(\theta_1) - \beta(\theta_2))\,(t)\|_V^2 + C_{\delta_2}\|1 * \bar{v}(t)\|^2_{L^2(\Gamma)} \\
&\qquad + \int_{\Sigma_t} |\bar{v}|^2 \,\mathrm{d}s\,\mathrm{d}\tau + \int_0^t \|1 * (\beta(\theta_1) - \beta(\theta_2))\,\|^2_{L^2(\Gamma)}\,\mathrm{d}\tau. \qquad (2.22)
\end{aligned}$$

Collecting now estimates (2.20)–(2.22), using Young's inequality once more in order to treat the first integral on the right hand side of (2.20), and choosing properly the constants δ_1 and δ_2, we infer that

$$\int_{Q_t} |\theta|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \|1 * (\beta(\theta_1) - \beta(\theta_2))\,(t)\|_V^2 + \|\bar{\varphi}(t)\|_{L^2(\Omega)}^2 + \int_{Q_t} |\nabla\bar{\varphi}|^2 \,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leq C_2 \left(\int_0^t \|\bar{\varphi}(\tau)\|_{L^2(\Omega)}^2 \mathrm{d}\tau + \int_0^t \|1 * (\beta(\theta_1) - \beta(\theta_2))\|_V^2 \,\mathrm{d}\tau + \|\bar{u}\|_{L^2(Q_t)}^2 + \|\bar{v}\|_{L^2(\Sigma_t)}^2 \right),$$

from which, using a standard Gronwall lemma, we deduce the desired (2.11).

Proof of estimate (2.14). In order to prove the regularity (2.13) and estimate (2.14), we can proceed formally testing (1.2) by $-\Delta\varphi_t$ and integrating by parts with the help of (1.4). This choice should be made rigorous in the framework of a regularized scheme, *e.g.*, of Faedo−Galerkin type, but we prefer to proceed formally here in order not to overburden the presentation. Making this formal computation and integrating the resulting equation over $(0,t)$, $t \in (0,T]$, we get

$$\frac{1}{2}\|\Delta\varphi(t)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\Delta\varphi_0\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla\varphi_t\|_{L^2(\Omega)}^2 \,\mathrm{d}\tau$$
$$\leq \int_0^t \left\|\frac{1}{\theta}\right\|_V \|\nabla\varphi_t\|_{L^2(\Omega)} \,\mathrm{d}\tau + \int_0^t \int_\Omega \nabla(\varphi^3 - \varphi)\nabla\varphi_t \,\mathrm{d}x\,\mathrm{d}\tau. \tag{2.23}$$

In order to estimate the first integral on the right hand side we can just use the Young inequality together with estimate (2.9). The last integral, instead, can be treated as follows:

$$\int_0^t \int_\Omega \nabla(\varphi^3 - \varphi)\nabla\varphi_t \,\mathrm{d}x\,\mathrm{d}\tau \leq C \int_0^t \left(\|\varphi\|_{L^6(\Omega)}^2 + 1 \right) \|\nabla\varphi\|_{L^6(\Omega)} \|\nabla\varphi_t\|_{L^2(\Omega)} \,\mathrm{d}\tau$$
$$\leq \frac{1}{4} \int_0^t \|\nabla\varphi_t\|_{L^2(\Omega)}^2 \,\mathrm{d}\tau + C \int_0^t (1 + \|\Delta\varphi\|_{L^2(\Omega)}^2) \,\mathrm{d}\tau, \tag{2.24}$$

where the Hölder and Young inequalities have been used together with the Gagliardo-Nirenberg inequality ([19], p. 125) and the previous estimate (2.10). By rearranging in (2.23) and using once more (2.10) for the boundedness of φ in $L^2(0,T;H^2(\Omega))$, we obtain the estimate

$$\|\Delta\varphi\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla\varphi_t\|_{L^2(Q)} \leq C,$$

which, together with (2.9)–(2.10), the standard elliptic regularity results and the continuous embedding of $L^\infty(0,T;H^2(\Omega))$ into $L^\infty(Q)$ in 3D, gives the desired (2.14).

Proof of estimate (2.17). We aim first to prove the $L^\infty(Q)$-bound for θ. In order to do that, we use a Moser-type technique. The procedure consists in testing (1.1) by $(p+1)\theta^p$, $p \in (1,\infty)$. This estimate is formal (*cf.* (2.7)); indeed, in order to perform it rigorously we would need to introduce a regularized (truncated) system and then pass to the limit. However, since the procedure is quite standard, we prefer to perform only the formal estimate here. Testing (1.1) by $(p+1)\theta^p$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta^{p+1}\,\mathrm{d}x + \frac{4p}{p+1}\int_\Omega \left|\nabla\theta^{\frac{p+1}{2}}\right|^2 \mathrm{d}x + \frac{4p(p+1)}{(p-1)^2}\int_\Omega \left|\nabla\theta^{\frac{p-1}{2}}\right|^2 \mathrm{d}x + (p+1)\int_\Gamma \alpha\theta^{p+1}\,\mathrm{d}s$$
$$= (p+1)\int_\Gamma \alpha\theta^{p-1}\,\mathrm{d}s + (p+1)\int_\Gamma \alpha v\theta^p\,\mathrm{d}s + (p+1)\int_\Omega (u - \varphi_t)\theta^p\,\mathrm{d}x.$$

Using now Assumptions (2.4) and (2.15), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta^{p+1}\,\mathrm{d}x + \frac{4p}{p+1}\int_\Omega \left|\nabla\theta^{\frac{p+1}{2}}\right|^2 \mathrm{d}x + (p+1)\alpha_m \int_\Gamma \left|\theta^{\frac{p+1}{2}}\right|^2 \mathrm{d}s$$
$$\leq (p+1)\alpha_M \int_\Gamma \theta^{p-1}\,\mathrm{d}s + (p+1)\alpha_M v_M \int_\Gamma \theta^p\,\mathrm{d}s + (p+1)\int_\Omega (u - \varphi_t)\theta^p\,\mathrm{d}x. \tag{2.25}$$

Owing to the Young inequality in the form $a \cdot b \le \epsilon \frac{a^q}{q} + \frac{1}{\epsilon^{q'/q}} \cdot \frac{b^{q'}}{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$, we estimate the two boundary terms as follows:

$$\alpha_M \int_\Gamma \theta^{p-1}\,\mathrm{d}s \le \frac{\alpha_m}{4}\frac{(p-1)}{p+1}\int_\Gamma \theta^{p+1}\,\mathrm{d}s + \left(\frac{4}{\alpha_m}\right)^{\frac{(p-1)}{2}} \frac{\alpha_M^{\frac{p+1}{2}}}{(p+1)} 2|\Gamma|, \tag{2.26}$$

$$\alpha_M v_M \int_\Gamma \theta^{p}\,\mathrm{d}s \le \frac{\alpha_m}{4}\frac{p}{(p+1)}\int_\Gamma \theta^{p+1}\,\mathrm{d}s + \left(\frac{4}{\alpha_m}\right)^{p} \frac{(\alpha_M v_M)^{p+1}}{(p+1)} |\Gamma|. \tag{2.27}$$

Thanks to estimates (2.26)–(2.27), (2.25) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta^{p+1}\,\mathrm{d}x + \delta\left(\int_\Omega \left|\nabla\theta^{\frac{p+1}{2}}\right|^2 \mathrm{d}x + \int_\Gamma \left|\theta^{\frac{p+1}{2}}\right|^2 \mathrm{d}s\right)$$
$$\le CR^{p+1} + (p+1)\int_\Omega |u-\varphi_t|\theta^p\,\mathrm{d}x,$$

where $\delta := \min\{\frac{3}{2}\alpha_m, \frac{8}{3}\} > 0$ and C, R are independent of p. Using then the continuous embedding of $V = H^1(\Omega)$ into $L^6(\Omega)$ in 3D, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta^{p+1}\,\mathrm{d}x + \delta'\left(\int_\Omega \theta^{3(p+1)}\,\mathrm{d}x\right)^{1/3} \le CR^{p+1} + (p+1)\int_\Omega |u-\varphi_t|\theta^p\,\mathrm{d}x, \tag{2.28}$$

for some $\delta' > 0$ always independent of p. Then, as (2.14) entails the boundedness of φ_t in $L^2(0,T;L^6(\Omega))$, we estimate the last integral as follows:

$$\begin{aligned}
&(p+1)\int_\Omega |u-\varphi_t|\theta^p\,\mathrm{d}x \\
&\le (p+1)\|u-\varphi_t\|_{L^6(\Omega)}\left(\int_\Omega \theta^{3(p+1)}\,\mathrm{d}x\right)^{1/6}\left(\int_\Omega \theta^{\frac{3}{2}\frac{(p-1)}{2}}\,\mathrm{d}x\right)^{2/3} \\
&\le \frac{\delta'}{2}\left(\int_\Omega \theta^{3(p+1)}\,\mathrm{d}x\right)^{1/3} + C_{\delta'}(p+1)^2\|u-\varphi_t\|^2_{L^6(\Omega)}\left(\int_\Omega \theta^{\frac{3}{4}(p-1)}\,\mathrm{d}x\right)^{4/3} \\
&\le \frac{\delta'}{2}\left(\int_\Omega \theta^{3(p+1)}\,\mathrm{d}x\right)^{1/3} + C_{\delta'}(p+1)^2\|u-\varphi_t\|^2_{L^6(\Omega)}\left(\int_\Omega \theta^{p-1}\,\mathrm{d}x\right),
\end{aligned} \tag{2.29}$$

where we have also used the inequality $\|\theta^{\frac{p-1}{2}}\|^2_{L^{3/2}(\Omega)} \le C(\Omega)\|\theta^{\frac{p-1}{2}}\|^2_{L^2(\Omega)}$. Choosing now $p=3$ in (2.28)–(2.29) and integrating with respect to time, with the help of (2.9) and (2.15) we obtain

$$\|\theta\|^4_{L^\infty(0,T;L^4(\Omega))} \le C\left(1 + \int_0^T \|u-\varphi_t\|^2_{L^6(\Omega)}\,\mathrm{d}\tau\right). \tag{2.30}$$

In general, integrating (2.28) from 0 to t, $t \in (0,T]$, and using (2.29) and (2.15), we infer that

$$\int_\Omega \theta^{p+1}(t)\,\mathrm{d}x \le C\left(R^{p+1} + (p+1)^2 \sup_{[0,T]}\left(\int_\Omega \theta^{\frac{3}{4}(p-1)}\,\mathrm{d}x\right)^{4/3}\right), \tag{2.31}$$

where C depends on the data but not on p. At this point, we can introduce the sequence $p_0 = 4$, $p_{k+1} = \frac{4}{3}p_k + 2$, $k \in \mathbb{N}$ and take $p = p_{k+1} - 1$ in (2.31), getting

$$\int_\Omega \theta^{p_{k+1}}(t)\,\mathrm{d}x \le C\left(R^{p_{k+1}} + (p_{k+1})^2 \sup_{[0,T]}\left(\int_\Omega \theta^{p_k}\,\mathrm{d}x\right)^{4/3}\right).$$

We can apply now ([13], Lem. A.1) with the choices $a = 4/3$, $b = c = 2$, $\delta_0 = 4$, $\delta_k = p_k$. Thus, we deduce that $\sup_{[0,T]} \|\theta\|_{L^{p_k}(\Omega)} \leq C$, where C is independent of k. Hence, letting k tend to ∞, we get

$$\|\theta\|_{L^\infty(Q)} \leq C. \tag{2.32}$$

Finally, we aim to prove the $L^\infty(Q)$-bound for $1/\theta$. Hence, let us call $h = 1/\theta$ and rewrite formally (1.2)–(1.3) as follows

$$h_t - h^2 \Delta \left(h - \frac{1}{h} \right) = -h^2 (u - \varphi_t), \quad \text{in } Q, \tag{2.33}$$

$$-\frac{\partial}{\partial \nu} \left(h - \frac{1}{h} \right) = \alpha \left(h - \frac{1}{h} + v \right), \quad \text{on } \Sigma. \tag{2.34}$$

Note that, due to the estimate (2.32), there exists a positive constant $\bar{C}$ (depending on the data) such that

$$h(t,x) \geq \bar{C} \quad \text{a.e. } (t,x) \in \bar{Q}.$$

Test now (2.33) by ph^{p-1}, $p \in (1, \infty)$, getting

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega h^p \,\mathrm{d}x + \frac{4p(p+1)}{(p+2)^2} \int_\Omega \left| \nabla h^{\frac{p+2}{2}} \right|^2 \mathrm{d}x + \frac{4(p+1)}{p} \int_\Omega \left| \nabla h^{\frac{p}{2}} \right|^2 \mathrm{d}x + p \int_\Gamma \alpha h^{p+2} \,\mathrm{d}s \\ &= p \int_\Gamma \alpha h^p \,\mathrm{d}s + p \int_\Gamma \alpha v h^{p+1} \,\mathrm{d}s - p \int_\Omega (u - \varphi_t) h^{p+1} \,\mathrm{d}x. \end{aligned}$$

