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CONTROL PROBLEM ON SPACE OF RANDOM VARIABLES AND MASTER EQUATION

Alain Bensoussan[1,2,*] and Sheung Chi Phillip Yam[3]

Abstract. In this article, we study a control problem in an appropriate space of random variables; in fact, in our set up, we can consider an arbitrary Hilbert space, yet we specialize only to a Hilbert space of square-integrable random variables. We see that the control problem can then be related to a mean field type control problem. We explore here a suggestion of Lions in (Lectures at College de France, http://www.college-de-france.fr) and (Seminar at College de France). Mean field type control problems are control problems in which functionals depend on probability measures of the underlying controlled process. Gangbo and Święch [*J. Differ. Equ.* **259** (2015) 6573–6643] considered this type of problem in the space of probability measures equipped with the Wasserstein metric and use the concept of Wasserstein gradient; their work provides a completely rigorous treatment, but it is quite intricate, because metric spaces are not vector spaces. The approach suggested by Lions overcomes this difficulty. Nevertheless, our present proposed approach also benefits from the useful concept of L-derivatives as introduced in a recent interesting treatise of Carmona and Delarue [Probabilistic Theory of Mean Field Games with Applications. Springer Verlag (2017)]. We also consider Bellman equation and the Master equation of mean field type control. We provide also some extension of the results of Gangbo and Święch [*J. Differ. Equ.* **259** (2015) 6573–6643].

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1. Introduction

We study first an abstract control problem where the state is in a generic Hilbert space. We then show how this model applies when the Hilbert space is the space of square-integrable random variables and for certain forms of the cost functionals. In particular, we are interested in cost functionals depending on probability measures of random variables. In this case, we can make the connection with mean field type control problems. In mean field type control problems, big issues concern Bellman equation and the Master equation; also see [6, 7]. They

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[1] International Center for Decision and Risk Analysis, Jindal School of Management, University of Texas at Dallas, Richardson, USA.
[2] Department of System Engineering and Engineering Management, College of Science and Engineering, City University of Hong Kong, Kowloon Tong, Hong Kong.
[3] Department of Statistics, The Chinese University of Hong Kong, Shatin, Hong Kong.

* Corresponding author: axb046100@utdallas.edu

are partial differential equations in infinite dimensional spaces. A rigorous approach could be quite involving and complex. In this paper, we specialize to the case studied by Gangbo and Święch [13], and show that the approach of the Hilbert space of square integrable random variables simplifies greatly the development. The idea of using the Hilbert space of square-integrable random variables has been suggested by Lions, and is exploited for interpretation purposes. Here, we want to exploit it completely, and make the comparison with the work of Gangbo and Święch [13]; namely, our proposed approach is *via* the idea of control problems in Hilbert spaces of square integrable random variables. It is worth to note that the involved technicality and mathematical theory are both greatly simplified, and this new framework is apparent to be readily generalized to alternative settings. The issue is then the interpretation, back to the mean field formulation, of the corresponding control problem which has been solved, by relatively standard methods in Control Theory, but now in a Hilbert space setting. Besides, to streamline our approach, we also adopt some very useful and recent results introduced in Carmona and Delarue [11], together with a more explicit formulation of the mean field type control problem.

2. An abstract control problem

2.1. Setting of the problem

We begin by defining an abstract control problem in a generic Hilbert space, without describing the application. We consider a Hilbert space $\mathcal{H}$, whose general elements are denoted by X. We identify $\mathcal{H}$ with its dual space $\mathcal{H}^*$. The inner(scalar) product is denoted by $((\cdot,\cdot))$ and the corresponding norm by $||\cdot||$. We then consider functionals, the running cost $\mathcal{F}(X)$, and the terminal cost $\mathcal{F}_T(X)$, which are assumed to be continuously (Fréchet) differentiable in $\mathcal{H}$. Also assume that the gradients $D_X\mathcal{F}(X)$ and $D_X\mathcal{F}_T(X)$ are both Lipschitz continuous:

$$\begin{aligned} ||D_X\mathcal{F}(X_1) - D_X\mathcal{F}(X_2)|| &\leq c||X_1 - X_2||, \\ ||D_X\mathcal{F}_T(X_1) - D_X\mathcal{F}_T(X_2)|| &\leq c||X_1 - X_2||. \end{aligned} \tag{2.1}$$

To simplify notation, we shall also assume that

$$||D_X\mathcal{F}(0)||,\ ||D_X\mathcal{F}_T(0)|| \leq c. \tag{2.2}$$

And therefore we can confine our functionals to satisfy the following growth conditions, *i.e.*

$$||D_X\mathcal{F}(X)||, ||D_X\mathcal{F}_T(X)|| \leq c(1 + ||X||), \tag{2.3}$$

and

$$|\mathcal{F}(X)|, |\mathcal{F}_T(X)| \leq C(1 + ||X||^2), \tag{2.4}$$

for some universal constant C which also depends on c. We also consider the admissible set of all controls which are functions $v(s)$'s belonging to $L^2(0,T;\mathcal{H})$. We associate to each control $v(\cdot)$ the state process $X(s)$ satisfying, for a $t > 0$,

$$\begin{cases} \frac{\mathrm{d}x(s)}{\mathrm{d}s} = v(s), \\ X(t) = X. \end{cases} \tag{2.5}$$

We may write it as $X^{X,t}(s)$ to emphasize the initial conditions and even $X^{X,t}(s;v)$ to emphasize the dependence on the control v; to avoid any cumbersome notations, we may omit the superscripts if there is no confusion caused. The function $X(\cdot) : [t,T] \to L^2(t,T;\mathcal{H})$ belongs to the Sobolev space $H^1(t,T;\mathcal{H})$. We then define the cost functional:

$$J_{X,t}(v) := \frac{\lambda}{2}\int_t^T ||v(s)||^2 \mathrm{d}s + \int_t^T \mathcal{F}(X(s))\mathrm{d}s + \mathcal{F}_T(X(T)), \tag{2.6}$$

and the value function

$$V(X,t) = \inf_v J_{X,t}(v). \tag{2.7}$$

2.2. Bellman equation

We first establish the solvability of the abstract control problem as follows:

Theorem 2.1. *Assume that (2.1), (2.2) and*

$$\lambda > cT(1+T). \tag{2.8}$$

Then the value function V defined in (2.7) is C^1 and also satisfies the growth conditions:

$$\begin{aligned} V(X,t) &\le C(1+||X||^2), \\ ||D_X V(X,t)|| \le C(1+||X||) \quad \text{and} \quad &\left|\frac{\partial V(X,t)}{\partial t}\right| \le C(1+||X||^2), \end{aligned} \tag{2.9}$$

for a universal constant $C>0$. Besides, both $D_X V(X,t)$ and $\dfrac{\partial V(X,t)}{\partial t}$ are Lipschitz continuous so that:

$$\begin{aligned} &||D_X V(X^1,t^1) - D_X V(X^2,t^2)|| \le C||X^1-X^2|| + C(1+||X^1||+||X^2||)|t^1-t^2|, \\ &\left|\frac{\partial V(X^1,t^1)}{\partial t} - \frac{\partial V(X^2,t^2)}{\partial t}\right| \le C(1+||X^1||+||X^2||)||X^1-X^2|| + C(1+||X^1||^2+||X^2||^2)|t^1-t^2|. \end{aligned} \tag{2.10}$$

Finally, the value function V is the unique solution (subject to (2.9) and (2.10)) of the following Bellman equation:

$$\begin{cases} \dfrac{\partial V}{\partial t} - \frac{1}{2\lambda}||D_X V||^2 + \mathcal{F}(X) = 0, \\ V(X,T) = \mathcal{F}_T(X). \end{cases} \tag{2.11}$$

Therefore, the control problem (2.5) and (2.6) has a unique solution.

We devote the rest of this section for the proof of this one of the main theorems in this article.

Proof. We begin by studying the properties of the cost functional $J_{X,t}(v)$. We first argue that $J_{X,t}(v)$ is Gâteaux differentiable over the space $L^2(t,T;\mathcal{H})$, for each fixed pair of (X,t). Define the forward-backward "deterministic" process $(X_v(s), Z_v(s))$ by

$$\begin{cases} \frac{\mathrm{d}x_v(s)}{\mathrm{d}s} = v(s) & \text{with} \;\; X_v(t) = X; \\ -\frac{\mathrm{d}z_v(s)}{\mathrm{d}s} = D_X\mathcal{F}(X_v(s)) & \text{with} \;\; Z_v(T) = D_X\mathcal{F}_T(X_v(T)), \end{cases}$$

then, by the first principle, we can prove easily that, for any $\bar{v} \in L^2(0,T;\mathcal{H})$,

$$\frac{\mathrm{d}}{\mathrm{d}\mu} J_{X,t}(v+\mu\bar{v})\bigg|_{\mu=0} = \int_t^T ((\lambda v(s) + Z_v(s), \bar{v}(s)))\,\mathrm{d}s. \tag{2.12}$$

Next, we claim that the functional $J_{X,t}(v)$ is strictly convex; indeed, let v_1 and v_2 are in $L^2(t,T;\mathcal{H})$, and we can write

$$\begin{aligned} J_{X,t}(\theta v_1 + (1-\theta)v_2) &= J_{X,t}(v_1 + (1-\theta)(v_2 - v_1)) \\ &= J_{X,t}(v_1) + \int_0^1 \frac{dJ_{X,t}}{\mathrm{d}\mu}(v_1 + \mu(1-\theta)(v_2-v_1))\,\mathrm{d}\mu. \end{aligned} \tag{2.13}$$

From formula (2.12), we also have

$$\frac{\mathrm{d}}{\mathrm{d}\mu} J_{X,t}(v+\mu\theta\bar{v}) = \theta \int_t^T ((\lambda(v(s)+\mu\theta\bar{v}) + Z_{v+\mu\theta\bar{v}}(s), \bar{v}(s)))\mathrm{d}s.$$

Therefore, we obtain an expression for the second term on the right hand side of (2.13)

$$\begin{aligned} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\mu} J_{X,t}(v_1 + \mu(1-\theta)(v_2 - v_1))\,\mathrm{d}\mu = (1-\theta)\int_0^1 \Bigg\{ \int_t^T ((\lambda(v_1(s) + \mu(1-\theta)(v_2(s) - v_1(s))) \\ + \; Z_1(s),\; v_2(s) - v_1(s)\,))\,\mathrm{d}s \Bigg\}\,\mathrm{d}\mu, \end{aligned}$$

where we denote $Z_1(s) = Z_{v_1+\mu(1-\theta)(v_2-v_1)}(s)$. Similarly, we can also write

$$J_{X,t}(\theta v_1 + (1-\theta)v_2) = J_{X,t}(v_2 + \theta(v_1 - v_2)) = J_{X,t}(v_2) + \int_0^1 \frac{\mathrm{d}J_{X,t}}{\mathrm{d}\mu}(v_2 + \mu\theta(v_1 - v_2))\,\mathrm{d}\mu, \tag{2.14}$$

while

$$\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\mu} J_{X,t}(v_2 + \mu\theta(v_1 - v_2))\,\mathrm{d}\mu = \theta \int_0^1 \Bigg\{ \int_t^T ((\lambda(v_2(s) + \mu\theta(v_1(s) - v_2(s))) + Z_2(s),\; v_1(s) - v_2(s)))\mathrm{d}s \Bigg\}\,\mathrm{d}\mu,$$

where we also denote $Z_2(s) = Z_{v_2+\mu\theta(v_1-v_2)}(s)$. Adding θ of (2.13) to $(1-\theta)$ of (2.14), we can write

$$\begin{aligned} J_{X,t}(\theta v_1 + (1-\theta)v_2) = \theta J_{X,t}(v_1) + (1-\theta)J_{X,t}(v_2) + \theta(1-\theta)\Bigg[-\frac{\lambda}{2}\int_t^T ||v_1(s) - v_2(s)||^2 \mathrm{d}s \\ + \int_0^1 \int_t^T ((Z_1(s) - Z_2(s), v_2(s) - v_1(s)\,))\mathrm{d}s\mathrm{d}\mu \Bigg]. \end{aligned} \tag{2.15}$$

Let $X_1(s)$ and $X_2(s)$ denote the states corresponding to the controls $v_1 + \mu(1-\theta)(v_2 - v_1)$ and $v_2 + \mu\theta(v_1 - v_2)$ respectively. One can easily check that

$$X_1(s) - X_2(s) = (1-\mu)\int_t^s (v_1(\sigma) - v_2(\sigma))\mathrm{d}\sigma,$$

and using the definitions of Z_1 and Z_2, we obtain

$$||Z_1(s) - Z_2(s)|| \leq c \left[||X_1(T) - X_2(T)|| + \int_s^T ||X_1(\sigma) - X_2(\sigma)||\mathrm{d}\sigma \right],$$

and combining formulas, we can have a simple estimate:

$$||Z_1(s) - Z_2(s)|| \leq c(1-\mu)(1+T) \int_t^T ||v_1(\sigma) - v_2(\sigma)||\mathrm{d}\sigma.$$

Applying Cauchy-Schwartz and also substituting the last inequality back to (2.15), we obtain easily that

$$J_{X,t}(\theta v_1 + (1-\theta)v_2) \leq \theta J_{X,t}(v_1) + (1-\theta)J_{X,t}(v_2) - \frac{\theta(1-\theta)}{2}(\lambda - cT(1+T)) \int_t^T ||v_1(s) - v_2(s)||^2 \mathrm{d}s, \quad (2.16)$$

and due to the assumption (2.8), $J_{X,t}(v(.))$ has to be strictly convex. Next, we write the mean value for $\mathcal{F}$,

$$\mathcal{F}(X(s)) - \mathcal{F}(X) = \int_0^1 ((D_X\mathcal{F}(X + \theta \int_t^s v(\sigma)\mathrm{d}\sigma), \int_t^s v(\sigma)\mathrm{d}\sigma))\mathrm{d}\theta,$$

hence, under the growth condition (2.3), we can have an estimate:

$$\begin{aligned} |\mathcal{F}(X(s)) - \mathcal{F}(X)| &\leq c(1 + ||X||) \left\| \int_t^s v(\sigma)\mathrm{d}\sigma \right\| + \frac{c}{2} \left\| \int_t^s v(\sigma)\mathrm{d}\sigma \right\|^2 \\ &\leq \frac{c^2(1+||X||)^2}{2\delta} + \frac{c+\delta}{2} \left\| \int_t^s v(\sigma)\mathrm{d}\sigma \right\|^2, \end{aligned}$$

for any $\delta > 0$. Using (2.4), we can assert that

$$|\mathcal{F}(X(s))| \leq C_\delta(1 + ||X||^2) + \frac{c+\delta}{2} T \int_t^T ||v(\sigma)||^2 \mathrm{d}\sigma,$$

for a constant $C_\delta > 0$ that increases as δ decreases. A similar estimate holds for $\mathcal{F}_T(X(T))$. Therefore, collecting results, we obtain

$$\left| \int_t^T \mathcal{F}(X(s))\mathrm{d}s + \mathcal{F}_T(X(T)) \right| \leq C_\delta(1 + ||X||^2)(1+T) + \frac{c+\delta}{2} T(1+T) \int_t^T ||v(s)||^2 \mathrm{d}s.$$

It follows that

$$J_{X,t}(v) \geq \frac{\lambda - (c+\delta)T(1+T)}{2} \int_t^T ||v(s)||^2 \mathrm{d}s - C_\delta(1 + ||X||^2)(1+T) \quad (2.17)$$

Since $\lambda - cT(1+T) > 0$, one can find a sufficiently small $\delta > 0$ so that $\lambda - (c+\delta)T(1+T) > 0$. This implies that $J_{X,t}(v) \to +\infty$ as $\int_t^T ||v(s)||^2 \mathrm{d}s \to +\infty$. This coercive property and the strict convexity imply that the functional $J_{X,t}(v)$ should pocesses a unique minimum; see for instance [1]. The Gâteaux derivative must vanish at this minimum, which we denot it by u. The corresponding state is denoted by $Y(\cdot)$. By using formula (2.12), we actually deduce the existence of a solution of the two-point boundary value problem, or forward-backward processes,

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}s} = -\frac{Z(s)}{\lambda}, \\ Y(t) = X; \\ -\frac{\mathrm{d}z}{\mathrm{d}s} = D_X\mathcal{F}(Y(s)), \\ Z(T) = D_X\mathcal{F}_T(Y(T)); \end{cases} \tag{2.18}$$

and the optimal control u is given by the formula

$$u(s) = -\frac{Z(s)}{\lambda}. \tag{2.19}$$

Remark 2.2. In fact, the forward-backward system (2.18) can be studied directly, and we can show directly that it has a unique solution; indeed, we can notice that there is a second order differential equation, since

$$\begin{cases} \frac{d^2Y}{\mathrm{d}s^2} = \frac{1}{\lambda}D_X\mathcal{F}(Y(s)), \\ Y(t) = X \quad \text{ and } \quad \frac{\mathrm{d}y}{\mathrm{d}s}(T) = -\frac{1}{\lambda}D_X\mathcal{F}_T(Y(T)). \end{cases} \tag{2.20}$$

We can write also (2.20) as an integral equation, for any $t < s < T$,

$$Y(s) = X - \frac{s-t}{\lambda}D_X\mathcal{F}_T(Y(T)) - \frac{1}{\lambda}\int_t^T D_X\mathcal{F}(Y(\sigma))\left((s\wedge\sigma) - t\right)\mathrm{d}\sigma, \tag{2.21}$$

and we can regard this equation as a fixed point equation in the space $C^0([t,T];\mathcal{H})$, namely $Y(\cdot) = \mathcal{K}(Y(\cdot))$, where $\mathcal{K}$ is the operator defined by the right hand side of (2.21). Due to the assumption (2.8), one can easily show that $\mathcal{K}$ is a contraction, and hence $Y(\cdot)$ is uniquely defined.

