WELL-POSEDNESS OF THE SUPERCRITICAL LANE–EMDEN HEAT FLOW IN MORREY SPACES

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Abstract. For any smoothly bounded domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, and any exponent $p > 2^* = 2n/(n-2)$ we study the Lane–Emden heat flow $u_t - \Delta u = |u|^{p-2}u$ on $\Omega \times [0,T]$ and establish local and global well-posedness results for the initial value problem with suitably small initial data $u|_{t=0} = u_0$ in the Morrey space $L^{2,\lambda}(\Omega)$ for suitable $T \leq \infty$, where $\lambda = 4/(p - 2)$. We contrast our results with results on instantaneous complete blow-up of the flow for certain large data in this space, similar to ill-posedness results of Galaktionov–Vazquez for the Lane–Emden flow on $\mathbb{R}^n$.

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1. Introduction

Let $\Omega$ be a smoothly bounded domain in $\mathbb{R}^n$, $n \geq 3$, and let $T > 0$. Given initial data $u_0$, we consider the Lane–Emden heat flow

$$u_t - \Delta u = |u|^{p-2}u \text{ on } \Omega \times [0,T], \quad u = 0 \text{ on } \partial \Omega \times [0,T], \quad u|_{t=0} = u_0$$

(1.1)

for a given exponent $p > 2^* = 2n/(n-2)$, that is, in the “supercritical” regime.

As observed by Matano–Merle [14], p. 1048, the initial value problem (1.1) may be ill-posed for certain data $u_0 \in H_0^1 \cap L^p(\Omega)$; see also our results in Section 4 below. However, as we had shown in two previous papers [4], Section 6.5, [5], Remark 3.3, the Cauchy problem (1.1) is globally well-posed for suitably small data $u_0$ belonging to the Morrey space $H^{1,\mu}_0 \cap L^{p,\mu}(\Omega)$, where $\mu = \frac{2p}{p-2} < n$. Here we go one step further and show that problem (1.1) even is well-posed for suitably small data $u_0 \in L^{2,\lambda}(\Omega) \supset L^{p,\mu}(\Omega)$, where $\lambda = \frac{2\mu}{p} = \frac{4}{p-2} = \mu - 2$, thus considerably improving on the results of Brezis–Cazenave [6] or Weissler [16] for initial data in $L^q$, $q \geq n(p-2)/2$.

Our results are similar to results of Taylor [15] who demonstrated local and global well-posedness of the Cauchy problem for the equation

$$u_t - \Delta u = DQ(u) \text{ on } \Omega \times [0,T],$$

for suitably small initial data $u|_{t=0} = u_0$ in a Morrey space, where $D$ is a linear differential operator of first order and $Q$ is a quadratic form in $u$ as in the Navier–Stokes system. However, similar to the work...
of Koch–Tataru [12] on the Navier–Stokes system, in our treatment of (1.1) we are able to completely avoid the use of pseudodifferential operators in favor of simple integration by parts and Banach’s fixed-point theorem.

The study of the initial value problem for (1.1) for non-smooth initial data is motivated by the question whether a solution $u$ of (1.1) blowing up at some time $T < \infty$ can be extended as a weak solution of (1.1) on a time interval $[0, T_1]$ for some $T_1 > T$. Note that if such a continuation is possible and if the extended solution still satisfies the monotonicity formula [5], Proposition 3.1, it follows that $u(T) \in L^{2,\lambda}(\Omega)$; see Remark 3.3. Hence, the regularity assumption $u_0 \in L^{2,\lambda}(\Omega)$ is necessary from this point of view and cannot be weakened. However, our results in Section 4 show that the condition $u(T) \in L^{2,\lambda}(\Omega)$ in general is not sufficient for continuation and that a smallness condition as in our Theorems 2.1, 2.2 below is needed.

Note that the question of continuation after blow-up only is of relevance in the supercritical case when $p > 2^*$. Indeed, as shown by Baras–Cohen [3], in the subcritical case $p < 2^*$ a classical solution $u \geq 0$ to (1.1) blowing up at some time $T < \infty$ always undergoes “complete blow-up” (see Sect. 4 for a definition), and $u$ cannot be continued as a (weak) solution to (1.1) after time $T$ in any reasonable way. In [9] Galaktionov und Vazquez extend the Baras–Cohen result to the critical case $p = 2^*$.

In the next section we state our well-posedness results, which we prove in Section 3. In Section 4 we then contrast these results with results on instantaneous complete blow-up of the flow for certain large data $u_0 \geq 0$. These results crucially use the scaling properties of equation (1.1) and the maximum principle by comparing our solution with a family of flow solutions blowing up in finite time, with the time of blow-up arbitrarily close to zero after suitable scaling, in a way similar to the ill-posedness results of Galaktionov–Vazquez for the Lane–Emden flow on $\mathbb{R}^n$; see for instance [9], Theorem 10.4. We conclude the paper with some open problems.

Note that in dimension $n = 2$ the limit case of Sobolev’s embedding is given by the Orlicz map

$$M_\alpha = \{ u \in H^1_0(\Omega); \| \nabla u \|_{L^2}^2 = \alpha \} \ni u \mapsto e^{u^2} \in L^1(\Omega)$$

when $\alpha = 4\pi$. In [13], Lamm–Robert–Struwe study a variant of the corresponding Lane–Emden type flow also in a range of super-critical “energies” $\alpha > 4\pi$.