Using now the Young inequality and the Assumption (2.15), we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega h^p \,\mathrm{d}x + \delta \left(\int_\Omega \left| \nabla h^{\frac{p+2}{2}} \right|^2 \mathrm{d}x + \int_\Gamma \left| h^{\frac{p+2}{2}} \right|^2 \mathrm{d}s \right) \leq CR^{p+2} + p \int_\Omega |u - \varphi_t| h^{p+1} \,\mathrm{d}x \,,$$

where δ and R are positive constants independent of p. By recalling the continuous embedding of $H^1(\Omega)$ into $L^6(\Omega)$ in 3D along with Hölder's inequality, we have that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega h^p \,\mathrm{d}x + \delta' \left(\int_\Omega |h|^{3(p+2)} \,\mathrm{d}x \right)^{1/3} \\ &\leq CR^{p+2} + p \int_\Omega |u - \varphi_t| h^{p+1} \,\mathrm{d}x \\ &\leq CR^{p+2} + p \|u - \varphi_t\|_{L^6(\Omega)} \|h^{\frac{p+2}{2}}\|_{L^6(\Omega)} \|h^{\frac{p}{2}}\|_{L^{3/2}(\Omega)} \\ &\leq CR^{p+2} + \frac{\delta'}{2} \|h^{\frac{p+2}{2}}\|^2_{L^6(\Omega)} + \frac{p^2}{2\delta'} \|u - \varphi_t\|^2_{L^6(\Omega)} \|h^{\frac{p}{2}}\|^2_{L^{3/2}(\Omega)} \,. \end{aligned}$$

Now, integrating over $(0,t)$, $t \in (0,T]$, and using the continuous embedding of $L^p(\Omega)$ into $L^{3p/4}(\Omega)$ as well as assumption (2.15), we infer that

$$\begin{aligned} \int_\Omega h^p(t) \,\mathrm{d}x + \delta' \left(\int_\Omega |h|^{3(p+2)} \,\mathrm{d}x \right)^{1/3} &\leq C \left(R^{p+2} + p^2 \int_0^t \|u - \varphi_t\|^2_{L^6(\Omega)} \|h\|^p_{L^{\frac{3p}{4}}(\Omega)} \,\mathrm{d}\tau \right) \\ &\leq C \left(R^{p+2} + p^2 \int_0^t \|u - \varphi_t\|^2_{L^6(\Omega)} \|h\|^p_{L^p(\Omega)} \,\mathrm{d}\tau \right), \end{aligned}$$

where C depends on the data, but not on p. Choosing now $p = 6$ and applying the Gronwall lemma, we obtain the starting point for an iterating procedure which is completely analogous to the one in ([14], p. 269). Hence, we obtain

$$\|h\|_{L^\infty(Q)} = \|1/\theta\|_{L^\infty(Q)} \leq C \tag{2.35}$$

and this concludes the proof of Theorem 2.2. □

The next result proves the existence of a solution to problem (P).

Theorem 2.3. *Assume that*

$$\theta_0 \in L^2(\Omega), \quad \theta_0 > 0 \quad a.e.\ in \quad \Omega, \quad \ln\theta_0 \in L^1(\Omega), \quad \varphi_0 \in H^1(\Omega), \tag{2.36}$$

and (1.8) *hold. Then* (P) *has at least one solution.*

Proof. Since $J(u,v,\eta) \geq 0$, it follows that J has an infimum d and this infimum is nonnegative. Let $(u_n, v_n, \eta_n)_{n\geq 1}$ be a minimizing sequence for J. This means that $u_n \in K_1$, $v_n \in K_2$, $\eta_n \in K_3$, (θ_n, φ_n) is the solution to (1.1)–(1.6) corresponding to u_n, v_n, η_n, and the following inequalities take place

$$d \leq \lambda_1 \int_Q (\theta_n - \theta_f)^2 \,\mathrm{d}x\mathrm{d}t + \lambda_2 \int_Q (\varphi_n - \eta_n)^2 \,\mathrm{d}x\mathrm{d}t \leq d + \frac{1}{n},\ n \geq 1. \tag{2.37}$$

Therefore, possibly taking subsequences (denoted still by the subscript n), we deduce that

$$u_n \to u \ \text{ weakly* in } L^\infty(Q), \quad v_n \to v \ \text{ weakly* in } L^\infty(\Sigma),$$
$$\eta_n \to \eta \ \text{ weakly* in } L^\infty(Q), \ \text{ as } n \to \infty,$$

and $u \in K_1$, $v \in K_2$, $\eta \in K_3$. By (2.9)–(2.10) we have

$$\theta_n \to \theta \ \text{ weakly in } L^2(0,T;V) \cap W^{1,2}([0,T];V'),\ \text{ as } n \to \infty,$$

$$\frac{1}{\theta_n} \to l \ \text{ weakly in } L^2(0,T;V),\ \text{ as } n \to \infty,$$

$$\varphi_n \to \varphi \text{ weakly in } L^2(0,T;H^2(\Omega)) \cap W^{1,2}([0,T];L^2(\Omega))$$
$$\text{and weakly* in } L^\infty(0,T;V),\ \text{ as } n \to \infty.$$

These facts imply, by the Aubin–Lions theorem (see [16], p. 57), that

$$\theta_n \to \theta \ \text{ strongly in } L^2(0,T;L^2(\Omega)),\ \text{ as } n \to \infty,$$
$$\varphi_n \to \varphi \ \text{ strongly in } L^2(0,T;V),\ \text{ as } n \to \infty.$$

Therefore, on a subsequence it results that

$$\theta_n \to \theta \ \text{ a.e. in } Q,\ \text{ as } n \to \infty,$$

whence

$$\frac{1}{\theta_n} \to \frac{1}{\theta} \ \text{ a.e. in } Q,\ \text{ as } n \to \infty,$$

entailing that $l = 1/\theta$ a.e. in Q (*cf.*, *e.g.*, [16], Lem. 1.3, p. 12). With the help of the Egorov theorem, we can also conclude that

$$\frac{1}{\theta_n} \to \frac{1}{\theta} \ \text{ strongly in } L^p(Q),\ \text{ for all } 1 \leq p < 2, \text{ as } n \to \infty.$$

On the other hand, in view of (1.7) the above convergences yield

$$\beta(\theta_n) \to \beta(\theta) \ \text{ weakly in } L^2(0,T;V) \text{ and a.e. in } Q,\ \text{ as } n \to \infty.$$

Next, since $\{\varphi_n\}$ is bounded in $L^\infty(0,T;V)$ we deduce that $\{\varphi_n^3\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and consequently

$$\varphi_n^3 \to l_1 \ \text{ weakly* in } L^\infty(0,T;L^2(\Omega)),\ \text{ as } n \to \infty.$$

But, there exists a subsequence such that $\varphi_n \to \varphi$ a.e. in Q. This implies that $\varphi_n^3 \to \varphi^3$ a.e. in Q and we conclude that $l_1 = \varphi^3$ a.e. in Q.

Now, we recall that $\eta_n \in H(\theta_n)$ a.e. in Q, $\theta_n \to \theta$ strongly in $L^2(Q)$ and $\eta_n \to \eta$ weakly* in $L^\infty(Q)$. On the basis of the maximal monotonicity of H, we deduce that $\eta \in H(\theta)$ a.e. in Q.

Moreover, since the trace operator is continuous from V to $L^2(\Gamma)$, we have that $\|\beta(\theta_n)\|_{L^2(0,T;L^2(\Gamma))} \leq C$ and so

$$\beta(\theta_n)|_\Gamma \to \beta(\theta)|_\Gamma \text{ weakly in } L^2(0,T;L^2(\Gamma)), \text{ as } n \to \infty.$$

Passing to the limit as $n \to \infty$ in the weak forms (*cf.* (2.7)–(2.8)), written for (θ_n, φ_n) and (u_n, v_n), we obtain by the previous convergences that (θ, φ) satisfies (2.7)–(2.8), which means that it is the solution to (1.1)−(1.6) corresponding to u and v.

Finally, we pass to the limit in (2.37) using the weakly lower semicontinuity property of the terms in J and get

$$J(u, v, \eta) = d.$$

This concludes the proof, by specifying that (u, v, η) and the corresponding states (θ, φ) are optimal in (P). □

3. Approximating control problem

In this section we consider an approximating problem (P_ε) and show its convergence in a suitable sense to (P). First, we introduce the convex function

$$j : \mathbb{R} \to \mathbb{R}, \quad j(r) = |r - \theta_c| \tag{3.1}$$

whose subdifferential is the graph H defined in (1.9). The conjugate of j is

$$j^*(\omega) = \sup_{r\in\mathbb{R}} (\omega r - j(r))$$

and it precisely reads

$$j^*(\omega) = \omega\theta_c + I_{[-1,1]}(\omega). \tag{3.2}$$

We mention that, if K is a closed convex set, we denote by I_K its indicator function, which is defined by $I_K(r) = 0$ if $r \in K$, $I_K(r) = +\infty$ otherwise.

Let us recall that two conjugate functions j and j^* satisfy the relations (see, *e.g.*, [2], p. 6)

$$j(r) + j^*(\omega) \geq r\omega \text{ for all } r,\, \omega \in \mathbb{R}; \quad j(r) + j^*(\omega) = r\omega \text{ iff } \omega \in \partial j(r), \tag{3.3}$$

where ∂j denotes the subdifferential of j. In our special case, (3.3) reduces to

$$\begin{aligned} j(r) + \omega\theta_c - \omega r \geq 0 \text{ for all } r \in \mathbb{R},\ \omega \in [-1,1]; \\ j(r) + \omega\theta_c - \omega r = 0 \text{ iff } \omega \in H(r). \end{aligned} \tag{3.4}$$

Then, we let $\varepsilon > 0$ and state the approximating problem as follows. Setting

$$J_\varepsilon(u, v, \eta) = \lambda_1 \int_Q (\theta - \theta_f)^2 \mathrm{d}x\,\mathrm{d}t + \lambda_2 \int_Q (\varphi - \eta)^2 \mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon} \int_Q (j(\theta) + \eta\theta_c - \eta\theta) \mathrm{d}x\,\mathrm{d}t,$$

we deal with the minimization problem

$$\text{Minimize } J_\varepsilon(u, v, \eta) \text{ for all } (u, v, \eta) \in K_1 \times K_2 \times K_{[-1,1]}, \tag{P_ε}$$

subject to (1.1)−(1.6), where

$$K_{[-1,1]} = \{\eta \in L^\infty(Q) \,:\, |\eta(t,x)| \leq 1 \text{ a.e. } (t,x) \in Q\}. \tag{3.5}$$

Proposition 3.1. *Let the assumptions* (2.36) *and* (1.8) *hold. Then* (P_ε) *has at least one solution.*

Proof. According to (3.4) and (3.5), we have that $d_\varepsilon = \inf\limits_{u,v,\eta} J_\varepsilon(u,v,\eta) \geq 0$. Let $(u_\varepsilon^n, v_\varepsilon^n, \eta_\varepsilon^n)_n$ be a minimizing sequence for J_ε, that is,

$$d_\varepsilon \leq J_\varepsilon(u_\varepsilon^n, v_\varepsilon^n, \eta_\varepsilon^n) \leq d_\varepsilon + \frac{1}{n}\cdot \tag{3.6}$$

As in Theorem 2.3 we obtain that (as $n \to \infty$)

$$u_\varepsilon^n \to u_\varepsilon^* \text{ weakly* in } L^\infty(Q), \quad v_\varepsilon^n \to v_\varepsilon^* \text{ weakly* in } L^\infty(\Sigma), \quad \eta_\varepsilon^n \to \eta_\varepsilon^* \text{ weakly* in } L^\infty(Q),$$

and $u_\varepsilon^* \in K_1$, $v_\varepsilon^* \in K_2$, $\eta_\varepsilon^* \in K_{[-1,1]}$. Also, for the corresponding state we infer that

$$\theta_\varepsilon^n \to \theta_\varepsilon^* \text{ weakly in } L^2(0,T;V) \cap W^{1,2}([0,T];V') \quad \text{and strongly in } L^2(Q),$$

$$\begin{aligned} \varphi_\varepsilon^n \to \varphi_\varepsilon^* \text{ weakly in } & L^2(0,T;H^2(\Omega)) \cap W^{1,2}([0,T];L^2(\Omega)), \\ & \text{weakly* in } L^\infty(0,T;V), \text{ and strongly in } L^2(0,T;V), \end{aligned}$$

and $\eta_\varepsilon^n \theta_\varepsilon^n \to \eta_\varepsilon^* \theta_\varepsilon^*$ weakly in $L^2(Q)$. In a similar way as proved in Theorem 2.3, we deduce that $(\theta_\varepsilon^*, \varphi_\varepsilon^*)$ is a solution to (1.1)–(1.6) corresponding to $(u_\varepsilon^*, v_\varepsilon^*)$.

