ERRATUM TO THE ARTICLE

HAMILTON–JACOBI EQUATIONS FOR OPTIMAL CONTROL ON JUNCTIONS AND NETWORKS

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Abstract. We correct a mistake which affects an intermediate result, namely the second part of Lemma 4.5. The main results of the article are unchanged.

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The second part of Lemma 4.5, concerning subsolutions, is not correct in the published version of the paper. Recall that we are interested in proving a comparison principle for sub and super solutions of

\[ \lambda u(x) + \sup_{(\zeta, \xi) \in FL(x)} \{-Du(x, \zeta) - \xi\} = 0 \quad \text{in} \ G. \]  

(3.1)

Lemma 4.5 must be modified as follows:

**Lemma 4.5.** Let \( v : G \to \mathbb{R} \) be a viscosity supersolution of (3.1) in \( G \). Then if \( x \in J_i \setminus \{0\} \), we have for all \( t > 0 \),

\[ v(x) \geq \inf_{\alpha_i(\cdot), \theta_i} \left( \int_0^{t \wedge \theta_i} \ell_i(y^i_s(s), \alpha_i(s))e^{-\lambda s} ds + v(y^i_s(t \wedge \theta_i))e^{-\lambda (t \wedge \theta_i)} \right), \]

(4.8)

where \( \alpha_i \in L^\infty(0, \infty; A_i) \), \( y^i_x \) is the solution of \( y^i_x(t) = x + \left( \int_0^t f_i(y^i_s(s), \alpha_i(s)) ds \right) e_i \) and \( \theta_i \) is such that \( y^i_x(\theta_i) = 0 \) and \( \theta_i \) lies in \([\tau_i, \bar{\tau}_i]\), where \( \tau_i \) is the exit time of \( y^i_x \) from \( J_i \setminus \{0\} \) and \( \bar{\tau}_i \) is the exit time of \( y^i_x \) from \( J_i \).

Remark. Concerning subsolutions, the comparison results of Barles–Perthame [2] imply the following sub-optimality principle for subsolutions that will not be needed in the sequel: let \( w \) be a continuous viscosity

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subsolution of (3.1) in $G$. If $x \in J_i \setminus \{0\}$, we have for all $t > 0$,

$$w(x) \leq \inf_{\alpha_i(\cdot)} \sup_{\theta_i} \left( \int_{0}^{t \land \theta_i} \ell_i(y_x(s), \alpha_i(s)) e^{-\lambda_s} ds + w(y_x(t \land \theta_i)) e^{-\lambda(t \land \theta_i)} \right),$$

(4.9)

where $\alpha_i \in L^{\infty}(0, \infty; A_i)$, $y_x^i$ is the solution of $y_x^i(t) = x + \left( \int_{0}^{t} f_i(y_x(s), \alpha_i(s)) ds \right) e_i$ and $\theta_i$ is such that $y_x^i(\theta_i) = 0$ and $\theta_i$ lies in $[\tau_i, \bar{\tau}_i]$, where $\tau_i$ is the exit time of $y_x^i$ from $J_i \setminus \{0\}$ and $\bar{\tau}_i$ is the exit time of $y_x^i$ from $J_i$.

Then, Theorem 4.6 should be very slightly modified as follows (the very minor changes in the proof do not need to be written):

**Theorem 4.6.** Assume [H0], [H1], [H2] and [H3]. Let $r > 0$ be given by Lemma 4.2: any bounded subsolution of (3.1) is Lipschitz continuous in $B(O, r) \cap G$. Let $v : G \to \mathbb{R}$ be a viscosity supersolution of (3.1), bounded from below by $-c|x| - C$ for two positive numbers $c$ and $C$. Either [A] or [B] below is true:

[A] There exists a sequence $(\eta_k)_{k \in \mathbb{N}}$ of positive real numbers such that $\lim_{k \to +\infty} \eta_k = \eta > 0$, an index $i \in \{1, \ldots, N\}$ and a sequence $x_k \in J_i$ such that $x_k \in J_i \setminus \{0\}$ and $\lim_{k \to +\infty} x_k = O$ satisfying the following: for any $k \in \mathbb{N}$, there exists a control law $\alpha_k^i$ such that the corresponding trajectory $y_{x_k}(s) \in J_i \cap B(O, r)$ in the time interval $[0, \eta_k]$, i.e. $y_{x_k}(s) \in J_i \cap B(O, r)$ for all $s \in [0, \eta_k]$, and is such that

$$v(x_k) \geq \int_{0}^{\eta_k} \ell_i(y_{x_k}(s), \alpha_k^i(s)) e^{-\lambda_s} ds + v(y_{x_k}(\eta)) e^{-\lambda \eta_k}$$

(4.10)

[B]

$$\lambda v(O) + H^{\alpha_i}_O \geq 0.$$  

(4.11)

A new lemma is needed to replace the second part of Lemma 4.5:

**Lemma 4.7.** Assume [H0], [H1], [H2] and [H3]. Let $r > 0$ be given by Lemma 4.2: any bounded subsolution of (3.1) is Lipschitz continuous in $B(O, r) \cap G$. Consider $i \in \{1, \ldots, N\}$, $x \in (J_i \setminus \{0\}) \cap B(O, r)$, $\alpha_i \in L^{\infty}(0, \infty; A_i)$. Let $\eta > 0$ be such that $y_x(t) = x + \left( \int_{0}^{t} f_i(y_x(s), \alpha_i(s)) ds \right) e_i$ belongs to $J_i \cap B(O, r)$ for any $t \in [0, \eta]$. For any bounded viscosity subsolution $v$ of (3.1),

$$v(x) \leq \int_{0}^{\eta} \ell_i(y_x(t), \alpha_i(t)) e^{-\lambda t} dt + v(y_x(\eta)) e^{-\lambda \eta}.$$  

(a)

**Proof.** Since $v$ is Lipschitz continuous in $B(O, r) \cap J_i$, the function $t \mapsto v(y_x(t)) e^{-\lambda t}$ is Lipschitz continuous in $[0, \eta]$. Let us define the sets $K_O = \{ t \in (0, \eta) : y_x(t) = O \}$ and $K_O^c = [0, \eta] \setminus K_O$. It is clear that $K_O$ is closed and that $K_O^c$ is an open subset of $[0, \eta]$. We first observe that, from Stampacchia’s theorem,

$$\int_{0}^{\eta} 1_{K_O}(t) \frac{d}{dt} \left( v(y_x(t)) e^{-\lambda t} \right) dt = -\lambda v(O) \int_{0}^{\eta} 1_{K_O}(t) e^{-\lambda t} dt.$$

Therefore, we deduce from Lemma 4.3 that

$$\int_{0}^{\eta} 1_{K_O}(t) \frac{d}{dt} \left( v(y_x(t)) e^{-\lambda t} \right) dt \geq H^T_O \int_{0}^{\eta} 1_{K_O}(t) dt \geq - \int_{0}^{\eta} \ell_i(O, \alpha_i(t)) 1_{K_O}(t) dt = - \int_{0}^{\eta} \ell_i(y_x(t), \alpha_i(t)) 1_{K_O}(t) dt.$$  

(b)

On the other hand, since $K_O^c$ is an open subset of $[0, \eta]$, there exists a countable family of disjoint intervals $(\omega_j)_{j \in J}$, $\omega_j \subset [0, \eta]$ such that $K_O^c = \bigcup_{j \in J} \omega_j$. Let $a_j < b_j$ be the lower and upper endpoints of $\omega_j$. We can assume that $[a_j, b_j] \cap [a_k, b_k] = \emptyset$ if $j \neq k$. 
From a classical suboptimality principle, see ([1], Thm. III.2.33), we see that for any $j \in J$,

$$v(y_x(b_j)) e^{-\lambda b_j} - v(y_x(a_j)) e^{-\lambda a_j} \geq - \int_{a_j}^{b_j} \ell_i(y_x(t), \alpha_i(t)) e^{-\lambda t} dt.$$ 

