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EXISTENCE OF OPTIMAL SHAPES UNDER A UNIFORM BALL CONDITION FOR GEOMETRIC FUNCTIONALS INVOLVING BOUNDARY VALUE PROBLEMS

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Abstract. In this article, we are interested in shape optimization problems where the functional is defined on the boundary of the domain, involving the geometry of the associated hypersurface (normal vector $\mathbf{n}$, scalar mean curvature H) and the boundary values of the solution u_Ω related to the Laplacian posed on the inner domain Ω enclosed by the shape. For this purpose, given $\varepsilon > 0$ and a large hold-all $B \subset \mathbb{R}^n$, $n \geqslant 2$, we consider the class $\mathcal{O}_\varepsilon(B)$ of admissible shapes $\Omega \subset B$ satisfying an ε-ball condition. The main contribution of this paper is to prove the existence of a minimizer in this class for problems of the form $\inf_{\Omega \in \mathcal{O}_\varepsilon(B)} \int_{\partial\Omega} j[u_\Omega(\mathbf{x}), \nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})]\, \mathrm{d}A(\mathbf{x})$. We assume the continuity of j in the set of variables, convexity in the last variable, and quadratic growth for the first two variables. Then, we give various applications such as existence results for the configuration of fluid membranes or vesicles, the optimization of wing profiles, and the inverse obstacle problem.

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1. Introduction

In mathematical engineering, many practical applications are modelled by minimization processes. It often happens that the quantity of interest is given by the shape of some optimal domains. Hence, a first natural question arises from this setting: is our problem well posed *i.e.* does such a design exist? To answer this question, it is therefore necessary to study the existence of minimizers to the following kind of shape optimization problems:

$$\inf_{\Omega \in \mathcal{A}} J(\Omega), \tag{1.1}$$

where $J : \Omega \mapsto J(\Omega)$ is a real-valued functional defined over a set $\mathcal{A}$ of admissible shapes $\Omega \subseteq \mathbb{R}^n$, that may also include some additional constraints. In the theory of partial differential equations (PDE), a typical range

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of functionals is given by:

$$J(\Omega) = \int_{\Omega} j\left[\mathbf{x}, u_{\Omega}(\mathbf{x}), \nabla u_{\Omega}(\mathbf{x})\right] \mathrm{d}V(\mathbf{x}), \tag{1.2}$$

where the integration on Ω is done with respect to the n-dimensional Lebesgue measure $V(\bullet)$, and where $u_{\Omega} : \mathbf{x} \mapsto u_{\Omega}(\mathbf{x})$ is the solution of a PDE posed over the domain Ω. For example, the map u_{Ω} can refer to the solution of the Dirichlet Laplacian posed on a bounded domain Ω:

$$-\Delta u_{\Omega} = f \ \text{ in } \Omega, \qquad\qquad u_{\Omega} = g \ \text{ on } \partial\Omega; \tag{1.3}$$

and similarly, to the one associated with a Neumann boundary condition

$$-\Delta u_{\Omega} + \lambda u_{\Omega} = f \ \text{ in } \Omega, \qquad\qquad \partial_{\mathbf{n}} u_{\Omega} = g \ \text{ on } \partial\Omega; \tag{1.4}$$

or a Robin boundary condition

$$-\Delta u_{\Omega} = f \ \text{ in } \Omega, \qquad\qquad \partial_{\mathbf{n}} u_{\Omega} + \lambda u_{\Omega} = g \ \text{ on } \partial\Omega, \tag{1.5}$$

where $\lambda > 0$ and $f, g : \mathbb{R}^n \to \mathbb{R}$ are given. In general, one cannot expect to obtain the existence of a minimizer to (1.1) for general functionals (1.2). Indeed, many counterexamples are available in the vast literature on the subject, see for instance [1, 2, 11, 16, 20, 22]. However, it is possible to recover the existence in some cases by relaxing formulation (1.1) using *e.g.* the homogenization method [1], by assuming that $J : \Omega \mapsto J(\Omega)$ is non-increasing for inclusion [5], by adding some topological requirements like convexity or a maximal number of connected components for the complement of two-dimensional shapes as in [23], or by imposing some uniform regularity conditions such as capacitary constraints [4] or an α-cone property [7].

In this article, we are interested in the existence of solutions to such shape optimization problems when the functional is now defined on the boundary of the domain and also depends on the first- and second-order geometric properties of the associated (hyper-)surface:

$$J(\Omega) = \int_{\partial\Omega} j\left[\mathbf{x}, u_{\Omega}(\mathbf{x}), \nabla u_{\Omega}(\mathbf{x}), \mathbf{n}(\mathbf{x}), H(\mathbf{x}), K(\mathbf{x})\right] \mathrm{d}A(\mathbf{x}), \tag{1.6}$$

where the integration on $\partial\Omega$ is done with respect to the $(n-1)$-dimensional Hausdorff measure $A(\bullet)$, where $\mathbf{n}$ refers to the unit normal vector field to $\partial\Omega$ pointing outwards Ω, and where $H := \mathrm{div}_{\Sigma}\mathbf{n}$ (respectively $K := \det[\mathrm{D}_{\Sigma}\mathbf{n}]$) denotes the scalar mean (resp. Gaussian) curvature of $\Sigma := \partial\Omega$.

In fact, the present paper can be seen as the continuation of the previous work [10], both coming from the original study [9]. Hence, we consider here functionals depending on the boundary values of the solution to a PDE whereas in [10] we have considered purely geometric functionals (and constraints) *i.e.* of the form given by (1.6) but where the integrand does not depend on u_{Ω} and ∇u_{Ω}. Under some rather mild assumptions, we have proved that there always exists a $C^{1,1}$-regular minimizer to (1.1) in the following class of admissible shapes.

Definition 1.1. Let $\varepsilon > 0$ and $B \subseteq \mathbb{R}^n$ be open, $n \geqslant 2$. We say that an open set $\Omega \subset B$ with a non-empty boundary $\partial\Omega := \overline{\Omega}\backslash\Omega$ satisfies the ε-ball condition and we write $\Omega \in \mathcal{O}_{\varepsilon}(B)$ if for any $\mathbf{x} \in \partial\Omega$, there exists a unit vector $\mathbf{d}_{\mathbf{x}}$ of $\mathbb{R}^n$ such that $\mathbb{B}_{\varepsilon}(\mathbf{x} - \varepsilon\mathbf{d}_{\mathbf{x}}) \subseteq \Omega$ and $\mathbb{B}_{\varepsilon}(\mathbf{x} + \varepsilon\mathbf{d}_{\mathbf{x}}) \subseteq B\backslash\overline{\Omega}$, where $\mathbb{B}_r(\mathbf{z}) := \{\mathbf{y} \in \mathbb{R}^n,\ |\mathbf{y} - \mathbf{z}| < r\}$ denotes the open ball of $\mathbb{R}^n$ centred at $\mathbf{z}$ and of radius r.

As the uniform cone property is characterizing the Lipschitz regularity of the boundary for a compact domain [7] ([11], Chap. 2 Sect. 6.4) ([16], Sect. 2.4), the ε-ball condition characterizes uniformly its $C^{1,1}$-regularity [8–10], a feature illustrated in Figure 1. Consequently, the class $\mathcal{O}_{\varepsilon}(\mathbb{R}^n)$ can also be described in terms of positive reach [13] and $C^{1,1}$-oriented distance functions ([11], Chap. 7). We refer to ([10]. Sect. 2) and [8] for precise

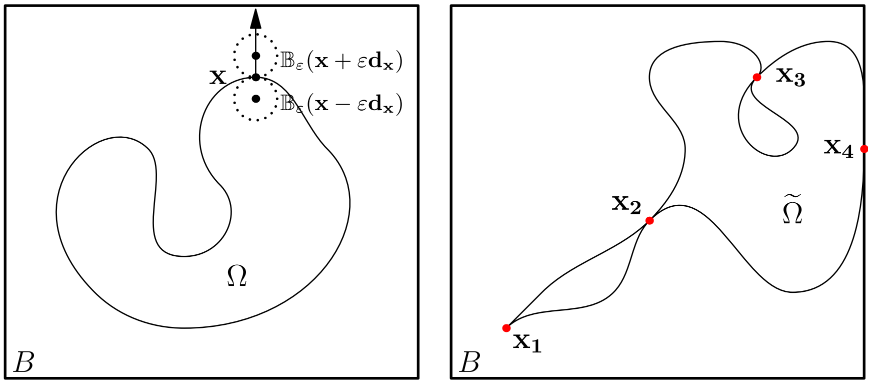

FIGURE 1. Illustration of an open set $\Omega \subset B$ satisfying the ε-ball condition whereas $\widetilde{\Omega} \subset B$ does not.

statements with proofs and further references about these well-known facts. We only recall here the following result.

Proposition 1.2 ([**10**], **Thm. 2.6**). *Let $\Omega \in \mathcal{O}_\varepsilon(B)$. Then, we have that:*

(i) Ω satisfies the $f^{-1}(\varepsilon)$-cone property ([16], Def. 2.4.1) with $f : \alpha \in]0, \frac{\pi}{2}[\mapsto \frac{2\alpha}{\cos\alpha} \in]0, +\infty[$;
(ii) the vector $\mathbf{d_x}$ of Definition 1.1 is the unit outer normal to the hypersurface at the point $\mathbf{x}$;
(iii) the Gauss map $\mathbf{n} : \mathbf{x} \in \partial\Omega \mapsto \mathbf{d_x} \in \mathbb{S}^{n-1}$ is well defined and $\frac{1}{\varepsilon}$-Lipschitz continuous.

Conversely, if $\Sigma \subset B$ is a non-empty compact $C^{1,1}$-hypersurface of $\mathbb{R}^n$, then there exists $\varepsilon > 0$ such that its inner domain $\Omega \in \mathcal{O}_\varepsilon(B)$.

Hence, equipped with this class of admissible shapes, the main contribution of this article is to extend the existence results of [10] for functionals of the form (1.6). We recall that the $C^{1,1}$-regularity is the minimum possible regularity to get existence in the case where the functionals depend on the principal curvatures of the domains, which are only L^∞ for $C^{1,1}$-domains. To our knowledge, such a study has not been carried out in its generality and these new results may have some potential applications in applied mathematics.

The paper is organized as follows. In Section 2, we precisely state our main existence result, namely Theorem 2.1. We firstly detail in Section 2.1 the principal ingredients of the proof of Theorem 2.1 (well-posedness of $J : \Omega \mapsto J(\Omega)$, compactness and continuity issues). In particular, we carefully distinguish what has already been established from what remains to be done. Roughly speaking, we end up with the following result: if we can prove that $\Omega \mapsto u_\Omega \circ X_{\partial\Omega}$ is continuous in some sense (with $X_{\partial\Omega}$ a certain local parametrization of $\partial\Omega$), then Theorem 2.1 holds true.

Although the condition of the last assertion can be shortly stated, its proof requires quite a lot of work. The next part of the article is thus devoted to this purpose. First, in Section 3, we prove some uniform *a priori* H^2-estimates for the three usual types of boundary conditions (Dirichlet, Neumann, and Robin). We follow the method described by Grisvard in ([15], Sect. 3.1) for dealing with convex C^2-domains. Then, in Section 4, we obtain the expected continuity of $\Omega \mapsto u_\Omega \circ X_{\partial\Omega}$. In order to lighten the reading, we also have decided to postpone some technical results to the Appendix. Finally, in Section 5, we show how Theorem 2.1 can be used for three types of problems: the configuration of fluid membranes, the optimization of wing profiles, and the inverse obstacle problems with impedance boundary condition. We conclude the paper by Section 6 where we discuss some possible extensions and suggest other interesting applications to study in future works.

2. Main existence result

Before stating our main theorem, we first need to recall some definitions. We say that two well-defined maps $f : \mathbb{R} \times (\mathbb{R}^n)^3 \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times (\mathbb{R}^n)^3 \to \mathbb{R}$ have quadratic growth in the first two variables if there exists positive continuous maps $c : (\mathbb{R}^n)^2 \times \mathbb{R} \to \mathbb{R}$ and $\tilde{c} : (\mathbb{R}^n)^2 \to \mathbb{R}$ such that:

$$\forall\,(s,\mathbf{z},\mathbf{x},\mathbf{y},t) \in \mathbb{R} \times (\mathbb{R}^n)^3 \times \mathbb{R}, \quad |f\,(s,\mathbf{z},\mathbf{x},\mathbf{y},t)\,| \;\leqslant\; c\,(\mathbf{x},\mathbf{y},t)\,(\,1 + s^2 + |\mathbf{z}|^2\,), \tag{2.1}$$

$$\forall\,(s,\mathbf{z},\mathbf{x},\mathbf{y}) \in \mathbb{R} \times (\mathbb{R}^n)^3, \quad |g\,(s,\mathbf{z},\mathbf{x},\mathbf{y})\,| \;\leqslant\; \tilde{c}\,(\mathbf{x},\mathbf{y})\,(\,1 + s^2 + |\mathbf{z}|^2\,). \tag{2.2}$$

Then, we say that f is convex in the last variable if the map $t \mapsto f(s,\mathbf{z},\mathbf{x},\mathbf{y},t)$ is convex for any $(s,\mathbf{z},\mathbf{x},\mathbf{y}) \in \mathbb{R} \times (\mathbb{R}^n)^3$. Finally, for any $l \in \{1,\dots,n-1\}$, let $H^{(l)} := \sum_{1\leqslant n_1<\ldots<n_l\leqslant n-1} \kappa_{n_1}\ldots\kappa_{n_l}$ denote the lth-order symmetric polynomial of the principal curvatures $(\kappa_l)_{1\leqslant l\leqslant n-1}$, which are the eigenvalues of the self-adjoint endomorphism $\mathrm{D}_\Sigma\mathbf{n}$, where we have set $\Sigma := \partial\Omega$. In particular, we have $H^{(1)} = H$, $H^{(n-1)} = K$, and $(H^{(l)})_{1\leqslant l\leqslant n-1}$ are the coefficients of the characteristic polynomial of $\mathrm{D}_\Sigma\mathbf{n}$. We also mention that they correspond to the curvature measures associated with the $C^{1,1}$-hypersurface Σ. They were originally introduced by Federer in the more general context of sets of positive reach [13], which is relevant here since we have $\mathrm{Reach}(\partial\Omega) = \sup\{\varepsilon > 0,\ \Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)\}$ for any $\Omega \subset \mathbb{R}^n$ such that $V(\partial\Omega) = 0$ ([10], Thm. 2.5). We are now in position to state our main existence result.

Theorem 2.1. *Let $n \geqslant 2$, $\varepsilon > 0$, and B be a non-empty bounded open subset of $\mathbb{R}^n$, large enough to ensure $\mathcal{O}_\varepsilon(B) \neq \emptyset$. We consider $(\widehat{C},\widetilde{C}) \in \mathbb{R}^2$, some continuous maps $j_0, f_0, g_0, g_l : \mathbb{R} \times (\mathbb{R}^n)^3 \to \mathbb{R}$ with quadratic growth (2.2) in the first two variables, and continuous maps $j_l, f_l : \mathbb{R} \times (\mathbb{R}^n)^3 \times \mathbb{R} \to \mathbb{R}$ with quadratic growth (2.1) in the first two variables and convex in the last variable, $l = 1,\dots,n-1$. Then, the following shape optimization problem has at least one solution:*

$$\inf \int_{\partial\Omega} j_0\left[u_\Omega(\mathbf{x}), \nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] \mathrm{d}A(\mathbf{x}) \;+\; \sum_{l=1}^{n-1} \int_{\partial\Omega} j_l\left[u_\Omega(\mathbf{x}), \nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H^{(l)}(\mathbf{x})\right] \mathrm{d}A(\mathbf{x}), \tag{2.3}$$

where $u_\Omega \in H^2(\Omega)$ refers to the unique solution of either (1.3) or (1.4) or (1.5) with $f \in L^2(B)$, $g \in H^2(B)$ and $\lambda > 0$ given, and where the infimum is taken among any $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying a finite number of constraints of the following form:

$$\int_{\partial\Omega} f_0\left[u_\Omega(\mathbf{x}), \nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] \mathrm{d}A(\mathbf{x}) + \sum_{l=1}^{n-1} \int_{\partial\Omega} f_l\left[u_\Omega(\mathbf{x}), \nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H^{(l)}(\mathbf{x})\right] \mathrm{d}A(\mathbf{x}) \;\leqslant\; \widehat{C}, \tag{2.4}$$

$$\int_{\partial\Omega} g_0\left[u_\Omega(\mathbf{x}), \nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] \mathrm{d}A(\mathbf{x}) + \sum_{l=1}^{n-1} \int_{\partial\Omega} H^{(l)}(\mathbf{x})\; g_l\left[u_\Omega(\mathbf{x}), \nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] \mathrm{d}A(\mathbf{x}) \;=\; \widetilde{C}. \tag{2.5}$$

In addition, let $\Omega_0 \in \mathcal{O}_\varepsilon(B)$ and Γ_0 be a measurable subset of $\partial\Omega_0$. Then, the existence result remains true if one adds the constraint $\Gamma_0 \subseteq \partial\Omega$ and if the domain of integration $\partial\Omega$ in the functional (2.3) and the constraints (2.4)–(2.5) are restricted to Γ_0 or to the complement $\partial\Omega\backslash\Gamma_0$.

Remark 2.2. Concerning the last assertion above, and as we will see in Section 2.1, the local parametrization used to prove Theorem 2.1 will be performed on the limit boundary denoted $\partial\Omega_\infty$. Here, since the Hausdorff convergence is stable for the inclusion, the constraint $\Gamma_0 \subseteq \partial\Omega_i$ will pass to the limit and we will have $\Gamma_0 \subseteq \partial\Omega_\infty$. Then, we can proceed as it will be done in Section 2.1.3 with a partition of unity only made on Γ_0 or $\partial\Omega\backslash\Gamma_0$.

In this article, we only consider shape optimization problems that involves functionals and constraints defined as boundary integrals. Indeed, the case where the domain of integration corresponds to the one of (1.2) is standard within the framework of the uniform cone property ([16], Sect. 4.3). Since the ε-ball condition implies an $\alpha(\varepsilon)$-cone property (Prop. 1.2 (i)), we have not considered such functionals in this paper. However, the class $\mathcal{O}_\varepsilon(B)$ may become interesting if some second-order partial derivatives of u_Ω appear in the integrand of (1.2). We can thus extend our main result as follows.

Corollary 2.3. *Consider the assumption of Theorem 2.1 and a map $j : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ measurable according to the first variable and continuous regarding the three last variables. We also assume that there exists a positive continuous map $\hat{c} : \mathbb{R}^n \to \mathbb{R}$ such that:*

$$\forall(\mathbf{x}, s, \mathbf{z}, Y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}, \quad |j\,(\mathbf{x}, s, \mathbf{z}, Y)\,| \leqslant \hat{c}\,(\mathbf{x})\,(\,1 + s^2 + |\mathbf{z}|^2 + \|Y\|^2\,), \tag{2.6}$$

where $\|Y\| := \sqrt{\mathrm{trace}([Y]^T Y)}$ refers here to the Frobenius norm over the set of $(n \times n)$-matrices. Then, Theorem 2.1 still holds true if the functionals in (2.3)–(2.5) contain terms of the form:

$$\int_\Omega j\,[\mathbf{x}, u_\Omega\,(\mathbf{x})\,, \nabla u_\Omega\,(\mathbf{x})\,, \mathrm{Hess}\; u_\Omega\,(\mathbf{x})]\;\mathrm{d}V(\mathbf{x}).$$

Proof. This will be a direct consequence of Corollary 4.13 and the quadratic growth assumption (2.6). We will only have to use similar arguments than in Lemma 2.8, see also ([12], Sect. 1.3, Thm. 4). □

Since we only assume measurability in the first variable, note that the above statement treats the case where the integration is not done on the whole domain Ω but only on a measurable part $\Omega_0 \subseteq \Omega$. Indeed, it suffices to introduce the characteristic function $\mathbf{1}_{\Omega_0}$ in the integrand j.

