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STABILITY RESULTS OF SOME FIRST ORDER VISCOUS HYPERBOLIC SYSTEMS

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Abstract. In this paper, we first introduce an abstract viscous hyperbolic problem for which we prove exponential decay under appropriated assumptions. We then give some illustrative examples, like the linearized viscous Saint-Venant system. In order to achieve the optimal decay rate, we also perform a detailed spectral analysis of our abstract problem under a natural assumption satisfied by various examples. We finally consider the boundary stabilizability of the linearized viscous Saint-Venant system on trees.

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1. Introduction

Stability of hyperbolic systems becomes a very important area of research due to various applications in fluid dynamics, electromagnetism, wave propagation, traffic flow, etc., see [3, 6, 14, 16, 29]. Obviously the stability property of a given system is closely related to the chosen dissipation law. In this paper, we concentrate on viscous dissipation laws that appear for instance for the linearized compressible Navier-Stokes system [5, 13, 41] or the viscous two-phase model [23]. In order to avoid repetitive proofs for each model, we perform an unified analysis by designing an abstract setting that includes these models and even allows to treat new ones like the linearized viscous Saint-Venant system on trees and the Maxwell system with a viscous damping.

More precisely, we first study an abstract evolution equation in the form

$$\begin{cases} u_t + Au + B^*\rho = 0, \\ \rho_t - Bu = 0, \\ u(0) = u_0, \rho(0) = \rho_0, \end{cases} \tag{1.1}$$

in an appropriate Hilbert setting with natural assumptions on A and B (see Sect. 2 for the details). First using semi-group theory we show that such a system is well-posed. Second using a frequency domain approach [27, 40], we show that this system is exponentially stable (in an appropriated subspace of the starting one).

Then we illustrate our theoretical results by four examples. In each case, we give the exact Hilbert setting and check the different assumptions. The first example is the linearized compressible Navier-Stokes systems on

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a bounded domain of $\mathbb{R}^d$ for which an exponential stability was proved in [41] by showing that the associated semi-group is analytic; here we give an alternative proof that is much simpler. The second example concerns the linearized viscous Saint-Venant (or compressible Navier-Stokes) system on trees, such a problem is an extension of the same problem set in an interval [5, 13], where the authors showed the exponential decay by performing a precise spectral analysis. There is a quite large literature on the Saint-Venant system on networks, see for instance [7, 19, 31, 37], but, as far as we know, the authors concentrate on nonlinear problems with boundary dissipation or on linear models with zero order dissipative terms [6, 7], but without viscosity. The third application is the study of a quite general one-dimensional hyperbolic system with a viscous damping that covers the case of the linearized viscous Saint-Venant system on an interval as well as the linearized viscous two-phase model [23]; to our best knowledge, no decay results are available for this last problem. The final example concerns the Maxwell system in a bounded domain of $\mathbb{R}^3$ with a viscosity term. Such a dissipation law is not considered in the literature, since usually either Ohm's law (an internal damping of the form σE) or a dissipative boundary condition (Silver-Müller boundary) are used, see [28, 38, 39].

The drawback of the frequency domain approach is that it does not furnish the exact decay of the energy. One possibility to overcome this difficulty and get the exact decay is to perform a detailed spectral analysis, namely by obtaining the eigenvalues and the (generalized) eigenvectors of the associated operator and by showing that these (generalized) eigenvectors form a Riesz basis of the energy space, see [5, 15]. In such a situation, the decay rate is equal to the spectral abscissa. For our system (1.1), with the assumption that B^*B is equal to A up to a positive multiplicative factor, we perform the full spectral analysis of the associated operator and therefore conclude that the decay rate of system (1.1) is equal to the spectral abscissa. The last three above examples enter in this framework and therefore a precise decay rate can be obtained for them.

Finally we come back to the linearized viscous Saint-Venant system on a tree, and inspired from [5], we are interested in the advection dominating case, for which the decay rate of the energy norm of the solution is slow due to the advection modes. Similarly to [5], our goal is to restore the optimal decay by building a Dirichlet control at all except one extremities of the tree. This method is based on an extension method and allows to obtain an exponential decay rate with, contrary to [5], an arbitrary initial datum in the energy space.

Our main contributions can be summarized as follows: In Theorem 2.7, we show the exponential decay of the abstract problem (1.1). We illustrate this result by various examples, in particular new ones, like the linearized viscous Saint-Venant system on trees, the linearized viscous two-phase model and the Maxwell system with a viscous damping, are considered. Under a realistic assumption between A and B, we perform a detailed spectral analysis that allows to show that the (generalized) eigenvectors of the (non-selfadjoint) operator associated with problem (1.1) form a Riesz basis and consequently that the decay rate coincides with the spectral abscissa (see Thm. 4.4). We extend the result from [5] to the linearized viscous Saint-Venant system on a tree in the advection dominating case.

The paper is organized as follows: In Section 2, we introduce the abstract setting, some notations and the general problem studied later on. Its well-posedness is proved and its exponential decay is obtained. Section 3 is devoted to the analysis of some illustrative examples. The spectral analysis of our abstract problem is performed in Section 4. We end up the paper with the boundary stabilizability of the linearized viscous Saint-Venant system on trees in Section 5.

Let us finish this section with some notations used in the remainder of the paper. For a bounded domain D, the usual norm and semi-norm of $H^s(D)$ $(s \geq 0)$ are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. For $s = 0$, we will drop the index s. Furthermore, the notation $A \lesssim B$ (resp. $A \gtrsim B$) means the existence of a positive constant C_1 (resp. C_2), which is independent of A and B such that $A \leq C_1 B$ (resp. $A \geq C_2 B$). The notation $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$ hold simultaneously.

2. An abstract framework

Let H_1 (resp. H_2, V) be a complex Hilbert space with norm and inner product denoted respectively by $\|\cdot\|_1$ and $(\cdot,\cdot)_1$ (resp. $\|\cdot\|_2$ and $(\cdot,\cdot)_2$, $\|\cdot\|_V$ and $(\cdot,\cdot)_V$) such that V is a dense subspace of H_1 and is compactly embedded into H_1. Denote by V' the dual space of V with respect to the pivot space H_1.

Let us now fix a sesquilinear (linear in the first variable and conjugate linear in the second one) and symmetric form a from $V \times V$ to $\mathbb{C}$ that is supposed to be strictly coercive, in the sense that

$$a(u,u) \gtrsim \|u\|_V^2, \forall u \in V. \tag{2.1}$$

Then we denote by A the operator from V into V' by

$$\langle Au, v\rangle_{V'-V} = a(u,v), \forall u, v \in V.$$

Let us further fix a sesquilinear and continuous form b from $V \times H_2$ into $\mathbb{C}$. Denote by B the linear and continuous operator from V into H_2 defined by

$$(Bu, \rho)_2 = b(u, \rho), \forall \rho \in H_2,$$

and let B^* be its adjoint operator (that is continuous from H_2 into V').

We finally assume that the next inf-sup condition is valid

$$\inf_{\rho \in H_3} \sup_{u \in V} \frac{|b(u,\rho)|}{\|u\|_V \|\rho\|_2} \gtrsim 1, \tag{2.2}$$

where

$$H_3 = (\ker B^*)^\perp = \{\rho \in H_2 : (\rho, \eta)_2 = 0, \forall \eta \in \ker B^*\}.$$

In this setting, we consider the following evolution system: find u and ρ solutions of

$$\begin{cases} u_t + Au + B^*\rho = 0 & \text{in } V', \\ \rho_t - Bu = 0 & \text{in } H_2, \\ u(0) = u_0, \rho(0) = \rho_0, \end{cases} \tag{2.3}$$

where u_t represents the time derivative, while u_0 and ρ_0 are the initial data.

The existence of a solution to (2.3) in an appropriated Hilbert setting is obtained using semigroup theory. Indeed let us introduce the Hilbert space $\mathcal{H} = H_1 \times H_2$ with its natural inner product and introduce the (unbounded) operator $\mathcal{A}$ from $\mathcal{H}$ into itself by

$$D(\mathcal{A}) = \{(u,\rho) \in V \times H_2 : Au + B^*\rho \in H_1\},$$
$$\mathcal{A}(u,\rho) = (-(Au + B^*\rho), Bu), \quad \forall (u,\rho) \in D(\mathcal{A}).$$

Theorem 2.1. *Under the above assumptions, the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $(T(t))_{t\geq 0}$ on $\mathcal{H}$. Therefore for all $U_0 \in \mathcal{H}$, the problem (2.3) has a weak solution $U \in C([0,\infty), \mathcal{H})$ given by $U = T(t)U_0$. If moreover $U_0 \in D(\mathcal{A})$, the problem (2.3) has a strong solution $U \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), \mathcal{H})$.*

Proof. It suffices to prove that $\mathcal{A}$ is a maximal dissipative operator, hence by Lumer-Phillips' theorem it generates a C_0-semigroup of contractions on $\mathcal{H}$.

Let us start with the dissipativity. For $U = (u,\rho) \in D(\mathcal{A})$, we have

$$(\mathcal{A}U, U)_{\mathcal{H}} = -(Au + B^*\rho, u)_1 + (Bu, \rho)_2 = -a(u,u) + (Bu,\rho)_2 - (\rho, Bu)_2.$$

Hence

$$\Re(\mathcal{A}U, U)_{\mathcal{H}} = -a(u,u) \leq 0. \tag{2.4}$$

Let us go on with the maximality. Let $\lambda > 0$ be fixed. Given $F = (f, g) \in \mathcal{H}$, we look for $U = (u, \rho) \in D(\mathcal{A})$ such that $(\lambda - \mathcal{A})U = F$, or equivalently

$$\begin{cases} \lambda u + Au + B^*\rho = f \text{ in } H_1, \\ \lambda\rho - Bu = g \text{ in } H_2. \end{cases} \tag{2.5}$$

Assume for the moment that such a U exists. Then the second identity is equivalent to

$$\rho = \frac{1}{\lambda}(g + Bu). \tag{2.6}$$

This expression in the first identity implies that

$$\lambda u + Au + \frac{1}{\lambda}B^*Bu = f - \frac{1}{\lambda}B^*g \text{ in } V'. \tag{2.7}$$

Taking the duality pairing with $v \in V$, we find that $u \in V$ satisfies

$$\lambda(u, v)_1 + a(u, v) + \frac{1}{\lambda}(Bu, Bv)_2 = (f, v)_1 - \frac{1}{\lambda}(g, Bv)_2, \forall v \in V.$$

Now this problem has a unique solution $u \in V$, by Lax-Milgram Lemma because the left-hand side is a continuous and coercive sesquilinear form on V, since

$$\lambda(u, u)_1 + a(u, u) + \frac{1}{\lambda}(Bu, Bu)_2 \geq a(u, u) \gtrsim \|u\|_V^2, \forall u \in V,$$

and because the right-hand side is a continuous form on V. But by the definition of A, we deduce that u is a solution of (2.7). Now defining ρ by (2.6), then the identity (2.7) means that

$$\lambda u + Au + \ B^*\rho = f \text{ in } V',$$

that leads to

$$Au + \ B^*\rho = f - \lambda u,$$

that clearly belongs to H_1. □

As usual the energy associated with (2.3) is defined by

$$E(t) = \frac{1}{2}(\|u\|_1^2 + \|\rho\|_2^2), \tag{2.8}$$

that is equal to 1/2 of the norm of (u, ρ) in $\mathcal{H}$.

Proposition 2.2. *The solution (u, ρ) of (2.3) with initial datum in $D(\mathcal{A})$ satisfies*

$$E'(t) = -a(u(t), u(t)) \leq 0,$$

therefore the energy is nonincreasing.

Proof. If $U_0 \in D(\mathcal{A})$, we can derive the energy (2.8) because $U \in C^1([0,\infty),\mathcal{H})$ and obtain

$$E'(t) = \Re(U, U_t)_{\mathcal{H}}.$$

Using problem (2.3), we get

$$E'(t) = \Re(\mathcal{A}U, U)_{\mathcal{H}}.$$

We conclude thanks to (2.4). □

Note that system (2.3) is not strongly stable in the whole space since the kernel of $\mathcal{A}$ is not necessarily reduced to $\{0\}$ as the next lemma shows.

Lemma 2.3. *It holds*

$$\ker \mathcal{A} = \{0\} \times \ker B^*.$$

Proof. Let $(u,\rho) \in \ker \mathcal{A}$, then by (2.4) we deduce that $a(u,u) = 0$. Hence by (2.1), we deduce that $u = 0$. From the definition of $\mathcal{A}$, $\mathcal{A}(0,\rho) = 0$ reduces to $B^*\rho = 0$, and the conclusion follows. □

Accordingly we denote by $\mathcal{A}_0$ the restriction of $\mathcal{A}$ to $\mathcal{H}_0 := H_1 \times H_3$, namely

$$D(\mathcal{A}_0) = D(\mathcal{A}) \cap \mathcal{H}_0, \tag{2.9}$$

and

$$\mathcal{A}_0(u,\rho) = \mathcal{A}(u,\rho), \forall (u,\rho) \in D(\mathcal{A}_0). \tag{2.10}$$

Note that $\mathcal{A}_0$ is well-defined from $D(\mathcal{A}_0)$ into $\mathcal{H}_0$ since the inf-sup condition (2.2) implies that $R(B) = (\ker B^*)^\perp = H_3$ (see Lem. I.4.1 of [25]).

At this stage, we want to prove the uniform stability of system (2.3) in $\mathcal{H}_0$. Our proof is based on a frequency domain approach, namely the exponential decay of the energy is deduced from the following result (see [40] or [27]):

Lemma 2.4. *A C_0 semigroup $(e^{t\mathcal{L}})_{t\geq 0}$ of contractions on a Hilbert space H is exponentially stable,* i.e., *satisfies*

$$||e^{t\mathcal{L}}U_0|| \leq C\, e^{-\omega t}||U_0||_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,$$

for some positive constants C and ω if and only if

$$\rho(\mathcal{L}) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{2.11}$$

and

$$\sup_{\beta\in\mathbb{R}} \|(i\beta - \mathcal{L})^{-1}\| < \infty, \tag{2.12}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

Let us check that $\mathcal{A}_0$ satisfies the first assumption of Lemma 2.4.