Next, since j is Lipschitz continuous and θ_ε^n converges strongly to θ_ε^* in $L^2(Q)$, we have

$$j(\theta_\varepsilon^n) \to j(\theta_\varepsilon^*) \text{ strongly in } L^2(Q), \text{ as } n \to \infty.$$

When passing to the limit in (3.6), in the third term of J_ε we exploit the weak lower semicontinuity. Then, we get $J_\varepsilon(u_\varepsilon^*, v_\varepsilon^*, \eta_\varepsilon^*) = d_\varepsilon$. In conclusion, $(u_\varepsilon^*, v_\varepsilon^*, \eta_\varepsilon^*)$ and the corresponding state $(\theta_\varepsilon^*, \varphi_\varepsilon^*)$ are optimal in (P_ε). □

The next theorem proves that (P_ε) converges to (P) in some sense as $\varepsilon \to 0$.

Theorem 3.2. *Under the hypotheses of Theorem* 2.3*, for any* $\varepsilon > 0$ *let the pair* $\{(u_\varepsilon^*, v_\varepsilon^*, \eta_\varepsilon^*), (\theta_\varepsilon^*, \varphi_\varepsilon^*)\}$ *be optimal in* (P_ε)*. Then, we have that*

$$u_\varepsilon^* \to u^* \text{ weakly* in } L^\infty(Q), \text{ as } \varepsilon \to 0, \tag{3.7}$$

$$v_\varepsilon^* \to v^* \text{ weakly* in } L^\infty(\Sigma), \text{ as } \varepsilon \to 0, \tag{3.8}$$

$$\eta_\varepsilon^* \to \eta^* \text{ weakly* in } L^\infty(Q), \text{ as } \varepsilon \to 0, \tag{3.9}$$

$$\theta_\varepsilon^* \to \theta^* \text{ weakly in } L^2(0,T;V) \cap W^{1,2}([0,T];V') \text{ and strongly in } L^2(Q), \text{ as } \varepsilon \to 0, \tag{3.10}$$

$$\begin{aligned} \varphi_\varepsilon^* \to \varphi^* \text{ weakly in } & L^2(0,T;H^2(\Omega)) \cap W^{1,2}([0,T];L^2(\Omega)), \\ & \text{weakly* in } L^\infty(0,T;V), \text{ and strongly in } L^2(0,T;V), \text{ as } \varepsilon \to 0, \end{aligned} \tag{3.11}$$

where (θ^*, φ^*) *is the solution to* (1.1)–(1.6) *corresponding to* (u^*, v^*, η^*) *and the pair* $\{(u^*, v^*, \eta^*), (\theta^*, \varphi^*)\}$ *is optimal in* (P)*. Furthermore, every triplet* (u^*, v^*, η^*) *obtained in* (3.7)–(3.9) *as weak* limits of subsequences of* $\{(u_\varepsilon^*, v_\varepsilon^*, \eta_\varepsilon^*)\}$ *yields an optimal solution to* (P)*.*

Proof. Let $\{(u_\varepsilon^*, v_\varepsilon^*, \eta_\varepsilon^*), (\theta_\varepsilon^*, \varphi_\varepsilon^*)\}$ be optimal in (P_ε). Then we can write

$$J_\varepsilon(u_\varepsilon^*, v_\varepsilon^*, \eta_\varepsilon^*) \leq J_\varepsilon(u, v, \eta),$$

for any $(u, v, \eta) \in K_1 \times K_2 \times K_{[-1,1]}$. In particular, we set $u = \widetilde{u}$, $v = \widetilde{v}$, $\eta = \widetilde{\eta}$, where $(\widetilde{u}, \widetilde{v}, \widetilde{\eta})$ is a solution to (P) with the corresponding state $\widetilde{\theta}$, $\widetilde{\varphi}$ solving (1.1)–(1.6). This entails that $\widetilde{\eta} \in H(\widetilde{\theta}) \equiv \partial j(\widetilde{\theta})$ a.e. in Q. The previous inequality reads

$$\lambda_1 \int_Q (\theta_\varepsilon^* - \theta_f)^2 \, \mathrm{d}x \, \mathrm{d}t + \lambda_2 \int_Q (\varphi_\varepsilon^* - \eta_\varepsilon^*)^2 \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{\varepsilon} \int_Q (j(\theta_\varepsilon^*) + \eta_\varepsilon^* \theta_c - \eta_\varepsilon^* \theta_\varepsilon^*) \, \mathrm{d}x \, \mathrm{d}t$$
$$\leq \lambda_1 \int_Q (\widetilde{\theta} - \theta_f)^2 \, \mathrm{d}x \, \mathrm{d}t + \lambda_2 \int_Q (\widetilde{\varphi} - \widetilde{\eta})^2 \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{\varepsilon} \int_Q (j(\widetilde{\theta}) + \widetilde{\eta}\theta_c - \widetilde{\eta}\widetilde{\theta}) \, \mathrm{d}x \, \mathrm{d}t, \tag{3.12}$$

and we see by (3.4) that the last term on the right-hand side is actually zero. Then the right-hand side is bounded by a constant.

In view of (1.11)–(1.12) and (3.5), by the boundedness of the optimal controllers $(u_\varepsilon^*, v_\varepsilon^*, \eta_\varepsilon^*)$ we obtain (3.7)−(3.9). Thanks to Theorem 2.2 (*cf.* especially (2.9)–(2.10)), it is straightforward to deduce (3.10)–(3.11). Then, writing the weak formulations (2.7)–(2.8) for the approximating state and passing to the limit as $\varepsilon \to 0$ we deduce that (θ^*, φ^*) is the solution to (1.1)–(1.6) corresponding to (u^*, v^*, η^*).

Finally, we have to show that $\eta^* \in H(\theta^*)$ a.e. in Q. We set

$$\zeta_\varepsilon = \int_Q (j(\theta_\varepsilon^*) + \eta_\varepsilon^* \theta_c - \eta_\varepsilon^* \theta_\varepsilon^*) \, \mathrm{d}x \, \mathrm{d}t$$

and remark that

$$0 \leq \gamma_\varepsilon := \frac{1}{\varepsilon} \zeta_\varepsilon \leq C$$

for some constant C (independent of ε), because of (3.12). Hence, we have that $\zeta_\varepsilon = \varepsilon\gamma_\varepsilon \to 0$, as $\varepsilon \to 0$. On the other hand, passing to the limit we infer that

$$0 \leq \int_Q (j(\theta^*) + \eta^* \theta_c - \eta^* \theta^*) \, \mathrm{d}x \, \mathrm{d}t \leq \lim_{\varepsilon \to 0} \zeta_\varepsilon = 0.$$

We deduce that $j(\theta^*(t,x)) + \eta^*(t,x)\theta_c - \eta^*(t,x)\theta^*(t,x) = 0$ a.e. $(t,x) \in Q$ and, thanks to (3.4), this implies that $\eta^*(t,x) \in \partial j(\theta^*(t,x)) = H(\theta^*(t,x))$ a.e. $(t,x) \in Q$.

Then, we pass to the limit in (3.12) as $\varepsilon \to 0$ and obtain

$$J(u^*, v^*, \eta^*) \leq J(\widetilde{u}, \widetilde{v}, \widetilde{\eta}),$$

for any $(\widetilde{u}, \widetilde{v}, \widetilde{\eta}) \in K_1 \times K_2 \times K_3$ (*cf.* (1.13)), with $(\widetilde{\theta}, \widetilde{\varphi})$ solution to (1.1)–(1.6). This shows that $\{(u^*, v^*, \eta^*), (\theta^*, \varphi^*)\}$ is optimal in (P). □

4. Optimality conditions

In this section we compute the optimality conditions for the problem (P_ε). We prove that whatever would be the optimal controllers u_ε^*, v_ε^*, η_ε^*, they are represented by the expressions given in Theorem 4.6. To this end, we use some intermediate results proved for a second approximating problem, introduced in order to regularize the function j. We recall that the Moreau−Yosida regularization is defined by

$$j_\sigma(r) = \inf_{s \in \mathbb{R}} \left\{ \frac{|r-s|^2}{2\sigma} + j(s) \right\}, \text{ for any } r \in \mathbb{R}, \ \sigma > 0, \tag{4.1}$$

and that it can be still written as

$$j_\sigma(r) = \frac{1}{2\sigma} \left| (I + \sigma H)^{-1} r - r \right|^2 + j((I + \sigma H)^{-1} r), \tag{4.2}$$

where I is the identity on $\mathbb{R}$. The function j_σ is convex, Lipschitz continuous along with its derivative, and it has the properties (see [2], p. 48):

$$0 \le j_\sigma(r) \le j(r) \;\text{ and }\; \lim_{\sigma\to 0} j_\sigma(r) = j(r), \;\text{for any } r \in \mathbb{R}. \tag{4.3}$$

Let $(u_\varepsilon^*, v_\varepsilon^*, \eta_\varepsilon^*)$ be optimal in (P_ε). Following a technique developed in [1], we introduce the approximating penalized problem:

$$\text{Minimize } J_{\varepsilon,\sigma}(u,v,\eta) \text{ for all } (u,v,\eta) \in K_1 \times K_2 \times K_{[-1,1]}, \tag{$P_{\varepsilon,\sigma}$}$$

subject to (1.1)–(1.6), where

$$J_{\varepsilon,\sigma}(u,v,\eta) = \lambda_1 \int_Q (\theta-\theta_f)^2 \mathrm{d}x\,\mathrm{d}t + \lambda_2 \int_Q (\varphi-\eta)^2 \mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_Q (j_\sigma(\theta) + \eta\theta_c - \eta\theta)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_Q (u-u_\varepsilon^*)^2\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma (v-v_\varepsilon^*)^2\,\mathrm{d}s\,\mathrm{d}t + \int_Q (\eta-\eta_\varepsilon^*)^2\,\mathrm{d}x\,\mathrm{d}t,$$

and $K_{[-1,1]}$ is defined by (3.5).

It is obvious that problem $(P_{\varepsilon,\sigma})$ has at least one solution, and the proof can be done arguing as in Proposition 3.1. Now, we shall show that in a formal way $(P_{\varepsilon,\sigma}) \to (P_\varepsilon)$ as $\sigma \to 0$.

Proposition 4.1. *Assume that* (2.36) *and* (1.8) *hold. Let* $(u_\varepsilon^*, v_\varepsilon^*, \eta_\varepsilon^*)$ *and* $(u_{\varepsilon,\sigma}^*, v_{\varepsilon,\sigma}^*, \eta_{\varepsilon,\sigma}^*)$ *be optimal in* (P_ε) *and* $(P_{\varepsilon,\sigma})$, *respectively. Then, we have that*

$$u_{\varepsilon,\sigma}^* \to u_\varepsilon^* \;\textit{ weakly* in } L^\infty(Q) \textit{ and strongly in } L^2(Q), \;\textit{as } \sigma \to 0, \tag{4.4}$$
$$v_{\varepsilon,\sigma}^* \to v_\varepsilon^* \;\textit{ weakly* in } L^\infty(\Sigma) \textit{ and strongly in } L^2(\Sigma), \;\textit{as } \sigma \to 0, \tag{4.5}$$
$$\eta_{\varepsilon,\sigma}^* \to \eta_\varepsilon^* \;\textit{ weakly* in } L^\infty(Q) \textit{ and strongly in } L^2(Q), \;\textit{as } \sigma \to 0, \tag{4.6}$$

and the corresponding states $(\theta_{\varepsilon,\sigma}^*, \varphi_{\varepsilon,\sigma}^*)$ *converge to the optimal states* $(\theta_\varepsilon^*, \varphi_\varepsilon^*)$ *that correspond to* $(u_\varepsilon^*, v_\varepsilon^*, \eta_\varepsilon^*)$ *in* (P_ε).