2.1. Proof of Theorem 2.1: what remains to be done

We now detail what are the main difficulties to overcome in order to prove Theorem 2.1, since an important part of the work has already been settled in [10]. We refer to Section 5 for applications.

2.1.1. Well-posedness of the shape functional

First, we recall from [10] that in the specific case where (1.6) does not depend on u_Ω and ∇u_Ω, we only need to assume the continuity of j in order to ensure that $J : \Omega \in \mathcal{O}_\varepsilon(B) \mapsto J(\Omega)$ is well defined. We now show that in general, measurability and the growth assumptions (2.1)–(2.2) of Theorem 2.1 yield the well-posedness of the shape functionals given in (2.3)–(2.5) over the class $\mathcal{O}_\varepsilon(B)$. Indeed, considering for example the term involving j_1 in (2.3), we have for any $\Omega \in \mathcal{O}_\varepsilon(B)$:

$$\left|\int_{\partial\Omega} j_1\,(u_\Omega, \nabla u_\Omega, \mathrm{Id}, \mathbf{n}, H)\;\mathrm{d}A\,\right| \leqslant \int_{\partial\Omega} c\,[\mathbf{x}, \mathbf{n}\,(\mathbf{x})\,, H\,(\mathbf{x})]\left[1 + u_\Omega\,(\mathbf{x})^2 + |\nabla u_\Omega\,(\mathbf{x})\,|^2\right]\mathrm{d}A(\mathbf{x}),$$

where $\mathrm{Id} : \mathbf{x} \mapsto \mathbf{x}$ is the identity map. From Proposition 1.2 (iii), the Gauss map $\mathbf{n}$ is $\frac{1}{\varepsilon}$-Lipschitz continuous. In particular, it is differentiable almost everywhere (by Rademacher's Theorem) with $\|\mathrm{D}_\Sigma \mathbf{n}\|_{L^\infty(\Sigma)} \leqslant \frac{1}{\varepsilon}$, where we have set $\Sigma := \partial\Omega$. We get $\|\kappa_l\|_{L^\infty(\Sigma)} \leqslant \frac{1}{\varepsilon}$ and $\|H\|_{L^\infty(\Sigma)} \leqslant \frac{n-1}{\varepsilon}$. Hence, $\mathbf{x} \mapsto (\mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x}))$ is always valued in the compact set $K := \overline{B} \times \mathbb{S}^{n-1} \times [-\frac{n-1}{\varepsilon}, \frac{n-1}{\varepsilon}]$. The continuity of c gives:

$$\left|\int_{\partial\Omega} j_1\,[u_\Omega\,(\mathbf{x})\,, \nabla u_\Omega\,(\mathbf{x})\,, \mathbf{x}, \mathbf{n}\,(\mathbf{x})\,, H\,(\mathbf{x})]\;\mathrm{d}A\,(\mathbf{x})\right| \leqslant \|c\|_{C^0(K)}[\,A(\partial\Omega) + \tilde{c}\,\|u_\Omega\|_{H^2(\Omega)}\,],$$

where $\tilde{c} > 0$ denotes the norm of the trace operator $H^2(\Omega) \to H^1(\partial\Omega)$. Since $\partial\Omega$ is a compact $C^{1,1}$-hypersurface, we have $A(\partial\Omega) < +\infty$. Moreover, it is well known ([15], Sect. 2.1 Thms. 2.4.2.5–7) that if $\lambda > 0$, $f \in L^2(\Omega)$ and $g \in H^{3/2}(\partial\Omega)$, then there always exists a unique solution $u_\Omega \in H^2(\Omega)$ satisfying either (1.3) or (1.4) or (1.5). Since we have assumed that $\lambda > 0$, $f \in L^2(B)$ and $g \in H^2(B)$, the map $\Omega \in \mathcal{O}_\varepsilon(B) \mapsto u_\Omega \in H^2(\Omega)$ is well defined. Similar arguments for the other terms in (2.3), and also in (2.4)–(2.5) conclude the proof of the following result.

Lemma 2.4. *Under the measurability and growth assumptions* (2.1)–(2.2) *of Theorem 2.1, any shape functional* $J : \Omega \in \mathcal{O}_\varepsilon(B) \mapsto J(\Omega) \in \mathbb{R}$ *of the form given in* (2.3)–(2.5) *is a well-defined map.*

Remark 2.5. If the measurability and quadratic growth are enough to define the functionals, the continuity and convexity assumptions of Theorem 2.1 will be essentially used to get their continuity.

2.1.2. Compactness and local continuity

Then, we prove Theorem 2.1 by following the classical method of Calculus of Variations, since we have now checked that the shape optimization problem (2.3) is well defined in the class $\mathcal{O}_\varepsilon(B)$. Hence, we consider any minimizing sequence $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$ associated with (2.3). The sequential compactness of $\mathcal{O}_\varepsilon(B)$ has already been studied in ([10], Prop. 3.2) and it states as follows.

Proposition 2.6 (**[10], Prop. 3.2**)**.** *There exists* $\Omega_\infty \in \mathcal{O}_\varepsilon(B)$ *such that a subsequence* $(\Omega_{i'})_{i\in\mathbb{N}}$ *converges to* Ω_∞ *for various modes of convergence (for the Hausdorff distance of the complements in* $\overline{B}$*, of the adherences, of the boundaries, for the* $L^p(B)$*-norm of the characteristic functions, for the* $W^{1,p}(B)$*-norm of the oriented distance functions,* $p \in [1, +\infty[$*, and in the sense of compact sets ([16], Sect. 2.2.4)).*

Therefore, for these modes of convergence, the existence result of Theorem 2.1 is achieved if we can get the lower-semicontinuity of the shape functionals appearing in (2.3) and in the inequality constraint (2.4) whereas we need full continuity in the equality constraint (2.5). Let us emphasize the fact that we are only able to get continuity if the integrands are linear in $H^{(l)}$, as in (2.5). The relaxation of the linearity into convexity has a price. Indeed, in this case, we can only obtain lower-semicontinuity of the functionals, but which is enough for passing to the limit in (2.3)–(2.4).

Our method of proof is similar to the one used in [10] for studying purely geometric functionals. It is based on localization and the study of convergence of graphs. We proved in ([10], Thm. 3.3) that if $(\Omega_i)_{i\in\mathbb{N}}$ converges to Ω_∞ as in Proposition 2.6, then for i large enough, the boundary $\partial\Omega_i$ can be locally parametrized by a $C^{1,1}$-graph in a local frame associated with $\partial\Omega_\infty$. The key point here is that the local frame does not depend on i. Moreover, we obtain the $C^{1,1-\delta}$-strong for any $\delta \in]0, 1]$ and $W^{2,\infty}$-weak-star convergence of these local graphs, where the limit graph is precisely the one associated with $\partial\Omega_\infty$. This local result is illustrated in Figure 2 and it states as follows.

Proposition 2.7 (**[10], Thm. 3.3**)**.** *Let* $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$ *converge to* $\Omega_\infty \in \mathcal{O}_\varepsilon(B)$ *in the sense of compact sets ([16], Sect. 2.2.4) and such that* $\partial\Omega_i \to \partial\Omega_\infty$ *for the Hausdorff distance. Then, for any* $\mathbf{x}_\infty \in \partial\Omega_\infty$*, there exists a direct orthonormal frame centred at* $\mathbf{x}_\infty$*, and also* $I \in \mathbb{N}$ *depending on* $\mathbf{x}_\infty$*,* ε*, and* $(\Omega_i)_{i\in\mathbb{N}\cup\{\infty\}}$*, such that inside this frame, for any* $i \in [\![I, \infty]\!] := \{i \in \mathbb{N} \cup \{\infty\}, i \geqslant I\}$*, there exists a continuously differentiable map* $\varphi_i : D_r(\mathbf{0}') \mapsto] - \varepsilon, \varepsilon[$ *such that:*

$$\left\{\begin{array}{rcl} \partial\Omega_i \cap C_{r,\varepsilon}(\mathbf{x}_\infty) & = & \{(\mathbf{x}', \varphi_i(\mathbf{x}')), \quad \mathbf{x}' \in D_r(\mathbf{0}')\} \\ \Omega_i \cap C_{r,\varepsilon}(\mathbf{x}_\infty) & = & \{(\mathbf{x}', x_n), \quad \mathbf{x}' \in D_r(\mathbf{0}') \text{ and } -\varepsilon < x_n < \varphi_i(\mathbf{x}')\}, \end{array}\right.$$

where $C_{r,\varepsilon}(\mathbf{x}_\infty)$ *refers to the cylinder* $D_r(\mathbf{0}') \times]-\varepsilon, \varepsilon[$ *with* $D_r(\mathbf{0}') := \{\mathbf{x}' \in \mathbb{R}^{n-1}, \quad |\mathbf{x}'| < r\}$ *denoting the open ball of* $\mathbb{R}^{n-1}$ *centred at the origin* $\mathbf{0}'$ *(identified with* $\mathbf{x}_\infty$*) and of radius* $r > 0$ *that only depends on* ε*. Moreover, any of the* $(\varphi_i)_{i\in[\![I,\infty]\!]}$ *has a unique* $C^{1,1}$*-extension to the closure* $\overline{D_r(\mathbf{0}')}$ *and we have:*

$$\left\{\begin{array}{ll} \varphi_i \to \varphi_\infty & \text{strongly in } C^{1,1-\delta}(\overline{D_r(\mathbf{0}')}) \text{ for any } \delta \in]0, 1], \\ \varphi_i \rightharpoonup \varphi_\infty & \text{weakly} - \text{star in } W^{2,\infty}(D_r(\mathbf{0}')). \end{array}\right. \tag{2.7}$$

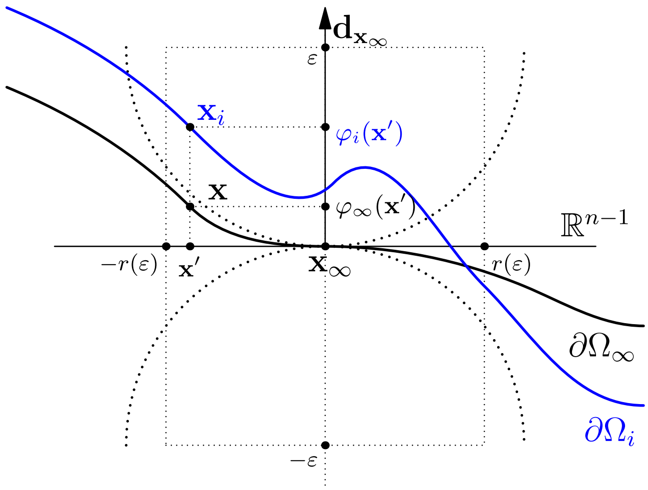

FIGURE 2. Illustration of Proposition 2.7 stating that there exists a fixed common local frame in which a converging sequence of elements in $\mathcal{O}_\varepsilon(B)$ can be simultaneously parametrized by $C^{1,1}$-graphs. The dotted half-circles correspond to the ε-ball condition satisfied by $\partial\Omega_\infty$ whereas the dotted square represents the cylinder $C_{r,\varepsilon}(\mathbf{x}_\infty)$.

2.1.3. A suitable partition of unity

With this previous local result in mind, it remains to build a suitable partition of unity in order to study properly the continuity of a shape functional defined on $\mathcal{O}_\varepsilon(B)$. Let us recall this procedure that was already detailed in the proof of ([10], Prop. 4.6). From now on, the notation $(\Omega_i)_{i\in\mathbb{N}}$ refers to the converging subsequence of Proposition 2.6 and Ω_∞ to its limit. Since $\partial\Omega_\infty$ is compact, there exists a finite number $K \geqslant 1$ of distinct points $(\mathbf{x}_k)_{1\leqslant k\leqslant K}$ of $\partial\Omega_\infty$ such that $\partial\Omega_\infty \subset \bigcup_{k=1}^K C_{\frac{r}{2},\frac{\varepsilon}{2}}(\mathbf{x}_k)$. First, we define $\delta := \frac{1}{2}\min(r,\varepsilon) > 0$, which only depends on ε. From the triangle inequality, the open neighbourhood $\mathcal{V}_\delta(\partial\Omega_\infty) := \{\mathbf{y} \in \mathbb{R}^n,\ d(\mathbf{y},\partial\Omega_\infty) < \delta\}$ has its closure embedded in $\bigcup_{k=1}^K C_{r,\varepsilon}(\mathbf{x}_k)$. Then, we introduce a partition of unity associated with this covering: there exists K smooth maps $\xi^k : \mathbb{R}^n \to [0,1]$ with compact support in $C_{r,\varepsilon}(\mathbf{x}_k)$ such that $\sum_{k=1}^K \xi^k = 1$ on $\mathcal{V}_\delta(\partial\Omega_\infty)$. Now applying Proposition 2.7 to the K points, there also exists integers $I_k \in \mathbb{N}$ and some maps $\varphi_i^k : D_r(\mathbf{x}_k) \mapsto]-\varepsilon,\varepsilon[$ such that for any $k \in \{1,\dots,K\}$ and $i \geqslant I_k$:

$$\left\{\begin{array}{lcl} \partial\Omega_i \cap C_{r,\varepsilon}(\mathbf{x}_k) & = & \left\{\left(\mathbf{x}',\varphi_i^k(\mathbf{x}')\right),\ \ \mathbf{x}' \in D_r(\mathbf{x}_k)\right\} \\ \Omega_i \cap C_{r,\varepsilon}(\mathbf{x}_k) & = & \left\{(\mathbf{x}',x_n),\ \ \mathbf{x}' \in D_r(\mathbf{x}_k)\ \ \text{and} -\varepsilon < x_n < \varphi_i^k(\mathbf{x}')\right\}. \end{array}\right.$$

Moreover, the K sequences of functions $(\varphi_i^k)_{i\geqslant I_k}$ and $(\nabla\varphi_i^k)_{i\geqslant I_k}$ converge uniformly on $\overline{D_r(\mathbf{x}_k)}$ respectively to the maps φ_∞^k and $\nabla\varphi_\infty^k$ associated with $\partial\Omega_\infty$ at $\mathbf{x}_k$, $k = 1,\dots,K$. Finally, from the Hausdorff convergence of the boundaries, there also exist $I_0 \in \mathbb{N}$ such that for any $i \geqslant I_0$, we have $\partial\Omega_i \in \mathcal{V}_\delta(\partial\Omega_\infty)$. We now set $I := \max_{k\in[\![0,K]\!]} I_k$ which thus only depends on ε and $(\Omega_i)_{i\in\mathbb{N}\cup\{\infty\}}$.

2.1.4. Global continuity issues

We are now in position to study the continuity of (2.3)–(2.5) by expressing the shape functionals in our previous parametrization. To illustrate our reasoning simply, we consider the term involving j_0 in (2.3) and we assume that u_Ω refers to the solution of the Dirichlet Laplacian (1.3) with $g \equiv 0$. In this specific case, observe that the dependence of j_0 in u_Ω can be dropped since $u_\Omega = 0$ on $\partial\Omega$. From the foregoing, we deduce that for any integer $i \geqslant I$, we have:

$$\begin{aligned} F(\Omega_i) &:= \int_{\partial\Omega_i} j_0\left(\nabla u_{\Omega_i}, \mathrm{Id}, \mathbf{n}\right) \mathrm{d}A \;=\; \int_{\partial\Omega_i \cap \mathcal{V}_\delta(\partial\Omega_\infty)} j_0\left(\nabla u_{\Omega_i}, \mathrm{Id}, \mathbf{n}\right) \mathrm{d}A \\ &= \int_{\partial\Omega_i} \left(\sum_{k=1}^{K} \xi^k \right) j_0\left(\nabla u_{\Omega_i}, \mathrm{Id}, \mathbf{n}\right) \mathrm{d}A \;=\; \sum_{k=1}^{K} \int_{\partial\Omega_i \cap C_{r,\varepsilon}(\mathbf{x}_k)} \xi^k \, j_0\left(\nabla u_{\Omega_i}, \mathrm{Id}, \mathbf{n}\right) \mathrm{d}A. \end{aligned}$$

Using the expression of a boundary integral parametrized by a local graph ([19], Prop. 5.13), we obtain that what we have defined above as $F(\Omega_i)$ is equal to:

$$\sum_{k=1}^{K} \int_{D_r(\mathbf{x}_k)} \xi^k \begin{pmatrix} \mathbf{x}' \\ \varphi_i^k(\mathbf{x}') \end{pmatrix} j_0 \left[\nabla u_{\Omega_i} \begin{pmatrix} \mathbf{x}' \\ \varphi_i^k(\mathbf{x}') \end{pmatrix}, \begin{pmatrix} \mathbf{x}' \\ \varphi_i^k(\mathbf{x}') \end{pmatrix}, \begin{pmatrix} \frac{-\nabla\varphi_i^k(\mathbf{x}')}{\sqrt{1+|\nabla\varphi_i^k(\mathbf{x}')|^2}} \\ \frac{1}{\sqrt{1+|\nabla\varphi_i^k(\mathbf{x}')|^2}} \end{pmatrix} \right] \sqrt{1+|\nabla\varphi_i^k(\mathbf{x}')|^2}\, \mathrm{d}\mathbf{x}'. \tag{2.8}$$

Hence, our strategy consists in letting correctly $i \to +\infty$ in (2.8). For this purpose, we aim to apply Lebesgue's Dominated Convergence Theorem. First, we prove that from the quadratic growth and the continuity of j_0 combined with the convergence properties of the K sequences $(\varphi_i^k)_{i \geqslant I}$, $k \in [\![1, K]\!] := \{1, \dots, K\}$, the following implication holds true.

Lemma 2.8. *If the map $v_i^k : \mathbf{x}' \mapsto \nabla u_{\Omega_i}(\mathbf{x}', \varphi_i^k(\mathbf{x}'))$ converges to $v_\infty^k : \mathbf{x}' \mapsto \nabla u_{\Omega_\infty}(\mathbf{x}', \varphi_\infty^k(\mathbf{x}'))$ strongly in $L^2(D_r(\mathbf{x}_k))$ for any $k \in [\![1, K]\!]$, then we can let $i \to +\infty$ in (2.8) and $F(\Omega_i) \to F(\Omega_\infty)$.*

Proof. We drop the index k to lighten the notation. We assume by contradiction that $F(\Omega_i)$ does not converge to $F(\Omega_\infty)$. Hence, there exists a subsequence $F(\Omega_{i'})$ remaining at a positive distance from $F(\Omega_\infty)$. Since $v_{i'}$ converges to v_∞ strongly in L^2, there also exists a subsequence $v_{i''}$ converging to v_∞ almost everywhere and dominated by an L^2-map. On the one hand, the quadratic growth of j_0 ensures that the integrand of (2.8) is also dominated. On the other hand, the continuity of j_0 yields the a.e convergence of the integrand of (2.8) to the right quantity. Applying Lebesgue's Dominated Convergence Theorem, we have proved $F(\Omega_{i''}) \to F(\Omega_\infty)$, which contradicts the definition of $F(\Omega_{i'})$. We conclude that the whole sequence is converging: $F(\Omega_i) \to F(\Omega_\infty)$. ☐

In fact, we can say a little bit more by introducing the previous local $C^{1,1}$-parametrizations $X_i^k : \mathbf{x}' \in D_r(\mathbf{x}_k) \mapsto (\mathbf{x}', \varphi_i^k(\mathbf{x}')) \in C_{r,\varepsilon}(\mathbf{x}_k) \cap \partial\Omega_i$. Indeed, on the one hand, we can follow the proof of Lemma 2.8. From Lebesgue's Dominated Convergence Theorem, we deduce that if $\nabla u_{\Omega_i} \circ X_i^k$ converges L^2-strongly to $\nabla u_{\Omega_\infty} \circ X_\infty^k$, then the integrand of (2.8) converges strongly in $L^1(D_r(\mathbf{x}_k))$. On the other hand, following ([10], Sect. 4.3), a direct computation of the scalar mean curvature in the local parametrization yields:

$$H_{\Omega_i} \circ X_i^k \;=\; -\sum_{p,q=1}^{n-1} \left[\delta_{pq} - \frac{\partial_p \varphi_i^k \, \partial_q \varphi_i^k}{1 + |\nabla\varphi_i^k|^2} \right] \frac{\partial_{pq}\varphi_i^k}{\sqrt{1+|\nabla\varphi_i^k|^2}}.$$

Using the convergence properties of $(\varphi_i^k)_{i \geqslant I}$ given in Proposition 2.7, we obtain that $(H_{\Omega_i} \circ X_i^k)_{i \geqslant I}$ converges to $H_{\Omega_\infty} \circ X_\infty^k$ weakly-star in $L^\infty(D_r(\mathbf{x}_k))$. More generally, we have proved in ([10], Sect. 4.4) that $(H_{\Omega_i}^{(l)} \circ X_i^k)_{i \geqslant I}$ converges L^∞-weakly-star to $H_{\Omega_\infty}^{(l)} \circ X_\infty^k$, for any $(k, l) \in [\![1, K]\!] \times [\![1, n-1]\!]$.

