ESAIM: COCV 25 (2019) 54
https://doi.org/10.1051/cocv/2019060

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

ERRATUM TO: A VARIATIONAL APPROACH TO A STATIONARY FREE BOUNDARY PROBLEM MODELING MEMS

Philippe Laurençot[1,*] and Christoph Walker[2]

Abstract. An incomplete argument in the proof of Theorem 3.4 from Ph. Laurençot and Ch. Walker [*ESAIM: COCV* **22** (2016) 417–438] is corrected.

Mathematics Subject Classification. 35J35, 35R35, 35Q74.

Received July 25, 2019. Accepted September 26, 2019.

We noticed a gap in the proof of Theorem 3.4 from [2] and the aim of this erratum is to provide a complete argument. Specifically, in Theorem 3.4 from [2], we derive the Euler-Lagrange equation satisfied by a minimizer u of the functional

$$\mathcal{E}_m(u) := \frac{\beta}{2}\|\partial_x^2 u\|_{L_2(I)}^2 + \frac{1}{2}\left(\tau + \frac{a}{2}\|\partial_x u\|_{L_2(I)}^2\right)\|\partial_x u\|_{L_2(I)}^2$$

on the set

$$\mathcal{A}_\rho := \{u \in H_D^2(I) \ : \ u \ \text{ is even with } \ -1 < u \le 0 \text{ and } \mathcal{E}_e(u) = \rho\},$$

where $I := (-1,1)$, $\rho \in (2,\infty)$, $H_D^2(I) := \{u \in H^2(I) \ : \ u(\pm 1) = \partial_x u(\pm 1) = 0\}$, and $\mathcal{E}_e$ is a non-negative nonlinear and nonlocal functional of u. The computation in [2] of the Euler-Lagrange equation, see equation (3.10) from [2], relies implicitly on the property that minimizers lie in the interior of $\mathcal{A}_\rho$, a property which is, however, not known *a priori*. Although knowing that minimizers are strictly greater than -1, it is actually not known whether minimizers are negative (even though this property can be shown *a posteriori*, which was the main reason to include it in the definition of $\mathcal{A}_\rho$). This issue can be remedied by changing slightly the admissible set $\mathcal{A}_\rho$ on which the functional $\mathcal{E}_m$ is minimized. In fact, the non-positivity assumption in $\mathcal{A}_\rho$ is not needed and our analysis works equally well in the set

$$\mathscr{A}_\rho := \{u \in H_D^2(I) \ : \ u \ \text{ is even with } \ -1 < u \text{ and } \ \mathcal{E}_e(u) = \rho\}. \tag{1}$$

Keywords and phrases: MEMS, stationary solution, constrained minimization.

[1] Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, 31062 Toulouse Cedex 9, France.
[2] Leibniz Universität Hannover, Institut für Angewandte Mathematik, Welfengarten 1, 30167 Hannover, Germany.

[*] Corresponding author: laurenco@math.univ-toulouse.fr

To be more precise, several results in [2] were derived for non-positive functions in

$$\mathcal{K}^s := \{u \in H^s_D(I) \,:\, -1 < u \le 0 \text{ on } I\}\,, \quad s \ge 1\,,$$

an assumption which is not required, as it suffices to work in

$$S^s := \{u \in H^s_D(I) \,:\, -1 < u \text{ on } I\}\,, \quad s \ge 1\,.$$

For $u \in S^1$, one shall then rather define the function b_u in equation (2.1) from [2] as

$$b_u(x,z) := \begin{cases} \dfrac{1+z}{1+u(x)} & \text{for} \quad (x,z) \in \overline{\Omega(u)}\,, \\ 1 & \text{for} \quad (x,z) \in \overline{\Omega(M_u)} \setminus \overline{\Omega(u)}\,, \end{cases}$$

where $\Omega(M_u) := I \times (-1, M_u + 1)$ with $M_u := \max\{0, \sup_I u\}$. Note that b_u belongs to $H^1(\Omega(M_u)) \cap C(\overline{\Omega(M_u)})$, which allows one to redefine $B_u \in H^{-1}(\Omega(M_u))$ (i.e. the dual space of $H^1_D(\Omega(M_u))$) in equation (2.2) from [2] by

$$\langle B_u, \vartheta \rangle := -\int_{\Omega(M_u)} \left[\varepsilon^2 \partial_x b_u \partial_x \vartheta + \partial_z b_u \partial_z \vartheta\right]\, \mathrm{d}(x,z)\,, \quad \vartheta \in H^1_D(\Omega(M_u))\,.$$

