

UNIQUENESS OF SOLUTION TO SYSTEMS OF ELLIPTIC OPERATORS AND APPLICATION TO ASYMPTOTIC SYNCHRONIZATION OF LINEAR DISSIPATIVE SYSTEMS

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Abstract. We show that under Kalman's rank condition on the coupling matrices, the uniqueness of solution to a complex system of elliptic operators can be reduced to the observability of a scalar problem. Based on this result, we establish the asymptotic stability and the asymptotic synchronization for a large class of linear dissipative systems.

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1. INTRODUCTION AND MAIN RESULTS

Let H and V be two separated Hilbert spaces such that $V \subset H$ with dense and compact imbedding. Let L be the duality mapping from V onto its dual V' , and γ be a linear continuous operator from V into V' .

Denote by \mathcal{V} and \mathcal{H} the product spaces:

$$\mathcal{V} = V^N, \quad \mathcal{H} = H^N. \quad (1.1)$$

Let $U = (u^{(1)}, \dots, u^{(N)})^T$. Define the vector operators \mathcal{L} , respectively, \mathcal{G} by

$$\mathcal{L}U = \begin{pmatrix} Lu^{(1)} \\ \vdots \\ Lu^{(N)} \end{pmatrix}, \quad \mathcal{G}U = \begin{pmatrix} \gamma u^{(1)} \\ \vdots \\ \gamma u^{(N)} \end{pmatrix}. \quad (1.2)$$

Let A and D be symmetric and positive semi-definite matrices. Consider the following second order evolution system

$$U'' + \mathcal{L}U + AU + D\mathcal{G}U' = 0, \quad (1.3)$$

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where “ $'$ ” denotes the time derivative.

Clearly, system (1.3) generates a semi-group of contractions in the space $\mathcal{V} \times \mathcal{H}$. Moreover, according to Foias' theory of spectral decomposition (see Thm. 3.4), the asymptotic stability of the dissipative system (1.3) is equivalent to the uniqueness of solution to the following over-determined system for the variable $\Phi = (\phi^{(1)}, \dots, \phi^{(N)})^T$:

$$\mathcal{L}\Phi + A\Phi = \beta^2\Phi \quad (1.4)$$

associated with the D -observation

$$D\mathcal{G}\Phi = 0. \quad (1.5)$$

We first observe that Kalman's rank condition

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N \quad (1.6)$$

is necessary for the uniqueness of solution to the over-determined system (1.4)–(1.5), therefore, necessary for the asymptotic stability of system (1.3) (see Props. 2.2 and 3.3). But, it is not sufficient in general as shown by an example in [16].

The first objective of the present work is to find an efficient characterization for the uniqueness of solution to the over-determined system (1.4)–(1.5). Since the matrix D in (1.6) is not invertible in general, the D -observation (1.5) cannot imply the nullity of the full observation

$$\gamma\phi^{(i)} = 0, \quad i = 1, \dots, N. \quad (1.7)$$

So, the following theorem is different from Carleman's classic unique continuation theorem on elliptic operators in [29], [5] or [10].

Theorem 1.1. *Let the pair (A, D) satisfy Kalman's rank condition (1.6). Assume that there exists a real number a such that $\|A - aI\|$ is small enough. Assume furthermore that there exists $c > 0$ independent of β and f , such that any solution $\phi \in V$ to the following over-determined scalar problem*

$$\beta^2\phi - L\phi = f \quad \text{and} \quad \gamma\phi = 0 \quad (1.8)$$

satisfies the uniform estimation:

$$\|\phi\|_H \leq c\|f\|_H. \quad (1.9)$$

Then, the over-determined system (1.4)–(1.5) has only the trivial solution $\Phi \equiv 0$.

Thus, “under Kalman's rank condition, the observability of a scalar problem implies the uniqueness of solution to a complex system.” Using this method, we may provide a direct and efficient approach to solve a seemingly difficult problem of uniqueness of a complex system.

When the matrices A and D do not satisfy Kalman's rank condition (1.6), the uniqueness theorem fails and system (1.3) is not asymptotically stable. We return to consider the asymptotic synchronization by p -groups, a weakened notion of stability, described as follows.

