

ON THE CONVEXITY OF PIECEWISE-DEFINED FUNCTIONS ^{*, **, ***}

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Abstract. Functions that are piecewise defined are a common sight in mathematics while convexity is a property especially desired in optimization. Suppose now a piecewise-defined function is convex on each of its defining components – when can we conclude that the entire function is convex? In this paper we provide several convenient, verifiable conditions guaranteeing convexity (or the lack thereof). Several examples are presented to illustrate our results.

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1. INTRODUCTION

Consider the function

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \mapsto \begin{cases} \frac{x^2 + y^2 + 2 \max\{0, xy\}}{|x| + |y|}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{otherwise.} \end{cases} \quad (1.1)$$

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Clearly, f is a piecewise-defined function with continuous components

$$f_1(x, y) := x + y \quad \text{on} \quad A_1 := \mathbb{R}_+ \times \mathbb{R}_+; \tag{1.2a}$$

$$f_2(x, y) := \frac{x^2 + y^2}{-x + y} \quad \text{on} \quad A_2 := \mathbb{R}_- \times \mathbb{R}_+; \tag{1.2b}$$

$$f_3(x, y) := x + y \quad \text{on} \quad A_3 := \mathbb{R}_- \times \mathbb{R}_-; \tag{1.2c}$$

$$f_4(x, y) := \frac{x^2 + y^2}{x - y} \quad \text{on} \quad A_4 := \mathbb{R}_+ \times \mathbb{R}_-. \tag{1.2d}$$

One may check that each f_i is a convex function (see Example 6.1 below for details). However, *whether or not f itself is convex* is not immediately clear. (As it turns out, f is convex.)

On the other hand, if $f_1(x) = x$ on $A_1 := \mathbb{R}_-$ and $f_2(x) = -x$ on $A_2 := \mathbb{R}_+$, then f_1 and f_2 are convex while the induced piecewise-defined function $f(x) = -|x|$ is *not* convex.

These and similar examples motivate the goal of this paper which is to present *verifiable conditions guaranteeing the convexity of a piecewise-defined function provided that each component is convex*. Special cases of our results have been known in the convex interpolation community (see Rem. 5.7). Moreover, our results have applications to computer-aided convex analysis (see Rem. 5.9).

(We mention in passing three different though related topics: The problem of guaranteeing the convexity of intersections of sublevel sets of functions was recently considered in [18]. Points of convexity, which localize the notion of convexity of a function, were investigated in [9, 11]. In [12], it is shown that the convex envelope of a function arises as the solution of a nonlinear obstacle problem.)

The remainder of this paper is organized as follows. In Section 2, we collect various auxiliary results concerning convexity and differentiability. We also require properties of collections of sets and of functions which we develop in Sections 3 and 4, respectively. Our main results guaranteeing convexity are presented in Section 5. Various examples illustrating convexity and the lack thereof are discussed in Sections 6 and 7, respectively.

Notation: Throughout, X is a Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. \mathbb{R} denotes the set of real numbers, $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$, and $\mathbb{R}_- := -\mathbb{R}_+$. For x and y in X , $[x, y] := \{(1-t)x + ty \mid 0 \leq t \leq 1\}$ is the line segment connecting x and y . Similarly, we set $]x, y[:= \{(1-t)x + ty \mid 0 < t < 1\}$, $[x, y[:= \{(1-t)x + ty \mid 0 \leq t < 1\}$, and $]x, y] := [y, x[$. For a subset A of X , $\text{conv } A$, $\text{cl } A$, $\text{int } A$, $\text{aff } A$, and $\text{ri } A$ respectively denote the *convex hull*, the *closure*, the *interior*, the *affine hull*, and the *relative interior* of A . Furthermore, ι_A is the *indicator function* of A defined by $\iota_A(x) = 0$, if $x \in A$; and $+\infty$ otherwise. Let $f: X \rightarrow]-\infty, +\infty] = \mathbb{R} \cup \{+\infty\}$. The *domain* of f is $D_f := \{x \in X \mid f(x) < +\infty\}$; f is said to be *proper* if $D_f \neq \emptyset$. The restriction of f on some subset A of X is denoted by $f|_A$. A set-valued mapping F from X to another Euclidean space Y is denoted by $F: X \rightrightarrows Y$; and its domain is $D_F := \{x \in X \mid F(x) \neq \emptyset\}$. For further background and notation, we refer the reader to [1, 10, 13, 14, 19].

2. CONVEXITY AND DIFFERENTIABILITY

Let $f: X \rightarrow]-\infty, +\infty]$ be proper. For every $x \in X$, the *subdifferential* (in the sense of convex analysis) of f at x , denoted by $\partial f(x)$, is the set of all vectors $x^* \in X$ such that

$$(\forall y \in X) \quad \langle x^*, y - x \rangle \leq f(y) - f(x). \tag{2.1}$$

The induced operator $\partial f: X \rightrightarrows X$ has domain $D_{\partial f} = \{x \in X \mid \partial f(x) \neq \emptyset\} \subseteq D_f$.

Let us now present some auxiliary results concerning the convexity of a function.

Lemma 2.1. *Let $f: X \rightarrow]-\infty, +\infty]$ and let z_1 and z_2 be in D_f . Set $x := (1-t)z_1 + tz_2$, where $t \in [0, 1]$, and assume that $x \in D_{\partial f}$. Then $f(x) \leq (1-t)f(z_1) + tf(z_2)$.*

Proof. Let $x^* \in \partial f(x)$. Then $\langle x^*, z_1 - x \rangle \leq f(z_1) - f(x)$ and $\langle x^*, z_2 - x \rangle \leq f(z_2) - f(x)$. Hence

$$(1 - t) \langle x^*, z_1 - x \rangle \leq (1 - t)(f(z_1) - f(x)); \quad (2.2a)$$

$$t \langle x^*, z_2 - x \rangle \leq t(f(z_2) - f(x)). \quad (2.2b)$$

Adding up the last two inequalities, we obtain $0 \leq (1 - t)f(z_1) + tf(z_2) - f(x)$. \square

Fact 2.2 (See [13], Thm. 6.1–6.3). *Let A be a nonempty convex subset of X . Then the following hold:*

- (1) $(\forall x \in \text{cl } A)(\forall y \in \text{ri } A)]x, y] \subseteq \text{ri } A$.
- (2) $\text{ri } A$ is nonempty and convex.
- (3) $\text{cl}(\text{ri } A) = \text{cl } A$.