Then Lemmas 2.1 and 2.2 from [2] remain true for $u \in S^1$ (instead of $u \in \mathcal{K}^1$) and Proposition 2.3 from [2] is actually valid for $u \in S^{2-\alpha}$ (instead of $u \in \mathcal{K}^{2-\alpha}$) when replacing equation (2.5) from [2] by

$$\frac{1+z}{1+M_u} \;\le\; \psi_u(x,z) \le 1\,, \quad (x,z) \in \Omega(u)\,.$$

Moreover, Propositions 2.6 and 2.7 from [2] are also true when replacing $\mathcal{K}^1$ by S^1. For later use, we note that Proposition 2.6 from [2] implies

$$\mathcal{E}_e(u) \le \mathcal{E}_e(0) = 2 \quad \text{for} \quad u \in S^1 \quad \text{with} \quad u \ge 0 \;\text{ in }\; I\,. \tag{2}$$

Also Lemma 2.8 from [2] remains true for $u \in S^1$ (instead of $u \in \mathcal{K}^1$), except that the lower bound on $\mathcal{E}_e(u)$ has to be replaced by

$$\mathcal{E}_e(u) \ge \int_{-1}^{1} \frac{\mathrm{d}x}{1+u(x)} \ge \frac{2}{1+M_u}\,.$$

All other statements of Section 2 from [2] are not affected by these changes.

The minimization of $\mathcal{E}_m$ in Section 3 from [2] is now performed on the set $\mathscr{A}_\rho$ defined in (1) for a given $\rho \in (2,\infty)$. The statement of Proposition 3.1 from [2] remains true, as it is easily checked that its proof only relies on the continuity of the map $t \mapsto \mathcal{E}_e(tv)$ for $v \in \mathscr{A}_\rho$ established in Proposition 2.7 from [2], but not on its monotonicity (which is only true when v is non-negative). Next, neither Proposition 3.2 from [2], nor Lemma 3.3 from [2] are affected by the change of $\mathcal{A}_\rho$ to $\mathscr{A}_\rho$. Therefore, in the proof of Theorem 3.4 from [2] we can use the same arguments to derive that, if $u \in \mathscr{A}_\rho$ is an arbitrary minimizer of $\mathcal{E}_m$ on $\mathscr{A}_\rho$, then $u \in H^4(D) \cap H^2_D(I)$, and there is a Lagrange multiplier $\lambda_u \in \mathbb{R}$ such that

$$\beta \partial_x^4 u - \big(\tau + a\|\partial_x u\|^2_{L_2(I)}\big)\partial_x^2 u = -\lambda_u g(u)\,, \qquad x \in I\,, \tag{3}$$

where $g(u) := \partial_u \mathcal{E}_e(u)$ is a non-negative functional of u, which belongs to $L_2(I)$. At this stage, since the non-positivity of u is not yet guaranteed, we need to employ a slightly different argument than in [2]. Indeed, we first assume for contradiction that $\lambda_u \le 0$. Then $-\lambda_u g(u)$ is non-negative and it follows of (3) and Theorem 1.1 from [1] that $u > 0$ in I. Hence $\rho = \mathcal{E}_e(u) \le \mathcal{E}_e(0) = 2$ by (2), contradicting $\rho \in (2,\infty)$. Consequently, $\lambda_u > 0$ and $-\lambda_u g(u)$ is negative, so that we infer of (3) and Theorem 1.1 from [1] that $u < 0$ in I. The remaining arguments in the proof of Theorem 3.4 from [2] are then the same.

Summarizing, the statement of Theorem 3.4 from [2] is correct, once $\mathcal{A}_\rho$ is replaced by $\mathscr{A}_\rho$. Thanks to the above analysis, Theorem 3.4 from [2] may be supplemented with the following result:

Corollary 1. *Consider $\rho \in (2,\infty)$ and let $u \in \mathscr{A}_\rho$ be an arbitrary minimizer of $\mathcal{E}_m$ in $\mathscr{A}_\rho$. Then $u < 0$ in I and $u \in \mathcal{A}_\rho$. In addition,*

$$\mathcal{E}_m(u) = \min_{v \in \mathscr{A}_\rho} \mathcal{E}_m(v) = \min_{v \in \mathcal{A}_\rho} \mathcal{E}_m(v)\ .$$

References

[1] Ph. Laurençot and Ch. Walker, Sign-preserving property for some fourth-order elliptic operators in one dimension or in radial symmetry. *J. Anal. Math.* **127** (2015) 69–89.

[2] Ph. Laurençot and Ch. Walker, A variational approach to a stationary free boundary problem modeling MEMS. *ESAIM: COCV* **22** (2016) 417–438.