Also note that if there is a solution of (2.18) while u is defined by (2.19), then this control u satisfies the necessary condition of optimality for the functional $J_{X,t}(v)$. Since this functional is convex, the necessary condition of optimality is also sufficient, and thus u is optimal. The value function is thus defined by the formula

$$V(X,t) = \frac{1}{2\lambda}\int_t^T ||Z(s)||^2\mathrm{d}s + \int_t^T \mathcal{F}(Y(s))\mathrm{d}s + \mathcal{F}_T(Y(T)). \tag{2.22}$$

We next study the properties of the value function. We begin by first establishing the property (2.9). Using (2.17), we obtain

$$V(X,t) \geq -C_\delta(1+T)(1+||X||^2);$$

on the other hand, we also have

$$V(X,t) \leq J_{X,t}(0) = (T-t)\mathcal{F}(X) + \mathcal{F}_T(X) \leq C(1+T)(1+||X||^2);$$

therefore, the first estimate in (2.9) is now obtained as demanded. We proceed on getting estimates for the solution $Y(\cdot)$ given in (2.21). We write

$$||Y(\cdot)|| = \sup_{t\leq s\leq T} ||Y(s)||.$$

Using the standard majorant method, we obtain that

$$\begin{cases} ||Y(\cdot)|| & \leq \frac{||X||\lambda+cT(T+1)}{\lambda-cT(T+1)}, \\ ||Z(\cdot)|| & \leq \frac{\lambda(1+T)c(1+||X||)}{\lambda-cT(T+1)}, \\ ||u(\cdot)|| & \leq \frac{(1+T)c(1+||X||)}{\lambda-cT(T+1)}. \end{cases} \tag{2.23}$$

We then study how these functions are sensitive to the pair (X,t). We recall that $Y(s)=Y_{X,t}(s)$. Let us consider two points (X_1,t_1) and (X_2,t_2), and denote $Y_1(s)=Y_{X_1,t_1}(s)$, $Y_2(s)=Y_{X_2,t_2}(s)$. Without loss of generality, we assume that $t_1<t_2$. For $s>t_2$, we have

$$\begin{aligned} Y_1(s)-Y_2(s) = X_1 - X_2 &- \frac{1}{\lambda}(D_X\mathcal{F}_T(Y_1(T))-D_X\mathcal{F}_T(Y_2(T)))(s-t_2) - \frac{1}{\lambda}D_X\mathcal{F}_T(Y_1(T))(t_2-t_1) \\ &-\frac{1}{\lambda}\int_{t_2}^T (D_X\mathcal{F}(Y_1(\sigma))-D_X\mathcal{F}(Y_2(\sigma)))(s\wedge\sigma-t_2)\mathrm{d}\sigma - \frac{1}{\lambda}\int_{t_1}^{t_2} D_X\mathcal{F}(Y_1(\sigma))(s\wedge\sigma-t_1)\mathrm{d}\sigma, \end{aligned}$$

from which we obtain

$$\begin{aligned} \sup_{t_2\leq s\leq T}||Y_1(s)-Y_2(s)|| \leq ||X_1-X_2|| &+ \frac{c}{\lambda}T(1+T)\sup_{t_2\leq s\leq T}||Y_1(s)-Y_2(s)|| \\ &+\frac{t_2-t_1}{\lambda}\left[||D_X\mathcal{F}_T(Y_1(T))|| + \int_{t_1}^T ||D_X\mathcal{F}(Y_1(s))||\mathrm{d}s\right]. \end{aligned}$$

Using the properties of $D_X\mathcal{F}$ and $D_X\mathcal{F}_T$ and the second estimate in (2.23), we can assert that

$$\sup_{t_2\leq s\leq T}||Y_1(s)-Y_2(s)|| \leq \frac{\lambda}{\lambda-cT(T+1)}\left(||X_1-X_2||+(t_2-t_1)(1+T)c\frac{1+||X_1||}{\lambda-cT(T+1)}\right).$$

More globally, we can further write

$$\sup_{\max(t_1,t_2)\leq s\leq T}||Y_{X_1,t_1}(s)-Y_{X_2,t_2}(s)||\leq \frac{\lambda}{\lambda-cT(T+1)}\left(||X_1-X_2||+|t_2-t_1|(1+T)c\frac{1+\max(||X_1||,||X_2||)}{\lambda-cT(T+1)}\right). \tag{2.24}$$

In particular, we have

$$\sup_{t\leq s\leq T}||Y_{X_1,t}(s)-Y_{X_2,t}(s)|| \leq \frac{\lambda||X_1-X_2||}{\lambda-cT(T+1)}. \tag{2.25}$$

Recalling that from the system (2.18), we have

$$Z(s)=\int_s^T D_X\mathcal{F}(Y(\sigma))\mathrm{d}\sigma + D_X\mathcal{F}_T(Y(T)), \tag{2.26}$$

and noting $Z(s)=Z_{X,t}(s)$, we deduce from (2.25) that

$$\sup_{t\leq s\leq T}||Z_{X_1,t}(s)-Z_{X_2,t}(s)|| \leq \frac{c(T+1)\lambda||X_1-X_2||}{\lambda-cT(T+1)}. \tag{2.27}$$

We next write

$$J_{X_1,t}(u_1) - J_{X_2,t}(u_1) \leq V(X_1,t) - V(X_2,t) \leq J_{X_1,t}(u_2) - J_{X_2,t}(u_2),$$

where u_1 and u_2 are the optimal controls for the problems with initial conditions (X_1,t) and (X_2,t) respectively. Denoting by $Y_{X_1,t}(s)$ and $Y_{X_2,t}(s)$ the optimal states, and by $Y_{X_1,t}(s;u_2)$ and $Y_{X_2,t}(s;u_1)$ the trajectories (though not optimal) when the control u_2 is used with the initial condition (X_1,t) and when the control u_1 is used with the initial condition (X_2,t) respectively, we have, by using the definition (2.5), for all $s \in (t,T)$,

$$Y_{X_1,t}(s;u_2) - Y_{X_2,t}(s) = Y_{X_1,t}(s) - Y_{X_2,t}(s;u_1) = X_1 - X_2.$$

Therefore, we have

$$V(X_1,t) - V(X_2,t) \leq \int_t^T (\mathcal{F}(Y_{X_2,t}(s) + X_1 - X_2) - \mathcal{F}(Y_{X_2,t}(s)))\mathrm{d}s + \mathcal{F}_T(Y_{X_2,t}(T) + X_1 - X_2) - \mathcal{F}_T(Y_{X_2,t}(T)),$$

and by applying the mean value argument as before, it follows that

$$V(X_1,t) - V(X_2,t) \leq \left(\left(\int_t^T D_X\mathcal{F}(Y_{X_2,t}(s))\mathrm{d}s + D_X\mathcal{F}_T(Y_{X_2,t}(T)), X_1 - X_2\right)\right) + \frac{c}{2}(1+T)||X_1 - X_2||^2,$$

which implies that, by using (2.26),

$$V(X_1,t) - V(X_2,t) \leq ((Z_{X_2,t}(t), X_1 - X_2)) + \frac{c}{2}(1+T)||X_1 - X_2||^2. \tag{2.28}$$

By interchanging the roles of X_1 and X_2, we also obtain

$$V(X_1,t) - V(X_2,t) \geq ((Z_{X_1,t}(t), X_1 - X_2)) - \frac{c}{2}(1+T)||X_1 - X_2||^2. \tag{2.29}$$

Using the estimate (2.27), we can also write

$$V(X_1,t) - V(X_2,t) \geq ((Z_{X_2,t}(t), X_1 - X_2)) - c(T+1)\left[\frac{\lambda}{\lambda - cT(T+1)} + \frac{1}{2}\right]||X_1 - X_2||^2. \tag{2.30}$$

Combining (2.28) and (2.30), we immediately get

$$|V(X_1,t) - V(X_2,t) - ((Z_{X_2,t}(t), X_1 - X_2))| \leq c(T+1)\left[\frac{\lambda}{\lambda - cT(T+1)} + \frac{1}{2}\right]||X_1 - X_2||^2, \tag{2.31}$$

which immediately shows that $V(X,t)$ is differentiable in X and that

$$D_X V(X,t) = Z(t) = -\lambda u(t). \tag{2.32}$$

Using the second estimate in (2.23), we immediately obtain the second estimate as stated in (2.9). We continue with the derivative in t. We first write the optimality principle:

$$V(X,t) = \frac{\lambda}{2}\int_t^{t+\epsilon} ||u(s)||^2\mathrm{d}s + \int_t^{t+\epsilon} \mathcal{F}(Y(s))\mathrm{d}s + V(Y(t+\epsilon), t+\epsilon), \tag{2.33}$$

which is a simple consequence of the definition of the value function and of the existence of an attainable optimal control. We note that, by definition, $Z_{Y(t+\epsilon),t+\epsilon}(t+\epsilon) = Z_{X,t}(t+\epsilon) = -\lambda u(t+\epsilon)$, since $u(s)$, on $(t+\epsilon, T)$, is optimal for the problem starting at the initial condition $(Y(t+\epsilon), t+\epsilon)$. Therefore, we substitute $X_2 = Y(t+\epsilon)$, $X_1 = X$ and $t = t+\epsilon$ in (2.31), we deduce that

$$V(Y(t+\epsilon), t+\epsilon) - V(X, t+\epsilon) \leq -\lambda((u(t+\epsilon), \int_t^{t+\epsilon} u(s)\mathrm{d}s)) + C\left\|\int_t^{t+\epsilon} u(s)\mathrm{d}s\right\|^2,$$

which can be combined with (2.33) to yield:

$$V(X,t) - V(X,t+\epsilon) \leq \frac{\lambda}{2}\int_t^{t+\epsilon} ||u(s)||^2\mathrm{d}s + \int_t^{t+\epsilon} \mathcal{F}(Y(s))\mathrm{d}s - \lambda((u(t+\epsilon), \int_t^{t+\epsilon} u(s)\mathrm{d}s)) + C\left\|\int_t^{t+\epsilon} u(s)\mathrm{d}s\right\|^2,$$

from which we obtain

$$\liminf_{\epsilon\to 0} \frac{V(X,t+\epsilon) - V(X,t)}{\epsilon} \geq \frac{\lambda}{2}||u(t)||^2 - \mathcal{F}(X). \tag{2.34}$$

Next, by the definition of the optimality of V, and using the fact that for any $s \in (t+\epsilon, T)$, $Y(s) - \int_t^{t+\epsilon} u(\sigma)\mathrm{d}\sigma = Y(s) - Y(t+\epsilon) + X$, we have

$$V(X, t+\epsilon) \leq \frac{\lambda}{2}\int_{t+\epsilon}^T ||u(s)||^2\mathrm{d}s + \int_{t+\epsilon}^T \mathcal{F}\left(Y(s) - \int_t^{t+\epsilon} u(\sigma)\mathrm{d}\sigma\right)\mathrm{d}s + \mathcal{F}_T\left(Y(T) - \int_t^{t+\epsilon} u(\sigma)\mathrm{d}\sigma\right),$$

therefore

$$\begin{aligned} V(X,t+\epsilon) - V(X,t) \leq & -\frac{\lambda}{2}\int_t^{t+\epsilon} ||u(s)||^2\mathrm{d}s - \int_t^{t+\epsilon} \mathcal{F}(Y(s))\mathrm{d}s \\ & + \int_{t+\epsilon}^T \left(\mathcal{F}\left(Y(s) - \int_t^{t+\epsilon} u(\sigma)\mathrm{d}\sigma\right) - \mathcal{F}(Y(s))\right)\mathrm{d}s \\ & + \mathcal{F}_T\left(Y(T) - \int_t^{t+\epsilon} u(\sigma)\mathrm{d}\sigma\right) - \mathcal{F}_T(Y(T)), \end{aligned}$$

and using assumptions on $\mathcal{F}$ and $\mathcal{F}_T$, it follows that

$$\begin{aligned} V(X,t+\epsilon) - V(X,t) \leq & -\frac{\lambda}{2}\int_t^{t+\epsilon} ||u(s)||^2\mathrm{d}s - \int_t^{t+\epsilon} \mathcal{F}(Y(s))\mathrm{d}s + \frac{c}{2}(1+T)\left\|\int_t^{t+\epsilon} u(\sigma)\mathrm{d}\sigma\right\|^2 \\ & - ((\int_{t+\epsilon}^T D_X\mathcal{F}(Y(s))\mathrm{d}s + D_X\mathcal{F}_T(Y(T)), \int_t^{t+\epsilon} u(\sigma)\mathrm{d}\sigma)), \end{aligned}$$

which means, by using (2.18) and (2.19),

$$\begin{aligned} V(X,t+\epsilon) - V(X,t) \leq & -\frac{\lambda}{2}\int_t^{t+\epsilon} ||u(s)||^2\mathrm{d}s - \int_t^{t+\epsilon} \mathcal{F}(Y(s))\mathrm{d}s \\ & + \lambda((u(t+\epsilon), \int_t^{t+\epsilon} u(\sigma)\mathrm{d}\sigma)) + \frac{c}{2}(1+T)\left\|\int_t^{t+\epsilon} u(\sigma)\mathrm{d}\sigma\right\|^2. \end{aligned}$$

We then obtain

$$\limsup_{\epsilon\to 0}\frac{V(X,t+\epsilon)-V(X,t)}{\epsilon}\leq\frac{\lambda}{2}||u(t)||^2-\mathcal{F}(X), \tag{2.35}$$

and combining with (2.34), it is now clear that $V(X,t)$ is differentiable in t, while the derivative is given by

$$\frac{\partial V}{\partial t}(X,t)=\frac{\lambda}{2}||u(t)||^2-\mathcal{F}(X). \tag{2.36}$$