Proof. We write that $(u_{\varepsilon,\sigma}^*, v_{\varepsilon,\sigma}^*, \eta_{\varepsilon,\sigma}^*)$ is optimal in $(P_{\varepsilon,\sigma})$, that is

$$\lambda_1 \int_Q (\theta_{\varepsilon,\sigma}^* - \theta_f)^2\,\mathrm{d}x\,\mathrm{d}t + \lambda_2 \int_Q (\varphi_{\varepsilon,\sigma}^* - \eta_{\varepsilon,\sigma}^*)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_Q (j_\sigma(\theta_{\varepsilon,\sigma}^*) + \eta_{\varepsilon,\sigma}^*\theta_c - \eta_{\varepsilon,\sigma}^*\theta_{\varepsilon,\sigma}^*)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_Q (u_{\varepsilon,\sigma}^* - u_\varepsilon^*)^2\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma (v_{\varepsilon,\sigma}^* - v_\varepsilon^*)^2\,\mathrm{d}s\,\mathrm{d}t + \int_Q (\eta_{\varepsilon,\sigma}^* - \eta_\varepsilon^*)^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \lambda_1 \int_Q (\theta-\theta_f)^2\,\mathrm{d}x\,\mathrm{d}t + \lambda_2 \int_Q (\varphi-\eta)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_Q (j_\sigma(\theta) + \eta\theta_c - \eta\theta)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_Q (u-u_\varepsilon^*)^2\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma (v-v_\varepsilon^*)^2\,\mathrm{d}s\,\mathrm{d}t + \int_Q (\eta-\eta_\varepsilon^*)^2\,\mathrm{d}x\,\mathrm{d}t, \tag{4.7}$$

for all $u \in K_1$, $v \in K_2$, $\eta \in K_{[-1,1]}$, with (θ, φ) denoting the corresponding solution to (1.1)–(1.6).

In particular, we set $u = u_\varepsilon^*$, $v = v_\varepsilon^*$, $\eta = \eta_\varepsilon^*$ in (4.7). This leads us to consider the corresponding solutions $\theta = \theta_\varepsilon^*$, $\varphi = \varphi_\varepsilon^*$ to (1.1)–(1.6) as well. It follows that the left-hand side in (4.7) is bounded independently of σ, because on the right-hand side the last three terms vanish and $j_\sigma(\theta_\varepsilon^*) \le j(\theta_\varepsilon^*)$ a.e. in Q, thanks to (4.3). Consequently, by selecting subsequences (still denoted by the subscript σ) we get

$$u_{\varepsilon,\sigma}^* \to u_\varepsilon \;\text{ weakly* in } L^\infty(Q), \quad v_{\varepsilon,\sigma}^* \to v_\varepsilon \;\text{ weakly* in } L^\infty(\Sigma),$$
$$\eta_{\varepsilon,\sigma}^* \to \eta_\varepsilon \;\text{ weakly* in } L^\infty(Q), \;\text{as } \sigma \to 0. \tag{4.8}$$

Relying on the estimates (2.9)–(2.10) for the state system we have

$$\theta^*_{\varepsilon,\sigma} \to \theta_\varepsilon \text{ weakly in } L^2(0,T;V) \cap W^{1,2}([0,T];V') \\ \text{and strongly in } L^2(Q), \text{ as } \sigma \to 0, \tag{4.9}$$

$$\varphi^*_{\varepsilon,\sigma} \to \varphi_\varepsilon \text{ weakly in } L^2(0,T;H^2(\Omega)) \cap W^{1,2}([0,T];L^2(\Omega)), \\ \text{weakly* in } L^\infty(0,T;V), \text{ and strongly in } L^2(0,T;V), \text{ as } \sigma \to 0, \tag{4.10}$$

where $(\theta_\varepsilon, \varphi_\varepsilon)$ is the solution to (1.1)–(1.6) corresponding to $(u_\varepsilon, v_\varepsilon, \eta_\varepsilon)$.

Next, we pass to the limit in (4.7) as $\sigma \to 0$. First, we assert that

$$\int_Q j(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \le \liminf_{\sigma\to 0} \int_Q j_\sigma(\theta^*_{\varepsilon,\sigma})\,\mathrm{d}x\,\mathrm{d}t, \tag{4.11}$$

where θ_ε is the limit of $\theta^*_{\varepsilon,\sigma}$. Indeed, by (4.2) we have

$$\frac{1}{2\sigma}\int_Q \left|(I+\sigma\partial j)^{-1}\theta^*_{\varepsilon,\sigma} - \theta^*_{\varepsilon,\sigma}\right|^2 \,\mathrm{d}x\,\mathrm{d}t \le \int_Q j_\sigma(\theta^*_{\varepsilon,\sigma})\,\mathrm{d}x\,\mathrm{d}t \le \text{ const.},$$

which implies that

$$\lim_{\sigma\to 0} \left\|(I+\sigma\partial j)^{-1}\theta^*_{\varepsilon,\sigma} - \theta^*_{\varepsilon,\sigma}\right\|_{L^2(Q)} = 0.$$

Therefore, we deduce that

$$(I+\sigma\partial j)^{-1}\theta^*_{\varepsilon,\sigma} \to \theta_\varepsilon \ \text{ strongly in } L^2(Q) \ \text{ as } \sigma \to 0. \tag{4.12}$$

Next, again by (4.2) we can infer that

$$\int_Q j(\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t = \lim_{\sigma\to 0}\int_Q j((I+\sigma\partial j)^{-1}\theta^*_{\varepsilon,\sigma})\,\mathrm{d}x\,\mathrm{d}t \le \liminf_{\sigma\to 0}\int_Q j_\sigma(\theta^*_{\varepsilon,\sigma})\,\mathrm{d}x\,\mathrm{d}t$$

by the Lipschitz continuity of j and (4.12). Then, passing to the limit in (4.7) as $\sigma \to 0$ we get

$$\begin{aligned} &\lambda_1 \int_Q (\theta_\varepsilon - \theta_f)^2\,\mathrm{d}x\,\mathrm{d}t + \lambda_2 \int_Q (\varphi_\varepsilon - \eta_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_Q (j(\theta_\varepsilon) + \eta_\varepsilon\theta_c - \eta_\varepsilon\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_Q (u_\varepsilon - u^*_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma (v_\varepsilon - v^*_\varepsilon)^2\,\mathrm{d}s\,\mathrm{d}t + \int_Q (\eta_\varepsilon - \eta^*_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t \\ \le\ &\lambda_1 \int_Q (\theta^*_\varepsilon - \theta_f)^2\,\mathrm{d}x\,\mathrm{d}t + \lambda_2 \int_Q (\varphi^*_\varepsilon - \eta^*_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_Q (j(\theta^*_\varepsilon) + \eta^*_\varepsilon\theta_c - \eta^*_\varepsilon\theta^*_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ \le\ &\lambda_1 \int_Q (\theta_\varepsilon - \theta_f)^2\,\mathrm{d}x\,\mathrm{d}t + \lambda_2 \int_Q (\varphi_\varepsilon - \eta_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_Q (j(\theta_\varepsilon) + \eta_\varepsilon\theta_c - \eta_\varepsilon\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

The second inequality can be written because $(u^*_\varepsilon, v^*_\varepsilon, \eta^*_\varepsilon)$ is optimal in (P_ε). Hence, it is not difficult to see that

$$u_\varepsilon = u^*_\varepsilon,\ v_\varepsilon = v^*_\varepsilon,\ \eta_\varepsilon = \eta^*_\varepsilon \ \text{ a.e. in } Q$$

and consequently $\theta_\varepsilon = \theta^*_\varepsilon$ and $\varphi_\varepsilon = \varphi^*_\varepsilon$ a.e. in Q. Actually, going back to (4.7) it follows that the convergences in (4.8) hold for the whole sequences and moreover

$$u^*_{\varepsilon,\sigma} \to u^*_\varepsilon \ \text{ strongly in } L^2(Q), \quad v^*_{\varepsilon,\sigma} \to v^*_\varepsilon \ \text{ strongly in } L^2(\Sigma), \\ \eta^*_{\varepsilon,\sigma} \to \eta^*_\varepsilon \ \text{ strongly in } L^2(Q), \quad \text{as } \sigma \to 0.$$

This ends the proof. □

4.1. Optimality conditions for $(\mathsf{P}_{\varepsilon,\sigma})$

For a later use we begin by proving the well-posedness of the problem

$$W_t - a(t,x)\Delta W + b(t,x)\Phi = \omega(t,x), \quad \text{in } Q, \tag{4.13}$$

$$\Phi_t - \Delta\Phi + c(t,x)\Phi + d(t,x)W_t = g(t,x), \quad \text{in } Q, \tag{4.14}$$

$$-\frac{\partial W}{\partial \nu} = \alpha(x)(W - \gamma(t,x)), \quad \text{on } \Sigma, \tag{4.15}$$

$$\frac{\partial \Phi}{\partial \nu} = 0, \quad \text{on } \Sigma, \tag{4.16}$$

$$W(0) = 0, \; \Phi(0) = 0, \; \text{in } \Omega. \tag{4.17}$$

Proposition 4.2. *Let the following conditions*

$$a,b,c,d \in L^\infty(Q), \quad 0 < a_0 \le a(t,x) \le \|a\|_{L^\infty(Q)} =: |a|_\infty\,, \; \textit{a.e. } (t,x) \in Q, \tag{4.18}$$

$$\omega, g \in L^2(Q), \quad \gamma \in W^{1,2}([0,T], L^2(\Gamma)), \quad \alpha \textit{ satisfies } (1.8) \tag{4.19}$$

hold. Then, the problem (4.13)–(4.17) *has a unique solution* (W,Φ) *with*

$$W \in L^\infty(0,T;V) \cap W^{1,2}([0,T];L^2(\Omega)),$$
$$\Phi \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;V) \cap W^{1,2}([0,T];L^2(\Omega)).$$

If $\gamma \equiv 0$ *in addition, we have that*

$$W \in L^2(0,T;H^2(\Omega)). \tag{4.20}$$

Proof. We use a fixed point argument. In (4.13)–(4.17) let us fix $\overline{\Phi} \in L^2(Q)$ and consider the equations and conditions

$$\overline{W}_t - a(t,x)\Delta\overline{W} + b(t,x)\overline{\Phi} = \omega(t,x), \quad \text{in } Q, \tag{4.21}$$

$$-\frac{\partial \overline{W}}{\partial \nu} = \alpha(x)(\overline{W} - \gamma(t,x)), \quad \text{on } \Sigma, \tag{4.22}$$

$$\overline{W}(0) = 0, \quad \text{in } \Omega, \tag{4.23}$$

$$\Phi_t - \Delta\Phi + c(t,x)\overline{\Phi} + d(t,x)\overline{W}_t = g(t,x), \quad \text{in } Q, \tag{4.24}$$

$$\frac{\partial \Phi}{\partial \nu} = 0, \quad \text{on } \Sigma, \tag{4.25}$$

$$\Phi(0) = 0, \quad \text{in } \Omega. \tag{4.26}$$

We first solve (4.21)–(4.23) and find $\overline{W}$, then we replace $\overline{W}$ in (4.24) and solve (4.24)–(4.26) by finding Φ. Thus, we construct a mapping

$$\Psi : C([0,T];L^2(\Omega)) \to C([0,T];L^2(\Omega)) \quad \text{such that} \quad \Psi(\overline{\Phi}) = \Phi. \tag{4.27}$$

We are going to show that a suitable power of Ψ is a contraction.

First of all, we claim that (4.21)–(4.23) has a unique solution

$$\overline{W} \in L^\infty(0,T;V) \cap W^{1,2}([0,T];L^2(\Omega)). \tag{4.28}$$

Let us outline the argument, without writing in detail all computations. By taking a partition of the interval $[0,T]$, setting $t_i = ih$, $i = 1,\dots,N$, with $h = T/N$, we consider the system of finite differences

$$\frac{1}{a_i(x)} \frac{w_i - w_{i-1}}{h} - \Delta w_i = f_i(x), \quad \text{in } \Omega, \tag{4.29}$$

$$-\frac{\partial w_i}{\partial \nu} = \alpha(x)(w_i - \gamma_i(x)), \quad \text{on } \Gamma, \tag{4.30}$$

for $i = 1,\dots,N$, with $w_0 = 0$. Here, a_i denotes the mean value of a on the time interval $((i-1)h,(ih))$, and the same definition can be set for f_i provided f is interpreted as $(\omega - b\overline{\Phi})/a$. On the other hand, in view of (4.19), γ_i can be defined as $\gamma_i(x) = \gamma(t_i,x)$, a.e. $x \in \Gamma$.