Lemma 2.5. *Under the previous assumptions, we have $i\mathbb{R} \subset \rho(\mathcal{A}_0)$.*

Proof. For any $z \in \mathbb{R}$ and an arbitrary $F = (f, g) \in \mathcal{H}_0$, we look for $U = (u, \rho) \in \mathcal{D}(\mathcal{A})$ solution of

$$izU - \mathcal{A}U = F, \tag{2.13}$$

or equivalently

$$\left\{ \begin{array}{l} izu + Au + B^*\rho = f, \\ iz\rho - Bu = g. \end{array} \right. \tag{2.14}$$

Now we distinguish the case $z = 0$ to the other cases.

First if $z \neq 0$, then in (2.14) we can eliminate η by $\frac{1}{iz}(Bu + g)$ and the first equation becomes (compare with the proof of Lem. 2.1)

$$izu + Au + \frac{1}{iz}B^*Bu = f - \frac{1}{iz}B^*g \text{ in } V'. \tag{2.15}$$

Taking the duality pairing with $v \in V$, we find that $u \in V$ satisfies

$$iz(u, v)_1 + a(u, v) + \frac{1}{iz}(Bu, Bv)_2 = (f, v)_1 - \frac{1}{iz}(g, Bv)_2, \forall v \in V.$$

Now this problem has a unique solution $u \in V$, by Lax-Milgram Lemma because the left-hand side is a continuous and coercive sesquilinear form on V, since

$$\Re\Big(iz(u, u)_1 + a(u, u) + \frac{1}{iz}(Bu, Bu)_2\Big) = a(u, u) \gtrsim \|u\|_V^2, \forall u \in V,$$

and because the right-hand side is a continuous form on V. As before we deduce that u is a solution of (2.15) and defining $\rho = \frac{1}{iz}(Bu + g)$, we deduce that (u, ρ) belongs to $D(\mathcal{A})$ and satisfies (2.13). Finally ρ is indeed in H_3 because Bu and g both belong to H_3.

In the case $z = 0$, problem (2.14) reduces to

$$\left\{ \begin{array}{l} Au + B^*\eta = f, \\ Bu = -g, \end{array} \right. \tag{2.16}$$

that is a standard saddle point problem (see [25], §I.4). By the coercivity assumption on a and the inf-sup condition (2.2), we deduce that this problem has a unique solution $(u, \rho) \in V \times H_3$ (see [25], Thm I.4, p. 59). This pair is clearly in $D(\mathcal{A}_0)$ since $Au + B^*\eta = f \in H_1$. □

Now we need to analyze the behaviour of the resolvent of $\mathcal{A}_0$ on the imaginary axis.

Lemma 2.6. *The resolvent of the operator of $\mathcal{A}_0$ satisfies condition* (2.12).

Proof. We use a contradiction argument, *i.e.*, we suppose that (2.12) is false. Then there exist a sequence of real numbers $\beta_n \to +\infty$ and a sequence of vectors $Z_n = (u_n, \rho_n)$ in $\mathcal{D}(\mathcal{A}_0)$ with $\|Z_n\|_{\mathcal{H}} = 1$ such that

$$\|(i\beta_n - \mathcal{A})Z_n\|_{\mathcal{H}} \to 0 \text{ as } n \to \infty. \tag{2.17}$$

By the definition of $\mathcal{A}$, this directly implies that

$$\|i\beta_n u_n + Au_n + B^*\rho_n\|_1 \to 0, \tag{2.18}$$

$$\|i\beta_n\rho_n - Bu_n\|_2 \to 0. \tag{2.19}$$

We first notice that by the dissipativeness of $\mathcal{A}$, we have

$$a(u_n, u_n) = \Re((i\beta_n - \mathcal{A})Z_n, Z_n)_{\mathcal{H}} \leq \|(i\beta_n - \mathcal{A})Z_n\|_{\mathcal{H}} \to 0,$$

that leads to

$$u_n \to 0 \text{ in } V, \tag{2.20}$$

hence in particular in H_1. But the property (2.19) can be rewritten as

$$i\beta_n \rho_n = Bu_n + g_n,$$

with

$$g_n \to 0 \text{ in } H_2.$$

As B is continuous from V into H_2, the property (2.20) allows to conclude that

$$i\beta_n \rho_n \to 0 \text{ in } H_2.$$

This proves that $(u_n, \rho_n) \to 0$ in $\mathcal{H}$ and yields a contradiction. □

These two lemmas directly imply the following energy decay.

Theorem 2.7. *There exists two positive constants C and ω such that for all $U_0 \in \mathcal{H}$,*

$$\|e^{t\mathcal{A}}U_0 - PU_0\|_{\mathcal{H}} \leq Ce^{-\omega t}\|U_0\|_{\mathcal{H}}, \forall t \geq 0,$$

where $P(u_0, \rho_0) = (0, Q\rho_0)$, Q being the orthogonal projection in H_2 on $\ker B^*$.

Proof. Lemmas 2.5 and 2.6 show that $\mathcal{A}_0$ satisfies the necessary and sufficient conditions from Lemma 2.4, therefore the semigroup generated by $\mathcal{A}_0$ is exponentially decaying.

But for $U_0 \in \mathcal{H}$, $U_0 - PU_0$ belongs to $\mathcal{H}_0$ and therefore

$$\|e^{t\mathcal{A}_0}(U_0 - PU_0)\|_{\mathcal{H}} \leq Ce^{-\omega t}\|U_0 - PU_0\|_{\mathcal{H}}, \forall t \geq 0.$$

The conclusion follows from the fact that

$$e^{t\mathcal{A}_0}(U_0 - PU_0) = e^{t\mathcal{A}}(U_0 - PU_0) = e^{t\mathcal{A}}U_0 - PU_0.$$

and the trivial estimate $\|U_0 - PU_0\|_{\mathcal{H}} \leq \|U_0\|_{\mathcal{H}}$. □

3. Some examples

In this section, we give some concrete examples that enter in our abstract framework.

3.1. The linearized compressible Navier-Stokes system on a bounded domain of $\mathbb{R}^d$

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \in \mathbb{N}^*$, with a Lipschitz boundary. On this domain, we examine the problem

$$\begin{cases} u_t - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + a\nabla\rho = 0 & \text{in } \Omega\times(0,\infty), \\ \rho_t + b \operatorname{div} u = 0 & \text{in } \Omega\times(0,\infty), \\ u = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(\cdot,0) = u_0, \rho(\cdot,0) = \rho_0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $u(x,t)$ (resp. $\rho(x,t)$) represents the velocity (resp. density) of the fluid at the point x and time t. As usual μ and λ are viscosity coefficients satisfying

$$\mu > 0, \lambda + \mu \geq 0,$$

while a and b are positive constants. This problem corresponds to the linearization of the compressible Navier-Stokes equation around a constant steady state $(0,\rho_0)$ with $\rho_0 > 0$, see [13, 26, 34, 41].

This system enters into our abstract framework in the following way. We take $H_1 = L^2(\Omega)^d$, $H_2 = L^2(\Omega)$ and $V = H_0^1(\Omega)^d$ and choose the sesquilinear forms [1]

$$a(u,v) = \int_\Omega \Big(\mu \sum_{i,j=1}^d \partial_i u_j \partial_i \bar{v}_j + (\lambda+\mu)\operatorname{div} u \operatorname{div} \bar{v}\Big)\,\mathrm{d}x, \quad \forall u,v \in V,$$

$$b(u,\rho) = -a\int_\Omega \operatorname{div} u\bar{\rho}\,\mathrm{d}x, \quad \forall u \in V, \rho \in L^2(\Omega).$$

The coerciveness of a is direct as

$$a(u,u) = \int_\Omega \Big(\mu \sum_{i,j=1}^d |\partial_i u_j|^2 + (\lambda+\mu)|\operatorname{div} u|^2\Big)\,\mathrm{d}x \geq \sum_{j=1}^d |u_j|_{1,\Omega}^2, \quad \forall u \in H_0^1(\Omega)^d.$$

On the other hand, it is easy to see that

$$B^*\rho = a\nabla\rho, \forall \rho \in L^2(\Omega),$$

and therefore

$$H_3 = L_0^2(\Omega) = \{\rho \in L^2(\Omega) : \int_\Omega \rho\,\mathrm{d}x = 0\}.$$

Then the inf-sup condition (2.2) is the standard inf-sup condition for the Stokes system (direct consequence of Cor. I.2.4 of [25], see [25], p. 81).

Finally, if H_1 is equipped with its natural inner product and H_2 with the inner product

$$(\rho_1,\rho_2)_2 = \frac{a}{b}\int_\Omega \rho_1\bar{\rho}_2\,\mathrm{d}x, \forall \rho_1,\rho_2 \in L^2(\Omega),$$

[1] here and below no confusion is possible between the constant a and the sesquilinear form $a(\cdot,\cdot)$

then it is easy to check that

$$Bu = -b \operatorname{div} u, \forall u \in V,$$

and that

$$D(\mathcal{A}) = \{(u,\rho) \in H_0^1(\Omega)^d \times L^2(\Omega) : -\mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + a\nabla\rho \in L^2(\Omega)^d\},$$

with

$$\mathcal{A}(u,\rho) = (\mu\Delta u + (\lambda+\mu)\nabla \operatorname{div} u - a\nabla\rho, -b \operatorname{div} u), \forall (u,\rho) \in D(\mathcal{A}).$$

According to Theorem 2.1, problem (3.1) is well-defined in a weak sense for initial data in $L^2(\Omega)^d \times L^2(\Omega)$ and in a strong sense for initial data in $D(\mathcal{A})$. Additionally, by Theorem 2.7, the solution (u,ρ) of problem (3.1) tends exponentially to the stationary solution $(0, c_0)$, where $c_0 = \frac{1}{|\Omega|}\int_\Omega \rho_0 \, \mathrm{d}x$, in other words, there exist two positive constants C and ω such that the solution (u,ρ) of (3.1) satisfies

$$\|u(\cdot,t)\|_\Omega + \|\rho(\cdot,t) - c_0\|_\Omega \leq Ce^{-\omega t}, \forall t \geq 0.$$

This decay is not new and is proved in Theorem 1.1 of [41] (see also [5], Cor. 1 or [13], Sect. 2.2 in the case $d = 1$) but our proof is much simpler. In dimension 1, the decay rate ω is explicit in Corollary 1 of [5], using the results from Section 4, we indeed recover this decay rate, this will be done below.

3.2. The linearized viscous Saint-Venant system on trees

We first recall the notion of C^2-networks, which is simply those of [9], we refer to [1, 2, 8, 10, 11, 32, 35] for more details.

All graphs considered here are non empty, finite and simple. Let G be a connected topological graph imbedded in $\mathbb{R}^m$, $m \in \mathbb{N}^*$, with n vertices $V = \{v_i : 1 \leq i \leq n\}$ and N edges $E = \{e_j : 1 \leq j \leq N\}$. Each edge e_j is a Jordan curve in $\mathbb{R}^m$ and is assumed to be parametrized by its arc length parameter x_j, such that the parametrization

$$\pi_j : [0, l_j] \to e_j : x_j \mapsto \pi_j(x_j)$$

is twice differentiable, *i.e.*, $\pi_j \in C^2([0,l_j], \mathbb{R}^m)$ for all $1 \leq j \leq N$.

We now define the C^2-network Γ associated with G as the union $\Gamma = E \cup V$.

The valency of each vertex v is denoted by $\gamma(v)$. For shortness, we later on denote by $V_{\text{ext}} = \{v \in V : \gamma(v) = 1\}$ the set of boundary (or exterior) vertices and $V_{\text{int}} = V \setminus V_{\text{ext}}$, corresponding to the set of interior vertices. For each vertex v, we also denote by $J_v = \{j \in \{1,\dots,N\} : v \in e_j\}$ the set of edges adjacent to v and let N_v be the cardinal of J_v. Note that if $v \in I_{ext}$ then N_v is a singleton that we write $\{j_v\}$. For each vertex v and $j \in N_v$, the normal vector in e_j at v is denoted by

$$\nu_j(v) = \begin{cases} 1 & \text{if} \quad \pi_j(l_j) = v, \\ -1 & \text{if} \quad \pi_j(0) = v. \end{cases}$$

For a function $u : \Gamma \to \mathbb{C}$, we set $u_j = u \circ \pi_j : [0, l_j] \to \mathbb{C}$, its "restriction" to the edge e_j and use the abbreviations:

$$u_j(v) = u_j(\pi_j^{-1}(v)), \quad u_j'(v) = \frac{du_j}{dx_j}(\pi_j^{-1}(v)), u_j''(v) = \frac{d^2u_j}{dx_j^2}(\pi_j^{-1}(v)), \partial_\nu u_j(v) = \nu_j(v)u_j'(v),$$

for a vertex $v \in e_j$. Finally, differentiations are carried out on each edge e_j with respect to the arc length parameter x_j.

For any $p \in [1, \infty)$, we denote by $L^p(\Gamma)$ the set of measurable functions on Γ such that $u_j \in L^p(0, l_j)$, for all $j = 1, \dots, J$. For shortness we write

$$\int_\Gamma u \, \mathrm{d}x = \sum_{j=1}^N \int_0^{l_j} u_j(x_j) \, \mathrm{d}x_j, \forall u \in L^1(\Gamma).$$

Now we denote by $PC(\Gamma)$ the set of piecewise continuous functions on Γ, which means that $u : \Gamma \to \mathbb{C}$ belongs to $PC(\Gamma)$ if and only if $u_j \in C([0, l_j])$, for all $j = 1, \dots, N$. Further $C(\Gamma)$ is the set of continuous functions on Γ, which means that $u \in C(\Gamma)$ if and only if $u \in PC(\Gamma)$ and

$$u_j(v) = u_k(v), \forall j, k \in J_v, v \in V_{\text{int}}.$$

Similarly we denote by $PH^1(\Gamma)$ the set of piecewise H^1 functions on Γ, in other words $u \in PH^1(\Gamma)$ if and only if $u_j \in H^1(0, l_j)$, for all $j = 1, \dots, N$. Further let us set $H^1(\Gamma) = PH^1(\Gamma) \cap C(\Gamma)$. These two spaces are clearly Hilbert spaces with their natural inner product.