Consequently, we deduce from the foregoing that any functional with a linear integrand in $H^{(l)}$ is continuous (L^1-strong *versus* L^∞-weakly-star convergence). For example, it is the case for the ones appearing in the equality constraint (2.5). Furthermore, considering the arguments used in the proof of ([10], Cor. 4.11), one can check that such functionals become lower semi-continuous if we relax the linearity assumption by convexity in $H^{(l)}$, as it is the case in (2.3)–(2.4). Finally, the previous arguments also work for the inhomogeneous Dirichlet, Neumann and Robin boundary conditions. We only have to additionally ensure that $u_{\Omega_i} \circ X_i^k$ converges to $u_{\Omega_\infty} \circ X_\infty^k$ strongly in $L^2(D_r(\mathbf{x}_k))$ for any $k \in [\![1, K]\!]$. Hence, the conclusion of our discussion can be summed up as follows.

Remark 2.9. Assume that the sequences $(u_{\Omega_i} \circ X_i^k)_{i\geqslant I}$ and $(\nabla u_{\Omega_i} \circ X_i^k)_{i\geqslant I}$ respectively converge to $u_{\Omega_\infty} \circ X_\infty^k$ and $\nabla u_{\Omega_\infty} \circ X_\infty^k$ strongly in $L^2(D_r(\mathbf{x}_k))$, for $k \in [\![1, K]\!]$. Then, Theorem 2.1 holds.

Therefore, the remaining part of the paper consists in proving the hypothesis of Remark 2.9 holds true. First, in Section 3, we establish some uniform *a priori* H^2-estimates, ensuring that the sequences are uniformly bounded. Then, in Section 4, we manage to uniquely determine the weak limit of a subsequence, next yielding the strong convergence of the entire sequence. We follow the method developed by Chenais in ([7], Thm. II.1): we consider a uniform H^2-extension of u_{Ω_i} in order to pass to the limit in the PDE and boundary conditions of (1.3)–(1.5). Finally, in Section 5, we give various applications and possible extensions of Theorem 2.1.

3. Uniform H^2-estimates for the Laplace operator

In this section, our goal is to control uniformly the constant appearing in *a priori* H^2-estimates of the Laplace operator for the three usual types of boundary conditions. We follow the method suggested by Grisvard in ([15], Sect. 3.1.2.2) and which is based on an identity (A.1) established in the case of convex C^2-domains ([15], Sect. 3.1.1.1). Indeed, it allows a uniform control in terms of curvatures, which is precisely what the uniform ball condition of Definition 1.1 provides (*cf.* Prop. 1.2).

In order to lighten the lecture, we have also decided to postpone to Appendix A the statement and proof of Grisvard's identity in the context of $C^{1,1}$-regularity (*cf.* Prop. A.2 for details). Similarly, for completeness, in Appendix B, we have made our statement more precise concerning the dependence of the constant appearing in *a priori* H^1-estimates, which is more or less standard in the framework of the uniform cone property. We start with Dirichlet boundary conditions.

3.1. Homogeneous Dirichlet boundary conditions

Proposition 3.1. *Let $n \geqslant 2$ and $\varepsilon > 0$. We also consider a non-empty open bounded set $B \subset \mathbb{R}^n$ with diameter d large enough to ensure that the class $\mathcal{O}_\varepsilon(B)$ given in Definition 1.1 is not empty. Then, there exists a constant $C > 0$, depending only on d, ε and n, such that:*

$$\forall \Omega \in \mathcal{O}_\varepsilon(B), \forall u \in H^2(\Omega) \cap H_0^1(\Omega), \quad \|u\|_{H^2(\Omega)} \leqslant C(d, \varepsilon, n) \|\Delta u\|_{L^2(\Omega)}.$$

Proof. Let $\Omega \in \mathcal{O}_\varepsilon(B)$ and $u \in H^2(\Omega) \cap H_0^1(\Omega)$. We set $\Sigma := \partial\Omega$. First, we have $u = 0$ on Σ so we deduce $\nabla_\Sigma u = 0$ and $\nabla u = \partial_{\mathbf{n}} u \, \mathbf{n}$. Applying Proposition A.2 to the map $\mathbf{v} = \nabla u$, we get from (A.1):

$$\sum_{i,j=1}^n \int_\Omega \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 = \int_\Omega (\Delta u)^2 - \int_\Sigma H |\nabla u|^2.$$

Then, from Proposition 1.2 (iii), the Gauss map $\mathbf{n} : \Sigma \to \mathbb{S}^{n-1}$ is $\frac{1}{\varepsilon}$-Lipschitz continuous. Hence, the eigenvalues $(\kappa_i)_{1\leqslant i\leqslant n-1}$ of its differential $\mathrm{D}_\Sigma \mathbf{n}$ (*i.e.* the principal curvatures) exist a.e. and are essentially bounded by $\frac{1}{\varepsilon}$. Recalling that $H := \mathrm{div}_\Sigma \mathbf{n}$, we get $\|H\|_{L^\infty(\Sigma)} \leqslant \frac{n-1}{\varepsilon}$. Moreover, from Proposition 1.2 (i), Ω satisfies the $\alpha(\varepsilon)$-cone property, where α only depends on ε. Combining these two observations with the inequality (B.1) of Corollary B.5, we obtain:

$$\sum_{i,j=1}^n \int_\Omega \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \leqslant \int_\Omega (\Delta u)^2 + \frac{(n-1)}{\varepsilon} C_0(\alpha, d, n) \left(\eta \sum_{i,j=1}^n \int_\Omega \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 + \frac{1}{\eta} \int_\Omega |\nabla u|^2 \right),$$

where we now set $\eta := \min(1, \frac{\varepsilon}{2(n-1)C_0})$. Indeed, we clearly have $\eta \in]0,1]$ but also $1-\eta\frac{(n-1)C_0}{\varepsilon} \geqslant \frac{1}{2}$. Using the H^1-estimates of Proposition B.1 and Corollary B.2, we deduce from the foregoing:

$$\|u\|_{H^2(\Omega)} \leqslant \left(2d\,(1+2d) + \sqrt{2+\left[\frac{2d}{\eta\,(\alpha\,(\varepsilon)\,,d,\varepsilon,n)}\right]^2}\right)\|\Delta u\|_{L^2(\Omega)}.$$

To conclude the proof of Proposition 3.1, the above constant only depends on ε, d and n. □

3.2. Neumann type of boundary conditions

It is well known that the solution of the Poisson equation with Neumann boundary conditions is defined up to a constant. Here, to avoid this technical issue, we consider instead H^2-estimates for the inhomogeneous Helmholtz equation. Moreover, as done in ([15], Sect. 3.1.2.3) for convex C^2-domains, we introduce a non-linear map $\beta : \mathbb{R} \to \mathbb{R}$ that allows to treat more general boundary conditions, and in particular simultaneously the Neumann ($\beta(x) = 0$) and Robin case ($\beta(x) = \mu^2 x$).

Proposition 3.2. *We consider the assumptions and notation of Proposition 3.1. Let $L > 0$, $\lambda > 0$, and β : $\mathbb{R} \to \mathbb{R}$ be a L-Lipschitz continuous map satisfying $\beta(x)x \geqslant 0$ for any $x \in \mathbb{R}$. Then, there exists a constant $C > 0$, depending only on d, ε, L, λ and n, such that for any $\Omega \in \mathcal{O}_\varepsilon(B)$ and any $u \in H^2(\Omega)$ satisfying $\partial_{\mathbf{n}} u + \beta(u) \in H^1(\partial\Omega)$, we have:*

$$\|u\|_{H^2(\Omega)} \leqslant C\,(d,\varepsilon,L,\lambda,n)\,(\,\|-\Delta u+\lambda u\|_{L^2(\Omega)} + \|\partial_{\mathbf{n}} u+\beta(u)\|_{H^1(\partial\Omega)}\,).$$

If we now assume that β is a non-decreasing Lipschitz continuous map satisfying $\beta(0) = 0$, then the same result holds but in this case, the constant C does **not** *depend on $L := \sup_{x\neq y}\frac{|\beta(x)-\beta(y)|}{|x-y|}$.*

Proof. Let $\lambda > 0$ and $\beta : \mathbb{R} \to \mathbb{R}$ be as in the statement. Consider $\Omega \in \mathcal{O}_\varepsilon(B)$ and $u \in H^2(\Omega)$ satisfying $\partial_{\mathbf{n}} u + \beta(u) \in H^1(\partial\Omega)$. We follow the arguments given in the proof of Proposition 3.1. Hence, we apply Proposition A.2 to the map $\mathbf{v} = \nabla u$. In particular, since $\partial_{\mathbf{n}} u + \beta(u) \in H^1(\partial\Omega)$, the brackets appearing in the right member of (A.1) can be written as an L^2-product. We have:

$$\begin{aligned}\sum_{i,j=1}^{n}\int_\Omega\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2 \;=\; & \int_\Omega(\Delta u)^2 + \int_{\partial\Omega}\left[\mathbf{II}\,(\nabla_\Sigma u,\nabla_\Sigma u) - H\,(\partial_{\mathbf{n}} u)^2\right] \\ & + 2\int_{\partial\Omega}[\,\langle\nabla_\Sigma\,(\partial_{\mathbf{n}} u+\beta\,(u))\;|\;\nabla_\Sigma u\rangle - \beta'\,(u)\,|\nabla_\Sigma u|^2\,],\end{aligned} \tag{3.1}$$

where we set $\Sigma := \partial\Omega$ in the tangential operators to avoid bulky notation. The right member of (3.1) is composed of three integrals that we are going to estimate separately. We denote them by I_1, I_2 and I_3 from left to right. The first term is evaluated as follows:

$$I_1 := \int_\Omega(\Delta u)^2 \leqslant 2\int_\Omega(-\Delta u+\lambda u)^2 + 2\lambda^2\int_\Omega u^2.$$

From Proposition 1.2 (i), Ω satisfies the $\alpha(\varepsilon)$-cone property, where α only depends on ε. In addition, using also the hypothesis on β, we can thus consider the relation (B.2) of Proposition B.6 to get:

$$I_1 \leqslant 2\left(1+\lambda^2 C_1\right)\left[\int_\Omega(-\Delta u+\lambda u)^2 + \int_{\partial\Omega}(\partial_{\mathbf{n}} u+\beta\,(u))^2\right], \tag{3.2}$$

where the constant $C_1(\alpha(\varepsilon), d, \lambda, n) > 0$ is the one of Proposition B.6. Then, the second term denoted as $I_2 := \int_{\partial\Omega}[\,\mathbf{II}(\nabla_\Sigma u, \nabla_\Sigma u) - H(\partial_{\mathbf{n}} u)^2\,]$ is decomposed as in (A.4). From Proposition 1.2 (iii), we recall that $\|\kappa_i\|_{L^\infty(\partial\Omega)} \leqslant \frac{1}{\varepsilon}$ so we simply deduce that:

$$I_2 \;\leqslant\; \frac{1}{\varepsilon}\sum_{i=1}^{n-1}\left[\int_{\partial\Omega} |\,\langle \nabla_\Sigma u \mid \mathbf{e}_i\rangle\,|^2 + \int_{\partial\Omega} |\,\langle \nabla u \mid \mathbf{n}\rangle\,|^2\right] \;=\; \frac{n-1}{\varepsilon}\int_{\partial\Omega} |\nabla u|^2,$$

where the last equality comes from the fact that $(\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}, \mathbf{n})$ forms an orthonormal basis of $\mathbb{R}^n$. Again, Ω satisfies the $\alpha(\varepsilon)$-cone property so we can apply Corollary B.5 with $\eta_0 := \min(1, \frac{\varepsilon}{4C_0(n-1)})$ and $C_0(\alpha(\varepsilon), d, n) > 0$ the constant of Proposition B.4. Consequently, we obtain from (B.1) and the H^1-estimate (B.2) of Proposition B.6:

$$I_2 \;\leqslant\; \frac{1}{4}\sum_{i,j=1}^{n}\int_\Omega \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \frac{C_1}{4\eta_0^2}\left[\int_\Omega (-\Delta u + \lambda u)^2 + \int_{\partial\Omega} (\partial_{\mathbf{n}} u + \beta(u))^2\right]. \tag{3.3}$$

It remains to treat the third term $I_3 := 2\int_{\partial\Omega}[\,\langle \nabla_\Sigma(\partial_{\mathbf{n}} u + \beta(u)) \,|\, \nabla_\Sigma u\rangle - \beta'(u)|\nabla_\Sigma u|^2\,]$. Denoting by L the Lipschitz modulus of continuity of β, the Cauchy-Schwarz inequality yields:

$$I_3 \;\leqslant\; \int_{\partial\Omega} |\nabla_\Sigma (\partial_{\mathbf{n}} u + \beta(u))|^2 + (1+2L)\int_{\partial\Omega} |\nabla_\Sigma u|^2.$$

Observing that $|\nabla_\Sigma u| \leqslant |\nabla u|$, we can combine another time the relation (B.1) of Corollary B.5 with the number $\eta_1 := \min(1, \frac{1}{4C_0(1+2L)})$ and the H^1-estimate (B.2) of Proposition B.6 in order end with:

$$I_3 \;\leqslant\; \frac{1}{4}\sum_{i,j=1}^{n}\int_\Omega \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \max\left(1, \frac{C_1}{4\eta_1^2}\right)\left(\|-\Delta u + \lambda u\|^2_{L^2(\Omega)} + \|\partial_{\mathbf{n}} u + \beta(u)\|^2_{H^1(\partial\Omega)}\right).$$

Finally, we insert in (3.1) the above estimation and (3.2)–(3.3). After simplification, we can use a last time Proposition B.6 to prove that the inequality of Proposition 3.2 holds true for the constant:

$$C(d, \varepsilon, L, \lambda, n) := \sqrt{C_1 + 2\left[2(1 + \lambda^2 C_1) + \frac{C_1}{4\eta_0^2} + \max\left(1, \frac{C_1}{4\eta_1^2}\right)\right]}.$$

We now assume that $\beta : \mathbb{R} \to \mathbb{R}$ is a non-decreasing Lipschitz continuous map satisfying $\beta(0) = 0$. Since we still have $\beta(x)x \geqslant 0$ in this case, the previous result remains true. However, looking again at (3.1), we now have $\beta'(u) \geqslant 0$ a.e. so the associated term can be dropped in the estimation of I_3. Consequently, the constant C does not depend on L, concluding the proof of Proposition 3.2. □

3.3. The case of Robin boundary conditions

Proposition 3.3. *We consider the assumptions and notation of Proposition 3.1. Let $\lambda > 0$. Then, there exists a constant $C > 0$, depending only on d, ε, λ and n, such that for any $\Omega \in \mathcal{O}_\varepsilon(B)$ and any $u \in H^2(\Omega)$ satisfying $\partial_{\mathbf{n}} u + \lambda\, u \in H^1(\partial\Omega)$, we have:*

$$\|u\|_{H^2(\Omega)} \;\leqslant\; C(d, \varepsilon, \lambda, n)\,(\,\|\Delta u\|_{L^2(\Omega)} + \|\partial_{\mathbf{n}} u + \lambda u\|_{H^1(\partial\Omega)}\,).$$

Proof. The proof is almost identical to the one of Proposition 3.2. Hence, we use the same notation and detail the main changes. First, starting from (3.1), the term I_1 does not have to be estimated in this case. Concerning the two other terms, roughly speaking, the main difference concerns the *a priori* H^1-estimate. Indeed, we now have to apply Proposition B.10 instead of Proposition B.6. From the notation viewpoint, this formally consists in setting $\lambda = 0$ in the previous proof, then considering $\beta(u) = \lambda\, u$, and replacing C_1 by the constant $C_2(d, \lambda)$ of (B.4). Finally, $\beta'(u) = \lambda > 0$ so the associated term can be dropped in the estimation of I_3, which is equivalent to set $L = 0$ in the proof. Hence, we can conclude that the inequality of Proposition 3.3 holds for the constant:

$$C\,(d, \varepsilon, \lambda, n) := \sqrt{C_2 + 2\left[1 + \frac{C_2}{4\eta_0^2} + \max\left(1, \frac{C_2}{4\eta_1^2}\right)\right]},$$

where we recall that $\eta_0 := \min(1, \frac{\varepsilon}{4C_0(n-1)})$ and $\eta_1 := \min(1, \frac{1}{4C_0})$, with $C_0(\alpha(\varepsilon), d, n)$ referring to the constant of the trace inequality of Proposition B.4. □

4. Continuity of PDE solutions in a local parametrization

In this section, our goal is to prove that the hypothesis of Remark 2.9 holds true. For this purpose, we consider the partition of unity introduced in Section 2.1.3 and we use the notation of Section 2.1.4. To lighten the lecture, we drop the dependence in the index k. We also set $\mathcal{C} := C_{r,\varepsilon}(\mathbf{x}_k)$ and $D := D_r(\mathbf{x}_k)$. First, we start by showing that the sequences $(u_{\Omega_i} \circ X_i)_{i \geqslant I}$ and $(\nabla u_{\Omega_i} \circ X_i)_{i \geqslant I}$ are uniformly bounded in $L^2(D)$. This will allow us to consider a weakly converging subsequence in the sequel.

4.1. A uniform L^2-bound for the sequence

Most of the work has already been done in Section 3. Here, we only gather the results and postpone to Appendix C the proof of a technical inequality, whose constant also needs to be controlled.

