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LOCAL BOUNDARY CONTROLLABILITY TO TRAJECTORIES FOR THE 1D COMPRESSIBLE NAVIER STOKES EQUATIONS

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Abstract. In this article, we show a local exact boundary controllability result for the 1d isentropic compressible Navier Stokes equations around a smooth target trajectory. Our controllability result requires a geometric condition on the flow of the target trajectory, which comes naturally when dealing with the linearized equations. The proof of our result is based on a fixed point argument in weighted spaces and follows the strategy already developed in [S. Ervedoza, O. Glass, S. Guerrero, J.-P. Puel, *Arch. Ration. Mech. Anal.* **206** (2012) 189–238] in the case of a non-zero constant velocity field. The main novelty of this article is in the construction of the controlled density in the case of possible oscillations of the characteristics of the target flow on the boundary.

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1. Introduction

Let us consider the 1d compressible Navier-Stokes equations stated in a bounded domain $(0, L)$ and in finite time horizon $T > 0$:

$$\begin{cases} \partial_t \rho_s + \partial_x(\rho_s u_s) = 0 & \text{in} \quad (0,T)\times(0,L), \\ \rho_s(\partial_t u_s + u_s \partial_x u_s) - \nu \partial_{xx} u_s + \partial_x p(\rho_s) = 0 & \text{in} \quad (0,T)\times(0,L). \end{cases} \tag{1.1}$$

Here, ρ_s is the density of the fluid, and u_s represents its velocity. The pressure term $p = p(\rho_s)$ is assumed to depend on the density ρ_s, according to a law which can take different forms. In this article, we will only require $p \in C^2(\mathbb{R}_+^*; \mathbb{R})$, which encompasses the cases of pressure of the form

$$p(\rho_s) = A\rho_s^\gamma, \quad \text{with} \quad A > 0 \quad \text{and} \quad \gamma \geq 1, \tag{1.2}$$

corresponding to the isentropic law for perfect gases. The constant $\nu > 0$ stands for the viscosity of the fluid.

To be well posed, system (1.1) should be completed with initial data at $t = 0$ and boundary conditions, but we will not make them precise yet.

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Instead, we assume that we have a solution $(\overline{\rho}, \overline{u})$ of (1.1) in $(0,T) \times (0,L)$, which enjoys the following regularity:

$$(\overline{\rho}, \overline{u}) \in C^2([0,T] \times [0,L]) \times C^2([0,T] \times [0,L]), \qquad \text{with} \qquad \inf_{[0,T]\times[0,L]} \overline{\rho}(t,x) > 0. \tag{1.3}$$

The question of local exact boundary controllability we aim at addressing is the following: if we consider a solution (ρ_s, u_s) of (1.1) starting at $t = 0$ from an initial data close to $(\overline{\rho}(0,\cdot), \overline{u}(0,\cdot))$, can we find boundary controls such that the corresponding solution of (1.1) reaches exactly $(\overline{\rho}(T,\cdot), \overline{u}(T,\cdot))$ at time T?

In order to state our result precisely, we introduce an extension of $(\overline{\rho}, \overline{u})$ on $[0,T] \times \mathbb{R}$, still denoted in the same way for sake of simplicity, enjoying the same conditions as in (1.3):

$$(\overline{\rho}, \overline{u}) \in C^2([0,T] \times \mathbb{R}) \times C^2([0,T] \times \mathbb{R}), \qquad \text{with} \qquad \inf_{[0,T]\times\mathbb{R}} \overline{\rho}(t,x) > 0. \tag{1.4}$$

These regularity assumptions allow to define the flow $\overline{X} = \overline{X}(t,\tau,x)$ corresponding to $\overline{u}$ defined for $(t,\tau,x) \in [0,T] \times [0,T] \times \mathbb{R}$ as follows:

$$\begin{cases} \dfrac{\mathrm{d}\overline{X}}{\mathrm{d}t}(t,\tau,x) = \overline{u}(t, \overline{X}(t,\tau,x)) \qquad \text{in} \qquad (0,T), \\ \overline{X}(\tau,\tau,x) = x. \end{cases} \tag{1.5}$$

Our main assumption then is the following one:

$$(\overline{X}, T) \text{ is such that } \forall x \in [0,L],\ \exists t_x \in (0,T) \text{ such that } \overline{X}(t_x, 0, x) \notin [0,L]. \tag{1.6}$$

Before going further, let us emphasize that Condition (1.6) is of geometric nature and does not depend on the choice of the extension of $(\overline{\rho}, \overline{u})$ on $\mathbb{R}$, but only depends on the original target trajectory $(\overline{\rho}, \overline{u})$ defined only on $[0,T] \times [0,L]$.

The main result of this article then states as follows:

Theorem 1.1. *Let $T > 0$, and $(\overline{\rho}, \overline{u}) \in (C^2([0,T] \times [0,L]))^2$ as in (1.3) be a solution of (1.1) in $(0,T) \times (0,L)$, and such that there exists an extension (still denoted the same) satisfying the regularity conditions (1.4). Assume that the flow $\overline{X}$ in (1.5) satisfies (1.6).*

Then there exists $\varepsilon > 0$ such that for all $(\rho_0, u_0) \in H^1(0,L) \times H^1(0,L)$ satisfying

$$\|(\rho_0, u_0)\|_{H^1(0,L)\times H^1(0,L)} \leq \varepsilon, \tag{1.7}$$

there exists a controlled trajectory (ρ_s, u_s) of (1.1) satisfying

$$(\rho_s(0,\cdot), u_s(0,\cdot)) = (\overline{\rho}(0,\cdot), \overline{u}(0,\cdot)) + (\rho_0, u_0) \qquad \text{in} \qquad (0,L), \tag{1.8}$$

and the control requirement

$$(\rho_s(T,\cdot), u_s(T,\cdot)) = (\overline{\rho}(T,\cdot), \overline{u}(T,\cdot)) \qquad \text{in} \qquad (0,L). \tag{1.9}$$

Furthermore, (ρ_s, u_s) enjoys the following regularity:

$$\rho_s \in C^0([0,T]; H^1(0,L)) \cap C^1([0,T]; L^2(0,L)) \text{ and } u_s \in H^1(0,T; L^2(0,L)) \cap L^2(0,T; H^2(0;L)). \tag{1.10}$$

Note that the control functions do not appear explicitly in Theorem 1.1. As said earlier, the controls are applied on the boundary of the domain, *i.e.* on $(0,T) \times \{0,L\}$. If one wants to make them appear explicitly, the equation (1.1)–(1.8) should be completed with

$$\begin{cases} \rho_s(t,0) = \overline{\rho}(t,0) + v_{0,\rho}(t) & \text{for} \quad t \in (0,T) & \text{with} \quad u_s(t,0) > 0, \\ \rho_s(t,L) = \overline{\rho}(t,L) + v_{L,\rho}(t) & \text{for} \quad t \in (0,T) & \text{with} \quad u_s(t,L) < 0, \\ u_s(t,0) = \overline{u}(t,0) + v_{0,u}(t) & \text{for} \quad t \in (0,T), & \\ u_s(t,L) = \overline{u}(t,L) + v_{L,u}(t) & \text{for} \quad t \in (0,T), & \end{cases}$$

where $v_{0,\rho}, v_{L,\rho}, v_{0,u}, v_{L,u}$ are the control functions. Our strategy will not make them appear explicitly, but we emphasize that these are the control functions used to control the fluid in Theorem 1.1.

Theorem 1.1 generalizes [13] where the same local boundary controllability result is proved when $(\overline{\rho}, \overline{u})$ is a constant trajectory with $\overline{u} \neq 0$ and $\|(\rho_0, u_0)\|_{H^3(0,L)\times H^3(0,L)} \leq \varepsilon$ instead of (1.7). In this case, the geometric condition (1.6) reduces to $T > L/|\overline{u}|$ and coincides with the geometric constraint in [13]. Note that even when considering the case of constant reference trajectories $(\overline{\rho}, \overline{u})$ with $\overline{u} \neq 0$, Theorem 1.1 is more precise than the result in [13] as the smallness of (ρ_0, u_0) is required in the $H^1(0,L) \times H^1(0,L)$ norm in (1.7), while it was required in the $H^3(0,L) \times H^3(0,L)$ norm in [13] due to the use of the work [24]. This improvement is obtained by using the Carleman estimates recently derived in [3] instead of the ones in [16].

The proof of Theorem 1.1 follows the one in [13] in the case of constant trajectory and is based on a fixed point argument on a kind of linearized system. The main novelty in our approach is on the control of the density, which is more delicate than in [13] due to possible cancellations on the target velocity $\overline{u}$ at $x = 0$ and $x = L$. The resulting difficulty is that the characteristics may be tangent to the boundary at some times. To overcome this difficulty which arises on the boundary, we transform our boundary control into a distributed control problem on an extended domain. This enlargement of the domain is carried out through a careful geometric discussion (Sect. 2.2) which constitutes the main difference with the approach in [13] and the main novelty of our work. This geometric consideration has important consequences on the construction of the controlled density and its estimates (Sect. 3). The second main interest of this construction is that it simplifies the proof in [13] in the sense that we now use only one parameter in the Carleman estimates we shall use in our fixed point argument, while [13] requires the use of two parameters in the Carleman estimates. Of course, the geometric condition (1.6) originates from the control of the density and it cannot be avoided when working on the linearized equations, as pointed out for instance in the recent work [22]. Such geometric conditions also appeared in [3] in order to obtain a local exact controllability result for the incompressible Navier-Stokes equations in 2 and 3d, but in that context, the authors did not require regularity conditions on the controlled density, while we need them in our problem.

Results on the local exact controllability of viscous compressible flows are very recent. The first one is due to [2] in the one-dimensional case when the equations are stated in Lagrangian forms and the density coincides with the target density at the time $t = 0$. The work [13] then obtained a similar result as Theorem 1.1 but in the context of constant trajectories $(\overline{\rho}, \overline{u})$ with $\overline{u} \neq 0$ and $T > L/|\overline{u}|$. This geometric condition appears naturally when dealing with the linearized equations and can already be found in [25] for a structurally damped wave equation, which is of similar nature as the 1d linearized compressible Navier-Stokes equations around the constant state $(\overline{\rho}, 0)$. This has later been thoroughly discussed in the work [8] on the stabilization of the linearized compressible Navier-Stokes equation when the actuator is located in the velocity equation. When the equations are linearized around constant trajectories $(\overline{\rho}, \overline{u})$ with $\overline{u} \neq 0$, some controllability results were obtained in the case of controls acting only on the velocity field [7]. These results are also deeply related to the ones obtained on the structurally damped wave equation with moving controls [6, 23].

The fact that we are dealing with the non-linear problem induces the use of a flexible tool to control the linearized equations. Therefore, we shall use Carleman estimates in the spirit of [16]. But in our context, the linearized equations encompass both parabolic and hyperbolic behaviors. We are therefore led to using Carleman estimates with a weight function traveling along the characteristics of the reference velocity, following an idea already used in several previous works, *e.g.* [1] for linear thermoelasticity, [3, 13] for non-homogeneous viscous fluids, or [6] for linear viscoelasticity with moving controls.

Let us also mention that other controllability results were derived in the case of compressible perfect fluids corresponding to $\nu = 0$. In that context, several results were obtained depending if one considers regular solutions [21] or BV solutions [18, 19]. But the methods developed in this context are very different from the ones used for viscous flows.

The literature is also very rich for what concerns incompressible fluids (with homogeneous density) in the two and three dimensional setting. With no aim at exhaustivity, we refer for instance to [10, 17] for the controllability of the incompressible Euler equations, and to [9,12,14,15,20] for the controllability of the incompressible Navier-Stokes equations.

Let us also mention that Theorem 1.1 states a result of local exact controllability to smooth trajectories under the geometric condition (1.6). But this geometric condition may not be necessary in order to get a local controllability result for the non-linear equations (1.1) based on the use of the non-linearity, for instance in the spirit of the return method, see *e.g.* [10,17] where this idea has been developed in the context of Euler equations, and [11] for several examples in which the non-linear effects help in controlling the equations at hand. To our knowledge, this idea has not been developed yet in the context of compressible Navier-Stokes equations.

The article is organized as follows. Section 2 presents the global strategy of the proof and introduces the mathematical framework. We will in particular present some geometrical considerations and we will define a fixed point map subsequently. Section 3 deals with the control of the density equation, which is the main contribution of this article. Section 4 is dedicated to the control of the velocity. Finally, Section 5 puts together the ingredients developed in the preceding sections and concludes the proof of Theorem 1.1. The appendix recalls a weighted Poincaré's inequality proved in ([13], Lem. 4.9).

2. Main steps of the proof of Theorem 1.1

The aim of this section is to give the structure of the proof of Theorem 1.1 and its main steps.

2.1. Reformulation of the problem

We start by introducing the variation (ρ, u) of (ρ_s, u_s) around the trajectory $(\overline{\rho}, \overline{u})$, defined by

$$(\rho, u) = (\rho_s, u_s) - (\overline{\rho}, \overline{u}) \qquad \text{in} \qquad (0,T) \times (0,L). \tag{2.1}$$

The new unknowns (ρ, u) then satisfy

$$\begin{cases} \partial_t \rho + (\overline{u} + u)\partial_x \rho + \overline{\rho}\partial_x u + \left(\dfrac{\overline{\rho}}{\nu} p'(\overline{\rho}) + \partial_x \overline{u}\right)\rho = f(\rho, u) & \text{in} \qquad (0,T) \times (0,L), \\ \overline{\rho}(\partial_t u + \overline{u}\partial_x u) - \nu \partial_{xx} u = g(\rho, u) & \text{in} \qquad (0,T) \times (0,L), \end{cases} \tag{2.2}$$

where

$$\begin{cases} f(\rho, u) = -\rho \partial_x u + \dfrac{\overline{\rho}}{\nu} p'(\overline{\rho})\rho - \partial_x \overline{\rho} u, \\ g(\rho, u) = -\rho\big(\partial_t(\overline{u} + u) + (\overline{u} + u)\partial_x(\overline{u} + u)\big) - \overline{\rho} u \partial_x(\overline{u} + u) - p'(\overline{\rho} + \rho)\partial_x(\overline{\rho} + \rho) + p'(\overline{\rho})\partial_x \overline{\rho}, \end{cases} \tag{2.3}$$

with initial data

$$(\rho(0, \cdot), u(0, \cdot)) = (\rho_0, u_0) \qquad \text{in} \qquad (0, L), \tag{2.4}$$

and controls acting on the boundary conditions, that we do not specify here. The control requirement (1.9) then reads as follows:

$$(\rho(T, \cdot), u(T, \cdot)) = (0, 0) \qquad \text{in} \qquad (0, L). \tag{2.5}$$

The strategy will then consist in finding a trajectory (ρ, u) solving (2.2)–(2.4) and satisfying the control requirement (2.5). This will be achieved by a fixed point argument that we will present in Section 2.4 and that will be detailed in Section 5.