Given $w_{i-1} \in L^2(\Omega)$, the system (4.29)–(4.30) has a unique variational solution $w_i \in H^1(\Omega)$ such that Δw_i lies in $L^2(\Omega)$ and the normal derivative $\frac{\partial w_i}{\partial \nu}$ is in $L^2(\Gamma)$. Then, thanks to well-known elliptic regularity results, the finite difference scheme (4.29)–(4.30), $i = 1,\dots,N$, has a unique solution $(w_1,\dots,w_N) \in X^N$ where

$$X = \{z \in H^{3/2}(\Omega);\ \Delta z \in L^2(\Omega)\}.$$

An *a priori* estimate is obtained by testing (4.29) by $w_i - w_{i-1}$, and summing with respect to i, from 1 to $k \le N$. Recalling that $w_0 = 0$, we obtain

$$\frac{h}{|a|_\infty} \sum_{i=1}^{k} \left\| \frac{w_i - w_{i-1}}{h} \right\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega |\nabla w_k|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Gamma \alpha_m |w_k|^2 \,\mathrm{d}s$$
$$\le \sum_{i=1}^{k} h \int_\Omega f_i \frac{w_i - w_{i-1}}{h} \,\mathrm{d}x + \sum_{i=1}^{k} h \int_\Gamma \alpha\, \gamma_i \frac{w_i - w_{i-1}}{h} \,\mathrm{d}s.$$

The last term on the right-hand side can be written as

$$\sum_{i=1}^{k} h \int_\Gamma \alpha\, \gamma_i \frac{w_i - w_{i-1}}{h} \,\mathrm{d}s = \int_\Gamma \alpha\, \gamma_k w_k \,\mathrm{d}s - \sum_{i=1}^{k} \int_\Gamma \alpha(\gamma_i - \gamma_{i-1}) w_{i-1} \,\mathrm{d}s$$

and standard computations involving the Young inequality and the discrete Gronwall's lemma along with Assumptions (4.18)–(4.19) lead to some estimates for the functions $\widehat{w}_h$ (piecewise linear in time interpolant) and $\overline{w}_h$ (piecewise constant in time interpolant). Namely, we have

$$\|\partial_t \widehat{w}_h\|_{L^2(0,T;L^2(\Omega))} + \|\overline{w}_h\|_{L^\infty(0,T;H^1(\Omega))} \le C.$$

Using this we can pass to the limit (by weak and weak* compactness) in the equations

$$\frac{1}{\overline{a}_h}\partial_t \widehat{w}_h - \Delta \overline{w}_h = \overline{f}_h, \quad \text{in } Q,$$

$$-\frac{\partial \overline{w}_h}{\partial \nu} = \alpha(\overline{w}_h - \overline{\gamma}_h), \quad \text{on } \Sigma,$$

$$\widehat{w}_h(0,\cdot) = 0, \quad \text{in } \Omega.$$

By comparison we also find the additional regularity

$$\|\Delta \overline{w}_h\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{\partial \overline{w}_h}{\partial \nu}\right\|_{L^2(0,T;L^2(\Gamma))} \le C.$$

We point out that if $\gamma \equiv 0$, in the system (4.29)–(4.30) one can recover that the normal derivative of the solution on the boundary, *i.e.* $\frac{\partial w_i}{\partial \nu}$, belongs to $H^{1/2}(\Gamma)$, and consequently $w_i \in H^2(\Omega)$, whenever the product $\alpha\, w_i$ lies in $H^{1/2}(\Gamma)$. Now, we have that α satisfies (1.8) and the trace w_i is in $H^1(\Gamma)$, on account of $w_i \in X$. Well, it is easy to check that the product of two elements of $H^1(\Gamma)$ belongs to $W^{1,p}(\Gamma)$ for all $1 \le p < \infty$ due to the Leibniz rule and to the fact that Γ is the (two-dimensional) boundary of a three-dimensional domain Ω. Hence, thanks to the $2D$ Sobolev embedding $W^{1,p}(\Gamma) \subset H^{1/2}(\Gamma)$ if $p \ge 4/3$, it turns out that

$$\frac{\partial w_i}{\partial \nu} = \alpha\, w_i \in H^{1/2}(\Gamma).$$

Then, if $\gamma \equiv 0$, it is not difficult to obtain (4.20) for $\overline{W}$.

Consequently to (4.28) it follows that the subsequent linear parabolic problem (4.24)–(4.26) has a unique solution

$$\Phi \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;V) \cap W^{1,2}([0,T];L^2(\Omega)).$$

At this point, we can write (4.21)–(4.23) for two functions $\overline{\Phi}_1, \overline{\Phi}_2 \in L^2(Q)$ getting the respective solutions $\overline{W}_1$, $\overline{W}_2$ which satisfy (4.28). Then, take the difference, divide by a, multiply the result by $(\overline{W}_1 - \overline{W}_2)_t$ and integrate over Q. After a few standard computations, we obtain

$$\begin{aligned} &\int_0^t \left\|(\overline{W}_1 - \overline{W}_2)_t(\tau)\right\|^2_{L^2(\Omega)} \mathrm{d}\tau + \left\|\nabla(\overline{W}_1 - \overline{W}_2)(t)\right\|^2_{L^2(\Omega)} \\ &+ \int_\Gamma \left|(\overline{W}_1 - \overline{W}_2)(t)\right|^2 \mathrm{d}s \le C_1 \int_0^t \left\|(\overline{\Phi}_1 - \overline{\Phi}_2)(\tau)\right\|^2_{L^2(\Omega)} \mathrm{d}\tau \end{aligned} \tag{4.31}$$

for some constant C_1 depending only on the data in (4.18)–(4.19). Next, we write (4.24)–(4.26) for $\overline{\Phi}_i$, $\overline{W}_i$, $i = 1, 2$, subtract, and test by $(\Phi_1 - \Phi_2)$, the difference of solutions. Hence, it is a standard matter to infer that

$$\begin{aligned} &\|(\Phi_1 - \Phi_2)(t)\|^2_{L^2(\Omega)} \\ &\le C_2 \left(\int_0^t \left\|(\overline{\Phi}_1 - \overline{\Phi}_2)(\tau)\right\|^2_{L^2(\Omega)} \mathrm{d}\tau + \int_0^t \left\|(\overline{W}_1 - \overline{W}_2)_t(\tau)\right\|^2_{L^2(\Omega)} \mathrm{d}\tau\right) \end{aligned} \tag{4.32}$$

for some positive constant C_2. Then, combining (4.32) with (4.31), we obtain

$$\|(\Phi_1 - \Phi_2)(t)\|^2_{L^2(\Omega)} \le C_3 \int_0^t \left\|(\overline{\Phi}_1 - \overline{\Phi}_2)(\tau)\right\|^2_{L^2(\Omega)} \mathrm{d}\tau \quad \text{for all } t \in [0,T]. \tag{4.33}$$

By observing that (*cf.* (4.27)) $\Phi_i = \Psi(\overline{\Phi}_i)$, $i = 1, 2$, is not difficult to check that relation (4.33) implies by recurrence that

$$\left\|\Psi^k(\overline{\Phi}_1) - \Psi^k(\overline{\Phi}_2)\right\|^2_{C([0,T];L^2(\Omega))} \le C_3 \frac{T^k}{k!} \left\|\overline{\Phi}_1 - \overline{\Phi}_2\right\|^2_{C([0,T];L^2(\Omega))}$$

for all $k \in \mathbb{N}$. Then for k large enough the above coefficient $C_3 \frac{T^k}{k!}$ is less than 1, and so Ψ^k has a unique fixed point Φ which also fulfils $\Phi = \Psi(\Phi)$. □

In the sequel, we will assume the further regularity conditions (2.12) and (2.15) (besides (2.4)) in order we can take advantage of uniform L^∞ estimates for both components of a solution to (1.1)–(1.6). Let us resume the computation of the optimality conditions for $(P_{\varepsilon,\sigma})$. Let $(u^*_{\varepsilon,\sigma}, v^*_{\varepsilon,\sigma}, \eta^*_{\varepsilon,\sigma})$ and $(\theta^*_{\varepsilon,\sigma}, \varphi^*_{\varepsilon,\sigma})$ be optimal in $(P_{\varepsilon,\sigma})$ and $\lambda \in (0,1)$. We introduce the variations

$$u^\lambda_{\varepsilon,\sigma} = (1-\lambda)u^*_{\varepsilon,\sigma} + \lambda u = u^*_{\varepsilon,\sigma} + \lambda\widetilde{u}, \quad u \text{ arbitrary in } K_1,$$

$$v^\lambda_{\varepsilon,\sigma} = (1-\lambda)v^*_{\varepsilon,\sigma} + \lambda v = v^*_{\varepsilon,\sigma} + \lambda\widetilde{v}, \quad v \text{ arbitrary in } K_2,$$

$$\eta^\lambda_{\varepsilon,\sigma} = (1-\lambda)\eta^*_{\varepsilon,\sigma} + \lambda \eta = \eta^*_{\varepsilon,\sigma} + \lambda\widetilde{\eta}, \quad \eta \text{ arbitrary in } K_{[-1,1]},$$

with

$$\widetilde{u} = u - u^*_{\varepsilon,\sigma}, \quad \widetilde{v} = v - v^*_{\varepsilon,\sigma}, \quad \widetilde{\eta} = \eta - \eta^*_{\varepsilon,\sigma}. \tag{4.34}$$

First of all, we note that the system (1.1)–(1.6) corresponding to $u^\lambda_{\varepsilon,\sigma}$ and $v^\lambda_{\varepsilon,\sigma}$ has a unique solution $(\theta^\lambda_{\varepsilon,\sigma}, \varphi^\lambda_{\varepsilon,\sigma})$, and

$$\theta^\lambda_{\varepsilon,\sigma} \to \theta^*_{\varepsilon,\sigma}, \quad \varphi^\lambda_{\varepsilon,\sigma} \to \varphi^*_{\varepsilon,\sigma} \text{ strongly in } L^2(Q), \text{ as } \lambda \to 0. \tag{4.35}$$

This can be obtained by the estimate (2.11) combined with weak* compactness. We set

$$\widetilde{\Theta}^\lambda = \frac{\theta^\lambda_{\varepsilon,\sigma} - \theta^*_{\varepsilon,\sigma}}{\lambda}, \quad \widetilde{\Phi}^\lambda = \frac{\varphi^\lambda_{\varepsilon,\sigma} - \varphi^*_{\varepsilon,\sigma}}{\lambda} \tag{4.36}$$

and claim that

$$\widetilde{\Theta}^\lambda \to Y \text{ and } \widetilde{\Phi}^\lambda \to \Phi \text{ weakly in } L^2(Q), \text{ as } \lambda \to 0, \tag{4.37}$$

where the limits Y and Φ solve the system in variations

$$Y_t - \Delta(\beta'(\theta^*_{\varepsilon,\sigma})Y) + \Phi_t = \widetilde{u}, \text{ in } Q, \tag{4.38}$$

$$\Phi_t - \Delta\Phi + (3(\varphi^*_{\varepsilon,\sigma})^2 - 1)\Phi = \frac{1}{(\theta^*_{\varepsilon,\sigma})^2}Y, \text{ in } Q, \tag{4.39}$$

$$-\frac{\partial}{\partial\nu}(\beta'(\theta^*_{\varepsilon,\sigma})Y) = \alpha(x)(\beta'(\theta^*_{\varepsilon,\sigma})Y - \widetilde{v}), \text{ on } \Sigma, \tag{4.40}$$

$$\frac{\partial\Phi}{\partial\nu} = 0, \text{ on } \Sigma, \tag{4.41}$$

$$Y(0) = 0, \ \Phi(0) = 0, \text{ in } \Omega. \tag{4.42}$$

The proof of (4.37) is done in Proposition 4.3, below. Before that, we define a (very weak) *solution* to (4.38)–(4.42) as a pair of functions $Y \in L^2(Q)$, $\Phi \in L^2(0,T;V)$ which satisfies the system

$$-\int_Q Y\psi_t \,\mathrm{d}x\,\mathrm{d}t - \int_Q \beta'(\theta^*_{\varepsilon,\sigma})Y\Delta\psi \,\mathrm{d}x\,\mathrm{d}t - \int_Q \Phi\psi_t \,\mathrm{d}x\,\mathrm{d}t = \int_Q \widetilde{u}\psi \,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma \alpha\widetilde{v}\psi \,\mathrm{d}s\,\mathrm{d}t, \tag{4.43}$$

$$-\int_Q \Phi(\psi_1)_t \,\mathrm{d}x\,\mathrm{d}t + \int_Q \nabla\Phi\cdot\nabla\psi_1 \,\mathrm{d}x\,\mathrm{d}t + \int_Q (3(\varphi^*_{\varepsilon,\sigma})^2 - 1)\Phi\psi_1 \,\mathrm{d}x\,\mathrm{d}t = \int_Q \frac{1}{(\theta^*_{\varepsilon,\sigma})^2}Y\psi_1 \,\mathrm{d}x\,\mathrm{d}t, \tag{4.44}$$

for all $\psi \in L^2(0,T;H^2(\Omega)) \cap W^{1,2}([0,T];L^2(\Omega))$ solving the problem

$$\psi_t + \Delta\psi = -f_Q, \text{ in } Q; \qquad \frac{\partial\psi}{\partial\nu} + \alpha\psi = 0, \text{ on } \Sigma; \qquad \psi(T) = 0, \text{ in } \Omega, \tag{4.45}$$

for a generic $f_Q \in L^2(Q)$, and for all $\psi_1 \in L^2(0,T;V) \cap W^{1,2}([0,T];L^2(\Omega))$ such that $\psi_1(T) = 0$.