Let $p \geq 1$ be an integer and

$$0 = n_0 < n_1 < n_2 < \dots < n_p = N, \quad (1.10)$$

be integers such that $n_r - n_{r-1} \geq 2$ for $1 \leq r \leq p$. We re-arrange the components of the state variable U into p groups

$$(u^{(1)}, \dots, u^{(n_1)}), (u^{(n_1+1)}, \dots, u^{(n_2)}), \dots, (u^{(n_{p-1}+1)}, \dots, u^{(n_p)}). \quad (1.11)$$

Definition 1.2. System (1.3) is asymptotically synchronizable by p -groups if for any given initial data $(U_0, U_1) \in \mathcal{V} \times \mathcal{H}$, the corresponding solution U to system (1.3) satisfies

$$(u^{(k)}(t) - u^{(l)}(t), (u^{(k)})'(t) - (u^{(l)})'(t)) \rightarrow (0, 0) \quad \text{in } V \times H \quad (1.12)$$

as $t \rightarrow +\infty$ for all $n_{r-1} + 1 \leq k, l \leq n_r$ and $1 \leq r \leq p$.

The exact boundary synchronization and the approximate boundary synchronization on a finite interval for a coupled system of wave equations with various boundary conditions were studied in the monograph [20].

The above asymptotic synchronization investigates the behavior of solutions on the *infinite horizon*. So, the notion of synchronizable state is no longer available as in the case of approximate boundary synchronization at a *finite horizon* (see [16, 17, 19]). This certainly raises interesting questions and needs new effective methods.

Now let us outline the main idea in the study of asymptotic synchronization.

Theorem 1.3. *Assume that system (1.3) is asymptotically synchronizable by p -groups. Then, we have the following rank condition*

$$\text{rank}(D, AD, \dots, A^{N-1}D) \geq N - p. \quad (1.13)$$

Moreover, we have

Theorem 1.4. *Assume that system (1.3) is asymptotically synchronizable by p -groups under the minimal rank condition:*

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N - p. \quad (1.14)$$

Then there exists α_{rs} such that the matrix $A = (a_{ij})$, respectively, the matrix $D = (d_{ij})$ satisfies the following row-sum conditions by blocks

$$\sum_{j=n_{s-1}+1}^{n_s} a_{ij} = \alpha_{rs}, \quad \text{respectively,} \quad \sum_{j=n_{s-1}+1}^{n_s} d_{ij} = 0 \quad (1.15)$$

for all $n_{r-1} + 1 \leq i \leq n_r$ and $1 \leq r, s \leq p$.

The above conditions will be called the conditions of C_p -compatibility, which are important ingredients in the study of asymptotic synchronization. In Proposition 4.6, we will show that these conditions guarantee the existence of symmetric and positive semi-definite matrices \bar{A}_p and \bar{D}_p of order $(N - p)$, such that the asymptotic synchronization by p -groups of system (1.3) is equivalent to the asymptotic stability of the following reduced system with an auxiliary variable $W = (w^{(1)}, \dots, w^{(N-p)})^T$:

$$W'' + \mathcal{L}W + \bar{A}_p W + \bar{D}_p G W' = 0. \quad (1.16)$$

The above approach is direct and efficient. However, the necessity of the conditions of compatibility is a delicate question. In fact, these conditions are usually imposed as a physically reasonable hypotheses for systems of ordinary differential equations. Theorem 1.4 clarifies that the necessity of these conditions of compatibility is

in fact the consequence of the minimality of the rank of Kalman's matrix. This makes the theory of asymptotic synchronization more complete for systems of partial differential equations.

Moreover, since the reduced matrices \bar{A}_p and \bar{D}_p are symmetric and positive semi-definite, the asymptotic stability of system (1.16) is equivalent to the uniqueness of the following over-determined system for the reduced variable $\Psi = (\psi^{(1)}, \dots, \psi^{(N-p)})^T$:

$$\mathcal{L}\Psi + \bar{A}_p\Psi = \beta^2\Psi \quad (1.17)$$

associated with the \bar{D}_p -observation

$$\bar{D}_p\mathcal{G}\Psi = 0. \quad (1.18)$$

The paper is organized as follows. Section 2 is devoted to the uniqueness theorem. Then, in Section 3, we apply this theorem to an abstract linear evolution equation and propose an easier way to get the asymptotic stability. Section 4 is devoted to the study of the asymptotic synchronization by p -groups. We first give a lower bound on the rank of Kalman's matrix, which is necessary for the asymptotic synchronization by p -groups, then we show that the necessity of the conditions of C_p -compatibility is in fact a consequence of the minimality of the rank of Kalman's matrix (Thms. 4.7 and 4.10). In Section 5, we give some applications to the system of wave equations with boundary damping or locally distributed damping. Moreover, we investigate a system of Kirchoff plate equations with boundary shear force damping. In Section 6, we raise some possible enlargements on the topic.