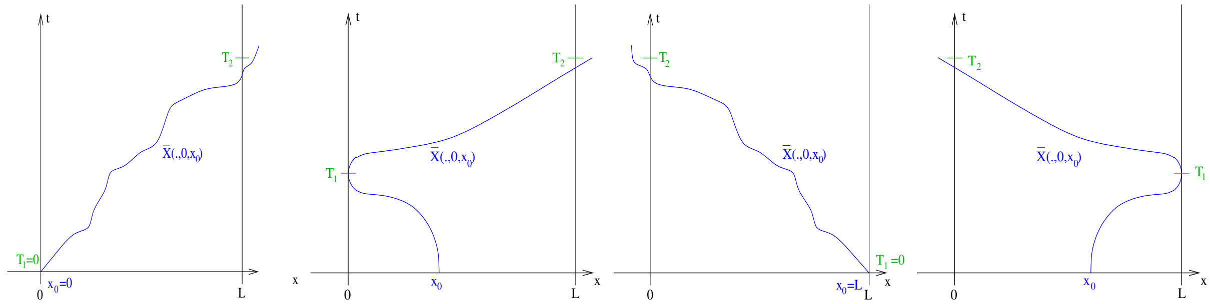

FIGURE 1. Some possible trajectories $t \mapsto \overline{X}(.,0,x_0)$.

2.2. Geometric considerations

To deal with system (2.2), we introduce a suitable extension of the spatial domain $(0, L)$ designed from the extension $\overline{u}$ on $\mathbb{R}$. This will allow us to pass from boundary controls to distributed controls which is easier to deal with from a theoretical point of view.

We first introduce the point $x_0 \in [0, L]$ defined by

$$x_0 = \sup\left\{x \in [0,L] \mid \exists t_x \in [0,T),\ \overline{X}(t_x,0,x) = 0 \quad \text{and} \quad \forall t \in (0,t_x),\ \overline{X}(t,0,x) \in (0,L)\right\}, \tag{2.6}$$

where $\overline{X}$ is the flow corresponding to $\overline{u}$ (recall (1.5)). This point x_0 has interesting properties due to the uniqueness of Cauchy−Lipschitz's theorem forbidding the crossing of characteristics (recall that $\overline{u}$ belongs to $C^2([0,T] \times \mathbb{R})$ by Assumption (1.4)):

- All trajectories $t \mapsto \overline{X}(t,0,x)$ starting from $x \in [0, x_0)$ first exit the domain $[0, L]$ through $x = 0$;
- According to the geometric assumption (1.6), all trajectories $t \mapsto \overline{X}(t,0,x)$ starting from $x \in (x_0, L]$ necessary exit the domain $[0, L]$ through $x = L$, and thus x_0 in (2.6) can also be defined as

$$x_0 = \inf\left\{x \in [0,L] \mid \exists t_x \in [0,T),\ \overline{X}(t_x,0,x) = L \quad \text{and} \quad \forall t \in (0,t_x),\ \overline{X}(t,0,x) \in (0,L)\right\}. \tag{2.7}$$

Besides, using again (1.6), there exists $T_2 < T$ such that $\overline{X}(T_2,0,x_0) \notin [0,L]$.

If $\overline{X}(T_2,0,x_0) > L$, by definition of x_0, there exists $T_1 \in [0,T_2)$ such that $\overline{X}(T_1,0,x_0) = 0$.

If $\overline{X}(T_2,0,x_0) < 0$, from (2.7), there exists $T_1 \in [0,T_2)$ such that $\overline{X}(T_1,0,x_0) = L$.

We are therefore in one of the configurations displayed in Figure 1.

Using the change of variables $x \mapsto L - x$ if needed, we now assume without loss of generality that

$$\overline{X}(T_2,0,x_0) > L \text{ and } T_1 \in [0,T_2) \text{ with } \overline{X}(T_1,0,x_0) = 0.$$

Using the continuity of $\overline{X}(T_2,0,\cdot)$, there exists $x_1 < x_0$ (x_1 may not belong to $[0,L]$ if $x_0 = 0$) such that $\overline{X}(T_2,0,x_1) > L$, while due to Cauchy Lipschitz's theorem, $\overline{X}(T_1,0,x_1) < 0$. By continuity of $\overline{X}(\cdot,0,x_1)$, we also have the existence of $T_1' \in (0,T_2)$ such that $\overline{X}(T_1',0,x_1) = \overline{X}(T_1,0,x_1)/2$. We then define

$$a = \frac{\overline{X}(T_1',0,x_1)}{2} \quad (< 0), \qquad b = \frac{\overline{X}(T_2,0,x_1) + L}{2} \quad (> L), \tag{2.8}$$

and we set

$$T_0 = \frac{T_1'}{2} \quad (> 0), \qquad T_L = T_2 \quad (\in (T_1', T)), \tag{2.9}$$

so that

$$\overline{X}(2T_0,0,x_1) < a \quad \text{and} \quad \overline{X}(T_L,0,x_1) > b. \tag{2.10}$$

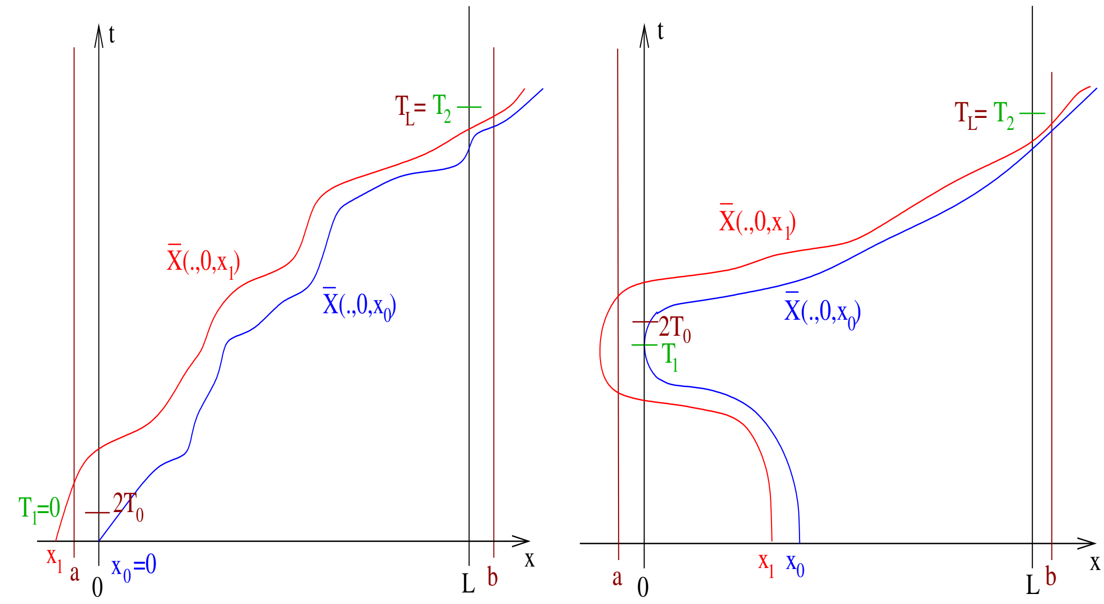

FIGURE 2. Choice of x_1 and construction of the extended domain (a, b) and the times (T_0, T_L) in two different configurations: Left, $x_0 = 0$, right, $x_0 \in (0, L)$.

Lastly, remark that due to the fact that two different characteristics do not cross each other due to the regularity of $\overline{u}$ in (1.4), we have the following counterpart of (1.6) on the domain (a, b):

$$\forall x \in [a, b], \quad \exists t_x \in (0, T_L), \quad \overline{X}(t_x, 0, x) \notin [a, b]. \tag{2.11}$$

We refer to Figure 2 for a summary of the notations introduced above.

For convenience, we then redefine the time horizon T by reducing it if necessary by setting

$$T := \min\{T, T_L + 1\}. \tag{2.12}$$

This can be done without loss of generality as a local exact controllability result on a given trajectory in some time obviously implies a local exact controllability result on the target trajectory in all larger times.

2.3. Carleman weights

Our fixed point argument will be done in suitable weighted spaces coming from the use of Carleman estimates, namely the ones in ([3], Thm. 2.5). In this section, we construct once for all the Carleman weights that will be used along the article.

Let us introduce a cut-off function $\eta \in C^\infty(\mathbb{R}; [0, 1])$ such that

$$\forall x \in \mathbb{R}, \quad \eta(x) = \begin{cases} 1 & \text{for} \quad x \in [0, L], \\ 0 & \text{for} \quad x \in \mathbb{R} \setminus \left(\dfrac{a}{2}, \dfrac{L+b}{2}\right). \end{cases} \tag{2.13}$$

We then construct a weight function ψ as follows:

Lemma 2.1. *Let η as in (2.13). There exists $\psi \in C^2([0,T]\times\mathbb{R})$ such that:*

$$\forall (t,x) \in [0,T]\times\mathbb{R}, \quad \partial_t\psi + \eta\overline{u}\partial_x\psi = 0, \tag{2.14}$$

$$\forall (t,x) \in [0,T]\times\mathbb{R}, \quad \psi(t,x) \in [0,1], \tag{2.15}$$

$$\forall t \in (0,T), \quad \partial_x\psi(t,2a) \geq 0, \tag{2.16}$$

$$\sup_{[0,T]\times[a,b]} \partial_x\psi < 0. \tag{2.17}$$

Proof of Lemma 2.1. Let $\psi_0 \in C^2(\mathbb{R};[0,1])$ such that $\partial_x\psi_0(2a) \geq 0$ and

$$\sup_{[a,b]} \partial_x\psi_0 < 0. \tag{2.18}$$

We then define ψ as the solution of the equation

$$\begin{cases} \partial_t\psi + \eta\overline{u}\partial_x\psi = 0 & \text{in} \quad (0,T)\times\mathbb{R}, \\ \psi(0,\cdot) = \psi_0 & \text{in} \quad \mathbb{R}. \end{cases} \tag{2.19}$$

It is easy to check that $\psi \in C^2([0,T]\times\mathbb{R})$ as $\overline{u} \in C^2([0,T]\times\mathbb{R})$ (recall (1.4)) and $\psi_0 \in C^2(\mathbb{R})$. The function ψ takes value in $[0,1]$ as $\psi_0 \in C^2(\mathbb{R};[0,1])$. In order to check conditions (2.16)–(2.17), we simply look at the equation satisfied by $\partial_x\psi$:

$$\begin{cases} \partial_t\partial_x\psi + \eta\overline{u}\partial_x(\partial_x\psi) + \partial_x(\eta\overline{u})\partial_x\psi = 0 & \text{in} \quad (0,T)\times\mathbb{R}, \\ \partial_x\psi(0,\cdot) = \partial_x\psi_0 & \text{in} \quad \mathbb{R}. \end{cases} \tag{2.20}$$

It follows that $\partial_x\psi(t,2a) = \partial_x\psi_0(2a)$ for all $t \in [0,T]$, so that (2.16) holds true, and that the critical points of $\psi(t,\cdot)$ are transported along the characteristics of $\eta\overline{u}$. As the velocity field $\eta\overline{u}$ vanishes outside $(a/2,(L+b/2))$, the fact that ψ_0 does not have any critical point in $[a,b]$ (recall (2.18)) implies that $\psi(t,\cdot)$ cannot have a critical point in $[a,b]$. Thus, $\partial_x\psi$ cannot vanish in $[0,T]\times[a,b]$, and the sign condition in (2.18) implies (2.17), therefore concluding the proof of Lemma 2.1. □

For $s \geq 1$ and $\lambda \geq 1$, we set $\mu = s\lambda^2 e^{2\lambda}$ and we define $\theta \in C^2([0,T);\mathbb{R})$ as follows:

$$\forall t \in [0,T), \quad \theta(t) = \begin{cases} 1 + \left(1 - \dfrac{t}{T_0}\right)^{\mu} & \text{for} \quad t \in [0,T_0], \\ 1 & \text{for} \quad t \in [T_0,T_L], \\ \text{nondecreasing} & \text{for} \quad t \in \left[T_L, \dfrac{T+T_L}{2}\right], \\ \dfrac{1}{T-t} & \text{for} \quad t \in \left[\dfrac{T+T_L}{2}, T\right), \end{cases} \tag{2.21}$$

where T_0 and T_L are defined in (2.9). Note that such construction is possible according to (2.12). We also define $\varphi = \varphi(t,x)$ by

$$\varphi(t,x) = \theta(t)\left(\lambda e^{12\lambda} - e^{\lambda(\psi(t,x)+6)}\right) \qquad \text{for all} \quad (t,x) \in [0,T]\times\mathbb{R}. \tag{2.22}$$

These functions θ and φ depend on the parameters $s \geq 1$ and $\lambda \geq 1$, but we shall omit these dependencies for simplicity of notations. Actually, the parameter $\lambda \geq 1$ in the Definitions (2.21)–(2.22) of θ and φ is chosen sufficiently large for Theorem 2.5 in [3], recalled in the present paper in Theorem 4.2, to be true and is fixed in all the article. Yet, the Carleman parameter $s \geq 1$ is not fixed and will be chosen at the end of the proof. Consequently, in all the article, constants C never depend on the parameter s, except if specifically said.

These functions will appear naturally as weights when using Carleman estimates, see Section 4. Our approach here is based on the Carleman estimate developed in ([3], Thm. 2.5), but it shares very close features to the ones developed in [16].

Here, we will use the Carleman estimates developed in ([3], Thm. 2.5) as they allow to avoid doing a lifting of the initial data and the use of technical results concerning the Cauchy problem (for instance [24] which was used in [13]). The Carleman estimate in ([3], Thm. 2.5) has also been developed for weight functions $\psi = \psi(t, x)$ depending on the time and space variables, which will be needed in our analysis to get estimates on the controlled density, see Section 3.2.3. In this step indeed, we shall strongly use that ψ solves the transport equation (2.14).

2.4. Definition of the fixed point map

We now make precise the fixed point map and the intermediate results which will be needed to conclude Theorem 1.1.