2. GLOBAL AND LOCAL WELL-POSEDNESS

Recall that for any $1 \leq p < \infty$, $0 < \lambda < n$ (in Adams’ [1] notation) a function $f \in L^p(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if

$$\|f\|_{L^{p,\lambda}(\Omega)}^p := \sup_{x_0 \in \mathbb{R}^n, r > 0} r^{-\lambda} \int_{B_r(x_0) \cap \Omega} |f|^p dx < \infty, \tag{2.1}$$

where $B_r(x_0)$ denotes the Euclidean ball of radius $r > 0$ centered at $x_0$. Moreover, we write $f \in L^{p,\lambda}_0(\Omega)$ whenever $f \in L^{p,\lambda}(\Omega)$ satisfies

$$\sup_{x_0 \in \mathbb{R}^n, 0 < r < r_0} r^{-\lambda} \int_{B_r(x_0) \cap \Omega} |f|^p dx \to 0 \text{ as } r_0 \downarrow 0.$$

Similarly, for any $1 \leq p < \infty$, $0 < \mu < n + 2$ a function $f \in L^p(E)$ on $E \subset \mathbb{R}^n \times \mathbb{R}$ belongs to the parabolic Morrey space $L^{p,\mu}(E)$ if

$$\|f\|_{L^{p,\mu}(E)}^p := \sup_{z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}, r > 0} r^{-\mu-(n+2)} \int_{P_r(z_0) \cap E} |f|^p dz < \infty,$$

where $P_r(x, t)$ denotes the backwards parabolic cylinder $P_r(x, t) = B_r(x) \times [t-r^2, t]$.

Note that in abuse of notation we use the symbol $L^{p,\mu}$ for both the standard and the parabolic Morrey space, where the latter is always meant on a space-time domain. For clarity, we write $\|u(t)\|_{L^{p,\mu}}$ for the standard Morrey norm of the function $u(t)$ at a fixed time $t$. 
Given \( p > 2^* \), we now fix the Morrey exponents \( \mu = \frac{2p}{p-2} \) and \( \lambda = \frac{4}{p-2} = \mu - 2 \), which are natural for the study of problem (1.1).

Throughout the following a function \( u \) will be called a smooth solution of (1.1) on \([0, T]\) if \( u \in C^1(\Omega \times [0, T]) \) with \( u_t \in L^2_{\text{loc}}(\Omega \times [0, T]) \) solves (1.1) in the sense of distributions and achieves the initial data in the sense of traces. By standard regularity theory then \( u \) also is of class \( C^2 \) with respect to \( x \) and satisfies (1.1) classically. Schauder theory, finally, yields even higher regularity to the extent allowed by smoothness of the nonlinearity \( g(v) = |v|^{p-2}v \). The function \( u \) will be called a global smooth solution of (1.1) if the above holds with \( T = \infty \).

Our results on local and global well-posedness are summarized in the following theorems.

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be a smoothly bounded domain, \( n \geq 3 \). There exists a constant \( \varepsilon_0 > 0 \) such that for any function \( u_0 \in L^{2, \lambda}(\Omega) \) satisfying \( \|u_0\|_{L^{2, \lambda}} < \varepsilon_0 \) there is a unique global smooth solution \( u \) to (1.1) on \( \Omega \times [0, \infty[ \).

The smallness condition can be somewhat relaxed.

**Theorem 2.2.** Let \( u_0 \in L^{2, \lambda}(\Omega) \) and suppose that there exists a number \( R > 0 \) such that

\[
\sup_{x_0 \in \mathbb{R}^n, 0 < r < R} \rho^{\lambda-n} \int_{B_r(x_0) \cap \Omega} |u_0|^2 \, dx \leq \varepsilon_0^2,
\]

where \( \varepsilon_0 > 0 \) is as determined in Theorem 2.1. Then there exists a unique smooth solution \( u \) to (1.1) on an interval \([0, T_0]\), where \( T_0 = C(\varepsilon_0/\|u_0\|_{L^{2, \lambda}}) > 0 \).

In particular, for any \( u_0 \in L^{2, \lambda}_0(\Omega) \) there exists a unique smooth solution \( u \) to (1.1) on some interval \([0, T]\), where \( T = T(u_0) > 0 \).

It is well-known that for smooth initial data \( u_0 \in C^4(\Omega) \) there exists a smooth solution \( u \) to the Cauchy problem (1.1) on some time interval \([0, T]\), \( T > 0 \). By the uniqueness of the solution to (1.1) constructed in Theorem 2.1 or 2.2, the latter solution coincides with \( u \) and hence is smooth up to \( t = 0 \) if \( u_0 \in C^4(\Omega) \).

### 3. Proof of Theorem 2.1

Let \( n \geq 3 \) and let

\[
G(x, t) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \ t > 0,
\]

be the fundamental solution to the heat equation on \( \mathbb{R}^n \) with singularity at \((0, 0)\). Given a domain \( \Omega \subset \mathbb{R}^n \) also let \( \Gamma = \Gamma(x, y, t) = \Gamma(y, x, t) \) be the corresponding fundamental solution to the heat equation on \( \Omega \) with homogeneous Dirichlet boundary data \( \Gamma(x, y, t) = 0 \) for \( x \in \partial \Omega \). Note that by the maximum principle for any \( x, y \in \Omega \), any \( t > 0 \) there holds \( 0 < \Gamma(x, y, t) \leq G(x - y, t) \).

For \( x \in \Omega \), \( r > 0 \) we let

\[
\Omega_r(x) = B_r(x) \cap \Omega;
\]

similarly, for \( x \in \Omega \), \( r, t > 0 \) we define

\[
Q_r(x, t) = P_r(x, t) \cap \Omega \times [0, \infty[.
\]

We sometimes write \( z = (x, t) \) for a generic point in space-time. The letter \( C \) will denote a generic constant, sometimes numbered for clarity.

