AN INTERNAL OBSERVABILITY ESTIMATE FOR STOCHASTIC HYPERBOLIC EQUATIONS

Xiaoyu Fu, Xu Liu, Qi Lü and Xu Zhang

Abstract. This paper is addressed to establishing an internal observability estimate for some linear stochastic hyperbolic equations. The key is to establish a new global Carleman estimate for forward stochastic hyperbolic equations in the $L^2$-space. Different from the deterministic case, a delicate analysis on the adaptedness for some stochastic processes is required in the stochastic setting.

Mathematics Subject Classification. 93B05, 93B07, 93C20.

Received June 6, 2016. Accepted June 7, 2016.

1. Introduction and main result

Let $T > 0$ and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space, on which a one-dimensional standard Brownian motion $\{B(t)\}_{t \geq 0}$ is defined such that $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by $B(\cdot)$, augmented by all the $\mathcal{P}$-null sets in $\mathcal{F}$. Let $\mathcal{H}$ be a Banach space, and let $C([0,T]; \mathcal{H})$ be the Banach space of all $\mathcal{H}$-valued strongly continuous functions defined on $[0,T]$. We denote by $L^2_\mathcal{F}(0,T; \mathcal{H})$ the Banach space consisting of all $\mathcal{H}$-valued $\mathcal{F}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(\|X(\cdot)\|_{L^2(0,T; \mathcal{H})}^2) < \infty$, with the canonical norm; by $L^\infty_\mathcal{F}(0,T; \mathcal{H})$ the Banach space consisting of all $\mathcal{H}$-valued $\mathcal{F}$-adapted essentially bounded processes; and by $L^2_{\mathcal{F}}(\Omega; C^m([0,T]; \mathcal{H}))$ for any positive integer $m$.

Let $G \subset \mathbb{R}^n$ (for some $n \in \mathbb{N}$) be a nonempty bounded domain with a $C^2$ boundary $\Gamma$. Set $Q = (0,T) \times G$ and $\Sigma = (0,T) \times \Gamma$. Assume that $b^{ij} \in C^2(\overline{G})$ ($i, j = 1, 2, \ldots, n$) satisfy

$$b^{ij}(x) = b^{ji}(x), \quad \forall x \in \overline{G},$$

Keywords and phrases. Stochastic hyperbolic equation, observability estimate, global Carleman estimate, adaptedness, optimal control.

* This work is partially supported by the NSF of China under grants 11231007, 11322110, 11771084 and 11471231, by the Fundamental Research Funds for the Central Universities under grants 212015JB011 and 215SCU04A02, by the Foundation for the Author of National Excellent Doctoral Dissertation of China under grant 201213, by the Program for New Century Excellent Talents in University under grant NCET-12-0812, by the Fok Ying Tong Education Foundation under grant 141001, and by the Chang Jiang Scholars Program from Chinese Education Ministry. The authors highly appreciate the anonymous referees for their constructive comments which led to this improved version.

1 School of Mathematics, Sichuan University, Chengdu 610064, P.R. China. xiaoyufu@scu.edu.cn; lu@scu.edu.cn; zhang_xu@scu.edu.cn

2 School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P.R. China. liux216@nenu.edu.cn

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and for some constant \( s_0 > 0 \),

\[
\sum_{i,j=1}^{n} b_{ij}(x)\xi_i \xi_j \geq s_0|\xi|^2, \quad \forall (x, \xi) = (x, \xi_1, \ldots, \xi_n) \in \overline{G} \times \mathbb{R}^n. \tag{1.2}
\]

Let us consider the following stochastic hyperbolic equation:

\[
\begin{aligned}
dy_t - \sum_{i,j=1}^{n} (b_{ij}y_{x_{i}})_{x_{j}} dt &= (b_1 y + f) dt + (b_2 y + g) dB(t) \quad \text{in } Q, \\
y(0) &= 0 \quad \text{on } \Sigma, \\
y(0) = y_0, \quad y_t(0) = y_1 \quad \text{in } G,
\end{aligned}
\tag{1.3}
\]

where \((y_0, y_1) \in L^2(G) \times H^{-1}(G), f, g \in L^2_{\mathbb{F}}(0, T; L^2(G)), b_1 \in L^\infty_{\mathbb{F}}(0, T; L^p(G))\) with \( p \in [n, \infty] \) and \( b_2 \in L^\infty_{\mathbb{F}}(0, T; L^\infty(G))\). \(\tag{1.4}\)

Also, set

\[
H_T = L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(G))) \bigcap L^2_{\mathbb{F}}(\Omega; C^1([0, T]; H^{-1}(G)))
\]

and

\[
\mathcal{H}_T = L^2_{\mathbb{F}}(\Omega; C^1([0, T]; L^2(G))) \bigcap L^2_{\mathbb{F}}(\Omega; C([0, T]; H^1(G))).
\]

Then \(H_T\) and \(\mathcal{H}_T\) are Banach spaces with the canonical norms. In this paper, we use the following notion of solution for equation (1.3).

**Definition 1.1.** A function \( y \in H_T \) is called a solution to equation (1.3), if the following conditions hold:

1. \( y(0) = y_0 \) and \( y_t(0) = y_1 \) in \( G, \mathcal{P}\)-a.s.
2. For any \( t \in (0, T) \) and \( \varphi \in C^2(\overline{G}) \cap C^1_0(G) \), it holds that

\[
\begin{align*}
\langle y(t), \varphi \rangle_{H^{-1}(G), H^1(G)} - \langle y(0), \varphi \rangle_{H^{-1}(G), H^1(G)} &= \\
&= \int_0^t \int_G \left\{ \sum_{i,j=1}^{n} \left[ b_{ij}(x)\varphi_{x_i}(x) \right]_{x_j} y(s, x) + \left[ b_1(s, x)y(s, x) + f(s, x) \right] \varphi(x) \right\} dxds \\
&\quad + \int_0^t \int_G \left[ b_2(s, x)y(s, x) + g(s, x) \right] \varphi(x) dxdB(s), \quad \mathcal{P}\text{-a.s.}
\end{align*}
\tag{1.5}
\]

For any initial value \((y_0, y_1) \in L^2(G) \times H^{-1}(G)\), it is easy to show that equation (1.3) admits a unique solution \( y \in H_T \).

Let \( \Gamma_0 \) be a part of the boundary of \( G \) satisfying certain conditions, which will be specified later. For any given constant \( \delta > 0 \), write

\[
\mathcal{O}_\delta(\Gamma_0) = \left\{ x \in G; \ \text{dist}(x, \Gamma_0) < \delta \right\}.
\]

Put

\[
\begin{align*}
r_1 &= |b_1|_{L^\infty_{\mathbb{F}}(0, T; L^p(G))} \quad \text{and} \quad r_2 = |b_2|_{L^\infty_{\mathbb{F}}(0, T; L^\infty(G))}.
\end{align*}
\tag{1.6}
\]

The main purpose of this paper is to establish the following inequality:

\[
\| (y_0, y_1) \|_{L^2(G) \times H^{-1}(G)} \leq Ce^{C(r_1^{1-p/r} + r_2^2)} \left[ \| y \|_{L^2_{\mathbb{F}}(0, T; L^2(\mathcal{O}_\delta(\Gamma_0)))} + |(f, g)|_{(L^2_{\mathbb{F}}(0, T; L^2(G)))}^2 \right],
\tag{1.7}
\]
where $y$ is the solution to (1.3) corresponding to any given initial value $(y_0, y_1)$. Here and henceforth, $C$ denotes a generic positive constant (which may be different from line to line), depending only on $G$, $T$, $I_0$, $b^{ij}$ ($i, j = 1, \ldots, n$), $\delta$, and $d(\cdot)$ and $\mu_0$ in Condition 1.2 (to be given later).

The inequality (1.7) is called an observability estimate for (1.3). For the case that $(f, g) = 0$ in (1.3), this inequality means that the initial energy of a solution in the time $t = 0$ can be bounded by its partial energy in the local observation domain $O_\delta(I_0)$ in the time duration $[0, T]$. Such kind of inequalities are closely related to control and state observation problems of deterministic/stochastic hyperbolic equations. For example, they can be applied to a study of the controllability (e.g. [1, 2, 4, 9, 15, 17]) and also inverse problems (e.g. [7, 8]) for deterministic hyperbolic equations. There exist numerous works devoted to observability estimates for deterministic hyperbolic equations. However, there are only a very few works addressed to similar problems but for stochastic hyperbolic equations [12, 13, 16].

Up to now, there are several methods to establish observability estimates for deterministic hyperbolic equations, such as the (Rellich-type) multiplier method [9], the non-harmonic Fourier series technique [14], the method of micro-local analysis [1] and the method of global Carleman estimate [15]. The multiplier method is only applicable to some very special hyperbolic equations. Indeed, even for a deterministic hyperbolic equation, it can be applied to a study of the controllability (e.g. [1, 2, 4, 9, 15, 17]) and also inverse problems (e.g. [7, 8]) for deterministic hyperbolic equations. There exist numerous works devoted to observability estimates for deterministic hyperbolic equations. However, there are only a very few works addressed to similar problems but for stochastic hyperbolic equations [12, 13, 16].

In [16], by means of a global Carleman estimate, a boundary observability estimate for equation (1.3) (with $(b^{ij})_{1 \leq i, j \leq n} = I_n$, the $n \times n$ identity matrix) was obtained:

$$\left| (y(T), y_t(T)) \right|_{L^2_T(\partial; H^1_0(G) \times L^2(G))} \leq C e^{C(r_1^2 + r_2^2)} \left( \left| \frac{\partial y}{\partial \nu} \right|_{L^2_T(0,T; L^2(\Gamma_0))} + \left| (f, g) \right|_{(L^2_T(0,T; L^2(G)))^2} \right),$$

(1.8)

where $y$ solved equation (1.3) associated to an initial data $(y_0, y_1) \in H^1_0(G) \times L^2(G)$, and $\nu = \nu(x) = (\nu^1, \nu^2, \ldots, \nu^n)$ denotes the unit outward normal vector of $G$ at $x \in \Gamma$. Also, in (1.8), $T$ was required to satisfy the condition:

$$\frac{4 + 5c}{9c} \min_{x \in \overline{G}} |x - x_0|^2 > e^2 T^2 > 4 \max_{x \in \overline{G}} |x - x_0|^2,$$

for some $c \in (0, 1)$ and $x_0 \in \mathbb{R}^n \setminus \overline{G}$.

In [12], by virtue of another global Carleman estimate, the result in [16] was improved to the following boundary observability inequality:

$$\left| (y_0, y_1) \right|_{H^1_0(G) \times L^2(G)} \leq C e^{C(r_1^2 + r_2^2 + r_3^2)} \left( \left| \frac{\partial y}{\partial \nu} \right|_{L^2_T(0,T; L^2(\Gamma_0))} + \left| (f, g) \right|_{(L^2_T(0,T; L^2(G)))^2} \right),$$

(1.9)

with $T > 2 \max_{x \in \overline{G}} |x - x_0|$ (for the case that $(b^{ij})_{1 \leq i, j \leq n} = I_n$). Notice that in (1.9), the power of $r_1$ is smaller than that in (1.8) (Indeed, $\frac{1}{3/2 - n/2} \leq 2$). Also, an internal observability estimate was established in [12]:

$$\left| (y_0, y_1) \right|_{H^1_0(G) \times L^2(G)} \leq C e^{C(r_1^2 + r_2^2 + r_3^2)} \left( \left| \nabla y \right|_{L^2_T(0,T; L^2(G_0(I_0)))} + \left| (f, g) \right|_{(L^2_T(0,T; L^2(G)))^2} \right).$$

(1.10)

The main difference between (1.7) and (1.10) is that the inequality (1.10) provides an observability estimate of the $H^1$-norm for solutions to equation (1.3), but the inequality (1.7) is an estimate of the $L^2$-norm type. Compared with the known inequality (1.10), the estimate (1.7) has more applications. For example, one application of (1.7) is the stabilization of stochastic hyperbolic equations (but the detailed analysis is beyond the
Remark 1.3. To be constructed to guarantee the adaptedness of the related stochastic processes. As we shall see later, a suitable auxiliary optimal control problem (different from the deterministic context) has some new difficulties. Actually, as we shall see later, a suitable auxiliary optimal control problem (different from the deterministic context) has to be constructed to guarantee the adaptedness of the related stochastic processes.

Before giving our main result, let us first introduce some assumptions on \((b^{ij})_{1 \leq i, j \leq n}(i, j = 1, \ldots, n)\) and \(T\).

**Condition 1.2.** There exists a positive function \(d(\cdot) \in C^2(\mathcal{G})\) with the property that \(\min_{x \in \mathcal{G}} |\nabla d(x)| > 0\) such that, for some constant \(\mu_0 > 0\), the following compatibility condition is satisfied:

\[
\sum_{i,j=1}^{n} \sum_{i',j'=1}^{n} \left[ 2b^{ij'}(b^{i'j} dx_{i'})_{x_{j'}} - b^{i'j'} b^{i'j} dx_{i'} \right] \xi^i \xi^j \geq \mu_0 \sum_{i,j=1}^{n} b^{ij} \xi^i \xi^j, \quad \forall (x, \xi^1, \ldots, \xi^n) \in \mathcal{G} \times \mathbb{R}^n.
\]

In the sequel, we shall choose the set \(\Gamma_0\) as follows:

\[
\Gamma_0 = \left\{ x \in \Gamma : \sum_{i,j=1}^{n} b^{ij}(x) d_{x_i}(x) \nu^j(x) > 0 \right\}, \tag{1.11}
\]

where the function \(d(\cdot)\) is given in Condition 1.2. Also, write

\[
G_0 = \mathcal{O}_5(\Gamma_0) \quad \text{and} \quad \Sigma_0 = \Gamma_0 \times (0, T). \tag{1.12}
\]

**Remark 1.3.** Notice that Condition 1.2 is a sufficient condition for establishing Carleman estimates for deterministic linear hyperbolic operators \(\partial^2_t - \sum_{i,j=1}^{n} \partial_{x_i}(b^{ij} \partial_{x_j})\). If \((b^{ij})_{1 \leq i, j \leq n} = I_n\), then \(d(x) = |x - x_0|^2\) satisfies Condition 1.2 with \(x_0\) being any given point in \(\mathbb{R}^n \setminus \mathcal{G}\). On the other hand, Condition 1.2 can also be regarded as a special case of the pseudo-convexity condition in [5]. In fact, for the wave operator \(\partial^2_t - \Delta\), if we set \(a(x, \xi) = |\xi|^2\) and \(d(x) = |x - x_0|^2\), then it is easy to check that

\[
\{a, \{a, d\}\}(x, \xi) = 4|\xi|^2 > 0, \quad \forall (x, \xi) \in \mathcal{G} \times (\mathbb{R}^n \setminus \{0\}),
\]

where \(\{a, d\}\) denotes the Poisson bracket of \(a\) and \(d\), i.e.,

\[
\{a, d\}(x, \xi) = \sum_{j=1}^{n} \left( \frac{\partial a}{\partial \xi_j} \frac{\partial d}{\partial x_j} - \frac{\partial a}{\partial x_j} \frac{\partial d}{\partial \xi_j} \right), \quad \forall (x, \xi) \in \mathcal{G} \times \mathbb{R}^n \setminus \{0\}.
\]

Moreover, it is easy to see that there is no critical point of the function \(d(\cdot)\) in \(\mathcal{G}\).

**Remark 1.4.** In [4], Condition 1.2 was used to establish an internal observability estimate for deterministic hyperbolic equations. We refer to [4,11] for more explanations on Condition 1.2 and some interesting nontrivial examples. Also, as ([11], Example 1.1) shows, there exists some example for which Condition 1.2 fails.

In what follows, set

\[
M_0 = \min_{x \in \mathcal{G}} \sum_{i,j=1}^{n} b^{ij} dx_i dx_j, \quad M_1 = \max_{x \in \mathcal{G}} \sum_{i,j=1}^{n} b^{ij} dx_i dx_j, \quad \text{and} \quad d_0 = \max_{x \in \mathcal{G}} \sum_{i,j=1}^{n} b^{ij} dx_i \nu^j, \tag{1.13}
\]
and define\footnote{From the proof of ([4], Thm. 5.1), it is easy to see that the number $d_0$ defined in (1.13) is positive. Hence, the set $T_0$ given by (1.11) is not empty.}

$$T_0 = \max \left\{ 2\sqrt{M_1}, 1 + \frac{24\sqrt{m_0}}{\min\{1, s_0\}}, \left(1 + \frac{1}{s_0^{3/2}} \sum_{i,j=1}^{n} |b_{ij}|_{C(\Gamma)} + \frac{1}{s_0^{3/2}} \right)^{1/2} \right\} ,$$

(1.14)

where $s_0$ is the constant appeared in (1.2).