Proposition 4.3. *Assume* (1.8), (2.12), (2.36) *and* (2.15). *Then the problem* (4.38)–(4.42) *has a unique solution* (Y, Φ) *with*

$$Y \in L^2(0,T;L^2(\Omega)), \tag{4.46}$$

$$\Phi \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;V) \cap W^{1,2}([0,T];L^2(\Omega)), \tag{4.47}$$

and the convergence properties in (4.37) *hold.*

Proof. First, we prove the existence and uniqueness of the solution to (4.38)–(4.42). Due to the hypotheses (2.15), we infer that the state system (1.1)–(1.6), written for $u = u^*_{\varepsilon,\sigma}$ and $v = v^*_{\varepsilon,\sigma}$, has the solution $(\theta^*_{\varepsilon,\sigma}, \varphi^*_{\varepsilon,\sigma})$ with both $\theta^*_{\varepsilon,\sigma}$ and $1/\theta^*_{\varepsilon,\sigma}$ bounded in $L^\infty(Q)$ (see (2.17)) and consequently (*cf.* (1.7)) $\beta'(\theta^*_{\varepsilon,\sigma}) \in L^\infty(Q)$. Moreover, in view of (2.12), by (2.14) we deduce the boundedness of $\varphi^*_{\varepsilon,\sigma}$ in $L^\infty(Q) \cap L^\infty(0,T;H^2(\Omega))$.

We integrate (4.38) and (4.40) with respect to τ on $(0,t)$. We obtain

$$Y(t,x) - \Delta \int_0^t \beta'(\theta^*_{\varepsilon,\sigma}(\tau,x)) Y(\tau,x)\mathrm{d}\tau + \Phi(t,x) = \int_0^t \widetilde{u}(\tau,x)\mathrm{d}\tau,$$
$$-\frac{\partial}{\partial\nu}\int_0^t (\beta'(\theta^*_{\varepsilon,\sigma}(\tau,x))Y(\tau,x))\mathrm{d}\tau = \alpha(x)\int_0^t (\beta'(\theta^*_{\varepsilon,\sigma}(\tau,x))Y(\tau,x) - \widetilde{v}(\tau,x))\mathrm{d}\tau$$

and then set $W(t,x) = \int_0^t \beta'(\theta^*_{\varepsilon,\sigma}(\tau,x))Y(\tau,x)\mathrm{d}\tau$ for $(t,x) \in Q$, so that

$$W_t(t,x) = \beta'(\theta^*_{\varepsilon,\sigma}(t,x))Y(t,x), \quad (t,x) \in Q. \tag{4.48}$$

Now, the system (4.38)–(4.42) can be replaced by

$$\left(\frac{1}{\beta'(\theta^*_{\varepsilon,\sigma})}W_t - \Delta W + \Phi\right)(t,x) = \int_0^t \widetilde{u}(\tau,x)\mathrm{d}\tau, \quad (t,x) \in Q, \tag{4.49}$$

$$\Phi_t - \Delta\Phi + (3\varphi^*_{\varepsilon,\sigma} - 1)\Phi = \frac{1}{(\theta^*_{\varepsilon,\sigma})^2 \beta'(\theta^*_{\varepsilon,\sigma})}W_t, \quad \text{in } Q, \tag{4.50}$$

$$-\frac{\partial W}{\partial\nu}(t,x) = \alpha(x)\left(W(t,x) - \int_0^t \widetilde{v}(\tau,x)\mathrm{d}\tau\right), \quad (t,x) \in \Sigma, \tag{4.51}$$

$$\frac{\partial\Phi}{\partial\nu} = 0, \quad \text{on } \Sigma, \tag{4.52}$$

$$W(0) = 0, \ \Phi(0) = 0, \quad \text{in } \Omega. \tag{4.53}$$

Here, we are allowed to apply Proposition 4.2 for

$$a(t,x) = b(t,x) = \beta'(\theta^*_\varepsilon(t,x)), \qquad \omega(t,x) = \beta'(\theta^*_\varepsilon(t,x))\int_0^t \widetilde{u}(\tau,x)\mathrm{d}\tau,$$
$$c(t,x) = (3\varphi^*_{\varepsilon,\sigma}(t,x) - 1), \quad d(t,x) = -\frac{1}{(\theta^*_{\varepsilon,\sigma})^2\beta'(\theta^*_{\varepsilon,\sigma})}, \quad g(t,x) = 0,$$

$(t,x) \in Q$, and $\gamma(t,x) = \displaystyle\int_0^t \widetilde{v}(\tau,x)\mathrm{d}\tau$, $(t,x) \in \Sigma$. Observing that such coefficients satisfy (4.18)–(4.19), we conclude that (4.49)–(4.53) has a unique solution (W, Φ) with W in the space $L^\infty(0,T;V) \cap W^{1,2}([0,T];L^2(\Omega))$. Then, owing to (4.48), it turns out that (4.46) holds.

Next, we have to show that if (Y,Φ) fulfils the variational equalities in (4.43) and (4.44), then the pair (W,Φ) with W specified by (4.48) just solves the system (4.49)–(4.53). Indeed, taking an arbitrary

$$\zeta \in H^2(\Omega) \ \text{ such that } \ \frac{\partial \zeta}{\partial \nu} + \alpha\zeta = 0, \quad \text{on } \Gamma, \tag{4.54}$$

according to (4.45) we can choose $\psi(t,x) = (T-t)\zeta(x)$, $(x,t) \in Q$, in (4.43). Then, if ζ also belongs to $\mathcal{D}(\Omega)$, integrating by parts in time it is not difficult to recover the equality (4.49) in the sense of distributions in Ω, for a.e. $t \in (0,T)$. Once (4.49) is proved, we can compare the terms and find additional regularity for W (in particular, $\Delta W \in L^2(0,T;L^2(\Omega))$ in order to be able to get back to (4.43) and this time still use $\psi(t,x) = (T-t)\zeta(x)$, but with an auxiliary function ζ as in (4.54) to find the boundary condition (4.51) as well. A similar approach can be used on (4.44) taking now $\psi_1(t,x) = (T-t)\zeta_1(x)$, $(x,t) \in Q$, with ζ_1 arbitrary first in $H^1_0(\Omega)$, then in $H^1(\Omega)$ in order to arrive at an integrated version of (4.50) and (4.52). Then, it suffices to examine the regularity of Φ and realize that (4.47), as well as (4.50) and (4.52) directly, are satisfied.

We prove now (4.37). As mentioned in Theorem 2.2, the solution to (1.1)–(1.6) is Lipschitz continuous with respect to the data. Relying on (2.11) and recalling (4.36), we can write

$$\left\|\widetilde{\Theta}^\lambda\right\|^2_{L^2(Q)} + \left\|\widetilde{\Phi}^\lambda\right\|^2_{C([0,T];L^2(\Omega))} + \left\|\widetilde{\Phi}^\lambda\right\|^2_{L^2(0,T;V)} \le \|\widetilde{u}\|^2_{L^2(Q)} + \|\widetilde{v}\|^2_{L^2(\Sigma)}\,. \tag{4.55}$$

It is also obvious that, for each λ, the functions $\widetilde{\Theta}^\lambda$ and $\widetilde{\Phi}^\lambda$ are in the same spaces as $\theta^*_{\varepsilon,\sigma}$ and $\varphi^*_{\varepsilon,\sigma}$ are, given by Theorem 2.2, and that they satisfy the system

$$\widetilde{\Theta}^\lambda_t - \Delta\frac{\beta(\theta^\lambda_{\varepsilon,\sigma}) - \beta(\theta^*_{\varepsilon,\sigma})}{\lambda} + \widetilde{\Phi}^\lambda_t = \widetilde{u}, \quad \text{in } Q, \tag{4.56}$$

$$\widetilde{\Phi}^\lambda_t - \Delta\widetilde{\Phi}^\lambda + \left((\varphi^\lambda_{\varepsilon,\sigma})^2 + \varphi^\lambda_{\varepsilon,\sigma}\varphi^*_{\varepsilon,\sigma} + (\varphi^*_{\varepsilon,\sigma})^2 - 1\right)\widetilde{\Phi}^\lambda = \frac{\widetilde{\Theta}^\lambda}{\theta^\lambda_{\varepsilon,\sigma}\theta^*_{\varepsilon,\sigma}}, \quad \text{in } Q, \tag{4.57}$$

$$-\frac{\partial}{\partial\nu}\frac{\beta(\theta^\lambda_{\varepsilon,\sigma}) - \beta(\theta^*_{\varepsilon,\sigma})}{\lambda} = \alpha(x)\left(\frac{\beta(\theta^\lambda_{\varepsilon,\sigma}) - \beta(\theta^*_{\varepsilon,\sigma})}{\lambda} - \widetilde{v}\right), \quad \text{on } \Sigma, \tag{4.58}$$

$$\frac{\partial\widetilde{\Phi}^\lambda}{\partial\nu} = 0, \quad \text{on } \Sigma, \tag{4.59}$$

$$\widetilde{\Theta}^\lambda = 0, \quad \widetilde{\Phi}^\lambda = 0, \quad \text{in } \Omega. \tag{4.60}$$

Thanks to (4.55), at least for a subsequence of λ we have that

$$\begin{aligned}
&\widetilde{\Theta}^\lambda \to \widetilde{Y} \ \text{ weakly in } L^2(Q), \ \text{ as } \lambda \to 0,\\
&\widetilde{\Phi}^\lambda \to \widetilde{\Phi} \ \text{ weakly in } L^2(0,T;V) \text{ and weakly* in } L^\infty(0,T;L^2(\Omega)), \ \text{ as } \lambda \to 0.
\end{aligned}$$

Let us test (4.56) by ψ given as in (4.45) and (4.57) by $\psi_1 \in W^{1,2}([0,T];L^2(\Omega)) \cap L^2(0,T;V)$ with $\psi_1(T) = 0$. Using the boundary conditions (4.58) and (4.59), we obtain

$$-\int_Q \widetilde{\Theta}^\lambda\psi_t\,\mathrm{d}x\,\mathrm{d}t - \int_Q \frac{\beta(\theta^\lambda_{\varepsilon,\sigma}) - \beta(\theta^*_{\varepsilon,\sigma})}{\lambda}\Delta\psi\,\mathrm{d}x\,\mathrm{d}t - \int_Q \widetilde{\Phi}^\lambda\psi_t\,\mathrm{d}x\,\mathrm{d}t = \int_Q \widetilde{u}\psi\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma \alpha\widetilde{v}\psi\,\mathrm{d}s\,\mathrm{d}t, \tag{4.61}$$

$$\begin{aligned}
&-\int_Q \widetilde{\Phi}^\lambda(\psi_1)_t\,\mathrm{d}x\,\mathrm{d}t + \int_Q \nabla\widetilde{\Phi}^\lambda\cdot\nabla\psi_1\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_Q ((\varphi^\lambda_{\varepsilon,\sigma})^2 + \varphi^\lambda_{\varepsilon,\sigma}\varphi^*_{\varepsilon,\sigma} + (\varphi^*_{\varepsilon,\sigma})^2 - 1)\widetilde{\Phi}^\lambda\psi_1\,\mathrm{d}x\,\mathrm{d}t = \int_Q \frac{\widetilde{\Theta}^\lambda}{\theta^\lambda_{\varepsilon,\sigma}\theta^*_{\varepsilon,\sigma}}\psi_1\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.62}$$

Now, we observe that

$$\frac{\beta(\theta^\lambda_{\varepsilon,\sigma}) - \beta(\theta^*_{\varepsilon,\sigma})}{\lambda} = \beta'(\overline{\theta}_\lambda)\widetilde{\Theta}^\lambda,$$

where $\overline{\theta}_\lambda$ is a measurable function taking intermediate values between $\theta^*_{\varepsilon,\sigma}$ and $\theta^\lambda_{\varepsilon,\sigma}$, a.e. in Q. Moreover, due to (4.35) we have that $\overline{\theta}_\lambda \to \theta^*_{\varepsilon,\sigma}$ strongly in $L^2(Q)$ as $\lambda \to 0$. Therefore, by the Lipschitz continuity of some restriction of β to a bounded subset of $(0, +\infty)$ we deduce that

$$\beta'(\overline{\theta}_\lambda) \to \beta'(\theta^*_{\varepsilon,\sigma}) \ \text{ strongly in } L^2(Q), \ \text{ as } \lambda \to 0,$$

whence

$$\frac{\beta(\theta^\lambda_{\varepsilon,\sigma}) - \beta(\theta^*_{\varepsilon,\sigma})}{\lambda} \to \beta'(\theta^*_{\varepsilon,\sigma})\widetilde{Y} \ \text{ weakly in } L^1(Q), \ \text{ as } \lambda \to 0,$$

first, and then weakly in $L^2(Q)$ due to the boundedness of $\beta'(\overline{\theta}_\lambda)\widetilde{\Theta}^\lambda$ in $L^2(Q)$. Analogously, in view of (4.35),we have that

$$((\varphi^\lambda_{\varepsilon,\sigma})^2 + \varphi^\lambda_{\varepsilon,\sigma}\varphi^*_{\varepsilon,\sigma} + (\varphi^*_{\varepsilon,\sigma})^2 - 1)\widetilde{\Phi} \to (3(\varphi^*_{\varepsilon,\sigma})^2 - 1)\widetilde{\Phi} \ \text{ and}$$
$$\frac{\widetilde{\Theta}^\lambda}{\theta^\lambda_{\varepsilon,\sigma}\theta^*_{\varepsilon,\sigma}} \to \frac{\widetilde{Y}}{(\theta^*_{\varepsilon,\sigma})^2} \ \text{ weakly in } L^2(Q), \ \text{ as } \lambda \to 0.$$