Fact 2.3 (See [13], Thm. 23.4). *Let $f: X \rightarrow]-\infty, +\infty]$ be convex and proper. Then $\text{ri } D_f \subseteq D_{\partial f}$.*

Fact 2.4 (See [19], Thm. 2.4.1(iii)). *Let $f: X \rightarrow]-\infty, +\infty]$ be proper. Assume that $D_f = D_{\partial f}$ is convex. Then f is convex.*

Proof. Take z_1 and z_2 in D_f and let $t \in [0, 1]$. Set $x := (1 - t)z_1 + tz_2$. Since $\partial f(x) \neq \emptyset$, Lemma 2.1 implies that $f(x) \leq (1 - t)f(z_1) + tf(z_2)$. Therefore, f is convex. \square

In the presence of continuity, Fact 2.4 admits the following extension.

Lemma 2.5. *Let $f: X \rightarrow]-\infty, +\infty]$ be proper. Assume that D_f is convex, that $f|_{D_f}$ is continuous, and that $\text{ri } D_f \subseteq D_{\partial f}$. Then f is convex.*

Proof. Fact 2.2 implies that $\text{ri } D_f$ is convex and $\text{cl}(\text{ri } D_f) = \text{cl } D_f$. Then the function $f + \iota_{\text{ri } D_f}$ is convex by Fact 2.4. Since $f|_{D_f}$ is continuous and $\text{ri } D_f$ is dense in D_f , we conclude that $f|_{D_f}$ is convex. \square

Given a nonempty subset A of X , we define the dimension of A to be the dimension of the linear subspace parallel to the affine hull of A , i.e., $\dim A := \dim(\text{aff } A - \text{aff } A)$. We then have the following result.

Lemma 2.6. *Let $f: X \rightarrow]-\infty, +\infty]$ be proper. Assume that $f|_{D_f}$ is continuous, that D_f is convex and at least 2-dimensional, and that there exists a finite subset E of X such that $f|_{[x,y]}$ is convex for every segment $[x, y]$ contained in $(\text{ri } D_f) \setminus E$. Then f is convex.*

Proof. Take two distinct points x and y in D_f , let $t \in [0, 1]$, and set $z := (1 - t)x + ty$. Then $z \in D_f$ because D_f is convex. It remains to show that

$$f(z) \leq (1 - t)f(x) + tf(y). \quad (2.3)$$

First, since D_f is convex and $\dim(D_f) \geq 2$, there exists $w \in (\text{ri } D_f) \setminus \text{aff}\{x, y\}$. For each $\varepsilon \in]0, 1[$, set

$$x_\varepsilon := (1 - \varepsilon)x + \varepsilon w, \quad y_\varepsilon := (1 - \varepsilon)y + \varepsilon w, \quad \text{and} \quad z_\varepsilon := (1 - t)x_\varepsilon + ty_\varepsilon. \quad (2.4)$$

Using Fact 2.2(1) and the finiteness of E , we have

$$z_\varepsilon \in [x_\varepsilon, y_\varepsilon] \subseteq (\text{ri } D_f) \setminus E, \quad \text{for every } \varepsilon > 0 \text{ sufficiently small.} \quad (2.5)$$

Because f is convex on $[x_\varepsilon, y_\varepsilon]$, this implies

$$f(z_\varepsilon) \leq (1 - t)f(x_\varepsilon) + tf(y_\varepsilon). \quad (2.6)$$

Letting $\varepsilon \rightarrow 0^+$, we obtain (2.3) by using the continuity of $f|_{D_f}$. \square

Remark 2.7 (The assumption on the dimension is important). Lemma 2.6 fails on \mathbb{R} in the following sense. Consider $f: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto -|x|$ and set $E := \{0\}$. Then all assumptions of Lemma 2.6 hold except that $D_f = \mathbb{R}$ is only 1-dimensional. Clearly, the conclusion of Lemma 2.6 is not true because f is not convex.

We now turn our attention to differentiability properties. Recall that $f: X \rightarrow]-\infty, +\infty]$ is differentiable at $x \in \text{int } D_f$ if there exists $\nabla f(x) \in X$ such that $(\forall y \in X) f(y) - f(x) - \langle \nabla f(x), y - x \rangle = o(\|y - x\|)$; f is differentiable on subset A of $\text{int } D_f$ if f is differentiable at every $x \in A$. We will require the following results.

Fact 2.8 (See [13], Thm. 25.1). *Let $f: X \rightarrow]-\infty, +\infty]$ be convex and proper, and assume that $x \in \text{int } D_f$. Then f is differentiable at x if and only if $\partial f(x)$ is a singleton.*

Fact 2.9 (See [13], Thm. 25.5). *Let $f: X \rightarrow]-\infty, +\infty]$ be convex and proper, and let Ω be the set of points where f is differentiable. Then Ω is a dense subset of $\text{int } D_f$, and its complement in $\text{int } D_f$ is a set of measure zero. Moreover, $\nabla f: \Omega \rightarrow X$ is continuous.*

Fact 2.10 (See [13], Thm. 25.6). *Let $f: X \rightarrow]-\infty, +\infty]$ be convex and proper such that D_f is closed with nonempty interior. Then*

$$(\forall x \in D_f) \quad \partial f(x) = \text{cl}(\text{conv } S(x)) + N_{D_f}(x), \tag{2.7}$$

where $N_{D_f}(x) := \{x^* \in X \mid (\forall y \in D_f) \langle x^*, y - x \rangle \leq 0\}$ is the normal cone to D_f at x and $S(x)$ is the set of all limits of sequences $(\nabla f(x_n))_{n \in \mathbb{N}}$ such that f is differentiable at every x_n and $x_n \rightarrow x$.

3. COMPATIBLE SYSTEMS OF SETS

In this section, we always assume that

$$I \text{ is a nonempty finite set;} \tag{3.1a}$$

$$\mathcal{A} := \{A_i\}_{i \in I} \text{ is a system of convex subsets of } X; \tag{3.1b}$$

$$A := \bigcup_{i \in I} A_i. \tag{3.1c}$$

Definition 3.1 (Compatible systems of sets). Assume (3.1). We say that \mathcal{A} is a *compatible system of sets* if

$$\left. \begin{array}{l} i \in I \\ j \in I \\ i \neq j \end{array} \right\} \Rightarrow \text{cl } A_i \cap \text{cl } A_j \cap \text{ri } A = A_i \cap A_j \cap \text{ri } A; \tag{3.2}$$

otherwise, we say that \mathcal{A} is *incompatible*.

Example 3.2. Every system of finitely many closed convex subsets of X is compatible.