**Remark 1.5.** It is easy to check that if $d(\cdot)$ satisfies Condition 1.2, then for any $a \geq 1$ and $b \in \mathbb{R}$, the function $\tilde{d}(\cdot) = ad(\cdot) + b$ still satisfies this condition when $\mu_0$ is replaced by $a\mu_0$. Therefore, throughout this paper, we may assume that $d(\cdot)$ and $\mu_0$ satisfy that

$$\mu_0 > \frac{9T_0^2}{M_0} \quad \text{and} \quad M_0 \geq \max_{x \in \Gamma} d(x).$$

(1.15)

The main result of this paper is stated as follows.

**Theorem 1.6.** Assume that Condition 1.2 holds. Then, for any $T > T_0$ (defined by (1.14)), the observability inequality (1.7) holds for any solution to equation (1.3).

**Remark 1.7.** The restriction on $T$ in Theorem 1.6 is a technical condition, and $T_0$ is not sharp. However, this condition plays a key role in our Proof of Theorem 1.6. It is reasonable to expect that it can be improved to a better one as that in [12] (for the estimates (1.9) and (1.10)), but this is an unsolved problem.

**Remark 1.8.** The condition (1.15) is relevant to the interior behavior/property of the diffusion, and it will play a key role in the estimates on the energy terms (see (2.13)–(2.15) in the proof of Thm. 3.1). On the other hand, the assumption on the time $T$ in Theorem 1.6 is relevant to the diffusion/reflection on the boundary. This assumption will play a key role in the estimates on the boundary term (see Step 4 in the proof of Thm. 3.1). If one considers a special case, i.e., $(b_{ij})_{1 \leq i,j \leq n} = I_n$, then $s_0 = 1$ and we take $d(x) = |x - x_0|^2$, the corresponding condition on $T$ is the following:

$$T > T_0 = \max \left\{ 4 \max_{x \in \Gamma} |x - x_0|, \ 1 + 48\sqrt{n}(n + 2) \max_{x \in \Gamma} [(x - x_0) \cdot \nu(x)] \right\}.$$ 

The rest of this paper is organized as follows. In Section 2, we present a key weighted identity for partial differential operators of second order with symmetric coefficients. Section 3 is devoted to establishing a Carleman estimate for deterministic hyperbolic equations in the $H^1$-space. In Section 4, an auxiliary optimal control problem is introduced and analyzed. In Section 5, a global Carleman estimate for stochastic hyperbolic equations in the $L^2$-space is derived. In Section 6, energy estimates for random hyperbolic equations and backward stochastic hyperbolic equations are given. Section 7 is devoted to a proof of our main result (i.e., Thm. 1.6). Finally, in Appendices A and B, we give the proofs of some technical results.

## 2. A WEIGHTED IDENTITY FOR PARTIAL DIFFERENTIAL OPERATORS OF SECOND ORDER WITH SYMMETRIC COEFFICIENTS

In this section, we show a pointwise weighted identity for partial differential operators of second order with symmetric coefficients, which will play a crucial role in the sequel.
Lemma 2.1. Assume that $a^{ij} = a^{ji} \in C^2(\mathbb{R}^m)$ $(i, j = 1, 2, \ldots, m)$ for some $m \in \mathbb{N}$, $u, \ell \in C^2(\mathbb{R}^m)$ and $\Psi \in C^1(\mathbb{R}^m)$. Set $\theta = e^{t}$ and $v = \theta u$. Then

$$\theta^2 \left| \sum_{i,j=1}^{m} (a^{ij} u_{x_i})_{x_j} \right|^2 + 2 \sum_{j=1}^{m} \left[ \sum_{i,i',j'=1}^{m} \left( 2a^{ij} a^{i'j'} \ell_{x_i}v_{x_i}v_{x_j} - a^{ij} a^{i'j'} \ell_{x_i}v_{x_j}v_{x_j} \right) + \Psi \sum_{i=1}^{m} a^{ij} v_i v - \Lambda \sum_{i=1}^{m} a^{ij} \ell_{x_i} v^2 \right]_{x_j}$$

$$\geq 2 \sum_{i,j=1}^{m} \left[ \sum_{i',j'=1}^{m} \left( 2a^{ij} (a^{i'j'} \ell_{x_i})_{x_j} - (a^{ij} a^{i'j'} \ell_{x_i})_{x_j} \right) + \Psi a^{ij} \right] v_{x_i} v_{x_j} + 2 \sum_{i,j=1}^{m} a^{ij} \Psi_{x_i} v v_{x_j} + B v^2,$n

where

$$A = - \sum_{i,j=1}^{m} (a^{ij} \ell_{x_i} \ell_{x_j} - a^{ij} \ell_{x_i} - a^{ij} \ell_{x_i} x_j) - \Psi \quad \text{and} \quad B = 2 A \Psi - 2 \sum_{i,j=1}^{m} (A a^{ij} \ell_{x_i})_{x_j}.$$

(2.1)

Proof of Lemma 2.1. Recalling $\theta = e^t$ and $v = \theta u$, we see that $\theta u_{x_i} = v_{x_i} - \ell_{x_i} v$ $(i = 1, 2, \ldots, m)$. Proceeding exactly as ([4], Thm. 4.1), we obtain that

$$-\theta \sum_{i,j=1}^{m} (a^{ij} u_{x_i})_{x_j} = I_1 + I_2,$$

where

$$I_1 = - \sum_{i,j=1}^{m} (a^{ij} v_{x_i})_{x_j} + A v \quad \text{and} \quad I_2 = 2 \sum_{i,j=1}^{m} a^{ij} \ell_{x_i} v_{x_j} + \Psi v.$$

(2.3)

This implies that

$$\theta^2 \left| \sum_{i,j=1}^{m} (a^{ij} u_{x_i})_{x_j} \right|^2 = |I_1|^2 + 2 I_1 I_2 + |I_2|^2 \geq 2 I_1 I_2.$$n

(2.4)

By virtue of ([4], Eq. (4.8)), a simple calculation shows that

$$I_1 I_2 = 2 \sum_{i,j=1}^{m} a^{ij} \ell_{x_i} v_{x_j} \left( - \sum_{i,j=1}^{m} (a^{ij} v_{x_i})_{x_j} + A v \right) + \Psi v \left( - \sum_{i,j=1}^{m} (a^{ij} v_{x_i})_{x_j} + A v \right)$$

$$= - \sum_{j=1}^{m} \left( 2 \sum_{i,i',j'=1}^{m} a^{ij} a^{i'j'} \ell_{x_i} v_{x_i} v_{x_j} - \sum_{i,i',j'=1}^{m} a^{ij} a^{i'j'} \ell_{x_i} v_{x_i} v_{x_j} - \Lambda \sum_{i=1}^{m} a^{ij} \ell_{x_i} v^2 \right)_{x_j}$$

$$+ \sum_{i,j,i',j'=1}^{m} \left( 2a^{ij} (a^{i'j'} \ell_{x_i})_{x_j} - (a^{ij} a^{i'j'} \ell_{x_i})_{x_j} \right) v_{x_i} v_{x_j} - \sum_{i,j=1}^{m} (A a^{ij} \ell_{x_i})_{x_j} v^2$$

$$- \sum_{i,j=1}^{m} (\Psi a^{ij} v v_{x_i})_{x_j} + \Psi \sum_{i,j=1}^{m} a^{ij} v v_{x_j} + \sum_{i,j=1}^{m} a^{ij} \Psi_{x_i} v v_{x_j} + B v^2.$$n

(2.5)

Combining (2.5) with (2.4), we obtain the desired inequality (2.1).
3. A Carleman estimate for deterministic hyperbolic equations in the $H^1$-norm

This section is addressed to deriving a Carleman estimate for the following (deterministic) hyperbolic equation:

\[
\begin{cases}
    u_{tt} - \sum_{i,j=1}^{n} (b^{ij} u_{x_i})_{x_j} = F \quad \text{in } Q, \\
    u = 0 \quad \text{on } \Sigma, \\
    u(0) = u_0, \quad u_t(0) = u_1 \quad \text{in } G,
\end{cases}
\]  

(3.1)

where $(u_0, u_1) \in H^1_0(G) \times L^2(G)$, $F \in L^2(Q)$, and $b^{ij} (i, j = 1, \ldots, n)$ satisfy (1.1), (1.2), and the Condition 1.2.

As in Theorem 1.6, we assume that $T > T_0$ (defined in (1.14)), and $G_0$, $\mu_0$ and $d(\cdot)$ are given in (1.12) and Condition 1.2. By (1.15), it is easy to see that $T_0 < \sqrt{\mu_0 M_0} 3 \sqrt{T}$. Hence, we can choose a constant $c_1 \in (T_0/T, \min\{1, \sqrt{\mu_0 M_0} 3 \sqrt{T}\})$. Now, for any given constant $c_0 \in (0, 1)$ and parameter $\lambda > 0$, we choose the weight function $\theta$ and the auxiliary function $\Psi$ (appeared in Lem. 2.1) as follows:

\[
\begin{cases}
    \theta(t, x) = e^{\ell(t, x)}, \\
    \ell(t, x) = \lambda \phi(t, x), \\
    \phi(t, x) = d(x) - c_1 (t - T/2)^2, \\
    \Psi(x) = \lambda \left( \sum_{i,j=1}^{n} (b^{ij} d_{x_i})_{x_j} - 2c_1 - c_0 \right).
\end{cases}
\]  

(3.2)

We have the following global Carleman estimate for equation (3.1).

**Theorem 3.1.** Assume that Condition 1.2 holds. Then, there is a positive constant $\lambda_0$, such that for any $T > T_0$ and $\lambda \geq \lambda_0$, any solution $u$ to (3.1) satisfies that

\[
\int_Q e^{2\lambda \phi} \left[ \lambda (u_t^2 + |\nabla u|^2) + \lambda^2 u^2 \right] \, dx dt \leq C \left[ \int_Q e^{2\lambda \phi} F^2 \, dx dt + \lambda^2 \int_0^T \int_{G_0} e^{2\lambda \phi} (u_t^2 + \lambda^2 u^2) \, dx dt \right].
\]  

(3.3)

**Remark 3.2.** Notice that Theorem 3.1 is an improvement of ([4], Thm. 5.1). Indeed, in ([4], Thm. 5.1), the global Carleman estimate (3.3) for a deterministic hyperbolic equation was established under the additional condition that $u(0) = u(T) = 0$ in $G$. However, this condition seems too strong to be satisfied in applications (see, for example Eq. (7.5) in [4]). Therefore, it is necessary to establish the global Carleman estimate (3.3) without this restriction.

In the rest of this section, we give a Proof of Theorem 3.1.

**Proof of Theorem 3.1.** The proof is long and therefore we divide it into four steps.

**Step 1. A pointwise inequality for hyperbolic operators.** In Lemma 2.1, we choose $m = n + 1$ and $(a^{ij})_{m \times m} = \left( \begin{array}{cc} -1 & 0 \\ 0 & (b^{ij})_{n \times n} \end{array} \right)$, and $\theta$, $\ell$, $\phi$ and $\Psi \equiv \Psi(x)$ being given as in (3.2). Then, by a simple calculation, we have the following weighted inequality for the hyperbolic operator, which is very similar to ([4], Cor. 4.2),
except some different lower order terms.

\[
e^{2\lambda_0}u_{tt} - \sum_{i,j=1}^{n} (b^{ij}u_{x_i})_{x_j}^2 + 2\sum_{j=1}^{n} \left( 2\sum_{i,i',j'=1}^{n} b^{ij}b^{i'j'} \ell_{x_i',x_j'}v_{x_i'}v_{x_j'} - \sum_{i=1}^{n} b^{ij}\Lambda \ell_{x_i}v^2 \right)
- \sum_{i,i',j'=1}^{n} b^{ij}b^{i'j'} \ell_{x_i}v_{x_i'}v_{x_j'} + \Psi v_0^2 - \Psi v_0 + \sum_{i=1}^{n} b^{ij}\ell_{x_i}v_{x_i} + 2M_t + 2\sum_{i,j=1}^{n} b^{ij}\psi_{x_i}v_{x_i} + Bv^2

\geq 2\left[ \ell_{tt} + \sum_{i,j=1}^{n} (b^{ij}\ell_{x_i})_{x_j} - \Psi \right] v^2 - 8\sum_{i,j=1}^{n} b^{ij}\ell_{x_i}v_{x_i}v_t + 2\sum_{i,j=1}^{n} b^{ij}\psi_{x_i}v_{x_i} + 2M_t
+ 2\sum_{i,j=1}^{n} \left[ b^{ij}\ell_{tt} - \sum_{i',j'=1}^{n} \left( 2b^{ij}(b^{i'j'}\ell_{x_{i'}})_{x_j'} - (b^{ij}b^{i'j'}\ell_{x_{i'}})_{x_j'} + \Psi b^{ij} \right) v_{x_{i'}}v_{x_{i'}} \right] v_{x_i}v_{x_j},
\]

where

\[
\begin{align*}
A &= \ell_t^2 - \ell_{tt} - \sum_{i,j=1}^{n} (b^{ij}\ell_{x_i}x_j - b^{ij}\ell_{x_i}x_i - b^{ij}\ell_{x_i}x_j) - \Psi, \\
M &= \ell_t \left( v^2 + \sum_{i,j=1}^{n} b^{ij}v_{x_i}v_{x_j} \right) - 2\sum_{i,j=1}^{n} b^{ij}\ell_{x_i}v_{x_i}v_t - \Psi v_0 + \Lambda \ell_t v^2, \\
B &= 2\left[ A\psi + (M_t)_t - \sum_{i,j=1}^{n} (b^{ij}\ell_{x_i})_{x_j} \right].
\end{align*}
\]

Step 2. Estimates on “the energy terms”. First, by the definitions of \( \Psi \) and \( \ell \), it is easy to show that

\[
2\left[ \ell_{tt} + \sum_{i,j=1}^{n} (b^{ij}\ell_{x_i})_{x_j} - \Psi \right] = -4\lambda_0 - 2\sum_{i,j=1}^{n} (\lambda b^{ij}d_{x_i})_{x_j} - 2\Psi = 2\lambda_0.
\]

Further,

\[
2\sum_{i,j=1}^{n} \left[ b^{ij}\ell_{tt} + \sum_{i',j'=1}^{n} \left[ 2b^{ij}(b^{i'j'}\ell_{x_{i'}})_{x_j'} - (b^{ij}b^{i'j'}\ell_{x_{i'}})_{x_j'} + \Psi b^{ij} \right] v_{x_{i'}}v_{x_{i'}} \right] v_{x_i}v_{x_j}
= 2\sum_{i,j=1}^{n} \left[ -2\lambda_0 b^{ij} + \sum_{i',j'=1}^{n} \lambda b^{ij}b^{i'j'}d_{x_{i'}}d_{x_{j'}} - 2\lambda_0 b^{ij} - c_0 b^{ij} \\
+ 2\lambda \sum_{i',j'=1}^{n} b^{ij}(b^{i'j'}d_{x_{i'}}d_{x_{j'}} - \lambda \sum_{i',j'=1}^{n} (b^{ij}b^{i'j'}d_{x_{i'}}d_{x_{j'}})_{x_j'} v_{x_i}v_{x_j} \right]
\geq 2\lambda_0 \sum_{i,j=1}^{n} b^{ij}v_{x_i}v_{x_j} - (8c_1 + c_0)\lambda \sum_{i,j=1}^{n} b^{ij}v_{x_i}v_{x_j} = 2\lambda(\mu_0 - 4c_1 - c_0) \sum_{i,j=1}^{n} b^{ij}v_{x_i}v_{x_j}.
\]

Further, by (3.2) and (3.5), we obtain that

\[
A = \ell_t^2 - \ell_{tt} - \sum_{i,j=1}^{n} (b^{ij}\ell_{x_i}x_j - b^{ij}\ell_{x_i}x_i - b^{ij}\ell_{x_i}x_j) - \Psi = \lambda^2 \left[ c_0^2(2t - T)^2 - \sum_{i,j=1}^{n} b^{ij}d_{x_i}d_{x_j} \right] + O(\lambda).
\]

Hence,

\[
B = 2\lambda^3 \left[ (4c_1 + c_0) \sum_{i,j=1}^{n} b^{ij}d_{x_i}d_{x_j} + \sum_{i,j=1}^{n} b^{ij}d_{x_i} \left( \sum_{i',j'=1}^{n} b^{i'j'}d_{x_{i'}}d_{x_{j'}} \right)_{x_j} \right] (8c_1 + c_0)c_0^2(2t - T)^2 + O(\lambda^2).
\]
Proceeding the same analysis as (11.6)–(11.8) in [4], we have that
\[
\sum_{i,j=1}^{n} \sum_{i',j'=1}^{n} b^{i}d_{x_i}(b^{i'}d_{x_{i'}},d_{x_{j'}}) \geq \mu_0 \sum_{i,j=1}^{n} b^{i}d_{x_i}d_{x_j}.
\] (3.9)

By (3.8), (3.9) and (1.15), noticing that \(c_1 < \frac{\sqrt{\mu_0}M_0}{4}\), we find that for any \(T > T_0\),
\[
B \geq 2\lambda^3(4c_1 + c_0) \sum_{i,j=1}^{n} b^{i}d_{x_i}d_{x_j} + O(\lambda^2).
\] (3.10)