The controllability of (2.2)–(2.4) will be studied in the extended domain (a, b), where a and b are defined in (2.8). Let $(\rho_0, u_0) \in H^1(0, L) \times H^1(0, L)$ satisfying (1.7) with $\varepsilon > 0$. This parameter ε will be chosen small enough within our proof of Theorem 1.1, see Section 5 and more precisely Lemma 5.1. We thus extend (ρ_0, u_0) such that $(\rho_0, u_0) \in H_0^1(a, b) \times H_0^1(a, b)$ and

$$\|(\rho_0, u_0)\|_{H_0^1(a,b) \times H_0^1(a,b)} \leq C\varepsilon, \tag{2.23}$$

where C is independent of ε. We also introduce the characteristic function of $(a, b) \setminus [0, L]$

$$\chi(x) = \begin{cases} 1 & \text{for} \quad x \in (a, 0) \cup (L, b), \\ 0 & \text{for} \quad x \in [0, L]. \end{cases} \tag{2.24}$$

Instead of considering the equations (2.2)–(2.4), we now consider the following system of equations:

$$\begin{cases} \partial_t \rho + (\overline{u} + u)\partial_x \rho + \overline{\rho}\partial_x u + \left(\dfrac{\overline{\rho}}{\nu} p'(\overline{\rho}) + \partial_x \overline{u}\right) \rho = f(\rho, u) + v_\rho \chi & \text{in} \quad (0, T) \times (a, b), \\ \overline{\rho}(\partial_t u + \overline{u}\partial_x u) - \nu \partial_{xx} u = g(\rho, u) & \text{in} \quad (0, T) \times (a, b), \end{cases} \tag{2.25}$$

with initial data

$$(\rho(0, \cdot), u(0, \cdot)) = (\rho_0, u_0) \quad \text{in } (a, b), \tag{2.26}$$

source terms $f(\rho, u)$, $g(\rho, u)$ as in (2.3), and with boundary conditions

$$\begin{cases} \rho(t, a) = 0 & \text{for} \quad t \in (0, T) \quad \text{with} \quad (\overline{u} + u)(t, a) > 0, \\ \rho(t, b) = 0 & \text{for} \quad t \in (0, T) \quad \text{with} \quad (\overline{u} + u)(t, b) < 0, \\ u(t, a) = v_u(t) & \text{for} \quad t \in (0, T), \\ u(t, b) = 0 & \text{for} \quad t \in (0, T). \end{cases} \tag{2.27}$$

In (2.25) and (2.27), the functions $v_\rho \in L^2((0, T) \times ((a, 0) \cup (L, b)))$ and $v_u \in L^2(0, T)$ are control functions and will be chosen such that the solution (ρ, u) of (2.25)–(2.27) satisfies the control requirement

$$(\rho(T, \cdot), u(T, \cdot)) = 0 \qquad \text{in} \qquad (a, b). \tag{2.28}$$

The existence of a controlled trajectory (ρ, u) satisfying (2.25)–(2.28) will be obtained through a fixed point argument that we now introduce.

Let us first describe the functional space in which we will define our fixed point map. For $s \geq 1$, we introduce the norms $\|.\|_{\mathscr{X}_s}$ and $\|.\|_{\mathscr{Y}_s}$ defined by

$$\|\rho\|_{\mathscr{X}_s} = s\|e^{s\varphi}\rho\|_{L^\infty(0,T;L^2(a,b))} + \|\theta^{-1}e^{s\varphi}\partial_x\rho\|_{L^\infty(0,T;L^2(a,b))} + \|e^{s\varphi/2}\rho\|_{L^\infty(0,T;L^\infty(a,b))} + \|\partial_t\rho\|_{L^2(0,T;L^2(a,b))}, \tag{2.29}$$

and

$$\begin{aligned}\|u\|_{\mathscr{Y}_s} &= s^{3/2}\|e^{s\varphi}u\|_{L^2(0,T;L^2(a,b))} + s^{1/2}\|\theta^{-1}e^{s\varphi}\partial_x u\|_{L^2(0,T;L^2(a,b))} + s^{-1/2}\|\theta^{-2}e^{s\varphi}\partial_{xx}u\|_{L^2(0,T;L^2(a,b))} \\ &\quad + s^{-1/2}\|\theta^{-2}e^{s\varphi}\partial_t u\|_{L^2(0,T;L^2(a,b))} + s^{1/2}\|\theta^{-1}e^{s\varphi}u\|_{L^\infty(0,T;L^2(a,b))},\end{aligned} \tag{2.30}$$

and the corresponding spaces

$$\mathscr{X}_s = \left\{\rho \in L^2(0,T;H_0^1(a,b)) \quad \text{with} \quad \|\rho\|_{\mathscr{X}_s} < \infty\right\}, \tag{2.31}$$

$$\mathscr{Y}_s = \left\{u \in L^2(0,T;L^2(a,b)) \quad \text{with} \quad \|u\|_{\mathscr{Y}_s} < \infty\right\}. \tag{2.32}$$

For $s \geq 1$ and $R_\rho, R_u > 0$, we also introduce the corresponding balls

$$\mathscr{X}_{s,R_\rho} = \{\rho \in \mathscr{X}_s \text{ with } \|\rho\|_{\mathscr{X}_s} \leq R_\rho\}, \qquad \mathscr{Y}_{s,R_u} = \{u \in \mathscr{Y}_s \text{ with } \|u\|_{\mathscr{Y}_s} \leq R_u\}. \tag{2.33}$$

The fixed point map is then constructed as follows: for $(\widehat{\rho}, \widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$, find control functions (v_ρ, v_u) such that the controlled trajectories (ρ, u) solving

$$\begin{cases} \partial_t\rho + (\overline{u} + u)\partial_x\rho + \overline{\rho}\partial_x u + \left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u}\right)\rho = f(\widehat{\rho}, \widehat{u}) + v_\rho\chi & \text{in} \quad (0,T)\times(a,b), \\ \overline{\rho}(\partial_t u + \overline{u}\partial_x u) - \nu\partial_{xx}u = g(\widehat{\rho}, \widehat{u}) & \text{in} \quad (0,T)\times(a,b), \end{cases} \tag{2.34}$$

with initial data (2.26), source terms

$$f(\widehat{\rho}, \widehat{u}) = -\widehat{\rho}\partial_x\widehat{u} + \frac{\bar{\rho}}{\nu}p'(\bar{\rho})\widehat{\rho} - \partial_x\bar{\rho}\widehat{u}, \tag{2.35}$$

$$g(\widehat{\rho}, \widehat{u}) = -\widehat{\rho}\big(\partial_t(\bar{u} + \widehat{u}) + (\bar{u} + \widehat{u})\partial_x(\bar{u} + \widehat{u})\big) - \bar{\rho}\widehat{u}\partial_x(\bar{u} + \widehat{u}) - p'(\bar{\rho} + \widehat{\rho})\partial_x(\bar{\rho} + \widehat{\rho}) + p'(\bar{\rho})\partial_x\bar{\rho}, \tag{2.36}$$

boundary conditions (2.27) (involving the control v_u), satisfies the controllability requirement(2.28).

Our main task is then to show that the above construction is well-defined for $(\widehat{\rho}, \widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ for a suitable choice of parameters $s \geq 1$, $R_\rho > 0$ and $R_u > 0$, that the corresponding controlled trajectory (ρ, u) can be constructed such that $(\rho, u) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ and then to apply Schauder's fixed point theorem.

System (2.34) is not properly speaking the linearized system of (2.25) due to the term $u\partial_x\rho$ in the equation $(2.34)_{(1)}$, which is quadratic. However, as in [13], this term cannot be handled as a source term due to regularity issues. But still, the controllability of (2.34) can be solved using subsequently two controllability results for linear equations. Indeed, one can first control the equation $(2.34)_{(2)}$ of the velocity and obtain u and v_u from $(\widehat{\rho}, \widehat{u})$, and once u is constructed, the equation $(2.34)_{(1)}$ of the density is a linear transport equation which can be controlled independently. Our approach will then follows this 2-step construction.

Let us start with a controllability result for the equation of the velocity:

Theorem 2.2. *There exist $C_u > 0$, $s_0 \geq 1$, $\lambda \geq 1$, such that for all $s \geq s_0$, for all $R_u \in (0,1)$, for all $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$, for all $u_0 \in H_0^1(a,b)$ and for all $(\widehat{\rho}, \widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$, there exists $v_u \in L^2(0,T)$ such that the solution u of*

$$\begin{cases} \overline{\rho}(\partial_t u + \overline{u}\partial_x u) - \nu\partial_{xx}u = g(\widehat{\rho}, \widehat{u}) & \textit{in} \quad (0,T)\times(a,b), \\ u(t,a) = v_u(t) & \textit{on} \quad (0,T), \\ u(t,b) = 0 & \textit{on} \quad (0,T), \\ u(0,\cdot) = u_0 & \textit{in} \quad (a,b), \end{cases} \tag{2.37}$$

where $g(\widehat{\rho},\widehat{u})$ *is defined as in* (2.36), *satisfies*

$$u(T,\cdot)=0 \qquad in \qquad (a,b). \tag{2.38}$$

Besides, $u\in\mathscr{Y}_s$ *and satisfies the estimate*

$$\|u\|_{\mathscr{Y}_s}\le C_u'(s)\|u_0\|_{H^1(a,b)}+C_uR_\rho+\frac{C_u}{s}R_u+C_uR_u^2, \tag{2.39}$$

where $C_u'(s)$ *depends on the parameter* s.

Theorem 2.2 does not present any significant new difficulty compared to [3, 13]. For sake of completeness, we shall nonetheless provide some details in Section 4.

In a second step, we analyze the controllability properties of the transport equation $(2.34)_{(1)}$:

Theorem 2.3. *Let* λ *and* s_0 *as in Theorem 2.2. There exist* $C_\rho>0$, $s_1\ge s_0$ *such that for all* $s\ge s_1$, *there exists* $\varepsilon(s)>0$ *such that for all* $\varepsilon\in(0,\varepsilon(s)]$, *for all* $(\rho_0,u_0)\in H_0^1(a,b)\times H_0^1(a,b)$ *satisfying* (2.23), *for all* $R_\rho\in(0,\min\{1,\min_{[0,T]\times\mathbb{R}}\overline{\rho}/2\})$, *for all* $R_u\in(0,1)$, *for all* $(\widehat{\rho},\widehat{u})\in\mathscr{X}_{s,R_\rho}\times\mathscr{Y}_{s,R_u}$ *and for* u *constructed in Theorem 2.2, there exists* $v_\rho\in L^2(0,T;L^2((a,0)\cup(L,b)))$ *such that the solution* ρ *of*

$$\begin{cases}\partial_t\rho+(\overline{u}+u)\partial_x\rho+\overline{\rho}\partial_xu+\left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho})+\partial_x\overline{u}\right)\rho=f(\widehat{\rho},\widehat{u})+v_\rho\chi & in \quad (0,T)\times(a,b),\\ \rho(t,a)=0 & for\ t\in(0,T)\quad with\quad (\overline{u}+u)(t,a)>0,\\ \rho(t,b)=0 & for\ t\in(0,T)\quad with\quad (\overline{u}+u)(t,b)<0,\\ \rho(0,\cdot)=\rho_0 & in\quad (a,b),\end{cases} \tag{2.40}$$

where $f(\widehat{\rho},\widehat{u})$ *is defined as in* (2.35), *satisfies*

$$\rho(T,\cdot)=0\ in\ (a,b). \tag{2.41}$$

Besides, $\rho\in\mathscr{X}_s$ *and satisfies the estimate*

$$\|\rho\|_{\mathscr{X}_s}\le C_\rho'(s)\varepsilon+\frac{C_\rho}{\sqrt{s}}(R_\rho+R_u)+C_\rho(R_\rho^2+R_u^2). \tag{2.42}$$

where $C_\rho'(s)$ *depends on the parameter* s.

The proof of Theorem 2.3 is developed in Section 3 and is the main contribution of our work.

The end of the proof of Theorem 1.1 then consists in putting together the aforementioned steps and show that Schauder's fixed point theorem applies. This last point will be explained in Section 5.

Remark 2.4. Theorems 2.2 and 2.3 are designed to be used later in the proof of Theorem 1.1 in Section 5. Therefore, we only focused on giving precise estimates of the controlled trajectories (see (2.39) and (2.42)). In particular, we did not provide precise regularity results on the controls v_u in Theorem 2.2 nor v_ρ in Theorem 2.3. One can check that the proofs of Theorems 2.2 and 2.3 provide controls which are more regular than simply L^2:

- The control v_u in Theorem 2.2 belongs to $H^{3/4}(0,T)$, the space of trace of $L^2(0,T;H^2(a,b))\cap H^1(0,T;L^2(a,b))$.
- The control v_ρ in Theorem 2.3 belongs to $L^\infty(0,T;L^2((a,0)\cup(L,b))$. Note that this regularity result on v_ρ is not sufficient to guarantee that the solution ρ of (2.40) belongs to $L^\infty(0,T;H^1(a,b))$. In fact, this regularity will be deduced during the construction of the controlled trajectory ρ.

Notations. For simplicity of notations, we shall often use the notations

$$\widehat{f}:=f(\widehat{\rho},\widehat{u}),\quad \widehat{g}:=g(\widehat{\rho},\widehat{u}), \tag{2.43}$$

where $f(\widehat{\rho},\widehat{u})$, $g(\widehat{\rho},\widehat{u})$ are respectively defined in (2.35) and (2.36).

3. Control of the density

The proof of Theorem 2.3 is divided into two steps: the first step presents the construction of the controlled trajectory and the second step is devoted to get estimates on it.

3.1. Construction of a controlled trajectory ρ

For the time being, let us fix $f \in L^2(0,T;L^2(a,b))$ and assume that u satisfies

$$u \in L^1(0,T;W^{1,\infty}(a,b)) \cap L^\infty(0,T;L^\infty(a,b)), \qquad \text{with} \quad \|u\|_{L^1(0,T;W^{1,\infty}(a,b))\cap L^\infty(0,T;L^\infty(a,b))} \leq \varepsilon_0, \tag{3.1}$$

for some $\varepsilon_0 > 0$ small enough that will be defined later as the one given by Lemma 3.1.

We then focus on the following controllability problem: find a control $v_\rho \in L^2(0,T;L^2((a,0)\cup(L,b))$ such that the solution ρ of

$$\begin{cases} \partial_t \rho + (\overline{u}+u)\partial_x \rho + \left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x \overline{u}\right)\rho = f - \overline{\rho}\partial_x u + v_\rho \chi & \text{in} \quad (0,T)\times(a,b), \\ \rho(t,a) = 0 & \text{for} \quad t\in(0,T) \quad \text{with } (\overline{u}+u)(t,a) > 0, \\ \rho(t,b) = 0 & \text{for} \quad t\in(0,T) \quad \text{with } (\overline{u}+u)(t,b) < 0, \\ \rho(0,\cdot) = \rho_0 & \text{in} \quad (a,b), \end{cases} \tag{3.2}$$

satisfies (2.41), where χ is as in (2.24).

The construction of such solution is done following the spirit of the construction in [13] by gluing forward and backward solutions of the transport equation. More precisely, using the function η in (2.13), we define ρ_f as the solution of

$$\begin{cases} \partial_t \rho_f + \eta(\overline{u}+u)\partial_x \rho_f + \left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x \overline{u}\right)\rho_f = \eta(f - \overline{\rho}\partial_x u) & \text{in} \quad (0,T)\times(a,b), \\ \rho_f(0,\cdot) = \rho_0 & \text{in} \quad (a,b), \end{cases} \tag{3.3}$$

and ρ_b as the solution of

$$\begin{cases} \partial_t \rho_b + \eta(\overline{u}+u)\partial_x \rho_b + \left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x \overline{u}\right)\rho_b = \eta(f - \overline{\rho}\partial_x u) & \text{in} \quad (0,T)\times(a,b), \\ \rho_b(T,\cdot) = 0 & \text{in} \quad (a,b). \end{cases} \tag{3.4}$$

Remark that in (3.3) and (3.4), we do not need to specify any boundary condition as $(\eta(\overline{u}+u))(t,a) = (\eta(\overline{u}+u))(t,b) = 0$. Besides, since $\rho_0 \in H_0^1(a,b)$,

$$\rho_f(t,a) = \rho_f(t,b) = 0 \qquad \text{for all} \qquad t \in (0,T). \tag{3.5}$$

Similarly, we also have

$$\rho_b(t,a) = \rho_b(t,b) = 0 \qquad \text{for all} \qquad t \in (0,T). \tag{3.6}$$

We now construct a suitable cut-off function $\tilde{\eta}$ traveling along the characteristics:

Lemma 3.1. *Let χ as in (2.24). There exist positive constants $C > 0$ and $\varepsilon_0 \in (0,1)$ such that for all u belonging to $L^1(0,T;W^{1,\infty}(a,b)) \cap L^\infty(0,T;L^\infty(a,b))$ and satisfying (3.1), there exists $\tilde{\eta} = \tilde{\eta}(t,x)$ satisfying the equation*

$$\partial_t \tilde{\eta} + (\overline{u}+u)\partial_x \tilde{\eta} = w\chi \qquad \text{in} \qquad (0,T)\times(a,b), \tag{3.7}$$

for some $w \in L^2(0,T;L^2((a,0)\cup(L,b))$, (recall χ in (2.24) is the indicator function of $(a,0)\cup(0,b)$), the conditions

$$\tilde{\eta}(t,x) = 1 \qquad \text{for all} \qquad (t,x) \in [0,T_0]\times[a,b], \tag{3.8}$$

$$\tilde{\eta}(t,x) = 0 \qquad \text{for all} \qquad (t,x) \in [T_L,T]\times[a,b], \tag{3.9}$$

and the bound

$$\|\tilde{\eta}\|_{W^{1,\infty}((0,T)\times(a,b))} \leq C, \tag{3.10}$$

for some constant C independent of ε_0.