Let us now fix a C^2-network Γ that is a tree (graph without cycle, so that V_{ext} is non empty) and two positive constants a and ν corresponding to the advection and viscosity coefficients respectively. With these assumptions, we consider the linearized viscous Saint-Venant system on Γ (see [5] for one interval)

$$\begin{cases} u_{jt} - \nu u_j'' + a\rho_j' = 0 \text{ in } Q_j, \forall j = 1, \dots, N, \\ \rho_{jt} + u_j' = 0 \text{ in } Q_j, \forall j = 1, \dots, N, \\ \sum_{j \in J_v} u_j(v,t)\nu_j(v) = 0, \forall v \in V_{\text{int}}, t > 0, \\ \nu u_j'(v,t) - a\rho_j(v,t) = \nu u_k'(v,t) - a\rho_k(v,t), \forall v \in V_{\text{int}}, j, k \in J_v, t > 0, \\ u_{j_v}(v,t) = 0, \forall v \in V_{\text{ext}}, t > 0, \\ u(\cdot, 0) = u_0, \rho(\cdot, 0) = \rho_0 \text{ in } \Gamma, \end{cases} \tag{3.2}$$

where $Q_j := (0, l_j) \times (0, \infty)$. As before u and ρ represent the velocity and the water height of the fluid respectively.

In the above system, the first transmission condition at interior nodes physically means the conservation of mass through the vertices, while the second one corresponds to the stress balance equation.

Remark 3.1. This system also corresponds to the linearized compressible Navier-Stokes system on trees, where in that case, ρ represent the density of the fluid, see [13]. Note that in [6, 7], the damping appears in the model as a zero order term; but such a case is outside the scope of our analysis. We may further mention that the derivation of water wave model from Navier-Stokes equations leads to the introduction of a non-local term [21, 30], this term is here neglected, since it requires additional investigations.

As before, this system is covered by our abstract setting with the next choices. Take the Hilbert spaces $H_1 = H_2 = L^2(\Gamma)$ and

$$V = \{u \in PH^1(\Gamma) : \sum_{j \in J_v} u_j(v)\nu_j(v) = 0, \forall v \in V_{\text{int}} \text{ and } \quad u_{j_v}(v) = 0, \forall v \in V_{\text{ext}}\},$$

equipped with the inner product

$$(u,v)_1 = \int_\Gamma u\bar{v}\,\mathrm{d}x, \forall u,v \in L^2(\Gamma),$$
$$(u,v)_2 = a\int_\Gamma u\bar{v}\,\mathrm{d}x, \forall u,v \in L^2(\Gamma),$$
$$(u,v)_V = \sum_{j=1}^N \int_0^{l_j} (u_j\bar{v}_j + u_j'\bar{v}_j')\,\mathrm{d}x_j, \forall u,v \in V.$$

Choose the sesquilinear forms

$$a(u,v) = \nu\sum_{j=1}^N \int_0^{l_j} u_j'\bar{v}_j'\,\mathrm{d}x_j, \quad \forall u,v \in V,$$
$$b(u,\rho) = -a\sum_{j=1}^N \int_0^{l_j} u_j'\bar{\rho}_j\,\mathrm{d}x_j, \quad \forall u \in V, \rho \in L^2(\Gamma).$$

The coerciveness of a is a direct consequence of the compact embedding of V into $L^2(\Gamma)$ that yields

$$a(u,u) = \nu|u|_{1,\Gamma}^2 \gtrsim \|u\|_{1,\Gamma}^2, \quad \forall u \in V,$$

since Γ is a tree (Poincaré type inequality on Γ [2]).

From the definition of b, we see that B is given by

$$Bu = -(u_j')_{j=1}^N,$$

and is clearly a linear continuous operator from V into $L^2(\Gamma)$. Moreover again due to Poincaré's inequality, B is injective and its range is closed. Consequently according to Theorem 2.19 of [12], we have $R(B) = (\ker B^*)^\perp$. This means that B is actually an isomorphism from V into H_3 and by Lemma I.4.1 of [25], the inf-sup condition (2.2) holds.

In order to have an explicit form of the space H_3, let us characterize the kernel of B^*.

Lemma 3.2. $\ker B^* = \mathbb{C}$.

Proof. Let ρ be in $\ker B^*$, then it satisfies

$$\sum_{j=1}^N \int_0^{l_j} u_j'\bar{\rho}_j\,\mathrm{d}x_j = 0, \quad \forall u \in V. \tag{3.3}$$

In a first step for any $j = 1,\dots,N$, we take $u \in V$ such that $u_k = 0$ for all $k \neq j$ and $u_j = \bar{\varphi} \in \mathcal{D}(0,l_j)$. Then (3.3) reduces to

$$\int_0^{l_j} \varphi'\rho_j\,\mathrm{d}x_j = 0, \quad \forall \varphi \in \mathcal{D}(0,l_j),$$

and consequently

$$\rho_j' = 0.$$

This implies in particular that ρ belongs to $PH^1(\Gamma)$.

In a second step, we come back to (3.3) and apply Green's formula on each edge to find

$$\sum_{j=1}^{N}[u_j\bar{\rho}_j]_0^{l_j} = 0, \quad \forall u \in V,$$

or equivalently

$$\sum_{v\in V_{\rm int}}\sum_{j\in J_v} u_j(v)\bar{\rho}_j(v)\nu_j(v) = 0, \quad \forall u \in V.$$

Since for any $v \in V_{\rm int}$ and any $X \in \mathbb{C}^{N_v}$ orthogonal to $(1,\dots,1)^\top$, there exists a function $u \in V$ such that $(u_j(v)\nu_j(v))_{j\in J_v} = X$, and $u_k(v') = 0$, for all $k \in J_{v'}, v' \in V_{\rm int}\setminus\{v\}$, by taking such a test function in the previous identity, we find that $(\rho_j)_{j\in J_v}$ is orthogonal to X. Since this holds for all X orthogonal to $(1,\dots,1)^\top$, we deduce that $(\rho_j)_{j\in J_v}$ is a multiple of $(1,\dots,1)^\top$ or equivalently

$$\rho_j = \rho_k, \forall j,k \in J_v.$$

As this holds for all interior vertices and Γ is connected, this proves the inclusion

$$\ker B^* \subset \mathbb{C}.$$

The other inclusion being a direct consequence of Green's formula, the proof is complete. □

A direct consequence of the previous lemma is that

$$H_3 = L_0^2(\Gamma) = \{\rho \in L^2(\Gamma) : \int_\Gamma \rho\,{\rm d}x = 0\},$$

exactly as in the previous subsection.

Let us further characterize the domain of the operator $\mathcal{A}$.

Lemma 3.3. *It holds*

$$D(\mathcal{A}) = \{(u,\rho) \in V \times L^2(\Gamma) : -\nu u' + a\rho \in H^1(\Gamma)\},$$

with

$$\mathcal{A}(u,\rho) = ((\nu u_j' - a\rho_j)')_{j=1}^N, -u_j')_{j=1}^N), \forall (u,\rho) \in D(\mathcal{A}),$$

and the equivalence

$$\|(u,\rho)\|_{D(\mathcal{A})} \sim \|(u,\rho)\|_D, \tag{3.4}$$

where

$$\|(u,\rho)\|_D = \|u\|_{1,\Gamma} + \|\rho\|_\Gamma + \|-\nu u' + a\rho\|_{H^1(\Gamma)}. \tag{3.5}$$

Proof. By its definition, (u,ρ) belongs to $D(\mathcal{A})$ if and only if $(u,\rho)\in V\times L^2(\Gamma)$ and satisfies

$$Au+B^*\rho\in L^2(\Gamma).$$

In other words, there exists $h\in L^2(\Gamma)$ such that

$$\langle Au+B^*\rho,v\rangle_{V'-V}=\sum_{j=1}^N\int_0^{l_j}h_j\bar{v}_j\,\mathrm{d}x,\forall v\in V.$$

From the definition of A and B, this equivalently means that

$$\sum_{j=1}^N\int_0^{l_j}(\nu u_j'-a\rho_j)\bar{v}_j'\,\mathrm{d}x=\sum_{j=1}^N\int_0^{l_j}h_j\bar{v}_j\,\mathrm{d}x,\forall v\in V.$$

As in Lemma 3.2, by taking first test functions equal to zero except at one edge where it coincides with a smooth function with a compact support, we deduce that

$$-(\nu u_j'-a\rho_j)'=h_j\text{ in }\mathcal{D}'(0,l_j),\forall j=1,\dots,N,$$

and consequently

$$\nu u_j'-a\rho_j\in H^1(0,l_j),\forall j=1,\dots,N,$$

and

$$(Au+B^*\rho)_j=-(\nu u_j'-a\rho_j)',\forall j=1,\dots,N. \tag{3.6}$$

The second step of the proof of Lemma 3.2 directly shows that $\nu u'-a\rho$ belongs to $H^1(\Gamma)$.
The equivalence (3.4) directly follows from the definition of $\mathcal{A}(u,\rho)$ and from (3.6). □

In conclusion according to Theorem 2.1, problem (3.2) is well-defined in a weak sense for initial data in $L^2(\Gamma)\times L^2(\Gamma)$ and in a strong sense for initial data in $D(\mathcal{A})$. Further by Theorem 2.7 and the above characterization of $\ker B^*$, the solution (u,ρ) tends exponentially to the stationary solution $(0,Q\rho_0)$, where $Q\rho_0=\frac{1}{|\Gamma|}\int_\Gamma\rho_0$, $|\Gamma|=\int_\Gamma\mathrm{d}x$ being the length of Γ. In other words, there exist two positive constants C and ω such that the solution (u,ρ) of (3.2) satisfies

$$\|u(\cdot,t)\|_\Gamma+\|\rho(\cdot,t)-Q\rho_0\|_\Gamma\le Ce^{-\omega t}(\|u_0\|_\Gamma+\|\rho_0\|_\Gamma),\forall t\ge 0.$$

To our best knowledge this result is new, the only drawback is that the decay rate is not explicit. This drawback will be set up by a precise spectral analysis, and is based on the property

$$B^*B=\frac{a}{\nu}A. \tag{3.7}$$

This property follows from the definition of B. Indeed for any $u,v\in V$, one has

$$\langle B^*Bu,v\rangle=(Bu,Bv)_{H_2}=a\sum_{j=1}^N\int_0^{l_j}u_j'\bar{v}_j'\,\mathrm{d}x_j=\frac{a}{\nu}a(u,v),$$

which shows that (3.7) holds.

3.3. An one-dimensional hyperbolic system with a viscous damping

On a real interval $(0,1)$, we consider the system

$$\begin{cases} u_t - Mu_{xx} + C^*\rho_x = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_t + Cu_x = 0 & \text{in } (0,1)\times(0,\infty), \\ u(0,\cdot) = u(1,\cdot) = 0 & \text{on } (0,\infty), \\ u(\cdot,0) = u_0, \rho(\cdot,0) = \rho_0 & \text{in } (0,1), \end{cases} \tag{3.8}$$

where $u(x,t)$ (resp. $\rho(x,t)$) represents the vectorial unknown functions at the point x and time t with values in $\mathbb{C}^n$ (resp. $\mathbb{C}^m$), M is a $n\times n$ symmetric and positive definite matrix and C is a $m\times n$ matrix. This system is a first order linear hyperbolic system if $M=0$ that is conservative [14], therefore the term $-Mu_{xx}$ corresponds to a viscous damping that will be responsable of the exponential decay of the system. If $n=m=1$, this system is nothing else than the linearized viscous Saint-Venant system in $(0,1)$ (with $a=1$) [5, 13]. If $n=1$ and $m=2$, it corresponds to a linearized viscous two-phase model where ρ_1 and ρ_2 are the density of phase 1 and 2 (with $\rho=(\rho_1,\rho_2)$), u is the common velocity, with the pressure law $p(\rho_1,\rho_2) = a_1^2\rho_1 + a_2^2\rho_2$, a_1, a_2 being two given real numbers, hence the matrix C^* is given by (a_1^2, a_2^2) and $M=\mu>0$, see system (1.11) of [23]. Note that the linearization of this system is made around the point $(m_0,n_0,0)$ with $m_0,n_0>0$.

Again, system (3.8) enters in our abstract setting with the next choices: Take the Hilbert spaces $H_1 = L^2(0,1)^n, H_2 = L^2(0,1)^m$ and $V = H_0^1(0,1)^n$ equipped with their natural inner product and choose the sesquilinear forms:

$$a(u,v) = \int_0^1 Mu_x\cdot\bar{v}_x\,\mathrm{d}x, \quad \forall u,v\in V,$$
$$b(u,\rho) = -\int_0^1 Cu_x\bar{\rho}\,\mathrm{d}x, \quad \forall u\in V, \rho\in L^2(0,1)^m.$$

The coerciveness of a is a direct consequence of Poincaré's inequality in $H_0^1(0,1)$ as

$$a(u,u) = \int_0^1 Mu_x\cdot\bar{u}_x\,\mathrm{d}x \gtrsim |u|^2_{1,(0,1)} \gtrsim \|u\|^2_{1,(0,1)}, \quad \forall u\in H_0^1(0,1)^n,$$

since M is supposed to be symmetric and positive definite.

From the definition of b, we see that B is given by

$$Bu = -Cu_x$$

and is a linear continuous operator from V into $L^2(0,1)^m$. If $\ker C = \{0\}$, then obviously B is injective, otherwise B is not injective but its kernel is clearly a closed subspace of V. Further we have the next characterization of $\ker B$.