Noting that

$$v(y_x(b_j)) e^{-\lambda b_j} - v(y_x(a_j)) e^{-\lambda a_j} = \int_0^\eta \frac{d}{dt} \left(v(y_x(t)) e^{-\lambda t}\right) 1_{(a_j, b_j)}(t) dt,$$

and summing over $j \in J$, we obtain that

$$\int_0^\eta 1_{K_0}^o(t) \frac{d}{dt} \left(v(y_x(t)) e^{-\lambda t}\right) dt \geq - \int_0^\eta \ell_i(y_x(t), \alpha_i(t)) 1_{K_0}^o(t) dt. \quad (c)$$

We get (a) by summing (b) and (c). \hfill \Box

The main comparison result holds but its proof is modified.

**Theorem 5.1.** Assume [H0], [H1], [H2] and [H3]. Let $u : G \to \mathbb{R}$ be a bounded viscosity subsolution of (3.1), and $v : G \to \mathbb{R}$ be a bounded viscosity supersolution of (3.1). Then $u \leq v$ in $G$.

**Proof.** It is a simple matter to check that there exists a positive real number $M$ such that the function $\psi(x) = -|x|^2 - M$ is a viscosity subsolution of (3.1). For $0 < \mu < 1$, $\mu$ close to 1, the function $u_\mu = \mu u + (1 - \mu) \psi$ is a viscosity subsolution of (3.1), which tends to $-\infty$ as $|x|$ tends to $+\infty$. Let $M_\mu$ be the maximal value of $u_\mu - v$ which is reached at some point $\bar{x}_\mu$.

We want to prove that $M_\mu \leq 0$.

1. If $\bar{x}_\mu \neq O$, then we introduce the function $u_\mu(x) - v(x) - d^2(x, \bar{x}_\mu)$, which has a strict maximum at $\bar{x}_\mu$, and we double the variables, i.e. for $0 < \varepsilon \ll 1$, we consider

$$u_\mu(x) - v(y) - d^2(x, \bar{x}_\mu) - \frac{d^2(x, y)}{\varepsilon^2}.$$ 

Classical arguments then lead to the conclusion that $u_\mu(\bar{x}_\mu) - v(\bar{x}_\mu) \leq 0$, thus $M_\mu \leq 0$.

2. If $\bar{x}_\mu = O$. We use Theorem 4.6; we have two possible cases:

   [B] $\lambda v(O) \geq -H^T_O$. From Lemma 4.3, $\lambda u(O) + H^T_O \leq 0$. Therefore, we obtain that $u_\mu(O) \leq v(O)$, thus $M_\mu \leq 0$.

   [A] With the notations of Theorem 4.6,

$$v(x_k) \geq \int_0^{\eta_k} \ell_i(y_{x_k}(s), \alpha_i^k(s)) e^{-\lambda s} ds + v(y_{x_k}(\eta_k)) e^{-\lambda \eta_k}.$$ 

Moreover, since $y_{x_k}(s) \in J_i \cap B(O, r)$ for all $s \in [0, \eta_k]$, Lemma 4.7 can be applied and yields that

$$u_\mu(x_k) \leq \int_0^{\eta_k} \ell_i(y_{x_k}(s), \alpha_i^k(s)) e^{-\lambda s} ds + u_\mu(y_{x_k}(\eta_k)) e^{-\lambda \eta_k}.$$ 

Therefore

$$u_\mu(x_k) - v(x_k) \leq (u_\mu(y_{x_k}(\eta_k)) - v(y_{x_k}(\eta_k))) e^{-\lambda \eta_k}.$$ 

Letting $k$ tend to $+\infty$, we find that $M_\mu \leq M_\mu e^{-\lambda \eta_k}$, which implies that $M_\mu \leq 0$.

We conclude by letting $\mu$ tend to 1. \hfill \Box
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