Before going into the proof of Lemma 3.1, let us briefly explain how we can conclude the construction of a controlled trajectory ρ satisfying (3.2) when $u \in L^1(0,T;W^{1,\infty}(a,b)) \cap L^\infty(0,T;L^\infty(a,b))$ satisfies (3.1) for $\varepsilon_0 \in (0,1)$ as in Lemma 3.1.

It simply consists in setting

$$\rho(t,x) = \tilde{\eta}(t,x)\rho_f(t,x) + (1-\tilde{\eta}(t,x))\rho_b(t,x) \qquad \text{for all} \qquad (t,x) \in [0,T]\times[a,b], \tag{3.11}$$

where $\tilde{\eta}$ is the cut-off function constructed in Lemma 3.1. Indeed, one easily checks that ρ defined in that way solves (3.2) with control function

$$v_\rho = (1-\eta)\big[(\overline{u}+u)\big(\tilde{\eta}\partial_x\rho_f + (1-\tilde{\eta})\partial_x\rho_b\big) - f + \overline{\rho}\partial_x u\big] + w(\rho_f-\rho_b)\chi, \tag{3.12}$$

and satisfies

$$\rho(t,a) = \rho(t,b) = 0, \qquad \text{for all } t \in (0,T). \tag{3.13}$$

Furthermore, using (3.8)–(3.9), we immediately get the two following identities:

$$\rho(t,x) = \rho_f(t,x) \qquad \text{for all} \qquad (t,x) \in [0,T_0]\times[a,b], \tag{3.14}$$

$$\rho(t,x) = \rho_b(t,x) \qquad \text{for all} \qquad (t,x) \in [T_L,T]\times[a,b]. \tag{3.15}$$

We now prove Lemma 3.1.

Proof of Lemma 3.1. We first extend u to $(0,T)\times\mathbb{R}$ such that $u \in L^1(0,T;W^{1,\infty}(\mathbb{R}))\cap L^\infty(0,T;L^\infty(\mathbb{R}))$ satisfies

$$\|u\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}))\cap L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C\varepsilon_0 \quad (\leq C \text{ as } \varepsilon_0 \leq 1). \tag{3.16}$$

For $(t,\tau,x) \in [0,T]\times[0,T]\times\mathbb{R}$, we define the flow X associated to $\overline{u}+u$ as the solution of:

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t}(t,\tau,x) = (\overline{u}+u)(t,X(t,\tau,x)) \text{ in } (0,T), \\ X(\tau,\tau,x) = x. \end{cases} \tag{3.17}$$

Writing

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\big(X-\overline{X}\big)(t,\tau,x) = \overline{u}(t,X(t,\tau,x)) - \overline{u}(t,\overline{X}(t,\tau,x)) + u(t,X(t,\tau,x)) \text{ in } (0,T), \\ \big(X-\overline{X}\big)(\tau,\tau,x) = 0, \end{cases}$$

using (1.4) and (3.16), Gronwall's lemma yields

$$\sup_{(t,\tau,x)\in[0,T]\times[0,T]\times\mathbb{R}} \big\{|X(t,\tau,x) - \overline{X}(t,\tau,x)|\big\} \leq C\varepsilon_0.$$

Therefore, using (2.10), for $\varepsilon_0 > 0$ small enough,

$$X(2T_0,0,x_1) < a \text{ and } X(T_L,0,x_1) > b. \tag{3.18}$$

We then perform the construction of $\tilde{\eta}$ in three steps, defining it separately on each time interval $(0,2T_0)$, $(2T_0,T_L)$ and (T_L,T).

On $[0,2T_0]$, we simply set

$$\tilde{\eta}(t,x) = 1 - \eta_1(t)\eta_2(x), \text{ for all } (t,x) \in [0,2T_0]\times\mathbb{R}, \tag{3.19}$$

where $\eta_1 = \eta_1(t) \in C^\infty([0, 2T_0])$ such that $\eta_1(t) = 0$ for all $t \in [0, T_0]$ and $\eta_1(2T_0) = 1$, and $\eta_2 = \eta_2(x) \in C^\infty(\mathbb{R})$ such that $\eta_2(x) = 0$ in $[0, L]$ and $\eta_2(x) = 1$ in $(-\infty, a] \cup [b, \infty)$. On $(2T_0, T_L)$, we solve the equation

$$\begin{cases} \partial_t \tilde{\eta} + (\overline{u} + u)\partial_x \tilde{\eta} = 0 & \text{in} \quad (2T_0, T_L) \times \mathbb{R}, \\ \tilde{\eta}(2T_0, \cdot) = 1 - \eta_1(2T_0)\eta_2(\cdot) & \text{in} \quad \mathbb{R}. \end{cases} \tag{3.20}$$

On the time interval $[T_L, T]$, we set

$$\tilde{\eta}(t, x) = 0, \quad \text{for all} \quad (t, x) \in [T_L, T] \times \mathbb{R}. \tag{3.21}$$

The above piecewise construction (3.19)–(3.20)–(3.21) defines $\tilde{\eta}$ on the whole time interval $[0, T]$. One easily checks that this $\tilde{\eta}$ solves (3.7) on $(0, T) \times (a, b)$ (with $w(t, x) = -1_{[0,2T_0]}(\partial_t \eta_1(t)\eta_2(x) + \eta_1(t)(\overline{u} + u)\partial_x \eta_2(x)))$ as $\tilde{\eta}(2T_0^-, x) = \tilde{\eta}(2T_0^+, x)$ for all $x \in (a, b)$ and $\tilde{\eta}(T_L^-, x) = 0$ for all $x \in (a, b)$ according to (3.18) and the fact that $\tilde{\eta}(2T_0, x) = 0$ for all $x \leq a$. Besides, $\tilde{\eta}$ obviously satisfies (3.8)–(3.9).

We then check that $\tilde{\eta}$ belongs to $W^{1,\infty}((0, T) \times (a, b))$. Of course, the only difficulty is on the time interval $(2T_0, T_L)$. But we can then look at the equation satisfied by $\partial_x \tilde{\eta}$, *i.e.*

$$\begin{cases} \partial_t(\partial_x \tilde{\eta}) + (\overline{u} + u)\partial_x(\partial_x \tilde{\eta}) + \partial_x(\overline{u} + u)\partial_x \tilde{\eta} = 0 & \text{in} \quad (2T_0, T_L) \times \mathbb{R}, \\ \partial_x \tilde{\eta}(2T_0, \cdot) = -\partial_x \eta_2(\cdot) & \text{in} \quad \mathbb{R}, \end{cases}$$

and solve it using the flow X in (3.17). The regularity (1.4) and the bound (3.16) then yields $\partial_x \tilde{\eta} \in L^\infty((0, T) \times (a, b))$ with an explicit bound independent of $\varepsilon_0 \in (0, 1)$. We then deduce that $\partial_t \tilde{\eta} \in L^\infty((0, T) \times (a, b))$ from the equation (3.7) and the fact that $w(t, x) = -1_{[0,2T_0]}(\partial_t \eta_1(t)\eta_2(x) + \eta_1(t)(\overline{u} + u)\partial_x \eta_2(x)) \in L^\infty((0, T) \times ((a, 0) \cup (0, b)))$. This concludes the proof of (3.10). □

3.2. Estimates on ρ

The purpose of this section is to prove that the controlled trajectory ρ constructed in (3.11) satisfies the estimate (2.42) claimed in Theorem 2.3.

We shall then put ourselves in the setting of Theorem 2.3. In particular, in the whole section we assume the following:

Assumption 3.2. Let $(\rho_0, u_0) \in H_0^1(a, b) \times H_0^1(a, b)$ satisfying (2.23) for $\varepsilon > 0$, $(\widehat{\rho}, \widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ for some $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$, $R_u \in (0, 1)$, and u the controlled trajectory given by Theorem 2.2.

Further assume the condition (3.1) for $\varepsilon_0 > 0$ small enough so that the construction in Section 3.1 can be done. The controlled trajectory ρ in (3.11) is constructed for $f = \widehat{f} = f(\widehat{\rho}, \widehat{u})$ defined in (2.35).

Note that due to the continuity of the embedding of $L^2(0, T; H^2(a, b)) \cap H^1(0, T; L^2(a, b))$ into $L^1(0, T; W^{1,\infty}(a, b)) \cap L^\infty(0, T; L^\infty(a, b))$ and $\inf\{s^{3/2}e^{s\varphi}, s^{1/2}\theta^{-1}e^{s\varphi}, s^{-1/2}\theta^{-2}e^{s\varphi}\} \geq Ce^s$, we have

$$\|u\|_{L^1(W^{1,\infty}) \cap L^\infty(L^\infty)} \leq Ce^{-s}\|u\|_{\mathscr{Y}_s} \leq Ce^{-s}\left(C_u'(s)\varepsilon + C_u + \frac{C_u}{s} + C_u\right),$$

with (2.39), (2.23) and $R_\rho, R_u \leq 1$. Thus, the condition (3.1) for $\varepsilon_0 > 0$ given in Lemma 3.1 can be imposed by choosing $s \geq s_1'$ large enough and $\varepsilon \leq 1/C_u'(s)$.

3.2.1. The effective velocity

In order to derive estimates on the controlled trajectory ρ constructed in (3.11), similarly as in [13], we introduce the following quantities, defined for $(t, x) \in [0, T] \times [a, b]$:

$$\mu_f = \eta u + \frac{\nu}{\overline{\rho}^2}\partial_x \rho_f, \qquad \text{and} \qquad \mu_b = \eta u + \frac{\nu}{\overline{\rho}^2}\partial_x \rho_b, \tag{3.22}$$

where η denotes the cut-off function in (2.13).

Tedious computations show that μ_f satisfies

$$\begin{cases} \partial_t \mu_f + \eta(\overline{u}+u)\partial_x \mu_f + k\mu_f = h - \dfrac{\nu}{\overline{\rho}^2}\partial_x\left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x \overline{u}\right)\rho_f & \text{in} \quad (0,T)\times(a,b), \\ \mu_f(0,\cdot) = \eta u_0 + \dfrac{\overline{\rho}^2}{\nu}\partial_x \rho_0 & \text{in} \quad (a,b), \end{cases} \tag{3.23}$$

and μ_b satisfies

$$\begin{cases} \partial_t \mu_b + \eta(\overline{u}+u)\partial_x \mu_b + k\mu_b = h - \dfrac{\nu}{\overline{\rho}^2}\partial_x\left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x \overline{u}\right)\rho_b & \text{in} \quad (0,T)\times(a,b), \\ \mu_b(T,\cdot) = 0 & \text{in} \quad (a,b), \end{cases} \tag{3.24}$$

where

$$k = \frac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x \overline{u} + \partial_x\big(\eta(\overline{u}+u)\big) + 2\frac{\partial_t \overline{\rho}}{\overline{\rho}} + 2\eta(\overline{u}+u)\frac{\partial_x \overline{\rho}}{\overline{\rho}}, \tag{3.25}$$

and

$$\begin{aligned} h = {} & \frac{\eta}{\overline{\rho}}\left(\frac{\nu}{\overline{\rho}}\partial_x \widehat{f} + \widehat{g}\right) + \frac{\nu}{\overline{\rho}^2}\partial_x \eta \widehat{f} + \left[\eta\big(\eta(\overline{u}+u) - \overline{u} - \frac{\nu}{\overline{\rho}^2}\partial_x\overline{\rho}\big) - \frac{\nu}{\overline{\rho}}\partial_x \eta\right]\partial_x u \\ & + \left[\frac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x \overline{u} + \partial_x\big(\eta(\overline{u}+u)\big) + 2\frac{\partial_t \overline{\rho}}{\overline{\rho}} + 2\eta(\overline{u}+u)\frac{\partial_x\overline{\rho}}{\overline{\rho}} + \partial_x \eta(\overline{u}+u)\right]\eta u, \end{aligned} \tag{3.26}$$

in which $\widehat{f} = f(\widehat{\rho},\widehat{u})$ and $\widehat{g} = g(\widehat{\rho},\widehat{u})$ have been defined in (2.35)–(2.36).

The quantities μ_f and μ_b in (3.22) correspond to what is known as the effective velocity, and which has been used for instance in [4,5] in the study of the well-posedness of some models of viscous compressible fluids. These quantities have also been used in [13] in order to get good estimates on the controlled density. However, the approach in [13] looks for a controlled density on the space interval $(0,L)$ and thus requires estimates on the trace of μ at $x=0$ and $x=L$, which cannot be achieved in our case as $t\mapsto \overline{u}(t,a)$ and $t\mapsto\overline{u}(t,b)$ may vanish for some values of time. Our approach avoids this difficulty by a careful discussion of the geometry of the problem. In particular, this allows us to avoid the use of the second parameter λ in the weight function φ in (2.22), contrarily to what is done in [13].

The interest of using the quantities μ_f and μ_b can be understood in terms of regularity. Indeed, when looking at the linearized version of (3.3), it seems that $\partial_x \rho_f$ in $L^\infty(L^2)$ can be estimated in terms of $\partial_{xx} u$ in $L^1(L^2)$. But considering the linearized version of (3.23) instead, it rather seems that μ_f in $L^\infty(L^2)$ can be estimated in terms of $\partial_x u$ in $L^1(L^2)$, and as $\partial_x \rho_f$ in $L^\infty(L^2)$ can be estimated immediately from μ_f and u in $L^\infty(L^2)$, this latter estimate seems better. The goal of the next sections is to make this argument completely rigorous.