The main results of this work were announced in [21].

2. UNIQUENESS THEOREM

We will reformulate Theorem 1.1 for a general matrix A of order N and a general matrix D of order $N \times M$. In particular, the matrix A is not necessarily of "cascade type" as in the literature.

Let H and V be two separated Hilbert spaces such that $V \subset H$ with dense and compact imbedding. Let L be the duality mapping from V onto its dual space V' , such that

$$\langle L\phi, \psi \rangle_{V',V} = (\phi, \psi)_V, \quad \forall \phi, \psi \in V. \quad (2.1)$$

Let γ be a linear continuous operator from V into V' , such that

$$\langle \gamma\phi, \psi \rangle_{V',V} = \langle \gamma\psi, \phi \rangle_{V',V} \quad (2.2)$$

and

$$\langle \gamma\phi, \phi \rangle_{V',V} \geq 0 \quad \text{and} \quad \langle \gamma\phi, \phi \rangle_{V',V} = 0 \quad \text{if and only if} \quad \gamma\phi = 0. \quad (2.3)$$

Recall that \mathcal{L} and \mathcal{G} are defined by (1.2). Consider the eigen-system for the variable $\Phi = (\phi^{(1)}, \dots, \phi^{(N)})^T$:

$$\mathcal{L}\Phi + A^T\Phi = \beta^2\Phi \quad (2.4)$$

associated with the D -observation

$$D^T\mathcal{G}\Phi = 0. \quad (2.5)$$

Let us recall the following generalized rank condition of Kalman's type, which will play an important role in the study of uniqueness.

Proposition 2.1. (Lem. 2.1 in [16]) Let $d \geq 0$ be an integer. The rank condition

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N - d \quad (2.6)$$

holds if and only if d is the largest dimension of the subspaces which are invariant for A^T and contained in $\text{Ker}(D^T)$.

Proposition 2.2. Assume that system (2.4)–(2.5) has only the trivial solution. Then the pair (A, D) necessarily satisfies Kalman's rank condition (1.6).

Proof. If (1.6) fails, by Proposition 2.1, there exist a number $\lambda \in \mathbb{R}$ and a non-trivial vector $E \in \mathbb{R}^N$, such that

$$A^T E = \lambda E \quad \text{and} \quad D^T E = 0.$$

Noting that L is self-adjoint and the imbedding from V into H is compact, by the spectral theory of compact self-adjoint operators, there exist $v \in V$ and $\alpha \in \mathbb{R}^+$ large enough, such that $Lv = \alpha v$. Defining

$$\beta^2 = \alpha + \lambda > 0 \quad \text{and} \quad \Phi = vE,$$

it is easy to check that Φ is a solution to (2.4). Moreover, noting that G is of diagonal form, we have

$$D^T \mathcal{G} \Phi = \gamma v D^T E = 0.$$

Thus, we get a non-trivial solution of (2.4)–(2.5), which contradicts the assumption. \square

Definition 2.3. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of A . The matrix A satisfies the ϵ -closing condition if there exists a number a such that

$$\sup_{1 \leq l \leq m} |\lambda_l - a| \leq \epsilon. \quad (2.7)$$

Definition 2.4. The operator L satisfies the c -gap condition if there exists a number $c > 0$, such that

$$|\alpha_n - \alpha_{n'}| \geq c \quad (2.8)$$

holds true for all distinct eigenvalues $\alpha_n \neq \alpha_{n'}$ of L .