For \( f \in L^1(\Omega) \) set

\[
(S_{\Omega f})(x, t) := \int_{\Omega} \Gamma(x, y, t)f(y) \, dy, \quad t > 0,
\]

so that \( v = S_{\Omega f} \) solves the equation

\[
v_t - \Delta v = 0 \text{ on } \Omega \times [0, \infty[ \quad (3.1)
\]
with boundary data \( v(x,t) = 0 \) for \( x \in \partial \Omega \) and initial data \( v|_{t=0} = f \) on \( \Omega \). See [7] for a thorough introduction to the concept of fundamental solutions.

Similar to [4, Proposition 4.3], by adapting the methods of Adams [1] we can show that \( S_\Omega \) is well-behaved on Morrey spaces. Recall that \( \mu = \frac{2n}{p-2} \) with \( 2 < \mu < n \), and \( \lambda = \mu - 2 = \frac{4}{p-2} > 0 \).

**Lemma 3.1.**

(i) *For any \( \lambda > 0 \), let \( v \in L^{2,\lambda}(\Omega) \) be well-defined and bounded. Moreover, we have the bounds*

\[
\|v(t)\|_{L^2} \leq C t^{\lambda/2} \|f\|_{L^{2,\lambda}}, \quad \|v(t)\|_{L^{2,\lambda}}^2 \leq C \|f\|_{L^{2,\lambda}}^2, \quad t > 0. \tag{3.2}
\]

(ii) *Let \( f \in L^{2,\lambda}(\Omega) \) and suppose that for a given \( \varepsilon_0 > 0 \) there exists a number \( R > 0 \) such that*

\[
\sup_{x_0 \in \Omega, 0 < r < R} \left( \int_{\Omega_r(x_0)} |f|^2 \right)^{1/2} \leq \varepsilon_0.
\]

*Then with a constant \( C > 0 \) for \( v = S_\Omega f \) there holds the estimate*

\[
\sup_{x_0 \in \Omega, 0 < r^2 \leq t_0} \left( \int_{Q_r(x_0,t_0)} |v|^p \right)^{1/p} \leq C \varepsilon_0,
\]

*where \( T_0/R^2 = C(\varepsilon_0/\|f\|_{L^{2,\lambda}(\Omega)}) > 0 \).*

**Proof.**

(i) *Let \( f \in L^{2,\lambda}(\mathbb{R}^n) \) and set \( v = S_\Omega f \) as above. Recall the definition of the fractional maximal functions*

\[
M_\alpha f(x) := \sup_{r > 0} M_{\alpha,r} f(x), \quad M_{\alpha,r} f(x) := \int_{\Omega_r(x)} |f(y)| \, dy, \quad \alpha > 0.
\]

*Note that Hölder’s inequality gives the uniform bound*

\[
(M_{\lambda/2} f)^2 \leq M_\lambda (|f|^2) \leq \|f\|^2_{L^{2,\lambda}}. \tag{3.3}
\]

*Following the scheme outlined by Adams [1], proof of Proposition 3.1, we first derive pointwise estimates for \( v \) and bounds on parabolic cylinders \( P_r(x_0,t_0) \) with radius \( r \) satisfying \( 0 < 2r^2 < t_0 \). Using the well-known estimate*

\[
G(x-y,t) \leq C (|x-y| + \sqrt{t})^{-n}
\]

*for the heat kernel and recalling that \( \Gamma(x,y,t) \leq G(x-y,t) \), for any \( t > 0 \) we can bound*

\[
|v(x,t)| \leq C \int_{\Omega} (|x-y| + \sqrt{t})^{-n} |f(y)| \, dy
\]

\[
\leq C \int_{\Omega_{\sqrt{t}(x)}} (|x-y| + \sqrt{t})^{-n} |f(y)| \, dy
\]

\[
+ C \sum_{k=1}^{\infty} \int_{\Omega_{2^k \sqrt{t}(x)} \setminus \Omega_{2^{k-1} \sqrt{t}(x)}} (|x-y| + \sqrt{t})^{-n} |f(y)| \, dy
\]

\[
\leq C \sum_{k=0}^{\infty} (2^k \sqrt{t})^{-n} (2^k \sqrt{t})^{-n/2} M_{\lambda/2,2^k \sqrt{t}} f(x) \leq Ct^{-\lambda/4} M_{\lambda/2} f(x).
\]
Hence by (3.3) with a uniform constant $C > 0$ for any $t > 0$ there holds
$$
\|v(t)\|_{L^\infty}^2 \leq Ct^{-\lambda/2}\|M_{\lambda/2}f\|_{L^\infty}^2 \leq Ct^{-\lambda/2}\|f\|_{L^{2,\lambda}}^2,
$$
as claimed in (3.2). Moreover, for any $x_0 \in \mathbb{R}^n$, any $t_0 > 0$ and any $0 < r < \sqrt{t_0/2}$ we obtain the bounds
$$
\|v(t_0)\|_{L^2(\Omega_r(x_0))}^2 \leq Cr^{n-\lambda/2}\|f\|_{L^{2,\lambda}}^2 \leq Cr^{n-\lambda}\|f\|_{L^{2,\lambda}}^2,
$$
which is (3.4). Similarly
$$
\|v\|_{L^p(\Omega_r(x_0))}^p \leq Cr^{n+2}\|f\|_{L^{2,\lambda}}^p \leq Cr^{n+2-\mu}\|f\|_{L^{2,\lambda}}^p,
$$
where we also used that $\mu = 2p\lambda/4$.

In order to derive (3.5) also for radii $r \geq \sqrt{t_0/2}$ we need to argue slightly differently. We may assume that $x_0 = 0$. Moreover, after enlarging $t_0$, if necessary, we may assume that $t_0 = 2v^2$. Let $\psi = \psi_0 = \psi_0(x)$ be a smooth cut-off function satisfying $\chi_{B_r(0)} \leq \psi \leq \chi_{B_{2r}(0)}$ and with $|\nabla \psi|^2 \leq 4r^{-2}$. Set $r = r_0$ and let $r_i = 2^i r_0$, $\psi_i(x) = \psi(2^{-i}x)$, $i \in \mathbb{N}$. For ease of notation in the following estimates we drop the index $i$.