Theorem 4.1. *We consider the assumptions and notation of Proposition 3.1. Let $f \in L^2(B)$, $g \in H^2(B)$ and $\lambda > 0$ be given. We introduce the well-defined map $\Omega \in \mathcal{O}_\varepsilon(B) \mapsto u_\Omega \in H^2(\Omega)$ where u_Ω is the unique solution of either (1.3) or (1.4) or (1.5). Then, there exists a constant $C > 0$ depending only on d, ε, λ (except for (1.3)), and n, such that:*

$$\forall \Omega \in \mathcal{O}_\varepsilon\,(B)\,, \quad \|u_\Omega\|_{H^2(\Omega)} \;\leqslant\; C\,(d, \varepsilon, \lambda, n)\,(\,\|f\|_{L^2(B)} \,+\, \|g\|_{H^2(B)}\,). \tag{4.1}$$

Proof. First, we consider ([15], Thm. 2.4.2.5) so the problem (1.3) always has a unique solution $u_\Omega \in H^2(\Omega)$. Since we have $u_\Omega - g \in H^2(\Omega) \cap H_0^1(\Omega)$ and $-\Delta u_\Omega = f$ a.e. on Ω, we can apply Proposition 3.1 to get $\|u_\Omega - g\|_{H^2(\Omega)} \leqslant C(d, \varepsilon, n)\|f + \Delta g\|_{L^2(\Omega)}$. The triangle and Cauchy-Schwarz inequalities gives (4.1) for the constant $\max(1, \sqrt{n}\, C(d, \varepsilon, n))$. Then, let $\lambda > 0$ and ([15], Thm. 2.4.2.6) (respectively [15], Thm. 2.4.2.7) ensures that (1.4) (resp. (1.5)) also has a unique solution in $H^2(\Omega)$. Consequently, we deduce from Proposition 3.2 with $\beta \equiv 0$ (resp. Prop. 3.3) that we have $\|u_\Omega\|_{H^2(\Omega)} \leqslant C(d, \varepsilon, \lambda, n)(\|f\|_{L^2(B)} + \|g\|_{H^1(\partial\Omega)})$. It remains to combine Proposition B.4 and Corollary B.5 with $\eta = 1$ and $u = g$ in order to obtain that (4.1) holds for the constant $\max(1, \sqrt{2C_0})C(d, \varepsilon, \lambda, n)$. We conclude the proof by recalling Proposition 1.2 (i) so the constant $C_0 > 0$ of Proposition B.4 only depends on $\alpha(\varepsilon)$, d and n. □

Corollary 4.2. *We consider the assumptions and notation of Theorem 4.1. Then, there exists a constant $C > 0$ depending only on d, ε, λ (except for (1.3)) and n, such that:*

$$\sqrt{\int_D |u_{\Omega_i} \circ X_i|^2 \,+\, \int_D |\nabla u_{\Omega_i} \circ X_i|^2} \;\leqslant\; C\,(d, \varepsilon, \lambda, n)\,(\,\|f\|_{L^2(B)} \,+\, \|g\|_{H^2(B)}\,). \tag{4.2}$$

In particular, the sequences $(u_{\Omega_i} \circ X_i)_{i \geqslant I}$ and $(\nabla u_{\Omega_i} \circ X_i)_{i \geqslant I}$ are uniformly bounded in $L^2(D)$.

Proof. First, we use Corollary C.2 to bound the left member of (4.2) by $\sqrt{C_3}\|u_{\Omega_i}\|_{H^1(\partial\Omega_i)}$, where Proposition 1.2 (i) ensures that the constant $C_3 > 0$ of Corollary C.2 only depends on $\alpha(\varepsilon)$ and n. Then, we combine

Proposition B.4 and Corollary B.5 with $\eta = 1$ to newly bound the left member of (4.2) by $\sqrt{2C_3C_0}\|u_{\Omega_i}\|_{H^2(\Omega_i)}$, where the constant $C_0 > 0$ of Proposition B.4 only depends on $\alpha(\varepsilon)$, d and n. Finally, we can conclude the proof of Corollary 4.2 by applying Theorem 4.1. □

4.2. Passing to the limit in the strong formulation

From Section 4.1, the sequences $(u_{\Omega_i} \circ X_i)_{i\geqslant I}$ and $(\nabla u_{\Omega_i} \circ X_i)_{i\geqslant I}$ are uniformly bounded in $L^2(D)$. Hence, they have a weakly converging subsequence and in order to identify the limiting map, we first need to pass to the limit in the PDE problems (1.3)–(1.5). Note that both the weak and strong formulations are accessible in our ε-ball framework, which is not the case with the uniform cone property.

However, in the latter context, a very convenient way to study the continuity of $\Omega \mapsto u_\Omega$ was proposed by Chenais in [7] and it consists here in considering a $H^2(B)$-extension of $u_\Omega \in H^2(\Omega)$. Recalling from Proposition 1.2 (i) that our class $\mathcal{O}_\varepsilon(B)$ of shapes satisfies the $\alpha(\varepsilon)$-cone condition, it is thus possible to extend the maps in a uniform way as follows.

Lemma 4.3 (**[7], Thm. II.1**). *We consider the assumptions and notation of Proposition 3.1. Then, there exists a constant $C > 0$ depending only on d, ε and n, such that for any $\Omega \in \mathcal{O}_\varepsilon(B)$, there exists a continuous linear extension operator $p_\Omega : H^2(\Omega) \to H^2(B)$ satisfying:*

$$p_\Omega\left(u_\Omega\right)|_\Omega = u_\Omega \qquad \text{and} \qquad |||p_\Omega||| := \sup_{\substack{v\in H^2(\Omega)\\ v\neq 0}} \frac{\|p_\Omega\left(v\right)\|_{H^2(B)}}{\|v\|_{H^2(\Omega)}} \leqslant C(d,\varepsilon,n).$$

Therefore, we now set $U_i := p_{\Omega_i}(u_{\Omega_i})$. Combining Lemma 4.3 with Theorem 4.1, we deduce that $(U_i)_{i\geqslant I}$ is uniformly bounded in $H^2(B)$. Hence, there exists $U \in H^2(B)$ such that a subsequence, still denoted (U_i) for the moment, converges to U weakly in $H^2(B)$ and also strongly in $H^1(B)$ (Rellich-Kondrachov Theorem). In this section, we thus aim to show that U is an extension of u_{Ω_∞}. The standard procedure consists in proving that U satisfies (1.3)–(1.5) for Ω_∞ and the uniqueness of a solution will ensure that $U|_{\Omega_\infty} = u_{\Omega_\infty}$. We start by easily passing to the limit in the PDE.

Lemma 4.4. *Let $\lambda \in \mathbb{R}$ and $f \in L^2(B)$. We assume that $-\Delta U_i + \lambda U_i = f$ on Ω_i for any $i \geqslant I$. Then, we have $-\Delta U + \lambda U = f$ on Ω_∞.*

Proof. We consider the notation of Lemma C.5 and we set $h = 1$, $w_i = -\Delta U_i + \lambda U_i - f$ and $v_i = v = w = -\Delta U + \lambda U - f$. Then, from the hypothesis, we can apply Lemma C.5 to obtain that $\int_{\Omega_i}(-\Delta U_i + \lambda U_i - f)(-\Delta U + \lambda U - f)$ converges to $\int_{\Omega_\infty}(-\Delta U + \lambda U - f)^2$. Assuming $-\Delta U_i + \lambda U_i = f$ a.e. on Ω_i, we conclude that $-\Delta U + \lambda U = f$ a.e. on Ω_∞. □

We now aim to pass to the limit in the boundary conditions. Let $(\lambda, \mu) \in \mathbb{R}^2$ and $g \in H^2(B)$. Following the strategy of the previous proof, we are going to prove that:

$$\int_{\partial\Omega_i} [\,\mu\,\partial_{\mathbf{n}_i}U_i + \lambda\,U_i - g\,]\,[\,\mu\,\partial_{\mathbf{n}_i}U + \lambda\,U - g\,] \underset{i\to+\infty}{\longrightarrow} \int_{\partial\Omega_\infty}[\,\mu\,\partial_{\mathbf{n}_\infty}U + \lambda\,U - g\,]^2. \tag{4.3}$$

In fact, we can exactly proceed to the same argumentation than the one developed in Section 2.1.4 by considering the partition of unity of Section 2.1.3. In particular, the result of Lemma 2.8 remains true if u_{Ω_i} is replaced by U_i and u_{Ω_∞} by U. Hence, (4.3) holds true if we can prove that $(g \circ X_i)_{i\geqslant I}$, $(U_i \circ X_i)_{i\geqslant I}$ and $(\nabla U_i \circ X_i)_{i\geqslant I}$ respectively converges to $g \circ X_\infty$, $U \circ X_\infty$ and $\nabla U \circ X_\infty$ in $L^2(D)$.

Proposition 4.5. *Let $g \in H^1(B)$. Then, $(g \circ X_i)_{i\geqslant I}$ converges to $g \circ X_\infty$ strongly in $L^2(D)$.*

Proof. First, for any $i \in [\![I, \infty]\!]$, we have $g \in H^1(\Omega_i)$ so the trace of g is in $L^2(\partial\Omega_i)$ and thus $g \circ X_i \in L^2(D)$. Let $h \in C^1(B)$. From the Cauchy-Schwarz inequality, we have:

$$\int_D (g \circ X_i - g \circ X_\infty)^2 \leqslant 3 \int_D [\,(g \circ X_i - h \circ X_i)^2 + (h \circ X_i - h \circ X_\infty)^2 + (h \circ X_\infty - g \circ X_\infty)^2\,].$$

Then, the boundary of a set satisfying the ε-ball condition must remain at a positive distance of the boundary of B (at least 2ε, *cf.* Fig. 1). Hence, there exists a compact set $K \subset B$ such that $\partial\Omega \subset K$ for any $\Omega \in \mathcal{O}_\varepsilon(B)$. Recalling from Proposition 1.2 (i) that $(\Omega_i)_{i \in [\![I,\infty]\!]}$ satisfies the $\alpha(\varepsilon)$-cone property, we can apply Corollary C.2 then Proposition B.4 with $\eta = 1$ to obtain:

$$\|g \circ X_i - g \circ X_\infty\|^2_{L^2(D)} \leqslant 3 \left[2C_3C_0\left(\alpha\left(\varepsilon\right), d, n\right) \|g - h\|^2_{H^1(B)} + \|h\|^2_{C^1(K)} \|\varphi_i - \varphi_\infty\|^2_{C^0(\overline{D})} \int_D 1 \right].$$

For any $\varepsilon > 0$, there exists a certain $h_\varepsilon \in C^\infty(B)$ satisfying $\|g - h_\varepsilon\|_{H^1(B)} \leqslant \varepsilon$ (see *e.g.* [12], Sect. 4.2.1). There also exists $J \geqslant I$ such that for any $i \geqslant J$, we have $\|\varphi_i - \varphi_\infty\|_{C^0(\overline{D})} \leqslant \varepsilon/(1 + \|h_\varepsilon\|_{C^1(K)})$. We deduce $\|g \circ X_i - g \circ X_\infty\|_{L^2(D)} \leqslant \varepsilon[3(2C_0C_3 + \int_D 1)]^{1/2}$, concluding the proof. ☐

Corollary 4.6. *The sequence $(U_i \circ X_i)_{i \geqslant I}$ converges to $U \circ X_\infty$ strongly in $L^2(D)$.*

Proof. First, we have $\int_D (U_i \circ X_i - U \circ X_\infty)^2 \leqslant 2[\int_D (U_i \circ X_i - U \circ X_i)^2 + \int_D (U \circ X_i - U \circ X_\infty)^2]$. As in the previous proof, we can apply Corollary C.2 then Proposition B.4 with $\eta = 1$ to obtain $\|U_i \circ X_i - U \circ X_\infty\|^2_{L^2(D)} \leqslant 2[C_3C_0(\alpha(\varepsilon), d, n)\|U_i - U\|^2_{H^1(B)} + \|U \circ X_i - U \circ X\|^2_{L^2(D)}]$. We conclude by using Proposition 4.5 with $g = U$ and the strong convergence of U_i to U in $H^1(B)$. ☐

Note that the previous proof, and more precisely Corollary C.2, cannot be used to obtain the strong convergence of $(\nabla U_i \circ X_i)_{i \geqslant I}$ to $\nabla U \circ X_\infty$ in $L^2(D, \mathbb{R}^n)$. Indeed, in this case, we only have the weak convergence of $(\nabla U_i)_{i \geqslant I}$ to ∇U in $H^1(B, \mathbb{R}^n)$. We try instead to modify the quantity of interest in order to apply Stokes' Theorem, converting the surface integral into a volume one.

Proposition 4.7. *Let $l \in [\![1, n]\!]$ and consider any $w \in W^{1,\infty}(B)$ with compact support in $\mathcal{C}$. Then, the quantity $\int_D [\partial_l U_i \circ X_i - \partial_l U \circ X_i]^2 (w \circ X_i)$ converges to zero as $i \to +\infty$.*

Proof. Let $l \in [\![1, n]\!]$ and w as in the statement. First, we have:

$$\delta_i := \int_D \left[\partial_l U_i \circ X_i - \partial_l U \circ X_i\right]^2 \,(w \circ X_i) \;=\; \int_{\partial\Omega_i} \left[\partial_l\left(U_i - U\right)\right]^2 \, w\, \langle \mathbf{d} \mid \mathbf{n}_i \rangle, \tag{4.4}$$

where $\mathbf{d} := \mathbf{d}_{\mathbf{x}_k}$ represents the unit vector of our local frame according to which the last coordinate is expressed. It also refers to the direction of our local cylinder $\mathcal{C}$ written $C_{r,\varepsilon}(\mathbf{x}_k)$ in Section 2.1.3 before dropping the index k. The last equality comes from the computation of the normal vector $\mathbf{n}_i$ in the local parametrization, as in (2.8) for instance. Indeed, we have $\langle \mathbf{d} \mid \mathbf{n}_i \rangle = (1 + |\varphi_i|^2)^{-1/2}$ and the infinitesimal surface element also satisfies $\mathrm{d}A = (1 + |\varphi_i(\mathbf{x}')|^2)^{1/2}\,\mathrm{d}\mathbf{x}'$. Then, we are now in position to apply Stokes' Theorem in the right member (4.4), from which we deduce that:

$$\delta_i \;\leqslant\; \|\nabla w\|_{L^\infty(B,\mathbb{R}^n)} \left\|\partial_l\left(U_i - U\right)\right\|^2_{L^2(B)} \;+\; \sum_{j=1}^n \left| \int_{\Omega_i} \partial_{jl}\left(U_i - U\right)\, \partial_l\left(U_i - U\right)\, 2w\mathbf{d}_j \right|.$$

Finally, we set $w_i = \partial_{jl}(U_i - U) \rightharpoonup 0$ and $v_i = \partial_l(U_i - U) \to 0$ in $L^2(B)$. Applying Lemma C.5 with $h = 2w\mathbf{d}_j \in L^\infty(B)$, we conclude from the last inequality that $\delta_i \to 0$ as $i \to +\infty$. ☐

Corollary 4.8. *The sequence $(\nabla U_i \circ X_i)_{i \geqslant I}$ converges to $\nabla U \circ X_\infty$ strongly in $L^2(D, \mathbb{R}^n)$.*

Proof. Let $l \in [\![1, n]\!]$. First, we can divide the integral thanks to the Cauchy-Schwarz inequality: $\int_D [\partial_l U_i \circ X_i - \partial_l U \circ X_\infty]^2 \leqslant 2\int_D [\partial_l U_i \circ X_i - \partial_l U \circ X_i]^2 + 2\int_D [\partial_l U \circ X_i - \partial_l U \circ X_\infty]^2$. The right member of the last inequality is composed of two integrals that we are going to estimate separately. We denote them I_1 and I_2 from left to right. On the one hand, we have $\partial_l U \in H^1(B)$ so Proposition 4.5 with $g = \partial_l U$ directly yields $I_2 \to 0$ as $i \to +\infty$. On the other hand, we can apply Proposition C.1 with $u = [\partial_l (U_i - U)]^2$ in order to obtain:

$$I_1 := \int_D (\partial_l U_i \circ X_i - \partial_l U \circ X_i)^2 \;\leqslant\; C_3 \left(\alpha\left(\varepsilon\right), n\right) \sum_{k=1}^{K} \int_{D_k} \left(\partial_l U_i \circ X_i^k - \partial_l U \circ X_i^k\right]^2 \xi^k \circ X_i^k,$$

where Proposition 1.2 (i) ensures that the constant $C_3 > 0$ of Proposition C.1 only depends on ε and n. Since ξ^k has compact support in C_k, $\xi^k \circ X_i^k$ has compact support in D_k. We conclude the proof of Corollary 4.8 by considering Proposition 4.7 with $w = \xi^k$ and we end with $I_1 \to 0$. □

4.3. The strong convergence of the entire sequences

First, let us resume the situation by shortly gathering the results we can deduce from Section 4.2.

Lemma 4.9. *Let $(\lambda, \mu) \in \mathbb{R}^2$ and $g \in H^1(B)$. We assume that $\mu\, \partial_{\mathbf{n}_i} U_i + \lambda\, U_i = g$ on $\partial\Omega_i$ for any $i \geqslant I$. Then, we have $\mu\, \partial_{\mathbf{n}_\infty} U + \lambda\, U = g$ on $\partial\Omega_\infty$.*

Proof. First, combining Proposition 4.5 with Corollaries 4.6 and 4.8, we have proved $(g \circ X_i)_{i\geqslant I}$, $(U_i \circ X_i)_{i\geqslant I}$ and $(\nabla U_i \circ X_i)_{i\geqslant I}$ respectively converges to $g \circ X_\infty$, $U \circ X_\infty$ and $\nabla U \circ X_\infty$ in $L^2(D)$. Then, considering the arguments given in the proof of Lemma 2.8, we can express the boundary integrals of (4.3) according to the partition of unity and parametrization given in Sections 2.1.3 and 2.1.4. We deduce that (4.3) holds true. Finally, if we assume that $\mu\, \partial_{\mathbf{n}_i} U_i + \lambda\, U_i = g$ a.e. on $\partial\Omega_i$, then we get from (4.3) that $\mu\, \partial_{\mathbf{n}_\infty} U + \lambda\, U = g$ a.e. on $\partial\Omega_\infty$, concluding the proof of Lemma 4.9. □

Theorem 4.10. *The sequences $(u_{\Omega_i} \circ X_i)_{i\geqslant I}$ and $(\nabla u_{\Omega_i} \circ X_i)_{i\geqslant I}$ respectively converge to $u_{\Omega_\infty} \circ X_\infty$ and $\nabla u_{\Omega_\infty} \circ X_\infty$ strongly in $L^2(D)$. In particular, the assumptions of Remark 2.9 holds true.*

Proof. Let $l \in [\![0, n]\!]$. We set the notation $\partial_0 f = f$ to treat all the cases simultaneously. First, we assume by contradiction that $u_i := \partial_l u_{\Omega_i} \circ X_i$ does not converge to $u := \partial_l u_{\Omega_\infty} \circ X_\infty$. Hence, there exists a subsequence $u_{i'}$ staying at a positive distance from u in $L^2(D)$. We now introduce the H^2-extension $U_{i'}$ of $u_{\Omega_{i'}}$ defined in Section 4.2. Since $U_{i'} = u_{\Omega_{i'}}$ on $\Omega_{i'}$ as H^2-maps, their traces and the ones of their gradients are equal on $\partial\Omega_{i'}$ as L^2-maps. Consequently, we can correctly replace $u_{\Omega_{i'}}$ by $U_{i'}$ in the formulations (1.3)–(1.5). Then, we combine Lemma 4.3 with Theorem 4.1 to ensure that $U_{i'}$ is uniformly bounded in $H^2(B)$. Hence, there exists $U \in H^2(B)$ such that a subsequence $U_{i''}$ converges to U weakly in $H^2(B)$ and also strongly in $H^1(B)$. Adjusting the values of λ and μ to fit the PDE and boundary conditions of (1.3)–(1.5), Lemmas 4.4 and 4.9 ensure that U satisfies (1.3)–(1.5) for Ω_∞. From the uniqueness of a solution, we get that $U = u_{\Omega_\infty}$ on Ω_∞ as H^2-maps. In particular, their traces and the ones of their gradients are equal on $\partial\Omega_\infty$ as L^2-maps. Finally, we have obtained that $u_{i''} = \partial_l U_{i''} \circ X_i$ converges to $u = \partial_l U \circ X_\infty$ strongly in $L^2(D)$, which contradicts the definition of $u_{i'}$. The entire sequence u_i converges to u, concluding the proof. □

Dealing with U_i instead of u_{Ω_i} allowed us to get the convergence of the boundary term in an easier way. Indeed, we could consider $U \circ X_i$ whereas $u_\Omega \circ X_i$ does not have any meaning. However, there is a price to pay: the limiting map is uniquely determined only on Ω_∞ and *not* on B so we can only work up to a subsequence. One usual way to recover the convergence of the entire sequence consists in fixing to zero the values of the maps outside the domain. This is possible in our ε-ball framework since $\mathbf{1}_{\Omega_i}$ converges to $\mathbf{1}_{\Omega_\infty}$ in $L^1(B)$. Our continuity result states as follows.