Lemma 3.4. *If $\ker C\neq\{0\}$, let $\{e_i\}_{i=1}^I$ (with $0<I\le n$) be a basis of $\ker C$. Then it holds*

$$\ker B = \{u = \sum_{i=1}^I \alpha_i e_i : \alpha_i\in H_0^1(0,1), i=1,\dots,I\}. \tag{3.9}$$

Proof. Let $u \in \ker B$, then equivalently, one has

$$(Cu)_x = 0,$$

and therefore

$$Cu = 0,$$

as $(Cu)(0) = 0$. This means that (recalling that $H_0^1(0,1)$ is continuously embedded into $C([0,1])$).

$$u \in \ker C \text{ in } (0,1),$$

and therefore there exist $\alpha_i \in H_0^1(0,1)$, for all $i = 1, \dots, I$ such that

$$u = \sum_{i=1}^{I} \alpha_i e_i,$$

because for all $x \in [0,1]$, $\alpha_i(x) = u(x) \cdot \bar{f}_i$, where $\{f_i\}_{i=1}^I$ is a biorthogonal basis of $\{e_i\}_{i=1}^I$ (*i.e.*, satisfying $e_i \cdot \bar{f}_j = \delta_{i,j}$). □

This lemma allows us to characterize the space $W = (\ker B)^\perp$, the orthogonal of $\ker B$ in V once it is equipped with the inner product

$$(u,v)_V = \int_0^1 u_x \cdot \bar{v}_x \, \mathrm{d}x, \forall u, v \in V.$$

Lemma 3.5. *If* $\ker C \neq \{0\}$*, then*

$$W = \{u \in V : u \in (\ker C)^\perp \text{ in } (0,1)\}, \tag{3.10}$$

where $(\ker C)^\perp$ *means the orthogonal of* $\ker C$ *in* $\mathbb{C}^n$*.*

Proof. By definition, $u \in W$ if and only if

$$\int_0^1 u_x \cdot \bar{v}_x \, \mathrm{d}x = 0, \forall v \in \ker B.$$

Hence according to Lemma 3.4, this is equivalent to

$$\sum_{i=1}^{I} \int_0^1 \bar{\alpha}_{i,x} u_x \cdot \bar{e}_i \, \mathrm{d}x = 0, \forall \alpha_i \in H_0^1(0,1), i = 1, \dots, I.$$

For a fixed $j \in \{1, \dots, I\}$, as test function in the above identity, we take $\alpha_j = \varphi \in \mathcal{D}(0,1)$ and $\alpha_i = 0$ for any $i \neq j$, and obtain

$$\int_0^1 \bar{\varphi}_x u_x \cdot \bar{e}_j \, \mathrm{d}x = 0, \forall \varphi \in \mathcal{D}(0,1).$$

This means that

$$(u_x \cdot \bar{e}_j)_x = 0 \text{ in } \mathcal{D}'(0,1),$$

or equivalently

$$(u \cdot \bar{e}_j)_{xx} = 0 \text{ in } \mathcal{D}'(0,1).$$

In other words, $u \cdot \bar{e}_j$ is a polynomial of degree 1, and due to the boundary conditions $(u \cdot \bar{e}_j)(0) = (u \cdot \bar{e}_j)(1) = 0$, we get

$$u \cdot \bar{e}_j = 0 \text{ in } (0,1).$$

As j is arbitrary in $\{1, \dots, I\}$, this exactly shows that $u \in (\ker C)^\perp$. □

We are ready to show a sort of Poincaré's inequality.

Lemma 3.6. *It holds*

$$\|u\|_V \lesssim \|Bu_x\|_{(0,1)}, \forall u \in W. \tag{3.11}$$

Proof. If $\ker C = \{0\}$, $W = V$, and (3.11) is the standard Poincaré inequality. If $\ker C \neq \{0\}$, then as all norms are equivalent in finite-dimensional spaces, we can say that there exists $\delta > 0$ such that

$$\delta |x| \leq |Cx|, \forall x \in (\ker C)^\perp.$$

For any $u \in W$, as (3.10) implies that u_x belongs to $(\ker C)^\perp$ almost everywhere in $(0,1)$, this estimate implies that

$$\delta^2 |u_x(x)|_2^2 \leq |Cu_x(x)|^2, \forall \text{ a.a. } x \in (0,1).$$

Integrating this estimate in $(0,1)$, we conclude by using the standard Poincaré inequality. □

At this stage, we consider the restriction of B to W, namely let

$$B_0 : W \to L^2(0,1)^m : u \to Bu,$$

that is still a linear continuous operator from W into $L^2(0,1)^m$. Due to the previous lemma, this mapping is injective and its range is closed, consequently according to Theorem 2.19 of [12], B_0 is actually an isomorphism from W into H_3, since $R(B_0) = R(B)$. Hence by Lemma I.4.1 of [25], the inf-sup condition

$$\inf_{\rho \in H_3} \sup_{u \in W} \frac{|b(u,\rho)|}{\|u\|_V \|\rho\|_2} \gtrsim 1$$

holds, hence (2.2) as well, since $W \subset V$.

As in the previous section, let us characterize the kernel of B^*.

Lemma 3.7. *Let $\{e_i^*\}_{i=1}^{I^*}$ be a basis of $\ker C^*$ and $\{e_i^*\}_{i=I^*+1}^m$ be a basis of $(\ker C^*)^\perp$ (in $\mathbb{C}^m$). Then it holds*

$$\ker B^* = \{\rho \in L^2(0,1)^m : \rho = \sum_{i=1}^m \alpha_i e_i^* \text{ with } \alpha_i \in L^2(0,1), \forall i \leq I^* \text{ and } \alpha_i \in \mathbb{C}, \forall i > I^*\}.$$

Proof. Let ρ be in $\ker B^*$, then it satisfies

$$\int_0^1 C u_x \bar{\rho} \, \mathrm{d}x = 0, \quad \forall u \in V. \tag{3.12}$$

This equivalently means that $C^*\rho$ satisfies

$$(C^*\rho)_x = 0 \text{ in } \mathcal{D}'(0,1)^n.$$

In other words, $C^*\rho$ is constant in $(0,1)$, which means that there exists $k \in R(C^*)$ such that

$$C^*\rho(x) = k, \forall x \in (0,1).$$

Therefore there exists $\rho_0 \in (\ker C^*)^\perp$ such that

$$C^*(\rho(x) - \rho_0) = 0, \forall x \in (0,1),$$

which shows that

$$\rho - \rho_0 \in \ker C^* \text{ a.e. in } (0,1),$$

hence the result. □

From the previous lemma, we see that $\ker B^*$ is finite-dimensional if $\ker C^*$ is reduced to $\{0\}$, its dimension being equal to m, otherwise it is an infinite dimensional space. In any case, its projection Q can be easily computed as follows: Without loss of generality, we can assume that the basis $\{e_i^*\}_{i=1}^m$ of $\mathbb{C}^m$ is orthonormal, then one has

$$Q\rho = \sum_{i=1}^{I^*} \left(\rho \cdot \bar{e}_j^*\right) e_i^* + \sum_{i=I^*+1}^{m} \left(\int_0^1 \rho \cdot \bar{e}_j^* \, \mathrm{d}x\right) e_i^*, \forall \rho \in L^2(0,1)^m. \tag{3.13}$$

Indeed for an arbitrary $\rho \in L^2(0,1)^m$, by the previous lemma

$$Q\rho = \sum_{i=1}^{m} \alpha_i e_i^*, \tag{3.14}$$

with $\alpha_i \in L^2(0,1)$, for all $i \leq I^*$ and $\alpha_i \in \mathbb{C}$ for all $i > I^*$ fixed such that

$$\int_0^1 (\rho - Q\rho) \cdot \left(\sum_{i=1}^{m} \bar{\beta}_i \bar{e}_i^*\right) \mathrm{d}x = 0,$$

for all $\beta_i \in L^2(0,1)$, if $i \leq I^*$ and all $\beta_i \in \mathbb{C}$ if $i > I^*$. By the orthogonality property of the e_i^*, this is equivalent to

$$\sum_{i=1}^{m} \int_0^1 \alpha_i \bar{\beta}_i \, \mathrm{d}x = \sum_{i=1}^{m} \int_0^1 \rho \cdot \bar{e}_i^* \bar{\beta}_i \, \mathrm{d}x,$$

for all $\beta_i \in L^2(0,1)$, if $i \leq I^*$ and all $\beta_i \in \mathbb{C}$ if $i > I^*$. In this property, first for any $j \leq I^*$, we fix $\beta_j = \beta$ arbitrary in $L^2(0,1)$ and $\beta_i = 0$ else, and find

$$\int_0^1 \alpha_j \bar{\beta}\, \mathrm{d}x = \int_0^1 \rho \cdot \bar{e}_j^* \bar{\beta}\, \mathrm{d}x, \forall \beta \in L^2(0,1),$$

which shows that

$$\alpha_j = \rho \cdot \bar{e}_j^*. \tag{3.15}$$

Second for any $j > I^*$, we fix $\beta_j = \beta$ arbitrary in $\mathbb{C}$ and $\beta_i = 0$ else, and find again (3.15)

$$\int_0^1 \alpha_j \bar{\beta}\, \mathrm{d}x = \int_0^1 \rho \cdot \bar{e}_j^* \bar{\beta}\, \mathrm{d}x, \forall \beta \in \mathbb{C},$$

that here reduces to

$$\alpha_j = \int_0^1 \rho \cdot \bar{e}_j^*\, \mathrm{d}x. \tag{3.16}$$

Inserting (3.15) and (3.16) into the expression (3.14), we find (3.13).

In conclusion according to Theorem 2.1, problem (3.8) is well-defined in a weak sense for initial data in $L^2(0,1)^n \times L^2(0,1)^m$ and in a strong sense for initial data in $D(\mathcal{A})$. By Theorem 2.7 and the above characterization of $\ker B^*$, the solution (u, ρ) tends exponentially to the stationary solution $(0, Q\rho_0)$, where

$$Q\rho_0 = \sum_{i=1}^{I^*} \left(\rho_0 \cdot \bar{e}_j^*\right) e_i^* + \sum_{i=I^*+1}^{m} \left(\int_0^1 \rho_0 \cdot \bar{e}_j^*\, \mathrm{d}x\right) e_i^*.$$

Recall that Theorem 2.7 does not furnish an explicit decay rate but under the assumption that

$$C^* C = \beta M, \tag{3.17}$$

for some positive real number β, then the results of the next section allow to set up this drawback since we then have

$$B^* B = \beta A. \tag{3.18}$$

Indeed from the definition of B, for any $u, v \in V$, one has

$$\langle B^* B u, v\rangle = (Bu, Bv)_{H_2} = \int_0^1 C u_x \cdot C v_x\, \mathrm{d}x = \int_0^1 C^* C u_x \cdot v_x\, \mathrm{d}x,$$

and from the property (3.17), one obtains

$$\langle B^* B u, v\rangle = \beta a(u, v),$$

which shows that (3.18) holds.

Note that (3.17) always holds for the two examples mentioned at the beginning of this section, namely if $n = m = 1$ and $C \neq 0$ or if $n = 1$, $m = 2$ and $C^* = (a_1^2, a_2^2)$ with $a_1^2 + a_2^2 > 0$.

3.4. The viscous Maxwell system in a bounded domain of $\mathbb{R}^3$

Let Ω be a bounded domain of $\mathbb{R}^3$ with a Lipschitz and simply connected boundary. On this domain, we consider the Maxwell system

$$\begin{cases} E_t - \operatorname{curl} H + \nu \operatorname{curl}\operatorname{curl} E = 0 & \text{in } \Omega \times (0,\infty), \\ H_t + \operatorname{curl} E = 0 & \text{in } \Omega \times (0,\infty), \\ E \times n = 0, H \cdot n = 0 & \text{on } \partial\Omega \times (0,\infty), \\ E(\cdot,0) = E_0, H(\cdot,0) = H_0 & \text{in } \Omega, \end{cases} \tag{3.19}$$

where $E(x,t)$ (resp. $H(x,t)$) represents the electric (resp. magnetic) field at the point x and time t and the boundary conditions are standard electric boundary conditions. Here ν is a viscosity coefficient that is supposed to be positive. The case $\nu = 0$ corresponds to the standard Maxwell system that is conservative [17, 22, 33], and consequently the term $\nu \operatorname{curl}\operatorname{curl} E$ is a viscous damping that will be responsable of the exponential decay of the system.

This system enters into our abstract framework in the following way (see [22, 36] in the case $\nu = 0$). We first recall that

$$\begin{aligned} H(\operatorname{div},\Omega) &= \{E \in L^2(\Omega)^3 : \operatorname{div} E \in L^2(\Omega)\}, \\ H_0(\operatorname{div},\Omega) &= \{E \in H(\operatorname{div},\Omega) : E \cdot n = 0 \text{ on } \partial\Omega\}, \\ H(\operatorname{curl},\Omega) &= \{E \in L^2(\Omega)^3 : \operatorname{curl} E \in L^2(\Omega)^3\}, \\ H_0(\operatorname{curl},\Omega) &= \{E \in H(\operatorname{curl},\Omega) : E \times n = 0 \text{ on } \partial\Omega\}, \\ X_N(\Omega) &= H(\operatorname{div},\Omega) \cap H_0(\operatorname{curl},\Omega), \end{aligned}$$

are Hilbert spaces with their natural inner product, see [4]. Now we take

$$\begin{aligned} H_1 = H(\operatorname{div} = 0,\Omega) &:= \{E \in L^2(\Omega)^3 : \operatorname{div} E = 0 \text{ in } \Omega\}, \\ H_2 = H_1 \cap H_0(\operatorname{div},\Omega) &:= \{H \in H(\operatorname{div} = 0,\Omega) : H \cdot n = 0 \text{ on } \partial\Omega\}, \end{aligned}$$

both being Hilbert spaces with the inner product of $L^2(\Omega)^3$. We further set

$$V = H(\operatorname{div} = 0,\Omega) \cap H_0(\operatorname{curl},\Omega),$$

that is a Hilbert space with the inner product (due to Friedrich's inequality, see [33], Cor. 4.8)

$$(E,F)_V = \int_\Omega \operatorname{curl} E \cdot \operatorname{curl} \bar{F}\, \mathrm{d}x, \forall E,F \in V.$$

We now choose the sesquilinear forms

$$\begin{aligned} a(E,F) &= \nu \int_\Omega \operatorname{curl} E \cdot \operatorname{curl} \bar{F}\, \mathrm{d}x, \quad \forall E,F \in V, \\ b(E,H) &= -\int_\Omega \operatorname{curl} E \cdot \bar{H}\, \mathrm{d}x, \quad \forall E \in V, H \in H_2. \end{aligned}$$

The coerciveness of a is direct, since

$$a(E,E) = \nu \int_\Omega |\operatorname{curl} E|^2\, \mathrm{d}x, \quad \forall E \in V.$$

Further as for $E \in V$, $\operatorname{curl} E$ belongs to H_2, we directly deduce that

$$BE = -\operatorname{curl} E, \forall E \in V.$$

As before, let us characterize the kernel of B^* and the domain of the operator $\mathcal{A}$.