Example 3.3 (Incompatible systems). Suppose that $X = \mathbb{R}^2$, that $I = \{1, 2\}$, that $A_1 =]0, 1[\times]0, 1[$, and that $A_2 = [-1, 0] \times]0, 1[$. Then $A = A_1 \cup A_2 = [-1, 1] \times]0, 1[$ and $\text{ri } A =]-1, 1[\times]0, 1[$. Thus, $\mathcal{A} = \{A_1, A_2\}$ is incompatible because

$$\text{cl } A_1 \cap \text{cl } A_2 \cap \text{ri } A = \{0\} \times]0, 1[\neq \emptyset = A_1 \cap A_2 \cap \text{ri } A. \tag{3.3}$$

Definition 3.4 (Colinearly ordered tuple). The tuple of vectors $(x_0, \dots, x_n) \in X^n$ is said to be *colinearly ordered* if the following hold:

- (1) $[x_0, x_n] = [x_0, x_1] \cup \dots \cup [x_{n-1}, x_n]$;
- (2) $0 \leq \|x_0 - x_1\| \leq \|x_0 - x_2\| \leq \dots \leq \|x_0 - x_n\|$.

Proposition 3.5. *Assume (3.1) and that \mathcal{A} is a compatible system of sets (Def. 3.1). Then for every segment $[x, y]$ contained in $\text{ri } A$, there exists a colinearly ordered tuple (x_0, \dots, x_n) and $\{A_{i_1}, \dots, A_{i_n}\} \subseteq \mathcal{A}$ such that*

$$x_0 = x; \quad x_n = y; \quad \text{and} \quad (\forall k \in \{1, \dots, n\}) \quad [x_{k-1}, x_k] \subseteq A_{i_k}. \tag{3.4}$$

Proof. Let $[x, y] \subseteq \text{ri } A$, with $x \in A_{i_1}$ for some $i_1 \in I$. Set $x_0 := x$. For every $t \in [0, 1]$, define

$$x(t) := (1 - t)x + ty. \quad (3.5)$$

Furthermore, set

$$t_1 := \sup \{t \in [0, 1] \mid x(t) \in A_{i_1}\} \quad \text{and} \quad x_1 := x(t_1). \quad (3.6)$$

Then $[x_0, x_1[\subseteq A_{i_1}$ and $x_1 \in \text{cl } A_{i_1}$. Note also that $x_1 \in \text{ri } A$.

Case 1. $t_1 = 1$. Then $x_1 = y \in \text{cl } A_{i_1}$. Suppose that $y \notin A_{i_1}$. Then, $y \in A_{i_2}$ for some $i_2 \in I$. It follows that $y \in \text{cl } A_{i_1} \cap \text{cl } A_{i_2} \cap \text{ri } A = A_{i_1} \cap A_{i_2} \cap \text{ri } A$, which is a contradiction. Therefore, $y \in A_{i_1}$ and we are done because $[x, y] \subseteq A_{i_1}$.

Case 2. $t_1 < 1$. Then there exist $\varepsilon \in]0, 1 - t_1]$ and $A_{i_2} \in \mathcal{A} \setminus \{A_{i_1}\}$ such that

$$]x_1, x_2] \subseteq A_{i_2} \quad \text{where} \quad x_2 := x(t_1 + \varepsilon). \quad (3.7)$$

Hence $x_1 \in \text{cl } A_{i_2}$. We then have

$$x_1 \in \text{cl } A_{i_1} \cap \text{cl } A_{i_2} \cap \text{ri } A = A_{i_1} \cap A_{i_2} \cap \text{ri } A. \quad (3.8)$$

So we have split $[x, y]$ into two line segments

$$[x_0, x_1] \subseteq A_{i_1} \cap \text{ri } A \quad \text{and} \quad [x_1, y] \subseteq \left(\bigcup_{i \in I \setminus \{i_1\}} A_i\right) \cap \text{ri } A. \quad (3.9)$$

Next, we repeat the above process for the segment $[x_1, y]$. Since \mathcal{A} is finite, we eventually obtain (3.4). \square

Remark 3.6 (Closedness is not necessary for compatibility). We note that there are compatible systems of sets that are not closed. For example, suppose that $X = \mathbb{R}^2$, that $I = \{1, 2\}$, that $A_1 = [0, 1[\times [0, 1]$, and that $A_2 =]-1, 0] \times [0, 1]$. Then neither A_1 nor A_2 is closed. However, since

$$\text{cl } A_1 \cap \text{cl } A_2 = A_1 \cap A_2 = \{0\} \times [0, 1], \quad (3.10)$$

we deduce that $\mathcal{A} = \{A_1, A_2\}$ is compatible.

Definition 3.7 (Active index set). Assume (3.1). For every $x \in X$, we define the *active index set* associated with \mathcal{A} by

$$I_{\mathcal{A}}(x) := \{i \in I \mid x \in A_i\}, \quad (3.11)$$

and we will write $I(x)$ if there is no cause for confusion.

Proposition 3.8. Assume (3.1) and that \mathcal{A} is a compatible system of sets (see Def. 3.1). Suppose that $x \in \text{int } A$ and that $I_{\mathcal{A}}(x) = \{i\}$. Then $x \in \text{int } A_i$.

Proof. Because $\text{int } A \neq \emptyset$, we have $\text{ri } A = \text{int } A$. Suppose to the contrary that $x \notin \text{int } A_i$. Then there exist $j \in I \setminus \{i\}$ and a sequence $(x_n)_{n \in \mathbb{N}}$ in A_j such that $x_n \rightarrow x$. It follows that

$$x \in \text{cl } A_i \cap \text{cl } A_j \cap \text{ri } A = A_i \cap A_j \cap \text{ri } A, \quad (3.12)$$

which is absurd because $I(x) = \{i\}$ by assumption. \square

4. COMPATIBLE SYSTEMS OF FUNCTIONS

In this section, we always assume that

$$I \text{ is a nonempty finite set;} \tag{4.1a}$$

$$\mathcal{F} := \{f_i\}_{i \in I} \text{ is a system of proper convex functions from } X \text{ to }]-\infty, +\infty]; \tag{4.1b}$$

$$f := \min_{i \in I} f_i \text{ is the piecewise-defined function associated with } \mathcal{F}; \tag{4.1c}$$

$$I_{\mathcal{F}}: X \rightarrow I: x \mapsto \{i \in I \mid x \in D_{f_i}\} \text{ is the active index set function.} \tag{4.1d}$$

We will write $I(x)$ instead of $I_{\mathcal{F}}(x)$ if there is no cause for confusion. Note that $D_f = \bigcup_{i \in I} D_{f_i}$.

Definition 4.1 (Compatible systems of functions). Assume (4.1). We say that \mathcal{F} is a *compatible system of functions* if $(\forall i \in I) f_i|_{D_{f_i}}$ is continuous and

$$\left. \begin{array}{l} i \in I \\ j \in I \\ i \neq j \\ D_{f_i} \cap D_{f_j} \neq \emptyset \end{array} \right\} \Rightarrow f_i|_{D_{f_i} \cap D_{f_j}} \equiv f_j|_{D_{f_i} \cap D_{f_j}}. \tag{4.2}$$

We start with a useful lemma.