Substituting (2.32) into (2.36), $V(X,t)$ verifies the HJB equation (2.11) as desired. The third estimate in (2.9) is an immediate consequence of the equation and the estimate on $||D_XV(X,t)||$ as shown before. We next turn to check the addtional estimates (2.10). Using (2.32), we have

$$D_XV(X_1,t_1)-D_XV(X_2,t_2)=Z_{X_1,t_1}(t_1)-Z_{X_2,t_2}(t_2).$$

Without loss of generality, we assume that $t_1<t_2$. We can then write

$$\begin{aligned}Z_{X_1,t_1}(t_1)-Z_{X_2,t_2}(t_2)=&\int_{t_1}^{t_2}D_X\mathcal{F}(Y_{X_1,t_1}(s))\,\mathrm{d}s+\int_{t_2}^{T}\left(D_X\mathcal{F}(Y_{X_1,t_1}(s))-D_X\mathcal{F}(Y_{X_2,t_2}(s))\right)\mathrm{d}s\\&+D_X\mathcal{F}_T(Y_{X_1,t_1}(T))-D_X\mathcal{F}_T(Y_{X_2,t_2}(T)).\end{aligned} \tag{2.37}$$

Applying the previously obtained estimates based on majorant method, through (2.3) and (2.23) to the first term, (2.24) to the second and the third terms, we can verify that

$$||Z_{X_1,t_1}(t_1)-Z_{X_2,t_2}(t_2)||\leq\frac{\lambda c(T+1)}{\lambda-cT(T+1)}\left(||X_1-X_2||+|t_2-t_1|\frac{\lambda+c(T+1)}{T+1}\frac{1+\max(||X_1||,||X_2||)}{\lambda-cT(T+1)}\right), \tag{2.38}$$

and then the first estimate in (2.10) follows immediately. The second estimate in (2.10) is a direct consequence of the HJB equation (2.36) and of the first estimate in (2.10). So the value function has the regularity as demended in the statement and satisfies the HJB equation. Let us show that such a solution is necessarily unique. This is a consequence of the verification property; indeed, consider any control $v\in L^2(t,T;\mathcal{H})$ and the state $X(s)$ solution of (2.5). Let $V(x,t)$ be a solution of the HJB equation which is C^1 and satisifies (2.9) and (2.10). Then, the function $V(X(s),s)$ is differentiable and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}V(X(s),s)&=\frac{\partial V}{\partial s}(X(s),s)+((D_XV(X(s),s),v(s)))\\&=-\mathcal{F}(X(s))+\frac{1}{2\lambda}||D_XV(X(s),s)||^2+((D_XV(X(s),s),v(s)))\\&\geq-\mathcal{F}(X(s))-\frac{\lambda}{2}||v(s)||^2,\end{aligned}$$

from which we get immediately by integration that $V(X,t)\leq J_{X,t}(v)$. Now, if we consider the equation

$$\frac{d\hat{X}(s)}{\mathrm{d}s}=-\frac{1}{\lambda}D_XV(\hat{X}(s),s),\ \ \hat{X}(t)=X, \tag{2.39}$$

it has a unique solution, since $D_XV(X,s)$ is uniformly Lipschitz in X. If we set $\hat{v}(s)=-\dfrac{1}{\lambda}D_XV(\hat{X}(s),s)$, we see easily that $V(X,t)=J_{X,t}(\hat{v})$. So $V(X,t)$ coincides with the value function, and thus we have only one possible solution. This completes the proof of the whole theorem. □

3. The Master equation

3.1. Further regularity assumption

We now assume that

$$\mathcal{F} \text{ and } \mathcal{F}_T \text{ are } C^2. \tag{3.1}$$

The operators $D_X^2\mathcal{F}(X)$ and $D_X^2\mathcal{F}_T(X)$ belong to $\mathcal{L}(\mathcal{H};\mathcal{H})$. According to the assumptions (2.1) we can assert that

$$||D_X^2\mathcal{F}(X)||,\ ||D_X^2\mathcal{F}_T(X)|| \leq c, \tag{3.2}$$

where the norm of the operators is the usual operator norm over $\mathcal{L}(\mathcal{H};\mathcal{H})$. Recalling the equation (2.21) for $Y(s)$, we differentiate formally with respect to X to obtain

$$D_XY(s) = I - \frac{s-t}{\lambda}D_X^2\mathcal{F}_T(Y(T))D_XY(T) - \frac{1}{\lambda}\int_t^T D_X^2\mathcal{F}(Y(\sigma))D_XY(\sigma)(s\wedge\sigma - t)\mathrm{d}\sigma, \tag{3.3}$$

and so, D_XY appears as the solution of a linear equation, and we see easily that it has one and only one solution verifying

$$\sup_{t\leq s\leq T}||D_XY(s)|| \leq \frac{\lambda}{\lambda - cT(T+1)}. \tag{3.4}$$

It is then easy to check that $D_XY(s)$ is indeed the gradient of $Y_{X,t}(s)$ with respect to X, and the estimate (3.4) is consistent with (2.10) and (2.25). Since $D_XV(X,t) = Z(t) = Z_{X,t}(t)$ with

$$Z(t) = \int_t^T D_X\mathcal{F}(Y(s))\mathrm{d}s + D_X\mathcal{F}_T(Y(T)),$$

we can differentiate with repsect to X to obtain

$$D_X^2V(X,t) = \int_t^T D_X^2\mathcal{F}(Y(s))D_XY(s)\mathrm{d}s + D_X^2\mathcal{F}_T(Y(T))D_XY(T), \tag{3.5}$$

and

$$||D_X^2V(X,t)|| \leq \frac{\lambda c(T+1)}{\lambda - cT(T+1)}, \tag{3.6}$$

which is consistent with (2.27).

3.2. Master equation

We obtain the Master equation, by simply differentiating the HJB equation (2.11) with respect to X. We set $\mathcal{U}(X,t) = D_XV(X,t) = Z_{X,t}(t)$. We know from the second estimate in (2.23) that

$$||\mathcal{U}(X,t)|| \leq \frac{\lambda(1+T)c(1+||X||)}{\lambda - cT(T+1)}. \tag{3.7}$$

The function $\mathcal{U}(X,t)$ maps $\mathcal{H}\times(0,T)$ into $\mathcal{H}$. From (3.6), we see that it is differentiable in X, with $D_X\mathcal{U}(X,t):\mathcal{H}\times(0,T)\to\mathcal{L}(\mathcal{H};\mathcal{H})$ and

$$||D_X\mathcal{U}(X,t)||\leq\frac{\lambda c(1+T)}{\lambda-cT(T+1)}. \tag{3.8}$$

From the HJB equation, we see that $\mathcal{U}(X,t)$ is differentiable in t and satisfies the Master equation

$$\begin{cases}\frac{\partial\mathcal{U}}{\partial t}-\frac{1}{\lambda}D_X\mathcal{U}(X,t)\mathcal{U}(X,t)+D_X\mathcal{F}(X)=0,\\ \mathcal{U}(X,T)=D_X\mathcal{F}_T(X).\end{cases} \tag{3.9}$$

As a consequence, we have the following result:

Proposition 3.1. *Under the assumptions of Theorem 2.1 and (3.1), the equation (3.9) has one and only one solution satisfying the estimates (3.7) and (3.8).*

Proof. We have only to prove uniqueness. Noting that

$$D_X\mathcal{U}(X,t)\mathcal{U}(X,t)=\frac{1}{2}D_X||\mathcal{U}(X,t)||^2,$$

which is then plugged in back to (3.9), and we see immediately that $\mathcal{U}(X,t)$ is a gradient by simply referring back to (3.9); so $\mathcal{U}(X,t)=D_X\tilde{V}(X,t)$ for some regular enough function $\tilde{V}$. Therefore, the equation (3.9) reads

$$\begin{cases}D_X\left(\frac{\partial\tilde{V}}{\partial t}-\frac{1}{2\lambda}||D_X\tilde{V}||^2+\mathcal{F}(X)\right)=0,\\ D_X\tilde{V}(X,T)=D_X\mathcal{F}_T(X).\end{cases}$$

We thus can write

$$\begin{cases}\frac{\partial\tilde{V}}{\partial t}-\frac{1}{2\lambda}||D_X\tilde{V}||^2+\mathcal{F}(X)=f(t),\\ \tilde{V}(X,T)=\mathcal{F}_T(X)+h,\end{cases}$$

where $f(t)$ is purely a function in t and h is a constant. If we introduce the function $\varphi(t)$ solution of

$$\frac{\partial\varphi}{\partial t}=f(t),\quad\varphi(T)=h,$$

the function $\tilde{V}(X,t)-\varphi(t)$ is solution of the HJB equation (2.11) and satisfies the regularity properties of Theorem 2.1. From the uniqueness of the solution of the HJB equation we have $\tilde{V}(X,t)-\varphi(t)=V(X,t)$, the value function, hence $\mathcal{U}(X,t)=D_XV(X,t)$, which proves the uniqueness. □

4. Space of random variables

4.1. General comments

We shall be interested in the Hilbert space $\mathcal{H}=L^2(\Omega,\mathcal{A},P;\mathbb{R}^n)$ of square-integrable random variables, with values in $\mathbb{R}^n$. We have in mind mean field type control problems, in which one has to work with functionals on probability measures. The simple idea, to connect a probability measure on $\mathbb{R}^n$ to a random variable, has been introduced by Lions in [17, 18], and it is quite promising. The natural functional space, associated to probability measures on $\mathbb{R}^n$ is the Wasserstein metric space. It was the objective of Gangbo and Święch [13] to develop a rigorous control theory in the Wasserstein metric space, to solve mean field type control problems. As stated in the introduction, the objective of our work is to explore the alternative represented by control theory in the Hilbert space of square-integrable random variables.

4.2. Wasserstein space

We consider the space $\mathcal{P}_2(\mathbb{R}^n)$ of all probability measures on $\mathbb{R}^n$ with finite second order moments, equipped with the Wasserstein metric $W_2(\mu,\nu)$ defined by:

$$W_2^2(\mu,\nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} \int_{\mathbb{R}^n\times\mathbb{R}^n} |\xi-\eta|^2\gamma(\mathrm{d}\xi,\mathrm{d}\eta), \tag{4.1}$$

where $\Gamma(\mu,\nu)$ denotes the set of all joint probability measures on $\mathbb{R}^n\times\mathbb{R}^n$ such that the marginals are μ and ν respectively. Consider a probability space $(\Omega,\mathcal{A},\mathbb{P})$ and all its L^2 random variables namely, $\mathcal{H} := L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^n)$. We then write, for any $X, Y \in \mathcal{H}$, $\mu = \mathcal{L}_X$, $\nu = \mathcal{L}_Y$, and so

$$W_2^2(\mu,\nu) = \inf_{X,Y\in\mathcal{H},\mathcal{L}_X=\mu,\mathcal{L}_Y=\nu} \mathbb{E}\left(|X-Y|^2\right).$$

Therefore the map $X \mapsto \mathcal{L}_X$ is continuous, and this map actually induces an equivalence relation in $\mathcal{H}$, and so the set of probability measures can be considered as the quotient set.

4.3. Lifting procedure

Consider a functional $F(m)$ defined on $\mathcal{P}_2(\mathbb{R}^n)$; we then associate it with a functional $\mathcal{F}(X)$ on $\mathcal{H}$ such that:

$$\mathcal{F}(X) = F(\mathcal{L}_X), \tag{4.2}$$

called the lifted functional, and this is the key idea as proposed by Lions. Obviously, if F is continuous with respect to the Wasserstein metric, the lifted functional $\mathcal{F}$ is also continuous over $\mathcal{H}$. The interesting issue now is about any convenient notion of derivatives; clearly, we do have a well-defined and obvious one for functionals over $\mathcal{H}$, while that for those over $\mathcal{P}_2(\mathbb{R}^n)$ is more delicate. Here it comes with an important suggestion by Lions; particularly, Carmona and Delarue [11] defines that $F(m)$ is said to be L-differentiable if its lifted functional $\mathcal{F}(X)$ is Fréchet differentiable. The practicality of the concept of L-differentiability lies in the fact that, for each fixed m in $\mathcal{P}_2(\mathbb{R}^n)$, there exists a measurable function, denoted by $\partial_m F(m)(x)$, in x such that

$$\partial_{\mathcal{L}_X} F(\mathcal{L}_X)(X) = D_X\mathcal{F}(X) \text{ a.s.}, \quad \forall X \in \mathcal{H}. \tag{4.3}$$

Note that $\mathbb{E}|D_X\mathcal{F}(X)|^2 = \int_{\mathbb{R}^n} |\partial_{\mathcal{L}_X} F(\mathcal{L}_X)(x)|^2 d\mathcal{L}_X(x)$, so we have for any $m \in \mathcal{P}_2(\mathbb{R}^n)$, $\partial_m F(m)(\cdot) \in L_m^2(\mathbb{R}^n;\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R}^n | \int_{\mathbb{R}^n} |f(x)|^2 \mathrm{d}m(x) < +\infty\}$. The function $\partial_m F(m)(x)$ is called the L-derivative of $F(m)$ at m. Moreover, if $D_X\mathcal{F}(X)$ is uniformly Lipschitz, then $\partial_m F(m)(x)$ has a continuous version which is the gradient of a continuously differentiable function, denoted by $\dfrac{\delta F(m)}{\delta m}(x)$, called the functional derivative of $F(m)$; that is

$$\partial_m F(m)(x) = D_x \frac{\delta F(m)}{\delta m}(x), \quad \forall x, \tag{4.4}$$

and here $\dfrac{\delta F(m)}{\delta m}(x)$ is C^1. In fact, we have the formula

$$\frac{\delta F(m)}{\delta m}(x) - \frac{\delta F(m)}{\delta m}(0) = \int_0^1 \partial_m F(m)(tx)\cdot x\mathrm{d}t. \tag{4.5}$$

All these important results can be found in the recent book of Carmona and Delarue [11]. Finally, the justification on the terminology "functional derivative" stems from the following result which is also proven in [11]:

Proposition 4.1. *Suppose that* $D_x \dfrac{\delta F(m)}{\delta m}(x)$ *is jointly continuous in* $\mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n$ *such that*

$$\left| D_x \frac{\delta F(m)}{\delta m}(x) \right| \leq C(m)(1 + |x|), \tag{4.6}$$

where $C(m)$ *is bounded on any bounded subsets of* $\mathcal{P}_2(\mathbb{R}^n)$. *Then, for any sequence* m_ϵ *with* $W_2(m_\epsilon, m) \to 0$ *as* $\epsilon \to 0$, *we have*

$$\frac{F(m_\epsilon) - F(m) - \left(\int_{\mathbb{R}^n} \frac{\delta F(m)}{\delta m}(x) \mathrm{d}m_\epsilon(x) - \int_{\mathbb{R}^n} \frac{\delta F(m)}{\delta m}(x) \mathrm{d}m(x) \right)}{(W_2(m_\epsilon, m))} \to 0. \tag{4.7}$$

Remark 4.2. In [11], Carmona and Delarue have shown that the L-derivative coincides with the Wasserstein gradient when they are both defined. But a direct use of the Wasserstein metric space and the Wasserstein gradient to develop control theory is far more complicated than using the approach of the Hilbert space of square integrable random variables. This is the major goal of our present work so as to demonstrate the great simplification resulted. Besides, in the appendix, we also provide with an alternative development on motivating the connection among the (Gâteaux) functional derivative, Wasserstein gradient and the usual derivative in $\mathcal{H}$.