Lemma 3.8.

$$\ker B^* = K_T(\Omega) := \{H \in H_2 : \operatorname{curl} H = 0\}.$$

Proof. Let H be in $\ker B^*$, then it satisfies

$$\int_\Omega \operatorname{curl} E \cdot H \, \mathrm{d}x = 0, \quad \forall E \in V. \tag{3.20}$$

As $\mathcal{D}(\Omega)^3$ is not dense in V, while it is dense in $X_0(\Omega)$ [4], p. 827, therefore in this last identity, we want to change the set of test functions into $X_0(\Omega)$. For that purpose, for an arbitrary $\tilde{E}$ in $X_0(\Omega)$, we consider the unique solution $\varphi \in H_0^1(\Omega)$ of

$$\int_\Omega \nabla\varphi \cdot \nabla\psi \, \mathrm{d}x = \int_\Omega \tilde{E} \cdot \nabla\psi \, \mathrm{d}x, \forall \psi \in H_0^1(\Omega).$$

This identity implying that $\tilde{E} - \nabla\varphi$ is divergence free, we deduce that $E = \tilde{E} - \nabla\varphi$ belongs to V. Consequently (3.20) yields

$$\int_\Omega \operatorname{curl} \tilde{E} \cdot H \, \mathrm{d}x = \int_\Omega \operatorname{curl} E \cdot H \, \mathrm{d}x = 0.$$

Therefore one equivalently gets that

$$\int_\Omega H \cdot \operatorname{curl} \varphi \, \mathrm{d}x = 0, \forall \varphi \in \mathcal{D}(\Omega)^3,$$

which means that $\operatorname{curl} H = 0$, as announced. This shows the inclusion

$$\ker B^* \subset \{H \in H_2 : \operatorname{curl} H = 0\}.$$

The converse inclusion is a consequence of the next Green's formula

$$\int_\Omega (\operatorname{curl} E \cdot F - E \cdot \operatorname{curl} F) \, \mathrm{d}x = 0, \forall E \in H_0(\operatorname{curl}, \Omega), F \in H(\operatorname{curl}, \Omega), \tag{3.21}$$

see the identity (3.3) of [36] or Lemma 2.5 of [18], page 91. □

From the previous lemma and Lemma 1.2 of [24] (see also [20] in the smooth case and [4], Prop. 3.14 for a different expression of the basis), we deduce that $\ker B^*$ is finite-dimensional, its dimension being equal to J, the number of cuts $\Sigma_j, j = 1, \ldots, J$, such that $\Omega^\circ = \Omega \setminus \cup_{j=1}^J \Sigma_j$ becomes simply connected. More precisely there exist J functions $q_j \in H^1(\Omega^\circ)$, $j = 1, \ldots, J$, such that

$$\ker B^* = \text{ Span } \{\tilde{\nabla} q_j\}_{j=1}^J,$$

where $\tilde{\nabla} q_j$ means that we take the gradient of q_j in Ω°, the properties of q_j (stated in Lem. 1.2 of [24]) implying that $\tilde{\nabla} q_j$ belongs to $K_T(\Omega)$.

As a consequence, we have

$$H_3 = \{H \in H_2 : \int_\Omega H \cdot \tilde{\nabla} \bar{q}_j \, \mathrm{d}x = 0, \forall j = 1, \dots, J\}. \tag{3.22}$$

Let us now check the inf-sup condition (2.2).

Lemma 3.9. *For all $H \in H_3$, we have*

$$\|H\|_\Omega \leq \sup_{E \in V} \frac{\left|\int_\Omega \operatorname{curl} E \cdot \bar{H} \, \mathrm{d}x\right|}{\|\operatorname{curl} E\|_\Omega}. \tag{3.23}$$

Proof. We first check that

$$H_3 = \{H \in H_2 : \langle H \cdot n, 1 \rangle_{\Sigma_j} = 0, \forall j = 1, \dots, J\}, \tag{3.24}$$

where $\langle \cdot, \cdot \rangle_{\Sigma_j}$ means the duality pairing between $H^{\frac{1}{2}}(\Sigma_j)'$ and $H^{\frac{1}{2}}(\Sigma_j)$ (which has a meaning due to [4], Lem. 3.10). Indeed by (3.22) $H \in H_2$ belongs to H_3 if and only if

$$\int_\Omega H \cdot \tilde{\nabla} \bar{q}_j \, \mathrm{d}x = 0, \forall j = 1, \dots, J.$$

By Lemma 3.10 of [4], we get equivalently

$$\sum_{k=1}^{J} \langle H \cdot n, [q_j]_k \rangle_{\Sigma_k} = 0, \forall j = 1, \dots, J,$$

where $[q_j]_k$ means the jump of q_j through Σ_k. As $[q_j]_k = \delta_{jk}$ (see Lem. 1.2 of [24]), we conclude that

$$\langle H \cdot n, 1 \rangle_{\Sigma_j} = 0, \forall j = 1, \dots, J.$$

Given $H \in H_3$, owing to (3.24), we can apply Theorem 3.17 of [4] that yields a unique $\psi \in V$ such that

$$\operatorname{curl} \psi = H,$$

that trivially implies

$$\|H\|_\Omega = \frac{\left|\int_\Omega \operatorname{curl} \psi \cdot \bar{H} \, \mathrm{d}x\right|}{\|\operatorname{curl} \psi\|_\Omega},$$

and proves (3.23). □

Let us further characterize the domain of the operator $\mathcal{A}$.

Lemma 3.10. *It holds*

$$D(\mathcal{A}) = \{(E, H) \in V \times H_2 : \nu \operatorname{curl} E - H \in H(\operatorname{curl}, \Omega)\},$$

with

$$\mathcal{A}(E,H) = (-\operatorname{curl}(\nu \operatorname{curl} E - H), -\operatorname{curl} E), \forall (E,H) \in D(\mathcal{A}). \tag{3.25}$$

Proof. By its definition, (E,H) belongs to $D(\mathcal{A})$ if and only if $(E,H) \in V \times H_2$ and satisfies

$$AE + B^*H \in H_1.$$

In other words, there exists $h \in H_1$ such that

$$\langle AE + B^*H, F\rangle_{V'-V} = \int_\Omega h \cdot \bar{F}\,\mathrm{d}x, \forall F \in V.$$

From the definition of A and B, this equivalently means that

$$\int_\Omega (\nu \operatorname{curl} E \cdot \operatorname{curl} \bar{F} - H \cdot \operatorname{curl} \bar{F})\,\mathrm{d}x = \int_\Omega h \cdot \bar{F}\,\mathrm{d}x, \forall F \in V. \tag{3.26}$$

Again as in Lemma 3.8, we can change the test functions to any element of $X_0(\Omega)$ since for any $\varphi \in H_0^1(\Omega)$, one has

$$\int_\Omega h \cdot \nabla \bar{\varphi}\,\mathrm{d}x = -\int_\Omega \operatorname{div} h\bar{\varphi}\,\mathrm{d}x = 0,$$

due to Green's formula and recalling that h is divergence free.

In other words (3.26) is equivalent to

$$\int_\Omega (\nu \operatorname{curl} E \cdot \operatorname{curl} \bar{F} - H \cdot \operatorname{curl} \bar{F})\,\mathrm{d}x = \int_\Omega h \cdot \bar{F}\,\mathrm{d}x, \forall F \in X_0(\Omega). \tag{3.27}$$

Consequently, we find equivalently that

$$\operatorname{curl}(\nu \operatorname{curl} E - H) = h$$

in the distributional sense and since h belongs to $L^2(\Omega)^3$, $\nu \operatorname{curl} E - H$ belongs to $H(\operatorname{curl}, \Omega)$.

The identity (3.25) directly follows from the previous considerations. □

In conclusion, Theorem 2.1 guarantees that problem (3.19) is well-defined in a weak sense for initial data in $H_1 \times H_2$ and in a strong sense for initial data in $D(\mathcal{A})$. By Theorem 2.7, the solution (E,H) tends exponentially to the stationary solution $(0, QH_0)$, where

$$QH_0 = \sum_{j=1}^{J} \alpha_j(H_0)\tilde{\nabla} q_j,$$

with $\alpha_j(H_0), j = 1,\ldots,J$, uniquely determined by the condition

$$\int_\Omega (H_0 - QH_0) \cdot \tilde{\nabla}\bar{q}_k\,\mathrm{d}x = 0, \forall k = 1,\ldots,J.$$

As already mentioned before, Theorem 2.7 does not furnish an explicit decay rate but for this model as we immediately check that

$$B^*B = \frac{1}{\nu}A,$$

the results of the next section allow to give an explicit decay rate.

4. A spectral analysis

Here we come back to the abstract setting from Section 2 and show that a spectral analysis of the operator $\mathcal{A}$ is possible under the assumption

$$B^*B = \beta A \text{ for some positive real number } \beta. \tag{4.1}$$

First as A can be seen as a positive selfadjoint operator (with a compact resolvent) from H_1 into itself with domain $D(A) := \{u \in V : Au \in H_1\}$, let us denote by $(\lambda_k^2)_{k\in\mathbb{N}^*}$ (with $\lambda_k > 0$) the set of eigenvalues of A repeated according to their multiplicity and let $\{\varphi_k\}_{k\in\mathbb{N}^*}$ be the corresponding set of orthonomalized eigenvectors (in H_1). With these notations, one can prove the following result.

Theorem 4.1. *Under the assumption (4.1), for all $k \in \mathbb{N}^*$, the complex numbers*

$$\lambda_{k\pm} = \frac{-\lambda_k^2 \pm \lambda_k\sqrt{\lambda_k^2 - 4\beta}}{2} \tag{4.2}$$

are eigenvalues of $\mathcal{A}$. If $\lambda_k^2 - 4\beta \neq 0$, then λ_{k+} and λ_{k-} are simple eigenvalue of $\mathcal{A}$ and the corresponding eigenvectors (up to a multiplicative factor) are given by

$$U_{k\pm} = (u_{k\pm}, \rho_{k\pm}) = (\varphi_k, \frac{1}{\lambda_{k\pm}}B\varphi_k). \tag{4.3}$$

On the contrary, if $\lambda_k^2 - 4\beta = 0$, for some $k \in \mathbb{N}^*$, then $\lambda_{k+} = \lambda_{k-}$ is an eigenvalue of $\mathcal{A}$ of geometric multiplicity one and algebraic multiplicity 2, namely

$$U_{k+} = (u_{k+}, \rho_{k+}) = (\varphi_k, \frac{1}{\lambda_{k+}}B\varphi_k) \tag{4.4}$$

is the corresponding eigenvector, while

$$U_{k-} = (u_{k-}, \rho_{k-}) = (0, \frac{1}{\lambda_{k+}^2}B\varphi_k) \tag{4.5}$$

is a generalized eigenvector, namely it satisfies

$$(\lambda_{k+} - \mathcal{A})U_{k-} = U_{k+}.$$

Proof. Since we have already characterized the kernel of $\mathcal{A}$ (see Lem. 2.3), it suffices to look for a complex number $\lambda \neq 0$ and a non zero solution $(u, \rho) \in D(\mathcal{A})$ of

$$\mathcal{A}(u, \rho) = \lambda(u, \rho),$$

or equivalently

$$\begin{cases} Au + B^*\rho = -\lambda u, \\ Bu = \lambda\rho. \end{cases}$$

From the second equation, we can eliminate ρ given by

$$\rho = \frac{1}{\lambda} Bu, \tag{4.6}$$

and inserting it in the first identity, we get

$$Au + \frac{1}{\lambda} B^* Bu = -\lambda u.$$

But recalling our assumption (4.1), we find that u is solution of

$$(1 + \frac{\beta}{\lambda})Au = -\lambda u. \tag{4.7}$$

Now we notice that the definition of the domain of $\mathcal{A}$ requires that $Au + B^*\rho$ has to be in H_1, but due to (4.6) and (4.1), we get

$$(1 + \frac{\beta}{\lambda})Au \in H_1.$$

We remark that $1 + \frac{\beta}{\lambda}$ cannot be equal to zero, otherwise by (4.7), u is zero and hence $\rho = 0$ by (4.6), which is not allowed. Consequently u belongs to $D(A)$ and satisfies

$$Au = -\frac{\lambda^2}{\lambda + \beta} u. \tag{4.8}$$

This means that $u \neq 0$ is an eigenvector of A with eigenvalue $-\frac{\lambda^2}{\lambda+\beta}$, and therefore

$$-\frac{\lambda^2}{\lambda + \beta} = \lambda_k^2,$$

for some $k \in \mathbb{N}^*$ and $u = \varphi_k$ (up to a multiplicative factor). As this identity is equivalent to the second order equation

$$\lambda^2 + \lambda_k^2 \lambda + \beta\lambda_k^2 = 0, \tag{4.9}$$

we find that $\lambda \in \{\lambda_{k+}, \lambda_{k-}\}$. By the identity $u = \varphi_k$ and (4.6), we find the expression (4.3) for the associated eigenvector, except if $\lambda_k^2 - 4\beta = 0$. In that last case, we see that the expression of U_{k+} and U_{k-} in (4.6) are the same and consequently, λ_{k+} has only one eigenvector. But simple calculations show that U_{k-} given by (4.5) is a generalized eigenvector. □

The next step is to show that the eigenvectors and generalized ones that we just found form a Riesz basis of $\mathcal{H}_0$. First we show that they generate the whole $\mathcal{H}_0$. For further uses, we introduce the exceptional set

$$N_0 = \{k \in \mathbb{N}^* : \lambda_k^2 = 4\beta\},$$

and the set

$$N_1 = \{k \in \mathbb{N}^* : \lambda_k^2 > 4\beta\}.$$

Note that for $k \in N_0 \cup N_1$, the eigenvalues λ_{k+} and λ_{k-} are real.