Lemma 4.2. Assume (4.1) and that \mathcal{F} is compatible system of functions (Def. 4.1). Then

$$(\forall x \in X) \quad \partial f(x) \subseteq \bigcap_{i \in I_{\mathcal{F}}(x)} \partial f_i(x). \tag{4.3}$$

Proof. Suppose that $x^* \in \partial f(x)$ and that $i \in I_{\mathcal{F}}(x)$. Then $f_i(x) = f(x)$ and $(\forall y \in X) f_i(y) - f_i(x) \geq f(y) - f(x) \geq \langle x^*, y - x \rangle$. Therefore, $x^* \in \partial f_i(x)$. \square

Lemma 4.3. Let $(a, b, c) \in X^3$ be colinearly ordered (Def. 3.4). Assume (4.1) with $I = \{1, 2\}$, that \mathcal{F} is compatible system of functions (Def. 4.1), that $D_{f_1} = [a, b]$, that $D_{f_2} = [b, c]$, and that

$$\partial f_1(b) \cap \partial f_2(b) \neq \emptyset. \tag{4.4}$$

Then f is convex and

$$(\forall x \in D_f) \quad \partial f(x) = \bigcap_{i \in I_{\mathcal{F}}(x)} \partial f_i(x) = \begin{cases} \partial f_1(x), & \text{if } x \in [a, b[; \\ \partial f_1(b) \cap \partial f_2(b), & \text{if } x = b; \\ \partial f_2(x), & \text{if } x \in]b, c]. \end{cases} \tag{4.5}$$

Proof. We assume that a, b, c are pairwise distinct since the other cases are trivial. First, we show that

$$(\forall x \in [a, b[) \quad \partial f_1(x) \subseteq \partial f(x). \tag{4.6}$$

Suppose that $x \in [a, b[$ and that $x^* \in \partial f_1(x)$. To establish (4.6), it suffices to show that

$$(\forall y \in [a, c]) \quad \langle x^*, y - x \rangle \leq f(y) - f(x). \tag{4.7}$$

Indeed, (4.7) is true for $y \in [a, b]$ by definition of $\partial f_1(x)$ and f . Now suppose that $y \in]b, c]$. By (4.4), there exists $b^* \in \partial f_1(b) \cap \partial f_2(b)$. Then

$$\langle x^*, y - x \rangle = \langle x^*, b - x \rangle + \langle x^*, y - b \rangle \tag{4.8a}$$

$$\leq f_1(b) - f_1(x) + \frac{\|y - b\|}{\|b - x\|} \langle x^*, b - x \rangle \tag{4.8b}$$

$$\leq f_1(b) - f_1(x) + \frac{\|y - b\|}{\|b - x\|} \langle b^*, b - x \rangle \tag{4.8c}$$

$$\leq f_1(b) - f_1(x) + \langle b^*, y - b \rangle \tag{4.8d}$$

$$\leq f_1(b) - f_1(x) + f_2(y) - f_2(b) \tag{4.8e}$$

$$= f_2(y) - f_1(x) \tag{4.8f}$$

$$= f(y) - f(x). \tag{4.8g}$$

Hence (4.7) holds, as does (4.6).

Switching the roles of f_1 and f_2 , we obtain analogously

$$(\forall x \in [c, b[) \quad \partial f_2(x) \subseteq \partial f(x). \tag{4.9}$$

Next, it is straightforward to check that

$$\partial f_1(b) \cap \partial f_2(b) \subseteq \partial f(b). \tag{4.10}$$

Since the reverse inclusions of (4.6), (4.9), and (4.10) follow from Lemma 4.2, we conclude that (4.5) holds. Using (4.4), (4.5) and Fact 2.3, we conclude that $\partial f(x) \neq \emptyset$ for all $x \in]a, c[= \text{ri } D_f$. Finally, it follows from Lemma 2.5 that f is convex. \square

Theorem 4.4. *Let $(x_0, \dots, x_n) \in X^{n+1}$ be colinearly ordered (Def. 3.4). Assume (4.1) with $I = \{1, \dots, n\}$ and that \mathcal{F} is a compatible system of functions (Def. 4.1) such that the following hold:*

- (1) $(\forall i \in \{1, \dots, n\}) \ D_{f_i} = [x_{i-1}, x_i]$.
- (2) $(\forall i \in \{1, \dots, n-1\}) \ \partial f_i(x_i) \cap \partial f_{i+1}(x_i) \neq \emptyset$.

Then f is convex and

$$(\forall x \in D_f) \quad \partial f(x) = \bigcap_{i \in I_{\mathcal{F}}(x)} \partial f_i(x). \tag{4.11}$$

Proof. If $n = 1$, then the result is trivial. For $n \geq 2$, the result follows by inductively applying Lemma 4.3. \square

5. MAIN RESULTS

We are now ready for our main results.

Theorem 5.1 (Main result I). *Assume (4.1), that \mathcal{F} is a compatible system of functions (Def. 4.1), and that the following hold:*

- (a) $D_f = \bigcup_{i \in I} D_{f_i}$ is convex and at least 2-dimensional.
- (b) $\{D_{f_i}\}_{i \in I}$ is a compatible system of sets (Def. 3.1).
- (c) There exists a finite subset E of X such that

$$\left. \begin{array}{l} x \in (\text{ri } D_f) \setminus E \\ \text{card } I(x) \geq 2 \end{array} \right\} \Rightarrow \bigcap_{i \in I(x)} \partial f_i(x) \neq \emptyset. \tag{5.1}$$

Then f is convex and

$$(\forall x \in \text{ri } D_f) \quad \emptyset \neq \partial f(x) \subseteq \bigcap_{i \in I(x)} \partial f_i(x). \tag{5.2}$$

Proof. Let $[x, y] \subseteq (\text{ri } D_f) \setminus E$. By the compatibility in (b) and Proposition 3.5, there exist a colinearly ordered tuple $(x_0, \dots, x_n) \in X^{n+1}$ and functions f_{i_1}, \dots, f_{i_n} in \mathcal{F} such that

$$x_0 = x; \quad x_n = y; \quad \text{and} \quad (\forall k \in \{1, \dots, n\}) \quad [x_{k-1}, x_k] \subseteq D_{f_{i_k}} \cap \text{ri } D_f. \tag{5.3}$$

Define

$$(\forall k \in \{1, \dots, n\}) \quad g_k := f_{i_k} + \iota_{[x_{k-1}, x_k]} = f + \iota_{[x_{k-1}, x_k]}. \tag{5.4}$$

Using (5.1), we see that, for every $k \in \{1, \dots, n-1\}$,

$$\partial g_k(x_k) \cap \partial g_{k+1}(x_k) \supseteq \partial f_{i_k}(x_k) \cap \partial f_{i_{k+1}}(x_k) \supseteq \bigcap_{i \in I(x_k)} \partial f_i(x_k) \neq \emptyset. \tag{5.5}$$