5. Interpertation of the abstract problem

5.1. Assumptions

Our objective is now to interpret the abstract problem, particularly for the space $\mathcal{H} = L^2(\Omega, \mathcal{A}, P; \mathbb{R}^n)$. We want to show how it relates to a mean field type control problem. This will be completed in the next section. In this section we develop all the preliminary steps. We first introduce functions $f(x, m)$ and $h(x, m)$ defined on $\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$ satisfying the regularity properties:

$$\begin{aligned} &m \mapsto f(x, m),\ m \mapsto h(x, m) \text{ are } L\text{-differentiable;} \\ &|f(x, m)|,\ |h(x, m)| \leq c\left(1 + |x|^2 + \int_{\mathbb{R}^n} |\xi|^2 \mathrm{d}m(\xi)\right), \text{ for some constant } c > 0. \end{aligned} \tag{5.1}$$

We denote by $\partial_m f(x, m)(\xi)$, $\partial_m h(x, m)(\xi)$ the L-derivative (x here is a parameter). Further assume that these L-derivatives are defined through the respective functional derivatives $\frac{\delta f}{\delta m}(x, m)(\xi)$ and $\frac{\delta h}{\delta m}(x, m)(\xi)$, that is:

$$\partial_m f(x, m)(\xi) = D_\xi \frac{\delta f}{\delta m}(x, m)(\xi),\ \partial_m h(x, m)(\xi) = D_\xi \frac{\delta h}{\delta m}(x, m)(\xi).$$

We then introduce

$$\begin{aligned} F(x, m) &= f(x, m) + \int_{\mathbb{R}^n} \frac{\delta f(x, m)}{\delta m}(\xi)\, \mathrm{d}m(\xi), \\ F_T(x, m) &= h(x, m) + \int_{\mathbb{R}^n} \frac{\delta h(x, m)}{\delta m}(\xi)\, \mathrm{d}m(\xi); \end{aligned} \tag{5.2}$$

based on which, we further assume that

$$|D_x F(x,m)| \leq \frac{c}{2}\left(1+|x|+\sqrt{\int_{\mathbb{R}^n}|\xi|^2 \mathrm{d}m(\xi)}\right),$$
$$|D_x F_T(x,m)| \leq \frac{c}{2}\left(1+|x|+\sqrt{\int_{\mathbb{R}^n}|\xi|^2 \mathrm{d}m(\xi)}\right). \tag{5.3}$$

If $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ belongs to $L^2_m(\mathbb{R}^n;\mathbb{R}^n)$, *i.e.* $\int_{\mathbb{R}^n}|\varphi(x)|^2 \mathrm{d}m(x) < \infty$, we consider the image measure of m pushed forward by φ, and this image measure is denoted $\varphi_{\#}m$, which may not have a density. We assume that $D_x F(x,\varphi_{\#}m)$ and $D_x F_T(x,\varphi_{\#}m)$ are well-defined for those φ's of interest, and assume the following Lipschitz property:

$$|D_x F(x_1,\varphi_{1\#}m) - D_x F(x_2,\varphi_{2\#}m)| \leq \frac{c}{2}\left(|x_1-x_2| + \int_{\mathbb{R}^n}|\varphi_1(\xi)-\varphi_2(\xi)|^2 \mathrm{d}m(\xi)\right),$$
$$|D_x F_T(x_1,\varphi_{1\#}m) - D_x F_T(x_2,\varphi_{2\#}m)| \leq \frac{c}{2}\left(|x_1-x_2| + \int_{\mathbb{R}^n}|\varphi_1(\xi)-\varphi_2(\xi)|^2 \mathrm{d}m(\xi)\right). \tag{5.4}$$

Suppose that we now set

$$F(m) = \int_{\mathbb{R}^n} f(x,m)\mathrm{d}m(x);$$
$$F_T(m) = \int_{\mathbb{R}^n} h(x,m)\mathrm{d}m(x), \tag{5.5}$$

both of them are L-differentiable, and we also have

$$\frac{\delta F(m)}{\delta m}(x) = F(x,m),\ \partial_m F(m)(x) = D_x F(x,m); \tag{5.6}$$

$$\frac{\delta F_T(m)}{\delta m}(x) = F_T(x,m),\ \partial_m F_T(m)(x) = D_x F_T(x,m). \tag{5.7}$$

Moreover, we also assume a property stronger than L-differentiability on these functionals F and F_T such that

$$\left|F(m_1) - F(m_2) - \left(\int_{\mathbb{R}^n}\frac{\delta F(m_2)}{\delta m}(x)\mathrm{d}m_1(x) - \int_{\mathbb{R}^n}\frac{\delta F(m_2)}{\delta m}(x)\mathrm{d}m_2(x)\right)\right| \leq C W_2^2(m_1,m_2), \tag{5.8}$$

and we also assume the same for for $F_T(m)$; note that the same assumptions have been adopted in the work of [13]. Defining $\mathcal{F}(X)$ in accordance with (4.2), then we can set

$$D_X\mathcal{F}(X) = D_x F(X,\mathcal{L}_X). \tag{5.9}$$

Similarly, we introduce

$$\mathcal{F}_T(X) = F_T(\mathcal{L}_X), \tag{5.10}$$

and thus

$$D_X\mathcal{F}_T(X) = D_x F_T(X,\mathcal{L}_X). \tag{5.11}$$

It follows that

$$\begin{aligned}||D_X\mathcal{F}(X_1)-D_X\mathcal{F}(X_2)|| &\le ||D_xF(X_1,\mathcal{L}_{X_1})-D_xF(X_2,\mathcal{L}_{X_1})||+||D_xF(X_2,\mathcal{L}_{X_1})-D_xF(X_2,\mathcal{L}_{X_2})||\\ &\le c||X_1-X_2||,\end{aligned}$$

and similar estimate for $\mathcal{F}_T$ also holds. Therefore, the set up in Section 2.1 is satisfied.

5.2. Interpretation of the solution of the system (2.18)

The key point of the proof of Theorem 2.1 is the study of the system (2.18) which has one and only one solution. We next interpret the system with the choices of coefficient and terminal functionals as specified in (4.2) and (5.10). Considering the initial conditions (m,t), in which m is in $\mathcal{P}_2(\mathbb{R}^n)$. We introduce the differential equation, and look for its solution pair:

$$\begin{cases}\frac{\mathrm{d}y_{x,m,t}(s)}{\mathrm{d}s}=-\frac{1}{\lambda}z_{x,m,t}(s),\\ y_{x,m,t}(t)=x;\end{cases}\tag{5.12}$$

and

$$\begin{cases}-\dfrac{\mathrm{d}z_{x,m,t}(s)}{\mathrm{d}s}=D_xF(y_{x,m,t}(s),y_{m,t}(s)_{\#}m),\\ z_{x,m,t}(T)=D_xF_T(y_{x,m,t}(T),y_{m,t}(T)_{\#}m);\end{cases}\tag{5.13}$$

here in (5.13), $y_{m,t}(s)_{\#}m$ stands for the push forward measure of m by the map $y_{\cdot,m,t}(s)$, and similarly $y_{m,t}(T)_{\#}m$ is the image probability of m under the map $y_{\cdot,m,t}(T)$. We now see how the above system leads to (2.18). Consider a random variable X in $\mathcal{H}$, and take $m=\mathcal{L}_X$ the probability measure of X. We then set $Y(s)=y_{X,\mathcal{L}_X,t}(s)$ and $Z(s)=z_{X,\mathcal{L}_X,t}(s)$; note that $y_{\mathcal{L}_X,t}(s)_{\#}\mathcal{L}_X=\mathcal{L}_{y_{X,\mathcal{L}_X,t}(s)}$. Writing $y_{X,\mathcal{L}_X,t}(s)=Y(s)$ is to emphasize that we are dealing with a random variable. We see that

$$D_xF(y_{X,\mathcal{L}_X,t}(s),y_{\mathcal{L}_X,t}(s)_{\#}\mathcal{L}_X)=D_xF(Y(s),\mathcal{L}_{Y(s)})=D_X\mathcal{F}(Y(s)),\tag{5.14}$$

$$D_xF_T(y_{X,\mathcal{L}_X,t}(T),y_{\mathcal{L}_X,t}(T)_{\#}\mathcal{L}_X)=D_xF_T(Y(T),\mathcal{L}_{Y(T)})=D_X\mathcal{F}_T(Y(T)),\tag{5.15}$$

and we see immediately that the pair $(Y(s),Z(s))$ is actually the solution of (2.18). Nevertheless, we can study the forward-backward system (5.12) and (5.13) directly, which we leave to the readers. We state the following result:

Proposition 5.1. *Under assumptions (5.3), (5.4), (5.8) and condition (2.8), for a given (m,t), there exists one and only one solution $(y_{x,m,t}(s),z_{x,m,t}(s))$ of the system (5.12) and (5.13) in the space $C^0(t,T;L^2_m(\mathbb{R}^n;\mathbb{R}^n))$.*

We can further check that the unique solution pair $(y_{x,m,t}(s),z_{x,m,t}(s))$ also satisfies the estimates

$$||y_{x,m,t}(\cdot)||_{C(t,T;L^2_m)}\le\frac{\lambda\sqrt{\int_{\mathbb{R}^n}|x|^2\mathrm{d}m(x)}+\frac{c}{2}T(T+1)}{\lambda-cT(T+1)};\tag{5.16}$$

$$||z_{x,m,t}(\cdot)||_{C(t,T;L^2_m)}\le\lambda c(1+T)\frac{\sqrt{\int_{\mathbb{R}^n}|x|^2\mathrm{d}m(x)}+1}{\lambda-cT(T+1)}.\tag{5.17}$$

Furthermore, using assumption (5.4), we obtain easily that

$$\sup_{t<s<T}|y_{x_1,m,t}(s)-y_{x_2,m,t}(s)|\leq\frac{\lambda|x_1-x_2|}{\lambda-cT(T+1)};\tag{5.18}$$

$$\sup_{t<s<T}|z_{x_1,m,t}(s)-z_{x_2,m,t}(s)|\leq\frac{c}{2}\frac{\lambda|x_1-x_2|(1+T)}{\lambda-cT(T+1)}.\tag{5.19}$$

Similarly, under assumptions (5.3), we also have

$$\sup_{t<s<T}|y_{x,m,t}(s)|\leq\lambda\frac{|x|+\dfrac{cT(1+T)\left(1+\sqrt{\int_{\mathbb{R}^n}|\xi|^2\mathrm{d}m(\xi)}\right)}{\lambda-cT(T+1)}}{\lambda-cT(T+1)}.\tag{5.20}$$

A similar estimate can be obtained for $\sup_{t<s<T}|z_{x,m,t}(s)|$.

5.3. Interpretation of the value function and Bellman equation

By applying the formula (2.22) and Proposition 5.1 in the preceding subsection, we immediately deduce that

$$V(X,t)=\frac{1}{2\lambda}\int_t^T\int_{\mathbb{R}^n}|z_{x,m,t}(s)|^2\mathrm{d}m(x)\mathrm{d}s+\int_t^T F(y_{m,t}(s)_\#m)\mathrm{d}s+F_T(y_{m,t}(T)_\#m)=:V(m,t),\tag{5.21}$$

here $m=\mathcal{L}_X$. Note that the functional $V(X,t)$ depends on X only through its probability measure, but not the actual position. We may then consider the right hand side of (5.21) as a functional of measure m, and then view $V(X,t)$ as the lifted functional of $V(m,t)$.

On the other hand, according to the condition (2.10), the lifted functional $V(X,t)$ is differentiable in X and $D_XV(X,t)$ is uniformly Lipschitz in X, in regard to the discussion leading to (4.4), $V(m,t)$ has both a L-derivative $\partial_mV(m,t)(x)$ and a functional derivative $\frac{\delta V}{\delta m}(m,t)(x)$ such that, using (4.3),

$$\partial_mV(m,t)(x)=D_x\frac{\delta V}{\delta m}(m,t)(x),\ D_XV(X,t)=D_x\frac{\delta V}{\delta m}(m,t)(X);\tag{5.22}$$

We can now interpret Bellman equation (2.11) for the abstract control problem as follows; also see [6, 16]; indeed, from Bellman equation (2.11), we obtain immediately that

Proposition 5.2. *Under the assumptions specified in Proposition 5.1. The function $V(m,t)$ is the solution of Bellman equation*

$$\begin{cases}\frac{\partial V}{\partial t}-\frac{1}{2\lambda}\displaystyle\int_{\mathbb{R}^n}\left|D_\xi\frac{\delta V(m,t)}{\delta m}(\xi)\right|^2\mathrm{d}m(\xi)+\int_{\mathbb{R}^n}f(\xi,m)\mathrm{d}m(\xi)=0,\\ V(m,T)=\int_{\mathbb{R}^n}h(\xi,m)\mathrm{d}m(\xi).\end{cases}\tag{5.23}$$

6. Mean field type control problem

6.1. Mean field type control interpretation of $V(m,t)$

According to the problem formulation of (2.5), (2.6) and (2.7), we recall that $V(X,t)$ is the corresponding value function of that control problem. In this subsection, we aim to provide a control interpretation of $V(m,t)$.

Let $m \in \mathcal{P}_2(\mathbb{R}^n)$ and $v(x,s)$ be a deterministic function with values in $\mathbb{R}^n$ in the pair (x,s), for $s \geq t$, such that

$$\int_t^T \int_{\mathbb{R}^n} |v(x,s)|^2 \mathrm{d}m(x)\mathrm{d}s < \infty. \tag{6.1}$$

The function v here is the control, and we then define the state process $x^v(x_0,s)$ by

$$x^v(x_0,s) = x_0 + \int_t^s v(x_0,\tau)\mathrm{d}\tau. \tag{6.2}$$

As before, we consider the image probability $x^v(\cdot,s)_{\#}m$, we then consider a payoff function given by:

$$J_{m,t}(v) := \frac{\lambda}{2}\int_t^T \int_{\mathbb{R}^n} |v(x,s)|^2 \mathrm{d}m(x)\mathrm{d}s + \int_t^T F(x^v(\cdot,s)_{\#}m)\mathrm{d}s + F_T(x^v(\cdot,T)_{\#}m). \tag{6.3}$$

The setting above introduces a mean-field type control problem. Indeed, consider a random variable ξ and use m as the probability law of ξ. A control $v(s)$, $s \geq t$, is a stochastic process but just adapted to $\sigma(\xi)$, the σ-algebra generated by ξ, for all s. Besides, we specify that $\mathbb{E}\int_t^T |v(s)|^2 \mathrm{d}s < \infty$. Necessarily, $v(s) = v(\xi,s)$ where $v(x,s)$ is deterministic and jointly measurable. As expected, the state process is

$$x(s) = \xi + \int_t^s v(\tau)\mathrm{d}\tau,$$

while the pay-off functional is defined by

$$J(v) := \frac{\lambda}{2}\int_t^T \mathbb{E}|v(s)|^2 \mathrm{d}s + \int_t^T F(\mathcal{L}_{x(s)})\mathrm{d}s + F_T(\mathcal{L}_{x(T)}),$$

which is equivalent to (6.3) since $x^v(\cdot,s)_{\#}m = \mathcal{L}_{x(s)}$. Clearly, this is not a standard stochastic control problem, but is a mean-field type control problem.

We can then relate the system of (6.2) and (6.3) to that of (2.5) and (2.6) as follows. To each $m \in \mathcal{P}_2(\mathbb{R}^n)$, we associate some $X \in \mathcal{H}$ such that $m = \mathcal{L}_X$, and we consider the stochastic process $v(s) := v(X,s)$, for $s \geq t$, which clearly belongs to $L^2(t,T;\mathcal{H})$; but, in fact, each of such v even belongs to the closed subspace of all stochastic processes adapted to $\sigma(X)$. Obviously, if we define $X(s) := x^v(X,s)$, then $X(s)$ is solution of (2.5) and $x^v(\cdot,s)_{\#}m$ is the probability measure of $X(s)$ with

$$F(x^v(\cdot,s)_{\#}m) = \mathcal{F}(X(s)),\ F_T(x^v(\cdot,T)_{\#}m) = \mathcal{F}_T(X(T)).$$

Therefore, we can obtain:

$$J_{m,t}(v) = \frac{\lambda}{2}\int_t^T \mathbb{E}|v(s)|^2 \mathrm{d}s + \int_t^T \mathcal{F}(X(s))\mathrm{d}s + \mathcal{F}_T(X(T)) =: J_{X,t}(v),$$

where this $J_{X,t}(v)$ is the lifted functional of $J_{m,t}(v)$. Denote by $L^2_{\sigma(X)}(t,T;\mathcal{H})$ the Hilbert subspace of $L^2(t,T;\mathcal{H})$ of all stochastic processes adapted to the σ-algebra $\sigma(X)$, then we can write:

$$\inf_{v \in L^2(t,T;L^2_m(\mathbb{R}^n;\mathbb{R}^n))} J_{m,t}(v) = \inf_{v \in L^2_{\sigma(X)}(t,T;\mathcal{H})} J_{X,t}(v). \tag{6.4}$$

Clearly, as a priori, $\inf_{v\in L^2_{\sigma(X)}(t,T;\mathcal{H})} J_{X,t}(v) \geq \inf_{v\in L^2(t,T;\mathcal{H})} J_{X,t}(v)$; nevertheless, according to (2.19), the optimal control for the latter problem $\inf_{v\in L^2(t,T;\mathcal{H})} J_{X,t}(v)$ is

$$u(s) = -\frac{1}{\lambda}Z(s) = -\frac{1}{\lambda}z_{X,\mathcal{L}_X,t}(s),$$

which implies that actually $u \in L^2_{\sigma(X)}(t,T;\mathcal{H})$; therefore $\inf_{v\in L^2_{\sigma(X)}(t,T;\mathcal{H})} J_{X,t}(v) = \inf_{v\in L^2(t,T;\mathcal{H})} J_{X,t}(v)$. Hence, using (5.21), we can conclude with the following:

Proposition 6.1. *Under the assumptions of Proposition 5.1, we first have*

$$V(m,t) = \inf_{v\in L^2(t,T;L^2_m(\mathbb{R}^n;\mathbb{R}^n))} J_{m,t}(v), \tag{6.5}$$

where $V(X,t)$ is also the lifted functional of $V(m,t)$. Moreover, the optimal control is given by $\hat{v}(x,s) = -\frac{1}{\lambda}z_{x,\mathcal{L}_X,t}(s)$ for $s \geq t$.