On the other hand, by (1.15), it is easy to see that
\[
\frac{\mu_0 M_0}{8c_1 + c_0} > \frac{\mu_0 M_0}{9} > T_0^2 \geq 4M_1 \geq 4M_0,
\]
which implies that
\[
\mu_0 - 4c_1 - c_0 > \mu_0 - 32c_1 - 4c_0 > 0.
\] (3.11)

Therefore, combining (3.6), (3.7) (3.10) and (3.11) with (3.4), we conclude that for any \(T > T_0\), there is a \(\lambda_1 > 0\) and \(c^* > 0\), such that for any \(\lambda \geq \lambda_1\),
\[
2 \left[ \ell_{tt} + \sum_{i,j=1}^{n} (b^{i}\ell_{x_i})x_j - \Psi \right] v_t^2 - 8 \sum_{i,j=1}^{n} b^{i}\ell_{x_i}v_x,v_t + 2 \sum_{i,j=1}^{n} b^{i}\Psi_{x_i}v_x + Bu^2
\]
\[
+ 2 \sum_{i,j=1}^{n} \left\{ b^{i}\ell_{tt} + \sum_{i',j'=1}^{n} \left[ 2b^{i'}(b^{i'}\ell_{x_{i'}})x_{j'} - (b^{i'}b^{i''}\ell_{x_{i'}})x_{j'} \right] + \Psi b^{i'} \right\} v_x,v_x
\]
\[
\geq c^*\lambda(v_t^2 + |\nabla v|^2 + \lambda^2 v^2).
\] (3.12)

Integrating (3.12) in \(Q\) and noting that \(v = e^{\lambda\phi}u\) on \(\Sigma\), by (3.1) and (3.4), we obtain that
\[
e^\lambda \int_{Q} (v_t^2 + |\nabla v|^2 + \lambda^2 v^2)dx dt \leq \int_{Q} e^{2\lambda\phi}|F|^2 dx dt + 2 \int_{Q} M_t dx dt + 2\lambda_0 \int_{\Sigma} \sum_{i,j=1}^{n} b^{i}\nu\nu^j e^{2\lambda\phi} \frac{\partial u}{\partial \nu}^2 d\Sigma.
\] (3.13)

Here we use the following identity:
\[
\int_{\Sigma} \sum_{i,j=1}^{n} \left\{ b^{i}\nu^j \ell_{x_{i'}}v_{x_{j'}},v_{x_{i'}} - b^{i'}\nu^j \ell_{x_{i'}}v_{x_{j'}} \right\} \cdot \nu^j d\Sigma = \lambda \int_{\Sigma} \sum_{i,j=1}^{n} b^{i}\nu\nu^j \sum_{i',j'=1}^{n} b^{i'}d_{x_{i'}}v_{x_{j'}} \frac{\partial u}{\partial \nu}^2 d\Sigma.
\]

**Step 3. Estimates on “the spatial boundary term”**. Let us estimate the last term of (3.13). Similar to the proof of (11.15) in [4], we choose functions \(h_0 \in C^1(\partial G;[0,1]^n)\) and \(\rho \in C^2(\partial G;[0,1])\), such that \(h_0 = \nu\) on \(\Gamma\), and for the same \(\delta\) appeared in (1.12),
\[
\rho(x) \equiv 1, \quad x \in \mathcal{O}_{\delta/3}(\Gamma_0) \cap G,
\]
\[
\rho(x) \equiv 0, \quad x \in G \setminus \mathcal{O}_{\delta/2}(\Gamma_0).
\]

Let \(h = h_0pe^{2\lambda\phi}\). Then, by ([4], Lem. 3.2) (with \(g\) and \(a^{ij}\) replaced by \(h\) and \(b^{ij}\), respectively),
\[
\int_{\Sigma} \sum_{i,j=1}^{n} b^{i}\nu\nu^j e^{2\lambda\phi} \frac{\partial u}{\partial \nu}^2 dx dt = -\int_{Q} \left\{ 2 \left[ Fh \cdot \nabla u - (u_h \cdot \nabla u_t) + u_t \cdot \nabla u - (\nabla \cdot h)u^2_t \right] \right.
\]
\[
- \sum_{i,j,k=1}^{n} b^{i}u_{x_i}u_{x_k} \frac{\partial h^k}{\partial x_j} + \sum_{i,j=1}^{n} u_{x_i}u_{x_j} \nabla \cdot (b^{ij}h) \right\} dx dt.
\]
Therefore, it is easy to check that

\[
\int_0^T \sum_{i,j=1}^n b^{ij} \nu^i \nu^j \rho e^{2\lambda \phi} \left| \frac{\partial u}{\partial \nu} \right|^2 \, dx \, dt \leq \frac{C}{\lambda} \int_Q e^{2\lambda \phi} F^2 \, dx \, dt + \lambda \int_0^T \int_{\mathcal{O}_{\delta/2}(\Gamma_0) \cap G} e^{2\lambda \phi} |\nabla u|^2 \, dx \, dt + 2 \int_G \partial_t \mathbf{u} \cdot \nabla \mathbf{u} \bigg|_0^T \\
+ 4\sqrt{n} \lambda \int_0^T \int_{\mathcal{O}_{\delta/2}(\Gamma_0) \cap G} e^{2\lambda \phi} c_1 T |u_t| |\nabla u| \, dx \, dt + C \lambda \int_0^T \int_{\mathcal{O}_{\delta/2}(\Gamma_0) \cap G} e^{2\lambda \phi} u_t^2 \, dx \, dt \\
+ \left(6\sqrt{n} \lambda |\nabla d| \sum_{i,j=1}^n |b^{ij}|_{C(\mathcal{O})} + C\right) \int_0^T \int_{\mathcal{O}_{\delta/2}(\Gamma_0) \cap G} e^{2\lambda \phi} |\nabla u|^2 \, dx \, dt
\]

By \((3.13)\) and the above inequality, we get that

\[
e^s \lambda \int_Q (u_t^2 + |\nabla v|^2 + \lambda^2 v^2) \, dx \, dt \\
\leq C \left(\int_Q e^{2\lambda \phi} F^2 \, dx \, dt + \lambda \int_0^T \int_{G_0} e^{2\lambda \phi} u_t^2 \, dx \, dt\right) + 4\lambda d_0 \int_G u_t h \cdot \nabla u \bigg|_0^T + 2 \int_G \tilde{M} \, dx \bigg|_0^T \\
+ 12\sqrt{n} \lambda d_0 \left(|\nabla d| \sum_{i,j=1}^n |b^{ij}|_{C(\mathcal{O})} + 1\right) \int_0^T \int_{\mathcal{O}_{\delta/2}(\Gamma_0) \cap G} e^{2\lambda \phi} |\nabla u|^2 \, dx \, dt,
\]

where \(\tilde{M} = M + 2\lambda d_0 u_t h \cdot \nabla u\).

Next, let us estimate the last term in \((3.14)\). Put \(\eta(t, x) = \rho_1^2 e^{2\lambda \phi}\), where \(\rho_1 \in C^2(\mathcal{O}; [0, 1])\) satisfies that

\[
\begin{align*}
\rho_1(x) &\equiv 1, \quad x \in \mathcal{O}_{\delta/2}(\Gamma_0) \cap G, \\
\rho_1(x) &\equiv 0, \quad x \in G \setminus G_0.
\end{align*}
\]

By \((3.1)\), we have that

\[
\int_Q \eta u F \, dx \, dt = \int_Q \eta u \left(\partial_t \mathbf{u} - \sum_{i,j=1}^n (b^{ij} u_x)_x\right) \, dx \, dt \\
= \int_Q (\eta u_t) \, dx \, dt - \int_Q u_t (\eta u + \eta u_t) \, dx \, dt + \int_Q \eta \sum_{i,j=1}^n b^{ij} u_x u_x \, dx \, dt + \int_Q u \sum_{i,j=1}^n b^{ij} u_x \eta_x \, dx \, dt.
\]

This implies that

\[
\int_0^T \int_{\mathcal{O}_{\delta/2}(\Gamma_0) \cap G} e^{2\lambda \phi} |\nabla u|^2 \, dx \, dt \leq C \left[\frac{1}{\lambda^2} \int_Q e^{2\lambda \phi} |F|^2 \, dx \, dt + \int_0^T \int_{G_0} e^{2\lambda \phi} (\lambda^2 u_t^2 + u_t^2) \, dx \, dt\right] - \frac{1}{s_0} \int_Q (\eta u_t) \, dx \, dt.
\]
Combining (3.14) with (3.15), we end up with
\[ C^\lambda \int_Q (v_t^2 + |\nabla v|^2 + \lambda^2 v^2)\,dx\,dt \leq C \int_Q e^{2\lambda t} |F|^2\,dx\,dt + \int_G \overline{M} dx \bigg|_0^T + C \lambda^2 \int_{G_0}^T \int_G e^{2\lambda t} (u_t^2 + \lambda^2 u^2)\,dx\,dt, \]
where
\[ \overline{M} = M + 2d_0 \lambda u_t h \cdot \nabla u - \frac{1}{s_0} \left[ 12 \sqrt{n} \lambda^2 d_0 \left( |\nabla d| \sum_{i,j=1}^n |b_{ij}|_{C(\overline{G})} + 1 \right) \right] \eta u_t. \]

**Step 4. Estimates on “the time boundary term”**. Let us estimate \( \overline{M}(0, x) \) and \( \overline{M}(T, x) \), respectively. By (1.15) and the definition of \( M \) in (3.5), we have that
\[
M(0, x) \geq \left[ \lambda c_1 T \left( v_t^2 + \sum_{i,j=1}^n b_{ij} v_{x_i} v_{x_j} \right) - 2\lambda \sum_{i,j=1}^n b_{ij} d_{x_i} v_{x_j} v_t \right]_{t=0} + \left[ O(\lambda^2) v^2 - v_t^2 + c_1 T \lambda^3 \left( c_1^2 T^2 - \sum_{i,j=1}^n b_{ij} d_{x_i} d_{x_j} \right) v^2 \right]_{t=0} \geq \lambda \left[ c_1 T - \left( \sum_{i,j=1}^n b_{ij} d_{x_i} d_{x_j} \right) \right] \left( v_t^2 + \sum_{i,j=1}^n b_{ij} v_{x_i} v_{x_j} \right)_{t=0} + \left[ O(\lambda^2) v^2 - v_t^2 + c_1 T \lambda^3 \left( c_1^2 T^2 - \sum_{i,j=1}^n b_{ij} d_{x_i} d_{x_j} \right) v^2 \right]_{t=0}.
\]
Noting that by (1.14) and \( c_1 > T_0/T \), we have
\[ c_1 T > 2\sqrt{M_1} \geq 2 \left( \sum_{i,j=1}^n b_{ij} d_{x_i} d_{x_j} \right)^{\frac{1}{2}} \geq 2\sqrt{s_0} |\nabla d|. \]
This implies that
\[ M(0, x) \geq \left[ \frac{1}{2} \lambda c_1 T \min\{1, s_0\} (v_t^2 + |\nabla v|^2) + \frac{3}{4} \lambda^2 c_1^3 T^3 v^2 + O(\lambda^2) v^2 - v_t^2 \right]_{t=0}. \]

Further,
\[
2d_0 \lambda u_t h \cdot \nabla u \bigg|_{t=0} = -\sqrt{n} d_0 \lambda e^{2\lambda t} (u_t^2 + |\nabla u|^2) \bigg|_{t=0} \geq -2\sqrt{n} d_0 \lambda \left( v_t^2 + \lambda^2 c_1^2 T^2 v^2 + |\nabla v|^2 + \lambda^2 |\nabla d|^2 v^2 \right) \bigg|_{t=0}.
\]

On the other hand,
\[
-\frac{1}{s_0} \left[ 12 \sqrt{n} \lambda^2 d_0 \left( |\nabla d| \sum_{i,j=1}^n |b_{ij}|_{C(\overline{G})} + 1 \right) \right] \eta u_t = -\frac{1}{s_0} \left[ 12 \sqrt{n} \lambda^2 d_0 \left( |\nabla d| \sum_{i,j=1}^n |b_{ij}|_{C(\overline{G})} + 1 \right) \right] \rho_t^2 \left( v_{tt} - \lambda c_1 T v^2 - \frac{v_t^2}{\lambda c_1 T} + \frac{v_t^2}{\lambda c_1 T} \right) \bigg|_{t=0} \geq -\frac{1}{s_0 c_1 T} \lambda \left[ 12 \sqrt{n} d_0 \left( |\nabla d| \sum_{i,j=1}^n |b_{ij}|_{C(\overline{G})} + 1 \right) \right] v_t^2 \bigg|_{t=0}.
\]
Therefore, by (3.18)–(3.20), we get that

\[
\overline{M}(0, x) \geq (\lambda F_{1} v_{t}^{2} + \lambda F_{2} |v|^{2} + \lambda^{3} F_{3} v^{2} + O(\lambda^{2}) v^{2} + O(1) v^{2}) \Big|_{t=0},
\]

where

\[
\begin{align*}
F_{1} &= \frac{1}{2} c_{1} T \min\{1, s_{0}\} - 2 \sqrt{n} d_{0} - \frac{1}{s_{0} c_{1} T} \left[ 12 \sqrt{n} d_{0} \left( |\nabla d| \sum_{i,j=1}^{n} |b^{ij}|_{C(\overline{G})} + 1 \right) \right], \\
F_{2} &= \frac{1}{2} c_{1} T \min\{1, s_{0}\} - 2 \sqrt{n} d_{0}, \quad F_{3} = \frac{3}{4} c_{1}^{2} T^{3} - 2 \sqrt{n} d_{0} (c_{1}^{2} T^{2} + |\nabla d|^{2}).
\end{align*}
\]  

By (3.17) and (1.14), for any \( T > T_{0} \), it holds that \( c_{1} T > 1 \),

\[
F_{1} \geq \frac{1}{2} c_{1} T \min\{1, s_{0}\} - 2 \sqrt{n} d_{0} - \frac{6 \sqrt{n} d_{0}}{s_{0}^{3/2}} \sum_{i,j=1}^{n} |b^{ij}|_{C(\overline{G})} - \frac{12 \sqrt{n} d_{0}}{s_{0}} > 0,
\]

and therefore, \( F_{2} > 0 \). Moreover,

\[
F_{3} \geq \frac{3}{4} c_{1}^{2} T^{3} - 2 \sqrt{n} d_{0} c_{1}^{2} T^{2} \left( 1 + \frac{1}{4 s_{0}} \right) = \frac{3}{4} c_{1}^{2} T^{2} \left[ c_{1} T - \frac{8}{3} \sqrt{n} d_{0} \left( 1 + \frac{1}{4 s_{0}} \right) \right] > 0,
\]

where we use the following fact:

\[
c_{1} T > \frac{4 \sqrt{n}}{\min\{1, s_{0}\}} d_{0} = \frac{8 \sqrt{n}}{3 \min\{1, s_{0}\}} d_{0} + \frac{4 \sqrt{n}}{\min\{1, s_{0}\}} d_{0} \geq \frac{8 \sqrt{n}}{3} d_{0} + 4 \sqrt{n} d_{0} = \frac{8 \sqrt{n}}{3} \left( 1 + \frac{1}{2 s_{0}} \right).
\]

Finally, by (1.14), one can find constants \( C > 0 \) and \( \lambda_{2} > 0 \) such that for any \( \lambda \geq \lambda_{2} \),

\[
\overline{M}(0, \cdot) \geq 0.
\]  

(3.23)

Meanwhile, noting that \( \ell(T, x) = -\ell(0, x) \), we have that there is a constant \( \lambda_{3} > 0 \) such that for any \( \lambda \geq \lambda_{3} \),

\[
\overline{M}(T, \cdot) \leq 0.
\]  

(3.24)

Combining (3.23) and (3.24) with (3.16), and noting that \( v = e^{\lambda \phi} u \), for any \( \lambda \geq \lambda_{0} = \max\{\lambda_{1}, \lambda_{2}, \lambda_{3}\} \), we end up with the desired estimate (3.3). This completes the Proof of Theorem 3.1. □

4. AN AUXILIARY OPTIMAL CONTROL PROBLEM

In this section, as a preliminary, we analyze an auxiliary optimal control problem. Some ideas are taken from ([6], pp. 190–199 and [4], Prop. 6.1).