Theorem 4.11. *Let us consider the assumptions and the notation of Theorem 4.1. Then, the sequence* $(\mathbf{1}_{\Omega_i} u_{\Omega_i}, \mathbf{1}_{\Omega_i} \nabla u_{\Omega_i})_{i\geqslant I}$ *converges to* $(\mathbf{1}_{\Omega_\infty} u_{\Omega_\infty}, \mathbf{1}_{\Omega_\infty} \nabla u_{\Omega_\infty})$ *strongly in* $L^2(B,\mathbb{R}^{n+1})$. *Moreover, the sequence* $(\mathbf{1}_{\Omega_i} \text{Hess } u_{\Omega_i})_{i\geqslant I}$ *converges to* $(\mathbf{1}_{\Omega_\infty} \text{Hess } u_{\Omega_\infty})$ *weakly in* $L^2(B,\mathbb{R}^{n\times n})$.

Proof. Let $l \in [\![0,n]\!]$. We again set the notation $\partial_0 f = f$ to treat more cases at the same time. First, we assume by contradiction that $u_i := \mathbf{1}_{\Omega_i}\partial_l u_{\Omega_i}$ does not converge to $u := \mathbf{1}_{\Omega_\infty}\partial_l u_{\Omega_\infty}$. Hence, there exists a subsequence $u_{i'}$ remaining at a positive distance from u in $L^2(B)$. However, we can consider the H^2-extension $U_{i'}$ of $u_{\Omega_{i'}}$ defined in Section 4.2. Following the arguments given in the proof of Theorem 4.10, the sequence is uniformly bounded in $H^2(B)$ so there exists a subsequence $U_{i''}$ converging to a certain U weakly in $H^2(B)$ and strongly in $H^1(B)$. Moreover, we also deduce that $U = u_{\Omega_\infty}$ on Ω_∞. We thus have:

$$\int_B (u_{i''} - u)^2 \;=\; \int_{\Omega_{i''}} (\partial_l U_{i''})^2 - \int_B \mathbf{1}_{\Omega_{i''}}\partial_l U_{i''}\, 2\partial_l U \mathbf{1}_{\Omega_\infty} + \int_{\Omega_\infty} (\partial_l U)^2. \tag{4.5}$$

On the one hand, we get $\int_{\Omega_{i''}}(\partial_l U_{i''})^2 \to \int_{\Omega_\infty}(\partial_l U)^2$ from Lemma C.5 with $v_i = w_i = \partial_l U_{i''}$ and $h = 1$. On the other hand, we consider Lemma C.4 with $w_i = \partial_l U_{i''}$ and $v = 2\partial_l U\mathbf{1}_{\Omega_\infty}$, which yields $\int_B \mathbf{1}_{\Omega_{i''}}\partial_l U_{i''}\, 2\partial_l U \mathbf{1}_{\Omega_\infty} \to 2\int_{\Omega_\infty}(\partial_l U)^2$. Letting $i \to +\infty$ in (4.5), we obtain $u_{i''} \to u$ strongly in $L^2(B)$, contradicting the definition of $u_{i'}$. Then, let $(l,j) \in [\![1,n]\!]^2$ and assume again by contradiction that $u_i := \mathbf{1}_{\Omega_i}\partial_{jl}u_{\Omega_i}$ does not weakly converge to $u := \mathbf{1}_{\Omega_\infty}\partial_{jl}u_{\Omega_\infty}$ in $L^2(B)$. Hence, there exists $\varepsilon > 0$ and $v \in L^2(B)$ such that a subsequence $u_{i'}$ satisfies $|\int_B (u_{i'} - u)v| > \varepsilon$. Following again the arguments given in the proof of Theorem 4.10, the extension $U_{i'}$ of Section 4.2 remains uniformly bounded in $H^2(B)$ so it has a subsequence $U_{i''}$ converging to a certain U weakly in $H^2(B)$ and strongly in $H^1(B)$. Moreover, we also obtain $U = u_{\Omega_\infty}$ on Ω_∞. Finally, as for (4.5), we consider Lemma C.5 with $w_i = \partial_{jl}U_i \rightharpoonup w = \partial_{jl}U$, $v_i = v$ and $h = 1$ in order to deduce $\int_B (u_{i''} - u)v \to 0$, contradicting the definition of $u_{i'}$ and concluding the proof of Theorem 4.11. □

We conclude this section by proving that the weak convergence of Theorem 4.11 is in fact a strong one. An equivalent property can be obtained in the framework of the uniform cone property by simply taking u_{Ω_i} in the weak PDE formulation and correctly pass to the limit. Here, we get the strong convergence by considering instead Grisvard's identity (A.1) with $\mathbf{v} = \nabla u_{\Omega_i}$. Indeed, it relates the $L^2(B)$-norm of $\mathbf{1}_{\Omega_i}\partial_{jl}u_{\Omega_i}$ with boundary and volume integrals involving lower-order terms thanks to the PDE and boundary conditions of (1.3)–(1.5). We thus aim to correctly let $i \to +\infty$ in (A.1). From the foregoing, note that there is only one boundary term whose convergence has not already been studied: the one involving the second fundamental form $\mathbf{II}(\bullet,\bullet) := -\langle D_{\partial\Omega}\mathbf{n}(\bullet) \mid \bullet\rangle$.

Proposition 4.12. *Let* $(\Omega_i)_{i\in\mathbf{N}} \subset \mathcal{O}_\varepsilon(B)$ *converge to* $\Omega_\infty \in \mathcal{O}_\varepsilon(B)$ *as in Proposition 2.6. Then, we have:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} \mathbf{II}\left[\nabla_{\Sigma_i}u_{\Omega_i}, \nabla_{\Sigma_i}u_{\Omega_i}\right] \mathrm{d}A \;=\; \int_{\partial\Omega_\infty} \mathbf{II}\left[\nabla_{\Sigma_\infty}u_{\Omega_\infty}, \nabla_{\Sigma_\infty}u_{\Omega_\infty}\right] \mathrm{d}A,$$

where we have set $\Sigma_i := \partial\Omega_i$, *in the tangential operators, for any* $i \in \mathbb{N}\cup\{\infty\}$.

Proof. We only have to express the functional $F : \Omega \in \mathcal{O}_\varepsilon(B) \to \int_{\partial\Omega}\mathbf{II}(\nabla_\Sigma u_\Omega, \nabla_\Sigma u_\Omega)\,\mathrm{d}A$ in the local parametrization of Sections 2.1.4 and 2.1.3, as it is done for (2.8). Using the same notation, we have that $F(\Omega_i)$ is equal to the sum from $k = 1$ to $k = K$ of the following quantity:

$$\int_{D_r(\mathbf{x}_k)} \xi^k \circ X_i^k \sum_{p,p',q,q'=1}^{n-1} g_i^{pqk} g_i^{p'q'k} \left\langle \nabla u_{\Omega_i} \circ X_i^k \mid \partial_q X_i^k \right\rangle \left\langle \nabla u_{\Omega_i} \circ X_i^k \mid \partial_{q'} X_i^k \right\rangle \partial_{pp'}\varphi_i^k, \tag{4.6}$$

where $g_i^{pqk} := \delta_{pq} - \frac{\partial_p\varphi_i^k\,\partial_q\varphi_i^k}{1+|\nabla\varphi_i^k|^2}$ is the inverse matrix of $\langle \partial_p X_i^k \mid \partial_q X_i^k\rangle$. Such an expression is obtained by decomposing $\nabla_{\Sigma_i}u_{\Omega_i} \circ X_i^k$ and the normal vector in the basis $(\partial_p X_i^k)_{1\leqslant p\leqslant n-1}$, as in the proof of Lemma A.1. We now use the convergence properties of the different terms in (4.6). First, from Proposition 2.7, the sequences $(g_i^{pqk})_{i\geqslant I}$, $(\xi^k \circ X_i^k)_{i\geqslant I}$ and $(\partial_p X_i^k)_{i\geqslant I}$ converges uniformly to g_∞^{pqk}, $\xi_k \circ X_\infty^k$ and $\partial_p X_\infty^k$, respectively. Then, Theorem 4.10 yields the L^1-strong convergence of the integrand, except $\partial_{pp'}\varphi_i^k$. Finally, since $(\partial_{pp'}\varphi_i^k)_{i\geqslant I}$ converges to $\partial_{pp'}\varphi_\infty^k$ weakly-star in $L^\infty(D_r(\mathbf{x}_k))$, we conclude that we can correctly let $i \to +\infty$ in (4.6) and $F(\Omega_i) \to F(\Omega_\infty)$. □

Corollary 4.13. *Consider the assumptions and notation of Theorem 4.1. Then, the sequence* $(\mathbf{1}_{\Omega_i} u_{\Omega_i}, \mathbf{1}_{\Omega_i}\nabla u_{\Omega_i}, \mathbf{1}_{\Omega_i}\text{Hess } u_{\Omega_i})_{i\geqslant I}$ *converges to* $(\mathbf{1}_{\Omega_\infty} u_{\Omega_\infty}, \mathbf{1}_{\Omega_\infty}\nabla u_{\Omega_\infty}, \mathbf{1}_{\Omega_\infty}\text{Hess } u_{\Omega_\infty})$ *strongly in* $L^2(B, \mathbb{R}^{1+n+n^2})$.

Proof. From Theorem 4.11, we only have to prove $\sum_{j,l=1}^n \int_{\Omega_i} (\partial_{jl} u_{\Omega_i})^2 \to \sum_{j,l=1}^n \int_{\Omega_\infty} (\partial_{jl} u_{\Omega_\infty})^2$. First, let u_Ω refer to the solution of the Dirichlet Laplacian (1.3). We have:

$$\delta_i := \sum_{j,l=1}^n \int_{\Omega_i} (\partial_{jl} u_{\Omega_i})^2 = \sum_{j,l=1}^n \int_{\Omega_i} [\partial_{jl}(u_{\Omega_i} - g)]^2 + 2\sum_{j,l=1}^n \left(\int_{\Omega_i} \partial_{jl} g \, \partial_{jl} u_{\Omega_i} - \int_{\Omega_i} (\partial_{jl} g)^2 \right).$$

We now apply Proposition A.2 with $\mathbf{v} = \nabla(u_{\Omega_i} - g)$. We set $\Sigma_i := \partial\Omega_i$. Since we have $u_{\Omega_i} - g \in H^2(\Omega_i) \cap H_0^1(\Omega_i)$, we deduce $\nabla_{\Sigma_i}(u_{\Omega_i} - g) = 0$ and $\nabla(u_{\Omega_i} - g) = \partial_{\mathbf{n}}(u_{\Omega_i} - g)\mathbf{n}_i$ on $\partial\Omega_i$. We get from (A.1) that:

$$\delta_i = \int_{\Omega_i} (f - \Delta g)^2 - \int_{\partial\Omega_i} H\,|\nabla u_{\Omega_i}|^2 + 2\sum_{j,l=1}^n \left(\int_{\Omega_i} \partial_{jl} g \partial_{jl} u_{\Omega_i} - \int_{\Omega_i} (\partial_{jl} g)^2 \right).$$

We combine Theorem 4.11 with Lemmas C.3–C.5 to obtain the convergence of the volume integrals, whereas the boundary integral converges because it has an integrand which is linear in H, quadratic in ∇u_{Ω_i}, and continuous with respect to its set of variables. Then, we assume that u_Ω refer to the solution of the Neumann Laplacian (1.4). Apply Proposition A.2 with $\mathbf{v} = \nabla(u_{\Omega_i})$, we have:

$$\delta_i = \int_{\Omega_i} (f - \lambda u)^2 + \int_{\partial\Omega_i} \langle \nabla_{\Sigma_i} u_{\Omega_i} \mid 2\nabla g - (H + 2\partial_{\mathbf{n}_i} g)\,\mathbf{n}_i \rangle + \int_{\partial\Omega_i} \mathbf{II}\,[\,\nabla_{\Sigma_i} u_{\Omega_i}, \nabla_{\Sigma_i} u_{\Omega_i}\,].$$

We can correctly let $i \to +\infty$ as in the previous case. The only difference here is the integral involving the second fundamental form, which is passed to the limit thanks to Proposition 4.12. Finally, similar arguments also work when u_Ω is the solution of the Robin Laplacian (1.5). □

5. Three direct applications of Theorem 2.1

In this section, we detail three direct applications of Theorem 2.1 that can be found in the literature. The first one is purely geometrical and has already been established in the previous work ([10], Prop. 4.27). It concerns a classical model for the configuration of fluid membranes, also called *vesicles* ([21], Sects. 2.4–2.5). The second example is related to the optimal design of a wing profile for a plane ([20], Sect. 2.3.4). It is a typical example of shape optimization problems involving PDE constraints. Finally, we consider the shape optimization formulation for the inverse obstacle problem with impedance boundary conditions. It can be seen as a particular case of the more general inverse problem described in ([6], Sect. 2.2).

5.1. On the configuration of fluid membranes

In biology, an aqueous media containing a large amount of phospholipids will result in the formation of bilayers, also called vesicles. They can be designed artificially and are quite studied because they can model, for example, the behaviour of red blood cells. Given $k_b > 0$, $(H_0, k_m, M_0) \in \mathbb{R}^3$, it can be shown ([21], Sect. 2) that the shape of a vesicle is minimizing the following energy:

$$E_1(\Omega) := k_b \int_{\partial\Omega} (H - H_0)^2 \, \mathrm{d}A + k_m \left(\int_{\partial\Omega} H \, \mathrm{d}A - M_0 \right)^2, \tag{5.1}$$

subject to an inner volume constraint $V(\Omega) = V_0$ and prescribed area $A(\partial\Omega) = A_0$. We recall that H refers to the scalar mean curvature while K denotes the Gaussian curvature (*cf.* below (1.6)). The spherical topology is also imposed thanks to the Gauss-Bonnet Theorem: $\int_{\partial\Omega} K \, \mathrm{d}A = 4\pi$. Moreover, we have to assume the isoperimetric inequality $A_0^3 > 36\pi V_0^2$ in order to have a non-empty class of admissible shapes satisfying the area and volume constraints. This formulation is referred to as the *area-difference-elasticity* model ([21], Sect. 2.5.5), where a so-called *spontaneous curvature* H_0 can be added to describe a potential asymmetry between the two layers. The existence and regularity of minimizers for such shape optimization problems remain open. Experiments show that singular behaviours can occur to vesicles such as the budding effect ([21], Fig. 11). Our existence result states as follows.

Proposition 5.1 (**[10], Prop. 4.27**). *Let $(k_b, H_0, k_m, M_0, A_0, V_0) \in \mathbb{R}^6$ with $k_b, V_0 > 0$ and $A_0^3 > 36\pi V_0^2$. We also consider $\varepsilon > 0$ and B as in Theorem 2.1. Then, there exists a minimizer to the shape optimization problem* $\inf E_1(\Omega)$ *where E_1 is defined by (5.1), and where the infimum is taken among all $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying $A(\partial\Omega) = A_0$, $V(\Omega) = V_0$, and $\int_{\partial\Omega} K \, \mathrm{d}A = 4\pi$.*

Finally, we mention that imposing a uniform ball condition here can also be motivated by the modelization of the equilibrium shapes of red blood cells. They are usually represented as vesicles on which is fixed a network of proteins playing the role of a skeleton inside the membrane, which prevents the membrane from bending too much locally ([18], Sect. 2.1).

5.2. On the optimal design of a wing

We now give a simplified example related to the optimal cross-sectional profile of an aircraft wing. It aims to illustrate the fact that cost functionals defined on the boundary of the domain can arise quite naturally in shape optimization problems with PDE constraints.

First, we detail the optimal design problem described in ([20], Sect. 2.3.4) and illustrated in Figure 3. Let B be a large fixed hold-all in $\mathbb{R}^2$. We consider a wing profile denoted by $\mathcal{S} \subset B$ and we are interested here in the region occupied by air (the complement of $\mathcal{S}$ in B). At moderate speed, the pressure on the wing is given by:

$$P := k_1 - k_2 |\nabla\Psi|^2,$$

where $k_1, k_2 > 0$ are given, and where Ψ is the solution of:

$$\Delta\Psi = 0 \quad \text{on } B \backslash \mathcal{S}, \qquad \Psi = 0 \quad \text{on } \partial\mathcal{S}, \qquad \Psi(x, y) = u_{0x} y - u_{0y} x + \lambda \quad \text{on } \partial B. \tag{5.2}$$

The vector $\mathbf{u}_0 := (u_{0x}, u_{0y})$ is prescribed and corresponds to the velocity of the wing. The parameter λ can be determined from the Joukowski condition stating that there exists only one $\lambda \in \mathbb{R}$ for which the flow does not turn around the trailing edge of $\mathcal{S}$ (represented by the point T in Fig. 3). The lift factor of the wing is proportional to λ and good wings have a boundary layer that separates from the profile very close to the trailing edge. Since this last property is related to the flatness of the wing pressure P, one may want to minimize the following energy:

$$\int_{\partial\mathcal{S}} \left| \, k - |\nabla\Psi|^2 \, \right| , \tag{5.3}$$

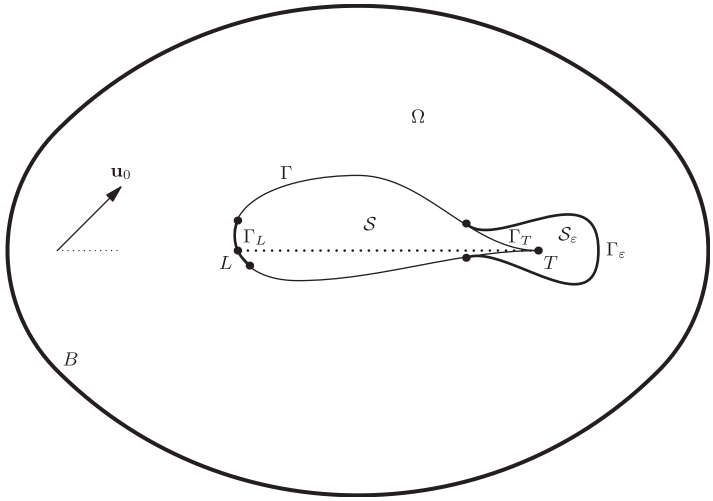

FIGURE 3. How to modify the original problem of optimizing wing profiles into a framework that fits into the ε-ball condition.

with $k := k_1/k_2$. Hence, we look for an optimal design by minimizing (5.3) among all wing profiles $\mathcal{S}$ given a trailing edge (the point T in Fig. 3) and a chord line (the horizontal dotted segment $[LT]$ in Fig. 3). However, in this simplified model, the wing profile must be singular at the trailing edge T, otherwise there is no lift (meaning the airfoil cannot fly). In particular, the approach does not fit directly in the $C^{1,1}$-regularity framework of the ε-ball condition, but we now show how an approximation of this problem can be solved with the help of Theorem 2.1.

Instead of imposing the trailing edge and the chord line, we are going to freeze a neighbourhood of the leading edge (denoted Γ_L in Fig. 3) and of the trailing edge (denoted Γ_T in Fig. 3). By doing so, only the part denoted $\Gamma := \partial\mathcal{S}\backslash(\Gamma_T \sqcup \Gamma_L)$ in Figure 3 is meant to be optimized.