We here adopt the notation $\hat{v}(x,s)$ instead of using $u(x,s)$ for the optimal control, so as to avoid any confusion with a function $u_{m,t}(x,s)$ to be introduced in the next section when we try to obtain a feedback.

6.2. Feedback nature of z in y

We aim to show here that $z_{x,m,t}(s)$ is a feedback function in $y_{x,m,t}(s)$. We introduce $u_{m,t}(x,s)$ solution of the eikonal equation

$$\begin{cases} -\dfrac{\partial u_{m,t}(x,s)}{\partial s} + \dfrac{1}{2\lambda}|Du_{m,t}(x,s)|^2 = F(x, y_{m,t}(s)_\# m), \\ u_{m,t}(x,T) = F_T(x, y_{m,t}(T)_\# m). \end{cases} \tag{6.6}$$

In fact, both functions $F(x, y_{m,t}(s)_\# m)$ and $F_T(x, y_{m,t}(T)_\# m)$ are completely determined in accordance with Proposition 5.1 in subsection 5.2, in which an explicit construction for $y_{x,m,t}(s)$ was provided; in particular, once the parameters (m,t) are fixed and the function $y_{x,m,t}(s)$ is well-defined by (5.12) and (5.13), so the right hand side of (6.6) and the terminal condition are fixed. Here $u_{m,t}(x,s)$ is simply the solution of a simple PDE. To simplify technicalities, we shall assume that the matrix $D^2u_{m,t}(x,s)$ is continuous; and this could be confirmed by differentiating the equation (6.6) with respect to x. We then state:

Lemma 6.2. *Assume that $D^2u_{m,t}(x,s)$ is continuous, we have the formula*

$$z_{x,m,t}(s) = Du_{m,t}(y_{x,m,t}(s), s). \tag{6.7}$$

Proof. We first note that (6.7) clearly holds at the time T; indeed, by definition of (5.13),

$$Du_{m,t}(y_{x,m,t}(T), T) = D_xF_T(y_{x,m,t}(T), y_{m,t}(T)_\# m) = z_{x,m,t}(T).$$

We next set $\zeta_{m,t}(x,s) = Du_{m,t}(y_{x,m,t}(s), s)$. We then have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\zeta_{m,t}(x,s) &= D^2u_{m,t}(y_{x,m,t}(s),s)\frac{\mathrm{d}}{\mathrm{d}s}y_{x,m,t}(s) + D\frac{\partial u_{m,t}}{\partial s}(y_{x,m,t}(s),s) \\ &= -\frac{1}{\lambda}D^2u_{m,t}(y_{x,m,t}(s),s)z_{x,m,t}(s) + D\frac{\partial u_{m,t}}{\partial s}(y_{x,m,t}(s),s). \end{aligned}$$

But from equation (6.6) we can write, after differentiation:

$$-D\frac{\partial u_{m,t}}{\partial s}(x,s)+\frac{1}{\lambda}D^2u_{m,t}(x,s)Du_{m,t}(x,s)=D_xF(x,y_{m,t}(s)_{\#}m),$$

and then we substitute $y_{x,m,t}(s)$ for x, we further obtain

$$-D\frac{\partial u_{m,t}}{\partial s}(y_{x,m,t}(s),s)+\frac{1}{\lambda}D^2u_{m,t}(y_{x,m,t}(s),s)\zeta_{m,t}(x,s)=D_xF(y_{x,m,t}(s),y_{m,t}(s)_{\#}m)=-\frac{\mathrm{d}}{\mathrm{d}s}z_{x,m,t}(s).$$

Combining the results, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\zeta_{m,t}(x,s)-z_{x,m,t}(s)\right)=\frac{1}{\lambda}D^2u_{m,t}(y_{x,m,t}(s),s)(\zeta_{m,t}(x,s)-z_{x,m,t}(s)).$$

But $D^2u_{m,t}(y_{mt}(x,s),s)$ is a bounded matrix; from the latest linear differential equation and the terminal condition, we obtain that $\zeta_{m,t}(x,s)-z_{x,m,t}(s)=0$ identically, which concludes our claim (6.7). □

It follows that the optimal control for the problem (6.2) and (6.3) is obtained by a feedback:

$$\hat{v}(x,s)=-\frac{1}{\lambda}Du_{m,t}(y_{x,m,t}(s),s)=-\frac{1}{\lambda}z_{x,m,t}(s), \tag{6.8}$$

and the optimal state $x^{\hat{v}}(x,s)=x+\int_t^s\hat{v}(x,\tau)\mathrm{d}\tau=y_{x,m,t}(s)$, namely the solution of the system

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}s}y_{x,m,t}(s)=-\frac{1}{\lambda}Du_{m,t}(y_{x,m,t}(s),s),\\ y_{x,m,t}(t)=x,\end{cases} \tag{6.9}$$

in other words, the evolution of the dynamics of the trajectory $y_{x,m,t}(s)$ is driven through the feedback function $Du_{m,t}(x,s)$.

6.3. System of HJB-FP equations

The pair $m_{m,t}(s):=y_{m,t}(s)_{\#}m$, in which $y_{x,m,t}(s)$ is the solution of (6.9), and the function $u_{m,t}(x,s)$, the solution of (6.6), forms a system called HJB-FP (Fokker Planck) equations. Equation (6.6) is a HJB equation, and $m_{m,t}(s)$ is the image probability of m under the map $x\mapsto y_{x,m,t}(s)$. If the probability measure m possesses a probability density function denoted by $m(x)$, then $m_{m,t}(s)$ also has a probability density, denoted by $m(x,s)$, which is the solution of the following Fokker-Planck equation:

$$\begin{cases}\dfrac{\partial m}{\partial s}(x,s)-\dfrac{1}{\lambda}\mathrm{div}\left(Du_{m,t}(x,s)m(x,s)\right)=0,\\ m(x,t)=m(x),\end{cases}$$

from now on, we denote the solution as $m_{m,t}(x,s)$. As a result, we finally obtain the pair of functions $u_{m,t}(x,s)$ and $m_{m,t}(x,s)$ is the solution of the following coupled system, called HJB-FP equations:

$$\begin{cases}-\dfrac{\partial u_{m,t}(x,s)}{\partial s}+\dfrac{1}{2\lambda}|Du_{m,t}(x,s)|^2=F(x,m_{m,t}(\cdot,s)),\\ \dfrac{\partial m_{m,t}}{\partial s}-\dfrac{1}{\lambda}\mathrm{div}\left(Du_{m,t}(x,s)m_{m,t}(x,s)\right)=0,\\ m_{m,t}(x,t)=m(x),\ u_{m,t}(x,T)=F_T(x,m_{m,t}(\cdot,T)).\end{cases} \tag{6.10}$$

This system expresses a necessary condition of the optimality; also see [5].

7. Bellman equation for the mean field type control problem

We have already seen in (5.22) that the functional $V(m,t)$ has a functional derivative and an L-derivative; our aim in this section is to characterize $\dfrac{\delta V}{\delta m}(m,t)(x)$; indeed, under some adequate mild conditions, we can see that it is equal to $u_{m,t}(x,t)$. We first proceed with giving a useful formula for $u_{m,t}(x,t)$.

7.1. A useful formula for $u_{m,t}(x,t)$

Lemma 7.1. *Under assumptions as specified in Proposition 5.1, we can give an explicit solution to the system (6.10):*

$$u_{m,t}(x,t) = \frac{1}{2\lambda}\int_t^T |z_{x,m,t}(s)|^2 \mathrm{d}s + \int_t^T F(y_{x,m,t}(s), y_{m,t}(s)_\# m)\mathrm{d}s + F_T(y_{x,m,t}(T), y_{m,t}(T)_\# m), \tag{7.1}$$

and $m_{m,t}(\cdot,s) = y_{m,t}(s)_\# m$.

Proof. Recall that $m_{m,t}(\cdot,s) = y_{m,t}(s)_\# m$. If we look at $F(x, m_{m,t}(\cdot,s))$ and $F_T(x, m_{m,t}(\cdot,T))$ in which $m_{m,t}(\cdot,\cdot)$ is frozen, the HJB equation appears as a standard one for a deterministic control problem, $\inf_v J_{x,t}(v)$, where $J_{x,t}(v) := \frac{\lambda}{2}\int_t^T |v(s)|^2\mathrm{d}s + \int_t^T F(x(s), m_{m,t}(\cdot,s))\mathrm{d}s + F_T(x(T), m_{m,t}(\cdot,T))$,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} = v(s), \\ x(t) = x, \end{cases}$$

in which the function $m_{m,t}(\cdot,\cdot)$ is frozen, but not arbitrary; this frozen function is the solution of the FP equation in (6.10). If we write down the necessary condition of the optimality, one can check easily that in view of the specific value of $m_{m,t}(\cdot,s)$, the optimal state is $y_{x,m,t}(s)$ while the optimal control is $-\dfrac{1}{\lambda}z_{x,m,t}(s)$; by plugging these expressions in the cost function, we obtain the formula (7.1). $\square$

7.2. Characterization of the functional derivative $\frac{\delta V}{\delta m}(m,t)(x)$

We first propose some convenient assumptions. We assume that for any $x \in \mathbb{R}^n$ and any probability measure $m \in \mathcal{P}_2(\mathbb{R}^n)$:

$$|F(x,m)|, |F_T(x,m)| \le C\left(1 + |x|^2 + \int_{\mathbb{R}^n} |\xi|^2 \mathrm{d}m(\xi)\right), \tag{7.2}$$

$$\left|\frac{\delta F(x,m)}{\delta m}(\xi)\right|, \left|\frac{\delta F_T(x,m)}{\delta m}(\xi)\right| \le C\left(1 + |x|^2 + |\xi|^2 + \int_{\mathbb{R}^n} |\eta|^2 \mathrm{d}m(\eta)\right), \tag{7.3}$$

for some constant $C > 0$, and all these functions are continuous in their own arguments. We also make a further "convexity" assumption, though not necessary, which can help simplify the proofs, so as to avoid unnecessarily technical difficulties: for any $m_1, m_2 \in \mathcal{P}_2(\mathbb{R}^n)$,

$$\begin{aligned} &\int_{\mathbb{R}^n} (F(x,m_1) - F(x,m_2))(\mathrm{d}m_1(x) - \mathrm{d}m_2(x)) \ge 0, \\ &\int_{\mathbb{R}^n} (F_T(x,m_1) - F_T(x,m_2))(\mathrm{d}m_1(x) - \mathrm{d}m_2(x)) \ge 0. \end{aligned} \tag{7.4}$$

[1]This assumption allows to obtain the following useful monotonicity result. Denote, for $i=1,2$, $u_i(x,s)=u_{m_i,t}(x,s)$ and $m_i(s)=y_{m_i,t}(t)_{\#}m_i=m_{m_i,t}(s)$ in light of the solutions, of the system HJB-FP, (6.6) and (6.9) respectively, with the initial condition $m=m_i$. We then state:

Proposition 7.2. *Assume the condition (7.4). We have the property:*

$$\int_{\mathbb{R}^n}(u_1(x,t)-u_2(x,t))(\mathrm{d}m_1(x)-\mathrm{d}m_2(x))\geq 0. \tag{7.5}$$

Proof. From the system HJB (6.6), we can write

$$\begin{cases}-\frac{\partial}{\partial s}(u_1-u_2)+\frac{1}{2\lambda}|Du_1|^2-\frac{1}{2\lambda}|Du_2|^2=F(x,m_1(s))-F(x,m_2(s)),\\ u_1(x,T)-u_2(x,T)=F_T(x,m_1(T))-F_T(x,m_2(T));\end{cases}$$

then a simple calculation, or otherwise by first showing the claim for measures with densities and then passing to limit, leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^n}(u_1(x,s)-u_2(x,s))(\mathrm{d}m_1(s)(x)-\mathrm{d}m_2(s)(x))=&-\int_{\mathbb{R}^n}(F(x,m_1(s))-F(x,m_2(s)))(\mathrm{d}m_1(s)(x)-\mathrm{d}m_2(s)(x))\\&-\frac{1}{2\lambda}\int_{\mathbb{R}^n}|Du_1(x,s)-Du_2(x,s)|^2(\mathrm{d}m_1(s)(x)+\mathrm{d}m_2(s)(x))\end{aligned}$$

and the result follows immediately by recalling that both m_1 and m_2 are positive and using the second assumption in (7.4). ☐

Proposition 7.3. *We make the assumptions of Proposition 5.1, (7.2)–(7.4), we then have*

$$U(x,m,t):=\frac{\delta V}{\delta m}(m,t)(x)=u_{m,t}(x,t). \tag{7.6}$$

Moreover, we have the estimate

$$|U(x,m,t)|\leq C\left(1+|x|^2+\int_{\mathbb{R}^n}|\xi|^2\mathrm{d}m(\xi)\right). \tag{7.7}$$

Remark 7.4. Note that

$$D_xU(x,m,t)=z_{x,m,t}(t).$$

Therefore, from the estimate (5.20), we can assert that

$$|D_xU(x,m,t)|\leq C\left[1+|x|+\sqrt{\int|\xi|^2\mathrm{d}m(\xi)}\right].$$

[1]Note that Ahuja studies in [2] a mean field game in the presence of a common noise, he proposes a similar monotonicity condition (see A(4) in [2]).