3.2.2. Estimates on the coefficients

We start by estimating the coefficients appearing in (3.23)–(3.24):

Lemma 3.3. *There exists $C>0$ independent of $s\geq 1$ and $R_\rho, R_u \in (0,1)$ such that the following estimates hold true for all $s\geq 1$, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \overline{\rho}/2\})$, $R_u\in(0,1)$, $(\widehat{\rho},\widehat{u}) \in \mathscr{X}_{s,R_\rho}\times\mathscr{Y}_{s,R_\rho}$:*

$$\|\theta^{-1}\mathrm{e}^{s\varphi}\widehat{f}\|_{L^2(0,T;L^2(a,b))} \leq \frac{C}{s}(R_\rho + R_u) + C(R_\rho^2 + R_u^2), \tag{3.27}$$

$$\|\theta^{-1}\mathrm{e}^{s\varphi}h\|_{L^2(0,T;L^2(a,b))} \leq C'(s)\|u_0\|_{H^1(a,b)} + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2), \tag{3.28}$$

$$\|k\|_{L^1(0,T;L^\infty(a,b))} \leq C_k'(s)\|u_0\|_{H^1(a,b)} + C, \tag{3.29}$$

where $\widehat{f} = f(\widehat{\rho},\widehat{u})$ is defined in (2.35), h in (3.26), k in (3.25) and $C'(s)$ and $C_k'(s)$ are constants depending on the parameter s.

Proof. In the proof below, we will often denote the norms by omitting the mention of the time and space intervals, *e.g.* $\|\cdot\|_{L^2(L^2)}$ for denoting $\|\cdot\|_{L^2(0,T;L^2(a,b))}$.
We deal with each estimate separately.

• *Proof of estimate* (3.27). We derive the estimate on $\widehat{f}=f(\widehat{\rho},\widehat{u})$ in (2.35) term by term:

$$\|\theta^{-1}\mathrm{e}^{s\varphi}\widehat{\rho}\partial_x\widehat{u}\|_{L^2(L^2)}\le\|\widehat{\rho}\|_{L^\infty(L^\infty)}\|\theta^{-1}\mathrm{e}^{s\varphi}\partial_x\widehat{u}\|_{L^2(L^2)}\le CR_\rho R_u,$$
$$\|\theta^{-1}\mathrm{e}^{s\varphi}\frac{\overline{\rho}}{\nu}p'(\overline{\rho})\widehat{\rho}\|_{L^2(L^2)}\le C\|\mathrm{e}^{s\varphi}\widehat{\rho}\|_{L^2(L^2)}\le\frac{C}{s}R_\rho,$$
$$\|\theta^{-1}\mathrm{e}^{s\varphi}\partial_x\overline{\rho}\widehat{u}\|_{L^2(L^2)}\le C\|\theta^{-1}\mathrm{e}^{s\varphi}\widehat{u}\|_{L^2(L^2)}\le\frac{C}{s^{3/2}}R_u.$$

Estimate (3.27) immediately follows.

• *Proof of estimate* (3.28). Taking the definition of h in (3.26) and using the bound (3.10), we remark that for all $(t,x)\in(0,T)\times(a,b)$,

$$|h|\le C\left[\left|\frac{\nu}{\overline{\rho}}\partial_x\widehat{f}+\widehat{g}\right|+|\widehat{f}|+|u|+|\partial_xu|+|u|^2+|u\partial_xu|\right].\tag{3.30}$$

The estimation for the term $\widehat{f}$ is already done, see (3.27). The terms involving u can be estimated as follows:

$$\begin{aligned}&\|\theta^{-1}\mathrm{e}^{s\varphi}(|u|+|\partial_xu|+|u|^2+|u\partial_xu|)\|_{L^2(L^2)}\\&\le\|\theta^{-1}\mathrm{e}^{s\varphi}u\|_{L^2(L^2)}+\|\theta^{-1}\mathrm{e}^{s\varphi}\partial_xu\|_{L^2(L^2)}+\|\theta^{-1}\mathrm{e}^{s\varphi}u^2\|_{L^2(L^2)}+\|\theta^{-1}\mathrm{e}^{s\varphi}u\partial_xu\|_{L^2(L^2)}\\&\le\frac{C}{s^{3/2}}\|u\|_{\mathscr{Y}_s}+\frac{C}{s^{1/2}}\|u\|_{\mathscr{Y}_s}+\|u\|_{L^\infty(L^\infty)}\|\theta^{-1}\mathrm{e}^{s\varphi}u\|_{L^2(L^2)}+\|u\|_{L^\infty(L^\infty)}\|\theta^{-1}\mathrm{e}^{s\varphi}\partial_xu|\|_{L^2(L^2)}\\&\le\frac{C}{s^{1/2}}\|u\|_{\mathscr{Y}_s}+C\|u\|^2_{\mathscr{Y}_s},\end{aligned}$$

where we used

$$\|u\|_{L^\infty(L^\infty)}\le\|u\|_{L^\infty(H^1)}\le C\|u\|^{1/2}_{H^1(L^2)}\|u\|^{1/2}_{L^2(H^2)}\le C\|u\|_{\mathscr{Y}_s}.\tag{3.31}$$

Consequently, the proof of (3.28) will follow first from the above estimates and (2.39) from one side, and second from an estimate on $\theta^{-1}\mathrm{e}^{s\varphi}(\nu\partial_x\widehat{f}/\overline{\rho}+\widehat{g})$, on which we will focus from now. Recalling (2.35)–(2.36), we obtain

$$\begin{aligned}\frac{\nu}{\overline{\rho}}\partial_x\widehat{f}+\widehat{g}=&-\frac{\nu}{\overline{\rho}}\partial_x\widehat{\rho}\partial_x\widehat{u}-\frac{\nu}{\overline{\rho}}\widehat{\rho}\partial_{xx}\widehat{u}+\frac{\nu}{\overline{\rho}}\partial_x\left(\frac{\overline{\rho}}{\nu}p'(\overline{\rho})\right)\widehat{\rho}+p'(\overline{\rho})\partial_x\widehat{\rho}-\frac{\nu}{\overline{\rho}}\partial_{xx}\overline{\rho}\widehat{u}-\frac{\nu}{\overline{\rho}}\partial_x\overline{\rho}\partial_x\widehat{u}\\&-\widehat{\rho}\left(\partial_t(\overline{u}+\widehat{u})+(\overline{u}+\widehat{u})\partial_x(\overline{u}+\widehat{u})\right)-\overline{\rho}\widehat{u}\partial_x(\overline{u}+\widehat{u})-p'(\overline{\rho}+\widehat{\rho})\partial_x\overline{\rho}+p'(\overline{\rho})\partial_x\overline{\rho}-p'(\overline{\rho}+\widehat{\rho})\partial_x\widehat{\rho}.\end{aligned}\tag{3.32}$$

We then use the following estimates:

$$\begin{aligned}&\|\theta^{-1}\mathrm{e}^{s\varphi}\frac{\nu}{\overline{\rho}}\partial_x\widehat{\rho}\partial_x\widehat{u}\|_{L^2(L^2)}\le C\|\theta^{-1}\mathrm{e}^{s\varphi}\partial_x\widehat{\rho}\|_{L^\infty(L^2)}\|\partial_x\widehat{u}\|_{L^2(L^\infty)}\le CR_\rho\|\partial_{xx}u\|_{L^2(L^2)}\le CR_\rho R_u,\\&\|\theta^{-1}\mathrm{e}^{s\varphi}\frac{\nu}{\overline{\rho}}\widehat{\rho}\partial_{xx}\widehat{u}\|_{L^2(L^2)}\le C\|\widehat{\rho}\|_{L^\infty(L^\infty)}\|\theta^{-1}\mathrm{e}^{s\varphi}\partial_{xx}\widehat{u}\|_{L^2(L^2)}\le CR_\rho R_u,\\&\|\theta^{-1}\mathrm{e}^{s\varphi}\frac{\nu}{\overline{\rho}}\partial_x\big(\frac{\overline{\rho}}{\nu}p'(\overline{\rho})\big)\widehat{\rho}\|_{L^2(L^2)}\le C\|\mathrm{e}^{s\varphi}\widehat{\rho}\|_{L^2(L^2)}\le\frac{C}{s}R_\rho,\\&\|\theta^{-1}\mathrm{e}^{s\varphi}(p'(\overline{\rho})\partial_x\widehat{\rho}-p'(\overline{\rho}+\widehat{\rho})\partial_x\widehat{\rho})\|_{L^2(L^2)}\le C\|\widehat{\rho}\|_{L^\infty(L^\infty)}\|\theta^{-1}\mathrm{e}^{s\varphi}\partial_x\widehat{\rho}\|_{L^2(L^2)}\le CR^2_\rho,\\&\|\theta^{-1}\mathrm{e}^{s\varphi}\frac{\nu}{\overline{\rho}}\partial_{xx}\overline{\rho}\widehat{u}\|_{L^2(L^2)}\le C\|\theta^{-1}\mathrm{e}^{s\varphi}\widehat{u}\|_{L^2(L^2)}\le\frac{C}{s^{3/2}}R_u,\\&\|\theta^{-1}\mathrm{e}^{s\varphi}\frac{\nu}{\overline{\rho}}\partial_x\overline{\rho}\partial_x\widehat{u}\|_{L^2(L^2)}\le C\|\theta^{-1}\mathrm{e}^{s\varphi}\partial_x\widehat{u}\|_{L^2(L^2)}\le\frac{C}{s^{1/2}}R_u,\\&\|\theta^{-1}\mathrm{e}^{s\varphi}(-p'(\overline{\rho}+\widehat{\rho})\partial_x\overline{\rho}+p'(\overline{\rho})\partial_x\overline{\rho})\|_{L^2(L^2)}\le C\|\theta^{-1}\mathrm{e}^{s\varphi}\widehat{\rho}\|_{L^2(L^2)}\le\frac{C}{s}R_\rho.\end{aligned}\tag{3.33}$$

For this last estimate, we have used $p' \in C^1(\mathbb{R}_+^*;\mathbb{R})$ and $\|\rho\|_{L^\infty(L^\infty)} \le R_\rho \le \min_{[0,T]\times\mathbb{R}} \overline{\rho}/2$. Using (3.31) for $\widehat{u}$, we also get

$$\begin{aligned}
&\|\theta^{-1}\mathrm{e}^{s\varphi}\widehat{\rho}\big(\partial_t(\overline{u}+\widehat{u})+(\overline{u}+\widehat{u})\partial_x(\overline{u}+\widehat{u})\big)\|_{L^2(L^2)}\\
&\le \frac{C}{s}R_\rho + C\|s^{1/2}\theta\widehat{\rho}\|_{L^\infty(L^\infty)}\|s^{-1/2}\theta^{-2}\mathrm{e}^{s\varphi}\partial_t\widehat{u}\|_{L^2(L^2)}\\
&\quad + C\|\widehat{\rho}\|_{L^\infty(L^\infty)}\left(\|\theta^{-1}\mathrm{e}^{s\varphi}(|\widehat{u}|+|\partial_x\widehat{u}|)\|_{L^2(L^2)} + \|\widehat{u}\|_{L^\infty(L^\infty)}\|\theta^{-1}\mathrm{e}^{s\varphi}\partial_x\widehat{u}\|_{L^2(L^2)}\right)\\
&\le \frac{C}{s}R_\rho + CR_\rho R_u + CR_\rho R_u^2,
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}
\|\theta^{-1}\mathrm{e}^{s\varphi}\overline{\rho}\widehat{u}\partial_x(\overline{u}+\widehat{u})\|_{L^2(L^2)} &\le C\|\theta^{-1}\mathrm{e}^{s\varphi}\widehat{u}\partial_x\overline{u}\|_{L^2(L^2)} + C\|\theta^{-1}\mathrm{e}^{s\varphi}\widehat{u}\partial_x\widehat{u}\|_{L^2(L^2)}\\
&\le C\|\theta^{-1}\mathrm{e}^{s\varphi}\widehat{u}\|_{L^2(L^2)} + C\|\widehat{u}\|_{L^\infty(L^\infty)}\|\theta^{-1}\mathrm{e}^{s\varphi}\partial_x\widehat{u}\|_{L^2(L^2)} \le \frac{C}{s^{3/2}}R_u + CR_u^2.
\end{aligned} \tag{3.35}$$

Combining the above estimates yields (3.28).

• *Proof of estimate* (3.29). From the definition of k in (3.25), we have

$$|k| \le C(1+|u|+|\partial_x u|).$$

Therefore,

$$\|k\|_{L^1(L^\infty)} \le C + \|u\|_{L^1(L^\infty)} + \|\partial_x u\|_{L^1(L^\infty)} \le C + \|u\|_{L^2(H^1)} + \|u\|_{L^2(H^2)} \le C + C\|u\|_{\mathscr{Y}_s}.$$

Using Theorem 2.2 and $R_\rho, R_u \le 1$, we deduce (3.29). ☐

Remark 3.4. Let us point out that the estimate on h in (3.28) is based on the fact that the combination of the terms $p'(\overline{\rho})\partial_x\widehat{\rho}$ and $p'(\overline{\rho}+\widehat{\rho})\partial_x\widehat{\rho}$ coming from the pressure in $\nu\partial_x\widehat{f}/\overline{\rho}+\widehat{g}$ cancels out at first order in $\widehat{\rho}$, see (3.33). This cancellation motivates the introduction of the term $\overline{\rho}p'(\overline{\rho})\rho/\nu$ in the left hand side of $(2.34)_{(1)}$ and $\overline{\rho}p'(\overline{\rho})\widehat{\rho}/\nu$ in the source term $f(\widehat{\rho},\widehat{u})$ in (2.35).

3.2.3. Energy Lemma

In order to get estimates on μ_f solving (3.23) and μ_b solving (3.24), we remark that both quantities μ_f and μ_b satisfy transport-type equation. Therefore, in this section we explain how to derive weighted estimates on μ_f and μ_b using weighted energy methods. It turns out that we will be able to get good $L^\infty(L^2)$ estimates on $\theta^{-1}e^{s\varphi}\mu_f$ only on the time interval $(0,T_L)$ and on $\theta^{-1}e^{s\varphi}\mu_b$ only on the time interval (T_0,T) as $\partial_t\theta$ has constant sign on each of these intervals (recall (2.21)).

We start with the estimates on the time interval $(0,T_L)$.

Lemma 3.5. *There exists $C>0$ such that for all $s\ge 1$, for all u with $\theta u \in L^1(0,T_L;W^{1,\infty}(a,b))$, for all $K\in L^1(0,T_L;L^\infty(a,b))$, for all $H\in L^2(0,T_L;L^2(a,b))$ with*

$$\|\theta^{-1}\mathrm{e}^{s\varphi}H\|_{L^2(0,T_L;L^2(a,b))} < +\infty, \tag{3.36}$$

and for all $c_0\in L^2(a,b)$, the solution c of

$$\begin{cases} \partial_t c + \eta(\overline{u}+u)\partial_x c + Kc = H & \text{in} \quad (0,T_L)\times(a,b),\\ c(0,\cdot) = c_0 & \text{in} \quad (a,b), \end{cases} \tag{3.37}$$

satisfies

$$\begin{aligned}
&\|\theta^{-1}\mathrm{e}^{s\varphi}c\|_{L^\infty(0,T_L;L^2(a,b))}\\
&\quad\le C\mathrm{e}^{C(1+\|\theta u\|_{L^1(0,T_L;W^{1,\infty}(a,b))}+\|K\|_{L^1(0,T_L;L^\infty(a,b))})}\left(\|\theta^{-1}\mathrm{e}^{s\varphi}H\|_{L^2(0,T_L;L^2(a,b))} + C'(s)\|c_0\|_{L^2(a,b)}\right),
\end{aligned} \tag{3.38}$$

where $C'(s)$ depends on the parameter s.