Upon multiplying (3.1) with $\psi^2$ we find the equation
$$
\frac{1}{2}\frac{d}{dt}(|v|^2\psi^2) - \text{div}(v\nabla v\psi^2) + |\nabla v|^2\psi^2 = -2v\nabla v\psi\Delta \psi \leq \frac{1}{2}|\nabla v|^2\psi^2 + 2|v|^2|\nabla \psi|^2.
$$

Integrating over $\Omega \times [0, t_1]$ and using the bound $|\nabla \psi|^2 \leq 4r^{-2}$, for any $0 < t_1 < t_0$ we obtain
$$
\int_{\Omega_r(0)} |v(t)|^2\psi^2 \, dx + \int_{\Omega_r(0) \times [0, t_1]} |\nabla v|^2\psi^2 \, dx \, dt \\
\leq \int_{\Omega_r(0)} |f|^2\psi^2 \, dx + 16r^{-2} \int_{\Omega_r(0) \times [0, t_1]} |v|^2 \, dx \, dt. \quad (3.6)
$$

For $r = r_i$, $i \in \mathbb{N}_0$, set
$$
\Psi(r) := \sup_{x_0 \in \Omega, 0 < t < t_0} r_i^{-n} \int_{\Omega_r(x_0)} |v(t)|^2 \, dx.
$$

Recalling that $\lambda = \mu - 2$, then from the previous inequality (3.6) with the uniform constants $C_1 = 2^{n-\lambda}$, $C_2 = 32C_1$ we obtain
$$
\Psi(r_i) \leq r_i^{-n} \left( \int_{\Omega_{2r_i}(0)} |f|^2 \, dx + 16t_0r_i^{-2} \sup_{0 < t < t_0} \int_{\Omega_{2r_i}(0)} |v(t)|^2 \, dx \right) \\
\leq C_1\|f\|_{L^{2,\lambda}}^2 + C_2 2^{-2\lambda}\Psi(r_{i+1}).
$$

By iteration, for any $k_0 \in \mathbb{N}$ there results
$$
\Psi(r_0) \leq C_1\|f\|_{L^{2,\lambda}}^2 + C_2\Psi(r_1) \leq C_1(1 + C_2)\|f\|_{L^{2,\lambda}}^2 + C_2^2 2^{-2\lambda}\Psi(r_2) \leq \ldots \\
\leq C_1 \sum_{k=0}^{k_0} C_2^k 2^{(1-k)k} \|f\|_{L^{2,\lambda}}^2 + C_2^{k_0+1} 2^{-k_0(k_0+1)}\Psi(r_{k_0+1}).
$$

Passing to the limit $k_0 \to \infty$, we obtain that $\Psi(r_1) \leq C\|f\|_{L^{2,\lambda}}^2$. Inserting this information into (3.6), where we again set $r = r_0$, then we find
$$
\Psi(r) + \sup_{x_0 \in \Omega} r^{\mu-2-n} \int_{\Omega_r(x_0) \times [0, t_0]} |\nabla v|^2 \, dx \, dt \leq C\|f\|_{L^{2,\lambda}}^2. \quad (3.7)
$$
In particular, together with (3.4) we have now shown the bound
\[ \|v(t)\|_{L^{2,\lambda}}^2 \leq C\|f\|_{L^{2,\lambda}}^2 \text{ for all } t > 0, \]
and thus have verified (3.2) completely.

To complete the proof of (3.5) for \( r = r_0 = \sqrt{t_0/2} \), let \( \psi = \psi_0 \) as above and let \( \tau(t) = \min\{t, t_0 - t\} \). Multiplying (3.1) with the function \( v|v|^{p-2}\tau^2 \) then we obtain
\[
\frac{1}{p} \frac{d}{dt} (|v|^p \tau^2) - \frac{1}{p} \frac{dr}{dt} |v|^p \tau^2 - \text{div}(|v|^{p-2} v \nabla \psi^2 \tau) + (p - 1) |\nabla v|^2 |v|^{p-2}\tau^2 = -2|v|^{p-2} \nabla \psi \nabla \psi \tau \geq -|\nabla v|^2 |v|^{p-2}\tau^2 - |v|^p |\nabla \psi|^2 \tau.
\]

Integrating over \( \Omega \times [0, t_0] \) and using that \( \frac{dr}{dt} = 1 \) for \( 0 < t < t_0/2 \), \( \frac{dr}{dt} = -1 \) for \( t_0/2 < t < t_0 \), as well as the fact that the region \( \Omega \) may be covered by a collection of at most \( L = L(n) \) cylinders \( Q_r(x_l, t_0) \), \( 1 \leq l \leq L \), we find
\[
\int_{Q_r(x_l, t_0/2)} |v|^p \, dz \leq \sup_{1 \leq l \leq L} \int_{Q_r(x_l, t_0)} |v|^p \, dz + Cr^{-2} \int_{\Omega \times [0, t_0]} |v|^p \tau \, dz \, dt + C \int_{\Omega \times [0, t_0]} |\nabla v|^2 |v|^{p-2} \tau \, dz \, dt.
\]

But by (3.2) we have \( |v|^{p-2}\tau \leq |v|^{p-2} \leq C\|f\|_{L^{2,\lambda}}^{p-2} \), and from (3.7) we obtain
\[
\int_{\Omega \times [0, t_0]} |v|^p \tau \, dz \, dt \leq C\|f\|_{L^{2,\lambda}}^{p-2} (r^{-n-\lambda} \Psi(2r) + \int_{\Omega \times [0, t_0]} |\nabla v|^2 \, dz \, dt) \leq Cr^{-\lambda} \|f\|_{L^{2,\lambda}}^p.
\]

Recalling that for each cylinder \( Q_r(x_l, t_0) \), \( 1 \leq l \leq L \), there holds (3.5), we then obtain
\[
\int_{Q_r(x_l, t_0/2)} |v|^p \, dz \leq \sup_{1 \leq l \leq L} \int_{Q_r(x_l, t_0)} |v|^p \, dz + Cr^{-n-\lambda} \|f\|_{L^{2,\lambda}}^p \leq Cr^{-\lambda} \|f\|_{L^{2,\lambda}}^p,
\]
and (3.5) follows since \( \lambda = \mu - 2 \).