Lemma 4.2. *Let the assumption (4.1) be satisfied. Denote by*

$$\mathcal{H}_{00} = \ \mathit{Span}\ \{U_{k\pm}\}_{k\in\mathbb{N}^*}.$$

Then $\mathcal{H}_{00}^{\perp} = \{0\}$ *in* $\mathcal{H}_0$.

Proof. Let $(u, \rho) \in \mathcal{H}_0$ be orthogonal to $\mathcal{H}_{00}$, namely satisfying

$$((u, \rho), U_{k\pm})_{\mathcal{H}} = 0, \forall k \in \mathbb{N}^*.$$

Hence by the definition of $U_{k\pm}$ and of the inner product in $\mathcal{H}$, we arrive at

$$(u, u_{k\pm})_{H_1} + (\rho, \rho_{k\pm})_{H_2} = 0, \forall k \in \mathbb{N}^*. \tag{4.10}$$

Now we need to distinguish between the case $k \in N_0$ or not:
1. For all $k \in \mathbb{N}^* \setminus N_0$, by the previous theorem, (4.10) is equivalent to

$$(u, \varphi_k)_{H_1} + (\rho, \frac{1}{\lambda_{k\pm}} B\varphi_k)_{H_2} = 0,$$

or in matrix form

$$\begin{pmatrix} \bar{\lambda}_{k+} & 1 \\ \bar{\lambda}_{k-} & 1 \end{pmatrix} \begin{pmatrix} (u, \varphi_k)_{H_1} \\ (\rho, B\varphi_k)_{H_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the determinant of this matrix is equal to $\bar{\lambda}_{k+} - \bar{\lambda}_{k+}$ and $\lambda_{k+} - \lambda_{k+} = \lambda_k \sqrt{\lambda_k^2 - 4\beta}$ is different from zero, we deduce that

$$(u, \varphi_k)_{H_1} = (\rho, B\varphi_k)_{H_2} = 0. \tag{4.11}$$

2. If $k \in N_0$, then by the previous theorem, (4.10) is now equivalent to

$$(u, \varphi_k)_{H_1} + (\rho, \frac{1}{\lambda_{k+}} B\varphi_k)_{H_2} = 0,$$
$$(\rho, \frac{1}{\lambda_{k+}^2} B\varphi_k)_{H_2} = 0.$$

The second identity directly implies that $(\rho, B\varphi_k)_{H_2} = 0$ and the first one becomes $(u, \varphi_k)_{H_1} = 0$. In other words, (4.11) is still valid in this case.

In summary we have shown that (4.11) holds for all $k \in \mathbb{N}^*$, or equivalently

$$(u, \varphi_k)_{H_1} = 0, \forall k \in \mathbb{N}^*,$$

as well as

$$\langle B^*\rho, \varphi_k\rangle_{V'-V} = 0, \forall k \in \mathbb{N}^*.$$

The first property implies that $u = 0$, since $\{\varphi_k\}_{k\in\mathbb{N}^*}$ is an orthonormal basis of H_1. The second property implies that

$$B^*\rho = 0,$$

since $\{\varphi_k\}_{k\in\mathbb{N}^*}$ is also an orthonormal basis of V. But as by assumption $\rho \in (\ker B^*)^\perp$, we conclude that $\rho = 0$. □

We are ready to prove the Riesz basis property.

Theorem 4.3. *Under the assumption (4.1), the set $\{\|U_{k\pm}\|_{\mathcal{H}}^{-1}U_{k\pm}\}_{k\in\mathbb{N}^*}$ is a Riesz basis of $\mathcal{H}_0$.*

Proof. For shortness, we denote by $V_{k\pm} = \kappa_{k\pm}U_{k\pm}$, with $\kappa_{k\pm} = \|U_{k\pm}\|_{\mathcal{H}}^{-1}$, so that $\|V_{k\pm}\|_{\mathcal{H}} = 1$. We first show that from the set $\{V_{k\pm}\}_{k\in\mathbb{N}^*}$ we can build an orthonormal basis of $\mathcal{H}_0$. For that purpose, we calculate their inner product by distinguish the following cases:
(a) For $k, k' \in \mathbb{N}^* \setminus N_0$, we have by (4.1)

$$\begin{aligned}(V_{k\pm}, V_{k'\pm})_{\mathcal{H}} &= \kappa_{k\pm}\kappa_{k'\pm}\Big((\varphi_k, \varphi_{k'})_{H_1} + \frac{1}{\lambda_{k\pm}\bar{\lambda}_{k'\pm}}(B\varphi_k, B\varphi_{k'})_{H_2}\Big)\\ &= \kappa_{k\pm}\kappa_{k'\pm}\Big((\varphi_k, \varphi_{k'})_{H_1} + \frac{\beta}{\lambda_{k\pm}\bar{\lambda}_{k'\pm}}a(\varphi_k, \varphi_{k'})\Big)\\ &= \kappa_{k\pm}\kappa_{k'\pm}\Big(1 + \frac{\beta\lambda_k^2}{\lambda_{k\pm}\bar{\lambda}_{k'\pm}}\Big)(\varphi_k, \varphi_{k'})_{H_1}.\end{aligned} \tag{4.12}$$

Hence if $k \neq k'$, we deduce that

$$(V_{k\pm}, V_{k'\pm})_{\mathcal{H}} = 0.$$

On the contrary if $k = k'$, we only deduce that

$$(V_{k+}, V_{k-})_{\mathcal{H}} = \kappa_{k+}\kappa_{k-}\Big(1 + \frac{\beta\lambda_k^2}{\lambda_{k+}\bar{\lambda}_{k-}}\Big),$$

that, as easily checked, is different from 0. But since V_{k+} and V_{k-} are linearly independent, we can orthonormalize them, by setting

$$\tilde{V}_{k-} = \frac{1}{\eta_k}(V_{k-} - \delta_k V_{k+}), \tag{4.13}$$

where

$$\delta_k = (V_{k+}, V_{k-})_{\mathcal{H}} = \kappa_{k+}\kappa_{k-}\Big(1 + \frac{\beta\lambda_k^2}{\lambda_{k+}\bar{\lambda}_{k-}}\Big), \tag{4.14}$$

$$\eta_k = \|V_{k-} - \delta_k V_{k+}\|_{\mathcal{H}} = \sqrt{1 - \delta_k^2}. \tag{4.15}$$

Note that for $k \in N_1$ from (4.9), we see that $\lambda_{k+}\lambda_{k-} = \beta\lambda_k^2$, and therefore

$$\delta_k = 2\kappa_{k+}\kappa_{k-}. \tag{4.16}$$

(b) For $k \in \mathbb{N}^* \setminus N_0$ and $k' \in N_0$, by the previous considerations, we clearly have

$$(V_{k\pm}, V_{k'+})_{\mathcal{H}} = 0,$$

while

$$(V_{k\pm}, V_{k'-})_{\mathcal{H}} = \frac{\kappa_{k+}\kappa_{k'-}\beta\lambda_k^2}{\lambda_{k\pm}\bar{\lambda}_{k'+}}(\varphi_k, \varphi_{k'})_{H_1} = 0.$$

(c) For $k, k' \in N_0$, with $k \neq k'$, by the previous considerations, we directly have

$$(V_{k+}, V_{k'+})_{\mathcal{H}} = (V_{k+}, V_{k'-})_{\mathcal{H}} = 0,$$

while

$$(V_{k-}, V_{k'-})_{\mathcal{H}} = \frac{\kappa_{k+}\kappa_{k'-}\beta\lambda_k^2}{\lambda_{k+}\bar{\lambda}_{k'+}}(\varphi_k, \varphi_{k'})_{H_1} = 0.$$

Obviously V_{k+} and V_{k-} are not orthogonal but as before, we can orthonormalize them by introducing $\tilde{V}_{k-}$ as in (4.13) with δ_k and η_k appropriately defined.

These three points show that the family $\{V_{k+}, \tilde{V}_{k-}\}_{k\in\mathbb{N}^*}$ is an orthonormal basis of $\mathcal{H}_0$. Consequently any $(u, \rho) \in \mathcal{H}_0$ can be written as

$$(u, \rho) = \sum_{k\in\mathbb{N}^*} (\tilde{\alpha}_k V_{k+} + \tilde{\beta}_k \tilde{V}_{k-}),$$

with $\tilde{\alpha}_k, \tilde{\beta}_k \in \mathbb{C}$ such that

$$\|(u, \rho)\|_{\mathcal{H}}^2 = \sum_{k\in\mathbb{N}^*} (|\tilde{\alpha}_k|^2 + |\tilde{\beta}_k|^2). \tag{4.17}$$

Coming back to the family $\{V_{k\pm}\}_{k\in\mathbb{N}^*}$, we have that

$$(u, \rho) = \sum_{k\in\mathbb{N}^*} (\alpha_k V_{k+} + \beta_k V_{k-}). \tag{4.18}$$

with

$$\alpha_k = \tilde{\alpha}_k - \frac{\delta_k}{\eta_k}\tilde{\beta}_k, \quad \beta_k = \frac{1}{\eta_k}\tilde{\beta}_k. \tag{4.19}$$

Hence $\{V_{k\pm}\}_{k\in\mathbb{N}^*}$ will be a Riesz basis, if we can show that

$$\|(u, \rho)\|_{\mathcal{H}}^2 \sim \sum_{k\in\mathbb{N}^*} (|\alpha_k|^2 + |\beta_k|^2). \tag{4.20}$$

According to (4.17), this equivalence holds if and only if

$$\sum_{k\in\mathbb{N}^*}(|\tilde{\alpha}_k|^2+|\tilde{\beta}_k|^2)\sim\sum_{k\in\mathbb{N}^*}(|\alpha_k|^2+|\beta_k|^2), \tag{4.21}$$

with the relations (4.19). To prove this last equivalence we split up the sum into two terms, the first one for k small and the second one for k large. Namely for K fixed later on, we show that there exist two positive constants C_1 and C_2 depending only on K such that

$$C_1\sum_{k=1}^{K}(|\tilde{\alpha}_k|^2+|\tilde{\beta}_k|^2)\le\sum_{k=1}^{K}(|\alpha_k|^2+|\beta_k|^2)\le C_2\sum_{k=1}^{K}(|\tilde{\alpha}_k|^2+|\tilde{\beta}_k|^2), \tag{4.22}$$

as well as

$$C_1(|\tilde{\alpha}_k|^2+|\tilde{\beta}_k|^2)\le|\alpha_k|^2+|\beta_k|^2\le C_2(|\tilde{\alpha}_k|^2+|\tilde{\beta}_k|^2),\forall k>K. \tag{4.23}$$

If these two equivalences hold for one fixed K, then by summing up (4.23) on $k>K$ and summing up with (4.22), we find that (4.21) is valid and the proof will be finished.

Hence it remains to prove the two equivalences above. The first equivalence is a simple consequence of the equivalence of norm in a finite dimensional space. Indeed it is clear that

$$\tilde{\alpha}_k-\frac{\delta_k}{\eta_k}\tilde{\beta}_k=\frac{1}{\eta_k}\tilde{\beta}_k=0$$

if and only if

$$\tilde{\alpha}_k=\tilde{\beta}_k=0,$$

and consequently the mid term in (4.22) is a norm in $\mathbb{C}^{2K}$.

For the equivalence (4.23), we look at the behavior of δ_k and η_k for k large. Hence without loss of generality we can assume that k is in N_1. For such a k, from (4.12), we have

$$\kappa_{k\pm}^2=\Big(1+\frac{\beta\lambda_k^2}{\lambda_{k\pm}^2}\Big)^{-1}.$$

Hence we need the asymptotic behavior of $\lambda_{k\pm}$. But from the expression (4.2), we have

$$\lambda_{k+}=\frac{-\lambda_k^2+\lambda_k\sqrt{\lambda_k^2-4\beta}}{2}=-\frac{2\beta\lambda_k^2}{\lambda_k^2+\lambda_k\sqrt{\lambda_k^2-4\beta}},$$

and therefore, one has

$$\lambda_{k+}\sim-\beta \text{ for } k \text{ large}. \tag{4.24}$$

This behavior shows that

$$\kappa_{k+}^2\sim(1+\frac{\lambda_k^2}{\beta})^{-1}\sim\frac{\beta}{\lambda_k^2} \text{ for } k \text{ large}.$$

Similarly

$$\lambda_{k-} = \frac{-\lambda_k^2 - \lambda_k\sqrt{\lambda_k^2 - 4\beta}}{2} \sim -\lambda_k^2 \text{ for } k \text{ large},$$

and therefore

$$\kappa_{k-}^2 \sim \left(1 + \frac{\beta}{\lambda_k^2}\right)^{-1} \sim 1 \text{ for } k \text{ large}.$$

These two equivalences in (4.16) yields

$$\delta_k = 2\kappa_{k+}\kappa_{k-} \sim 2\frac{\sqrt{\beta}}{\lambda_k} \text{ for } k \text{ large}.$$

By (4.15), we deduce that

$$\eta_k = \sqrt{1 - \delta_k^2} \sim 1 \text{ for } k \text{ large}.$$

Using (4.19), we can write

$$|\alpha_k|^2 + |\beta_k|^2 = \left(M_k \begin{pmatrix} \tilde{\alpha}_k \\ \tilde{\beta}_k \end{pmatrix}, \begin{pmatrix} \tilde{\alpha}_k \\ \tilde{\beta}_k \end{pmatrix}\right)_{\mathbb{C}^2},$$

where $(\cdot,\cdot)_{\mathbb{C}^2}$ means the standard inner product in $\mathbb{C}^2$ and the matrix M_k is given by

$$M_k = \begin{pmatrix} 1 & -\frac{\delta_k}{\eta_k} \\ -\frac{\delta_k}{\eta_k} & \frac{1+\delta_k^2}{\eta_k^2} \end{pmatrix}.$$

As the equivalences on δ_k and η_k imply that M_k is similar to the identity matrix for k large, we deduce that (4.23) holds for $k > K$ with K large enough. □

The Riesz basis property allows us to give an explicit decay rate of the semi-group generated by $\mathcal{A}_0$ defined by (2.9)–(2.10).