Let \( y \in L^{2}_{p}(\Omega; C([0, T]; L^{2}(G))) \) satisfy \( y(0) = y(T) = 0 \) in \( G \), \( \mathcal{P}\)-a.s. For any \( K > 1 \), let \( \varphi \equiv \varphi^{K}(x) \in C^{2}(\overline{G}) \), such that \( \min_{x \in \overline{G}} \varphi(x) = 1 \) and

\[
\varphi(x) = \begin{cases} 
1 & \text{if } x \in G_{0}, \\
K & \text{if } \text{dist}(x, G_{0}) \geq \frac{1}{K}. 
\end{cases}
\]  

(4.1)

For any integer \( m \geq 3 \), let \( h = \frac{T}{m} \), and set

\[
y_{m}^{j} \equiv y_{m}^{j}(x) = y(jh, x) \quad \text{and} \quad \phi_{m}^{j} \equiv \phi_{m}^{j}(x) = \phi(jh, x), \quad j = 0, 1, \ldots, m,
\]  

(4.2)
where $\phi(t, x) = d(x) - c_1(t - T/2)^2$ (see (3.2)). Consider the following system:

$$
\begin{aligned}
\mathbb{E}\left(\frac{z_{m}^{j+1} - 2z_m^j + z_m^{j-1}}{h^2}\right| F_{jh}) - \sum_{j_1, j_2 = 1}^n \partial_{x_{j_2}}(h^{j_1} \partial_{x_{j_1}} z_m^j) \\
= \mathbb{E}\left(\frac{r_{m}^{j+1} - r_m^j}{h}\right| F_{jh}) + r_{2m}^j + \lambda y_m^j e^{2\lambda \phi_m^j} + r_m^j \quad (1 \leq j \leq m - 1) \text{ in } G, \\
z_m^j = 0 \quad (0 \leq j \leq m) \\
\end{aligned}
$$

(4.3)

Here $(r_1^j, r_2^j, r_m^j) \in \left(L^2_{F_{jh}}(\Omega; L^2(G))\right)^3 \quad (j = 0, 1, \ldots, m)$ are controls. The set of admissible sequences for (4.3) is defined by

$$
\mathcal{A}_{ad} = \{(z_m^j, r_1^j, r_2^j, r_m^j)_{j=0}^m : (z_m^j, r_1^j, r_2^j, r_m^j) \in L^2_{F_{jh}}(\Omega; H^1_0(G)) \times \left(L^2_{F_{jh}}(\Omega; L^2(G))\right)^3 \quad \text{and} \quad (z_m^j, r_1^j, r_2^j, r_m^j)_{j=0}^m \text{ solves (4.3)}\}.
$$

Since $\{(0, 0, 0, -\lambda y_m^j e^{2\lambda \phi_m^j})\}_{j=0}^m \in \mathcal{A}_{ad}$, it follows that $\mathcal{A}_{ad} \neq \emptyset$.

Next, define a cost functional as follows:

$$
J((z_m^j, r_1^j, r_2^j, r_m^j)_{j=0}^m) = \frac{h}{2} \int_G \phi(r_1^j_{m}) + e^{-2\lambda \phi_m^j}dx + h \sum_{j=1}^{m-1} \int_G |z_m^{j-1}|^2 e^{-2\lambda \phi_m^j}dx \\
+ \int_G \phi \left(\frac{|r_1^j|^2}{2} + \frac{|r_2^j|^2}{\lambda^2}\right) e^{-2\lambda \phi_m^j}dx + K \int_G |r_m^j|^2dx,
$$

(4.4)

and consider the following optimal control problem: find a $\{(z_m^j, r_1^j, r_2^j, r_m^j)_{j=0}^m \in \mathcal{A}_{ad}, \text{ such that} \}

$$
J((z_m^j, r_1^j, r_2^j, r_m^j)_{j=0}^m) = \min_{\{(z_m^j, r_1^j, r_2^j, r_m^j)_{j=0}^m \in \mathcal{A}_{ad}\}} J((z_m^j, r_1^j, r_2^j, r_m^j)_{j=0}^m).
$$

(4.5)

Notice that for any $\{(z_m^j, r_1^j, r_2^j, r_m^j)_{j=0}^m \in \mathcal{A}_{ad}, \text{ by the standard regularity results for elliptic equations, we have} \}

$z_m^j \in L^2_{F_{jh}}(\Omega; H^2(G) \cap H^1_0(G))$.

We have the following result.

**Proposition 4.1.** For any $K > 1$ and $m \geq 3$, the problem (4.5) admits a unique solution $\{(z_m^j, r_1^j, r_2^j, r_m^j)_{j=0}^m \in \mathcal{A}_{ad}, \text{ (which depends on} \ K)\}$. Furthermore, define

$$
p_m^j \equiv p_m^j(x) \triangleq K \hat{r}_m^j(x), \quad 0 \leq j \leq m.
$$

(4.6)

Then,

$$
\begin{aligned}
z_m^0 = z_m^j = p_m^j = p_m^0 = 0 \text{ in } G, \\
\hat{z}_m^j, \ p_m^j \in L^2_{F_{jh}}(\Omega; H^2(G) \cap H^1_0(G)), \quad 1 \leq j \leq m - 1.
\end{aligned}
$$

(4.7)

Also, the following optimality conditions hold:

$$
\begin{aligned}
\frac{p_m^j - p_m^{j-1}}{h} + \phi \frac{r_1^{3m}}{2\lambda^2} e^{-2\lambda \phi_m} = 0 \text{ in } G, \\
p_m^j - \phi \frac{r_2^{3m}}{4\lambda^4} e^{-2\lambda \phi_m} = 0 \text{ in } G,
\end{aligned}
$$

(4.8)
First, recall the functions

\[
\begin{align*}
\frac{p_{m+1}^j - 2p_m^j + p_m^{j-1}}{h^2} - \sum_{j_1, j_2=1}^n \partial_{x_{j_1} x_{j_2}} (h^{j_1 j_2} \partial_{x_{j_1} p_m^j}) + e^{-2\lambda \phi} \bar{z}_m^j = 0 \quad \text{in} \; G, \quad \text{on} \; \Gamma,
\end{align*}
\]

Moreover, there is a constant \( C = C(K, \lambda) > 0 \), independent of \( m \), such that

\[
\begin{align*}
\mathbb{E} \sum_{j=1}^{m-1} \int_G |\bar{z}_m^j|^2 + |\bar{r}_1^m|^2 + |\bar{r}_2^m|^2 \ dx + h \mathbb{E} \int_G |\bar{r}_1^m|^2 \ dx &\leq C, \quad (4.10) \\
\mathbb{E} \sum_{j=0}^{m-1} \int_G \left( \frac{\bar{z}_m^j - \bar{z}^j_{m-1}}{h^2} + \frac{\mathbb{E}(\bar{r}_1^{j+1} - \bar{r}_1^j | \mathcal{F}_{jh})^2}{h^2} + \frac{\mathbb{E}(\bar{r}_2^{j+1} - \bar{r}_2^j | \mathcal{F}_{jh})^2}{h^2} + K \frac{(\bar{r}_1^{j+1} - \bar{r}_1^j)^2}{h^2} \right) \ dx &\leq C. \quad (4.11)
\end{align*}
\]

We refer to Appendix A for a proof of this proposition.

5. **Global Carleman estimate for stochastic hyperbolic equations**

**IN THE L²-SPACE**

We define a formal differential operator \( \mathcal{A} \) by

\[
\mathcal{A} \triangleq \frac{\partial^2}{\partial t^2} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( b_{ij} \frac{\partial}{\partial x_j} \right).
\]

In order to prove Theorem 1.6, we need the following global Carleman estimate for stochastic hyperbolic equations.

**Theorem 5.1.** Assume that the Condition 1.2 holds. Let \( T_0 \) be given by (1.14). Then there exists a \( \lambda_0^* > 0 \) such that for any \( T > T_0, \lambda \geq \lambda_0^* \), and any \( y \in L^2_\mathcal{E}(\Omega; \mathbb{C}[0,T]; L^2(G)) \) satisfying \( y(0) = y(T) = 0 \) in \( G \) and

\[
\mathbb{E}(y, \mathcal{A}y)_{L^2(\Omega)} = \mathbb{E}(b_1 y + f, \eta)_{H^{-1}(\Omega), H^1_0(\Omega)}, \quad \forall \; \eta \in L^2_\mathcal{E}(\Omega; H^1_0(\Omega)) \text{ with } \mathcal{A} \eta \in L^2_\mathcal{E}(0,T; L^2(G)),
\]

it holds that

\[
\lambda \mathbb{E} \int_Q e^{2\lambda \phi} y^2 \ dx dt \leq C \left( \mathbb{E} |e^{\lambda \phi} f|^2_{H^{-1}(\Omega)} + |e^{\lambda \phi} b_1 y|^2_{L^2_\mathcal{E}(0,T; H^{-1}(G))} + \lambda^2 \mathbb{E} \int_0^T \int_{G_0} e^{2\lambda \phi} y^2 \ dx dt \right). \quad (5.2)
\]

**Proof of Theorem 5.1.** We borrow some idea from [4, 6, 10]. The whole proof is divided into six steps.

**Step 1.** First, recall the functions \( \{ (\bar{z}_m^j, \bar{r}_1^m, \bar{r}_2^m, \bar{r}_m^j) \}_{j=0}^m \) in Proposition 4.1. For \( m = 2^i \) \( (i = 2, 3, \ldots) \), we define

\[
\begin{align*}
\bar{z}_m^j(t, x) = \frac{1}{h} \sum_{j=0}^{m-1} \mathbb{E} \left( \left( t - jh \right) \bar{z}_m^{j+1}(x) - \left( t - (j+1)h \right) \bar{z}_m^j(x) \right) | \mathcal{F}_{jh} \chi(jh,(j+1)h)(t), \\
\bar{r}_1^m(t, x) = \frac{1}{h} \sum_{j=0}^{m-1} \mathbb{E} \left( \left( t - jh \right) \bar{r}_1^{j+1}(x) - \left( t - (j+1)h \right) \bar{r}_1^j(x) \right) | \mathcal{F}_{jh} \chi(jh,(j+1)h)(t), \\
\bar{r}_2^m(t, x) = \frac{1}{h} \sum_{j=0}^{m-1} \mathbb{E} \left( \left( t - jh \right) \bar{r}_2^{j+1}(x) - \left( t - (j+1)h \right) \bar{r}_2^j(x) \right) | \mathcal{F}_{jh} \chi(jh,(j+1)h)(t), \\
\bar{r}_m^j(t, x) = \frac{1}{h} \sum_{j=0}^{m-1} \mathbb{E} \left( \left( t - jh \right) \bar{r}_m^{j+1}(x) - \left( t - (j+1)h \right) \bar{r}_m^j(x) \right) | \mathcal{F}_{jh} \chi(jh,(j+1)h)(t).
\end{align*}
\]
By (4.10) and (4.11), there is a subsequence of \( \{ (\tilde{z}^m, \tilde{r}_1^m, \tilde{r}_2^m, \tilde{r}^m) \}_{m=2}^\infty \) (still denoted by itself), such that for some \((\tilde{z}, \tilde{r}_1, \tilde{r}_2, \tilde{r}) \in (L^2_\bar{p}(\Omega; H^1(0, T; L^2(G)))^4, as m \to \infty,
\begin{equation}
(\tilde{z}^m, \tilde{r}_1^m, \tilde{r}_2^m, \tilde{r}^m) \to (\tilde{z}, \tilde{r}_1, \tilde{r}_2, \tilde{r}) \text{ weakly in } (L^2_\bar{p}(\Omega; H^1(0, T; L^2(G)))^4.
\end{equation}
Also, by (4.3), \( \tilde{z} \in L^2_\bar{p}(\Omega; H^1(0, T; L^2(G))) \) is the weak solution to the following random hyperbolic equation:
\begin{equation}
\begin{aligned}
A\tilde{z} &= \tilde{r}_1 t + \tilde{r}_2 + \lambda ye^{2\lambda \phi} + \tilde{r} \quad \text{in } Q, \\
\tilde{z} &= 0 \quad \text{on } \Sigma, \\
\tilde{z}(0) &= \tilde{z}(T) = 0 \quad \text{in } G.
\end{aligned}
\tag{5.5}
\end{equation}
This implies that
\begin{equation}
\tilde{z} \in L^2_\bar{p}(\Omega; C([0, T]; H^1_0(\Omega))) \cap L^2_\bar{p}(\Omega; C^1([0, T]; L^2(\Omega))).
\tag{5.6}
\end{equation}
The proof of (5.6) is given in the Appendix B. For any constant \( K > 1 \), put
\[ \tilde{p} \triangleq K\tilde{r}. \]
By (4.8)–(4.11), it is easy to see that \( \tilde{p} \) is the solution to the following system:
\begin{equation}
\begin{aligned}
A\tilde{p} + \tilde{z}e^{-2\lambda \phi} &= 0 \quad \text{in } Q, \\
\tilde{p} &= 0 \quad \text{on } \Sigma, \\
\tilde{p}(0) &= \tilde{p}(T) = 0 \quad \text{in } G, \\
\tilde{p}_t + \rho \frac{\tilde{r}_1}{\lambda^2} e^{-2\lambda \phi} &= 0 \quad \text{in } Q, \\
\tilde{p} - \rho \frac{\tilde{r}_2}{\lambda^2} e^{-2\lambda \phi} &= 0 \quad \text{in } Q.
\end{aligned}
\tag{5.7}
\end{equation}
Noting that \((\tilde{r}_1, \tilde{r}_2) \in (L^2_\bar{p}(\Omega; H^1(0, T; L^2(G)))^2, similar to the proof of (5.6), we can also deduce that
\begin{equation}
\tilde{p} \in L^2_\bar{p}(\Omega; C([0, T]; H^1_0(\Omega))) \cap L^2_\bar{p}(\Omega; C^1([0, T]; L^2(\Omega))).
\end{equation}

**Step 2.** Applying Theorem 3.1 to \( \tilde{p} \) in (5.7), we obtain that
\begin{equation}
\begin{aligned}
\lambda \mathbb{E} \int_Q (\lambda^2 \tilde{p}_t^2 + \tilde{p}_t^2 + |\nabla \tilde{p}|^2)e^{2\lambda \phi}dxdt &\leq C \left[ \mathbb{E} \int_Q \tilde{z} e^{-2\lambda \phi} dxdt + \lambda^2 \mathbb{E} \int_0^T \int_{G_0} (\lambda^2 \tilde{p}_t^2 + \tilde{p}_t^2)e^{2\lambda \phi}dxdt \right] \\
&\leq C \left[ \mathbb{E} \int_Q \tilde{z} e^{-2\lambda \phi} dxdt + \mathbb{E} \int_0^T \int_{G_0} \left( \frac{\tilde{r}_1^2}{\lambda^2} + \frac{\tilde{r}_2^2}{\lambda^2} \right) e^{-2\lambda \phi}dxdt \right].
\tag{5.8}
\end{aligned}
\end{equation}

Here and hereafter, \( C \) denotes a constant, independent of \( K \) and \( \lambda \). Moreover, by (5.7) again, \( \tilde{p}_t \) satisfies
\begin{equation}
\begin{aligned}
A\tilde{p}_t + (\tilde{z} e^{-2\lambda \phi})_t &= 0 \quad \text{in } Q, \\
\tilde{p}_t &= 0 \quad \text{on } \Sigma, \\
\tilde{p}_{tt} + \rho \left( \frac{\tilde{r}_1}{\lambda} - 2\phi_t \tilde{r}_1 \right) e^{-2\lambda \phi} &= 0 \quad \text{in } Q, \\
\tilde{p}_t - \rho \left( \frac{\tilde{r}_2}{\lambda} - 2\phi_t \tilde{r}_2 \right) e^{-2\lambda \phi} &= 0 \quad \text{in } Q.
\end{aligned}
\tag{5.9}
\end{equation}
Applying Theorem 3.1 to \( \tilde{p}_t \), by (5.9), we obtain that
\[
\lambda \mathbb{E} \int_Q \left( \frac{\lambda^2}{x^2} \tilde{p}_t^2 + \tilde{p}_t^2 + |\nabla \tilde{p}_t|^2 \right) e^{2\lambda \phi} \, dx \, dt
\]
\[
\leq C \left[ \mathbb{E} \left| e^{\lambda \phi} (e^{-2\lambda \phi} \tilde{z}) \right|^2_{L^2(Q)} + \lambda^2 \mathbb{E} \int_0^T \int_{G_0} \left( \frac{\lambda^2}{x^2} \tilde{p}_t^2 + \tilde{p}_t^2 \right) e^{2\lambda \phi} \, dx \, dt \right]
\]
\[
\leq C \left[ \mathbb{E} \int_Q (\tilde{z}_t^2 + \lambda^2 \tilde{x}^2) e^{-2\lambda \phi} \, dx \, dt + \mathbb{E} \int_0^T \int_{G_0} \left( \frac{\tilde{r}_1^2}{\lambda^2} + \frac{\tilde{r}_2^2}{\lambda^4} + \tilde{r}_1^2 + \frac{\tilde{r}_2^2}{\lambda^2} \right) e^{-2\lambda \phi} \, dx \, dt \right].
\] (5.10)