Proof. Multiplying the equation (3.37) by $\theta^{-2}\mathrm{e}^{2s\varphi}c$ and integrating in space, we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^b \theta^{-2}\mathrm{e}^{2s\varphi}|c|^2\mathrm{d}x\right) = \int_a^b \left(\partial_t(\theta^{-2}\mathrm{e}^{2s\varphi}) + \partial_x\left(\theta^{-2}\mathrm{e}^{2s\varphi}\eta(\overline{u}+u)\right)\right)|c|^2\mathrm{d}x$$
$$- 2\int_a^b \theta^{-2}\mathrm{e}^{2s\varphi}K|c|^2\mathrm{d}x + 2\int_a^b \theta^{-2}\mathrm{e}^{2s\varphi}Hc\mathrm{d}x. \tag{3.39}$$

Using then the choice of the weight function ψ in (2.1), which satisfies the transport equation (2.14), θ in (2.21) and φ in (2.22),

$$\partial_t(\theta^{-2}\mathrm{e}^{2s\varphi}) + \partial_x\left(\theta^{-2}\mathrm{e}^{2s\varphi}\eta(\overline{u}+u)\right) = \partial_t(\theta^{-2}\mathrm{e}^{2s\varphi}) + \eta(\overline{u}+u)\partial_x\left(\theta^{-2}\mathrm{e}^{2s\varphi}\right) + \partial_x(\eta(\overline{u}+u))\theta^{-2}\mathrm{e}^{2s\varphi}$$
$$= 2\partial_t\theta\theta^{-3}\mathrm{e}^{2s\varphi}(-1+s\varphi) - 2s\lambda\mathrm{e}^{\lambda(\psi+6)}u\partial_x\psi\theta^{-1}\mathrm{e}^{2s\varphi} + \partial_x(\eta(\overline{u}+u))\theta^{-2}\mathrm{e}^{2s\varphi}$$
$$\leq C(1+\|\theta u(t)\|_{W^{1,\infty}(a,b)})\theta^{-2}\mathrm{e}^{2s\varphi},$$

as $\partial_t\theta(t) \leq 0$ for $t \in (0, T_L)$ and $s\varphi \geq 2$ for $s \geq 1$.

Therefore, (3.39) yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^b \theta^{-2}\mathrm{e}^{2s\varphi}|c|^2\mathrm{d}x\right) \leq C\left(1+\|\theta u(t)\|_{W^{1,\infty}(a,b)} + \|K(t)\|_{L^\infty(a,b)}\right)\int_a^b \theta^{-2}\mathrm{e}^{2s\varphi}|c|^2\mathrm{d}x + \int_a^b \theta^{-2}\mathrm{e}^{2s\varphi}H^2\mathrm{d}x. \tag{3.40}$$

The estimate (3.38) easily follows from Gronwall's Lemma. □

Using now that $\partial_t\theta(t) \geq 0$ for all $t \in (T_0, T)$ by construction, recall (2.21), we get the following counterpart of Lemma 3.5:

Lemma 3.6. *There exists $C > 0$ such that for all $s \geq 1$, for all u with $\theta u \in L^1(T_0, T; W^{1,\infty}(a,b))$, for all $K \in L^1(T_0, T; L^\infty(a,b))$, for all $H \in L^2(T_0, T; L^2(a,b))$ with*

$$\|\theta^{-1}\mathrm{e}^{s\varphi}H\|_{L^2(T_0,T;L^2(a,b))} < +\infty, \tag{3.41}$$

the solution c of

$$\begin{cases} \partial_t c + \eta(\overline{u}+u)\partial_x c + Kc = H & \text{in} \quad (T_0, T)\times(a,b), \\ c(T,\cdot) = 0 & \text{in} \quad (a,b), \end{cases} \tag{3.42}$$

satisfies

$$\|\theta^{-1}\mathrm{e}^{s\varphi}c\|_{L^\infty(T_0,T;L^2(a,b))} \leq C\mathrm{e}^{C(1+\|\theta u\|_{L^1(T_0,T;W^{1,\infty}(a,b))}+\|K\|_{L^1(T_0,T;L^\infty(a,b))})}\|\theta^{-1}\mathrm{e}^{s\varphi}H\|_{L^2(T_0,T;L^2(a,b))}. \tag{3.43}$$

Proof. The proof is exactly the same as Lemma 3.5. The only minor difference is that φ is singular at the time $t = T$. In order to avoid this difficulty, we introduce $\theta_\delta(t) = \theta(t-\delta)$ for $t \in [T_0+\delta, T]$ and $\theta_\delta(t) = 1$ for $t \in [T_0, T_0+\delta]$. We thus prove the estimate (3.43) with φ replaced by $\varphi_\delta(t,x) = \theta_\delta(t)(\lambda\mathrm{e}^{12\lambda} - \mathrm{e}^{\lambda(\psi(t,x)+6)})$, uniformly with respect to the parameter $\delta > 0$, and we pass to the limit $\delta \to 0$. □

3.2.4. Proof of Theorem 2.3

We are then in position to prove Theorem 2.3.

We start by choosing $\varepsilon \in (0, \varepsilon(s))$ with

$$\varepsilon(s) = \min\{1/C'_u(s), 1/C'_k(s)\},$$

where $C'_u(s)$ and $C'_k(s)$ are the constants in (2.39) and (3.29), so that $\|u\|_{\mathscr{Y}_s} \leq C$ and $\|k\|_{L^1(0,T;L^\infty(a,b))} \leq C$ where k is defined in (3.25). Remark that the continuity of the embedding $H^1(0,T;L^2(a,b)) \cap L^2(0,T;H^2(a,b))$ into $L^1(0,T;W^{1,\infty}(a,b)) \cap L^\infty(0,T;L^\infty(a,b))$ implies

$$\|\theta u\|_{L^1(W^{1,\infty})\cap L^\infty(L^\infty)} \leq C. \tag{3.44}$$

Indeed, introduce

$$\forall t \in [0,T), \quad \theta_0(t) = \begin{cases} 1 & \text{for} \quad t \in [0,T_0], \\ \theta(t) & \text{for} \quad t \in [T_0,T]. \end{cases}$$

Remark that $\theta_0 \leq \theta$ and $0 \leq \partial_t \theta_0 \leq C\theta_0^2$ where C is independent of s. We can then easily check that $\|e^{\theta_0} u\|_{L^2(H^2)\cap H^1(L^2)} \leq C\|u\|_{\mathscr{Y}_s}$. Inequality (3.44) follows from the continuity of the embedding $H^1(0,T;L^2(a,b)) \cap L^2(0,T;H^2(a,b))$ into $L^1(0,T;W^{1,\infty}(a,b)) \cap L^\infty(0,T;L^\infty(a,b))$ and $e^{\theta_0} \geq \theta$.

Recall then that ρ_f and ρ_b have been constructed as solutions of (3.3) and (3.4) respectively, with source term $f = \widehat{f}$. Using then Lemma 3.3 and Lemma 3.5 for ρ_f, or Lemma 3.6 for ρ_b, we get that $\theta^{-1}e^{s\varphi}\rho_f \in L^\infty(0,T_L;L^2(a,b))$ and $\theta^{-1}e^{s\varphi}\rho_b \in L^\infty(T_0,T;L^2(a,b))$.

Using Lemma 3.3, we can apply Lemma 3.5 to the solution μ_f of (3.23) with $K = k$ in (3.25),

$$H = h - \frac{\nu}{\overline{\rho}^2}\partial_x\left(\frac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x \overline{u}\right)\rho_f,$$

where h is defined in (3.26) and $c = \mu_f$ in (3.23):

$$\begin{aligned}\|\theta^{-1}e^{s\varphi}\mu_f\|_{L^\infty(0,T_L;L^2(a,b))} &\leq C\left(\|\theta^{-1}e^{s\varphi}h\|_{L^2(0,T_L;L^2(a,b))} + \|\theta^{-1}e^{s\varphi}\rho_f\|_{L^2(0,T_L;L^2(a,b))} + C'(s)\|\mu_f(0,\cdot)\|_{L^2(a,b)}\right) \\ &\leq C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2) + \|\theta^{-1}e^{s\varphi}\rho_f\|_{L^2(0,T_L;L^2(a,b))}.\end{aligned}$$

Using the definition of μ_f in (3.22), we deduce

$$\|\theta^{-1}e^{s\varphi}\partial_x\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} \leq \frac{C}{\sqrt{s}}\|u\|_{\mathscr{Y}_s} + C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2) + \|\theta^{-1}e^{s\varphi}\rho_f\|_{L^2(0,T_L;L^2(a,b))}.$$

We then use the weighted Poincaré inequality in Lemma A.1 for ρ_f (recall that ρ_f vanishes at $x = a$ and $x = b$, see (3.5)): for $s \geq s_1$ large enough,

$$\begin{aligned}&s\|e^{s\varphi}\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + \|\theta^{-1}e^{s\varphi}\partial_x\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} \\ &\qquad \leq \frac{C}{\sqrt{s}}\|u\|_{\mathscr{Y}_s} + C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2) \leq C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2),\end{aligned}$$

where the last estimate comes from (2.39).

Using then the equation (3.3) satisfied by ρ_f and the estimates (3.27) on $f = \widehat{f}$ and (2.39) on $\partial_x u$, we deduce

$$\begin{aligned}&s\|e^{s\varphi}\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + \|\theta^{-1}e^{s\varphi}\partial_x\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + \|\partial_t\rho_f\|_{L^2(0,T_L;L^2(a,b))} \\ &\qquad \leq \frac{C}{\sqrt{s}}\|u\|_{\mathscr{Y}_s} + C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2) \leq C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2),\end{aligned}$$

We also have

$$\begin{aligned}\|e^{s\varphi/2}\rho_f\|_{L^\infty(0,T_L;L^\infty(a,b))} &\leq C\|e^{s\varphi/2}\rho_f\|_{L^\infty(0,T_L;H^1(a,b))} \\ &\leq Cs\|\theta e^{s\varphi/2}\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + C\|e^{s\varphi/2}\partial_x\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} \\ &\leq Cs\|e^{s\varphi}\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + C\|\theta^{-1}e^{s\varphi}\partial_x\rho_f\|_{L^\infty(0,T_L;L^2(a,b))}.\end{aligned}$$

Combining the above estimates on ρ_f we get

$$\begin{aligned}&s\|e^{s\varphi}\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + \|\theta^{-1}e^{s\varphi}\partial_x\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + \|e^{s\varphi/2}\rho_f\|_{L^\infty(0,T_L;L^\infty(a,b))} \\ &\qquad + \|\partial_t\rho_f\|_{L^2(0,T_L;L^2(a,b))} \leq C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2). \quad (3.45)\end{aligned}$$

Similar computations can be done for μ_b based on Lemma 3.3, Lemma 3.6, the boundary conditions (3.6) satisfied by ρ_b and the weighted Poincaré inequality in Lemma A.1. That way, we achieve:

$$s\|\mathrm{e}^{s\varphi}\rho_b\|_{L^\infty(T_0,T;L^2(a,b))} + \|\theta^{-1}\mathrm{e}^{s\varphi}\partial_x\rho_b\|_{L^\infty(T_0,T;L^2(a,b))} + \|\mathrm{e}^{s\varphi/2}\rho_b\|_{L^\infty(T_0,T;L^\infty(a,b))} + \|\partial_t\rho_b\|_{L^2(T_0,T;L^2(a,b))} \leq C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2). \quad (3.46)$$

We then recall that ρ is given by (3.11) with $\tilde{\eta}$ constructed in Lemma 3.1 and satisfying the bound (3.10). Using then the above estimates (3.45)–(3.46) and (3.14)–(3.15), we get (2.42).

4. Control of the velocity

The purpose of this section is to present the main ingredients of the proof of Theorem 2.2, which mainly consists in a suitable adaptation of ([3], Thms. 2.5 and 2.6).

4.1. Construction of a controlled trajectory u

For the time being let us fix $g \in L^2(0,T;L^2(a,b))$ and $u_0 \in H_0^1(a,b)$, and consider the following controllability problem: Find a control function $v_u \in L^2(0,T)$ such that the solution u of

$$\begin{cases} \overline{\rho}\partial_t u + \overline{\rho u}\partial_x u - \nu\partial_{xx}u = g & \text{in} \quad (0,T)\times(a,b), \\ u(t,a) = v_u & \text{on} \quad (0,T), \\ u(t,b) = 0 & \text{on} \quad (0,T), \\ u(0,\cdot) = u_0 & \text{in} \quad (a,b), \end{cases} \quad (4.1)$$

satisfies the controllability requirement (2.38).

We then claim the following result, strongly inspired in ([3], Thms. 2.5 and 2.6):

Theorem 4.1. *There exist $C > 0$ and $s_0 \geq 1$ such that for all $s \geq s_0$, for all $g \in L^2(0,T;L^2(a,b))$ with*

$$\|\theta^{-3/2}\mathrm{e}^{s\varphi}g\|_{L^2(0,T;L^2(a,b))} < +\infty, \quad (4.2)$$

for all $u_0 \in H_0^1(a,b)$, there exists a controlled trajectory u of (4.1) satisfying (2.38) with the estimate

$$\|u\|_{\mathscr{Y}_s} \leq C'(s)\|u_0\|_{H_0^1(a,b)} + C\|\theta^{-3/2}\mathrm{e}^{s\varphi}g\|_{L^2(0,T;L^2(a,b))}. \quad (4.3)$$

where $C'(s)$ depends on the parameter s.

Sketch of the proof. As in ([13], Sect. 3), we first extend the domain (a,b) into $(2a,b)$ (recall $a<0$), with g and u_0 both extended by 0 on $(2a,a)$, and instead of solving the controllability problem (4.1), (2.38) directly, we consider a distributed control v supported in space in $(2a,a)$. We therefore focus on the following system:

$$\begin{cases} \overline{\rho}\partial_t u + \overline{\rho u}\partial_x u - \nu\partial_{xx}u = g + v\chi_{(2a,a)} & \text{in} \quad (0,T)\times(2a,b), \\ u(t,2a) = u(t,b) = 0 & \text{in} \quad (0,T), \\ u(0,\cdot) = u_0 & \text{in} \quad (2a,b). \end{cases} \quad (4.4)$$

Here, v is the control function and $\chi_{(2a,a)} = \chi_{(2a,a)}(x)$ is the indicator of the space interval $(2a,a)$: $\chi_{(2a,a)}(x) = 1$ for $x \in (2a,a)$ and $\chi_{(2a,a)}(x) = 0$ for $x \in (a,b)$.

Our purpose now is to find $v \in L^2(0,T;L^2(2a,a))$ such that the solution u of (4.4) satisfies

$$u(T,\cdot) = 0 \quad \text{in} \quad (2a,b). \quad (4.5)$$

If we have such a controlled trajectory, then u restricted to (a,b) provides a solution of the controllability problem (4.1), (2.38).