By applying Theorem 4.4 to the system $\{g_k\}_{k \in \{1, \dots, n\}}$, we see that $g = \min_{k \in \{1, \dots, n\}} g_k = f + \iota_{[x, y]}$ is convex and hence so is $f|_{[x, y]}$. In view of Lemma 2.6, we obtain the convexity of f . Finally, for all $x \in \text{ri } D_f$, we have $\partial f(x) \neq \emptyset$ by Fact 2.3. Therefore, (5.2) follows from Lemma 4.2. \square

Remark 5.2. We note an interesting feature in Theorem 5.1. In assumption (c), we require the non-emptiness of the subdifferential intersection (5.1) at all relative interior points except for finitely many points. Since our conclusion says that f is convex, the subdifferential intersection is nonempty at every point in $\text{ri } D_f$ (see Fact 2.3). That means, in order to check the convexity of f , we are allowed to ignore verifying (5.1) at finitely many points in $\text{ri } D_f$. This turns out to be very convenient since checking (5.1) at certain points may not be obvious (see, for example, Example 6.1).

Remark 5.3 (Compatibility on the system of domains is essential). Theorem 5.1 fails if the domain compatibility assumption (b) is omitted: indeed, suppose that $X = \mathbb{R}^2$, that $I = \{1, 2\}$, and that

$$f_1 = \iota_{[0,1] \times [0,1]} \quad \text{and} \quad f_2 = \iota_{[-1,0] \times [0,1]} + 1. \tag{5.6}$$

Then \mathcal{F} is a compatible system of functions. Even though $D_f = [-1, 1] \times [0, 1]$ is convex, $\{D_{f_i} \cap \text{ri } D_f\}_{i \in I}$ is not a compatible system of sets. So, Theorem 5.1(b) is violated. Clearly, f is not convex.

We thank a referee for a comment that caused us to point out the following result, which is complementary to Theorem 5.1. It deals with the case when D_f is 1-dimensional. For notational simplicity we state the result when $X = \mathbb{R}$.

Theorem 5.4. *Assume that $X = \mathbb{R}$, that (4.1) holds, that \mathcal{F} is a compatible system of functions (Def. 4.1), and that the following hold:*

- (a) $I = \{1, 2, \dots, n\}$ for some $n \geq 2$ and $(\forall i \in I) \text{int } D_{f_i} \neq \emptyset$.
- (b) $(\forall i \in \{1, \dots, n-1\}) \ x_i := \max D_{f_i} = \min D_{f_{i+1}}$ and $\partial f_i(x_i) \cap \partial f_{i+1}(x_i) \neq \emptyset$.
- (c) $\{D_{f_i}\}_{i \in I}$ is a compatible system of sets (Def. 3.1).

Then f is convex and $(\forall x \in D_f) \ \partial f(x) = \bigcap_{i \in I_{\mathcal{F}}(x)} \partial f_i(x)$.

Proof. Take $x_0 \in D_{f_1}$ and $x_n \in D_{f_n}$ such that $x_0 < x_1 \leq x_{n-1} < x_n$. By Theorem 4.4, the result holds for $f + \iota_{[x_0, x_n]}$. Since x_0 and x_n were chosen arbitrarily, we are done. \square

Theorem 5.5 (Main result II). *Assume (4.1), that \mathcal{F} is a compatible system of functions (Def. 4.1), that each f_i is differentiable on $\text{int } D_{f_i} \neq \emptyset$, and that the following hold:*

- (a) $D_f = \bigcup_{i \in I} D_{f_i}$ is convex and at least 2-dimensional.
- (b) $\{D_{f_i}\}_{i \in I}$ is a compatible system of sets (Def. 3.1).

(c) *There exists a finite subset E of X such that*

$$\left. \begin{array}{l} x \in (\text{int } D_f) \setminus E \\ \{i, j\} \subseteq I(x) \end{array} \right\} \Rightarrow \lim_{\substack{z \rightarrow x \\ z \in \text{int } D_{f_i}}} \nabla f_i(z) = \lim_{\substack{z \rightarrow x \\ z \in \text{int } D_{f_j}}} \nabla f_j(z) \text{ exists.} \quad (5.7)$$

Then f is convex; moreover, it is continuously differentiable on $(\text{int } D_f) \setminus E$.

Proof. We will prove the convexity of f by using Theorem 5.1. Note that it suffices to verify assumption (c) of Theorem 5.1. To this end, let $x \in \text{int}(D_f) \setminus E$ such that $\text{card } I(x) \geq 2$ and denote by u_x^* the limit in (5.7). Fact 2.10 and Lemma 4.2 imply

$$u_x^* \in \bigcap_{i \in I(x)} \partial f_i(x). \quad (5.8)$$

So assumption (c) in Theorem 5.1 holds. Thus, we conclude that f is convex.

Turning towards the differentiability statement, let Ω be the set of points at which f is differentiable. Then $\bigcup_{i \in I} \text{int } D_{f_i} \subseteq \Omega \subseteq \text{int } D_f$.

Now let $x \in (\text{int } D_f) \setminus E$. We consider two cases.

Case 1. $\text{Card } I(x) = 1$. Then Proposition 3.8 implies that $x \in \text{int } D_{f_i}$ for some $i \in I$, which implies that $x \in \Omega$.

Case 2. $\text{Card } I(x) \geq 2$. Since $x \in \text{int } D_f$, we obtain $N_{D_f}(x) = \{0\}$. Hence, by Fact 2.10, we have

$$\partial f(x) = \text{cl conv} \left\{ \lim_{n \in \mathbb{N}} \nabla f(z_n) \mid \Omega \ni z_n \rightarrow x \right\}. \quad (5.9)$$

Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in Ω such that $z_n \rightarrow x$, and let $(\varepsilon_n)_{n \in \mathbb{N}}$ be in \mathbb{R}_{++} such that $\varepsilon_n \rightarrow 0^+$. By Fact 2.9, $\nabla f|_{\Omega}$ is the continuous and because $\bigcup_{i \in I} \text{int } D_{f_i}$ is dense in D_f , there exists a sequence $(y_n)_{n \in \mathbb{N}}$ in $\bigcup_{i \in I} \text{int } D_{f_i}$ such that

$$(\forall n \in \mathbb{N}) \quad \|y_n - z_n\| \leq \varepsilon_n \quad \text{and} \quad \|\nabla f(y_n) - \nabla f(z_n)\| \leq \varepsilon_n. \quad (5.10)$$

Combining with (5.7), we deduce that

$$y_n \rightarrow x \quad \text{and} \quad \lim_{n \in \mathbb{N}} \nabla f(z_n) = \lim_{n \in \mathbb{N}} \nabla f(y_n) = u_x^*. \quad (5.11)$$

So, (5.9) becomes $\partial f(x) = \{u_x^*\}$. Thus, f is differentiable at x by Fact 2.8. □

Corollary 5.6. *Assume (4.1), that \mathcal{F} is a compatible system of functions (Def. 4.1), and that the following hold:*

- (a) $D_f = \bigcup_{i \in I} D_{f_i}$ is convex and at least 2-dimensional.
- (b) $\{D_{f_i}\}_{i \in I}$ is a compatible system of sets (Def. 3.1).
- (c) f is continuously differentiable on $\text{int } D_f$.