**Step 3.** By (5.7), we have that
\[
-\mathbb{E} \int_Q (\tilde{r}_1, t + \tilde{r}_2) \tilde{p} \, dx \, dt = \mathbb{E} \int_Q (\tilde{r}_1 \tilde{p}_t - \tilde{r}_2 \tilde{p}_t) \, dx \, dt = -\mathbb{E} \int_Q (\frac{\tilde{r}_1^2}{\lambda^2} + \frac{\tilde{r}_2^2}{\lambda^4}) e^{-2\lambda \phi} \, dx \, dt.
\]
This implies that
\[
0 = \mathbb{E} \left( A \tilde{z} - \tilde{r}_1, t - \tilde{r}_2 - \lambda \phi e^{2\lambda \phi} - \tilde{r}, \tilde{p} \right)_{L^2(Q)}
\]
\[
= -\mathbb{E} \int_Q \tilde{z} e^{-2\lambda \phi} \, dx \, dt - \mathbb{E} \int_Q \phi \left( \frac{\tilde{r}_1^2}{\lambda^2} + \frac{\tilde{r}_2^2}{\lambda^4} \right) e^{-2\lambda \phi} \, dx \, dt - \mathbb{E} \int_Q y \tilde{p} e^{2\lambda \phi} \, dx \, dt - \mathbb{E} \int_Q \tilde{r} \tilde{p} \, dx \, dt.
\]
Hence,
\[
\mathbb{E} \int_Q \tilde{z} e^{-2\lambda \phi} \, dx \, dt + \mathbb{E} \int_Q \phi \left( \frac{\tilde{r}_1^2}{\lambda^2} + \frac{\tilde{r}_2^2}{\lambda^4} \right) e^{-2\lambda \phi} \, dx \, dt + \mathbb{E} \int_Q \tilde{r} \tilde{p} \, dx \, dt = -\mathbb{E} \int_Q y \tilde{p} e^{2\lambda \phi} \, dx \, dt.
\] (5.11)

Combining (5.8) and (5.11), we arrive at
\[
\mathbb{E} \int_Q \tilde{z} e^{-2\lambda \phi} \, dx \, dt + \mathbb{E} \int_Q \phi \left( \frac{\tilde{r}_1^2}{\lambda^2} + \frac{\tilde{r}_2^2}{\lambda^4} \right) e^{-2\lambda \phi} \, dx \, dt + \mathbb{E} \int_Q \tilde{r} \, dx \, dt \leq \frac{C}{\lambda} \mathbb{E} \int_Q y^2 e^{2\lambda \phi} \, dx \, dt.
\] (5.12)

**Step 4.** Using (5.5) and (5.9) again, and noting \( \tilde{p}_t(0) = \tilde{p}_t(T) = 0 \) in \( G \), we find that
\[
0 = \mathbb{E} \left( A \tilde{z} - \tilde{r}_1, t - \tilde{r}_2 - \lambda \phi e^{2\lambda \phi} - \tilde{r}, \tilde{p}_t \right)_{L^2(Q)}
\]
\[
= -\mathbb{E} \int_Q \tilde{z} e^{-2\lambda \phi} \, dx \, dt - \mathbb{E} \int_Q (\tilde{r}_1, t + \tilde{r}_2) \tilde{p}_t \, dx \, dt - \mathbb{E} \int_Q \tilde{r} \tilde{p}_t \, dx \, dt.
\] (5.13)

Notice that
\[
-\mathbb{E} \int_Q \tilde{z} e^{-2\lambda \phi} \, dx \, dt = \mathbb{E} \int_Q \left( \tilde{z} e^{-2\lambda \phi} \, dx \, dt - \frac{\tilde{z}^2}{2} (e^{-2\lambda \phi})_{tt} \right) \, dx \, dt = \mathbb{E} \int_Q \left( \tilde{z}^2 + \lambda \phi \tilde{z}^2 - 2\lambda \phi \tilde{z}^2 \right) e^{-2\lambda \phi} \, dx \, dt.
\] (5.14)

Further, in view of the third and fourth equalities in (5.9), it follows that
\[
-\mathbb{E} \int_Q (\tilde{r}_1, t + \tilde{r}_2) \tilde{p}_t \, dx \, dt = -\mathbb{E} \int_Q (\tilde{r}_1, t - \tilde{r}_2, t \tilde{p}_t) \, dx \, dt
\]
\[
= \mathbb{E} \int_Q \tilde{r}_1, t \phi \left( \frac{\tilde{r}_1}{\lambda^2} - 2\phi \tilde{r}_1 \right) e^{-2\lambda \phi} \, dx \, dt + \mathbb{E} \int_Q \tilde{r}_2, t \phi \left( \frac{\tilde{r}_2}{\lambda^2} - 2\phi \tilde{r}_2 \right) e^{-2\lambda \phi} \, dx \, dt
\]
\[
= \mathbb{E} \int_Q \phi \left( \frac{\tilde{r}_1^2}{\lambda^2} + \frac{\tilde{r}_2^2}{\lambda^2} - 2\phi \tilde{r}_1 \tilde{r}_1, t - \frac{\tilde{r}_2}{\lambda^2} \phi \tilde{r}_2 \right) e^{-2\lambda \phi} \, dx \, dt.
\] (5.15)
Further, by $\tilde{p} = K\tilde{r}$ and integration by parts,

$$-E \int_Q \tilde{r} \tilde{p}_t \, dx \, dt = KE \int_Q \tilde{r}_t^2 \, dx \, dt. \tag{5.16}$$

Therefore, by (5.13)–(5.16), we get that

$$E \int_Q \phi \left( \frac{\tilde{r}_t^2}{\lambda^2} + \frac{\tilde{r}_z^2}{\lambda^2} - \frac{2}{\lambda^3} \phi_t \phi_{t} \tilde{r}_2 \right) e^{-2\lambda \phi} \, dx \, dt + KE \int_Q \tilde{r}_t^2 \, dx \, dt$$

$$+ E \int_Q \left( \tilde{z}_t^2 + \lambda \phi_{tt} \tilde{z}^2 - 2\lambda^2 \phi_t \tilde{z} \phi_{t} \right) e^{-2\lambda \phi} \, dx \, dt = \lambda E \int_Q y \tilde{p}_t e^{2\lambda \phi} \, dx \, dt. \tag{5.17}$$

Now, by (5.17) and (5.12), using the Cauchy–Schwarz inequality and noting (5.10), we obtain that

$$E \int_Q (\tilde{z}_t^2 + \lambda \tilde{z}^2) e^{-2\lambda \phi} \, dx \, dt + E \int_Q \phi \left( \frac{\tilde{r}_t^2}{\lambda^2} + \frac{\tilde{r}_z^2}{\lambda^2} + \frac{\tilde{r}_1^2}{\lambda^2} \right) e^{-2\lambda \phi} \, dx \, dt \leq C \lambda E \int_Q y^2 e^{2\lambda \phi} \, dx \, dt. \tag{5.18}$$

**Step 5.** By (5.7), we find that

$$E(\tilde{r}_{1,t} + \tilde{r}_2 + \lambda y e^{2\lambda \phi} + \tilde{r}, \tilde{z} e^{-2\lambda \phi})_{L^2(Q)} = E(\tilde{A} \tilde{z}, \tilde{z} e^{-2\lambda \phi})_{L^2(Q)}$$

$$= -E \int_Q (\tilde{z}_t e^{-2\lambda \phi})_t \, dx \, dt + \sum_{j,k=1}^n E \int_Q b_{jk} \tilde{z}_{x_j} (\tilde{z} e^{-2\lambda \phi})_{x_k} \, dx \, dt$$

$$= -E \int_Q (\tilde{z}_t^2 + \lambda \phi_{tt} \tilde{z}^2 - 2\lambda^2 \phi_t \tilde{z} \phi_{t}) e^{-2\lambda \phi} \, dx \, dt + \sum_{j,k=1}^n E \int_Q b_{jk} \tilde{z}_{x_j} \tilde{z}_{x_k} e^{-2\lambda \phi} \, dx \, dt. \tag{5.19}$$

This yields that

$$E \int_Q |\nabla \tilde{z}|^2 e^{-2\lambda \phi} \, dx \, dt \leq C E \int_Q \left[ |\tilde{r}_{1,t} + \tilde{r}_2 + \tilde{r}| \tilde{z} e^{-2\lambda \phi} + \lambda |y \tilde{z}| + (\tilde{z}_t^2 + \lambda \tilde{z}^2) e^{-2\lambda \phi} \right] \, dx \, dt$$

$$\leq C E \int_Q \left[ y^2 e^{2\lambda \phi} + \left( \frac{\tilde{r}_{1,t}^2}{\lambda^2} + \frac{\tilde{r}_z^2}{\lambda^2} + \frac{\tilde{r}_1^2}{\lambda^2} \right) + \tilde{z}_t^2 + \lambda \tilde{z}^2 \right] e^{-2\lambda \phi} \, dx \, dt. \tag{5.20}$$

By (5.12), (5.18) and (5.20), we choose a constant $K$ in (5.12) so that

$$K \geq Ce^{2\lambda \max_{(t, x) \in Q} |\phi|} \tag{5.21}$$

(to absorb the term $CE \int_Q \tilde{r}_t^2 e^{-2\lambda \phi} \, dx \, dt$ in (5.20)). Then we deduce that

$$E \int_Q |\nabla \tilde{z}|^2 + \tilde{z}_t^2 + \lambda \tilde{z}^2 \right) e^{-2\lambda \phi} \, dx \, dt + E \int_Q \phi \left( \frac{\tilde{r}_{1,t}^2}{\lambda^2} + \frac{\tilde{r}_z^2}{\lambda^2} + \frac{\tilde{r}_1^2}{\lambda^2} \right) e^{-2\lambda \phi} \, dx \, dt \leq C \lambda E \int_Q y^2 e^{2\lambda \phi} \, dx \, dt. \tag{5.22}$$

**Step 6.** Recall that $(\tilde{z}, \tilde{r}_1, \tilde{r}_2, \tilde{r})$ depend on $K$. Now, we fix $\lambda$ and let $K$ tend to infinity. By (5.12) and (5.22), we conclude that there exists a subsequence of

$$(\tilde{z}, \tilde{r}_1, \tilde{r}_2, \tilde{r}) \in (L^2_t(\Omega; H^1_0(0, T; L^2(G))) \cap L^2_t(\Omega; L^2(0, T; H^1_0(G)))) \times (L^2_t(\Omega; H^1(0, T; L^2(G))))^2 \times L^2_t(0, T; L^2(G)),$$

where the limit function satisfies the desired properties.
which converges weakly to some \((\bar{z}, \bar{r}_1, \bar{r}_2, 0)\), with \(\operatorname{supp} \bar{r}_j \subset (0, T) \times G_0\) (\(j = 1, 2\)), since \(\varrho(x) \equiv \varrho^K(x) \to \infty\) for any \(x \notin G_0\), as \(K \to \infty\). By (5.7), we deduce that \((\bar{z}, \bar{r}_1, \bar{r}_2)\) satisfies

\[
\begin{cases}
\mathcal{A}\bar{z} = \bar{r}_1 + \bar{r}_2 + \lambda y e^{2\lambda \varphi} & \text{in } Q, \\
\bar{z} = 0 & \text{on } \partial Q.
\end{cases}
\]

(5.23)

Using (5.22) again, we find that

\[
\mathbb{E} \int_0^T E \left[ (\bar{r}_1^2 + \bar{r}_2^2) e^{-2\lambda \varphi} dt \right] \leq C \int_Q y^2 e^{2\lambda \varphi} dx dt.
\]

(5.24)

By (5.1), with \(\eta\) replaced by \(\bar{z}\) above, one gets that

\[
\mathbb{E} \left( y, \bar{r}_1 + \bar{r}_2 + \lambda y e^{2\lambda \varphi} \right)_{L^2(Q)} = \mathbb{E} \left( b_1 y + f, \bar{z} \right)_{L^2(Q)}.
\]

Hence, for any \(\varepsilon > 0\),

\[
\lambda \int_Q y^2 e^{2\lambda \varphi} dx dt = \mathbb{E} (f, \bar{z})_{L^2(Q)} + \mathbb{E} (b_1 y, \bar{z})_{L^2(Q)} - \mathbb{E} (y, \bar{r}_1 + \bar{r}_2)_{L^2((0, T) \times G_0)}
\]

\[
\leq C \left\{ \frac{1}{\varepsilon} \left[ \mathbb{E} \left| e^{\lambda \varphi} f \right|_{H^{-1}(Q)}^2 + \mathbb{E} \left| e^{\lambda \varphi} b_1 y \right|_{L^2((0, T); H^{-1}(G))}^2 + \lambda^2 \mathbb{E} \int_0^T \int_{G_0} y^2 e^{2\lambda \varphi} dx dt \right] 
\]

\[
+ \varepsilon \left[ \mathbb{E} \left| e^{-\lambda \varphi} z \right|_{H^0_0(Q)}^2 + \mathbb{E} \left| e^{-\lambda \varphi} \bar{z} \right|_{L^2((0, T); H^0_0(G))}^2 + \frac{1}{\lambda^2} \mathbb{E} \int_0^T \int_{G_0} (\bar{r}_1^2 + \bar{r}_2^2) e^{-2\lambda \varphi} dx dt \right] \right\}.
\]

(5.25)

Finally, choosing \(\varepsilon\) in (5.25) sufficiently small and noting (5.24), we arrive at the desired estimate (5.2). This completes the proof of Theorem 5.1.

\[\square\]

6. An Energy Estimate for Backward Stochastic Hyperbolic Equations

In this section, we establish energy estimates respectively for a random hyperbolic equation and a backward stochastic hyperbolic equation, which will play important roles in the proof of Theorem 1.6.

First, set \(\bar{T} \in [0, T)\) and consider the following backward hyperbolic equation:

\[
\begin{cases}
\theta_{tt} - \sum_{i,j=1}^n (b^{ij} \partial_{x_i})_{x_j} dt = b_1 \theta & \text{in } (\bar{T}, T) \times G, \\
\partial_t \theta = 0 & \text{on } (\bar{T}, T) \times \partial G, \\
\theta(\bar{T}) = \bar{\theta}, & \theta_t(\bar{T}) = \bar{\theta}_1 \quad \text{in } G.
\end{cases}
\]

(6.1)

It is easy to see that for any \((\bar{\theta}_0, \bar{\theta}_1) \in L^2(\Omega; \mathcal{F}_{\bar{T}}; \mathcal{P}; H^0_0(G)) \times L^2(\Omega; \mathcal{F}_{\bar{T}}; \mathcal{P}; L^2(G))\), (6.1) admits a unique solution

\[
\theta \in L^2(\Omega; C([0, T]; H^0_0(G))) \cap L^2(\Omega; C^1([0, T]; L^2(G))).
\]

Furthermore, we have the following energy estimate.

**Proposition 6.1.** There is a constant \(C > 0\), depending only on \(T, G\) and \(b^{ij}\) (\(1 \leq i, j \leq n\)), such that for any solution \(\theta\) to (6.1) and for all \(t, s\) satisfying \(\bar{T} \leq t \leq s \leq T\), it holds that

\[
\mathbb{E} \int_G (|\theta_t(s, x)|^2 + |\nabla \theta(s, x)|^2) dx \leq C e^{C_T \int_{\bar{T}}^{t-} \mathbb{E} \left| \theta_{tt}(x,s) \right|^2} \mathbb{E} \int_G (|\theta_{tt}(t, x)|^2 + |\nabla \theta(t, x)|^2) dx.
\]

(6.2)
Lemma 6.3. For any $\alpha$ to the system (6.3), there is a constant

\[ E \in \tilde{T} \times T \times \Omega \times \sum_{i,j=1}^n (b^i)_{x_i}x_j \int_0^\infty \alpha_0(t) \, dt + b_2 \alpha_0 + \zeta dB(t) \, dt \quad \text{in} \quad (0, \tilde{T}) \times G, \]

(6.3)

Proposition 6.4. (1) $\alpha(\tilde{T}) = \alpha_0$ and $\beta(\tilde{T}) = \beta_0$ in $G$, $P$-a.s.
(2) For any $(T, \eta, \zeta) \in \mathbb{H}_T$, we shall use the following notion of solution for the system (6.3).

Proof of Proposition 6.4. It is easy to show the following well-posedness result for (6.3) (and hence we omit the proof).

Next, let $\tilde{T} \in (0, T]$. We consider the following backward stochastic hyperbolic equation:

\[
\begin{cases}
  d\alpha = \beta dt + \eta dB(t) & \text{in} \quad (0, \tilde{T}) \times G, \\
  d\beta - \sum_{i,j=1}^n (b^i \alpha_{x_j}) x_j dt = b_1 \alpha dt + b_2 \eta dt + \zeta dB(t) & \text{in} \quad (0, \tilde{T}) \times G, \\
  \alpha = 0, \quad \beta = 0 & \text{on} \quad (0, \tilde{T}) \times \Gamma, \\
  \alpha(\tilde{T}) = \alpha_0, \quad \beta(\tilde{T}) = \beta_0 & \text{in} \quad G.
\end{cases}
\]

Set

\[ H \tilde{T} = L^2_0(\Omega; C([0, \tilde{T}]; L^2(G))) \times L^2_0(\Omega; C([0, \tilde{T}]; L^2(G))) \times L^2_0(\Omega; H^1(G)) \times L^2_0(\Omega; L^2(G)). \]

We shall use the following notion of solution for the system (6.3).