We then use the following weighted observability result, obtained in ([3], Thm. 2.5):

Theorem 4.2. *There exist $C > 0$, $s_0 \geq 1$ and $\lambda \geq 1$ large enough such that for all $z \in C^\infty([0,T]\times[2a,b])$ with $z(t,2a) = z(t,b) = 0$ for all $t \in (0,T)$ and for all $s \geq s_0$, we have*

$$s^3 \int\int_{(0,T)\times(2a,b)} \theta^3 \mathrm{e}^{-2s\varphi}|z|^2 \,\mathrm{d}t\mathrm{d}x + s^2 \int_{2a}^{b} \mathrm{e}^{-2s\varphi(0,\cdot)}|z(0,\cdot)|^2 \mathrm{d}x \\ + s\int\int_{(0,T)\times(2a,b)} \theta \mathrm{e}^{-2s\varphi}|\partial_x z|^2 \,\mathrm{d}t\mathrm{d}x \leq C \int\int_{(0,T)\times(2a,b)} \mathrm{e}^{-2s\varphi}|(-\overline{\rho}\partial_t - \nu\partial_{xx})z|^2 \,\mathrm{d}t\mathrm{d}x \\ + Cs^3 \int\int_{(0,T)\times(2a,a)} \theta^3 \mathrm{e}^{-2s\varphi}|z|^2 \,\mathrm{d}t\mathrm{d}x. \quad (4.6)$$

As it is done classically for control problems, see *e.g.* [16], we then use duality. From now, our approach follows very closely the one in ([3], Thm. 2.6 and Appendix A.2). Namely, for $z \in C^\infty([0,T]\times[2a,b])$ with $z(t,2a) = z(t,b) = 0$ for all $t \in (0,T)$, we define

$$J(z) = \frac{1}{2}\int\int_{(0,T)\times(2a,b)} \mathrm{e}^{-2s\varphi}|(-\partial_t(\overline{\rho}z) - \partial_x(\overline{\rho u}z) - \nu\partial_{xx}z|^2 \mathrm{d}t\mathrm{d}x \\ + \frac{s^3}{2}\int\int_{(0,T)\times(2a,a)} \theta^3\mathrm{e}^{-2s\varphi}|z|^2\mathrm{d}t\mathrm{d}x - \int_{2a}^{b} \overline{\rho}u_0 z(0)\mathrm{d}x - \int\int_{(0,T)\times(2a,b)} gz\mathrm{d}t\mathrm{d}x. \quad (4.7)$$

According to Theorem 4.2, the assumptions (1.4) and condition (4.2), for s large enough, the functional J can be extended as a continuous, strictly convex and coercive function on the set

$$Y_{\mathrm{obs}} = \overline{\{z \in C^\infty([0,T]\times[2a,b]) \text{ with } z(t,2a) = z(t,b) = 0 \text{ for all } t \in (0,T)\}}^{\|\cdot\|_{\mathrm{obs}}}$$

where $\|\cdot\|_{\mathrm{obs}}$ is given by

$$\|z\|^2_{\mathrm{obs}} = \int\int_{(0,T)\times(2a,b)} \mathrm{e}^{-2s\varphi}|-\partial_t(\overline{\rho}z) - \partial_x(\overline{\rho u}z) - \nu\partial_{xx}z|^2 + s^3\int\int_{(0,T)\times(2a,a)} \theta^3\mathrm{e}^{-2s\varphi}|z|^2.$$

Consequently, J admits a minimum $z_{\min}$ on Y_{obs}. Using the Euler-Lagrange equation for J at $z_{\min}$, (u,v) defined by

$$\begin{cases} u = \mathrm{e}^{-2s\varphi}(-\partial_t(\overline{\rho}z_{\min}) - \partial_x(\overline{\rho u}z_{\min}) - \nu\partial_{xx}z_{\min}) & \text{in} \quad (0,T)\times(2a,b), \\ v = -s^3\theta^3\mathrm{e}^{-2s\varphi}z_{\min} & \text{in} \quad (0,T)\times(2a,a), \end{cases} \quad (4.8)$$

solves (4.4)–(4.5), (see [3], Thm. 2.6).

The coercivity of the functional J in (4.7) immediately yields an estimate on the $L^2(L^2)$-norm of $\mathrm{e}^{s\varphi}u$ and $\theta^{-3/2}\mathrm{e}^{s\varphi}v$ in terms of the $L^2(L^2)$-norm of $\theta^{-3/2}\mathrm{e}^{s\varphi}g$ and the L^2 norm of $\mathrm{e}^{s\varphi(0,\cdot)}u_0$:

$$s^{3/2}\|\mathrm{e}^{s\varphi}u\|_{L^2(0,T;L^2(2a,b))} + \|\theta^{-3/2}\mathrm{e}^{s\varphi}v\|_{L^2(0,T;L^2(2a,a))} \leq C\|\theta^{-3/2}\mathrm{e}^{s\varphi}g\|_{L^2(0,T;L^2(2a,b))} + Cs^{1/2}\|\mathrm{e}^{s\varphi(0,\cdot)}u_0\|_{L^2(2a,b)}.$$

The computations to get the $L^2(L^2)$ estimates on $\theta^{-1}\mathrm{e}^{s\varphi}\partial_x u$, $\theta^{-2}\mathrm{e}^{s\varphi}\partial_{xx}u$ and $\theta^{-2}\mathrm{e}^{s\varphi}\partial_t u$ closely follows the ones in ([3], Appendix A.2). The only difference is that [3] considers an initial datum $u_0 = 0$ while we are not. This introduces boundary terms in time $t = 0$ when doing the weighted energy estimates ([3], Appendix A.2), which are all bounded by the $H^1_0(a,b)$-norm of $u_0\mathrm{e}^{s\varphi(0,\cdot)}$. To be more precise, following ([3], Appendix A.2), we get

$$s^{3/2}\|\mathrm{e}^{s\varphi}u\|_{L^2(0,T;L^2(2a,b))} + s^{1/2}\|\theta^{-1}\mathrm{e}^{s\varphi}\partial_x u\|_{L^2(0,T;L^2(a,b))} \\ + \|\theta^{-3/2}\mathrm{e}^{s\varphi}v\|_{L^2(0,T;L^2(2a,a))} + s^{-1/2}\|\theta^{-2}\mathrm{e}^{s\varphi}(\partial_t u, \partial_{xx}u)\|_{L^2(0,T;L^2(a,b))} \\ \leq C\|\theta^{-3/2}\mathrm{e}^{s\varphi}g\|_{L^2(0,T;L^2(2a,b))} + Cs^{1/2}\|\mathrm{e}^{s\varphi(0,\cdot)}u_0\|_{H^1_0(2a,b)}. \quad (4.9)$$

In order to conclude (4.3), we shall also get an $L^\infty(0,T;L^2(a,b))$ norm of $\theta^{-1}\mathrm{e}^{s\varphi}u$ which has not been derived in ([3], Thm. 2.6), though this estimate can also be obtained by weighted energy estimates. Indeed, if we multiply (4.4) by $s\theta^{-2}\mathrm{e}^{2s\varphi}u$ and integrate on $(0,t)\times(2a,b)$, we easily obtain:

$$\begin{aligned} s\int_{2a}^{b}\overline{\rho}(t)\theta(t)^{-2}\mathrm{e}^{2s\varphi(t)}|u(t)|^2\mathrm{d}x &= s\int\int_{(0,t)\times(2a,b)}\partial_t\big(\overline{\rho}\theta^{-2}\mathrm{e}^{2s\varphi}\big)u^2 - 2s\int\int_{(0,t)\times(2a,b)}\overline{\rho u}\theta^{-2}\mathrm{e}^{2s\varphi}u\partial_x u \\ &+ 2s\nu\int\int_{(0,t)\times(2a,b)}\overline{\rho}\theta^{-2}\mathrm{e}^{2s\varphi}u\partial_{xx}u + 2s\int\int_{(0,t)\times(2a,b)}\theta^{-2}\mathrm{e}^{2s\varphi}(g+v\chi_{(2a,a)})u \\ &+ s\int_{2a}^{b}\overline{\rho}(0,\cdot)\theta(0)^{-2}\mathrm{e}^{2s\varphi(0,\cdot)}|u_0|^2\mathrm{d}x. \end{aligned} \tag{4.10}$$

Using then (4.9) and the fact that

$$\partial_t(\theta^{-2}\mathrm{e}^{2s\varphi}) \le Cs\mathrm{e}^{2s\varphi} \text{ in } (0,T)\times(2a,b),$$

we easily conclude (4.3). □

4.2. Estimates on $\widehat{g} = g(\widehat{\rho},\widehat{u})$ in (2.36)

We wish to apply Theorem 4.1 to $g = \widehat{g} = g(\widehat{\rho},\widehat{u})$ defined in (2.36). We shall therefore show that for $(\widehat{\rho},\widehat{u}) \in \mathscr{X}_{s,R_\rho}\times\mathscr{Y}_{s,R_u}$, $\widehat{g}$ satisfies (4.2):

Lemma 4.3. *There exists $C>0$ such that for all $s\ge 1$, $R_\rho\in(0,\min\{1,\min_{[0,T]\times\mathbb{R}}\overline{\rho}/2\})$, $R_u\in(0,1)$ and $(\widehat{\rho},\widehat{u})\in\mathscr{X}_{s,R_\rho}\times\mathscr{Y}_{s,R_u}$, $\widehat{g}=g(\widehat{\rho},\widehat{u})$ in (2.36) satisfies*

$$\|\theta^{-3/2}\mathrm{e}^{s\varphi}\widehat{g}\|_{L^2(0,T;L^2(a,b))} \le CR_\rho + \frac{C}{s}R_u + CR_u^2. \tag{4.11}$$

Proof. Some terms in $\widehat{g}$ were already estimated in (3.34)–(3.35):

$$\begin{aligned} &\|\theta^{-3/2}\mathrm{e}^{s\varphi}\widehat{\rho}\big(\partial_t(\overline{u}+\widehat{u})+(\overline{u}+\widehat{u})\partial_x(\overline{u}+\widehat{u})\big)\|_{L^2(L^2)} + \|\theta^{-3/2}\mathrm{e}^{s\varphi}\overline{\rho}\widehat{u}\partial_x(\overline{u}+\widehat{u})\|_{L^2(L^2)} \\ &\le \|\theta^{-1}\mathrm{e}^{s\varphi}\widehat{\rho}\big(\partial_t(\overline{u}+\widehat{u})+(\overline{u}+\widehat{u})\partial_x(\overline{u}+\widehat{u})\big)\|_{L^2(L^2)} + \|\theta^{-1}\mathrm{e}^{s\varphi}\overline{\rho}\widehat{u}\partial_x(\overline{u}+\widehat{u})\|_{L^2(L^2)} \\ &\le \frac{C}{s}(R_\rho+R_u)+CR_u^2. \end{aligned}$$

We then only have to estimate the remaining terms in $\widehat{g}$. Since $p'\in C^1(\mathbb{R}_+^*;\mathbb{R})$ and $\|\widehat{\rho}\|_{L^\infty(L^\infty)}\le R_\rho\le\min_{[0,T]\times\mathbb{R}}\overline{\rho}/2$, one has

$$\begin{aligned} &\|\theta^{-3/2}\mathrm{e}^{s\varphi}\left(-p'(\overline{\rho}+\widehat{\rho})\partial_x(\overline{\rho}+\widehat{\rho})+p'(\overline{\rho})\partial_x\overline{\rho}\right)\|_{L^2(L^2)} \\ &\le \|\theta^{-3/2}\mathrm{e}^{s\varphi}p'(\overline{\rho}+\widehat{\rho})\partial_x\widehat{\rho}\|_{L^2(L^2)} + \|\theta^{-3/2}\mathrm{e}^{s\varphi}(p'(\overline{\rho})-p'(\overline{\rho}+\widehat{\rho}))\partial_x\overline{\rho})\|_{L^2(L^2)} \\ &\le C(1+\|\widehat{\rho}\|_{L^\infty(L^\infty)})\|\theta^{-3/2}\mathrm{e}^{s\varphi}\partial_x\widehat{\rho}\|_{L^2(L^2)} + C\|\theta^{-3/2}\mathrm{e}^{s\varphi}\widehat{\rho}\|_{L^2(L^2)} \\ &\le CR_\rho + CR_\rho^2 \le CR_\rho. \end{aligned}$$

Combining the above estimates yields (4.11) and concludes the proof of Lemma 4.3. □

4.3. End of the proof of Theorem 2.2

For $s\ge1$, $R_\rho\in(0,\min\{1,\min_{[0,T]\times\mathbb{R}}\overline{\rho}/2\})$, $R_u\in(0,1)$, $(\widehat{\rho},\widehat{u})\in\mathscr{X}_{s,R_\rho}\times\mathscr{Y}_{s,R_u}$, and $\widehat{g}=g(\widehat{\rho},\widehat{u})$ in (2.36), applying Lemma 4.3, $\widehat{g}$ satisfies (4.2). We can then apply Theorem 4.1 to $g=\widehat{g}$ and concludes Theorem 2.2 simply by putting together estimates (4.3) and (4.11).

5. The fixed point argument

Theorems 2.2 and 2.3 allow to define, for $s \geq s_1$ large enough, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$, $R_u \in (0,1)$ and $\varepsilon \in (0, \varepsilon(s))$ in (2.23) small enough, a map $\mathscr{F} : (\widehat{\rho}, \widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u} \mapsto (\rho, u) \in \mathscr{X}_s \times \mathscr{Y}_s$, where

- u is the solution of the control problem (2.37)–(2.38) given by Theorem 2.2 with $g(\widehat{\rho}, \widehat{u})$ defined in (2.36),
- ρ is the solution of the control problem (2.40)–(2.41) given by Theorem 2.3 with $f(\widehat{\rho}, \widehat{u})$ defined in (2.35).