Now, we can pass to the limit in (4.61)–(4.62) and find out that

$$-\int_Q \widetilde{Y}\psi_t \,\mathrm{d}x\,\mathrm{d}t - \int_Q \beta'(\theta^*_{\varepsilon,\sigma})\widetilde{Y}\Delta\psi \,\mathrm{d}x\,\mathrm{d}t - \int_Q \widetilde{\Phi}\psi_t \,\mathrm{d}x\,\mathrm{d}t = \int_Q \widetilde{u}\psi \,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma \alpha\widetilde{v}\psi \,\mathrm{d}s\,\mathrm{d}t,$$

$$-\int_Q \widetilde{\Phi}(\psi_1)_t \,\mathrm{d}x\,\mathrm{d}t + \int_Q \nabla\widetilde{\Phi}\cdot\nabla\psi_1 \,\mathrm{d}x\,\mathrm{d}t + \int_Q (3(\varphi^*_{\varepsilon,\sigma})^2 - 1)\widetilde{\Phi}\psi_1 \,\mathrm{d}x\,\mathrm{d}t = \int_Q \frac{\widetilde{Y}}{(\theta^*_{\varepsilon,\sigma})^2}\psi_1 \,\mathrm{d}x\,\mathrm{d}t,$$

which means that $\widetilde{Y}$, $\widetilde{\Phi}$ yield a solution to (4.38)–(4.42) (see (4.43)–(4.44)). Since this solution is unique we obtain $\widetilde{Y} = Y$, $\widetilde{\Phi} = \Phi$ and it is the whole family to converge in (4.37) as $\lambda \to 0$. □

Next, let us denote by $p_{\varepsilon,\sigma}$ and $q_{\varepsilon,\sigma}$ the dual variables and introduce the dual system

$$(p_{\varepsilon,\sigma})_t + \beta'(\theta^*_{\varepsilon,\sigma})\Delta p_{\varepsilon,\sigma} + \frac{1}{(\theta^*_{\varepsilon,\sigma})^2}q_{\varepsilon,\sigma} = -I^\sigma_{1,\varepsilon}, \ \text{ in } Q, \tag{4.63}$$

$$(q_{\varepsilon,\sigma})_t + \Delta q_{\varepsilon,\sigma} - (3(\varphi^*_{\varepsilon,\sigma})^2 - 1)q_{\varepsilon,\sigma} + (p_{\varepsilon,\sigma})_t = -I^\sigma_{2,\varepsilon}, \ \text{ in } Q, \tag{4.64}$$

$$\frac{\partial p_{\varepsilon,\sigma}}{\partial\nu} + \alpha(x)p_{\varepsilon,\sigma} = 0, \ \ \frac{\partial q_{\varepsilon,\sigma}}{\partial\nu} = 0, \ \text{ on } \Sigma, \tag{4.65}$$

$$p_{\varepsilon,\sigma}(T) = 0, \ \ q_{\varepsilon,\sigma}(T) = 0, \ \text{ in } \Omega, \tag{4.66}$$

where

$$I^\sigma_{1,\varepsilon} = 2\lambda_1(\theta^*_{\varepsilon,\sigma} - \theta_f) + \frac{1}{\varepsilon}(\xi^*_{\varepsilon,\sigma} - \eta^*_{\varepsilon,\sigma}), \quad I^\sigma_{2,\varepsilon} = 2\lambda_2(\varphi^*_{\varepsilon,\sigma} - \eta^*_{\varepsilon,\sigma}), \quad \xi^*_{\varepsilon,\sigma} = j'_\sigma(\theta^*_{\varepsilon,\sigma}).$$

We note that

$$I^\sigma_{1,\varepsilon},\ I^\sigma_{2,\varepsilon},\ \xi^*_{\varepsilon,\sigma} \in L^\infty(Q).$$

Proposition 4.4. *Assume* (1.8), (2.12), (2.36) *and* (2.15). *Then the dual system* (4.63)–(4.66) *has a unique solution* $(p_{\varepsilon,\sigma}, q_{\varepsilon,\sigma})$ *with*

$$p_{\varepsilon,\sigma},\ q_{\varepsilon,\sigma} \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;V) \cap W^{1,2}([0,T];L^2(\Omega)) \tag{4.67}$$

and such that the estimates

$$\|(p_{\varepsilon,\sigma})_t\|^2_{L^2(Q)} + \|p_{\varepsilon,\sigma}\|^2_{L^\infty(0,T;V)} + \int_0^T \|\Delta p_{\varepsilon,\sigma}(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \le C, \tag{4.68}$$

$$\|(q_{\varepsilon,\sigma})_t\|^2_{L^2(Q)} + \|q_{\varepsilon,\sigma}\|^2_{L^\infty(0,T;V)} + \int_0^T \|\Delta q_{\varepsilon,\sigma}(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \le C \tag{4.69}$$

hold independently of $\sigma > 0$.

Proof. First, we make in (4.63)–(4.66) the variable transformation $t' = T - t$. Then, the thesis follows from Proposition 4.2, by setting in the transformed system

$$a = \beta'(\theta^*_{\varepsilon,\sigma}), \quad b = -\frac{1}{(\theta^*_{\varepsilon,\sigma})^2}, \quad \omega = I^\sigma_{1,\varepsilon},$$
$$c = (3(\varphi^*_{\varepsilon,\sigma})^2 - 1), \quad d = -1, \quad g = I^\sigma_{2,\varepsilon}, \quad \text{and} \quad \gamma = 0.$$

The estimates (4.68)–(4.69) can be obtained by standard computations, testing (4.63) by $-(p_{\varepsilon,\sigma})_t$ and (4.64) by $-(q_{\varepsilon,\sigma})_t$, integrating, combining the resulting equalities, and so on. Finally, a comparison of terms in (4.63) and (4.64) yields the desired estimates also for $\Delta p_{\varepsilon,\sigma}$ and $\Delta q_{\varepsilon,\sigma}$. □

Proposition 4.5. *Under the assumptions* (1.8), (2.12), (2.36) *and* (2.15), *the optimality conditions for* $(P_{\varepsilon,\sigma})$ *read*

$$-\big(p_{\varepsilon,\sigma} + 2(u^*_{\varepsilon,\sigma} - u^*_\varepsilon)\big) \in \partial I_{K_1}(u^*_{\varepsilon,\sigma}), \tag{4.70}$$

$$-\big(\alpha p_{\varepsilon,\sigma} + 2(v^*_{\varepsilon,\sigma} - v^*_\varepsilon)\big) \in \partial I_{K_2}(v^*_{\varepsilon,\sigma}), \tag{4.71}$$

$$-\big(I^\sigma_{3,\varepsilon} + 2(\eta^*_{\varepsilon,\sigma} - \eta^*_\varepsilon)\big) \in \partial I_{K_{[-1,1]}}(\eta^*_{\varepsilon,\sigma}), \tag{4.72}$$

where

$$I^\sigma_{3,\varepsilon} = -2\lambda_2(\varphi^*_{\varepsilon,\sigma} - \eta^*_{\varepsilon,\sigma}) + \frac{1}{\varepsilon}(\theta_c - \theta^*_{\varepsilon,\sigma}). \tag{4.73}$$

Proof. Due to (4.63)–(4.66), it is straightforward to realize that one can take $\psi = p_{\varepsilon,\sigma}$ in (4.43) (*cf.* (4.45)) and $\psi_1 = q_{\varepsilon,\sigma}$ in (4.44). Then, by adding the equalities and integrating by parts in one term, we obtain

$$\begin{aligned}
&-\int_Q \left\{(p_{\varepsilon,\sigma})_t + \beta'(\theta^*_{\varepsilon,\sigma})\Delta p_{\varepsilon,\sigma} + \frac{1}{(\theta^*_{\varepsilon,\sigma})^2} q_{\varepsilon,\sigma}\right\} Y\,\mathrm{d}x\,\mathrm{d}t \\
&-\int_Q \left\{(q_{\varepsilon,\sigma})_t + \Delta q_{\varepsilon,\sigma} - (3(\varphi^*_{\varepsilon,\sigma})^2 - 1)q_{\varepsilon,\sigma} + (p_{\varepsilon,\sigma})_t\right\} \Phi\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q \widetilde{u}\, p_{\varepsilon,\sigma}\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma \alpha\,\widetilde{v}\, p_{\varepsilon,\sigma}\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

Hence, with the help of (4.63)–(4.64) we have

$$\int_Q I^\sigma_{1,\varepsilon} Y\,\mathrm{d}x\,\mathrm{d}t + \int_Q I^\sigma_{2,\varepsilon}\Phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q \widetilde{u}\, p_{\varepsilon,\sigma}\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma \alpha\,\widetilde{v}\, p_{\varepsilon,\sigma}\,\mathrm{d}s\,\mathrm{d}t. \tag{4.74}$$

Then we write the optimality condition

$$J_{\varepsilon,\sigma}(u^*_{\varepsilon,\sigma}, v^*_{\varepsilon,\sigma}, \eta^*_{\varepsilon,\sigma}) \le J_{\varepsilon,\sigma}(\widehat{u}, \widehat{v}, \widehat{\eta}), \quad \text{for any } (\widehat{u}, \widehat{v}, \widehat{\eta}) \in K_1 \times K_2 \times K_{[-1,1]}.$$

In particular, taking $\widehat{u} = u^\lambda_{\varepsilon,\sigma}$, $\widehat{v} = v^\lambda_{\varepsilon,\sigma}$, $\widehat{\eta} = \eta^\lambda_{\varepsilon,\sigma}$, making some computations, dividing by λ and letting λ go to 0 lead to the inequality

$$\begin{aligned} &2\lambda_1 \int_Q (\theta^*_{\varepsilon,\sigma} - \theta_f) Y \,\mathrm{d}x\,\mathrm{d}t + 2\lambda_2 \int_Q (\varphi^*_{\varepsilon,\sigma} - \eta^*_{\varepsilon,\sigma})(\Phi - \widetilde{\eta}) \,\mathrm{d}x\,\mathrm{d}t \\ &+ \frac{1}{\varepsilon} \int_Q (\xi^*_{\varepsilon,\sigma} Y + \widetilde{\eta}\theta_c - \widetilde{\eta}\theta^*_{\varepsilon,\sigma} - \eta^*_{\varepsilon,\sigma} Y) \,\mathrm{d}x\,\mathrm{d}t \\ &+ 2\int_Q (u^*_{\varepsilon,\sigma} - u^*_\varepsilon)\widetilde{u} \,\mathrm{d}x\,\mathrm{d}t + 2\int_\Sigma (v^*_{\varepsilon,\sigma} - v^*_\varepsilon)\widetilde{v} \,\mathrm{d}s\,\mathrm{d}t + 2\int_Q (\eta^*_{\varepsilon,\sigma} - \eta^*_\varepsilon)\widetilde{\eta} \,\mathrm{d}x\,\mathrm{d}t \ge 0. \end{aligned}$$

With the previous notation, this yields

$$\begin{aligned} &\int_Q I^\sigma_{1,\varepsilon} Y \,\mathrm{d}x\,\mathrm{d}t + \int_Q I^\sigma_{2,\varepsilon} \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_Q I^\sigma_{3,\varepsilon} \widetilde{\eta} \,\mathrm{d}x\,\mathrm{d}t \\ &+ 2\int_Q (u^*_{\varepsilon,\sigma} - u^*_\varepsilon)\widetilde{u} \,\mathrm{d}x\,\mathrm{d}t + 2\int_\Sigma (v^*_{\varepsilon,\sigma} - v^*_\varepsilon)\widetilde{v} \,\mathrm{d}s\,\mathrm{d}t + 2\int_Q (\eta^*_{\varepsilon,\sigma} - \eta^*_\varepsilon)\widetilde{\eta} \,\mathrm{d}x\,\mathrm{d}t \ge 0. \end{aligned} \tag{4.75}$$