**Definition 6.2.** $(\alpha, \beta, \eta, \zeta) \in \mathbb{H}_T$ is called a solution to the system (6.3), if

1. $\alpha(\tilde{T}) = \alpha_0$ and $\beta(\tilde{T}) = \beta_0$ in $G$, $P$-a.s.
2. For any $t \in (0, \tilde{T})$ and $\varphi \in C_0^1(G)$, it holds that

\[
\alpha(\tilde{T}) - \alpha(t) = \int_t^{\tilde{T}} \beta(s) ds + \int_t^{\tilde{T}} \eta(s) dB(s) \quad \text{in} \quad G, \quad P\text{-a.s.} \tag{6.4}
\]

and

\[
\langle \beta(T), \varphi \rangle_{L^2(G)} - \langle \beta(t), \varphi \rangle_{L^2(G)} = \int_t^{\tilde{T}} \int_G \left[ - \sum_{i,j=1}^n (b^i)_{x_i}x_j \varphi_{x_j}(x, s, x) + b_1 \alpha(s, x) \varphi(x) \right] dx ds
\]

\[
+ \int_t^{\tilde{T}} \int_G [b_2(s, x) \eta(s, x) \varphi(x) dx ds + \zeta(s, x) \varphi(x) dx dB(s)], \quad P\text{-a.s.} \tag{6.5}
\]

It is easy to show the following well-posedness result for (6.3) (and hence we omit the proof).

**Lemma 6.3.** For any $(\alpha_0, \beta_0) \in L^2_{p^0}((\Omega; C^1(G))) \times L^2_{p^0}((\Omega; L^2(G)))$, there is a unique solution $(\alpha, \beta, \eta, \zeta) \in \mathbb{H}_T$ to the system (6.3).

Furthermore, we have the following energy estimate.

**Proposition 6.4.** There is a constant $C > 0$, depending only on $T$, $G$ and $b^{ij}$ ($1 \leq i, j \leq n$), such that for any solution $(\alpha, \beta, \eta, \zeta)$ to (6.3), and for all $s, t$ satisfying $0 \leq s \leq t \leq \tilde{T}$, it holds that

\[
\mathbb{E} \int_G (|\beta(s, x)|^2 + |
abla \alpha(s, x)|^2) dx \leq C e^{C(r_1^{\frac{1}{n/p}} + r_2^{1/n/p})} \tilde{T} \mathbb{E} \int_G (|\beta(t, x)|^2 + |
abla \alpha(t, x)|^2) dx, \tag{6.6}
\]

and

\[
\mathbb{E} \int_0^{\tilde{T}} \int_G (|\eta|^2 + |\nabla \eta|^2) dx dt \leq C e^{C(r_1^{\frac{1}{n/p}} + r_2^{1/n/p})} \tilde{T} \mathbb{E} \int_G (|\alpha_0|, |\beta_0|^2)_{L^2_p(G)} \times L^2_p((\Omega; L^2(G))). \tag{6.7}
\]

**Proof of Proposition 6.4.** Define a (modified) energy of the system (6.3) as follows:

\[
\tilde{E}(t) = \frac{1}{2} \mathbb{E} \int_G \left( |\beta(t, x)|^2 + \sum_{i,j=1}^n (b^i_{x_i} \alpha_{x_j}(t, x) \alpha_{x_j}(t, x) + r_1^{\frac{1}{n/p}} |\alpha(t, x)|^2) dx, \quad t \in [0, T]. \tag{6.8}
\]
Then, by Itô’s formula, we get that
\[
\tilde{\mathcal{E}}(t) - \tilde{\mathcal{E}}(s) = \mathbb{E} \int_{s}^{t} \mathbb{E} \left( b_1(\tau) + b_2(\eta) \right) d\tau + \frac{1}{2} \mathbb{E} \int_{s}^{t} \mathbb{E} \left( \sum_{i,j=1}^{n} b_{ij}(\eta, \eta_{ij}) + \zeta^2 \right) d\tau
\]
\[+ r_1^{\frac{2}{2-n/p}} \mathbb{E} \int_{s}^{t} \mathbb{E} \left( \frac{1}{2} r_1^{\frac{2}{2-n/p}} - r_2^{2-n/p} \right) \mathbb{E} +(t) \mathbb{E}_s \mathbb{E} d\tau + \mathbb{E} (s) \leq \tilde{\mathcal{E}}(t) + C \left( r_1^{\frac{1}{n/p}} + r_2^{2-n/p} \right) \int_{s}^{t} \tilde{\mathcal{E}}(\tau) d\tau.
\] (6.10)

By a similar argument, we can also obtain that
\[
\tilde{\mathcal{E}}(t) - \tilde{\mathcal{E}}(s) = \mathbb{E} \int_{s}^{t} \mathbb{E} \left( b_1(\tau) + b_2(\eta) \right) d\tau + \frac{1}{2} \mathbb{E} \int_{s}^{t} \mathbb{E} \left( \sum_{i,j=1}^{n} b_{ij}(\eta, \eta_{ij}) + \zeta^2 \right) d\tau
\]
\[+ r_1^{\frac{2}{2-n/p}} \mathbb{E} \int_{s}^{t} \mathbb{E} \left( \frac{1}{2} r_1^{\frac{2}{2-n/p}} - r_2^{2-n/p} \right) \mathbb{E} +(t) \mathbb{E}_s \mathbb{E} d\tau + \mathbb{E} (s) \leq \tilde{\mathcal{E}}(t) + C \left( r_1^{\frac{1}{n/p}} + r_2^{2-n/p} \right) \int_{s}^{t} \tilde{\mathcal{E}}(\tau) d\tau.
\] (6.11)

Further, for a sufficiently small \( \epsilon > 0 \),
\[
\left| \mathbb{E} \int_{s}^{t} \mathbb{E} \left( \int_{G} b_2(\tau, x) \beta(\tau, x) \eta(\tau, x) d\tau \right) d\tau \right| \leq C(\epsilon) r_1^{\frac{1}{n/p}} \int_{s}^{t} \tilde{\mathcal{E}}(\tau) d\tau + \epsilon \int_{s}^{t} \int_{G} \nabla \eta(\tau, x)^2 d\tau d\tau.
\] (6.12)

By (6.9)–(6.12), we find that
\[
\frac{1}{2} \mathbb{E} \int_{s}^{t} \int_{G} \left( |\zeta|^2 + r_1^{\frac{2}{2-n/p}} |\eta|^2 + \frac{1}{2} \sum_{i,j=1}^{n} b_{ij}(\eta, \eta_{ij}) \right) d\tau d\tau + \tilde{\mathcal{E}}(s) \leq \tilde{\mathcal{E}}(t) + C \left( r_1^{\frac{1}{n/p}} + r_2^{2-n/p} \right) \int_{s}^{t} \tilde{\mathcal{E}}(\tau) d\tau.
\] (6.13)

This, together with Gronwall’s inequality, implies that
\[
\tilde{\mathcal{E}}(s) \leq e^{C \left( r_1^{\frac{1}{n/p}} + r_2^{2-n/p} \right) \tilde{\mathcal{E}}(t)},
\] (6.14)

which implies (6.6) and (6.7).
7. Proof of Theorem 1.6

In this section, we shall prove Theorem 1.6.

Proof of Theorem 1.6. We borrow some ideas from [3]. The whole proof is divided into four steps.

Step 1. Note that the solution $y$ to (1.3) may not be zero at $t = 0$ and $t = T$. To apply Theorem 5.1, we need to choose a suitable cutoff function. Set

$$T_j = \frac{T}{2} - \varepsilon_j T, \quad T'_j = \frac{T}{2} + \varepsilon_j T, \quad R_0 = \min_{x \in \Omega} \sqrt{d(x)} \quad \text{and} \quad R_1 = \max_{x \in \Omega} \sqrt{d(x)},$$

where $j = 0, 1, 2$ and $0 < \varepsilon_0 < \varepsilon_1 < \frac{1}{2}$. By (3.2), (1.14) and (1.15), for any $T > T_0$, we have that

$$\phi(0, x) = \phi(T, x) \leq R^2_1 - \frac{c_1 T^2}{8} < 0, \quad \forall \ x \in G. \quad (7.2)$$

Therefore, there exists an $\varepsilon_1 \in (0, \frac{1}{2})$, which is close to $\frac{1}{2}$, such that

$$\phi(t, x) \leq \frac{R^2_1}{2} - \frac{c_1 T^2}{8} < 0, \quad \forall \ (t, x) \in [(0, T_1) \cup (T'_1, T_1)] \times G. \quad (7.3)$$

On the other hand, it follows from (3.2) that

$$\phi \left( \frac{T}{2}, x \right) = d(x) \geq R^2_0, \quad \forall \ x \in G. \quad (7.4)$$

Therefore, there is an $\varepsilon_0 \in (0, \frac{1}{2})$, which is close to 0, such that

$$\phi(t, x) \geq \frac{R^2_0}{2}, \quad \forall \ (t, x) \in (T_0, T'_0) \times G. \quad (7.5)$$

Furthermore, choose a nonnegative function $\xi \in C_0^\infty(0, T)$ such that

$$\xi(t) = 1 \quad \text{in} \ (T_1, T'_1). \quad (7.6)$$

Step 2. In this step, we prove that there is a $\lambda_1 > 0$, such that for any $\lambda \geq \lambda_1$,

$$\lambda \mathbb{E} \int_Q e^{2\lambda \phi} y^2 \, dx \, dt \leq C \left( \lambda^2 \mathbb{E} \int_0^T \int_{G_0} e^{2\lambda \phi} y^2 \, dx \, dt + \mathbb{E} |y|_{L^2(J \times G)}^2 + \mathbb{E} \int_Q e^{2\lambda \phi} f^2 \, dx \, dt \right), \quad (7.7)$$

where $J = (0, T_1) \cup (T'_1, T)$.

To this aim, set $\tilde{y} = \xi y$. Then $\tilde{y}$ satisfies the following forward stochastic hyperbolic equation:

$$\begin{cases}
\tilde{y}_t - \sum_{i,j=1}^n (b_{ij} \tilde{y}_x)_x \, dt = (b_{11} \tilde{y} + \tilde{f}) \, dt + (b_{21} \tilde{y} + \xi \tilde{y}) \, dB(t) \quad \text{in} \ Q, \\
\tilde{y} = 0 \quad \text{on} \ \Sigma, \\
\tilde{y}(0) = \tilde{y}(T) = 0 \quad \text{in} \ G,
\end{cases} \quad (7.8)$$

with $\tilde{f} = \xi f + \xi u y + 2 \xi t y$. By Theorem 5.1, for any $\lambda \geq \lambda_0$, we have that

$$\lambda \mathbb{E} \int_Q e^{2\lambda \phi} \tilde{y}^2 \, dx \, dt \leq C \left( \mathbb{E} |e^{\lambda \phi} \tilde{f}|_{H^2-J}^2 + |e^{\lambda \phi} b_1 y|_{L^2(0, T; H^2-J)}^2 + \lambda^2 \mathbb{E} \int_0^T \int_{G_0} e^{2\lambda \phi} \tilde{y}^2 \, dx \, dt \right). \quad (7.9)$$
By the definition of \( \tilde{f} \), we find that
\[
\mathbb{E}|e^{\lambda \phi} \tilde{f}|_{H^{-1}(Q)}^2 = \mathbb{E}|e^{\lambda \phi}(\xi f + \xi u y + 2\xi_i y_k)|_{H^{-1}(Q)}^2
\]
\[
= \sup_{|h|_{L^2((0,T);H^1_0(Q))} = 1} \left| \mathbb{E}(e^{\lambda \phi}(\xi f + \xi u y + 2\xi_i y_k), h)_{H^{-1}(Q), H^1_0(Q)} \right|^2
\leq |e^{\lambda \phi}f|_{L^2_2(0,T;L^2(G))}^2 + C\lambda^2|e^{\lambda \phi}y|_{L^2_2(J;L^2(G))}^2
\leq |e^{\lambda \phi}f|_{L^2_2(0,T;L^2(G))}^2 + C\lambda^2(e^{(R^2_1 - cT^2/4)}|y|_{L^2_2(J;L^2(G))}^2).
\]

Further, recalling the definition of \( r_1 \) and noting the embedding \( L^{2p/p+2}(G) \hookrightarrow H^{-1}(G) \), we get that
\[
|e^{\lambda \phi}b_1\tilde{y}|_{L^2_2(0,T;H^{-1}(G))} \leq C|e^{\lambda \phi}b_1\xi y|_{L^2_2(0,T;L^{2p/p+2}(G))} \leq C r_1 |e^{\lambda \phi}y|_{L^2_2(0,T;L^2(G))}.
\]

Further, by (7.3) and (7.5),
\[
|e^{\lambda \phi}y|_{L^2_2(0,T;L^2(G))}^2 = |e^{\lambda \phi}y|_{L^2_2(0,T;L^2(G))}^2 - \mathbb{E}\int_Q e^{2\lambda \phi}(1 - \xi^2)y^2dxdt
\geq |e^{\lambda \phi}y|_{L^2_2(0,T;L^2(G))}^2 - Ce^{R^2_1 - cT^2/4} |y|_{L^2_2(J;L^2(G))}^2.
\]

Therefore, by (7.8)–(7.11), there is a constant \( C_1 = C_1(T,G) \), independent of \( \lambda \) and \( r_1 \), such that
\[
|e^{\lambda \phi}y|_{L^2_2(0,T;L^2(G))} \leq C_1 \left[ \frac{r_1^2}{\lambda} |e^{\lambda \phi}y|_{L^2_2(0,T;L^2(G))}^2 + \lambda \mathbb{E}\int_{0}^{T} \int_{G_0} e^{2\lambda \phi}y^2dxdt
\right.
\]
\[
\left. + e^{(R^2_1 - cT^2/4)}(1 + \lambda)|\mathbb{E}|y|_{L^2_2(J;L^2(G))}^2 + \frac{1}{\lambda} |e^{\lambda \phi}f|_{L^2_2(0,T;L^2(G))}^2 \right].
\]

Since \( R^2_1 - cT^2/4 < 0 \), one may find a sufficiently large \( \lambda_1 > 0 \), such that for any \( \lambda > \lambda_1 \), (7.6) holds.

**Step 3.** We establish an energy estimate for solutions to (1.3). Set
\[
\mathcal{E}(t) \triangleq \frac{1}{2}(\mathbb{E}|y(t, \cdot)|_{L^2_2(G)}^2 + \mathbb{E}|y_t(t, \cdot)|_{H^{-1}(G)}^2).
\]

Then by the classical energy estimate, for any \( S_0 \in (T_0, \frac{T}{2}) \) and \( S_0' \in (\frac{3T}{2}, T_0') \),
\[
\int_{S_0}^{S_0'} \mathcal{E}(t)dt \leq C(1 + r_1 + r_2)\mathbb{E}\int_{S_0}^{S_0'} \int_{G} y^2dxdt + C\mathbb{E}\int_{0}^{T} \int_{G} (f^2 + g^2)dxdt.
\]

On the other hand, we claim that there exists a constant \( C > 0 \), such that
\[
\mathcal{E}(t) \leq Ce^{\left(\frac{t}{\lambda_1} - cT^2/4 + R^2_1\right)} (\mathcal{E}(s) + |(f,g)|_{(L^2_2(0,T;L^2(G)))^2}^2), \quad \forall \ t, \ s \in [0,T].
\]

In the following, we only prove the case of \( t \geq s \). The other case can be also proved by a similar technique and Proposition 6.1. By Itô’s formula, let \( \bar{T} = t \) in (6.3) and \( T = t \) in (1.3). Then it follows that
\[
\mathbb{E}|y(t, \beta_0)|_{L^2_2(G)} + \mathbb{E}|y_t(t, -\alpha_0)_{H^{-1}(G), H^1_0(G)}
\]
\[
= \mathbb{E}|y(s, \beta(s))|_{L^2_2(G)} + \mathbb{E}|y_t(s, -\alpha(s))_{H^{-1}(G), H^1_0(G)} - \mathbb{E}\int_{s}^{t} \int_{G} (\alpha f + \eta g)dxdt.
\]
Denote by $S$ the unit sphere of the space $L^2_F(\Omega; H^1_0(G)) \times L^2_F(\Omega; L^2(G))$. By (7.16), (6.6) and (6.7), we obtain that

$$
\sqrt{2\mathcal{E}(t)} = \sup_{(\alpha, \beta) \in S} \left| \mathbb{E}\left( \langle y(t), \beta \rangle_{L^2(G)} + \langle y(t), -\alpha \rangle_{H^{-1}(G), H^1_0(G)} \right) \right| \\
= \sup_{(\alpha, \beta) \in S} \mathbb{E} \left[ \langle y(s), \beta(s) \rangle_{L^2(G)} + \langle y(s), -\alpha(s) \rangle_{H^{-1}(G), H^1_0(G)} - \int_s^t \int_G (\alpha f + \eta g) dx dt \right] \\
\leq C \sqrt{\mathcal{E}(s)} \sup_{(\alpha(s), \beta(s)) \in S} \| (\alpha(s), \beta(s)) \|_{L^2_F(\Omega; H^1_0(G)) \times L^2_F(\Omega; L^2(G))} \\
+ \sup_{(\alpha, \beta) \in S} |(\alpha, \eta)|_{(L^2(s, t; L^2(G)))^2} |(f, g)|_{(L^2(s, t; L^2(G)))^2} \\
\leq C e^{C(r_1^{-\frac{1}{7\gamma} + r_2^2})} \left[ \sqrt{\mathcal{E}(s)} + |(f, g)|_{(L^2(s, t; L^2(G)))^2} \right].
$$

This implies our claim (7.15).