Finally, for \( t_0 \leq r^2 \) and any \( x_0 \in \Omega \) equation (3.6) yields the gradient bound
\[
\int_{Q_{r}(x_0, t_0)} |\nabla v|^2 \, dz \leq \int_{\Omega_{2r}(0)} |f|^2 \psi^2 \, dx + 16r^{-2} \int_{\Omega_{2r}(0) \times [0, t_0]} |v|^2 \, dz \, dt \leq Cr^{-n-\lambda} \|f\|_{L^{2,\lambda}}^2 \leq Cr^{-\lambda} \|f\|_{L^{2,\lambda}}^2.
\]

In view of (3.2) the same bound also holds for \( t_0 > r^2 \) as can be seen by shifting time by \( t_0 - r^2 \) and replacing \( f \) with the function \( \tilde{f}(x) = v(x, t_0 - r^2) \in L^{2,\lambda}(\Omega) \). With \( \lambda = \mu - 2 \) we obtain the bound \( \|\nabla v\|_{L^{2,n}} \leq C\|f\|_{L^{2,\lambda}} \), as desired.

(ii) Set \( L_0 := \|f\|_{L^{2,\lambda}} \). As before, for any \( x \in \Omega \) we have the bound
\[
|v(x, t)| \leq C \sum_{k=0}^{\infty} (2^k \sqrt{t})^{-\lambda/2} M_{\lambda/2,2^k}\sqrt{f(x)}.
\]
By assumption for \( r = 2^k \sqrt{t} \leq R \) we can estimate
\[
M_{\lambda/2,r}(|f|)(x) \leq (M_{\lambda,r}(|f|^2)(x))^{1/2} \leq \varepsilon_0,
\]
whereas for any $r > 0$ we have
\[
M_{\lambda/2,r}(|f|(x)) \leq (M_{\lambda,r}(|f|^2)(x))^{1/2} \leq \|f\|_{L^2,\lambda} = L_0.
\]

Let $k_0 \in \mathbb{N}$ such that $2^{-k_0\lambda/2}L_0 \leq \varepsilon_0$. Then for $0 < t < T := 2^{-2k_0}R^2$ we find the uniform estimate
\[
|v(x,t)| \leq Ct^{-\lambda/4}\left(\sum_{k=0}^{k_0} 2^{-k\lambda/2}\varepsilon_0 + \sum_{k=k_0+1}^{\infty} 2^{-k\lambda/2}L_0\right) \leq Ct^{-\lambda/4}\varepsilon_0.
\]

Proceeding as in part (i) of the proof, for any $0 < t < T$, any $x_0 \in \Omega$, and any $0 < r < \sqrt{t/2}$ we then obtain the bound
\[
\|v(t)\|^2_{L^2(\Omega_r(x_0))} \leq C r^n t^{-\lambda/2} \varepsilon_0^2 \leq C r^n \varepsilon_0^2,
\]
similarly, we find
\[
\|v(t)\|^p_{L^p(Q_r(x_0,t_0))} \leq C r^{n+2} t^{-p\lambda/4} \varepsilon_0^p \leq C r^{n+2} \varepsilon_0^p
\]
whenever $0 < 2r^2 < t_0 < T$. In order to derive the latter bound also for radii $r > 0$ with $t_0/2 \leq r^2 \leq t_0 \leq T$ as in i) we may assume that $x_0 = 0$ and fix some numbers $0 < t_0 < T$, $r_0 \geq \sqrt{t_0/2}$. Setting
\[
\Psi(r) := \sup_{0 < t < t_0} r^{\lambda-n} \int_{B_r(0)} |v(t)|^2 \, dx, \quad r > 0,
\]
for $r = r_i = 2^ir_0$, $i \in \mathbb{N}_0$, from (3.6) we obtain the bound
\[
\Psi(r_i) \leq r_i^{\lambda-n} \int_{B_{r_i}(0)} |f|^2 \, dx + 16C_1 t_0 r_i^{-2}\Psi(2r_i)
\]
\[
\leq C_1 M_{\lambda,r_i+1}(|f|^2)(0) + C_2 2^{-2i}\Psi(r_{i+1})
\]
for any $i \in \mathbb{N}$, with constants $C_1 = 2^{n-\lambda}$, $C_2 = 32C_1$ as before. Suppose that $r_{i_0} \leq R$ for some $i_0 \in \mathbb{N}$. Bounding $M_{\lambda,r_i}(|f|^2)(x) \leq \varepsilon_0^2$ for $i \leq i_0$ and $M_{\lambda,r_i}(|f|^2)(x) \leq L_0^2$ else, by iteration we then obtain
\[
\Psi(r_0) \leq C_1 \varepsilon_0^2 + C_2 \Psi(r_1) \leq C_1 (1 + C_2) \varepsilon_0^2 + C_2^2 2^{-2} \Psi(r_2) \leq \ldots
\]
\[
\leq C_1 \sum_{i=0}^{i_0-1} C_2^2 (1-i)^2 \varepsilon_0^2 + C_1 \sum_{i=i_0}^{k} C_2^i 2^{(1-i)i} L_0^2 + C_2^{k+1} 2^{-k(k+1)} \Psi(r_{k+1}).
\]

Thus, if $i_0$ is such that $C_2 2^{1-i_0} \leq (\varepsilon_0/L_0)^2 \leq 1/2$, that is, if
\[
\sqrt{2t_0} \leq 2r_0 = 2^{1-i_0}r_{i_0} \leq 2^{1-i_0}R \leq C_2^{-1}(\varepsilon_0/L_0)^2 R,
\]
upon passing to the limit $k \to \infty$ we obtain $\Psi(r_0) \leq C \varepsilon_0^2$ and the analogue of (3.7) with $\varepsilon_0$ in place of $\|f\|_{L^{2,\lambda}}$.