Then, let $\varepsilon > 0$ be fixed and small enough. From the knowledge of Γ_T and ε, it is possible to build an approximate portion Γ_ε, as close as possible to Γ_T, and so that the boundary $\Gamma_L \sqcup \Gamma \sqcup \Gamma_\varepsilon$ encloses a domain denoted $\mathcal{S}_\varepsilon$ and satisfies the ε-ball condition. Note that if we decrease the value of ε, then we can choose around T a smaller portion Γ_T to be frozen, and consequently, the *approximated ε-hull* Γ_ε will be smaller and will better approximate Γ_T. As shown in Figure 3, we now define $\Omega := B\backslash\mathcal{S}_\varepsilon$ and consider the solution u_Ω of:

$$\Delta u_\Omega = 0 \quad \text{on } \Omega, \qquad u_\Omega = 0 \quad \text{on } \partial\mathcal{S}_\varepsilon, \qquad u_\Omega(x,y) = u_{0x}y - u_{0y}x \quad \text{on } \partial B. \tag{5.4}$$

Moreover, we introduce the approximate energy to minimize:

$$E_2(\Omega) := \int_{\partial\Omega\backslash\Gamma_0} \left| k - |\nabla u_\Omega|^2 \right| , \tag{5.5}$$

where we have set $\Gamma_0 := \partial B \sqcup \Gamma_\varepsilon$. This formulation now fits into the framework of Theorem 2.1.

Proposition 5.2. *There exists a minimizer to the shape optimization problem* $\inf E_2(\Omega)$ *where* E_2 *is defined by* (5.5)*, and where the infimum is taken among all* $\Omega \in \mathcal{O}_\varepsilon(B)$ *such that* $\Gamma_0 \subset \partial\Omega$ *and* $\Gamma_L \subset \partial\Omega$.

Finally, let us denote by $\mathcal{S}_\varepsilon^*$ the optimal profile obtain with Proposition 5.2. We can now completely reverse the previous construction process and we end up with a wing profile $\mathcal{S}^*$ (containing Γ_T and Γ_L) that can be seen as a good candidate to approximate a minimizer of (5.3). We mention that $\partial\mathcal{S}^*$ satisfies the exterior sphere condition as in ([14], I Sect. 2.8 (2.34)) so that the regularity theory of harmonic functions ([14], I, Sect. 2.8 Thm. 1.4) ensures that problem (5.2) has a well-defined solution Ψ^* associated with $\mathcal{S}^*$.

We emphasize the fact that our approximate wing profile $\mathcal{S}^*$ depends on ε and it would be very interesting to investigate if the shape $\mathcal{S}^*$ converges to a minimizer of (5.3) as ε tends to zero. This seems to be a delicate question that would lead us beyond the scope of this article.

5.3. On the inverse obstacle problem

Here, we consider a particular case of the problem considered in [6]. It deals with the reconstruction of the shape of an object living in a larger bounded domain from boundary measurements. The method followed adopts the shape optimization viewpoint. We refer to [6] for further references on this vast topic. Let $\varepsilon > 0$ and B be as in Theorem 2.1, also satisfying the ε-ball condition. Then, for any inclusion $\omega \in O_\varepsilon(B)$, we introduce the solution u_Ω to the following PDE over $\Omega := B\backslash\overline{\omega}$:

$$\Delta u = 0 \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\omega, \qquad u = g_D \quad \text{on } \partial B,$$

where g_D is given and corresponds to the imposed impedance boundary conditions. Measuring the response g_N of the system over a fixed subregion $\Gamma \subset \partial B$, one may want to identify the shape of the inclusion ω by minimizing the following energy:

$$E_3\left(\Omega\right) := \int_\Gamma \left(\partial_{\mathbf{n}} u_\Omega - g_N\right)^2. \tag{5.6}$$

We are now in position to state our existence result.

Proposition 5.3. *Let $(g_D, g_N) \in H^2(B) \times H^2(B)$. Then, there exists a minimizer to the shape optimization problem* $\inf E_3(\Omega)$ *where E_3 is defined by (5.6), and where the infimum is taken among all $\omega \in \mathcal{O}_\varepsilon(B)$ such that $\int_\omega K \, \mathrm{d}A = 4\pi$.*

Note that the last condition simply ensures that ω has a spherical topology (Gauss-Bonnet theorem). In particular, the set $\Omega = B\backslash\overline{\omega}$ is constrained to remain simply connected.

6. Conclusion

In this article, we have obtained quite general existence results for shape optimization problems where the functional is defined over the boundary of the domain, involving the geometry of the associated hypersurface and the boundary values of the PDE solution associated with the Laplace operator. We have given three applications and we are quite confident in extending the method developed in this paper to some other operators. For example, it would be very interesting to study similar shape optimization problems for the biLaplacian operator, such as the Kirchhoff-Love model for a thin plate described in ([3], (1.4)). Another possible extension concerns some eigenvalue problems related to the Dirac operator, such as the fundamental state for the MIT-bag model in relativistic quantum mechanics [17]. To our knowledge, these two problems are open and could fit within the framework of the ε-ball condition, since they both involve PDE and the geometry of the boundary in the functional.

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position at the ICS[3] and then at the CMM[4]. Finally, the author wants to warmly thank the referees who gave several important suggestions to wisely improve the quality of the paper.

Appendix A. Grisvard's identity for $C^{1,1}$-domains

In ([15], Sect. 3.1.1.1), Grisvard proves an identity for C^2-domains, which is then used to get *a priori* H^2-estimates for the Laplacian in convex sets. From our shape optimization perspective, this method is interesting because a uniform control in terms of curvature can be done for the constant of such estimates, as suggested in ([15], Sect. 3.1.2.2). In this section, our only contribution is to prove that the identity still holds for domains with $C^{1,1}$-boundary, which firstly requires a similar version for Stokes' theorem. It is also the occasion to introduce some notation used thereafter in the proofs. We mention that our proof of Stokes' theorem exactly follows the one given in ([16], Lem. 5.4.10), checking that C^2-regularity can be relaxed by the $C^{1,1}$-one. For very low regular vector fields (*i.e.* in $W^{1,1}$), we believe it is the minimal regularity that can be assumed over the hypersurface so that Stokes' theorem holds.

Lemma A.1 (**Stokes' theorem for $C^{1,1}$-hypersurfaces**). *Let Σ be a compact $C^{1,1}$-hypersurface of $\mathbb{R}^n$. Then, for any map $\mathbf{v} \in W^{1,1}(\Sigma, \mathbb{R}^n)$ such that $\langle \mathbf{v} \mid \mathbf{n} \rangle = 0$, we have $\int_\Sigma \mathrm{div}_\Sigma \mathbf{v} = 0$.*

Proof. Pick any $\mathbf{x}_0 \in \Sigma$. Since Σ be a compact $C^{1,1}$-hypersurface of $\mathbb{R}^n$, there exists a cylinder $C(\mathbf{x}_0)$ centred at $\mathbf{x}_0$ and in which Σ can be written as the graph of a $C^{1,1}$-map φ. We can thus introduce the local $C^{1,1}$-parametrization $X : \mathbf{x}' \in D(\mathbf{x}_0) \mapsto (\mathbf{x}', \varphi(\mathbf{x}')) \in \Sigma \cap C(\mathbf{x}_0)$, where the disk $D(\mathbf{x}_0)$ centred at $\mathbf{x}_0$ refers to the basis of $C(\mathbf{x}_0)$. First, we assume that $\mathbf{v} : \Sigma \to \mathbb{R}^n$ is a smooth map with compact support in $\Sigma \cap C(\mathbf{x}_0)$. Expressing $\mathbf{v} \circ X$ in the basis $(\partial_1 X, \ldots, \partial_{n-1} X, \mathbf{n} \circ X)$ and using the hypothesis $\langle \mathbf{v} \mid \mathbf{n} \rangle = 0$, we get:

$$\mathbf{v} \circ X \ = \ \sum_{i,j=1}^{n-1} g^{ij} \langle \mathbf{v} \circ X \mid \partial_j X \rangle \, \partial_i X \ + \ \langle \mathbf{v} \circ X \mid \mathbf{n} \circ X \rangle \, \mathbf{n} \circ X \ = \ \sum_{i,j=1}^{n-1} g^{ij} \, v_j \, \partial_i X,$$

where $g^{ij} := \delta_{ij} - \frac{\partial_i \varphi \, \partial_j \varphi}{1 + |\nabla \varphi|^2}$ is the inverse matrix of $g_{ij} := \langle \partial_i X \mid \partial_j X \rangle$, and where $v_j := \langle \mathbf{v} \circ X \mid \partial_j X \rangle$. In this decomposition, note that $\mathbf{v} \circ X$ is Lipschitz continuous. In particular, it is differentiable almost everywhere so we can compute a.e. $(\mathrm{div}_\Sigma \mathbf{v}) \circ X := \sum_{i,j=1}^{n-1} g^{ij} \langle \partial_j (\mathbf{v} \circ X) \mid \partial_i X \rangle$. Direct standard (but tedious) calculations yield the following relation a.e.

$$(\mathrm{div}_\Sigma \mathbf{v}) \circ X = \frac{1}{\sqrt{\det(g)}} \sum_{i=1}^{n-1} \partial_i \left(\sqrt{\det(g)} \sum_{j=1}^{n-1} g^{ij} \, v_j \right).$$

Since $\mathbf{v}$ has compact support in $\Sigma \cap C(\mathbf{x}_0)$, so does $v_j := \langle \mathbf{v} \circ X \mid \partial_j X \rangle$ in $D(\mathbf{x}_0)$. We thus obtain successively: $\int_\Sigma \mathrm{div}_\Sigma \mathbf{v} = \int_{D(\mathbf{x}_0)} \sqrt{\det(g)} \mathrm{div}_\Sigma \mathbf{v} \circ X = \sum_{i=1}^{n-1} \int_{D(\mathbf{x}_0)} \partial_i (\sqrt{\det(g)} \sum_{j=1}^{n-1} g^{ij} v_j) = 0$. Hence, Lemma A.1 is established for smooth maps $\mathbf{v} : \Sigma \to \mathbb{R}^n$ with compact support in $\Sigma \cap C(\mathbf{x}_0)$ for any $\mathbf{x}_0 \in \Sigma$. Then, we assume $\mathbf{v} \in C^\infty(\Sigma, \mathbb{R}^n)$. Since Σ is compact, there are a finite number of points of $\partial\Omega$ denoted $\mathbf{x}_1, \ldots, \mathbf{x}_K$ and such that $\Sigma \subset \cup_{k=1}^K C(\mathbf{x}_k)$. Thanks to a partition of unity, there exists K smooth maps $\xi_k : \mathbb{R}^n \to [0,1]$ with compact support in $C(\mathbf{x}_k)$ such that $\sum_{k=1}^K \xi_k = 1$ on Σ. Since $\xi_k \mathbf{v}$ is a smooth map with compact support in $C(\mathbf{x}_k)$, $k = 1, \ldots, K$, we deduce:

$$\int_\Sigma \mathrm{div}_\Sigma \mathbf{v} \ = \ \int_\Sigma \mathrm{div}_\Sigma \left(\sum_{k=1}^K \xi_k \mathbf{v} \right) \ = \ \sum_{k=1}^K \int_\Sigma \mathrm{div}_\Sigma \left(\xi_k \mathbf{v} \right) \ = \ 0.$$

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Finally, we can extend the result to $W^{1,1}(\Sigma,\mathbb{R}^n)$ by density. Indeed, we recall that $W^{1,1}(\Sigma)$ is the completion of the space of smooth maps $v:\Sigma\to\mathbb{R}$ with respect to the norm $\int_\Sigma[\,|v|+|\nabla_\Sigma v|\,]$. Considering any $\mathbf{v}:=(\mathbf{v}_1,\ldots,\mathbf{v}_n)\in W^{1,1}(\Sigma,\mathbb{R}^n)$, there exists a sequence of maps $\mathbf{v}^m\in C^\infty(\Sigma,\mathbb{R}^n)$ strongly converging to $\mathbf{v}$ in $W^{1,1}(\Sigma,\mathbb{R}^n)$. We thus get from the foregoing:

$$\left|\int_\Sigma \mathrm{div}_\Sigma\mathbf{v}\right| = \left|\int_\Sigma \mathrm{div}_\Sigma\left(\mathbf{v}-\mathbf{v}^m\right)\right| \leqslant \sum_{i=1}^n\int_\Sigma\left|\nabla_\Sigma\left(\mathbf{v}_i-\mathbf{v}_i^m\right)\right| \underset{m\to+\infty}{\longrightarrow} 0.$$

To conclude, we have proved that $\int_\Sigma \mathrm{div}_\Sigma\mathbf{v}=0$ for any $\mathbf{v}\in W^{1,1}(\Sigma,\mathbb{R}^n)$ such that $\langle\mathbf{v}\mid\mathbf{n}\rangle=0$. □

Proposition A.2 (Grisvard's identity for $C^{1,1}$-domains). *Let Ω be a non-empty bounded open subset of $\mathbb{R}^n$ with $C^{1,1}$-boundary $\Sigma:=\partial\Omega$. Then, for any $\mathbf{v}:=(\mathbf{v}_1,\ldots,\mathbf{v}_n)\in H^1(\Omega,\mathbb{R}^n)$, we have:*

$$\begin{aligned}\sum_{i,j=1}^n\int_\Omega\frac{\partial\mathbf{v}_i}{\partial x_j}\frac{\partial\mathbf{v}_j}{\partial x_i} - \int_\Omega|\,\mathrm{div}\,\mathbf{v}|^2 \;=\;& 2\left\langle\nabla_\Sigma v_{\mathbf{n}})\mid\mathbf{v}_\Sigma\right\rangle_{H^{-\frac12}(\Sigma,\mathbb{R}^n),H^{\frac12}(\Sigma,\mathbb{R}^n)}\\ &+\int_\Sigma\left[\mathbf{II}\left(\mathbf{v}_\Sigma,\mathbf{v}_\Sigma\right)-H\left(v_{\mathbf{n}}\right)^2\right],\end{aligned}\tag{A.1}$$

where $v_{\mathbf{n}}:=\langle\mathbf{v}\mid\mathbf{n}\rangle$ and $\mathbf{v}_\Sigma:=\mathbf{v}-v_{\mathbf{n}}\mathbf{n}$ refers to the normal and tangential part of $\mathbf{v}$, respectively, and where $\mathbf{II}$ denotes the second fundamental form associated with the $C^{1,1}$-hypersurface Σ.

Proof. First, we assume that $\mathbf{v}=(\mathbf{v}_1,\ldots,\mathbf{v}_n)\in C^\infty(\overline{\Omega},\mathbb{R}^n)$ and we get from two integrations by parts:

$$\sum_{i,j=1}^n\int_\Omega\frac{\partial\mathbf{v}_i}{\partial x_j}\frac{\partial\mathbf{v}_j}{\partial x_i} - \int_\Omega|\,\mathrm{div}\,\mathbf{v}|^2 = \int_\Sigma\langle\mathrm{D}\mathbf{v}\left(\mathbf{v}\right)\mid\mathbf{n}\rangle - \int_\Sigma v_{\mathbf{n}}\,\mathrm{div}\,\mathbf{v}.\tag{A.2}$$

We now show that the right member of (A.2) equals the one of (A.1). For this purpose, we decompose the operators $\mathrm{D}\mathbf{v}=\mathrm{D}_\Sigma\mathbf{v}+\mathrm{D}\mathbf{v}(\mathbf{n})\,[\mathbf{n}]^{\mathrm{T}}$ and $\mathrm{div}\,\mathbf{v}=\mathrm{div}_\Sigma\mathbf{v}+\langle\mathrm{D}\mathbf{v}(\mathbf{n})\mid\mathbf{n}\rangle$ into their normal and tangential parts, from which we deduce that:

$$\langle\mathrm{D}\mathbf{v}(\mathbf{v})\mid\mathbf{n}\rangle - v_{\mathbf{n}}\,\mathrm{div}\,\mathbf{v}=\langle\mathrm{D}_\Sigma\mathbf{v}(\mathbf{v})\mid\mathbf{n}\rangle-v_{\mathbf{n}}\,\mathrm{div}_\Sigma\mathbf{v}.$$

Note that the right member only involves tangential operators. Hence, everything can be expressed in a local parametrization X associated with Σ, as in the proof of Lemma A.1. Indeed, $\mathbf{v}\circ X$ and $\mathbf{n}\circ X$ (thus $v_{\mathbf{n}}\circ X$ and $\mathbf{v}_\Sigma\circ X$) are Lipschitz continuous. In particular, they are differentiable a.e. so we can correctly decompose $\mathbf{v}$ as $\mathbf{v}_\Sigma+v_{\mathbf{n}}\mathbf{n}$. We emphasize the fact that this is only possible here because tangential operators are involved thus $C^{1,1}$-regularity of the surface is enough (otherwise Σ must be at least of class C^2 to properly extend the normal field). Direct calculations yield a.e.

$$\langle\mathrm{D}\mathbf{v}\left(\mathbf{v}\right)\mid\mathbf{n}\rangle-v_{\mathbf{n}}\,\mathrm{div}\,\mathbf{v} \;=\; \langle\mathrm{D}_\Sigma\mathbf{v}\left(\mathbf{v}_\Sigma\right)\mid\mathbf{n}\rangle-v_{\mathbf{n}}\,\mathrm{div}_\Sigma\mathbf{v}_\Sigma-H(v_{\mathbf{n}})^2.$$

Using the integration-by-part technique *i.e.* $\partial_i f\,g=\partial_i(f\,g)-f\,\partial_i g$ in the local parametrization, and recalling that $\mathrm{II}(\mathbf{v}_\Sigma,\mathbf{v}_\Sigma):=-\langle\mathrm{D}_\Sigma\mathbf{n}(\mathbf{v}_\Sigma)\mid\mathbf{v}_\Sigma\rangle$, we obtain a.e.

$$\langle\mathrm{D}\mathbf{v}\left(\mathbf{v}\right)\mid\mathbf{n}\rangle-v_{\mathbf{n}}\,\mathrm{div}\,\mathbf{v} \;=\; 2\,\langle\nabla_\Sigma v_{\mathbf{n}}\mid\mathbf{v}_\Sigma\rangle+\mathbf{II}\left(\mathbf{v}_\Sigma,\mathbf{v}_\Sigma\right)-H\left(v_{\mathbf{n}}\right)^2-\mathrm{div}_\Sigma(v_{\mathbf{n}}\,\mathbf{v}_\Sigma).$$

Integrating on Σ the last equality and combining it with (A.2), we then use Lemma A.1 in order to ensure that $\int_\Sigma \mathrm{div}_\Sigma(v_{\mathbf{n}}\,\mathbf{v}_\Sigma) = 0$ and deduce that:

$$\sum_{i,j=1}^{n} \int_\Omega \frac{\partial \mathbf{v}_i}{\partial x_j}\frac{\partial \mathbf{v}_j}{\partial x_i} - \int_\Omega |\,\mathrm{div}\,\mathbf{v}|^2 \;=\; 2\int_\Sigma \langle \nabla_\Sigma v_{\mathbf{n}} \mid \mathbf{v}_\Sigma\rangle + \int_\Sigma \Big[\mathbf{II}\,(\mathbf{v}_\Sigma,\mathbf{v}_\Sigma) - H\,(v_{\mathbf{n}})^2\Big]. \tag{A.3}$$