Then f is convex.

Proof. This follows from Theorem 5.5 with $E = \emptyset$. □

Remark 5.7 (Convex interpolation). When $X = \mathbb{R}^2$, then Corollary 5.6 is known and of importance in the convex interpolation of data; (see, e.g., [5] Thm. 1, [6] Prop. 2.2, [15] Thm. 3.1, [2] Prop. 5.1 and the related [16] Thm. 3.6) for further details.

Corollary 5.8. *Assume (4.1), that \mathcal{F} is a compatible system of functions (Def. 4.1), that each D_{f_i} is closed, and that $(\forall i \in I)(\forall x \in D_{f_i}) f_i(x) = \frac{1}{2} \langle x, A_i x \rangle + \langle b_i, x \rangle + \gamma_i$, where $A_i: X \rightarrow X$ is linear with $A_i = A_i^* \succeq 0$, $b_i \in X$, and $\gamma_i \in \mathbb{R}$. Furthermore, assume that $D_f = \bigcup_{i \in I} D_{f_i}$ is convex and at least 2-dimensional, and that*

$$\left. \begin{array}{l} \{i, j\} \subseteq I \\ i \neq j \\ x \in D_{f_i} \cap D_{f_j} \end{array} \right\} \Rightarrow A_i x + b_i = A_j x + b_j. \quad (5.12)$$

Then f is convex; moreover, it is continuously differentiable on $\text{int } D_f$.

Proof. This follows from Example 3.2 and Theorem 5.5 with $E = \emptyset$ since $\nabla f_i(x) = A_i x + b_i$ when $x \in \text{int } D_{f_i}$. \square

Remark 5.9 (Piecewise linear-quadratic function). Consider Corollary 5.8 with the additional assumption that each D_{f_i} is a polyhedral set. Then Corollary 5.8 provides a sufficient condition for checking the convexity of f which in this case is a *piecewise linear-quadratic function*. These functions play a role in computer-aided convex analysis (see [7], [14] Sect. 10.E) and also [17] for further information on functions of this type). Moreover, we thus partially answer an open question from ([8], Sect. 23.4.2). Our results also enhance our understanding of how nonconvexity occurs and form another step towards building a nonconvex toolbox that extends current bivariate computational convex analysis algorithms [3, 4].

6. CHECKING CONVEXITY

We start with an application of Theorem 5.5.

Example 6.1. The function

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \mapsto \begin{cases} \frac{x^2 + y^2 + 2 \max\{0, xy\}}{|x| + |y|}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0) \end{cases} \tag{6.1}$$

is convex, and differentiable on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Proof. First, set $I := \{1, \dots, 4\}$ and

$$f_1(x, y) := \begin{cases} x + y, & \text{if } (x, y) \in A_1 := \mathbb{R}_+^2, \\ +\infty, & \text{otherwise;} \end{cases} \tag{6.2a}$$

$$f_2(x, y) := \begin{cases} \frac{x^2 + y^2}{-x + y}, & \text{if } (x, y) \in A_2 := \mathbb{R}_- \times \mathbb{R}_+, \\ +\infty, & \text{otherwise;} \end{cases} \tag{6.2b}$$

$$f_3(x, y) := \begin{cases} -x - y, & \text{if } (x, y) \in A_3 := \mathbb{R}_-^2, \\ +\infty, & \text{otherwise;} \end{cases} \tag{6.2c}$$

$$f_4(x, y) := \begin{cases} \frac{x^2 + y^2}{x - y}, & \text{if } (x, y) \in A_4 := \mathbb{R}_+ \times \mathbb{R}_-, \\ +\infty, & \text{otherwise.} \end{cases} \tag{6.2d}$$

Then $\{f_i\}_{i \in I}$ is a compatible system of functions (Def. 4.1) with f being the corresponding piecewise-defined function. Moreover, $\{D_{f_i}\}_{i \in I} = \{A_i\}_{i \in I}$ is a compatible system of sets.

Each f_i is differentiable on $\text{int } A_i$ with the gradient given by

$$\nabla f_1(x, y) = (1, 1) \quad \text{for } (x, y) \in \text{int } A_1; \tag{6.3a}$$

$$\nabla f_2(x, y) = \left(\frac{-x^2 + 2xy + y^2}{(x - y)^2}, \frac{-x^2 - 2xy + y^2}{(x - y)^2} \right) \quad \text{for } (x, y) \in \text{int } A_2; \tag{6.3b}$$

$$\nabla f_3(x, y) = (-1, -1) \quad \text{for } (x, y) \in \text{int } A_3; \tag{6.3c}$$

$$\nabla f_4(x, y) = \left(\frac{x^2 - 2xy - y^2}{(x - y)^2}, \frac{x^2 + 2xy - y^2}{(x - y)^2} \right) \quad \text{for } (x, y) \in \text{int } A_4. \tag{6.3d}$$

One readily checks that the Hessian of each f_i is positive semi-definite on $\text{int } A_i$; hence, by the continuity of $f_i|_{D_{f_i}}$, we have that f_i is convex.

Moreover,

$$(\forall a > 0) \quad \lim_{\substack{(x,y) \rightarrow (a,0) \\ (x,y) \in \text{int } A_1}} \nabla f_1(x,y) = \lim_{\substack{(x,y) \rightarrow (a,0) \\ (x,y) \in \text{int } A_4}} \nabla f_4(x,y) = (1, 1); \quad (6.4a)$$

$$(\forall a < 0) \quad \lim_{\substack{(x,y) \rightarrow (a,0) \\ (x,y) \in \text{int } A_2}} \nabla f_2(x,y) = \lim_{\substack{(x,y) \rightarrow (a,0) \\ (x,y) \in \text{int } A_3}} \nabla f_3(x,y) = (-1, -1); \quad (6.4b)$$

$$(\forall b > 0) \quad \lim_{\substack{(x,y) \rightarrow (0,b) \\ (x,y) \in \text{int } A_1}} \nabla f_1(x,y) = \lim_{\substack{(x,y) \rightarrow (0,b) \\ (x,y) \in \text{int } A_2}} \nabla f_2(x,y) = (1, 1); \quad (6.4c)$$

$$(\forall b < 0) \quad \lim_{\substack{(x,y) \rightarrow (0,b) \\ (x,y) \in \text{int } A_3}} \nabla f_3(x,y) = \lim_{\substack{(x,y) \rightarrow (0,b) \\ (x,y) \in \text{int } A_4}} \nabla f_4(x,y) = (-1, -1). \quad (6.4d)$$

Now set $E := \{(0,0)\}$. From the above computations, we observe that all assumptions of Theorem 5.5 are satisfied. Thus, we conclude that f is a convex function that is also continuously differentiable away from the origin. \square

In fact, the function defined by (6.1) is actually a norm since it is clearly positively homogeneous. An analogous use of Theorem 5.5 allows for a systematic proof of the convexity of the function considered next.