We then choose the parameters $s \geq s_1$, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$, $R_u \in (0,1)$ and $\varepsilon \in (0, \varepsilon(s)]$ in (2.23) where $\varepsilon(s)$ is given in Theorem 2.3, such that $\mathscr{F}$ maps $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ into itself. This can be done according to the following lemma:

Lemma 5.1. *Let $C_u, C_u'(s)$ and $C_\rho, C_\rho'(s), s_1, \varepsilon(s)$ be the constants in Theorems 2.2 and 2.3 respectively. There exist $s \geq s_1$, $R_u \in (0,1)$, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$, $\varepsilon \in (0, \varepsilon(s)]$ such that*

$$C_u'(s)\varepsilon + C_u R_\rho + \frac{C_u}{s} R_u + C_u R_u^2 \leq R_u, \quad \textit{and} \quad C_\rho'(s)\varepsilon + \frac{C_\rho}{\sqrt{s}}(R_\rho + R_u) + C_\rho(R_\rho^2 + R_u^2) \leq R_\rho. \tag{5.1}$$

Proof. We set $C_0 = \max\{C_\rho, C_u, 1, (2\min_{[0,T]\times\mathbb{R}} \bar{\rho})^{-1}\}$ and $C_0'(s) = \max\{C_\rho'(s), C_u'(s)\}$, and we look for parameters s, R_ρ, R_u and ε such that

$$C_0'(s)\varepsilon + C_0 R_\rho + \frac{C_0}{s} R_u + C_0 R_u^2 \leq R_u, \quad \text{and} \quad C_0'(s)\varepsilon + \frac{C_0}{\sqrt{s}}(R_\rho + R_u) + C_0(R_\rho^2 + R_u^2) \leq R_\rho.$$

We thus choose

$$R_u = \frac{1}{12C_0^2 + 3}, \quad R_\rho = \frac{1}{4C_0} R_u.$$

so that $R_u \in (0,1)$, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$ and satisfies:

$$C_0 R_u^2 \leq \frac{R_u}{4}, \qquad C_0 R_\rho \leq \frac{R_u}{4}, \qquad C_0(R_\rho^2 + R_u^2) \leq \frac{R_\rho}{3}\cdot$$

We then choose

$$s = \max\{s_1, 4C_0, 9C_0^2(1+4C_0)^2\},$$

which guarantees

$$\frac{C_0}{s} R_u \leq \frac{R_u}{4}, \qquad \frac{C_0}{\sqrt{s}}(R_\rho + R_u) \leq \frac{R_\rho}{3}\cdot$$

Lastly, we choose $\varepsilon > 0$ as follows:

$$\varepsilon = \min\left\{\varepsilon(s), \frac{R_\rho}{3C_0'(s)}, \frac{R_u}{4C_0'(s)}\right\}.$$

One then easily checks that the inequalities (5.1) are satisfied with these choices of parameters, and this concludes the proof of Lemma 5.1. □

We thus fix the parameters $s \geq s_1$, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$, $R_u \in (0,1)$ and $\varepsilon \in (0, \varepsilon(s)]$ such that the inequalities (5.1) are satisfied. Using Theorems 2.2 and 2.3, we then have that $\mathscr{F}$ maps $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ into itself.

We are then left to check that $\mathscr{F}$ satisfies the assumptions of Schauder's fixed point theorem. In order to do that, we endow the set $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ with the $L^2(0,T;L^2(a,b))$-topology, which makes this set compact (for the $L^2(0,T;L^2(a,b))$-topology) by Aubin−Lions' theorem, see [26].

It thus remains to prove that the map $\mathscr{F}$ is continuous on $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ for the $L^2(0,T;L^2(a,b))$ topology. This can be done as in ([13], Sect. 5.3), but we recall the main ingredients for the convenience of the reader. Let us then consider a sequence $(\widehat{\rho}_n, \widehat{u}_n)$ in $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ converging in $L^2(0,T;L^2(a,b))$ towards some $(\widehat{\rho}, \widehat{u})$, and set $(\rho_n, u_n) = \mathscr{F}(\widehat{\rho}_n, \widehat{u}_n)$, $(\rho, u) = \mathscr{F}(\widehat{\rho}, \widehat{u})$. As the sequence $(\widehat{\rho}_n, \widehat{u}_n)$ is bounded in $\mathscr{X}_s \times \mathscr{Y}_s$, we then also have the following weak convergences:

$$\widehat{\rho}_n \underset{n\to\infty}{\rightharpoonup} \widehat{\rho} \quad \text{weakly * in } L^\infty(0,T;H^1(a,b)) \cap H^1(0,T;L^2(a,b)), \tag{5.2}$$

$$\widehat{u}_n \underset{n\to\infty}{\rightharpoonup} \widehat{u} \quad \text{weakly in } L^2(0,T;H^2(a,b)) \cap H^1(0,T;L^2(a,b)). \tag{5.3}$$

Using interpolations of L^p and H^1 spaces (see [27], Sect. 4.3.1, Thm. 1), the functional space $L^p(0,T;H^1(a,b)) \cap H^1(0,T;L^2(a,b))$ is continuously embedded into $W^{1/4,q}(0,T;H^{3/4}(a,b))$ (see [27], Sects. 2.3.1 and 4.3.1]) with q given by

$$\frac{1}{q} = \frac{3}{4} \times \frac{1}{p} + \frac{1}{4} \times \frac{1}{2} = \frac{3}{4p} + \frac{1}{8}\cdot$$

But, for $p > 6$, we have $q > 4$ and thus $W^{1/4,q}(0,T)$ embeds into some Hölder spaces $C^{0,\alpha(q)}(0,T)$ with $\alpha(q) > 0$ (see [27], Sect. 4.6.1). Therefore, using the compact embedding of the spaces of Hölder spaces into the space of continuous function (Ascoli's theorem), the space $L^\infty(0,T;H^1(a,b)) \cap H^1(0,T;L^2(a,b))$ is compactly embedded into the set of continuous functions $C^0([0,T] \times [a,b])$. Furthermore, using Aubin Lions' Lemma, we also have a compact embedding of $L^2(0,T;H^2(a,b)) \cap H^1(0,T;L^2(a,b))$ in $L^\infty(0,T;L^\infty(a,b))$. We finally obtain the following strong convergences:

$$\widehat{\rho}_n \underset{n\to\infty}{\to} \widehat{\rho} \qquad \text{strongly in} \quad L^\infty(0,T;L^\infty(a,b)), \tag{5.4}$$

$$\widehat{u}_n \underset{n\to\infty}{\to} \widehat{u} \qquad \text{strongly in} \quad L^\infty(0,T;L^\infty(a,b)). \tag{5.5}$$

We then easily show that

$$(f(\widehat{\rho}_n, \widehat{u}_n), g(\widehat{\rho}_n, \widehat{u}_n)) \underset{n\to\infty}{\rightharpoonup} (f(\widehat{\rho}, \widehat{u}), g(\widehat{\rho}, \widehat{u})) \qquad \text{in} \quad (\mathscr{D}'((0,T) \times (a,b)))^2. \tag{5.6}$$

The control process in Theorem 4.1 is linear in (u_0, g), and therefore, following the construction done in Section 4, u_n weakly converges to u in $\mathscr{D}'((0,T) \times (a,b))$. As $\mathscr{F}$ maps $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ into itself, u_n is bounded in $\mathscr{Y}_s$ and we can therefore also deduce the convergences

$$u_n \underset{n\to\infty}{\rightharpoonup} u \quad \text{weakly in } L^2(0,T;H^2(a,b)) \cap H^1(0,T;L^2(a,b)), \tag{5.7}$$

$$u_n \underset{n\to\infty}{\to} u \quad \text{strongly in } L^\infty(0,T;L^\infty(a,b)). \tag{5.8}$$

We then focus on the construction of ρ_n, ρ performed in Section 3.1 and its continuity with respect to u_n and $f(\widehat{\rho}_n, \widehat{u}_n)$. We then introduce $\rho_{f,n}$ the solution of (3.3) with u_n instead of u and $f = f(\widehat{\rho}_n, \widehat{u}_n)$. Due to (3.45), $\rho_{f,n}$ is uniformly bounded in $H^1((0,T_L) \times (a,b))$ and therefore weakly converges to some ρ_f^* in $H^1((0,T_L) \times (a,b))$. Using the convergences (5.8) and (5.6), we can pass to the limit in the equation satisfied by $\rho_{f,n}$ and then obtain that ρ_f^* is the solution ρ_f of (3.3) with u and $f = f(\widehat{\rho}, \widehat{u})$. Similarly, the solutions $\rho_{b,n}$ of (3.4) with u_n instead of u and $f = f(\widehat{\rho}_n, \widehat{u}_n)$ weakly converge in $H^1((T_0,T) \times (a,b))$ to the solution ρ_b of (3.4) with u and $f = f(\widehat{\rho}, \widehat{u})$. It is then easy to check that the construction of the cut-off function $\tilde{\eta}$ in Lemma 3.1 is continuous with respect to u. Indeed, if we call $\tilde{\eta}_n$ the cut-off functions constructed in Lemma 3.1 corresponding to u_n, as the sequence $\tilde{\eta}_n$ is uniformly bounded in $H^1((0,T) \times (a,b))$, recall (3.10), we can pass to the limit in the construction and get that $\tilde{\eta}_n$ weakly converges in $H^1((0,T) \times (a,b))$ to $\tilde{\eta}$, the cut-off function constructed in Lemma 3.1 corresponding to u, and thus strongly converges in $L^2(0,T;L^2(a,b))$ according to Aubin−Lions' Lemma. Therefore, the sequence $\rho_n = \tilde{\eta}_n \rho_{f,n} + (1 - \tilde{\eta}_n)\rho_{b,n}$ weakly converges to $\rho = \tilde{\eta}\rho_f + (1 - \tilde{\eta})\rho_b$ in $\mathscr{D}'((0,T) \times (0,L))$.

We have thus shown that $\mathscr{F}(\widehat{\rho}_n, \widehat{u}_n)$ weakly converges as $n \to \infty$ towards $\mathscr{F}(\widehat{\rho}, \widehat{u})$ in $(\mathscr{D}'((0,T) \times (a,b)))^2$. Moreover, the sequence $\mathscr{F}(\widehat{\rho}_n, \widehat{u}_n)$ is bounded in $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$. Recall then that this set is compact for the $(L^2(0,T;L^2(a,b)))^2$ topology, so that the sequence $\mathscr{F}(\widehat{\rho}_n, \widehat{u}_n)$ strongly converges in $(L^2(0,T;L^2(a,b)))^2$ to $\mathscr{F}(\widehat{\rho}, \widehat{u})$. This proves that $\mathscr{F}$ is continuous on $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ endowed with the $(L^2(0,T;L^2(a,b)))^2$ topology.

We can then use Schauder's fixed point theorem for $\mathscr{F}$ defined on the set $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ endowed with the $(L^2(0,T;L^2(a,b)))^2$ topology. Let (ρ, u) be a fixed point of $\mathscr{F}$. By construction, $(\rho, u) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ and solves the controllability problem (2.25)–(2.26)–(2.28) for some $v_\rho \in L^2(0,T;L^2((a,0) \cup (L,b))$ and $v_u \in L^2(0,T)$. The restriction of (ρ, u) on the space interval $(0,L)$ provides a solution of (2.2)–(2.4)–(2.5). Therefore, (ρ_s, u_s) in (2.1) is a controlled solution of (1.1) satisfying the initial condition (1.8) and the controllability requirement (1.9) with the regularity stated in (1.10). Besides, remark that the positivity of the density ρ_s is assured by Lemma 5.1 since $R_\rho < \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2$. This concludes the proof of Theorem 1.1.

Appendix A. Weighted Poincaré inequality

Here, we recall the following result, proved for instance in ([13], Lem. 4.9):

Lemma A.1 (A weighted Poincaré inequality [13], Lem. 4.9)**.** *Let φ as in (2.22) with θ and ψ as in (2.21) and Lemma 2.1. There exist constants $C > 0$ and $s_2 \geq 1$ such that for all $s \geq s_2$, for all $t \in [0,T)$ and for all $f \in H_0^1(a,b)$,*

$$s \|e^{s\varphi(t,\cdot)} f\|_{L^2(a,b)} \leq C \|\theta^{-1}(t) e^{s\varphi(t,\cdot)} \partial_x f\|_{L^2(a,b)}. \tag{A.1}$$

The proof of Lemma A.1 is not difficult and simply requires that ψ in Lemma 2.1 does not have any critical point in $[a,b]$, *i.e.* Assumption (2.17). We refer the interested reader to ([13], Lem. 4.9) for a detailed proof.

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References

[1] P. Albano and D. Tataru, Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system. *Electron. J. Differ. Equ.* **22** (2000) 15.

[2] E.V. Amosova, Exact local controllability for the equations of viscous gas dynamics. *Differ. Uravneniya* **47** (2011) 1754–1772.

[3] M. Badra, S. Ervedoza and S. Guerrero, Local controllability to trajectories for non-homogeneous incompressible Navier-Stokes equations. *Ann. Inst. Henri Poincaré (C) Non Lin. Anal.* (2014).

[4] D. Bresch and B. Desjardins, Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. *Commun. Math. Phys.* **238** (2003) 211–223.

[5] D. Bresch, B. Desjardins and C.-K. Lin, On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Commun. Partial Differ. Equ.* **28** (2003) 843–868.

[6] F.W. Chaves-Silva, L. Rosier and E. Zuazua, Null controllability of a system of viscoelasticity with a moving control. *J. Math. Pures Appl.* **101** (2014) 198–222.

[7] S. Chowdhury, M. Debanjana, M. Ramaswamy and M. Renardy, Null controllability of the linearized compressible Navier Stokes system in one dimension. *J. Differ. Equ.* **257** (2014) 3813–3849.

[8] S. Chowdhury, M. Ramaswamy and J.-P. Raymond, Controllability and stabilizability of the linearized compressible Navier-Stokes system in one dimension. *SIAM J. Control Optim.* **50** (2012) 2959–2987.

[9] J.-M. Coron, On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions. *ESAIM: COCV* **1** (1996) 35–75.

[10] J.-M. Coron, On the controllability of 2-D incompressible perfect fluids. *J. Math. Pures Appl.* **75** (1996) 155–188.

[11] J.-M. Coron, Control and nonlinearity, Vol. 136 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI (2007).

[12] J.-M. Coron and A.V. Fursikov, Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary. *Russian J. Math. Phys.* **4** (1996) 429–448.

[13] S. Ervedoza, O. Glass, S. Guerrero and J.-P. Puel, Local exact controllability for the one-dimensional compressible Navier-Stokes equation. *Arch. Ration. Mech. Anal.* **206** (2012) 189–238.

[14] E. Fernández-Cara, S. Guerrero, O.Y. Imanuvilov and J.-P. Puel, Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl.* **83** (2004) 1501–1542.

[15] A.V. Fursikov and O. Yu. Èmanuilov, Exact controllability of the Navier-Stokes and Boussinesq equations. *Uspekhi Mat. Nauk* **54** (1999) 93–146.

[16] A.V. Fursikov and O.Y. Imanuvilov, Controllability of evolution equations, Vol. 34 of *Lecture Notes Series*. Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul (1996).

[17] O. Glass, Exact boundary controllability of 3-D Euler equation. *ESAIM: COCV* **5** (2000) 1–44.

[18] O. Glass, On the controllability of the 1-D isentropic Euler equation. *J. Eur. Math. Soc. (JEMS)* **9** (2007) 427–486.

[19] O. Glass, On the controllability of the non-isentropic 1-D Euler equation. *J. Differ. Equ.* **257** (2014) 638–719.

[20] O.Y. Imanuvilov, Remarks on exact controllability for the Navier-Stokes equations. *ESAIM: COCV* **6** (2001) 39–72.

[21] T.-T. Li and B.-P. Rao, Exact boundary controllability for quasi-linear hyperbolic systems. *SIAM J. Control Optim.* **41** (2003) 1748–1755.

[22] D. Maity, Some controllability results for linearized compressible Navier-Stokes system. *ESAIM: COCV* **21** (2015) 1002–1028.

[23] P. Martin, L. Rosier and P. Rouchon, Null controllability of the structurally damped wave equation with moving control. *SIAM J. Control Optim.* **51** (2013) 660–684.

[24] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.* **20** (1980) 67–104.

[25] L. Rosier and P. Rouchon, On the controllability of a wave equation with structural damping. *Int. J. Tomogr. Stat.* **5** (2007) 79–84.

[26] J. Simon, Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl.* **146** (1987) 65–96.

[27] H. Triebel, Interpolation theory, function spaces, differential operators. North-Holland Publishing Co. Amsterdam New York (1978).