More precisely, (A.3) holds for any $\mathbf{v} \in C^\infty(\overline{\Omega},\mathbb{R}^n)$ with compact support in the cylinder associated with any local parametrization of $\partial\Omega$. As in the proof of Lemma A.1, (A.3) remains true for any $\mathbf{v} \in C^\infty(\overline{\Omega},\mathbb{R}^n)$ thanks to a suitable partition of unity (consider *e.g.* $\sqrt{\xi_k}\mathbf{v}$ in (A.3) and sum on k). Finally, we extend the result on $H^1(\Omega,\mathbb{R}^n)$ by density. Let $\mathbf{v} \in H^1(\Omega,\mathbb{R}^n)$. Since Σ is Lipschitz, there exists a sequence of smooth maps $\mathbf{v}^m \in C^\infty(\overline{\Omega},\mathbb{R}^n)$ converging to $\mathbf{v}$ in $H^1(\Omega,\mathbb{R}^n)$ ([12], Sect. 4.2.1). From the foregoing, (A.3) holds for any $\mathbf{v}^m$ and we now prove how we can let $m \to +\infty$. The volume integrals in the left member of (A.3) are passed to the limit without difficulty, whereas the continuity of the two operators $(\bullet)_\Sigma : H^1(\Omega,\mathbb{R}^n) \to H^{1/2}(\Sigma,\mathbb{R}^n)$ and $(\bullet)_{\mathbf{n}} : H^1(\Omega,\mathbb{R}^n) \to H^{1/2}(\Sigma)$ are combined with the one of $\nabla_\Sigma(\bullet) : H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma,\mathbb{R}^n)$ in order to deduce:

$$\int_\Sigma \langle \nabla_\Sigma v_{\mathbf{n}}^m \mid \mathbf{v}_\Sigma^m\rangle \underset{m\to+\infty}{\longrightarrow} \langle \nabla_\Sigma v_{\mathbf{n}} \mid \mathbf{v}_\Sigma\rangle_{H^{-\frac{1}{2}}(\Sigma,\mathbb{R}^n),H^{\frac{1}{2}}(\Sigma,\mathbb{R}^n)}.$$

Since Σ is a compact $C^{1,1}$-hypersurface, the Gauss map $\mathbf{n} : \Sigma \to \mathbb{S}^{n-1}$ is Lipschitz continuous. In particular, the eigenvalues $(\kappa_i)_{1\leqslant i\leqslant n-1}$ of its differential (called principal curvatures) exist a.e. and are essentially bounded. Introducing also the associated principal directions, the so-called eigenvectors $(\mathbf{e}_1,\ldots,\mathbf{e}_{n-1})$ forms an orthonormal basis of the tangent hyperplane to Σ. We thus have:

$$\int_\Sigma \Big[\mathbf{II}\,(\mathbf{v}_\Sigma^m,\mathbf{v}_\Sigma^m) - H\,(v_{\mathbf{n}}^m)^2\Big] = -\sum_{i=1}^{n-1}\int_\Sigma \kappa_i\,(\,|\,\langle \mathbf{v}_\Sigma^m \mid \mathbf{e}_i\rangle\,|^2 \;+\; |\,\langle \mathbf{v}^m \mid \mathbf{n}\rangle\,|^2\,). \tag{A.4}$$

Considering the above expression (A.4), we use the continuity of the trace $H^1(\Omega,\mathbb{R}^n) \to L^2(\Sigma,\mathbb{R}^n)$ with the fact that $\kappa_i \in L^\infty(\Sigma)$ to correctly let $m \to +\infty$. Hence, we get from the foregoing that relation (A.1) holds true for any $\mathbf{v} \in H^1(\Omega,\mathbb{R}^n)$, concluding the proof of Proposition A.2. ☐

Appendix B. Uniform H^1-estimates for the Laplacian

If we want to get uniform H^2-estimates from Grisvard's identity (A.1), we first need to control the constants coming from *a priori* H^1-estimates. It is well known that a uniform Lipschitz regularity is enough to control such constants. We recall here how to treat the Laplace operator for three usual types of boundary conditions.

B.1 Homogeneous Dirichlet boundary conditions

For the Dirichlet case, the constant only depends on diameter thanks to the Poincaré inequality.

Proposition B.1 (**Poincaré inequality**). *Let Ω be a non-empty bounded open subset of $\mathbb{R}^n$ with diameter $d := \sup_{(\mathbf{x},\mathbf{y})\in\Omega\times\Omega}|\mathbf{x}-\mathbf{y}|$. Then, for any $u \in H_0^1(\Omega)$, $\|u\|_{L^2(\Omega)} \leqslant 2d\,\|\nabla u\|_{L^2(\Omega,\mathbb{R}^n)}$. Moreover, for any $u \in H^2(\Omega)\cap H_0^1(\Omega)$, we also have $\|\nabla u\|_{L^2(\Omega,\mathbb{R}^n)} \leqslant 2d\|\Delta u\|_{L^2(\Omega)}$.*

Proof. The result is standard ([15], Sect. 1.4.3.4) so we only sketch the proof. Introducing the extension $\tilde{u}$ of $u \in H_0^1(\Omega)$ by zero outside Ω, we use the Cauchy-Schwarz inequality to get successively:

$$\int_\Omega u^2 \;=\; \int_{\mathbb{R}^{n-1}}\int_{-d}^{d}\left(\int_{-d}^{x_n}\frac{\partial\tilde{u}}{\partial x_n}\right)^2 \;\leqslant\; 4d^2\int_{\mathbb{R}^{n-1}}\int_{-d}^{d}\left(\frac{\partial\tilde{u}}{\partial x_n}\right)^2 \;\leqslant\; 4d^2\int_\Omega|\nabla u|^2.$$

If in addition, we assume $u \in H^2(\Omega)$, then an integration by parts is possible, which yields:

$$\int_\Omega |\nabla u|^2 \;=\; -\int_\Omega u\,\Delta u \;\leqslant\; 2d^2 \int_\Omega |\Delta u|^2 + \frac{1}{8d^2}\int_\Omega u^2 \;\leqslant\; 2d^2 \int_\Omega |\Delta u|^2 + \frac{1}{2}\int_\Omega |\nabla u|^2.$$

After simplification, we conclude the second inequality holds: $\|\nabla u\|_{L^2(\Omega,\mathbb{R}^n)} \leqslant 2d\|\Delta u\|_{L^2(\Omega)}$. □

Corollary B.2. *Let B be a non-empty bounded open subset of $\mathbb{R}^n$ with diameter d. Then, for any non-empty open set $\Omega \subseteq B$ and $u \in H^2(\Omega)\cap H_0^1(\Omega)$, $\|u\|_{H^1(\Omega)} \leqslant 2d(1+2d)\|\Delta u\|_{L^2(\Omega)}$.*

Proof. Use the triangle inequality and the fact that the diameter of Ω is not greater than d. □

B.2 Neumann type of boundary conditions

We still follow the strategy explained at the beginning of Section B with Neumann boundary conditions. For this purpose, we need to make the statement more precise concerning the dependence of the constant appearing in the trace inequality. As shown in ([15], Sect. 1.5.1.10), it depends on the Lipschitz modulus of continuity of the local graph but also on the C^1-norm of any partition of unity associated with a finite open covering of the hypersurface. Therefore, we first need to build a partition of unity for which we can control uniformly the number of maps and the C^0-norm of their gradient. Here, we adapt to our case the construction and arguments given in ([7], Prop. II.2).

Lemma B.3. *Let $h > 0$ and B be a non-empty bounded open subset of $\mathbb{R}^n$ with diameter d. Then, there exists an integer $N \geqslant 1$ and a constant $C > 0$, both depending only on d, h and n, such that for any non-empty open set $\Omega \subseteq B$, there exists K distinct points $(\mathbf{x}_k)_{1\leqslant k\leqslant K}$ of $\partial\Omega$ with $1 \leqslant K \leqslant N(d,h,n)$, such that the set $\mathcal{V}_{\frac{h}{4}}(\partial\Omega) := \{\mathbf{x}\in\mathbb{R}^n,\ d(\mathbf{x},\partial\Omega) < \frac{h}{4}\}$ has its closure embedded in $\cup_{k=1}^K \mathbb{B}_h(\mathbf{x}_k)$, and there exists K smooth maps $\xi_k : \mathbb{R}^n \to [0,1]$ with compact support in $\mathbb{B}_h(\mathbf{x}_k)$, such that $\sum_{k=1}^K \xi_k = 1$ on $\overline{\mathcal{V}_{\frac{h}{4}}(\partial\Omega)}$ and also $\sum_{k=1}^K \sum_{i=1}^n \|\partial_i \xi_k\|_{C^0(\mathbb{R}^n)} \leqslant C(d,h,n)$.*

Proof. Considering a non-empty open set $\Omega \subseteq B$ and $\mathbf{x}\in B$, the set Ω belongs to the open ball $\mathbb{B}_d(\mathbf{x})$ of radius d centred at $\mathbf{x}$. In particular, Ω is included in the open cube of length d centred at $\mathbf{x}$. We set:

$$a := \frac{h}{2\sqrt{n}} \qquad \text{and} \qquad N(d,h,n) := \left(1 + \left\lfloor \frac{d}{a} \right\rfloor\right)^n,$$

where $\lfloor \bullet \rfloor$ refers to the integer part. Hence, the larger cube of length $a(1+\lfloor \frac{d}{a}\rfloor) > d$ centred at $\mathbf{x}$ is divided into $N(d,h,n)$ small cubes of length a. We introduce their centres $(\mathbf{y}_k)_{1\leqslant k\leqslant N}$. Note that with our choice of a, their diameter equals $\frac{h}{2}$ so they are themselves contained in balls of radius $\frac{h}{4}$ centred at $\mathbf{y}_k$. Consequently, we have $\mathbb{B}_d(\mathbf{x}) \subset \cup_{k=1}^N \mathbb{B}_{\frac{h}{4}}(\mathbf{y}_k)$, from which we deduce:

$$\partial\Omega \quad\subseteq \bigcup_{\substack{1\leqslant k\leqslant N\\ \partial\Omega\cap\mathbb{B}_{\frac{h}{4}}(\mathbf{y}_k)\neq\emptyset}} \overline{\mathbb{B}_{\frac{h}{4}}(\mathbf{y}_k)}.$$

Therefore, we can relabel the points $(\mathbf{y}_k)_{1\leqslant k\leqslant N}$ and there exists a positive integer $K \leqslant N$ satisfying $\partial\Omega \subseteq \cup_{k=1}^K \overline{\mathbb{B}_{\frac{h}{4}}(\mathbf{y}_k)}$ and $\partial\Omega \cap \overline{\mathbb{B}_{\frac{h}{4}}(\mathbf{y}_k)} \neq \emptyset$ for $k = 1,\ldots,K$, which means that there exists K points $(\mathbf{x}_k)_{1\leqslant k\leqslant K}$ of $\partial\Omega$ such that $\|\mathbf{x}_k - \mathbf{y}_k\| \leqslant \frac{h}{4}$. The triangle inequality ensures that $\partial\Omega \subseteq \cup_{k=1}^K \overline{\mathbb{B}_{\frac{h}{2}}(\mathbf{x}_k)}$ and $\overline{\mathcal{V}_{\frac{h}{4}}(\partial\Omega)} \subseteq \cup_{k=1}^K \overline{\mathbb{B}_{\frac{3h}{4}}(\mathbf{x}_k)}$.

We now build the partition of unity by introducing the maps:

$$w: \begin{array}{ccl} \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ \mathbf{x} & \longmapsto & \begin{cases} \exp\left[\frac{-|\mathbf{x}|^2}{\left(\frac{h}{16}\right)^2 - |\mathbf{x}|^2}\right] & \text{if } |\mathbf{x}| < \frac{h}{16} \\ 0 & \text{otherwise} \end{cases} \end{array} \qquad \Psi_k: \begin{array}{ccl} \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ \mathbf{x} & \longmapsto & \dfrac{\displaystyle\int_{\mathbb{B}_{\frac{13h}{16}}(\mathbf{x}_k)} w(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}}{\displaystyle\int_{\mathbb{R}^n} w(\mathbf{y})\,\mathrm{d}\mathbf{y}}. \end{array}$$

One can check that $\Psi_k \in C^\infty(\mathbb{R}^n, [0,1])$ with compact support in $\mathbb{B}_h(\mathbf{x}_k)$, and $\Psi_k = 1$ on $\overline{\mathbb{B}_{\frac{3h}{4}}(\mathbf{x}_k)}$. Moreover, explicit calculations allow to bound uniformly $\partial_i \Psi_k$ by a constant $c(h,n)$ which only depends on h and n. Finally, we set $\xi_1 = \Psi_1$ and recursively $\xi_k = \Psi_k \prod_{j=1}^{k-1}(1-\Psi_j)$ for $2 \leqslant k \leqslant K$. We get that $\xi_k \in C^\infty(\mathbb{R}^n, [0,1])$ has compact support in $\mathbb{B}_h(\mathbf{x}_k)$, and $\sum_{k=1}^K \xi_k = 1$ on $\cup_{k=1}^K \overline{\mathbb{B}_{\frac{3h}{4}}(\mathbf{x}_k)}$ thus also on the closure of $\mathcal{V}_{\frac{h}{4}}(\partial\Omega)$. Furthermore, we have:

$$\sum_{k=1}^{K}\sum_{i=1}^{n} \|\partial_i \xi_k)\|_{C^0(\mathbb{R}^n)} \leqslant \sum_{k=1}^{K}\sum_{i=1}^{n}\sum_{j=1}^{k} \|\partial_i \Psi_j\|_{C^0(\mathbb{R}^n)} \leqslant c(h,n)\, n\, N(d,h,n)^2 := C(d,h,n),$$

which concludes the proof of Lemma B.3. ☐

Proposition B.4 (**Trace inequality**). *Let $\alpha \in]0, \frac{\pi}{2}[$ and B as in Lemma B.3. We define $\mathfrak{O}_\alpha(B)$ as the class of non-empty open subsets of B satisfying the α-cone property. We assume that the diameter d of B is large enough to ensure $\mathfrak{O}_\alpha(B) \neq \emptyset$. Then, there exists a constant $C_0 > 0$, depending only on α, d and n, such that:*

$$\forall \Omega \in \mathfrak{O}_\alpha(B),\ \forall \eta \in]0,1],\ \forall u \in H^1(\Omega), \quad \int_{\partial\Omega} u^2 \leqslant C_0(\alpha,d,n)\left(\eta \int_\Omega |\nabla u|^2 + \frac{1}{\eta}\int_\Omega u^2\right).$$

Proof. We only sketch the proof since it is very similar to ([15], Sect. 1.5.1.9–10). We pick Ω in the class $\mathfrak{O}_\alpha(B)$ of the statement. We also define $C_{r,a}(\mathbf{x})$ as a cylinder of radius $r > 0$, half height $a > 0$, centred at $\mathbf{x}$. From the uniform cone property, $\partial\Omega$ has a Lipschitz boundary, which means that for any $\mathbf{x}_0 \in \partial\Omega$, there exists $C_{r,a}(\mathbf{x}_0)$ of direction a unit vector $\mathbf{d}_{\mathbf{x}_0}$ in which $\partial\Omega$ is the graph of a L-Lipschitz continuous map (and where Ω remains below this graph). Moreover, the constants $r > 0$, $a > 0$, and $L > 0$ only depend on α. Consequently, we apply Lemma B.3 with $h := \min(r,a)$ thus only depending on α. Using the same notation, we set $\mathbf{m} = \sum_{k=1}^K \xi_k \mathbf{d}_{\mathbf{x}_k} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and one can check the relation $\langle \mathbf{m} \mid \mathbf{n}\rangle \geqslant [1+L^2]^{-\frac{1}{2}}$ a.e. on $\partial\Omega$. Let $u \in H^1(\Omega)$ and $\eta \in]0,1]$. From the foregoing, we obtain successively:

$$\begin{aligned} \int_{\partial\Omega} u^2 &\leqslant \sqrt{1+L^2}\int_{\partial\Omega} u^2 \langle \mathbf{m} \mid \mathbf{n}\rangle = \sqrt{1+L^2}\int_\Omega \operatorname{div}\left(u^2\,\mathbf{m}\right) \\ &\leqslant \sqrt{1+L^2}\left[\eta\int_\Omega |\nabla u|^2 + \frac{1}{\eta}\int_\Omega u^2 + \sum_{k=1}^{K}\sum_{i=1}^{n}\|\partial_i \xi_k\|_{C^0(\mathbb{R}^n)}\int_\Omega u^2\right] \\ &\leqslant [1 + C(h,d,n)]\sqrt{1+L^2}\left(\eta\int_\Omega |\nabla u|^2 + \frac{1}{\eta}\int_\Omega u^2\right). \end{aligned}$$

To conclude the proof of Proposition B.4, the above constant only depends on α, d and n. ☐

Corollary B.5. *Considering the assumptions and notation of Proposition B.4, we also have for any $\Omega \in \mathfrak{O}_\alpha(B)$ and any $\eta \in]0,1]$:*

$$\forall u \in H^2(\Omega), \quad \int_{\partial\Omega} |\nabla u|^2 \leqslant C_0(\alpha,d,n)\left(\eta \sum_{i,j=1}^{n} \int_\Omega \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \frac{1}{\eta}\int_\Omega |\nabla u|^2\right). \tag{B.1}$$

Proof. Apply Proposition B.4 to each $\partial_i u \in H^1(\Omega)$ and sum the n inequalities obtained. □

Proposition B.6. *Let $\lambda > 0$, $\alpha \in]0, \frac{\pi}{2}[$, and B be as in Lemma B.3. We introduce a Lipschitz continuous map $\beta : \mathbb{R} \to \mathbb{R}$ satisfying $\beta(x)x \geqslant 0$ for any $x \in \mathbb{R}$. Then, considering the class $\mathfrak{O}_\alpha(B)$ of Proposition B.4, there exists a constant $C_1 > 0$ depending only on α, d, λ and n such that for any $\Omega \in \mathfrak{O}_\alpha(B)$ and any $u \in H^2(\Omega)$, we have:*

$$\int_\Omega u^2 + \int_\Omega |\nabla u|^2 \leqslant C_1(\alpha,d,\lambda,n)\left(\int_\Omega (-\Delta u + \lambda u)^2 + \int_{\partial\Omega} (\partial_{\mathbf{n}} u + \beta(u))^2\right). \tag{B.2}$$

Proof. First, we get from the hypothesis that $\partial_{\mathbf{n}} u + \beta(u) \in L^2(\partial\Omega)$ and an integration by parts can be performed:

$$\int_\Omega |\nabla u|^2 + \lambda\int_\Omega u^2 + \int_{\partial\Omega} u\beta(u) = \int_\Omega u(-\Delta u + \lambda u) + \int_{\partial\Omega} u(\partial_{\mathbf{n}} u + \beta(u)). \tag{B.3}$$

Then, combining the fact that $\int_{\partial\Omega} u\beta(u) \geqslant 0$ with inequalities of the form $xy \leqslant \mu x^2 + \frac{1}{4\mu} y^2$ valid for any $\mu > 0$ and $(x,y) \in \mathbb{R}\times\mathbb{R}$, we deduce from Proposition B.4 that for any $\eta \in]0,1]$:

$$\left[\min\left(1,\frac{\lambda}{2}\right) - \eta C_0\right]\left(\int_\Omega u^2 + \int_\Omega |\nabla u|^2\right) \leqslant \frac{1}{2\lambda}\int_\Omega (-\Delta u + \lambda u)^2 + \frac{1}{4\eta}\int_{\partial\Omega} (\partial_{\mathbf{n}} u + \beta(u))^2.$$

Choosing $\eta := \frac{1}{2C_0}\min(1, \frac{\lambda}{2}, 2C_0)$, we have $\eta \in]0,1]$ and $\min(1,\frac{\lambda}{2}) - \eta C_0 > 0$. From the foregoing, we conclude that (B.2) holds true for a constant $C_1 > 0$ that only depends on α, d, λ and n. □

Remark B.7. First, observe that Neumann boundary conditions occur for $\beta(x) = 0$ and that Robin boundary conditions are achieved for $\beta(x) = \mu x$ with $\mu > 0$. Then, since β is continuous, the condition $\beta(x)x \geqslant 0$ implies $\beta(0) = 0$. Finally, in the specific case of homogeneous boundary conditions, note that the previous proof does not require the use of the trace inequality. As a consequence, the constant only depends on λ and we do not need to impose a uniform cone condition.