Recalling the definition $T = 2^{-2k_0}R^2$ with $k_0 \in \mathbb{N}$ satisfying $2^{-k_0\lambda/2}L_0 \leq \varepsilon_0$, we see that these bounds hold true for
\[
0 < t_0/2 \leq r_0^2 \leq t_0 \leq T_0 := R^2 \cdot \min\{(\varepsilon_0/L_0)^{4/\lambda}, C_2^{-2}(\varepsilon_0/L_0)^4\}.
\]

Using (3.8), the remainder of the proof of (3.5) in part i) now may be copied unchanged to yield the claim.

The assertions of Theorems 2.1 and 2.2 now are a consequence of the following result.
Lemma 3.2.

(i) For any $p > 2^*$ there exists a constant $\varepsilon_0 > 0$ such that for any $u_0 \in L^{2,\lambda}(\Omega)$ with $\|u_0\|_{L^{2,\lambda}} \leq \varepsilon_0$ there exists a unique solution $u \in L^{p,\mu}(\Omega \times ]0,\infty[)$ to the Cauchy problem (1.1) such that

$$\|u\|_{L^{p,\nu}} \leq C\|u_0\|_{L^{2,\lambda}}. \tag{3.9}$$

(ii) Let $u_0 \in L^{2,\lambda}(\Omega)$ and suppose that there exists a number $R > 0$ such that

$$\sup_{x_0 \in \Omega, 0 < r < R} r^{\lambda - n} \int_{B_r(x_0)} |u_0|^2 \, dx \leq \varepsilon_0^2,$$

where $\varepsilon_0 > 0$ is as determined in (i). Then there exists a unique smooth solution $u$ to (1.1) on an interval $]0, T_0[,$ where $T_0/R^2 = C(\varepsilon_0^{-1}\|u_0\|_{L^{2,\lambda}(\Omega)}) > 0$.

Proof. For $u_0 \in L^{2,\lambda}(\mathbb{R}^n)$ set $w_0 = S_\Omega u_0$. For suitable $a > 0$ let

$$X := \{v \in L^{p,\mu}(\Omega \times ]0, T_0[); \|v\|_{L^{p,\mu}} \leq a\},$$

where $T_0 > 0$ in the case of the assumptions in (i) may be chosen arbitrarily large and otherwise is as in assertion (ii) of Lemma 3.1.

Then $X$ is a closed subset of the Banach space $L^{p,\mu} = L^{p,\mu}(\Omega \times ]0, T_0[)$. Moreover, for any $v \in X$ we have $|v|^{p-2}v \in L^{p/(p-1),\mu}$. By Lemma 4.1 in [4] there exists a unique solution $w = S(v|v|^{p-2}) \in L^{p,\mu}$ of the Cauchy problem

$$w_t - \Delta w = |v|^{p-2}v \text{ on } \Omega \times ]0, T_0[; \ w|_{t=0} = 0,$$

with

$$\|w\|_{L^{p,\mu}} \leq C\|v\|_{L^{p,\mu}}^{p-1} \leq Ca^{p-1}.$$

For sufficiently small $\varepsilon_0, a > 0$ then the map

$$\Phi: X \ni v \mapsto w_0 + w \in X,$$

and for $v_{1,2} \in X$ with corresponding $w_i = S(v_i|v_i|^{p-2}), i = 1, 2,$ we can estimate

$$\|\Phi(v_1) - \Phi(v_2)\|_{L^{p,\mu}} = \|w_1 - w_2\|_{L^{p,\mu}} \leq C\|v_1|v_1|^{p-2} - v_2|v_2|^{p-2}\|_{L^{p/(p-1),\mu}}.$$

The latter can be bounded

$$\|v_1|v_1|^{p-2} - v_2|v_2|^{p-2}\|_{L^{p/(p-1),\mu}} \leq C\left(\|v_1\|_{L^{p,\mu}}^{p-2} + \|v_2\|_{L^{p,\mu}}^{p-2}\right)\|v_1 - v_2\|_{L^{p,\mu}}.$$

Thus for sufficiently small $a > 0$ we find

$$\|\Phi(v_1) - \Phi(v_2)\|_{L^{p,\mu}} \leq Ca^{p-2}\|v_1 - v_2\|_{L^{p,\mu}} \leq \frac{1}{2}\|v_1 - v_2\|_{L^{p,\mu}}.$$

By Banach’s theorem the map $\Phi$ has a unique fixed point $u \in X,$ and $u$ solves the initial value problem (1.1) in the sense of distributions. Finally, for sufficiently small $a, \varepsilon_0 > 0$ we can invoke Proposition 4.1 in [4] to show that $u,$ in fact, is a smooth global solution of (1.1). \qed