Definition 2.5. The pair (L, γ) is observable if there exists a constant $c > 0$, independent of $\beta \in \mathbb{R}$ and $f \in H$, such that the observability inequality

$$c \|\phi\|_H \leq \|f\|_H \quad (2.9)$$

holds for any given solution $\phi \in V$ to the following over-determined scalar problem

$$\beta^2 \phi - L\phi = f \quad \text{with} \quad \gamma\phi = 0. \quad (2.10)$$

Theorem 2.6. Assume that conditions (2.1)–(2.3) hold true. Assume furthermore that the pair (A, D) satisfies Kalman's rank condition (1.6). Then, system (2.4)–(2.5) has only the trivial solution $\Phi = 0$ in any one of the following cases:

(i) operator γ is global, namely,

$$\gamma\phi = 0 \implies \phi = 0; \quad (2.11)$$

(ii) matrix A satisfies the ϵ -closing condition (2.7) with $\epsilon > 0$ small enough, and L satisfies the c -gap condition (2.8) and the over-determined scalar problem

$$\beta^2\phi = L\phi \quad \text{and} \quad \gamma\phi = 0 \quad (2.12)$$

has only the trivial solution $\phi \equiv 0$;

(iii) matrix A satisfies the ϵ -closing condition (2.7) with $\epsilon > 0$ small enough, and the pair (L, γ) is observable.

In order to clarify the main idea, we first make some algebraic preliminaries.

First, since Kalman's rank condition (1.6) is invariant under invertible linear transformation (see [18]), we may assume that A is a diagonal by Jordan blocks. Let λ be an eigenvalue of A associated with a Jordan chain of length r :

$$AE_k = \lambda E_k + E_{k-1}, \quad 1 \leq k \leq r \quad \text{with} \quad E_0 = 0. \quad (2.13)$$

For $k = 1, \dots, r$, multiplying system (2.4) by E_k and setting $\phi_k = (E_k, \Phi)$, we get

$$L\phi_k + \lambda\phi_k + \phi_{k-1} = \beta^2\phi_k, \quad 1 \leq k \leq r \quad \text{with} \quad \phi_0 = 0. \quad (2.14)$$

Since L is self-adjoint, from the 1st equation of (2.14)

$$L\phi_1 + \lambda\phi_1 = \beta^2\phi_1,$$

we deduce that λ must be a real number.

Taking the inner product of the 1st equation of (2.14) with ϕ_2 and that of the 2nd one with ϕ_1 , respectively, we get

$$(L\phi_1, \phi_2)_H + \lambda(\phi_1, \phi_2)_H = \beta^2(\phi_1, \phi_2)_H \quad (2.15)$$

and

$$(L\phi_2, \phi_1)_H + \lambda(\phi_2, \phi_1)_H + (\phi_1, \phi_1)_H = \beta^2(\phi_2, \phi_1)_H. \quad (2.16)$$

Since L is self-adjoint and λ is a real number, it follows from (2.15) and (2.16) that $\phi_1 = 0$.

Repeating the same procedure for 2nd and 3rd equations of (2.14), we get $\phi_2 = 0$ and so on, until the sub-system (2.14) is reduced to a single equation

$$\beta^2\phi_r - L\phi_r - \lambda\phi_r = 0.$$

Without loss of generality, A can be written as

$$A = \text{diag}(\overbrace{\lambda_1, \dots, \lambda_1}^{\sigma_1}, \dots, \overbrace{\lambda_m, \dots, \lambda_m}^{\sigma_m}),$$

where λ_l is an eigenvalue of A with multiplicity σ_l ($l = 1, \dots, m$).

Accordingly, defining

$$\mu_r = \mu_{r-1} + \sigma_r, \quad r = 1, \dots, m \quad \text{with} \quad \mu_0 = 0,$$

we regroup the components of Φ into

$$(\phi^{(1)}, \dots, \phi^{(\mu_1)}), (\phi^{(\mu_1+1)}, \dots, \phi^{(\mu_2)}), \dots, (\phi^{(\mu_{m-1}+1)}, \dots, \phi^{(\mu_m)}).$$

On the other hand, if we replace A by $A + bI$, and β^2 by $\beta^2 + b$ for any given $b > 0$ in (2.4), this will not modify anything in Theorem 2.6. So, without loss of generality, we may assume that $\lambda_l > 0$ for $1 \leq l \leq m$.