**Step 4.** First, it follows from (7.4) that

$$
\mathbb{E} \int_Q e^{2\lambda \phi} y^2 dx dt \geq e^{R_0^2 \lambda} \int_{T_0} G y^2 dx dt.
$$

Also, (7.15) implies that

$$
\mathbb{E}[y]_{L^2(J \times G)}^2 \leq C e^{C(r_1^{-\frac{1}{7\gamma} + r_2^2})} \left( \mathcal{E}(0) + |(f, g)|_{(L^2(0, T; L^2(G)))^2} \right)
$$

and

$$
C e^{-C(r_1^{-\frac{1}{7\gamma} + r_2^2})} \mathcal{E}(0) \leq \int_{S_0}^T \mathcal{E}(t) dt + |(f, g)|_{(L^2(0, T; L^2(G)))^2}.
$$

By (7.17), (7.14) and (17.9), we get that

$$
\mathcal{E}(0) \leq C e^{C(r_1^{-\frac{1}{7\gamma} + r_2^2})} \left( e^{-R_0^2 \lambda} \int_Q e^{2\lambda \phi} y^2 dx dt + |(f, g)|_{(L^2(0, T; L^2(G)))^2} \right).
$$

Combining the above estimate with (7.6) and (17.8), we find that

$$
\mathcal{E}(0) \leq C e^{C(r_1^{-\frac{1}{7\gamma} + r_2^2})} \left[ e^{-R_0^2 \lambda} \left( \mathbb{E} \int_0^T \int_G e^{2\lambda \phi} y^2 dx dt + \frac{1}{\lambda} \mathbb{E} [y]_{L^2(J \times G)} \right) + |(f, g)|_{(L^2(0, T; L^2(G)))^2} \right] \\
\leq C e^{C(r_1^{-\frac{1}{7\gamma} + r_2^2})} \left[ e^{-R_0^2 \lambda} \left( \mathbb{E} \int_0^T \int_G e^{2\lambda \phi} y^2 dx dt + \frac{1}{\lambda} \mathcal{E}(0) \right) + |(f, g)|_{(L^2(0, T; L^2(G)))^2} \right].
$$

Therefore, there exists a sufficiently large constant $\lambda_3 > 0$, such that for any $\lambda > \lambda_3$, the desired observability inequality (1.7) holds. \hfill \Box

**Appendix A. Proof of Proposition 4.1**

In this section, we give a proof of Proposition 4.1. To this aim, we need the following known result.

**Lemma A.1** [4], Prop. 3.5. For any $h > 0$, $m = 3, 4, \ldots$, and $q^j_m, w^j_m \in \mathbb{C}$, we have

$$
- \sum_{j=1}^{m-1} q^j_m \frac{w^{j+1}_m - 2w^j_m + w^{j-1}_m}{h^2} = \sum_{j=0}^{m-1} q^j_m \frac{q^j_m - w^{j+1}_m - w^{j-1}_m}{h} = \sum_{j=1}^{m} q^j_m \frac{q^j_m - q^{j-1}_m}{h} \frac{w^j_m - w^{j-1}_m}{h}.
$$

(A.1)
Proof of Proposition 4.1. The whole proof is divided into four steps.

Step 1. Let \( \{(z_{j,k}^m, r_{1,m}^k, r_{2,m}^k, r_{m}^j)^{m}_{j=0}\}_{k=1}^{\infty} \subset A_{ad} \) be a minimizing sequence of \( J(\cdot) \). Thanks to the coercivity of the cost functional, since \( z_{j,k}^m \) solves an elliptic equation, it can be shown that \( \{(z_{j,k}^m, r_{1,m}^k, r_{2,m}^k, r_{m}^j)^{m}_{j=0}\}_{k=1}^{\infty} \) is bounded in \( A_{ad} \). Therefore, there exists a subsequence of \( \{(z_{j,k}^m, r_{1,m}^k, r_{2,m}^k, r_{m}^j)^{m}_{j=0}\}_{k=1}^{\infty} \) converging weakly to some \( \{(z_{j,k}^l, r_{1,l}^k, r_{2,l}^k, r_{m}^j)^{l}_{j=0}\}_{k=1}^{\infty} \) in \( (L^2_{\mathcal{F}_m}(\Omega; L^2(G)) \times (L^2_{\mathcal{F}_m}(\Omega; L^2(G)))^3 \). Since the functional \( J \) is strictly convex, this element is the unique solution to (4.5). By (4.6) and the definition of \( A_{ad} \), it is obvious that \( z_{0}^m = \hat{z}_{m}^m = \hat{p}_{0}^m = \hat{p}_{m}^m = 0 \) in \( G \).

Step 2. Fix \( \delta^{3}_{0m} \in L^2_{\mathcal{F}_m}(\Omega; H^2(G) \cap H^1_{0}(G)) \), \( \delta^{1}_{1m} \in L^2_{\mathcal{F}_m}(\Omega; L^2(G)) \), and \( \delta^{1}_{2m} \in L^2_{\mathcal{F}_m}(\Omega; L^2(G)) \) (\( j = 0, 1, 2, \ldots, m \)) with \( \delta^{0}_{0m} = \delta^{3}_{0m} = \delta^{0}_{2m} = \delta^{m}_{2m} = 0 \) and \( \delta^{1}_{1m} = \delta^{1}_{1m} \). For \( (\mu_0, \mu_1, \mu_2) \in \mathbb{R}^3 \), put

\[
\begin{align*}
\{ r_{j,k}^m \triangleq \mathbb{E}
& \left( \frac{\delta_{0m}^{j+1} - 2\delta_{0m}^{j} + \delta_{0m}^{j-1}}{h^2} \right) | \mathcal{F}_{j,k}, \right) + \mathbb{E} \left( \frac{\delta_{0m}^{j+1} - 2\delta_{0m}^{j} + \delta_{0m}^{j-1}}{h^2} \right) | \mathcal{F}_{j,k} \right) \mu_0 ,
- \sum_{j_1, j_2 = 1}^{n} \partial_{x_{j_2}} \left( b_{j_1 j_2} \partial_{x_{j_1}} \left( z_{j,k}^m + \mu_0 \delta^{3}_{0m} \right) \right) - \mathbb{E} \left( \frac{\delta_{0m}^{j+1} - \delta_{0m}^{j}}{h} \right) | \mathcal{F}_{j,k} \right) \mu_1 - \delta_{2m}^{j} - \mu_2 \delta_{2m}^{j} - \lambda y_{j,m} \mu_2 e^{2\lambda \phi_{m}} ,
\end{align*}
\]

Then \( \{(z_{j,k}^m + \mu_0 \delta^{3}_{0m}, \delta^{1}_{1m}, \delta^{1}_{2m}, \mu_2 \delta_{2m}^{j}, r_{m}^j)^{m}_{j=0} \} \in A_{ad} \). Define a function \( g(\cdot, \cdot, \cdot) \) in \( \mathbb{R}^3 \) by

\[
g(\mu_0, \mu_1, \mu_2) = J \left( \{(z_{j,k}^m + \mu_0 \delta^{3}_{0m}, \delta^{1}_{1m}, \delta^{1}_{2m}, \mu_2 \delta_{2m}^{j}, r_{m}^j)^{m}_{j=0} \} \right).
\]

Since \( g \) has a minimum at \( (0, 0, 0) \), we get that \( \nabla g(0, 0, 0) = 0 \).

By \( \frac{\partial g(0, 0, 0)}{\partial \mu_1} = 0 \), noting that \( \{(z_{j,k}^m, \delta^{3}_{1m}, \delta^{3}_{2m}, \delta^{j}_{m})^{m}_{j=0} \} \) satisfy the first equation of (4.3), we find that

\[
-K \sum_{j=1}^{m-1} \int_{G} \frac{\delta_{1m}^{j+1} - \delta_{1m}^{j}}{h} \mid \mathcal{F}_{j,k} \right) \ dx + \mathbb{E} \sum_{j=1}^{m} \int_{G} \frac{\delta_{1m}^{j+1} - \delta_{1m}^{j}}{h} \mid \mathcal{F}_{j,k} \right) \ dx = 0.
\]

and

\[
-K \sum_{j=1}^{m-1} \int_{G} \frac{\delta_{2m}^{j+1} - \delta_{2m}^{j}}{h} \mid \mathcal{F}_{j,k} \right) \ dx + \mathbb{E} \sum_{j=1}^{m} \int_{G} \frac{\delta_{2m}^{j+1} - \delta_{2m}^{j}}{h} \mid \mathcal{F}_{j,k} \right) \ dx = 0.
\]

which, combined with (4.6), gives (4.8).

On the other hand, by \( \frac{\partial g(0, 0, 0)}{\partial \mu_0} = 0 \), we have that

\[
\mathbb{E} \sum_{j=1}^{m} \int_{G} K \hat{r}_{m}^{j} \left[ \mathbb{E} \left( \frac{\delta_{0m}^{j+1} - 2\delta_{0m}^{j} + \delta_{0m}^{j-1}}{h^2} \right) | \mathcal{F}_{j,k} \right) \ dx = 0,\]

(A.2)
which, combined with \( p_m^0 = p_{m-1}^0 = \delta_{m}^0 = \delta_{0m} = 0 \) in \( G \), implies that (4.9) holds. By means of the regularity theory for elliptic equations of second order, one finds that \( \tilde{\alpha}_m, \tilde{\beta}_m^i \in L^2_{F_{j_n}}(\Omega; H^2(G) \cap H^1_0(\Omega)), 1 \leq j \leq m - 1 \).

**Step 3.** Recalling that \( \{(\tilde{\zeta}_m^j, \tilde{r}_m^i, \tilde{r}_m^j)\}_{j=0}^m \) satisfy (4.3), and noting (4.8)–(4.9) and \( \tilde{p}_m^j = K \tilde{r}_m^j \), one gets

\[
0 = E \sum_{j=1}^{m-1} \int_G \left[ E \left( \frac{\tilde{\zeta}_m^j - 2 \tilde{\zeta}_m^j}{h^2} + \frac{\tilde{z}_m^j}{h^2} \right) \right] \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} - \sum_{j_1, j_2 = 1}^n \partial_{x_{j_1}} \left( b^{j_1 j_2} \partial_{x_{j_1}} \tilde{z}_m^j \right) \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx 
\]

\[
= E \sum_{j=1}^{m-1} \int_G \left[ E \left( \frac{\tilde{\beta}_m^j(x, y) - \tilde{r}_m^j}{h^2} - \tilde{r}_m^j - \lambda \tilde{\phi}_m^j e^{2 \lambda \tilde{\phi}_m^j - \tilde{r}_m^j} \right) \right] \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx 
\]

\[
= E \sum_{j=1}^{m-1} \int_G \left[ E \left( \frac{\tilde{p}_m^j - \tilde{r}_m^j}{h^2} \right) \right] \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx - E \sum_{j=1}^{m-1} \int_G \left[ \tilde{r}_m^j + \lambda \tilde{\phi}_m^j e^{2 \lambda \tilde{\phi}_m^j - \tilde{r}_m^j} \right] \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx 
\]

\[
+ E \sum_{j=1}^{m} \int_G \left[ \tilde{r}_m^j \right] \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx - E \sum_{j=1}^{m-1} \int_G \left[ \tilde{r}_m^j + \lambda \tilde{\phi}_m^j e^{2 \lambda \tilde{\phi}_m^j - \tilde{r}_m^j} \right] \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx 
\]

By (4.8) and (A.3), there is a constant \( C = C(K, \lambda) > 0 \), such that

\[
E \sum_{j=1}^{m-1} \int_G \left[ \tilde{\beta}_m^j(x, y) \right] \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx + E \int_G \left( \frac{\tilde{r}_m^j}{\lambda^2} \right) \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx + K \int_G \left[ \tilde{r}_m^j \right] \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx 
\]

\[
E \int_G \left[ \tilde{r}_m^j \right] \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx \leq C \sum_{j=1}^{m-1} \int_G \left[ \tilde{\beta}_m^j(x, y) \right] \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx. 
\]

This implies (4.10).

**Step 4.** Noting that (4.9) holds and \( p_m^0 = z_m^0 = p_{m-1}^0 = \delta_{m}^0 = \delta_{0m} = 0 \), we obtain that

\[
\left\{ \begin{array}{l}
E(p_m^3 \mid F_h) - 4E(p_m^2 \mid F_h) + 5p_m^1 - \sum_{j_1, j_2 = 1}^n \partial_{x_{j_1}} \left( b^{j_1 j_2} \partial_{x_{j_1}} \frac{p_m^2 + p_m^0}{h^2} \right) \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx 
\end{array} \right.
\]

\[
+ \frac{E(\tilde{\zeta}_m^2 \mid F_h) e^{-2 \lambda \tilde{\phi}_m^2 - 2 \tilde{\zeta}_m^2} e^{-2 \lambda \tilde{\phi}_m^0 + \tilde{\zeta}_m^0} e^{-2 \lambda \tilde{\phi}_m^0}}{h^2} = 0 \quad \text{in } G,
\]

\[
+ \frac{4p_m^3 + E(p_m^3 \mid F_h) - 4p_m^2 + p_m^3 - \sum_{j_1, j_2 = 1}^n \partial_{x_{j_1}} \left( b^{j_1 j_2} \partial_{x_{j_1}} \frac{p_m^3 + p_m^0}{h^2} \right) \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx}{h^2} = 0 \quad \text{in } G,
\]

and for \( j = 2, \ldots, m - 2, \)

\[
\frac{E(p_m^{j+2} \mid F_h) - 4E(p_m^{j+1} \mid F_h) + 5p_m^j + E(p_m^j \mid F_{(j-1)h}) - 4p_m^{j-1} + p_m^{j-2}}{h^4} 
\]

\[
- \sum_{j_1, j_2 = 1}^n \partial_{x_{j_1}} \left( b^{j_1 j_2} \partial_{x_{j_1}} \frac{p_m^{j+1} + p_m^{j-1}}{h^2} \right) \frac{\tilde{\beta}_m^j(x, y)}{\tilde{\beta}_m^j(x, y)} dx 
\]

\[
+ \frac{E(\tilde{\zeta}_m^{j+1} \mid F_h) e^{-2 \lambda \tilde{\phi}_m^{j+1} - 2 \tilde{\zeta}_m^{j+1} e^{-2 \lambda \tilde{\phi}_m^0 + \tilde{\zeta}_m^0} e^{-2 \lambda \tilde{\phi}_m^0}}}{h^2} = 0 \quad \text{in } G.
\]
By (4.3),

\[
0 = \mathbb{E} \sum_{j=1}^{m-1} \int_{G} \left[ \mathbb{E} \left( \frac{\hat{z}^{j+1}_{m} - 2\hat{z}^{j}_{m} + \hat{z}^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) - \sum_{j_{1}, j_{2}=1}^{n} \partial_{x_{j_{1}}} (p^{j_{1}}_{m} \partial_{x_{j_{2}}} \hat{z}^{j}_{m}) \right] dx
\]

\[
- \mathbb{E} \left( \frac{\hat{z}^{j+1}_{1m} - \hat{z}^{j}_{1m}}{h} \big| \mathcal{F}_{jh} \right) - \mathbb{E} \left( \lambda y^{j}_{m} e^{2\lambda \phi^{j}_{m}} - \hat{r}^{j}_{m} \big| \mathcal{F}_{jh} \right) \mathbb{E} \left( \frac{p^{j+1}_{m} - 2p^{j}_{m} + p^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) dx. \tag{A.6}
\]