Theorem 4.4. *Under the assumption (4.1), the semi-group $(e^{t\mathcal{A}_0})_{t\geq 0}$ generated by $\mathcal{A}_0$ satisfies*

$$\|e^{t\mathcal{A}_0}\|_{\mathcal{L}(\mathcal{H}_0)} \leq Ce^{-\tau t}, \forall t \geq 0, \tag{4.25}$$

for a positive constant C and

$$\tau = \min\{\beta, \frac{\lambda_1^2}{2}\}.$$

Proof. Owing to Theorem 4.3, any $U \in \mathcal{H}_0$ can be written as

$$U = \sum_{k\in\mathbb{N}^*} (c_{k+}V_{k+} + c_{k-}V_{k-}),$$

with $c_{k+}, c_{k-} \in \mathbb{C}$ (recalling that $V_{k\pm} = \kappa_{k\pm} U_{k\pm}$) such that

$$\|U\|_{\mathcal{H}}^2 \sim \sum_{k\in\mathbb{N}^*} (|c_{k+}|^2 + |c_{k-}|^2).$$

Therefore $e^{t\mathcal{A}_0}U$ is given by

$$\begin{aligned} e^{t\mathcal{A}_0}U = &\sum_{k\in\mathbb{N}^*\setminus N_0} (c_{k+}e^{t\lambda_{k+}}V_{k+} + c_{k-}e^{t\lambda_{k-}}V_{k-}) \\ &+ \sum_{k\in N_0} e^{t\lambda_{k+}}((c_{k+} + \frac{\kappa_{k-}}{\kappa_{k+}}c_{k-}t)V_{k+} + c_{k-}V_{k-}). \end{aligned}$$

Theorem 4.3 then yields

$$\begin{aligned} \|e^{t\mathcal{A}_0}U\|_{\mathcal{H}}^2 \sim &\sum_{k\in\mathbb{N}^*\setminus N_0} (|c_{k+}|^2 e^{2t\Re\lambda_{k+}} + |c_{k-}|^2 e^{2t\Re\lambda_{k-}}) \\ &+ \sum_{k\in N_0} e^{2t\lambda_{k+}}(|c_{k+} + \frac{\kappa_{k-}}{\kappa_{k+}}c_{k-}t|^2 + |c_{k-}|^2). \end{aligned}$$

The conclusion will follow if we can show that

$$\Re\lambda_{k\pm} \le -\tau, \forall k \in \mathbb{N}^* \setminus N_0, \tag{4.26}$$

and

$$\lambda_{k+} < -\tau, \forall k \in N_0. \tag{4.27}$$

But the second situation occurs if and only if there exists $k \in \mathbb{N}^*$ such that $\lambda_k^2 = 4\beta$ and in that case

$$\lambda_{k+} = -\frac{\lambda_k^2}{2} = -2\beta < -\beta \le -\tau.$$

In the first situation, we distinguish two cases:
1. If $k \in \mathbb{N}^* \setminus (N_0 \cup N_1)$, then this is equivalent to the condition that $\lambda_k^2 < 4\beta$ and in such a case, we have

$$\Re\lambda_{k\pm} = -\frac{\lambda_k^2}{2} \le -\frac{\lambda_1^2}{2} \le -\tau.$$

2. If $k \in N_1$ (equivalent to $\lambda_k^2 > 4\beta$), then clearly

$$\lambda_{k-} < -\frac{\lambda_k^2}{2} \le -\frac{\lambda_1^2}{2} \le -\tau.$$

On the other hand, it is easy to check that λ_{k+} is a non decreasing function of k, and therefore, by (4.24), we have

$$\lambda_{k+} \le \lim_{\ell\to\infty} \lambda_{\ell+} = -\beta \le -\tau.$$

as requested. □

In conclusion if we define the decay rate of (2.3) as (see for instance [15], p. 214)

$$\omega_{\text{opt}} = \inf\{\omega : \exists C(\omega) > 0 \text{ s. t. } E(t) \leq C(\omega)e^{2\omega t}E(0), \forall t \geq 0 \text{ and every finite energy sol. of (2.3)}\},$$

under the assumption (4.1), we have just shown that the decay rate is equal to the spectral abscissa of $\mathcal{A}$ defined by

$$\sup\{\Re\lambda : \lambda \in \sigma(\mathcal{A})\}$$

that is here equal to $-\tau$, in other words, one has

$$\omega_{\text{opt}} = -\tau.$$

5. Boundary stabilizability of the linearized viscous Saint-Venant system on trees

If we come back to the linearized viscous Saint-Venant system (3.2) on a tree Γ, according to Remark (3.7), the assumption (4.1) holds with $\beta = \frac{a}{\nu}$. Consequently the decay of the semigroup generated by $\mathcal{A}_0$ is given by $-\tau$ with

$$\tau = \min\{\frac{a}{\nu}, \frac{\lambda_1^2}{2}\},$$

where λ_1^2 is the first eigenvalue of the Laplace operator (up to the factor ν) on Γ with boundary and transmission conditions associated with V (compare with Cor. 1 of [5] in the case of an interval of length π).

As in [5], we are here interested in the advection dominating case, that here corresponds to the case when $\frac{\lambda_1^2}{2} < \frac{a}{\nu}$. In that case, the decay rate of the energy norm of the solution of (3.2) is $-\frac{\lambda_1^2}{2}$ and is then slow due to the advection modes (corresponding to the complex eigenvalues of $\mathcal{A}$ with a real part smaller than $\frac{a}{\nu}$). Inspired from [5], our goal is to restore the optimal decay $-\frac{a}{\nu}$ for any tree by building a Dirichlet control at all except one extremities of the tree. This method allows to obtain an exponential decay rate $-\sigma$ with σ as close as we want from $\frac{a}{\nu}$ and with an arbitrary initial datum in the energy space.

From now on we then fix a C^2-network Γ that is a tree and fix $V_{\text{ext}}^{\text{Diss}}$ a subset of V_{ext} made of all vertices of V_{ext}, except one, say r (the root of the tree), namely $V_{\text{ext}}^{\text{Diss}} = V_{\text{ext}} \setminus \{r\}$. As in Section 3.2, we suppose given two positive constants a and ν corresponding to the advection and viscosity coefficients respectively. Then we consider the non-homogeneous problem

$$\begin{cases} u_{jt} - \nu u_j'' + a\rho_j' = 0 \text{ in } Q_j, \forall j = 1, \dots, N, \\ \rho_{jt} + u_j' = 0 \text{ in } Q_j, \forall j = 1, \dots, N, \\ \sum_{j \in J_v} u_j(v,t)\nu_j(v) = 0, \forall v \in V_{\text{int}}, t > 0, \\ \nu u_j'(v,t) - a\rho_j(v,t) = \nu u_k'(v,t) - a\rho_k(v,t), \forall v \in V_{\text{int}}, j, k \in J_v, t > 0, \\ u_{j_r}(r,t) = 0, \forall v \in V_{\text{ext}}, t > 0, \\ u_{j_v}(v,t) = q_v(t), \forall v \in V_{\text{ext}}^{\text{Diss}}, t > 0, \\ u(\cdot,0) = u_0, \rho(\cdot,0) = \rho_0 \text{ in } \Gamma, \end{cases} \tag{5.1}$$

where (u_0, ρ_0) belongs to (compare with Lem. 3.3)

$$D = \{(u,\rho) \in H_r^1(\Gamma) \times L^2(\Gamma) : -\nu u' + a\rho \in H^1(\Gamma)\},$$

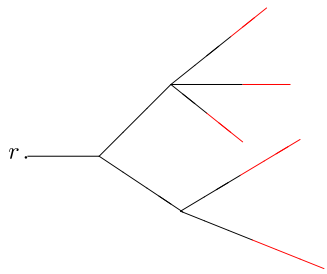

FIGURE 1. An original tree Γ and its extension $\tilde{\Gamma}$.

with

$$H_r^1(\Gamma) = \{v \in H^1(\Gamma) : v(r) = 0\},$$

that is a Hilbert space with the norm (3.5), and for all $v \in V_{\text{ext}}^{\text{Diss}}$, $q_v \in C([0,\infty))$ satisfies the compatibility condition

$$q_v(0) = u_0(v), \forall v \in V_{\text{ext}}^{\text{Diss}}. \tag{5.2}$$

The existence and decay of the solution to this problem will be given below by using an extension method (as in the proof of Thm. 4.1 of [5]). But before, let us introduce some notations and useful properties.

First, for all $v \in V_{\text{ext}}^{\text{Diss}}$, we extend the edge e_{j_v} of Γ having v as extremity into a longer edge $\tilde{e}_{j_v}$ (of length $\tilde{l}_{i_v}$ strictly larger than l_{j_v}). Denote by $\tilde{\Gamma}$ the new tree obtained from Γ by simply replacing the edges e_{j_v}, with $v \in V_{\text{ext}}^{\text{Diss}}$, by $\tilde{e}_{j_v}$, see Fig. 1, where Γ is in black and the extension in red. Denote by $\tilde{V}_{\text{ext}}$ the set of exterior vertices of $\tilde{\Gamma}$ (note that $\tilde{\Gamma}$ has the same set V_{int} of interior vertices as Γ). On this new tree $\tilde{\Gamma}$, we now denote by $\tilde{\lambda}_k^2$, $k \in \mathbb{N}^*$, the eigenvalue of the operator $\tilde{A}$ (that corresponds to the operator A but defined in $\tilde{\Gamma}$) of associated eigenfunctions $\tilde{\varphi}_k$, $\tilde{\lambda}_{k\pm}$ the associated eigenvalues of $\tilde{\mathcal{A}}$ (that again corresponds to the operator $\mathcal{A}$ but defined in $\tilde{\Gamma}$).

The energy space associated with (3.2) set on $\tilde{\Gamma}$ is

$$\tilde{\mathcal{H}} = L^2(\tilde{\Gamma}) \times L_0^2(\tilde{\Gamma}),$$

equipped with the inner product

$$((y,\rho),(y_1,\rho_1))_{\tilde{\mathcal{H}}} = \int_{\tilde{\Gamma}} (y\bar{y}_1 + a\rho\bar{\rho}_1)\,\mathrm{d}x, \forall (y,\rho),(y_1,\rho_1) \in \tilde{\mathcal{H}}.$$

We first build an extension operator from Γ to $\tilde{\Gamma}$.

Lemma 5.1. *There exists a continuous operator E_1 from $\mathcal{H}$ to $\tilde{\mathcal{H}}$ such that*

$$E_1 Y = Y \text{ in } \Gamma, \forall Y \in \mathcal{H},$$

and that is also continuous from D to $D(\tilde{\mathcal{A}})$, i.e., *satisfying*

$$\|E_1 Y\|_{D(\tilde{\mathcal{A}})} \lesssim \|Y\|_D, \forall Y \in D. \tag{5.3}$$

Proof. With loss of generality, for all $v \in V_{\text{ext}}^{\text{Diss}}$, we can assume that the parametrization of the edge e_{j_v} is such that v corresponds to l_{j_v} and that the edge $\tilde{e}_{j_v}$ is parametrized by $(0, \tilde{l}_{j_v})$. Fix a function $\eta \in H^1(\Gamma)$ (that plays the rule of a cut-off function) such that $\eta_j \in C^\infty([0, l_j])$, for all $j = 1, \dots, N$ and satisfying, for all $v \in V_{\text{ext}}^{\text{Diss}}$,

$$\eta_{j_v}(x) = 1, \forall x \in [l_{j_v} - \varepsilon, l_{j_v}],$$

and

$$\eta_{j_v}(x) = 0, \forall x \in [0, l_{j_v} - 2\varepsilon],$$

with $\varepsilon > 0$ fixed small enough so that

$$0 < l_{j_v} - 4\varepsilon < l_{j_v} + 4\varepsilon < \tilde{l}_{j_v}.$$

Furthermore for all $v \in V_{\text{ext}}^{\text{Diss}}$, we fix a cut-off function $\chi_{j_v} \in \mathcal{D}(l_{j_v}, l_{j_v} + 4\varepsilon)$ such that

$$\int_{l_{j_v}}^{l_{j_v}+4\varepsilon} \chi_{j_v}(x)\,\mathrm{d}x = 1.$$

For $Y = (u, \rho) \in \mathcal{H}$, we take $E_1 Y = Y$ on Γ and for all $v \in V_{\text{ext}}^{\text{Diss}}$, we set

$$\begin{aligned}
(E_1 Y)_{j_v}(x) &= -3\eta_{j_v}(2l_{j_v} - x) Y_{j_v}(2l_{j_v} - x) + 4\eta_{j_v}(\frac{3l_{j_v} - x}{2}) Y_{j_v}(\frac{3l_{j_v} - x}{2}) \\
&\quad - \Big(\int_0^{l_{j_v}} \eta_{j_v}(t) Y_{j_v}(t)\,\mathrm{d}t\Big)\chi_{j_v}(x),\ \forall x \in (l_{j_v}, l_{j_v} + 4\varepsilon], \\
(E_1 Y)_{j_v}(x) &= (0, 0),\ \forall x \in (l_{j_v} + 4\varepsilon, \tilde{l}_{j_v}].
\end{aligned}$$

The continuity property of E_1 from $\mathcal{H}$ to $\tilde{\mathcal{H}}$ follows by simple changes of variable, while its continuity property from D to $D(\tilde{\mathcal{A}})$ is a simple consequence of Leibniz' rule and again simple changes of variable. □

We go on with a multiplicative result.