Denote by ϵ_i ($i = 1, \dots, N$) the canonical basis vectors in \mathbb{R}^N and by d_i the i th column-vector of the matrix D^T . Noting that $D^T \epsilon_i = d_i$ and the subspace $\text{Span}\{\epsilon_{\mu_{l-1}+1}, \dots, \epsilon_{\mu_l}\}$ is invariant for A , by Proposition 2.1, Kalman's rank condition (1.6) implies that $\text{Span}\{\epsilon_{\mu_{l-1}+1}, \dots, \epsilon_{\mu_l}\} \cap \text{Ker}(D^T) = \{0\}$, namely,

$$\sum_{i=\mu_{l-1}+1}^{\mu_l} \alpha_i d_i = 0 \text{ if and only if } \alpha_i = 0 \text{ for } i = \mu_{l-1} + 1, \dots, \mu_l.$$

Therefore, for any given l with $1 \leq l \leq m$, the vectors $d_{\mu_{l-1}+1}, \dots, d_{\mu_l}$ are linearly independent.

Now we proceed the proof of Theorem 2.6.

Case (i) From (1.2) and (2.11), we have

$$D^T \mathcal{G}\Phi = \mathcal{G}(D^T \Phi) = 0 \implies D^T \Phi \equiv 0.$$

Then, applying D^T to (2.4), it follows that

$$D^T A \Phi \equiv 0,$$

namely,

$$\sum_{l=1}^m \sum_{i=\mu_{l-1}+1}^{\mu_l} \lambda_l \phi^{(i)} d_i = 0.$$

We write (2.4) as

$$L\phi^{(i)} = (\beta^2 + \lambda_l)\phi^{(i)}, \quad \mu_{l-1} + 1 \leq i \leq \mu_l, \quad 1 \leq l \leq m. \quad (2.17)$$

Since L is self-adjoint, the eigen-spaces

$$\text{Span}\{\phi^{(\mu_{l-1}+1)}, \dots, \phi^{(\mu_l)}\}, \quad 1 \leq l \leq m$$

are mutually orthogonal, then we have

$$\sum_{i=\mu_{l-1}+1}^{\mu_l} \lambda_l \phi^{(i)} d_i = 0, \quad 1 \leq l \leq m.$$

Noting $\lambda_l > 0$ and the linear independence of $d_{\mu_{l-1}+1}, \dots, d_{\mu_l}$, it follows that

$$\phi^{(i)} = 0, \quad \mu_{l-1} + 1 \leq i \leq \mu_l, \quad 1 \leq l \leq m,$$

namely, $\Phi \equiv 0$.

Case (ii) Assume that there exist l and k with $l \neq k$, such that $\phi^{(i)} \neq 0$ for $\mu_{l-1} + 1 \leq i \leq \mu_l$, and $\phi^{(i')} \neq 0$ for $\mu_{k-1} + 1 \leq i' \leq \mu_k$. There exist α_n and $\alpha_{n'}$ such that

$$\beta^2 + \lambda_l = \alpha_n \quad \text{and} \quad \beta^2 + \lambda_k = \alpha_{n'},$$

then

$$\lambda_l - \lambda_k = \alpha_n - \alpha_{n'}.$$

However, because of the ϵ -closing condition (2.7) and the c -gap condition (2.8), the above equality cannot be satisfied for $\epsilon > 0$ small enough. Therefore, there exists at most an integer k with $1 \leq k \leq m$, such that

$$\phi^{(i)} = 0, \quad \mu_{l-1} + 1 \leq i \leq \mu_l, \quad l \neq k. \quad (2.18)$$

Then, (2.5) becomes

$$D^T \mathcal{G} \Phi = \sum_{l=1}^m \sum_{i=\mu_{l-1}+1}^{\mu_l} \gamma \phi^{(i)} d_i = \sum_{i=\mu_{k-1}+1}^{\mu_k} \gamma \phi^{(i)} d_i = 0.$$

Noting (2.17) and the linear independence of $d_{\mu_{k-1}+1}, \dots, d_{\mu_k}$, it follows that

$$L\phi^{(i)} = (\beta^2 + \lambda_k)\phi^{(i)} \quad \text{with} \quad \gamma \phi^{(i)} = 0 \quad \text{for} \quad \mu_{k-1} + 1 \leq i \leq \mu_k. \quad (2.19)$$

Then the uniqueness of solution to the scalar problem (2.12) implies that

$$\phi^{(i)} = 0, \quad \mu_{k-1} + 1 \leq i \leq \mu_k,$$

which, combining with (2.18), leads to

$$\phi^{(i)} = 0, \quad \mu_{l-1} + 1 \leq i \leq \mu_l, \quad 1 \leq l \leq m,$$

namely, $\Phi \equiv 0$.