Proof. We first recall some basic facts about the HJB-FP system (6.6) and (6.9); while the optimal control is given by (6.8) and the value function can be written as:

$$V(m,t)=J_{m,t}(\hat{v})=\frac{1}{2\lambda}\int_t^T\int_{\mathbb{R}^n}|Du_{m,t}(y_{x,m,t}(s),s)|^2\mathrm{d}m(x)\mathrm{d}s+\int_t^T F(y_{m,t}(s)_{\#}m)\mathrm{d}s+F_T(y_{m,t}(T)_{\#}m). \quad (7.8)$$

Consider two arbitrary elements $m_1,m_2\in\mathcal{P}_2(\mathbb{R}^n)$. For $i=1,2$, we denote $u_i(x,s)=u_{m_i,t}(x,s)$, $\hat{v}_i(x,s)=-\frac{1}{\lambda}Du_i(y_{x,m_i,t}(s),s)$, $m_i(s)=y_{m_i,t}(s)_{\#}m_i=m_{m_i,t}(s)$, and we then further have:

$$V(m_i,t)=J_{m_i,t}(\hat{v}_i)=\frac{1}{2\lambda}\int_t^T\int_{\mathbb{R}^n}|Du_i(\eta,s)|^2\mathrm{d}m_i(s)(\eta)\mathrm{d}s+\int_t^T F(m_i(s))\mathrm{d}s+F_T(m_i(T)).$$

The probability measure at time s with the initial condition at time t being equal to m_2 and feedback $\hat{v}_1$ being $y_{m_1,t}(s)_{\#}m_2$ is denoted by $m_{12}(s)=m_{12}(s)$. We can then write

$$J_{m_2,t}(\hat{v}_1)=\frac{1}{2\lambda}\int_t^T\int_{\mathbb{R}^n}|Du_1(x,s)|^2\mathrm{d}m_{12}(s)(x)\mathrm{d}s+\int_t^T F(m_{12}(s))\mathrm{d}s+F_T(m_{12}(T));$$

clearly $V(m_2,t)\le J_{m_2,t}(\hat{v}_1)$. Therefore, we have the inequality

$$\begin{aligned}V(m_2,t)-V(m_1,t)&\le J_{m_2,t}(\hat{v}_1)-J_{m_1,t}(\hat{v}_1) \quad (7.9)\\&=\frac{1}{2\lambda}\int_t^T\int_{\mathbb{R}^n}|Du_1(x,s)|^2(\mathrm{d}m_{12}(s)(x)-\mathrm{d}m_1(s)(x))\mathrm{d}s\\&\quad+\int_t^T(F(m_{12}(s))-F(m_1(s)))\mathrm{d}s+(F_T(m_{12}(T))-F_T(m_1(T))),\end{aligned}$$

where in accordance with the definitions and (6.6), we note that

$$\begin{cases}-\frac{\partial}{\partial s}u_1+\frac{1}{2\lambda}|Du_1|^2=F(x,m_1(s)),\\u_1(x,T)=F_T(x,m_1(T)),\end{cases} \quad (7.10)$$

and $\int_{\mathbb{R}^n}u_1(x,s)(\mathrm{d}m_{12}(s)(x)-\mathrm{d}m_1(s)(x))=\int_{\mathbb{R}^n}u_1(y_{x,m_1,t}(s),s)(\mathrm{d}m_2(x)-\mathrm{d}m_1(x))$. On the other hand, by the application of the simple chain rule and (6.9), we can then obtain:

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^n}u_1(x,s)(\mathrm{d}m_{12}(s)(x)-\mathrm{d}m_1(s)(x))=\int_{\mathbb{R}^n}\left[\frac{\partial u_1}{\partial s}(y_{x,m_1,t}(s),s)-\frac{1}{\lambda}|Du_1(y_{x,m_1,t}(s),s)|^2\right](\mathrm{d}m_2(x)-\mathrm{d}m_1(x)).$$

Therefore, simply using the PDE (7.10), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^n}u_1(x,s)(\mathrm{d}m_{12}(s)(x)-\mathrm{d}m_1(s)(x))&=-\int_{\mathbb{R}^n}\left[\frac{1}{2\lambda}|Du_1(y_{x,m_1,t}(s),s)|^2+F(y_{x,m_1,t}(s),m_1(s))\right]\\&\qquad\times(\mathrm{d}m_2(x)-\mathrm{d}m_1(x))\\&=-\int_{\mathbb{R}^n}\left[\frac{1}{2\lambda}|Du_1(x,s)|^2+F(x,m_1(s))\right](\mathrm{d}m_{12}(s)(x)-\mathrm{d}m_1(s)(x)).\end{aligned}$$

Integrating from t to T, we further obtain

$$\int_{\mathbb{R}^n} u_1(x,t)(\mathrm{d}m_2(x)-\mathrm{d}m_1(x)) = \int_t^T \int_{\mathbb{R}^n} \left[\frac{1}{2\lambda}|Du_1(x,s)|^2 + F(x,m_1(s))\right](\mathrm{d}m_{12}(s)(x)-\mathrm{d}m_1(s)(x)) \\ + \int_{\mathbb{R}^n} F_T(x,m_1(T))(\mathrm{d}m_{12}(T)(x)-\mathrm{d}m_1(T)(x)). \quad (7.11)$$

Combining with (7.9), we can write

$$V(m_2,t) - V(m_1,t) \le \int_{\mathbb{R}^n} u_1(x,t)(\mathrm{d}m_2(x)-\mathrm{d}m_1(x)) \\ + \int_t^T \left[F(m_{12}(s)) - F(m_1(s)) - \int_{\mathbb{R}^n} F(x,m_1(s))(\mathrm{d}m_{12}(s)(x)-\mathrm{d}m_1(s)(x))\right]\mathrm{d}s \\ + F_T(m_{12}(T)) - F_T(m_1(T)) - \int_{\mathbb{R}^n} F_T(x,m_1(T))(\mathrm{d}m_{12}(T)(x)-\mathrm{d}m_1(T)(x)). \quad (7.12)$$

By recalling (5.6) subject to the assumption (5.8), we have

$$\left|F(m_{12}(s)) - F(m_1(s)) - \int_{\mathbb{R}^n} F(x,m_1(s))(\mathrm{d}m_{12}(s)(x)-\mathrm{d}m_1(s)(x))\right| \le CW_2^2(m_{12}(s),m_1(s)), \quad (7.13)$$

and a similar inequality also holds for F_T with F_T in place of F and T intead of s. Besides, by the definitions of $m_{12}(s)$ and $m_1(s)$, we should have

$$W_2^2(m_{12}(s),m_1(s)) \le \mathbb{E}|y_{X_2,m_1,t}(s) - y_{X_1,m_1,t}(s)|^2,$$

for any $X_1, X_2 \in \mathcal{H}$ such that $\mathcal{L}_{X_1} = m_1$ and $\mathcal{L}_{X_2} = m_2$. Further applying the estimate (5.18), we deduce that $\mathbb{E}|y_{X_2,m_1,t}(s) - y_{X_1,m_1,t}(s)|^2 \le C'\mathbb{E}|X_2 - X_1|^2$ for some universal constant $C' > 0$. Finally, since X_1 and X_2 are arbitrary random variables with respective probability laws $\mathcal{L}_{X_1}$ and $\mathcal{L}_{X_2}$, we therefore obtain $W_2^2(m_{12}(s),m_1(s)) \le C'W_2^2(m_1,m_2)$ for all $s \in [t,T]$. From (7.12) and (7.13), it follows that

$$V(m_2,t) - V(m_1,t) \le \int_{\mathbb{R}^n} u_1(x,t)(\mathrm{d}m_2(x)-\mathrm{d}m_1(x)) + (T+1)C'W_2^2(m_1,m_2). \quad (7.14)$$

Interchanging the roles of m_1 and m_2, we also have

$$V(m_1,t) - V(m_2,t) \le \int_{\mathbb{R}^n} u_2(x,t)(\mathrm{d}m_1(x)-\mathrm{d}m_2(x)) + (T+1)C'W_2^2(m_1,m_2) \\ = \int_{\mathbb{R}^n} u_1(x,t)(\mathrm{d}m_1(x)-\mathrm{d}m_2(x)) \\ + \int_{\mathbb{R}^n} (u_2(x,t) - u_1(x_1,t))(\mathrm{d}m_1(x)-\mathrm{d}m_2(x)) + (T+1)C'W_2^2(m_1,m_2).$$

Under assumption (7.4), we then have Proposition 7.2 which further implies that the second integral is negative, therefore we get

$$V(m_1,t) - V(m_2,t) \le \int_{\mathbb{R}^n} u_1(x,t)(\mathrm{d}m_1(x)-\mathrm{d}m_2(x)) + (T+1)C'W_2^2(m_1,m_2),$$

or

$$V(m_2,t)-V(m_1,t)\geq \int_{\mathbb{R}^n} u_1(x,t)(\mathrm{d}m_2(x)-\mathrm{d}m_1(x))-(T+1)C'W_2^2(m_1,m_2);$$

and combining this with (7.14), we can assert that

$$\left|V(m_2,t)-V(m_1,t)-\int_{\mathbb{R}^n} u_1(x,t)(\mathrm{d}m_2(x)-\mathrm{d}m_1(x))\right|\leq (T+1)C'W_2^2(m_1,m_2). \tag{7.15}$$

Under condition (5.3), by applying the estimates (5.17) and (5.20) to the explicit formula (7.1), we now have

$$|u_{m,t}(x,t)|\leq C\left(1+|x|^2+\int_{\mathbb{R}^n}|\xi|^2\mathrm{d}m(\xi)\right). \tag{7.16}$$

From the estimate (7.15), we get immediately the claim (7.6). The proof is now complete. □

7.3. Scalar Master equation

More details of the alternative of this section can be found in [6, 7]. We recall the Master equation of the abstract control problem, see (3.9), and our present objective is to interpret it for the mean field type control problem (6.2) and (6.3). In fact, we shall get two Master equations, one is called the scalar Master equation, while another one is called the vector Master equation, which will correspond to the Master equation of the abstract control problem itself. The scalar Master equation is the equation for the function $U(x,m,t)=\frac{\delta V}{\delta m}(m,t)(x)=u_{m,t}(x,t)$. The vector Master equation corresponds to the equation for $\mathcal{U}(x,m,t)=D_xU(x,m,t)$. We recall the equivalence $\mathcal{U}(X,t)=D_XV(X,t)=D_x\frac{\delta V}{\delta m}(m,t)(X)$, where m is the probability law of X. With reference to Bellman equation (5.23) in Proposition 5.2, and consider the functional $\int_{\mathbb{R}^n}|D_x\frac{\delta V}{\delta m}(m,t)(x)|^2\mathrm{d}m(x)$ in m, which clearly has a lifted functional $||D_XV(X,t)||^2$. Under the assumptions (3.1) and (3.2), both the second order derivative $D_X^2V(X,t)$ is well-defined and the bound (3.6) holds; and hence, we deduce that $||D_XV(X,t)||^2$ is differentiable such that

$$D_X||D_XV(X,t)||^2=2D_X^2V(X,t)D_XV(X,t).$$

Furthermore, based on the discussion leading to (4.4), we can now assert that $\int_{\mathbb{R}^n}|D_x\frac{\delta V}{\delta m}(m,t)(x)|^2\mathrm{d}m(x)$ has both a functional derivative and a L-derivative. Therefore, from (5.23), we obtain the follwing equation:

$$\begin{cases}\frac{\partial U(x,m,t)}{\partial t}-\frac{1}{2\lambda}\frac{\delta}{\delta m}\left(\int_{\mathbb{R}^n}|D_\xi U(\xi,m,t)|^2\mathrm{d}m(\xi)\right)(x)+F(x,m)=0,\\ U(x,m,T)=F_T(x,m),\end{cases} \tag{7.17}$$

which is the scalar Master equation. We can then state

Proposition 7.5. *Under assumptions (3.1), (3.2) and (7.4), $U(x,m,t)$ is the solution of the scalar Master equation (7.17).*

Remark 7.6. One can also establish that $U(x,m,t)$ possesses a functional derivative which further has the symmetric property, namely:

$$\frac{\delta U}{\delta m}(x,m,t)(\xi)=\frac{\delta U}{\delta m}(\xi,m,t)(x);$$

also see [6] and [7]. By expanding the functional derivative in (7.17) and applying this symmetry property, we obtain the equation:

$$\begin{cases} \frac{\partial U}{\partial t} - \frac{1}{\lambda}\int_{\mathbb{R}^n} D_\xi \frac{\delta U}{\delta m}(x,m,t)(\xi)\cdot D_\xi U(\xi,m,t)\mathrm{d}m(\xi) - \frac{1}{2\lambda}|D_x U(x,m,t)|^2 + F(x,m) = 0, \\ U(x,m,T) = F_T(x,m), \end{cases} \tag{7.18}$$

which is equivalent to the equation (1.1) in [13]. Except that their equation is only defined on a compact torus $\mathbb{T}^n$, which corresponds to the periodic nature of the underlying functions; while ours remains applicable for most regular enough functions on $\mathbb{R}^n$, with the former as an interesting example. This function allows to decouple the system of HJB-FP equations (6.10); indeed, we first solve the FP equation in which we replace u by U, *i.e.*

$$\begin{cases} \frac{\partial m}{\partial s} - \frac{1}{\lambda}\mathrm{div}(DU\, m) = 0, \\ m(x,t) = m(x), \end{cases} \tag{7.19}$$

then $u(x,s) = U(x,m(s),s)$ is solution of the HJB equation in (6.10), as easily checked. In particular, we now have

$$u(x,t) = U(x,m,t). \tag{7.20}$$

7.4. Vector MASTER equation

We next consider $\mathcal{U}(x,m,t) = D_x U(x,m,t)$. Differentiating (7.18), we can write

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} - \frac{1}{2\lambda} D_x \frac{\delta}{\delta m}\left(\int_{\mathbb{R}^n} |\mathcal{U}(\xi,m,t)|^2 \mathrm{d}m(\xi)\right)(x) + D_x F(x,m) = 0, \\ \mathcal{U}(x,m,T) = D_x F_T(x,m); \end{cases} \tag{7.21}$$

or equivalently,

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} - \frac{1}{\lambda}\int_{\mathbb{R}^n} D_\xi \frac{\delta \mathcal{U}}{\delta m}(x,m,t)(\xi)\mathcal{U}(\xi,m,t)m(\xi)\mathrm{d}\xi - \frac{1}{\lambda} D_x\mathcal{U}(x,m,t)\mathcal{U}(x,m,t) + D_x F(x,m) = 0, \\ \mathcal{U}(x,m,T) = D_x F_T(x,m). \end{cases}$$

The lifted equation of (7.21) is clearly:

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} - \frac{1}{2\lambda} D_X ||\mathcal{U}(X,t)||^2 + D_X \mathcal{F}(X) = 0, \\ \mathcal{U}(X,T) = D_X \mathcal{F}_T(X). \end{cases} \tag{7.22}$$

8. Quadratic case

8.1. Assumptions

We shall provide a complete explicit formulae in the quadratic case. To simplify notations, we denote $\bar{x} := \int_{\mathbb{R}^n} x\mathrm{d}m(x)$ for any $m \in \mathcal{P}_2(\mathbb{R}^n)$. Fix a matrix S, and symmetric matrices $Q, \bar{Q}, Q_T, \bar{Q}_T$, we take:

$$\begin{cases} f(x,m) = \frac{1}{2}(x - S\bar{x})^* \bar{Q}(x - S\bar{x}) + \frac{1}{2}x^* Q x, \\ h(x,m) = \frac{1}{2}(x - S_T\bar{x})^* \bar{Q}_T(x - S_T\bar{x}) + \frac{1}{2}x^* Q_T x. \end{cases} \tag{8.1}$$

We also define:

$$\begin{aligned} F(m) &= \int_{\mathbb{R}^n} f(x,m)\mathrm{d}m(x) \\ &= \frac{1}{2}\int_{\mathbb{R}^n} x^*(Q+\bar{Q})x\mathrm{d}m(x) + \frac{1}{2}\left(\bar{x}^*S^*\bar{Q}S\bar{x}\right) - \frac{1}{2}\bar{x}^*(\bar{Q}S+S^*\bar{Q})\bar{x}. \end{aligned} \tag{8.2}$$

Similarly, we define:

$$F_T(m) = \frac{1}{2}\int_{\mathbb{R}^n} x^*(Q_T+\bar{Q}_T)x\mathrm{d}m(x) + \frac{1}{2}\left(\bar{x}^*S_T^*\bar{Q}_TS_T\bar{x}\right) - \frac{1}{2}\bar{x}^*(\bar{Q}_TS_T+S_T^*\bar{Q}_T)\bar{x}.$$

In the space $\mathcal{H}$, we have the correspondence $\mathcal{F}(X) = F(m)$ and $\mathcal{F}_T(X) = F_T(m)$:

$$\begin{aligned} \mathcal{F}(X) &= \frac{1}{2}\mathbb{E}\left(X^*(Q+\bar{Q})X\right) + \frac{1}{2}\mathbb{E}X^*(S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q})\mathbb{E}X, \\ \mathcal{F}_T(X) &= \frac{1}{2}\mathbb{E}\left(X^*(Q_T+\bar{Q}_T)X\right) + \frac{1}{2}\mathbb{E}X^*(S_T^*\bar{Q}_TS_T - \bar{Q}_TS_T - S_T^*\bar{Q}_T)\mathbb{E}X. \end{aligned} \tag{8.3}$$

Note that $\mathcal{F}(X) = \mathbb{E}f(X,\mathcal{L}_X)$ by using (8.1).