Using \( \hat{z}^{0}_{m} = \hat{z}^{m}_{m} = p^{0}_{m} = p^{m}_{m} = 0 \) again, we get that

\[
\mathbb{E} \sum_{j=1}^{m-1} \int_{G} \mathbb{E} \left( \frac{\hat{z}^{j+1}_{m} - 2\hat{z}^{j}_{m} + \hat{z}^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) \mathbb{E} \left( \frac{p^{j+1}_{m} - 2p^{j}_{m} + p^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) dx
\]

\[
= \mathbb{E} \sum_{j=1}^{m-1} \int_{G} \mathbb{E} \left( \frac{\hat{z}^{j+1}_{m} - 2\hat{z}^{j}_{m} + \hat{z}^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) \mathbb{E} \left( \frac{p^{j+2}_{m} - 2p^{j+1}_{m} + p^{j}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) dx
\]

\[
- 2\mathbb{E} \sum_{j=1}^{m-1} \int_{G} \mathbb{E} \left( \frac{p^{j+1}_{m} - 2p^{j}_{m} + p^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) dx + \mathbb{E} \sum_{j=1}^{m-1} \int_{G} \mathbb{E} \left( \frac{p^{j+2}_{m} - 2p^{j+1}_{m} + p^{j}_{m}}{h^{2}} \big| \mathcal{F}_{(j+1)h} \right) dx
\]

\[
= \mathbb{E} \left[ \int_{G} \left( p^{j+1}_{m} \big| \mathcal{F}_{jh} \right) - 2\mathbb{E} \left( \frac{p^{j}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) + 5p^{j}_{m} + 4p^{m-1}_{m} + \mathbb{E} \left( \frac{p^{j}_{m}}{h^{2}} \big| \mathcal{F}_{(j-1)h} \right) - 4p^{m-1}_{m} + p^{j-2}_{m} \right] \hat{z}^{j}_{m} dx.
\]

Therefore,

\[
\mathbb{E} \sum_{j=1}^{m-1} \int_{G} \left( \frac{\hat{z}^{j+1}_{m} - 2\hat{z}^{j}_{m} + \hat{z}^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) \mathbb{E} \left( \frac{p^{j+1}_{m} - 2p^{j}_{m} + p^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) dx
\]

\[
= \mathbb{E} \sum_{j=1}^{m-1} \int_{G} \mathbb{E} \left( \frac{\hat{z}^{j+1}_{m} - 2\hat{z}^{j}_{m} + \hat{z}^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) \mathbb{E} \left( \frac{p^{j+1}_{m} - 2p^{j}_{m} + p^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) dx
\]

\[
- (\hat{z}^{j+1}_{m} \big| \mathcal{F}_{jh}) e^{-2\lambda \phi^{j+1}_{m}} - 2\hat{z}^{j}_{m} e^{-2\lambda \phi^{j}_{m}} + \hat{z}^{j-1}_{m} e^{-2\lambda \phi^{j-1}_{m}} \right) \hat{z}^{j}_{m} dx.
\]

Noting that \( \hat{z}^{j}_{m} \big| \Gamma = p^{j}_{m} \big| \Gamma = 0 \) (\( j = 0, 1, \ldots, m \)), one has

\[
\mathbb{E} \sum_{j=1}^{m-1} \int_{G} \sum_{j_{1}, j_{2}=1}^{n} \partial_{x_{j_{1}}} (b^{j_{1}} \partial_{x_{j_{2}}} \hat{z}^{j}_{m}) \mathbb{E} \left( \frac{p^{j+1}_{m} - 2p^{j}_{m} + p^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) dx
\]

\[
= \mathbb{E} \sum_{j=1}^{m-1} \int_{G} \sum_{j_{1}, j_{2}=1}^{n} \partial_{x_{j_{1}}} (b^{j_{1}} \partial_{x_{j_{2}}} \hat{z}^{j}_{m}) \mathbb{E} \left( \frac{p^{j+1}_{m} - 2p^{j}_{m} + p^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) dx
\]

Then by (A.6), we obtain that

\[
0 = -\mathbb{E} \sum_{j=1}^{m-1} \int_{G} \left[ \frac{\hat{z}^{j+1}_{m}}{h} \mathbb{E} \left( \frac{\hat{z}^{j}_{m}}{h} \big| \mathcal{F}_{jh} \right) - \hat{r}^{j}_{1m} + \hat{r}^{j}_{2m} + \lambda y^{j}_{m} e^{2\lambda \phi^{j}_{m}} - \hat{r}^{j}_{m} \right] \mathbb{E} \left( \frac{p^{j+1}_{m} - 2p^{j}_{m} + p^{j-1}_{m}}{h^{2}} \big| \mathcal{F}_{jh} \right) dx. \tag{A.7}
\]
It follows from Lemma A.1 that

\[-E \sum_{j=1}^{m-1} \int_{G} \left( \frac{\hat{z}_{m}^j E(\hat{z}_{m}^{j+1} | \mathcal{F}_{jh}) e^{-2\lambda_{j+1}^m} - 2\hat{z}_{m}^j e^{-2\lambda_{j}^m} + \hat{z}_{m}^{j-1} e^{-2\lambda_{j-1}^m}}{h^2} \right) dx \]

\[= \mathbb{E} \sum_{j=0}^{m-1} \int_{G} \left( \frac{\hat{z}_{m}^{j+1} - \hat{z}_{m}^j}{h} \right) \left( \frac{\hat{z}_{m}^{j+1} e^{-2\lambda_{j+1}^m} - \hat{z}_{m}^{j} e^{-2\lambda_{j}^m}}{h} \right) dx \quad (A.8) \]

\[= \mathbb{E} \sum_{j=0}^{m-1} \int_{G} \left[ \frac{(\hat{z}_{m}^{j+1} - \hat{z}_{m}^{j})^2}{h^2} e^{-2\lambda_{j+1}^m} + \frac{\hat{z}_{m}^{j+1} - \hat{z}_{m}^{j} e^{-2\lambda_{j+1}^m}}{h} - \frac{e^{-2\lambda_{j}^m}}{h} \right] dx. \]

By Lemma A.1 and $p_{m}^j = K\hat{r}_{m}^j$, we have that

\[-E \sum_{j=1}^{m-1} \int_{G} \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} \right) \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} - \frac{p_{m}^j}{h} \right) | \mathcal{F}_{jh} \right) dx = K \mathbb{E} \sum_{j=0}^{m-1} \int_{G} \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} \right)^2 dx. \quad (A.9) \]

Further, by (4.8) and Lemma A.1, we get that

\[-E \sum_{j=1}^{m-1} \int_{G} \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} \right) \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} - \frac{p_{m}^j}{h} \right) | \mathcal{F}_{jh} \right) dx = \mathbb{E} \sum_{j=1}^{m-1} \int_{G} \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} \right) \left( \frac{\hat{r}_{m}^{j+1} e^{-2\lambda_{j+1}^m} - \hat{r}_{m}^{j} e^{-2\lambda_{j}^m}}{h} \right) | \mathcal{F}_{jh} \right) dx \]

\[= \mathbb{E} \sum_{j=1}^{m-1} \int_{G} \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} \right) \left( \frac{\hat{r}_{m}^{j+1} e^{-2\lambda_{j+1}^m} - \hat{r}_{m}^{j} e^{-2\lambda_{j}^m}}{h} \right) \mathbb{E}(\hat{r}_{m}^{j+1} | \mathcal{F}_{jh}) dx \quad (A.10) \]

Therefore,

\[-E \sum_{j=1}^{m-1} \int_{G} \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} \right) \left( \frac{\hat{r}_{m}^{j+1} e^{-2\lambda_{j+1}^m} - \hat{r}_{m}^{j} e^{-2\lambda_{j}^m}}{h} \right) \mathbb{E}(\hat{r}_{m}^{j+1} | \mathcal{F}_{jh}) dx \]

\[= \mathbb{E} \sum_{j=1}^{m-1} \int_{G} \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} \right) \left( \frac{\hat{r}_{m}^{j+1} e^{-2\lambda_{j+1}^m} - \hat{r}_{m}^{j} e^{-2\lambda_{j}^m}}{h} \right) \mathbb{E}(\hat{r}_{m}^{j+1} | \mathcal{F}_{jh}) dx \quad (A.11) \]

and

\[-E \sum_{j=1}^{m-1} \int_{G} \left( \lambda y_{m}^j e^{2\lambda_{j+1}^m} + \hat{r}_{2m}^j \right) \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} \right) \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} - \frac{p_{m}^j}{h} \right) | \mathcal{F}_{jh} \right) dx \]

\[= \mathbb{E} \sum_{j=1}^{m-1} \int_{G} \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} \right) \left( \frac{\hat{r}_{m}^{j+1} e^{-2\lambda_{j+1}^m} - \hat{r}_{m}^{j} e^{-2\lambda_{j}^m}}{h} \right) \mathbb{E}(\hat{r}_{m}^{j+1} | \mathcal{F}_{jh}) dx + \mathbb{E} \sum_{j=0}^{m-1} \int_{G} \left( \frac{\hat{r}_{2m}^j - \hat{r}_{2m}^j}{h} \right) \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} \right) dx \]

\[= \lambda \mathbb{E} \sum_{j=1}^{m-1} \int_{G} \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} \right) \left( \frac{\hat{r}_{m}^{j+1} e^{-2\lambda_{j+1}^m} - \hat{r}_{m}^{j} e^{-2\lambda_{j}^m}}{h} \right) \mathbb{E}(\hat{r}_{m}^{j+1} | \mathcal{F}_{jh}) dx \]

\[+ \mathbb{E} \sum_{j=0}^{m-1} \int_{G} \left( \frac{\hat{r}_{2m}^j - \hat{r}_{2m}^j}{h} \right) \left( \frac{\hat{r}_{m}^{j+1} - \hat{r}_{m}^j}{h} \right) \left( \frac{\hat{r}_{m}^{j+1} e^{-2\lambda_{j+1}^m} - \hat{r}_{m}^{j} e^{-2\lambda_{j}^m}}{h} \right) \hat{r}_{2m}^j dx. \]
By (A.8)–(A.11), it follows that
\[
\begin{align*}
\mathbb{E} \sum_{j=0}^{m-1} \int_G \left\{ \left( \frac{\hat{z}_m^{j+1} - \hat{z}_m^j}{h^2} \right)^2 e^{-2\lambda \phi_m} + \frac{\theta}{\lambda^2} \left[ \mathbb{E}(\hat{r}_m^{j+1} - \hat{r}_m^j | \mathcal{F}_{jh}) \right]^2 e^{-2\lambda \phi_m} \\
+ \frac{\theta}{\lambda^4} \left( \frac{\hat{r}_m^{j+1} - \hat{r}_m^j}{h^2} \right)^2 e^{-2\lambda \phi_m} + K \left( \frac{\hat{r}_m^{j+1} - \hat{r}_m^j}{h^2} \right)^2 \right\} dx \\
= -\mathbb{E} \sum_{j=0}^{m-1} \int_G \frac{\hat{z}_m^{j+1} - \hat{z}_m^j}{h} \left( e^{-2\lambda \phi_m} \right) \hat{z}_m^j dx \\
-\mathbb{E} \sum_{j=1}^{m-1} \int_G \frac{\theta}{\lambda^2} \left( \frac{\hat{r}_m^{j+1} - \hat{r}_m^j}{h} \right) \left[ \mathbb{E}(\hat{r}_m^{j+1} | \mathcal{F}_{jh}) \right] \\
-\lambda \mathbb{E} \sum_{j=0}^{m-1} \int_G \frac{\theta}{\lambda^4} y_m \left[ e^{-2\lambda \phi_m} \right] \hat{r}_m^{j+1} dx.
\end{align*}
\]
By Hölder’s inequality, there is a positive constant \( C = C(K, \lambda) \), independent of \( m \), such that
\[
\begin{align*}
\mathbb{E} \sum_{j=0}^{m-1} \int_G \left\{ \left( \frac{\hat{z}_m^{j+1} - \hat{z}_m^j}{h^2} \right)^2 e^{-2\lambda \phi_m} + \frac{\theta}{\lambda^2} \left[ \mathbb{E}(\hat{r}_m^{j+1} - \hat{r}_m^j | \mathcal{F}_{jh}) \right]^2 e^{-2\lambda \phi_m} \\
+ \frac{\theta}{\lambda^4} \left( \frac{\hat{r}_m^{j+1} - \hat{r}_m^j}{h^2} \right)^2 e^{-2\lambda \phi_m} + K \left( \frac{\hat{r}_m^{j+1} - \hat{r}_m^j}{h^2} \right)^2 \right\} dx \\
\leq C \mathbb{E} \left[ \sum_{j=1}^{m-1} \int_G \left| \hat{z}_m^j \right|^2 + \left| \hat{r}_m^j \right|^2 + \left| \hat{r}_m^{2j} \right|^2 + K \left| \hat{r}_m^{2j+1} \right|^2 + \left| y_m \right|^2 \right] dx + \int_G \left| \hat{r}_m^m \right|^2 dx.
\end{align*}
\] (A.12)

Finally, by (A.12) and (4.10), recalling that \( y \in L^2_\Omega(\mathbb{R}, C([0, T]; L^2(G))) \), we get the desired estimate (4.11). This completes the Proof of Proposition 4.1. \( \square \)

**APPENDIX B. PROOF OF (5.6)**

This appendix is addressed to proving (5.6). By (5.5), for a.e. \( \omega \in \Omega \), \( z_\omega = \tilde{z}(\omega) \in H^1(0, T; L^2(G)) \) is a weak solution to the following random equation:
\[
\left\{ \begin{array}{ll}
A z_\omega = \tilde{r}_1, \omega, t + \tilde{r}_2, \omega + \lambda y_\omega e^{2\lambda \phi} + \tilde{r}_\omega & \text{in } Q, \\
z_\omega = 0 & \text{on } \Sigma, \\
z_\omega(0) = z_\omega(T) = 0 & \text{in } G.
\end{array} \right.
\] (B.1)

Here \( \tilde{r}_1, \omega, t = \tilde{r}_1(\omega), \tilde{r}_2, \omega = \tilde{r}_2(\omega), y_\omega = y(\omega) \) and \( \tilde{r}_\omega = \tilde{r}(\omega) \). Also, set \( h_\omega = \tilde{r}_1, \omega, t + \tilde{r}_2, \omega + \lambda y_\omega e^{2\lambda \phi} + \tilde{r}_\omega \).

In the following, without loss of generality, we assume that \( z_\omega \) is smooth and give a uniform estimate for it. Let \( 0 < t_1 < t_2 < T \). Multiplying the first equation of (B.1) by \( \tilde{t}^2(T - t)^2 z_\omega \) and integrating it in \( (0, T) \times G \), we get that
\[
\int_{t_1}^{t_2} \int_G |\nabla z_\omega|^2 dx dt \leq C \int_0^T \int_G (|h_\omega|^2 + |z_\omega, t|^2) dx dt.
\] (B.2)
Put
\[ E(t) = \frac{1}{2} \int_G \left( |z_{\omega,t}(t)|^2 + |\nabla z_{\omega}(t)|^2 \right) dx. \]

By the usual energy estimate for the first equation of (B.1) and noting the time reversibility of (B.1), we have that
\[ E(t) \leq C \left( E(s) + \int_{t_1}^{t_2} \int_G |h_{\omega}(\tau, x)|^2 d\tau dx \right), \quad \forall \, t, s \in [t_1, t_2]. \] (B.3)

Integrating (B.3) with respect to \( s \) from \( t_1 \) to \( t_2 \), we obtain
\[ E(t) \leq C \left( \int_{t_1}^{t_2} E(s) ds + \int_{t_1}^{t_2} \int_G |h_{\omega}(\tau, x)|^2 d\tau dx \right), \quad \forall \, t \in [t_1, t_2]. \] (B.4)

By (B.2) and (B.4), for any \( t \in [t_1, t_2] \), there is a constant \( C > 0 \) such that
\[ |z_{\omega,t}(t)|_{L^2(G)}^2 + |z_{\omega}(t)|_{H^1_0(G)}^2 \leq C \int_0^T \int_G (|h_{\omega}|^2 + |z_{\omega,t}|^2) dx dt. \] (B.5)

Applying the usual energy estimate to the first equation of (B.1) and noting the time reversibility of (B.1) again, similar to the proof of (B.3), we find that
\[ |z_{\omega}|_{C([0,T]; H^1_0(G) \cap C^1([0,T]; L^2(G)))}^2 \leq C \left( |z_{\omega,t}(t)|_{L^2(G)}^2 + |z_{\omega}(t)|_{H^1_0(G)}^2 + |h_{\omega}|_{L^2(0,T; L^2(G))}^2 \right). \]

This, together with (B.5), implies that
\[ |z_{\omega}|_{C([0,T]; H^1_0(G) \cap C^1([0,T]; L^2(G)))}^2 \leq C \int_0^T \int_G (|h_{\omega}|^2 + |z_{\omega,t}|^2) dx dt. \]

It follows that
\[ E|z_{\omega}|_{C([0,T]; H^1_0(G) \cap C^1([0,T]; L^2(G)))}^2 \leq CE \int_0^T \int_G (|h_{\omega}|^2 + |z_{\omega,t}|^2) dx dt. \]

This, together with \( \tilde{z} \in L^2(\Omega; H^1(0, T; L^2(G))) \), implies that
\[ \tilde{z} \in L^2(\Omega; C([0, T]; H^1_0(G))) \cap L^2(\Omega; C^1([0, T]; L^2(G))). \]

\[ \Box \]

REFERENCES


