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ERRATUM TO THE ARTICLE HAMILTON–JACOBI EQUATIONS FOR OPTIMAL CONTROL ON JUNCTIONS AND NETWORKS

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Abstract. We correct a mistake which affects an intermediate result, namely the second part of Lemma 4.5. The main results of the article are unchanged.

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The second part of Lemma 4.5, concerning subsolutions, is not correct in the published version of the paper. Recall that we are interested in proving a comparison principle for sub and super solutions of

$$\lambda u(x) + \sup_{(\zeta,\xi)\in \mathrm{FL}(x)} \{-Du(x,\zeta) - \xi\} = 0 \quad \text{in } \mathcal{G}. \tag{3.1}$$

Lemma 4.5 must be modified as follows:

Lemma 4.5. *Let $v : \mathcal{G} \to \mathbb{R}$ be a viscosity supersolution of (3.1) in $\mathcal{G}$. Then if $x \in J_i\backslash\{0\}$, we have for all $t > 0$,*

$$v(x) \geq \inf_{\alpha_i(\cdot),\theta_i} \left(\int_0^{t\wedge\theta_i} \ell_i(y_x^i(s), \alpha_i(s))\mathrm{e}^{-\lambda s}\mathrm{d}s + v(y_x^i(t\wedge\theta_i))\mathrm{e}^{-\lambda(t\wedge\theta_i)} \right), \tag{4.8}$$

where $\alpha_i \in L^\infty(0,\infty;A_i)$, y_x^i is the solution of $y_x^i(t) = x + \left(\int_0^t f_i(y_x^i(s),\alpha_i(s))\mathrm{d}s\right) e_i$ and θ_i is such that $y_x^i(\theta_i) = 0$ and θ_i lies in $[\tau_i, \bar\tau_i]$, where τ_i is the exit time of y_x^i from $J_i\backslash\{O\}$ and $\bar\tau_i$ is the exit time of y_x^i from J_i.

Remark. Concerning subsolutions, the comparison results of Barles−Perthame [2] imply the following sub-optimality principle for subsolutions that will not be needed in the sequel: let w be a continuous viscosity

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subsolution of (3.1) in $\mathcal{G}$. If $x \in J_i\backslash\{0\}$, we have for all $t > 0$,

$$w(x) \le \inf_{\alpha_i(\cdot)} \sup_{\theta_i} \left(\int_0^{t\wedge\theta_i} \ell_i(y_x^i(s), \alpha_i(s)) \mathrm{e}^{-\lambda s} \mathrm{d}s + w(y_x^i(t\wedge\theta_i)) \mathrm{e}^{-\lambda(t\wedge\theta_i)} \right), \tag{4.9}$$

where $\alpha_i \in L^\infty(0,\infty; A_i)$, y_x^i is the solution of $y_x^i(t) = x + \left(\int_0^t f_i(y_x^i(s), \alpha_i(s))\mathrm{d}s\right) e_i$ and θ_i is such that $y_x^i(\theta_i) = 0$ and θ_i lies in $[\tau_i, \bar{\tau}_i]$, where τ_i is the exit time of y_x^i from $J_i\backslash\{O\}$ and $\bar{\tau}_i$ is the exit time of y_x^i from J_i.

Then, Theorem 4.6 should be very slightly modified as follows (the very minor changes in the proof do not need to be written):

Theorem 4.6. *Assume* [H0], [H1], [H2] *and* [H3]. *Let $r > 0$ be given by Lemma 4.2: any bounded subsolution of* (3.1) *is Lipschitz continuous in $B(O,r) \cap \mathcal{G}$. Let $v : \mathcal{G} \to \mathbb{R}$ be a viscosity supersolution of* (3.1), *bounded from below by $-c|x| - C$ for two positive numbers c and C. Either* [A] *or* [B] *below is true:*