We also note that $F(x,m) = \dfrac{\delta F(m)}{\delta m}(x)$, therefore, we can explicitly write:

$$\begin{aligned} F(x,m) &= \frac{1}{2}x^*(Q+\bar{Q})x + \bar{x}^*(S^*\bar{Q}S\,m_1 - \bar{Q}S - S^*\bar{Q})x + \frac{1}{2}\bar{x}^*S^*\bar{Q}S\bar{x}, \\ D_xF(x,m) &= (Q+\bar{Q})x - (\bar{Q}S + S^*\bar{Q} - S^*\bar{Q}S)\bar{x}, \end{aligned}$$

so that the following regularity conditions are also satisfied:

$$\begin{aligned} |D_xF(x,m)| &\le \frac{c}{2}\left(1+|x| + \left(\int |\xi|^2\mathrm{d}m(\xi)\right)^{\frac{1}{2}}\right), \\ |D_xF(x_1,m_1) - D_xF(x_2,m_2)| &\le \frac{c}{2}\left(|x_1-x_2| + W_2(m_1,m_2)\right). \end{aligned}$$

On the other hand, we note that

$$D_X\mathcal{F}(X) = (Q+\bar{Q})X + (S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q})\mathbb{E}X. \tag{8.4}$$

in the Hilbert space $\mathcal{H}$.

8.2. Bellman equation

Bellman equation (5.23) can now be written as follows; also see Section 8 in [7].

$$\begin{cases} \frac{\partial V}{\partial t} - \frac{1}{2\lambda}\displaystyle\int_{\mathbb{R}^n}\left|D_\xi\frac{\delta V(m,t)}{\delta m}(\xi)\right|^2\mathrm{d}m(\xi) + \frac{1}{2}\int_{\mathbb{R}^n}\xi^*(Q+\bar{Q})\xi\mathrm{d}m(\xi) + \frac{1}{2}\bar{x}^*S^*\bar{Q}S\bar{x} - \frac{1}{2}\bar{x}^*(\bar{Q}S+S^*\bar{Q})\bar{x} = 0, \\ V(m,T) = \frac{1}{2}\int_{\mathbb{R}^n}\xi^*(Q_T+\bar{Q_T})\xi\mathrm{d}m(\xi) + \frac{1}{2}\bar{x}^*S_T^*\bar{Q}_TS_T\bar{x} - \frac{1}{2}\bar{x}^*(\bar{Q}_TS_T + S_T^*\bar{Q}_T)\bar{x}. \end{cases} \tag{8.5}$$

The solution is in the form:

$$V(m,t) = \frac{1}{2}\int_{\mathbb{R}^n} \xi^* P(t)\xi \mathrm{d}m(\xi) + \frac{1}{2}\bar{x}^*\Sigma(t)\bar{x}, \tag{8.6}$$

where

$$\begin{cases} \frac{dP}{\mathrm{d}t} - \frac{P^2}{\lambda} + Q + \bar{Q} = 0, \\ P(T) = Q_T + \bar{Q}_T; \end{cases} \tag{8.7}$$

$$\begin{cases} \frac{\mathrm{d}\sigma}{\mathrm{d}t} - \frac{1}{\lambda}(\Sigma P + P\Sigma) - \frac{1}{\lambda}\Sigma^2 + S^*\bar{Q}S - (\bar{Q}S + S^*\bar{Q}) = 0, \\ \Sigma(T) = S_T^*\bar{Q}_T S_T - (\bar{Q}_T S_T + S_T^*\bar{Q}_T). \end{cases} \tag{8.8}$$

8.3. Master equation

The scalar Master equation (7.17) now reads

$$\begin{cases} \frac{\partial U}{\partial t} - \frac{1}{2\lambda}\frac{\delta}{\delta m}\left(\int_{\mathbb{R}^n} |D_\xi U(\xi,m,t)|^2 \mathrm{d}m(\xi)\right)(x) + \frac{1}{2}x^*(Q+\bar{Q})x + \bar{x}^*(S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q})x + \frac{1}{2}\bar{x}^*S^*\bar{Q}S\bar{x} = 0, \\ U(x,m,T) = \frac{1}{2}x^*(Q_T + \bar{Q}_T)x + \bar{x}^*(S_T^*\bar{Q}_T S_T - \bar{Q}_T S_T - S_T^*\bar{Q}_T)x + \frac{1}{2}\bar{x}^*S_T^*\bar{Q}_T S_T\bar{x}. \end{cases} \tag{8.9}$$

Its solution is

$$U(x,m,t) = \frac{\delta V(m,t)}{\delta m}(x) = \frac{1}{2}x^*P(t)x + \bar{x}^*\Sigma(t)x. \tag{8.10}$$

We then turn to the vector Master equation (7.21) which reads

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} - \frac{1}{2\lambda}D_x\frac{\delta}{\delta m}\left(\int_{\mathbb{R}^n} |\mathcal{U}(\xi,m,t)|^2 \mathrm{d}m(\xi)\right)(x) + (Q+\bar{Q})x + (S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q})\bar{x} = 0, \\ \mathcal{U}(x,m,T) = (Q_T + \bar{Q}_T)x + (S_T^*\bar{Q}_T S_T - \bar{Q}_T S_T - S_T^*\bar{Q}_T)\bar{x}, \end{cases} \tag{8.11}$$

whose solution is

$$\mathcal{U}(x,m,t) = D_x U(x,m,t) = P(t)x + \Sigma(t)\bar{x}. \tag{8.12}$$

This statement can easily be verified.

8.4. Formulation in the Hilbert space

We can also formulate Bellman equation and the Master equation in the Hilbert space $\mathcal{H}$. First, we have the Bellman equation:

$$\begin{cases} \frac{\partial V}{\partial t} - \frac{1}{2\lambda}||D_X V||^2 + \frac{1}{2}\mathbb{E}\left(X^*(Q+\bar{Q})X\right) + \frac{1}{2}\mathbb{E}(X)^*(S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q})\mathbb{E}(X) = 0, \\ V(X,T) = \frac{1}{2}\mathbb{E}\left(X^*(Q_T + \bar{Q}_T)X\right) + \frac{1}{2}\mathbb{E}(X)^*(S_T^*\bar{Q}_T S_T - \bar{Q}_T S_T - S_T^*\bar{Q}_T)\mathbb{E}(X), \end{cases} \tag{8.13}$$

whose solution is

$$V(X,t) = \frac{1}{2}\mathbb{E}\left(X^*P(t)X\right) + \frac{1}{2}\mathbb{E}(X)^*\Sigma(t)\mathbb{E}(X). \tag{8.14}$$

Second, the Master equation reads

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} - \frac{1}{\lambda} D_X \mathcal{U}(X,t)\mathcal{U}(X,t) + (Q+\bar{Q})X + (S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q})\mathbb{E}(X) = 0, \\ \mathcal{U}(X,T) = (Q_T + \bar{Q}_T)X + (S_T^*\bar{Q}_T S_T - \bar{Q}_T S_T - S_T^*\bar{Q}_T)\mathbb{E}(X), \end{cases} \tag{8.15}$$

whose solution is $\mathcal{U}(X,t) = D_X V(X,t) = P(t)X + \Sigma(t)\mathbb{E}(X)$. Note that $D_X\mathcal{U}(X,t)Z = P(t)Z + \Sigma(t)\mathbb{E}(Z)$. The state equation is the solution of

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}s} = -\frac{1}{\lambda}(P(s)Y(s) + \Sigma(s)\mathbb{E}(Y(s))), \\ Y(t) = X, \end{cases} \tag{8.16}$$

hence its explicit solution formula is

$$Y(s) = \exp\left(-\frac{1}{\lambda}\int_t^s P(\sigma)\mathrm{d}\sigma\right) X - \int_t^s \exp\left(-\frac{1}{\lambda}\int_\sigma^s P(\tau)\mathrm{d}\tau\right) \Sigma(\sigma) \exp\left(-\frac{1}{\lambda}\int_t^\sigma (P(\tau)+\Sigma(\tau))\mathrm{d}\tau\right) \mathrm{d}\sigma\, \mathbb{E}(X). \tag{8.17}$$

Appendix A

A.1 Functionals of probability measures

A.1.1 General comments

In this paper, we have seen how a general control problem in a Hilbert space, when specialized to a Hilbert space of square-integrable random variables, can lead to the solution of a mean field type control problem. A mean field type control problem is a control problem in which the pay off functionals also depend on probability measures. We have explored the simple and fruitful idea, first introduced by Lions in [17, 18], to consider a functional of probability measures as a functional of random variables whose probability laws are the former probability measures. To make a more convenient connection with mean field type control problems, we can assume that probability measures have densities on $\mathbb{R}^n$, which belong to $L^2(\mathbb{R}^n)$. This is to keep working with a Hilbert space, namely $L^2(\mathbb{R}^n)$, instead of $\mathcal{H} = L^2(\Omega, \mathcal{A}, P; \mathbb{R}^n)$. Although many results can be proven rigorously, it is not the optimal way to proceed. First of all, we reduce considerably the probability measures, and more importantly, this approach raises substantial technical difficulties, which are not worth considering. This is why the approach suggested by Lions is the right one, indeed. The alternative would be to work with the space of probability measures directly. This is the approach considered in Gangbo and Święch [13]. The difficulty is that the space of probability measures is not a Hilbert space but only a metric space. The key issue is to define the concept of gradient. Then one can proceed with the study of a control problem in a metric space. This has been done remarkably by Gangbo and Święch [13]. But the paper is quite complex and has limitations. The purpose of this appendix is to show that the Wasserstein gradient introduced on the space of probability measures is, in fact, equivalent to the gradient in the sense of the Hilbert space of random variables; also see [7] for details.

A.1.2 Wasserstein gradient

We consider the space $\mathcal{P}_2(\mathbb{R}^n)$ of all probability measures on $\mathbb{R}^n$ with finite second order moments, equipped with the Wasserstein metric $W_2(\mu,\nu)$ defined by:

$$W_2^2(\mu,\nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} \int_{\mathbb{R}^n\times\mathbb{R}^n} |\xi-\eta|^2 \gamma(\mathrm{d}\xi, \mathrm{d}\eta), \tag{A.1}$$

where $\Gamma(\mu,\nu)$ denotes the set of all joint probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ such that the marginals are μ and ν respectively. It is convenient to consider an atomless probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and all its L^2 random variables namely, $\mathcal{H} := L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$. We then write, for any $X, Y \in \mathcal{H}$, $\mu = \mathcal{L}_X$, $\nu = \mathcal{L}_Y$, and so

$$W_2^2(\mu,\nu) = \inf_{X,Y\in\mathcal{H},\mathcal{L}_X=\mu,\mathcal{L}_Y=\nu} \mathbb{E}\left(|X-Y|^2\right).$$

When the probability law has a density with respect to Lebesgue measure, say a positive $m(x)$ belonging to $L^1(\mathbb{R}^n)$, we then replace the law by its density. Note that $\int |x|^2 m(x)\mathrm{d}x < +\infty$. We call $L^2_m(\mathbb{R}^n;\mathbb{R}^n)$ the space of functions $f:\mathbb{R}^n \to \mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |f(x)|^2 m(x)\mathrm{d}x < +\infty$. We consider functionals $F(\mu)$ on $\mathcal{P}_2(\mathbb{R}^n)$; without the cause of ambiguity, if μ has a density m we also write $F(m)$. If $m \in L^2(\mathbb{R}^n)$, we say that $F(m)$ has a Gâteaux derivative at m, denoted by $\dfrac{\partial F(m)}{\partial m}(x)$, if we have

$$\lim_{\theta\to 0} \frac{F(m+\theta\mu)-F(m)}{\theta} = \int_{\mathbb{R}^n} \frac{\partial F(m)}{\partial m}(x)\mu(x)\mathrm{d}x,\ \forall \mu \in L^2(\mathbb{R}^n), \tag{A.2}$$

and this $\dfrac{\partial F(m)}{\partial m}(\cdot) \in L^2(\mathbb{R}^n)$. For probability densities, we shall extend this concept as follows. We say that $\dfrac{\partial F(m)}{\partial m}(\cdot) \in L^1_m(\mathbb{R}^n)$ is the *functional derivative* of F at m if for any sequence of probability densities m_ϵ in $\mathcal{P}_2(\mathbb{R}^n)$ such that $W_2(m_\epsilon, m) \to 0$, then $\dfrac{\partial F(m)}{\partial m}(\cdot) \in L^1_{m_\epsilon}(\mathbb{R}^n)$ and

$$\frac{F(m_\epsilon) - F(m) - \displaystyle\int_{\mathbb{R}^n} \frac{\partial F(m)}{\partial m}(x)(m_\epsilon(x) - m(x))\mathrm{d}x}{W_2(m_\epsilon, m)} \to 0 \ \text{ as } \ \epsilon \to 0. \tag{A.3}$$

The function $\dfrac{\partial F(m)}{\partial m}(\cdot)$ is called the *functional derivative* of $F(m)$ at the point m; let us see its connection with the concept of Wasserstein gradient on the metric space $\mathcal{P}_2(\mathbb{R}^n)$. We shall simply give the definition and the expression of the gradient. For a detailed discussion, one can refer to Ambrosio-Gigli-Savaré [3], Benamou-Brenier [4], Brenier [8], Jordan-Kinderlehrer-Otto [14], Otto [19], and Villani [20]. We first remind the notion of push-forward, $T_\#$, of a (Borel) map $T:\mathbb{R}^n \to \mathbb{R}^n$, so that for any $\mu \in \mathcal{P}_2(\mathbb{R}^n)$, the measure $T_\#\mu \in \mathcal{P}_2(\mathbb{R}^n)$ is defined by

$$T_\#\mu := \mu(T^{-1}(A)),\ \text{for any Borel set } A \subset \mathbb{R}^n.$$

We next recall some concept of optimal transport map, also called Brenier's map. Given a probability density $m \in L^2(\mathbb{R}^n)$ and a probability $\nu \in \mathcal{P}_2(\mathbb{R}^n)$, it is well-known that the Monge problem,

$$\inf_{\{T\,|\,T_\#(m\,\mathrm{d}x)=\nu\}} \int_{\mathbb{R}^n} |x - T(x)|^2 m(x)\mathrm{d}x,$$

has a unique optimal solution, denoted by T^ν_m, which is a gradient $\nabla\Phi$ of a convex function $\Phi:\mathbb{R}^n \to \mathbb{R}^n$, and is called the Brenier's map; noted that the result remains valid even when ν does not possess a density, see [3]. As a result, the following property holds

$$W_2^2(m,\nu) = \inf_{\phi \text{ is convex}} \int_{\mathbb{R}^n} |x - \nabla\phi(x)|^2 m(x)\mathrm{d}x = \int_{\mathbb{R}^n} |x - \nabla\Phi(x)|^2 m(x)\mathrm{d}x. \tag{A.4}$$

Later we shall see, based on (A.10), (A.4) motivates the definition of tangent space $\mathcal{T}(m)$ of the metric space $\mathcal{P}_2(\mathbb{R}^n)$ at the point m as

$$\mathcal{T}(m) := \overline{\{\nabla\Phi | \Phi \in C_c^\infty(\mathbb{R}^n)\}}. \tag{A.5}$$

We next consider curves in $\mathcal{P}_2(\mathbb{R}^n)$, defined by densities $m(t) \equiv m(t)(x) = m(x,t)$, so that the evolution of $m(t)$ is defined by a continuously differentiable and uniformly bounded velocity vector field $v(t) \equiv v(t)(x) = v(x,t)$ so that $m(x,t)$ is the solution of the continuity equation

$$\begin{cases} \frac{\partial m}{\partial t}(x,t) + \text{div } (v(x,t)m(x,t)) = 0, \\ m(x,0) = m(x). \end{cases} \tag{A.6}$$