Remark 3.3. As already pointed out in the introduction, the assumption $u_0 \in L^{2,\lambda}(\Omega)$ is natural in the context of weak continuations of the flow (1.1). Indeed, suppose that a solution $u$ of (1.1) blowing up at some time $T < \infty$ can be extended as a weak solution of (1.1) on a time interval $]0, T_1[ \>$ for some $T_1 > T$ and assume that the extended solution still satisfies the monotonicity formula [5], Proposition 3.1. In the notation of [5],
for any \( x_1 = 0 \in \Omega \) and any \( 0 < T < t_1 < T_1 \) choose \( (x_1, t_1) \) as center of scaling and integrate the scaled energy function \( \bar{H}^\varphi \) given by (2.13) in [5]. Using that \( rF_2^\varphi(r) \to 0 \) as \( r \downarrow 0 \), for any \( 0 < R \leq R_1 \leq \sqrt{T_1} \) similar to (4.7) in [5] we then obtain the inequality

\[
F_2^\varphi(R) + \frac{1}{R} \int_0^R \left( D^\varphi(r) + F_2^\varphi(r) \right) dr \leq CH^\varphi(R_1) + C \int_0^{R_1} \frac{|B^\varphi(r)|}{r} dr + C_0 \delta(\rho, R_1),
\]

where the integral involving \( B^\varphi(r) \) on the right can be bounded uniformly in \((x_1, t_1)\) by means of [5], Lemmas 4.1 and 4.3. Choosing \( R = \sqrt{T_1} - T \), for sufficiently small \( t_1 > T \) we have \( \varphi \equiv 1 \) on \( B_R(0) \) and thus we are able to bound

\[
R^{\lambda-n} \int_{\Omega_n(x_1)} |u(T)|^2 dx \leq C F_2^\varphi(R) \leq C
\]

with constants \( C > 0 \) independent of \( x_1 \) and \( R > 0 \); that is, \( u(T) \in L^{2,\lambda}(\Omega) \).

## 4. Ill-posedness for “large” data

### 4.1. Minimal solutions for non-negative initial data

In order to obtain a notion of solution of (1.1) on \( \Omega \times [0, \infty[ \) for arbitrary nonnegative initial data \( u_0 \geq 0 \), following Baras–Cohen [3] for \( n \in \mathbb{N} \) we solve the initial value problem

\[
u_{n,t} - \Delta u_n = f_n(u_n) = \min\{u_n^{p-1}, n^{p-1}\} \text{ on } \Omega \times [0, \infty], \quad u = 0 \text{ on } \partial \Omega \times [0, \infty],
\]

with initial data

\[
u_n(x, 0) = u_{n0}(x) := \min\{u_0(x), n\} \geq 0.
\]

As the right-hand side \( f_n(u_n) \) in (4.1) is uniformly bounded, for any \( n \in \mathbb{N} \) there exists a unique global solution of (4.1), (4.2). By the maximum principle, positivity of the initial data is preserved and \( u_n \) is monotonically increasing in \( n \). Hence, the pointwise limit \( u(x, t) := \lim_{n \to \infty} u_n(x, t) \leq \infty \) exists. Inspired by Baras and Cohen [3] we call this limit the minimal solution of problem (1.1) for the given data \( u_0 \). Moreover, similar to their Proposition 2.1 we have \( u \leq v \) for any \( v \) which is an integral solution \( v \) of (1.1) in the sense that

\[
v(t) = S_t u_0 + \int_0^t S_{t-s} v^{p-1}(s) ds,
\]

where for brevity we now write \( (S_t)_{t \geq 0} \) for the heat semigroup on \( \Omega \), defined by

\[
S_t w(x) = \int_\Omega \Gamma(x, y, t) w(y) dy,
\]

with \( \Gamma > 0 \) denoting the fundamental solution of the heat equation on \( \Omega \).

Indeed, by Duhamel’s principle the \( u_n \) satisfy the integral equation

\[
u_n(t) = S_t u_{n0} + \int_0^t S_{t-s} f_n(u_n(s)) ds.
\]

Recalling that the sequence \( u_n \) is monotonically increasing in \( n \), from Beppo–Levi’s theorem on monotone convergence we find that \( u \) satisfies (4.3). On the other hand, for each \( n \) and any integral solution \( v \) of (1.1) clearly there holds \( u_n \leq v \).

With these prerequisites we now show that there are initial data \( u_0 \in L^{p,\mu}(\Omega) \) with even \( \nabla u_0 \in L^{2,\mu} \) such that the minimal solution \( u \) to (1.1) satisfies \( u \equiv \infty \) on \( \Omega \times [0, \infty] \), that is, undergoes complete instantaneous blow-up. The following arguments are modelled on corresponding results on complete instantaneous blow-up by Galaktionov and Vazquez [9] in the case when \( \Omega = \mathbb{R}^n \).
4.2. Complete instantaneous blow-up

It is well-known that on a bounded domain $\Omega$ equation (1.1) may be interpreted as the negative gradient flow of the energy

$$E(u) = E_\Omega(u) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p} |u|^p \right) \, dx.$$  

As observed by Ball (2, Thm. 3.2), sharpening an earlier result of Kaplan (11), for data $u_0$ with $E(u_0) < 0$ the solution to (1.1) blows up in finite time. Indeed, Ball (2, Theorem 3.2), observes that testing equation (1.1) with $u$ leads to the differential inequality

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 = -\int_{\Omega \times \{t\}} (|\nabla u|^2 - |u|^p) \, dx = -2E(u(t)) + \frac{p-2}{p} \|u(t)\|_{L^p}^p \geq -2E(u_0) + c_0 \|u(t)\|_{L^2}^p \geq c_0 \|u(t)\|_{L^2}^p$$

for some constant $c_0 > 0$. Hence we find

$$\|u(t)\|_{L^2} \geq \left( \|u_0\|_{L^2}^{(2-p)/2} - c_0(p-2)t \right)^{-2/(p-2)},$$

and $u(t)$ must blow up at the latest at time $T = c_0^{-1}(p-2)^{-1}\|u_0\|_{L^2}^{(2-p)/2}$.