[A] *There exists a sequence $(\eta_k)_{k\in\mathbb{N}}$ of positive real numbers such that $\lim_{k\to+\infty} \eta_k = \eta > 0$, an index $i \in \{1,\dots,N\}$ and a sequence $x_k \in J_i$ such that $x_k \in J_i \setminus \{O\}$ and $\lim_{k\to+\infty} x_k = O$ satisfying the following: for any $k \in \mathbb{N}$, there exists a control law α_i^k such that the corresponding trajectory y_{x_k} remains in $J_i \cap B(O,r)$ in the time interval $[0,\eta_k]$,* i.e. *$y_{x_k}(s) \in J_i \cap B(O,r)$ for all $s \in [0,\eta_k]$, and is such that*

$$v(x_k) \ge \int_0^{\eta_k} \ell_i(y_{x_k}(s), \alpha_i^k(s)) \mathrm{e}^{-\lambda s} \mathrm{d}s + v(y_{x_k}(\eta)) \mathrm{e}^{-\lambda \eta_k} \tag{4.10}$$

[B]

$$\lambda v(O) + H_O^T \ge 0. \tag{4.11}$$

A new lemma is needed to replace the second part of Lemma 4.5:

Lemma 4.7. *Assume* [H0], [H1], [H2] *and* [H3]. *Let $r > 0$ be given by Lemma 4.2: any bounded subsolution of* (3.1) *is Lipschitz continuous in $B(O,r) \cap \mathcal{G}$. Consider $i \in \{1,\dots,N\}$, $x \in (J_i \setminus \{O\}) \cap B(O,r)$, $\alpha_i \in L^\infty(0,\infty; A_i)$. Let $\eta > 0$ be such that $y_x(t) = x + \left(\int_0^t f_i(y_x(s), \alpha_i(s))\mathrm{d}s\right) e_i$ belongs to $J_i \cap B(O,r)$ for any $t \in [0,\eta]$. For any bounded viscosity subsolution v of* (3.1),

$$v(x) \le \int_0^\eta \ell_i(y_x(t), \alpha_i(t)) \mathrm{e}^{-\lambda t} \mathrm{d}t + v(y_x(\eta)) \mathrm{e}^{-\lambda \eta}. \tag{a}$$

Proof. Since v is Lipschitz continuous in $B(O,r) \cap J_i$, the function $t \mapsto v(y_x(t))\mathrm{e}^{-\lambda t}$ is Lipschitz continuous in $[0,\eta]$. Let us define the sets $K_O = \{t \in (0,\eta) : y_x(t) = O\}$ and $K_O^c = [0,\eta]\backslash K_O$. It is clear that K_O is closed and that K_O^c is an open subset of $[0,\eta]$. We first observe that, from Stampacchia's theorem,

$$\int_0^\eta 1_{K_O}(t) \frac{\mathrm{d}}{\mathrm{d}t} \left(v(y_x(t))\mathrm{e}^{-\lambda t}\right) \mathrm{d}t = -\lambda v(O) \int_0^\eta 1_{K_O}(t) \mathrm{e}^{-\lambda t} \mathrm{d}t.$$

Therefore, we deduce from Lemma 4.3 that

$$\int_0^\eta 1_{K_O}(t) \frac{\mathrm{d}}{\mathrm{d}t} \left(v(y_x(t))\mathrm{e}^{-\lambda t}\right) \mathrm{d}t \ge H_O^T \int_0^\eta 1_{K_O}(t) \mathrm{d}t \ge -\int_0^\eta \ell_i(O, \alpha_i(t)) 1_{K_O}(t) \mathrm{d}t = -\int_0^\eta \ell_i(y_x(t), \alpha_i(t)) 1_{K_O}(t) \mathrm{d}t. \tag{b}$$

On the other hand, since K_O^c is an open subset of $[0,\eta]$, there exists a countable family of disjoint intervals $(\omega_j)_{j\in J}$, $\omega_j \subset [0,\eta]$ such that $K_O^c = \bigcup_{j\in J} \omega_j$. Let $a_j < b_j$ be the lower and upper endpoints of $\bar{\omega}_j$. We can assume that $[a_j, b_j] \cap [a_k, b_k] = \emptyset$ if $j \ne k$.