Example 6.2. The function

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}: (x,y) \mapsto \begin{cases} \sqrt{x^6 + y^4}, & \text{if } xy \geq 0; \\ |x|^3 + y^2, & \text{otherwise,} \end{cases} \quad (6.5)$$

is a convex and continuously differentiable.

We conclude this section with an application of Theorem 5.1.

Example 6.3. The function

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}: (x,y) \mapsto f(x,y) := \begin{cases} \sqrt{x^4 + y^2}, & \text{if } xy \geq 0; \\ x^2 + |y|, & \text{otherwise,} \end{cases} \quad (6.6)$$

is convex.

Proof. First, set $I := \{1, \dots, 4\}$ and

$$f_1(x) := \begin{cases} \sqrt{x_1^4 + x_2^2}, & \text{if } x \in A_1 := \mathbb{R}_+^2, \\ +\infty, & \text{otherwise;} \end{cases} \quad (6.7a)$$

$$f_2(x) := \begin{cases} x_1^2 + x_2, & \text{if } x \in A_2 := \mathbb{R}_- \times \mathbb{R}_+, \\ +\infty, & \text{otherwise;} \end{cases} \quad (6.7b)$$

$$f_3(x) := \begin{cases} \sqrt{x_1^4 + x_2^2}, & \text{if } x \in A_3 := \mathbb{R}_-^2, \\ +\infty, & \text{otherwise;} \end{cases} \quad (6.7c)$$

$$f_4(x) := \begin{cases} x_1^2 - x_2, & \text{if } x \in A_4 := \mathbb{R}_+ \times \mathbb{R}_-, \\ +\infty, & \text{otherwise.} \end{cases} \quad (6.7d)$$

Then $\{f_i\}_{i \in I}$ is a compatible system of functions (Def. 4.1) with f being the corresponding piecewise function. Moreover, $\{D_{f_i}\}_{i \in I} = \{A_i\}_{i \in I}$ is a compatible system of sets.

Each f_i is differentiable on $\text{int } A_i$ with the gradient given by

$$\nabla f_1(x) = \left(\frac{2x_1^3}{\sqrt{x_1^4 + x_2^2}}, \frac{x_2}{\sqrt{x_1^4 + x_2^2}} \right) \quad \text{for } x \in \text{int } A_1; \tag{6.8a}$$

$$\nabla f_2(x) = (2x_1, 1) \quad \text{for } x \in \text{int } A_2; \tag{6.8b}$$

$$\nabla f_3(x) = \left(\frac{2x_1^3}{\sqrt{x_1^4 + x_2^2}}, \frac{x_2}{\sqrt{x_1^4 + x_2^2}} \right) \quad \text{for } x \in \text{int } A_3; \tag{6.8c}$$

$$\nabla f_4(x) = (2x_1, -1) \quad \text{for } x \in \text{int } A_4. \tag{6.8d}$$

Next, since the Hessian of f_i is positive semidefinite on $\text{int } A_i$, we deduce that each f_i is convex.

Now set $E := \{(0, 0)\}$. We will verify (5.1). Note that simple computations show the following:

For $x = (0, x_2) \in (A_1 \cap A_2) \setminus E$,

$$\lim_{\substack{z \rightarrow x \\ z \in \text{int } A_1}} \nabla f_1(z) = \lim_{\substack{z \rightarrow x \\ z \in \text{int } A_2}} \nabla f_2(z) = (0, 1) \in \partial f_1(x) \cap \partial f_2(x). \tag{6.9}$$

For $x = (0, x_2) \in (A_3 \cap A_4) \setminus E$,

$$\lim_{\substack{z \rightarrow x \\ z \in \text{int } A_3}} \nabla f_3(z) = \lim_{\substack{z \rightarrow x \\ z \in \text{int } A_4}} \nabla f_4(z) = (0, -1) \in \partial f_3(x) \cap \partial f_4(x). \tag{6.10}$$

For $x = (x_1, 0) \in (A_2 \cap A_3) \setminus E$,

$$\lim_{\substack{z \rightarrow x \\ z \in \text{int } A_2}} \nabla f_2(z) = (2x_1, 1) \quad \text{and} \quad N_{A_2}(x) = \{0\} \times \mathbb{R}_-; \tag{6.11a}$$

$$\lim_{\substack{z \rightarrow x \\ z \in \text{int } A_3}} \nabla f_3(z) = (2x_1, 0) \quad \text{and} \quad N_{A_3}(x) = \{0\} \times \mathbb{R}_+. \tag{6.11b}$$

Then, using Fact 2.10, we conclude that $\partial f_2(x) \cap \partial f_3(x) \neq \emptyset$.

For $x = (x_1, 0) \in (A_1 \cap A_4) \setminus E$,

$$\lim_{\substack{z \rightarrow x \\ z \in \text{int } A_1}} \nabla f_1(z) = (2x_1, 0) \quad \text{and} \quad N_{A_1}(x) = \{0\} \times \mathbb{R}_-; \tag{6.12a}$$

$$\lim_{\substack{z \rightarrow x \\ z \in \text{int } A_4}} \nabla f_4(z) = (2x_1, -1) \quad \text{and} \quad N_{A_4}(x) = \{0\} \times \mathbb{R}_+. \tag{6.12b}$$

Then, using Fact 2.10, we conclude that $\partial f_1(x) \cap \partial f_4(x) \neq \emptyset$.

So, we have verified that assumption (c) in Theorem 5.1 holds. Therefore, f is convex by Theorem 5.1. \square

7. DETECTING THE LACK OF CONVEXITY

The nonempty subdifferential intersection condition (5.1) is indeed crucial for the check of convexity: we will see in the following result that the violation of (5.1) leads to nonconvexity.

Theorem 7.1 (Detecting lack of convexity). *Assume (4.1), that \mathcal{F} is a compatible system of functions (Def. 4.1), and that*

$$(\exists x \in \text{ri } D_f) \quad \bigcap_{i \in I(x)} \partial f_i(x) = \emptyset. \tag{7.1}$$

Then f is not convex.