Corollary B.8. *Let $\lambda > 0$. Consider $\beta : \mathbb{R} \to \mathbb{R}$ as in Proposition B.6 and a non-empty bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary. Then, for any $u \in H^2(\Omega)$ such that $\partial_{\mathbf{n}} u + \beta(u) = 0$, we have: $\int_\Omega u^2 + \int_\Omega |\nabla u|^2 \leqslant \frac{1}{\lambda}(1 + \frac{1}{\lambda})\int_\Omega(-\Delta u + \lambda u)^2$.*

Proof. Starting again from (B.3), we combine the hypothesis $\partial_{\mathbf{n}} u + \beta(u) = 0$ and $\int_{\partial\Omega} u\beta(u) \geqslant 0$ with the Cauchy-Schwarz inequality to obtain successively $\|u\|_{L^2(\Omega)} \leqslant \frac{1}{\lambda}\| - \Delta u + \lambda u\|_{L^2(\Omega)}$ and $\|\nabla u\|^2_{L^2(\Omega,\mathbb{R}^n)} \leqslant \|u\|_{L^2(\Omega)}\| - \Delta u + \lambda u\|_{L^2(\Omega)} \leqslant \frac{1}{\lambda}\| - \Delta u + \lambda u\|^2_{L^2(\Omega)}$, concluding the proof. □

B.3 The case of Robin boundary conditions

The study of Robin boundary conditions firstly requires a sort of reverse inequality for the trace.

Lemma B.9. *Let $\Omega \subset \mathbb{R}^n$ be non-empty, bounded, open, with Lipschitz boundary and diameter d. Then, for any $u \in H^1(\Omega)$, we have $\int_\Omega u^2 \leqslant 2d(1+2d)[\int_{\partial\Omega} u^2 + \int_\Omega |\nabla u|^2]$.*

Proof. Pick any $\mathbf{x}_0 \in \Omega$ and $u \in H^1(\Omega)$. First, we can perform an integration by parts:

$$n \int_\Omega u^2(\mathbf{x})\,\mathrm{d}V(\mathbf{x}) \;=\; \int_{\partial\Omega} u^2(\mathbf{x})\,\langle \mathbf{x}-\mathbf{x}_0 \mid \mathbf{n}(\mathbf{x})\rangle\,\mathrm{d}A(\mathbf{x}) \;-\; 2\int_\Omega u(\mathbf{x})\,\langle \nabla u(\mathbf{x}) \mid \mathbf{x}-\mathbf{x}_0\rangle\,\mathrm{d}V(\mathbf{x}).$$

Therefore, introducing the diameter d of Ω, we deduce from the Cauchy-Schwarz inequality:

$$\int_\Omega u^2 \;\leqslant\; n\int_\Omega u^2 \;\leqslant\; d\int_{\partial\Omega} u^2 \;+\; d\left(\frac{1}{2d}\int_\Omega u^2 \;+\; 2d\int_\Omega |\nabla u|^2\right).$$

After simplification, one can check we obtain the inequality of Lemma B.9, concluding the proof. □

Proposition B.10. *Let $\lambda > 0$ and d be the diameter of a non-empty bounded open set $B \subset \mathbb{R}^n$. Then, there exists a constant $C_2 > 0$ depending only on d and λ, such that for any non-empty open set $\Omega \subseteq B$ with Lipschitz boundary, we have:*

$$\forall u \in H^2(\Omega), \quad \int_\Omega u^2 + \int_\Omega |\nabla u|^2 \;\leqslant\; C_2(d,\lambda)\left(\int_{\partial\Omega} (\partial_{\mathbf{n}} u + \lambda\, u)^2 \;+\; \int_\Omega (\Delta u)^2\right). \tag{B.4}$$

Proof. Consider a non-empty open set $\Omega \subseteq B$ with Lipschitz boundary and $u \in H^2(\Omega)$. First, since we have $\partial_{\mathbf{n}} u + \lambda\, u \in L^2(\partial\Omega)$, we can again perform an integration by parts:

$$\lambda\int_{\partial\Omega} u^2 \;+\; \int_\Omega |\nabla u|^2 \;=\; \int_{\partial\Omega} u\,(\partial_{\mathbf{n}} u + \lambda\, u) \;-\; \int_\Omega u\,\Delta u.$$

Then, the inequality $xy \leqslant \frac{\lambda}{2}x^2 + \frac{1}{2\lambda}y^2$ with $x = u$ and $y = \partial_{\mathbf{n}} u + \lambda u$ yields after simplifications:

$$\frac{\lambda}{2}\int_{\partial\Omega} u^2 \;+\; \int_\Omega |\nabla u|^2 \;\leqslant\; \frac{1}{2\lambda}\int_{\partial\Omega} (\partial_{\mathbf{n}} u + \lambda\, u)^2 \;+\; \int_\Omega |u\,\Delta u|. \tag{B.5}$$

We can now apply Lemma B.9 on the left member of (B.5) while on its right member, we use an inequality of the form $|u\,\Delta u| \leqslant \mu u^2 + \frac{1}{4\mu}(\Delta u)^2$. We obtain:

$$\mu\int_\Omega u^2 \;=\; \left(\frac{\min\left(1,\frac{\lambda}{2}\right)}{2d\,(1+2d)} - \mu\right)\int_\Omega u^2 \;\leqslant\; \frac{1}{2\lambda}\int_{\partial\Omega} (\partial_{\mathbf{n}} u + \lambda\, u)^2 \;+\; \frac{1}{4\mu}\int_\Omega (\Delta u)^2,$$

where we have set $\mu := \frac{1}{4d(1+2d)}\min(1,\frac{\lambda}{2}) > 0$ that only depends on d and λ. We can now get back to (B.5) with this last estimation in mind and we deduce:

$$\int_\Omega |\nabla u|^2 \;\leqslant\; \frac{1}{2\lambda}\int_{\partial\Omega} (\partial_{\mathbf{n}} u + \lambda\, u)^2 \;+\; \mu\int_\Omega u^2 \;+\; \frac{1}{4\mu}\int_\Omega (\Delta u)^2 \;\leqslant\; \frac{1}{\lambda}\int_{\partial\Omega} (\partial_{\mathbf{n}} u + \lambda\, u)^2 + \frac{1}{2\mu}\int_\Omega (\Delta u)^2.$$

From the foregoing, we conclude that (B.4) holds, concluding the proof of Proposition B.10. □

Appendix C. Other miscellaneous results

This section gathers some results that are used in Section 4 and postponed here for readability. A first one concerns the dependence of the constant appearing in an inequality involving norms defined on the boundary. This technical statement firstly requires some notation. Let $\alpha \in]0, \frac{\pi}{2}[$ and B be a non-empty bounded open subset of $\mathbb{R}^n$ with diameter large enough to ensure that the class $\mathfrak{O}_\alpha(B)$ defined in Proposition B.4 is not empty. Consider any $\Omega \in \mathfrak{O}_\alpha(B)$ and any $\mathbf{x}_0 \in \partial\Omega$.

From the uniform cone property, $\partial\Omega$ has a Lipschitz boundary so there exists an open cylinder $C_{r,a}(\mathbf{x}_0)$ centred at $\mathbf{x}_0$, of radius r, half height a and direction a unit vector $\mathbf{d}_{\mathbf{x}_0}$, in which $\partial\Omega$ can be written as the graph of a L-Lipschitz continuous map (and where Ω remains below this graph). We denote this local map by φ_0 and it can be defined on the open disk $D_r(\mathbf{x}_0)$ of radius r, centred at $\mathbf{x}_0$ and orthogonal to $\mathbf{d}_{\mathbf{x}_0}$. Moreover, the constants $r > 0$, $a > 0$, and $L > 0$ only depend on α.

We also introduce $X_0 : \mathbf{x}' \in D_r(\mathbf{x}_0) \mapsto (\mathbf{x}', \varphi_0(\mathbf{x}')) \in C_{r,a}(\mathbf{x}_0) \cap \partial\Omega$, which is the local parametrization associated with $\mathbf{x}_0$. Finally, from the compactness of $\partial\Omega$, we can consider a finite number $K \geqslant 1$ of distinct points of $\partial\Omega$ denoted by $\mathbf{x}_1, \dots, \mathbf{x}_K$ and such that $\partial\Omega \subset \cup_{k=1}^{K} C_{r,a}(\mathbf{x}_k)$. Building a partition of unity on this covering, there exists K smooth maps $\xi_k : \mathbb{R}^n \to [0,1]$ with compact support in $C_{r,a}(\mathbf{x}_k)$ and such that $\sum_{k=1}^{K} \xi_k = 1$ on $\partial\Omega$. The result states as follows.

Proposition C.1. *There exists a constant $C_3 > 0$ depending only on α and n, such that we have:*

$$\forall \Omega \in \mathfrak{O}_\alpha(B), \, \forall \mathbf{x}_0 \in \partial\Omega, \, \forall u \in L^1(\partial\Omega), \int_{D_r(\mathbf{x}_0)} |u \circ X_0| \;\leqslant\; C_3(\alpha, n) \sum_{k=1}^{K} \int_{D_r(\mathbf{x}_k)} |u \circ X_k| \, \xi_k \circ X_k,$$

Proof. Let $\Omega \in \mathfrak{O}_\alpha(B)$, $\mathbf{x}_0 \in \partial\Omega$ and $u \in L^1(\partial\Omega)$. To lighten the notation, we set $\mathcal{C}_0 := C_{r,a}(\mathbf{x}_0)$, $D_0 := D_r(\mathbf{x}_0)$, and $\mathcal{C}_k := C_{r,a}(\mathbf{x}_k)$, $k = 1, \dots, K$. We also need to introduce the projector $\Pi_0 : (\mathbf{x}', x_n) \in \mathcal{C}_0 \cap \partial\Omega \mapsto \mathbf{x}' \in D_0$ which is precisely the inverse of the map X_0. First, we have:

$$\int_{D_0} |u \circ X_0| \;=\; \sum_{k=1}^{K} \int_{\Pi_0(\partial\Omega \cap \mathcal{C}_0)} \xi_k \circ X_0 \, |u \circ X_0| \;=\; \sum_{\substack{k \in [\![1,K]\!] \\ \partial\Omega \cap \mathcal{C}_0 \cap \mathcal{C}_k \neq \emptyset}} \int_{\Pi_0(\partial\Omega \cap \mathcal{C}_0 \cap \mathcal{C}_k)} |(\xi_k u) \circ X_0|,$$

where the last equality comes from the fact that $\xi_k(\mathbf{x}) = 0$ for any $\mathbf{x} \notin \mathcal{C}_k$. Then, given $k \in [\![1, K]\!]$ such that $\partial\Omega \cap \mathcal{C}_0 \cap \mathcal{C}_k \neq \emptyset$, we define the map $T_{k0} := \Pi_k \circ X_0 : \Pi_0(\partial\Omega \cap \mathcal{C}_0 \cap \mathcal{C}_k) \to \Pi_k(\partial\Omega \cap \mathcal{C}_0 \cap \mathcal{C}_k)$. Note that the map T_{k0} is Lipschitz continuous since we get for any $(\mathbf{x}', \mathbf{y}') \in \Pi_0(\partial\Omega \cap \mathcal{C}_0 \cap \mathcal{C}_k)^2$:

$$|T_{k0}(\mathbf{x}') - T_{k0}(\mathbf{y}')| \;\leqslant\; |X_0(\mathbf{x}') - X_0(\mathbf{y}')| = |(\mathbf{x}' - \mathbf{y}', \varphi_0(\mathbf{x}') - \varphi_0(\mathbf{y}'))| \;\leqslant\; \sqrt{1 + L^2}\, |\mathbf{x}' - \mathbf{y}'|.$$

Moreover, one can check that T_{k0} is a bijective map by observing that its inverse map is precisely given by $T_{0k} := \Pi_0 \circ X_k : \Pi_k(\partial\Omega \cap \mathcal{C}_0 \cap \mathcal{C}_k) \to \Pi_0(\partial\Omega \cap \mathcal{C}_0 \cap \mathcal{C}_k)$, which is also a $\sqrt{1+L^2}$-Lipschitz continuous map. In other words T_{k0} is a bi-Lipschitz change of coordinates. From Rademacher's Theorem, we deduce that T_{k0} and T_{0k} are differentiable a.e. and furthermore, the Jacobian of T_{0k} is uniformly L^∞-bounded. Indeed, we have for any $\mathbf{x}' \in \Pi_0(\partial\Omega \cap \mathcal{C}_0 \cap \mathcal{C}_k)$:

$$\mathrm{Jac}(T_{0k})(\mathbf{x}') \;:=\; |\det[\mathrm{D}_{\mathbf{x}'}(T_{0k})]| \;\leqslant\; \sum_{\sigma \in \mathcal{S}^n} \prod_{i=1}^{n} |\,[\mathrm{D}_{\mathbf{x}'}(T_{0k})]_{m,\sigma(m)}| \;\leqslant\; n!\,(1 + L^2)^{\frac{n}{2}}.$$

Finally, we perform a change of variables valid with the Lipschitz map T_{k0} ([12], Sect. 3.3.3). We obtain:

$$\begin{aligned}\int_{D_0} |u \circ X_0| &= \sum_{\substack{k \in [\![1,K]\!] \\ \partial\Omega \cap \mathcal{C}_0 \cap \mathcal{C}_k \neq \emptyset}} \int_{\Pi_0(\partial\Omega\cap\mathcal{C}_0\cap\mathcal{C}_k)} |(\xi_k u) \circ X_k \circ T_{k0}| \,\mathrm{Jac}\,(T_{k0})\ \mathrm{Jac}\,(T_{0k}) \\ &\leqslant n!\,(1+L)^{\frac{n}{2}} \sum_{\substack{k \in [\![1,K]\!] \\ \partial\Omega \cap \mathcal{C}_0 \cap \mathcal{C}_k \neq \emptyset}} \int_{\Pi_k(\partial\Omega\cap\mathcal{C}_0\cap\mathcal{C}_k)} |(\xi_k u) \circ X_k| \\ &\leqslant n!\,(1+L)^{\frac{n}{2}} \sum_{k=1}^{K} \int_{D_k} |(\xi_k u) \circ X_k|.\end{aligned}$$

To conclude the proof of Proposition C.1, the above constant only depends on n and α via L. ☐

Corollary C.2. *There exists a constant $C_3 > 0$ depending only on α and n, such that we have:*

$$\forall \Omega \in \mathfrak{O}_\alpha(B),\ \forall \mathbf{x}_0 \in \partial\Omega,\ \forall u \in L^1(\partial\Omega), \qquad \int_{D_r(\mathbf{x}_0)} |u \circ X_0| \ \leqslant\ C_3(\alpha, n) \int_{\partial\Omega} |u|,$$

with $X_0 : \mathbf{x}' \in D_r(\mathbf{x}_0) \mapsto (\mathbf{x}', \varphi_0(\mathbf{x}')) \in C_{r,a}(\mathbf{x}_0) \cap \partial\Omega$ the local parametrization associated with $\mathbf{x}_0$.

Proof. Considering the last inequality appearing in the proof of Proposition C.1, we deduce that we have: $\int_{D_0} |u \circ X_0| \ \leqslant\ n!(1+L)^{n/2} \sum_{k=1}^{K} \int_{D_k} |(\xi_k u) \circ X_k| \sqrt{1 + |\nabla\varphi_k|^2} \ =\ n!(1+L)^{n/2} \int_{\partial\Omega} |u|$. ☐

C.1 Some convergence properties of the characteristic functions

Given any measurable set $X \subseteq \mathbb{R}^n$, we define $\mathbf{1}_X$ as the L^∞-map valued one on X otherwise zero. Let B be a non-empty bounded open subset of $\mathbb{R}^n$. We consider a sequence $(\Omega_i)_{i\in\mathbb{N}}$ of measurable subsets of B and Ω also refers to a measurable subset of B. We now give some convergence results.

Lemma C.3. *Let $v \in L^2(B)$ and assume that $(\mathbf{1}_{\Omega_i})_{i\in\mathbb{N}}$ converges to $\mathbf{1}_\Omega$ strongly in $L^1(B)$. Then, the sequence $(v\mathbf{1}_{\Omega_i})_{i\in\mathbb{N}}$ converges to $v\mathbf{1}_\Omega$ strongly in $L^2(B)$.*

Proof. Let $v \in L^2(B)$ and $w \in L^\infty(B)$. First, note that $(\mathbf{1}_{\Omega_i} - \mathbf{1}_\Omega)^2 = |\mathbf{1}_{\Omega_i} - \mathbf{1}_\Omega| \leqslant 1$. Hence, we have: $\|v\mathbf{1}_{\Omega_i} - v\mathbf{1}_\Omega\|^2_{L^2(B)} \leqslant 2(\|w\|_{L^\infty(B)}\|\mathbf{1}_{\Omega_i} - \mathbf{1}_\Omega\|_{L^1(B)} + \|v - w\|^2_{L^2(B)})$. Then, for any $\varepsilon > 0$, one can find $w_\varepsilon \in C^\infty_c(B)$ such that $\|v - w_\varepsilon\|_{L^2(B)} \leqslant \varepsilon$. There also exists $I_\varepsilon \in \mathbb{N}$ such that $\|\mathbf{1}_{\Omega_i} - \mathbf{1}_\Omega\|_{L^1(B)} \leqslant \varepsilon^2/(1 + \|w_\varepsilon\|_{L^\infty(B)})$ for any integer $i \geqslant I_\varepsilon$. We thus get $\|v\mathbf{1}_{\Omega_i} - v\mathbf{1}_\Omega\|_{L^2(B)} \leqslant 2\varepsilon$, concluding the proof. ☐

Lemma C.4. *Let $v, w, (w_i)_{i\in\mathbb{N}} \subset L^2(B)$. We assume $w_i \to w$ strongly in $L^2(B)$ and $\mathbf{1}_{\Omega_i} \to \mathbf{1}_\Omega$ strongly in $L^1(B)$. Then, the sequence $(vw_i\mathbf{1}_{\Omega_i})_{i\in\mathbb{N}}$ converges to $vw\mathbf{1}_\Omega$ strongly in $L^1(B)$.*

Proof. We deduce $\|vw_i\mathbf{1}_{\Omega_i} - vw\mathbf{1}_\Omega\|_{L^1(B)} \leqslant \|v\|_{L^2(B)}\|w_i - w\|_{L^2(B)} + \|w\|_{L^2(B)}\|v\mathbf{1}_{\Omega_i} - v\mathbf{1}_\Omega\|_{L^2(B)}$ from the Cauchy-Schwarz inequality, and the result follows from Lemma C.3 and the fact that $w_i \to w$ as $i \to +\infty$. ☐

Lemma C.5. *Let $v, w, (v_i, w_i)_{i\in\mathbb{N}} \subset L^2(B)$. We assume $v_i \to v$ strongly in $L^2(B)$, $w_i \rightharpoonup w$ weakly in $L^2(B)$, and $\mathbf{1}_{\Omega_i} \to \mathbf{1}_\Omega$ strongly in $L^1(B)$. Then, $\int_B v_i w_i \mathbf{1}_{\Omega_i} h \to \int_B vw\mathbf{1}_\Omega h$ for any $h \in L^\infty(B)$.*

Proof. Let $h \in L^\infty(B)$. We get $|\int_B v_i w_i \mathbf{1}_{\Omega_i} h - \int_B vw\mathbf{1}_\Omega h| \leqslant \|w_i\|_{L^2(B)}\|v_i\mathbf{1}_{\Omega_i} h - v\mathbf{1}_\Omega h\|_{L^2(B)} + |\int_B (w_i - w)v\mathbf{1}_\Omega h|$. Since (w_i) weakly converges, it is bounded in $L^2(B)$. Moreover, B is bounded so we deduce $h \in L^2(B)$ and Lemma C.4 ensures the L^2-strong convergence of $(v_i\mathbf{1}_{\Omega_i} h)$ to $v\mathbf{1}_\Omega h$. We conclude by letting $i \to +\infty$ in the last inequality, using also the L^2-weak convergence of (w_i). ☐

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