From a classical suboptimality principle, see ([1], Thm. III.2.33), we see that for any $j \in J$,

$$v(y_x(b_j))\mathrm{e}^{-\lambda b_j} - v(y_x(a_j))\mathrm{e}^{-\lambda a_j} \geq -\int_{a_j}^{b_j} \ell_i(y_x(t), \alpha_i(t))\mathrm{e}^{-\lambda t}\mathrm{d}t.$$

Noting that

$$v(y_x(b_j))\mathrm{e}^{-\lambda b_j} - v(y_x(a_j))\mathrm{e}^{-\lambda a_j} = \int_0^{\eta} \frac{\mathrm{d}}{\mathrm{d}t}\left(v(y_x(t))\mathrm{e}^{-\lambda t}\right) 1_{(a_j,b_j)}(t)\mathrm{d}t,$$

and summing over $j \in J$, we obtain that

$$\int_0^{\eta} 1_{K_O^c}(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(v(y_x(t))\mathrm{e}^{-\lambda t}\right)\mathrm{d}t \geq -\int_0^{\eta} \ell_i(y_x(t), \alpha_i(t))1_{K_O^c}(t)\mathrm{d}t. \tag{c}$$

We get (a) by summing (b) and (c). □

The main comparison result holds but its proof is modified.

Theorem 5.1. *Assume* [H0], [H1], [H2] *and* [H3]. *Let* $u : \mathcal{G} \to \mathbb{R}$ *be a bounded viscosity subsolution of* (3.1), *and* $v : \mathcal{G} \to \mathbb{R}$ *be a bounded viscosity supersolution of* (3.1). *Then* $u \leq v$ *in* $\mathcal{G}$.

Proof. It is a simple matter to check that there exists a positive real number M such that the function $\psi(x) = -|x|^2 - M$ is a viscosity subsolution of (3.1.). For $0 < \mu < 1$, μ close to 1, the function $u_\mu = \mu u + (1-\mu)\psi$ is a viscosity subsolution of (3.1), which tends to $-\infty$ as $|x|$ tends to $+\infty$. Let M_μ be the maximal value of $u_\mu - v$ which is reached at some point $\bar{x}_\mu$.

We want to prove that $M_\mu \leq 0$.

(1) If $\bar{x}_\mu \neq O$, then we introduce the function $u_\mu(x) - v(x) - d^2(x, \bar{x}_\mu)$, which has a strict maximum at $\bar{x}_\mu$, and we double the variables, i.e. for $0 < \varepsilon \ll 1$, we consider

$$u_\mu(x) - v(y) - d^2(x, \bar{x}_\mu) - \frac{d^2(x,y)}{\varepsilon^2}.$$

Classical arguments then lead to the conclusion that $u_\mu(\bar{x}_\mu) - v(\bar{x}_\mu) \leq 0$, thus $M_\mu \leq 0$.

(2) If $\bar{x}_\mu = O$. We use Theorem 4.6; we have two possible cases:

[B] $\lambda v(O) \geq -H_O^T$.
From Lemma 4.3, $\lambda u(O) + H_O^T \leq 0$. Therefore, we obtain that $u_\mu(O) \leq v(O)$, thus $M_\mu \leq 0$.

[A] With the notations of Theorem 4.6,

$$v(x_k) \geq \int_0^{\eta_k} \ell_i(y_{x_k}(s), \alpha_i^k(s))\mathrm{e}^{-\lambda s}\mathrm{d}s + v(y_{x_k}(\eta_k))\mathrm{e}^{-\lambda \eta_k}.$$

Moreover, since $y_{x_k}(s) \in J_i \cap B(O,r)$ for all $s \in [0, \eta_k]$, Lemma 4.7 can be applied and yields that

$$u_\mu(x_k) \leq \int_0^{\eta_k} \ell_i(y_{x_k}(s), \alpha_i^k(s))\mathrm{e}^{-\lambda s}\mathrm{d}s + u_\mu(y_{x_k}(\eta_k))\mathrm{e}^{-\lambda \eta_k}.$$

Therefore

$$u_\mu(x_k) - v(x_k) \leq (u_\mu(y_{x_k}(\eta_k)) - v(y_{x_k}(\eta_k)))\mathrm{e}^{-\lambda \eta_k}.$$

Letting k tend to $+\infty$, we find that $M_\mu \leq M_\mu \mathrm{e}^{-\lambda \eta}$, which implies that $M_\mu \leq 0$

We conclude by letting μ tend to 1. □

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