In order to obtain data $u_0 \in L^{p,\mu}$ leading to instantaneous complete blow-up, we combine this observation with the following well-known scaling property of equation (1.1): whenever $u$ is a solution of (1.1) on $\Omega$, then for any $R > 0$, any $x_0 \in \mathbb{R}^n$ the function

$$u_{R,x_0}(x,t) = R^{-\alpha} u(R^{-1}(x-x_0), R^{-2}t)$$

(4.5)

with $\alpha = \frac{2}{p-2}$ is a solution of (1.1) on the scaled domain

$$\Omega_{R,x_0} := \{ x \in \mathbb{R}^n ; R^{-1}(x-x_0) \in \Omega \}.$$  

Clearly we may assume that $0 \in \Omega$.

Theorem 4.1. Let $0 \leq w_0 \in C^\infty_c(B_1(0))$ with $E_{B_1(0)}(w_0) < 0$. Set

$$M = M_{w_0} = \sup_{\|y\| \leq 1} (|y|^\alpha w_0(y)),$$

where $\alpha = \frac{2}{p-2}$ as above. Then for every initial data $0 \leq u_0 \in C^0(\Omega \setminus \{0\})$ satisfying

$$\liminf_{x \to 0} (u_0(x) - M|x|^{-\alpha}) > 0$$

the minimal solution $u$ to (1.1) blows up completely instantaneously.

Proof. By Ball’s above result, the solution $w$ to (1.1) on $B_1(0) \times [0,T]$ with initial data $w(0) = w_0$ blows up after some finite time $T$ at a point $y_0$.

Fix $R_0 > 0$ with $BR_0(0) \subset \Omega$ and such that

$$u_0(x) > M|x|^{-\alpha} \text{ for } |x| \leq R_0.$$

For $R < R_0$ and $x_0 \in \Omega$ with $|x_0| \leq R_0 - R$ consider the rescaled solutions

$$w_{R,x_0}(x,t) := R^{-\alpha} w(R^{-1}(x-x_0), R^{-2}t)$$

on $B_R(x_0) \times [0,R^2T]$ that blow up at time $R^2T$.  


Since by assumption we have

\[ w_{R,0}(x,0) = R^{-\alpha}w_0(R^{-1}x) \leq M|x|^{-\alpha} < u_0(x) \text{ on } B_R(0), \]

by continuity of \( u_0 \) away from \( x = 0 \) and continuity of \( w_0 \) there is a number \( \delta = \delta(R) > 0 \) such that

\[ w_{R,x_0}(x,0) < u_0(x) \text{ on } B_R(x_0) \]

for all \( x_0 \) with \( |x_0| < \delta \). Since in addition \( u \geq 0 = w_{R,x_0} \) on \( \partial B_R(x_0) \times [0, R^2T] \), by the maximum principle for any \( \varepsilon > 0 \), any \( n \geq \|w_{R,x_0}\|_{L^\infty(\partial B_R(x_0) \times [0, R^2T - \varepsilon])} \) there holds

\[ u(x,t) \geq u_n(x,t) \geq w_{R,x_0}(x,t) \text{ on } B_R(x_0) \times [0, R^2T - \varepsilon], \]

where \( u_n \) solves (4.1) for each \( n \in \mathbb{N} \). Passing to the limit \( \varepsilon \to 0 \), we then find

\[ u(x_0 + R_0, R^2T) = \left( S_{R^2T} u_n \right)(x_0 + R_0), \]

and

\[ = \lim_{n \to \infty} \left( S_{R^2T} u_n + \int_0^{R^2T} S_{R^2T-s} f(u_n(s)) ds \right)(x_0 + R_0) \]

for all \( x_0 \in B_0(0). \)

Since \( R > 0 \) may be chosen arbitrarily small, we conclude that for any sufficiently small \( t > 0 \) there holds \( \mathcal{L}^n(\{x \in \Omega; u(x,t) = \infty\}) > 0 \). But then positivity of \( \Gamma \) and Duhamel’s principle (4.3) yield

\[ u(x,t) = \left( S_t u_0 \right) + \int_0^t S_{t-s} u^{p-1}(s) ds \to \infty, \]

for any \( t > 0 \) and any \( x \in \Omega \).

5. Open problems

An obvious question to be investigated is whether the pathological situation that leads to instantaneous complete blow-up of the flow (1.1) can arise under “natural” hypotheses. In particular, is it possible that a smooth solution \( u \) of (1.1) on \([0, T]\) blowing up at time \( T > 0 \) with bounded energy \( |E(u(t))| \leq C < \infty \) for \( 0 < t < T \) has a “trace” \( u(T) \) giving rise to instantaneous complete blow-up? Of course, it would also be of interest to quantify the smallness conditions in Theorems 2.1 and 2.2.

Conversely, one might try to determine the smallest number \( M > 0 \) so that the conclusion of Theorem 4.1 holds true. Can one show that at least for exponents \( p \) strictly less than the Joseph–Lundgren [10] exponent

\[ p_{JL} = 2 + \frac{4}{n - 4 - 2\sqrt{n - 1}} \text{ if } n \geq 11, \quad p_{JL} = \infty \text{ if } n \leq 10, \]

we have \( M = \alpha(n - 2 - \alpha) =: c_\alpha \), where \( c_\alpha \) appears as coefficient in the singular solution \( u_\alpha(x) := c_\alpha|x|^{-\alpha} \) of the time-independent equation (1.1) on \( \mathbb{R}^n \)? The significance of the exponent \( p_{JL} \) is illustrated for instance in Lemma 9.3 of [9].

Hopefully, we will be able to answer some of these questions in the future.

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