Proof. Using Lemma 4.2, we have $\partial f(x) \subseteq \bigcap_{i \in I(x)} \partial f_i(x) = \emptyset$. Therefore, by Fact 2.3, f is not convex. \square

Using Theorem 7.1, we will now illustrate that the finiteness assumption on E is important for our main results (Thms. 5.1 and 5.5).

Example 7.2. Suppose that $X = \mathbb{R}^2$, set

$$f_1(x, y) := \begin{cases} \max\{-x, y\}, & \text{if } (x, y) \in A_1 := \mathbb{R}_+ \times \mathbb{R}; \\ +\infty, & \text{otherwise,} \end{cases} \quad (7.2a)$$

$$f_2(x, y) := \begin{cases} \max\{x, y\}, & \text{if } (x, y) \in A_2 := \mathbb{R}_- \times \mathbb{R}; \\ +\infty, & \text{otherwise,} \end{cases} \quad (7.2b)$$

$\mathcal{F} := \{f_1, f_2\}$, $f := \min\{f_1, f_2\}$, i.e.,

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \mapsto \max\{-|x|, y\}, \quad (7.3)$$

and $E := \{0\} \times \mathbb{R}_-$. Then one checks the following:

- (1) \mathcal{F} is a compatible system of functions.
- (2) $\{A_1, A_2\}$ is a compatible system of sets.
- (3) For every $(\bar{x}, \bar{y}) \in \mathbb{R}^2 \setminus E$ with $I_{\mathcal{F}}(\bar{x}, \bar{y}) = \{1, 2\}$, we must have $(\bar{x}, \bar{y}) \in \{0\} \times \mathbb{R}_{++}$, i.e., $\bar{x} = 0$ and $\bar{y} > 0$. Then $f(x, y) = y$ locally around $(0, \bar{y})$ and thus

$$\partial f_1(0, \bar{y}) \cap \partial f_2(0, \bar{y}) \supseteq \partial f(0, \bar{y}) = \{(0, 1)\}. \quad (7.4)$$

So, all assumptions in Theorems 5.1 and 5.5 are satisfied except that E is infinite. However, for $(0, \bar{y}) \in E \setminus \{(0, 0)\}$, we have $\bar{y} < 0$; thus, $f_1(x, y) = -x + \iota_{A_1}(x, y)$ locally around $(0, \bar{y})$. It follows that

$$\partial f_1(0, \bar{y}) = (-1, 0) + N_{A_1}(0, \bar{y}) =]-\infty, -1] \times \{0\} \quad (7.5)$$

and similarly that

$$\partial f_2(0, \bar{y}) = (1, 0) + N_{A_2}(x, y) = [1, +\infty[\times \{0\}. \quad (7.6)$$

Hence $\partial f_1(0, y) \cap \partial f_2(0, y) = \emptyset$ and so f is not convex by applying Theorem 7.1 or by direct inspection.

In the previous example, the set E was infinite, but unbounded. In the next (slightly more involved) example, we provide a case where E is bounded.

Example 7.3. Suppose that $X = \mathbb{R}^2$, set $I := \{1, \dots, 6\}$,

$$A_1 := \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \geq 1, x \geq 0, y \geq 0\}, \quad (7.7a)$$

$$A_2 := \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \geq 1, x \leq 0, y \geq 0\}, \quad (7.7b)$$

$$A_3 := \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \geq 1, x \leq 0, y \leq 0\}, \quad (7.7c)$$

$$A_4 := \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \geq 1, x \geq 0, y \leq 0\}, \quad (7.7d)$$

$$A_5 := \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1, x \geq 0\}, \quad (7.7e)$$

$$A_6 := \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1, x \leq 0\}, \quad (7.7f)$$

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \mapsto \max\{1 - |x|, |y|\}, \quad (7.8)$$

$$\mathcal{F} := \{f_i\}_{i \in I}, \quad \text{where } (\forall i \in I) \quad f_i := f + \iota_{A_i}, \quad (7.9)$$

and $E := (\{0\} \times [-1, 1]) \cup \{(\pm 1, 0)\}$. Then one checks the following:

(1) Each f_i is convex and continuous on D_{f_i} because

$$f_i(x, y) = y + \iota_{A_i}(x, y) \quad \text{for } i \in \{1, 2\}; \quad (7.10a)$$

$$f_i(x, y) = -y + \iota_{A_i}(x, y) \quad \text{for } i \in \{3, 4\}; \quad (7.10b)$$

$$f_5(x, y) = -x + 1 + \iota_{A_5}(x, y); \quad \text{and} \quad (7.10c)$$

$$f_6(x, y) = x + 1 + \iota_{A_6}(x, y). \quad (7.10d)$$

Consequently, \mathcal{F} is a compatible system of functions.

(2) $\{A_i\}_{i \in I}$ is a compatible system of sets.

(3) f is the piecewise-defined function associated with \mathcal{F} .

(4) Take $(\bar{x}, \bar{y}) \in \mathbb{R}^2 \setminus E$ with $\text{card } I_{\mathcal{F}}(\bar{x}, \bar{y}) \geq 2$. Then

$$I_{\mathcal{F}}(\bar{x}, \bar{y}) \in \{\{1, 2\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 6\}, \{3, 4\}, \{3, 6\}, \{4, 5\}\}. \quad (7.11)$$

Suppose, for instance, that $I_{\mathcal{F}}(\bar{x}, \bar{y}) = \{1, 5\}$. Then $\bar{x} > 0$, $\bar{y} > 0$, and $\bar{x} + \bar{y} = 1$. We have

$$\partial f_1(\bar{x}, \bar{y}) = (0, 1) + N_{A_1}(\bar{x}, \bar{y}) = (0, 1) + \mathbb{R}_+(-1, -1) \quad (7.12)$$

and

$$\partial f_5(\bar{x}, \bar{y}) = (0, 1) + N_{A_5}(\bar{x}, \bar{y}) = (-1, 0) + \mathbb{R}_+(1, 1). \quad (7.13)$$

Then

$$\left(-\frac{1}{2}, \frac{1}{2}\right) \in \partial f_1(\bar{x}, \bar{y}) \cap \partial f_5(\bar{x}, \bar{y}); \quad (7.14)$$

similarly, one obtains nonemptiness for the other cases.

We observe that all assumptions in Theorems 5.1 and 5.5 are satisfied except that E is infinite and bounded. However, for every $(0, \bar{y}) \in \{0\} \times]-1, 1[\subseteq E$, we have $f(x, y) = \min\{f_5(x, y), f_6(x, y)\} = -|x| + 1$ locally around $(0, \bar{y})$. Clearly, f is not convex.

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