In general, we can interpret this continuity equation in the sense of distributions, that is to look for the solution μ_t such that

$$\int_0^T \int_{\mathbb{R}^n} \frac{\partial \zeta}{\partial t} + v(x,t) \cdot \nabla \zeta \mathrm{d}\mu_t \mathrm{d}t = 0, \forall \zeta \in C_c^\infty(\mathbb{R}^n \times (0,T)), \tag{A.7}$$

and it is sufficient to only assume that $\int_0^T \int_{\mathbb{R}^n} |v(x,t)|^2 \mathrm{d}\mu_t \mathrm{d}t < +\infty$, $\forall T < +\infty$. This evolution model has a broad spectrum and turns out to be equivalent to the property that $m(t)$ is absolutely continuous in the sense that

$$W_2(m(s), m(t)) \leq \int_s^t \rho(\sigma) \mathrm{d}\sigma, \ \forall s < t, \tag{A.8}$$

with some locally L^2 function $\rho(\cdot)$. [2]Now, for a given absolutely continuous curve $m(t)$, the corresponding velocity field is not necessarily unique. Define

$$\mathcal{V} := \left\{ v \in L^2_{m_{(\cdot,\cdot)}}(\mathbb{R}^n \times (0,T)) \mid \text{(A.7) holds for all } \zeta \in C_c^\infty(\mathbb{R}^n \times (0,T)) \text{ and } m(x,0) = m(x) \right\},$$

which is a convex and closed subset in $L^2_{m_{(\cdot,\cdot)}}(\mathbb{R}^n \times (0,T))$. We define the velocity field with minimum norm, $\hat{v}(x,t)$, which is the solution of

$$\inf_v \left\{ \int_0^T \int_{\mathbb{R}^n} |v(x,t)|^2 m(x,t) \mathrm{d}x \mathrm{d}t \ : v \in \mathcal{V} \right\}. \tag{A.9}$$

[2]Indeed, for a.e. $0 \leq s < t \leq T$, the derivative of the Wasserstein distance is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} W_2^2(m(s), m(t)) &= \int_{\mathbb{R}^n} v\left(T_{m(s)}^{m(t)}(x), t\right) \cdot \left(x - T_{m(s)}^{m(t)}(x)\right) m(x,s) \mathrm{d}x \\ &\leq \left[\int_{\mathbb{R}^n} v^2 \left(T_{m(s)}^{m(t)}(x), t\right) m(x,s) \mathrm{d}x\right]^{\frac{1}{2}} \cdot \left[\int_{\mathbb{R}^n} \left|x - T_{m(s)}^{m(t)}(x)\right|^2 m(x,s) \mathrm{d}x\right]^{\frac{1}{2}}, \end{aligned}$$

therefore, by taking $\rho(t) := \frac{1}{2}\left[\int_{\mathbb{R}^n} v^2 \left(T_{m(s)}^{m(t)}(x), t\right) m(x,s) \mathrm{d}x\right]^{\frac{1}{2}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} W_2(m(s), m(t)) \leq \rho(t) \implies W_2(m(s), m(t)) \leq \int_s^t \rho(\sigma) d\sigma.$$

The Euler equation for this minimization problem is

$$\int_0^T \int_{\mathbb{R}^n} \hat{v}(x,t)\cdot v(x,t)m(x,t)\mathrm{d}x\mathrm{d}t = 0, \quad \forall v \text{ such that } \mathrm{div}(v(x,t)m(x,t)) = 0 \text{ a.e.}(x,t); \tag{A.10}$$

a simple application of divergence theorem implies immediately that $\hat{v}$ is a gradient of some function, and so $\hat{v}(\cdot,t) \in \mathcal{T}(m(t))$ a.e. t. Consequently, for every absolutely continuous curve $m(t)$, we can associate a unique velocity field $\hat{v}(\cdot,t)$ in the tangent space $\mathcal{T}(m(t))$ a.e. t. It is called the tangent vector field to the curve $m(t)$, and this tangent field can be expressed by the following formula

$$\hat{v}(x,t) = \lim_{\epsilon\to 0} \frac{T_{m(t)}^{m(t+\epsilon)}(x) - x}{\epsilon}, \tag{A.11}$$

where the limit is understood in $L^2_{m(t)}(\mathbb{R}^n;\mathbb{R}^n)$. As a Brenier's map, the function $T_{m(t)}^{m(t+\epsilon)}(x)$ is uniquely defined for each $\epsilon > 0$. By (A.4), $||T_{m(t)}^{m(t+\epsilon)}(x) - x||_{L^2_{m(t)}} = W_2(m(t), m(t+\epsilon))$; by (A.8), we also see that, for any absolutely continuous curve

$$W_2(m(t), m(t+\epsilon)) \le C(t)\epsilon. \tag{A.12}$$

In the definition of the functional derivative in (A.3), we can write

$$\frac{F(m_\epsilon) - F(m) - \displaystyle\int_{\mathbb{R}^n} \frac{\partial F(m)}{\partial m}(x)(m_\epsilon(x) - m(x))\mathrm{d}x}{\epsilon} = o(1)\frac{W_2(m_\epsilon, m)}{\epsilon} = C(0)o(1) \to 0 \text{ as } \epsilon \to 0, \tag{A.13}$$

provided that the sequence m_ϵ is absolutely continuous.

Consider the curve corrresponding to a gradient $\nabla\Phi(x)$ for some regular enough $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that (A.7) is solvable in the sense of distributions, *i.e.* the curve $m(t)$ is defined by

$$\begin{cases} \frac{\partial m}{\partial t}(x,t) + \mathrm{div}\ (\nabla\Phi(x)m(x,t)) = 0, \\ m(x,0) = m(x). \end{cases} \tag{A.14}$$

Since it is a gradient, according to the discussion after (A.9), $\nabla\Phi(x)$ has the minimal norm, and we can claim from (A.11) that

$$\nabla\Phi(x) = \lim_{\epsilon\to 0} \frac{T_m^{m(\epsilon)}(x) - x}{\epsilon} \text{ in } L^2_m(\mathbb{R}^n;\mathbb{R}^n). \tag{A.15}$$

We now consider a functional $F(m)$ on $\mathcal{P}_2(\mathbb{R}^n)$, and limit ourselves to densities. We say that $F(m)$ is weakly Wasserstein-differentiable at m, in the Wasserstein space, if there exists a function $\Gamma(\cdot,m)$ belonging to the tangent space $\mathcal{T}(m)$ with the property that

$$\frac{F(m(\epsilon)) - F(m) - \displaystyle\int_{\mathbb{R}^n} \Gamma(x,m)\cdot(T_m^{m(\epsilon)}(x) - x)m(x)\mathrm{d}x}{W_2(m, m(\epsilon))} \to 0 \text{ as } \epsilon \to 0, \tag{A.16}$$

and the function $\Gamma(x,m)$ is called the *Wasserstein gradient* and denoted $\nabla_m F(m)(x)$. If we apply the property (A.16), *via* the absolute continuity (A.12) and (A.15), to the map $m(t)$ defined by (A.14), this is equivalent to

$$\frac{F(m(\epsilon))-F(m)}{\epsilon} \to \int_{\mathbb{R}^n} \Gamma(x,m)\cdot\nabla\Phi(x)m(x)\mathrm{d}x.$$

From the continuity equation (A.14), using the regularity of Φ, we can state that

$$\frac{m(x,\epsilon)-m(x)}{\epsilon} \to -\mathrm{div}\,(\nabla\Phi(x)m(x)) \text{ as } \epsilon\to 0, \text{ in the sense of distributions.}$$

On the other hand, if $F(m)$ has a functional derivative, we then have

$$\frac{F(m(\epsilon))-F(m)}{\epsilon} \to -\int_{\mathbb{R}^n} \frac{\partial F(m)}{\partial m}(x)\mathrm{div}\,(\nabla\Phi(x)m(x))\mathrm{d}x.$$

Therefore, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^n} \Gamma(x,m)\cdot\nabla\Phi(x)m(x)\mathrm{d}x &= -\int_{\mathbb{R}^n} \frac{\partial F(m)}{\partial m}(x)\mathrm{div}\,(\nabla\Phi(x)m(x))\mathrm{d}x \\ &= \int_{\mathbb{R}^n} D_x\frac{\partial F(m)}{\partial m}(x)\cdot\nabla\Phi(x)m(x)\mathrm{d}x.\end{aligned}$$

If we assume that $D_x\frac{\partial F(m)}{\partial m}(x)\in L^2_m(\mathbb{R}^n;\mathbb{R}^n)$, we can replace $\nabla\Phi(x)$ by any element of $\mathcal{T}(m)$. Since $\Gamma(x,m)$ and $D_x\frac{\partial F(m)}{\partial m}(x)$ (as a gradient) belong to $\mathcal{T}(m)$, it follows that

$$\nabla_m F(m)(x) = D_x\frac{\partial F(m)}{\partial m}(x), \tag{A.17}$$

hence the Wasserstein gradient is simply the Euclidean gradient of the functional derivative $\frac{\partial F}{\partial m}(m)(x)$.

Remark A.1. The concept of functional derivative defined in (A.3) uses a sequence of probability densities $m_\epsilon \to m$, so it is not equivalent to the concept of Gâteaux derivative in $L^2(\mathbb{R}^n)$, which requires to remove the assumptions of positivity of m and its unity, *i.e.* $\int_{\mathbb{R}^n} m(x)\mathrm{d}x = 1$. We have developed the differences in examples in which explicit formulas are available in Section 8.

We next assume that $F(m)$ can be extended as a functional on $L^2(\mathbb{R}^n)$, and we assume that:

$$F(m) \text{ has a Gâteaux derivative, } \frac{\partial F(m)}{\partial m}(x), \text{ on } L^2(\mathbb{R}^n), \tag{A.18}$$

here the function $x\mapsto \dfrac{\partial F(m)}{\partial m}(x)$ belongs to $L^2(\mathbb{R}^n)$. We also assume that, for the argument m in (4.2), the finiteness of

$$\int_{\mathbb{R}^n} \left|D_x\frac{\partial F(m)}{\partial m}(x)\right|^2 m(x)\mathrm{d}x < \infty. \tag{A.19}$$

Next, we assume that there exists a dense subspace of $\mathcal{H}$, of random variables Y, so that

$$(X,Y) \text{ has a joint probability density } \pi(x,y) \text{ such that } \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\pi^2(x,y)}{\varpi(y)}\,\mathrm{d}x\,\mathrm{d}y < \infty, \tag{A.20}$$

where $\varpi(y)=\int_{\mathbb{R}^n}\pi(x,y)\mathrm{d}x$ is the marginal probability density of Y. This is clearly satisfied whenever X and Y are independent of each other. We then have the following simple but crucial formula:

Lemma A.2. *Under assumptions (4.2) and (A.18) to (A.20), the functional $\mathcal{F}(X)$ has a gradient in $\mathcal{H}$ at X, given by*

$$D_X\mathcal{F}(X)=D_x\frac{\partial F(m)}{\partial m}(X). \tag{A.21}$$

Proof. We now take a Y satisfying (A.20), then necessarily $m(x)=\int_{\mathbb{R}^n}\pi(x,y)\mathrm{d}y$. We then consider the parametrized random variable $X+tY$. Its probability density is given by

$$m(x,t)=\int_{\mathbb{R}^n}\pi(x-ty,y)\mathrm{d}y,$$

while it satisfies the continuity equation by simple calculus,

$$\frac{\partial m}{\partial t}=-\mathrm{div}_x\left(\int_{\mathbb{R}^n}\pi(x-ty,y)\,y\mathrm{d}y\right).$$

Moreover, from (A.20), $x\mapsto m(x,t)$ belongs to $L^2(\mathbb{R}^n)$; this function is denoted by $m(t)$. Therefore, if we identify $\mathcal{F}(X+tY)=F(m(t))$, we next have

$$\lim_{t\to 0}\frac{\mathcal{F}(X+tY)-\mathcal{F}(X)}{t}=((D_X\mathcal{F}(X),Y)),$$

and using divergence theorem,

$$\begin{aligned}\lim_{t\to}\frac{F(m(t))-F(m)}{t}&=-\int_{\mathbb{R}^n}\frac{\partial F(m(\cdot,0))}{\partial m}(x)\mathrm{div}\,\left(\int_{\mathbb{R}^n}\pi(x-0\cdot y,y)y\mathrm{d}y\right)\mathrm{d}x\\&=\int_{\mathbb{R}^n}D_x\frac{\partial F(m)}{\partial m}(x)\cdot\left(\int_{\mathbb{R}^n}\pi(x,y)y\mathrm{d}y\right)\mathrm{d}x\\&=((D_x\frac{\partial F(m)}{\partial m}(X),Y)).\end{aligned}$$

Since Y is arbitrary, using (A.17), thus it is necessarily that

$$D_x\frac{\partial F(m)}{\partial m}(X)=\nabla_m F(m)(X)=D_X\mathcal{F}(X).$$

□

Remark A.3. All the assumptions in Lemma A.2 are made to justify the steps of the proof. In practice formula (A.21) should be considered more as a **rule of correspondence** between concepts which can be defined independently; indeed, $D_X\mathcal{F}(X)$ and $D_x\dfrac{\partial F(m)}{\partial m}(x)$ are defined independently. We shall define control problems, which can be studied independently, that can interpreted as those respectively studied in Sections 2 and 3, by using the rule of correspondence.

A.2 Examples from [13]

We have already discussed the quadratic examples in Section 8. We here give the example considered in [13].

$$f(x,m)=\frac{1}{2}\int_{\mathbb{R}^n}K(x,\xi)m(\xi)\mathrm{d}\xi, \tag{A.22}$$

such that $K(x,\xi)=K(\xi,x)$ and

$$|K(x_1,\xi_1)-K(x_2,\xi_2)|\le C(1+|x_1|+|x_2|+|\xi_1|+|\xi_2|)(|x_1-x_2|+|\xi_1-\xi_2|); \tag{A.23}$$

$$\begin{cases}|D_xK(x_1,\xi_1)-D_xK(x_2,\xi_2)|\le \frac{c}{2}(|x_1-x_2|+|\xi_1-\xi_2|);\\ |D_xK(0,0)|\le \frac{c}{2};\end{cases} \tag{A.24}$$

note that our present examples include the convolution (1.3) in [13] as an interesting example. Our theory developed in this article should correspond to equation (1.4) in [13]. On the other hand, we now have

$$\frac{\partial f}{\partial m}(\xi,m)(x)=\frac{1}{2}K(\xi,x)=\frac{1}{2}K(x,\xi),$$

and hence $\int_{\mathbb{R}^n}\frac{\partial f}{\partial m}(\xi,m)(x)m(\xi)\mathrm{d}\xi=f(x,m)$ which implies that

$$F(x,m)=2f(x,m)=\int_{\mathbb{R}^n}K(x,\xi)m(\xi)\mathrm{d}\xi. \tag{A.25}$$

We thus have

$$\begin{aligned}|D_xF(x,m)||&\le\int_{\mathbb{R}^n}|D_xK(x,\xi)|m(\xi)\mathrm{d}\xi\\ &\le\frac{c}{2}\left(1+|x|+\int|\xi|m(\xi)\mathrm{d}\xi\right)\\ &\le\frac{c}{2}\left(1+|x|+\left(\int|\xi|^2m(\xi)\mathrm{d}\xi\right)^{\frac{1}{2}}\right).\end{aligned}$$

If we take two densities m_1 and m_2, we may consider two random variables Ξ_1 and Ξ_2 with the probabilities m_1 and m_2 respectively. Therefore, we have

$$\begin{aligned}|D_xF(x_1,m_1)-D_xF(x_2,m_2)|&\le\left|\int D_x(K(x_1,\xi)-K(x_2,\xi))m_1(\xi)\mathrm{d}\xi\right|+|\mathbb{E}\left[D_x(K(x_2,\Xi_1)-K(x_2,\Xi_2))\right]|\\ &\le\frac{c}{2}|x_1-x_2|+\frac{c}{2}\sqrt{\mathbb{E}|\Xi_1-\Xi_2|^2},\end{aligned}$$

and since Ξ_1 and Ξ_2 are arbitrary, with the respective marginals m_1 and m_2, we can choose a couple so that the L^2 is the minimal one, which implies the property:

$$|D_xF(x_1,m_1)-D_xF(x_2,m_2)|\le\frac{c}{2}\left(|x_1-x_2|+W_2(m_1,m_2)\right).$$

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