By comparing (4.74) and (4.75) we easily obtain

$$\begin{aligned} \int_Q \widetilde{u} \left\{ p_{\varepsilon,\sigma} + 2(u^*_{\varepsilon,\sigma} - u^*_\varepsilon) \right\} \mathrm{d}x\,\mathrm{d}t + \int_\Sigma \widetilde{v} \left\{ \alpha p_{\varepsilon,\sigma} + 2(v^*_{\varepsilon,\sigma} - v^*_\varepsilon) \right\} \mathrm{d}s\,\mathrm{d}t \\ + \int_Q \widetilde{\eta} \left\{ I^\sigma_{3,\varepsilon} + 2(\eta^*_{\varepsilon,\sigma} - \eta^*_\varepsilon) \right\} \mathrm{d}x\,\mathrm{d}t \ge 0. \end{aligned}$$

Therefore, recalling (4.34) we finally have

$$\begin{aligned} &\int_Q (u^*_{\varepsilon,\sigma} - u) \left\{ -\left(p_{\varepsilon,\sigma} + 2(u^*_{\varepsilon,\sigma} - u^*_\varepsilon) \right) \right\} \mathrm{d}x\,\mathrm{d}t \\ &+ \int_\Sigma (v^*_{\varepsilon,\sigma} - v) \left\{ -\left(\alpha p_{\varepsilon,\sigma} + 2(v^*_{\varepsilon,\sigma} - v^*_\varepsilon) \right) \right\} \mathrm{d}s\,\mathrm{d}t \\ &+ \int_Q (\eta^*_{\varepsilon,\sigma} - \eta) \left\{ -\left(I^\sigma_{3,\varepsilon} + 2(\eta^*_{\varepsilon,\sigma} - \eta^*_\varepsilon) \right) \right\} \mathrm{d}x\,\mathrm{d}t \ge 0, \end{aligned}$$

for any $(u, v, \eta) \in K_1 \times K_2 \times K_{[-1,1]}$. This implies (4.71), as claimed. □

Theorem 4.6. *Assume* (2.36), (1.8), (2.12), (2.15) *and let* $\{(u^*_\varepsilon, v^*_\varepsilon, \eta^*_\varepsilon), (\theta^*_\varepsilon, \varphi^*_\varepsilon)\}$ *be optimal in* (P_ε). *Then, the optimality conditions for* (P_ε) *read*

$$\begin{cases} u^*_\varepsilon(t,x) = u_m, & \text{on } \{(t,x) \in Q : \ p_\varepsilon(t,x) > 0\} \\ u_m \le u^*_\varepsilon(t,x) \le u_M, & \text{on } \{(t,x) \in Q : \ p_\varepsilon(t,x) = 0\}, \\ u^*_\varepsilon(t,x) = u_M, & \text{on } \{(t,x) \in Q : \ p_\varepsilon(t,x) < 0\} \end{cases} \tag{4.76}$$

$$\begin{cases} v^*_\varepsilon(t,x) = v_m, & \text{on } \{(t,x) \in \Sigma : \ p_\varepsilon(t,x) > 0\} \\ v_m \le v^*_\varepsilon(t,x) \le v_M, & \text{on } \{(t,x) \in \Sigma : \ p_\varepsilon(t,x) = 0\}, \\ v^*_\varepsilon(t,x) = v_M, & \text{on } \{(t,x) \in \Sigma : \ p_\varepsilon(t,x) < 0\} \end{cases} \tag{4.77}$$

$$\begin{cases} \eta^*_\varepsilon(t,x) = -1, & on\ \{(t,x)\in Q:\ I_{3,\varepsilon}(t,x) > 0\} \\ -1 \le \eta^*_\varepsilon(t,x) \le 1, & on\ \{(t,x)\in Q:\ I_{3,\varepsilon}(t,x) = 0\}, \\ \eta^*_\varepsilon(t,x) = 1, & on\ \{(t,x)\in Q:\ I_{3,\varepsilon}(t,x) < 0\} \end{cases} \tag{4.78}$$

where $(p_\varepsilon, q_\varepsilon)$ *is the solution to the problem*

$$(p_\varepsilon)_t + \beta'(\theta^*_\varepsilon)\Delta p_\varepsilon + \frac{1}{(\theta^*_\varepsilon)^2} q_\varepsilon = -I_{1,\varepsilon}, \quad in\ Q, \tag{4.79}$$

$$(q_\varepsilon)_t + \Delta q_\varepsilon - (3(\varphi^*_{\varepsilon,\sigma})^2 - 1)q_\varepsilon + (p_\varepsilon)_t = -I_{2,\varepsilon}, \quad in\ Q, \tag{4.80}$$

$$\frac{\partial p_\varepsilon}{\partial \nu} + \alpha(x) p_\varepsilon = 0, \quad \frac{\partial q_\varepsilon}{\partial \nu} = 0, \quad on\ \Sigma, \tag{4.81}$$

$$p_\varepsilon(T) = 0, \quad q_\varepsilon(T) = 0, \quad in\ \Omega, \tag{4.82}$$

and where

$$I_{1,\varepsilon} = 2\lambda_1(\theta^*_\varepsilon - \theta_f) + \frac{1}{\varepsilon}(\xi^*_\varepsilon - \eta^*_\varepsilon), \quad with \quad \xi^*_\varepsilon \in \partial j(\theta^*_\varepsilon) \ \ a.e.\ in\ Q,$$

$$I_{2,\varepsilon} = 2\lambda_2(\varphi^*_\varepsilon - \eta^*_\varepsilon), \qquad I_{3,\varepsilon} = -2\lambda_2(\varphi^*_\varepsilon - \eta^*_\varepsilon) + \frac{1}{\varepsilon}(\theta_c - \theta^*_\varepsilon).$$

Proof. Under the hypotheses, problem $(P_{\varepsilon,\sigma})$ has a minimizer $(u^*_{\varepsilon,\sigma}, v^*_{\varepsilon,\sigma}, \eta^*_{\varepsilon,\sigma})$ with the corresponding pair $(\theta^*_{\varepsilon,\sigma}, \varphi^*_{\varepsilon,\sigma})$ solving (1.1)–(1.6). We pass to the limit in $(P_{\varepsilon,\sigma})$. According to Proposition 4.1, we have the convergences in (4.4)–(4.6) and (4.9)–(4.10), in which however the actual limits are θ^*_ε and φ^*_ε. By the estimates (4.68)–(4.69) in Proposition 4.4, at least for a subsequence we have that

$$p_{\varepsilon,\sigma} \to p_\varepsilon, \ \ q_{\varepsilon,\sigma} \to q_\varepsilon \text{ weakly in } L^2(0,T;H^2(\Omega)) \cap W^{1,2}([0,T];L^2(\Omega)),$$
$$\text{weakly* in } L^\infty(0,T;V), \text{ and strongly in } L^2(0,T;V), \ \textit{as } \sigma \to 0.$$

Then, recalling (4.70) and passing to the limit we find that

$$-\big(p_{\varepsilon,\sigma} + 2(u^*_{\varepsilon,\sigma} - u^*_\varepsilon)\big) \to -p_\varepsilon \ \text{ strongly in } L^2(Q), \text{ as } \sigma \to 0,$$

which, along with (4.4), yields

$$-p_\varepsilon \in \partial I_{K_1}(u^*_\varepsilon), \quad \text{a.e. in } Q \tag{4.83}$$

for ∂I_{K_1} is maximal monotone and so strongly-weakly closed. The same argument works for the other two controllers in (4.71) and (4.72), hence we obtain

$$-\alpha p_\varepsilon \in \partial I_{K_2}(v^*_\varepsilon), \quad \text{a.e. on } \Sigma, \tag{4.84}$$

and

$$-I_{3,\varepsilon} \in \partial I_{K_{[-1,1]}}(\eta^*_\varepsilon), \quad \text{a.e. in } Q, \tag{4.85}$$

because of the convergence

$$-(I^\sigma_{3,\varepsilon} + 2(\eta^*_{\varepsilon,\sigma} - \eta^*_\varepsilon)) = 2\lambda_2(\varphi^*_{\varepsilon,\sigma} - \eta^*_{\varepsilon,\sigma}) - \frac{1}{\varepsilon}(\theta_c - \theta^*_{\varepsilon,\sigma}) - 2(\eta^*_{\varepsilon,\sigma} - \eta^*_\varepsilon)$$
$$\to \ 2\lambda_2(\varphi^*_\varepsilon - \eta^*_\varepsilon) - \frac{1}{\varepsilon}(\theta_c - \theta^*_\varepsilon) = -I_{3,\varepsilon} \ \text{ strongly in } L^2(Q), \text{ as } \sigma \to 0.$$

It is easily seen that

$$-I^{\sigma}_{2,\varepsilon} = 2\lambda_2(\varphi^*_{\varepsilon,\sigma} - \eta^*_{\varepsilon,\sigma}) \to -2\lambda_2(\varphi^*_{\varepsilon} - \eta^*_{\varepsilon}) = -I_{2,\varepsilon} \ \text{ strongly in } L^2(Q), \text{ as } \sigma \to 0.$$

Letting ξ^*_ε denote the weak* limit in $L^\infty(Q)$ of some subsequence of $\{\xi^*_{\varepsilon,\sigma}\}$, it turns out that $\xi^*_\varepsilon \in \partial j(\theta^*_\varepsilon)$ a.e. in Q. Indeed, recalling that $\xi^*_{\varepsilon,\sigma} = j'_\sigma(\theta^*_{\varepsilon,\sigma})$, we can write

$$\int_Q (j_\sigma(\theta^*_{\varepsilon,\sigma}) - j_\sigma(z))\,\mathrm{d}x\,\mathrm{d}t \le \int_Q j'_\sigma(\theta^*_{\varepsilon,\sigma})(\theta^*_{\varepsilon,\sigma} - z)\,\mathrm{d}x\,\mathrm{d}t$$

for any $z \in L^2(Q)$, and pass to the limit as $\sigma \to 0$ taking (4.11) into account. Thus, we deduce that

$$\int_Q (j(\theta^*_\varepsilon) - j(z))\,\mathrm{d}x\,\mathrm{d}t \le \int_Q \xi^*_\varepsilon(\theta^*_\varepsilon - z)\,\mathrm{d}x\,\mathrm{d}t,$$

which implies $\xi^*_\varepsilon \in \partial j(\theta^*_\varepsilon)$ a.e. in Q. Consequently, we have that

$$\begin{aligned} I^\sigma_{1,\varepsilon} &= 2\lambda_1(\theta^*_{\varepsilon,\sigma} - \theta_f) + \frac{1}{\varepsilon}(\xi^*_{\varepsilon,\sigma} - \eta^*_{\varepsilon,\sigma}) \\ &\to\ 2\lambda_1(\theta^*_\varepsilon - \theta_f) + \frac{1}{\varepsilon}(\xi^*_\varepsilon - \eta^*_\varepsilon) = I_{1,\varepsilon} \ \text{ weakly in } L^2(Q), \text{ as } \sigma \to 0. \end{aligned}$$

The above arguments prove that the solution to (4.63)–(4.66) converges to the solution to (4.79)–(4.82) as $\sigma \to 0$. In fact, due to the uniform boundedness properties ensured by assumptions (2.12) and (2.15), we also point out that

$$\beta'(\theta^*_{\varepsilon,\sigma}) \to \beta'(\theta^*_\varepsilon), \quad \frac{1}{(\theta^*_{\varepsilon,\sigma})^2} \to \frac{1}{(\theta^*_\varepsilon)^2}, \quad (3(\varphi^*_{\varepsilon,\sigma})^2 - 1) \to (3(\varphi^*_\varepsilon)^2 - 1)$$
$$\text{weakly* in } L^\infty(Q) \text{ and strongly in } L^2(Q), \text{ as } \sigma \to 0.$$

Note that the selection ξ^*_ε from $\partial j(\theta^*_\varepsilon) = H(\theta^*_\varepsilon)$ which is present in $I_{1,\varepsilon}$ is not uniquely determined unless $\theta^*_\varepsilon \neq \theta_c$ a.e. in Q. On the other hand, the pair $(p_\varepsilon, q_\varepsilon)$ turns out to be the unique solution of the problem (4.79)–(4.82) once ξ^*_ε is fixed in $I_{1,\varepsilon}$.

Now, in order to conclude the proof it suffices to notice that, *e.g.*, $\partial I_{K_1}(u^*_\varepsilon)$ is exactly $N_{K_1}(u^*_\varepsilon)$, the normal cone to K_1 at u^*_ε. Then, it is straightforward to derive (4.76)–(4.78), as claimed. □

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