Lemma 5.2. *Fix a function $\varphi \in H^1(\tilde{\Gamma})$ such that $\varphi = 0$ on Γ, $\varphi > 0$ on $\tilde{\Gamma} \setminus \bar{\Gamma}$ and $\varphi_{j_v} \in C^2([0, \tilde{l}_{j_v}])$, for all $v \in V_{\text{ext}}^{\text{Diss}}$. Then for any $Y \in D(\tilde{\mathcal{A}})$, φY also belongs to $D(\tilde{\mathcal{A}})$ and*

$$\|\varphi Y\|_{D(\tilde{\mathcal{A}})} \lesssim \|Y\|_{D(\tilde{\mathcal{A}})}.$$

Proof. Simple consequence of Leibniz' rule. □

For any $\sigma > 0$, we finally introduce the finite set

$$\tilde{N}_\sigma = \{k \in \mathbb{N}^* : \frac{\tilde{\lambda}_k^2}{2} < \sigma\}.$$

We finally need the next technical lemmas.

Lemma 5.3. *For any $\sigma > 0$, the eigenfunctions $\tilde{\varphi}_k, k \in \tilde{N}_\sigma$, are linearly independent on $\tilde{\Gamma} \setminus \bar{\Gamma}$.*

Proof. Assume that there exist coefficients c_k such that

$$\sum_{k\in\tilde{N}_\sigma} c_k\tilde{\varphi}_k = 0 \text{ on } \tilde{\Gamma}\setminus\bar{\Gamma}. \tag{5.4}$$

Then by grouping the sum by packet corresponding to the same eigenvalue, we can write

$$\psi = \sum_{k\in\tilde{N}_\sigma} c_k\tilde{\varphi}_k,$$

as

$$\psi = \sum_{k'\in K} \psi_{k'}, \tag{5.5}$$

where K is a subset of N_σ such that $\lambda_k \neq \lambda_{k'}$ for all $k, k' \in K$ such that $k \neq k'$ and

$$\psi_{k'} = \sum_{k\in\tilde{N}_\sigma:\lambda_k=\lambda_{k'}} c_k\tilde{\varphi}_k.$$

Now for all $v \in V_{\text{ext}}^{\text{Diss}}$, we notice that $\psi_{k'}$ can be written as

$$\psi_{k'}(x) = \alpha_{k'}\cos(\frac{\lambda_{k'}}{\nu}x) + \beta_{k'}\sin(\frac{\lambda_{k'}}{\nu}x) \text{ on } (0, l_{j_v}), \tag{5.6}$$

with $\alpha_{k'}, \beta_{k'} \in \mathbb{C}$. With these notations, (5.4) is equivalent to

$$\psi = 0 \text{ on } \tilde{\Gamma}\setminus\bar{\Gamma}.$$

Since the functions $\cos(\frac{\lambda_{k'}}{\nu}\cdot)$ and $\sin(\frac{\lambda_{k'}}{\nu}\cdot)$, with $k' \in K$, are linearly independent (because the λ_k are two by two disjoint), we deduce that

$$\alpha_{k'} = \beta_{k'} = 0, \forall k' \in K.$$

Going back to (5.6), this means that for any $k' \in K$, one has

$$\psi_{k'} = 0 \text{ on } e_{j_v}, \forall v \in V_{\text{ext}}^{\text{Diss}}. \tag{5.7}$$

We now show from generation to generation that this implies that $\psi_{k'}$ is zero on $\tilde{\Gamma}$, which automatically implies that $c_k = 0$, for all k. Indeed we first notice that

$$\psi''_{k'} - \frac{\lambda_{k'}^2}{\nu}\psi_{k'} = 0 \text{ on } e_j, \forall j \in \{1, \dots, N\}.$$

Then restricting this identity to an edge j of the last but one generation, as for the interior vertex v in common with the last generation, (5.7) and the transmission conditions

$$\sum_{j\in J_v} \psi_j(v)\nu_j(v) = 0, \forall v \in V_{\text{int}},$$
$$\psi'_j(v) = \psi'_k(v), \forall v \in V_{\text{int}}, j, k \in J_v,$$

imply that

$$\psi_{k'}(v) = \psi'_{k'}(v) = 0,$$

by Holmgren uniqueness theorem, we deduce that

$$\psi_{k'} = 0 \text{ on } e_j.$$

In other words, $\psi_{k'} = 0$ on all edges of the last but one generation and by iteration, we conclude. □

Lemma 5.4. *Let $\sigma \in (0, \frac{2a}{\sigma}]$ and φ be the function fixed in Lemma 5.2. Then the functions $\sqrt{\varphi}\tilde{U}_{k\pm}, k \in \tilde{N}_\sigma$, and $\sqrt{\varphi}(0,1)$, are linearly independent on $\tilde{\Gamma} \setminus \bar{\Gamma}$.*

Proof. Since $\sqrt{\varphi}$ is different from zero on $\tilde{\Gamma} \setminus \bar{\Gamma}$, the requested property is equivalent to the linear independence of $\tilde{U}_{k\pm}, k \in N_\sigma$, and $(0,1)$. Now assume that there exist coefficients $c_{k\pm}$ and d such that

$$\sum_{k \in \tilde{N}_\sigma} (c_{k+}\tilde{U}_{k+} + c_{k-}\tilde{U}_{k-}) + d(0,1) = 0 \text{ on } \tilde{\Gamma} \setminus \bar{\Gamma}. \tag{5.8}$$

Then taking the first component of this identity we find

$$\sum_{k \in \tilde{N}_\sigma} (c_{k+} + c_{k-})\tilde{\varphi}_k = 0 \text{ on } \tilde{\Gamma} \setminus \bar{\Gamma}.$$

Lemma 5.3 leads to

$$c_{k+} + c_{k-} = 0, \forall k \in \tilde{N}_\sigma. \tag{5.9}$$

Restricting the identity (5.8) to the second component we find

$$-\sum_{k \in \tilde{N}_\sigma} (\frac{c_{k+}}{\tilde{\lambda}_{k+}} + \frac{c_{k-}}{\tilde{\lambda}_{k-}})\tilde{\varphi}'_k + d = 0 \text{ on } \tilde{\Gamma} \setminus \bar{\Gamma}.$$

By deriving in space this identity we find that

$$-\sum_{k \in \tilde{N}_\sigma} (\frac{c_{k+}}{\tilde{\lambda}_{k+}} + \frac{c_{k-}}{\tilde{\lambda}_{k-}})\tilde{\varphi}''_k = 0 \text{ on } \tilde{\Gamma} \setminus \bar{\Gamma},$$

or equivalently

$$\sum_{k \in \tilde{N}_\sigma} (\frac{c_{k+}}{\tilde{\lambda}_{k+}} + \frac{c_{k-}}{\tilde{\lambda}_{k-}})\lambda_k^2\tilde{\varphi}_k = 0 \text{ on } \tilde{\Gamma} \setminus \bar{\Gamma}.$$

Again using Lemma 5.3 we find

$$\frac{c_{k+}}{\tilde{\lambda}_{k+}} + \frac{c_{k-}}{\tilde{\lambda}_{k-}} = 0, \forall k \in \tilde{N}_\sigma. \tag{5.10}$$

Combining this property with (5.9), we find that

$$c_{k+} = c_{k-} = 0, \forall k \in \tilde{N}_\sigma, \tag{5.11}$$

since $\tilde{\lambda}_{k+} \neq \tilde{\lambda}_{k-}$ (recalling that $\tilde{\lambda}_k^2 < 2\sigma \leq \frac{4a}{\nu}$ and using Thm. 4.1). Coming back to (5.8), we are reduced to

$$d(0,1) = 0 \text{ on } \tilde{\Gamma} \setminus \bar{\Gamma},$$

and we directly conclude that $d = 0$. □

We are ready to state our stability result (compare with Thm. 4.1 of [5] in the case of an interval).

Theorem 5.5. *Assume that $\frac{\lambda_1^2}{2} < \frac{a}{\nu}$ and let σ be arbitrarily in $(0, \frac{a}{\nu})$. Then there exists a positive constant C such that for all initial datum $(u_0, \rho_0) \in D$, there exist controls $q_v \in C([0,\infty))$, $v \in V_{\text{ext}}^{\text{Diss}}$, satisfying (5.2) for which problem (5.1) has a unique solution $(u, \rho) \in C([0,\infty), D) \cap C^1([0,\infty), \mathcal{H})$ that satisfies*

$$\|(u(t), \rho(t))\|_{L^2(\Gamma)^2} \leq Ce^{-\sigma t}\|(u_0, \rho_0)\|_{L^2(\Gamma)^2}, \forall t > 0. \tag{5.12}$$

Proof. As in the proof of Theorem 4.1 of [5], we use an extension method. The trick consists in building an appropriated extension EY_0 to $\tilde{\Gamma}$ of the initial datum $Y_0 = (u_0, \rho_0) \in D$ so that the new advection modes are cancelled. The difference with [5] is that our construction also allows to cancel the orthogonality conditions on the initial data.

With the help of Lemma 5.1, we consider

$$EY_0 = E_1Y_0 + E_2Y_0,$$

with $E_2Y_0 \in D(\tilde{\mathcal{A}})$ defined in such a way that

$$(EY_0, (0,1))_{\tilde{\mathcal{H}}} = 0, \tag{5.13}$$

$$(EY_0, \tilde{U}_{k\pm})_{\tilde{\mathcal{H}}} = 0, \forall k \in \tilde{N}_\sigma. \tag{5.14}$$

Hence we look for E_2Y_0 in the form[2]

$$E_2Y_0 = \varphi\Big(\sum_{k\in\tilde{N}_\sigma} (c_{k+}\tilde{U}_{k+} + c_{k-}\tilde{U}_{k-}) + d(0,1)\Big), \tag{5.15}$$

where φ is the function fixed in Lemma 5.2 (that allows to conclude that E_2Y_0 belongs to $D(\tilde{\mathcal{A}})$) and the coefficients $c_{k\pm}$ and d are fixed in order to satisfy (5.13) and (5.14). Indeed these two conditions are equivalent to

$$(E_2Y_0, (0,1))_{\tilde{\mathcal{H}}} = -(E_1Y_0, (0,1))_{\tilde{\mathcal{H}}},$$
$$(E_2Y_0, \tilde{U}_{k\pm})_{\tilde{\mathcal{H}}} = -(E_1Y_0, \tilde{U}_{k\pm})_{\tilde{\mathcal{H}}}, \forall k \in \tilde{N}_\sigma.$$

From the expression (5.15) of E_2Y_0, this system is equivalent to a square linear system

$$MX = C,$$

[2]Let us notice that the assumption $\frac{\lambda_1^2}{2} < \frac{a}{\nu}$ implies that $\tilde{N}_\sigma$ is non empty if σ is sufficiently close to $\frac{a}{\nu}$. Indeed as $\tilde{\Gamma}$ is larger than Γ, the space V defined on Γ can be viewed as a closed subspace of the space V defined on $\tilde{\Gamma}$ and by Rayleigh quotient techniques, one has $\tilde{\lambda}_1 \leq \lambda_1$.

where the unknown vector X is equal to $\big((c_{k+})_{k\in\tilde{N}_\sigma},(c_{k-})_{k\in\tilde{N}_\sigma},d\big)$ and the coefficients of the matrix M are simply the inner product in $L^2(\tilde{\Gamma}\setminus\bar{\Gamma})^2$ of the functions $\sqrt{\varphi}\tilde{U}_{k\pm}, k\in\tilde{N}_\sigma$ and $\sqrt{\varphi}(0,1)$. Since, by Lemma 5.4, these functions are linearly independent on $\tilde{\Gamma}\setminus\bar{\Gamma}$, the matrix M is invertible, which guarantees the existence and uniqueness of the coefficients $c_{k\pm}$ and d.

Once $EY_0=(\tilde{u}_0,\tilde{\rho}_0)\in D(\tilde{\mathcal{A}})$ satisfying (5.13) and (5.14) is given, we deduce that the next problem in $\tilde{\Gamma}$

$$\begin{cases} \tilde{u}_{jt}-\nu\tilde{u}_j''+a\tilde{\rho}_j'=0 \text{ in } \tilde{Q}_j:=(0,\tilde{l}_j)\times(0,\infty),\forall j=1,\dots,N,\\ \tilde{\rho}_{jt}+\tilde{u}_j'=0 \text{ in } \tilde{Q}_j,\forall j=1,\dots,N,\\ \tilde{u}_j(v,t)=\tilde{u}_k(v,t),\forall v\in V_{\text{int}},j,k\in J_v,t>0,\\ \sum_{j\in J_v}(\nu\tilde{u}_j'-a\tilde{\rho}_j)(v,t)\nu_j(v)=0,\forall v\in V_{\text{int}},t>0,\\ \tilde{u}_{j_v}(v,t)=0,\forall v\in\tilde{V}_{\text{ext}},t>0,\\ \tilde{u}(\cdot,0)=\tilde{u}_0,\rho(\cdot,0)=\tilde{\rho}_0 \text{ in } \tilde{\Gamma}, \end{cases} \tag{5.16}$$

has a unique solution $(\tilde{u},\tilde{\rho})\in C([0,\infty),D(\tilde{\mathcal{A}})\cap C([0,\infty),L^2(\tilde{\Gamma})^2)$. Furthermore the arguments of the proof of Theorem 4.4 lead to the estimate

$$\|(\tilde{u}(t),\tilde{\rho}(t))\|_{L^2(\tilde{\Gamma})^2}\le Ce^{-\sigma t}\|EY_0\|_{L^2(\tilde{\Gamma})^2},\forall t>0, \tag{5.17}$$

for some positive constant C (independent of EY_0). Hence the restriction (u,ρ) of $(\tilde{u},\tilde{\rho})$ to Γ clearly belongs to $C([0,\infty),D)\cap C^1([0,\infty),\mathcal{H})$ and satisfies (5.1) with

$$q_v(t)=\tilde{u}_{j_v}(v,t),\forall v\in V_{\text{ext}}^{\text{Diss}},t>0,$$

that by the Sobolev embedding theorem belongs to $C([0,\infty))$. Finally the estimate (5.12) is a simple consequence of (5.17) and of the property

$$\|EY_0\|_{L^2(\tilde{\Gamma})^2}\lesssim\|Y_0\|_{L^2(\Gamma)^2},$$

that follows from the continuity of E_1 from $\mathcal{H}$ to $\tilde{\mathcal{H}}$ and the definition of E_2 (and Lem. 5.2). □

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