Case (iii) Applying D^T to (2.4) and noting $W = D^T \Phi$, we get

$$(\beta^2 - a)W - \mathcal{L}W = D^T A \Phi - aW. \quad (2.20)$$

On the other hand, noting the diagonal form of \mathcal{G} in (1.2), condition (2.5) leads to

$$\mathcal{G}W = \mathcal{G}D^T \Phi = D^T \mathcal{G} \Phi = 0. \quad (2.21)$$

For $1 \leq i \leq N, 1 \leq j \leq M$, setting

$$W = (w_j), \quad D^T A \Phi - aW = (f_j) \quad \text{and} \quad D = (d_{ij}),$$

we have

$$w_j = \sum_{i=1}^m d_{ji} \phi^{(i)} = \sum_{l=1}^m \sum_{i=\mu_{l-1}+1}^{\mu_l} d_{ji} \phi^{(i)}$$

and

$$f_j = \sum_{l=1}^m (\lambda_l - a) \sum_{i=\mu_{l-1}+1}^{\mu_l} d_{ji} \phi^{(i)}.$$

Then, taking the j th component of (2.20) and (2.21), we get

$$(\beta^2 - a)w_j - Lw_j = f_j \quad (2.22)$$

associated with the D -observation

$$\gamma w_j = 0. \quad (2.23)$$

If $\beta^2 - a \leq 0$, multiplying (2.22) by w_j , we get

$$-(\beta^2 - a)\|w_j\|_H^2 + \|w_j\|_V^2 = -(f_j, w_j)_H \leq \|f_j\|_H \|w_j\|_H.$$

It follows that

$$c\|w_j\|_H \leq \|f_j\|_H. \quad (2.24)$$

If $\beta^2 - a > 0$, then w_j satisfies the scalar problem (2.9). Since (L, γ) is observable, we get again (2.24). On the other hand, noting that L is self-adjoint, we have

$$(\phi^{(i)}, \phi^{(i')})_H = 0, \quad \mu_{l-1} + 1 \leq i \leq \mu_l, \quad \mu_{k-1} + 1 \leq i' \leq \mu_k, \quad l \neq k.$$

Then it follows from (2.22) that

$$\|f_j\|_H^2 \leq \sup_{1 \leq l \leq m} |a - \lambda_l|^2 \sum_{l=1}^m \left\| \sum_{i=\mu_{l-1}+1}^{\mu_l} d_{ji} \phi^{(i)} \right\|_H^2 = \sup_{1 \leq l \leq m} |a - \lambda_l|^2 \|w_j\|_H^2,$$

hence, by the ϵ -closing condition (2.7) we get

$$\|f_j\|_H \leq \sup_{1 \leq l \leq m} |a - \lambda_l| \|w_j\|_H \leq \epsilon \|w_j\|_H.$$

Thus, it follows from (2.24) that

$$\sum_{i=\mu_{l-1}+1}^{\mu_l} d_{ij} \phi^{(i)} = 0, \quad 1 \leq j \leq M, \quad 1 \leq l \leq m,$$

provided that $\epsilon < c$, namely, we have

$$\sum_{i=\mu_{l-1}+1}^{\mu_l} d_i \phi^{(i)} = 0, \quad 1 \leq l \leq m.$$

Then, the linear independence of $d_{\mu_{l-1}+1}, \dots, d_{\mu_l}$ implies that

$$\phi^{(i)} = 0, \quad \mu_{l-1} + 1 \leq i \leq \mu_l, \quad 1 \leq l \leq m.$$

namely, $\Phi \equiv 0$. The proof is then complete.

Theorem 2.6 shows that a seemingly difficult uniqueness of solution to a complex system can be established through the observability of an over-determined scalar problem. This idea will be developed for studying the asymptotic stability and the asymptotic synchronization of second order evolution systems.

The first case (i) corresponds to the globally distributed observation. In this case, similarly to the finite dimension case, without any additional assumptions on the matrix A either on the operator L , only the Kalman's rank condition is sufficient for the uniqueness of solution to the over-determined system. In the second case (ii), the condition $\gamma\phi = 0$ is not *a priori* assumed to yield $\phi = 0$. However, thanks to the c -gap condition (2.8), the uniqueness of solution to the scalar problem (2.12), which is weaker than the observability inequality, implies also the uniqueness of solution to the over-determined system. In the last case (iii), the observability inequality (2.9) does not come from the well-posedness, but rather from Carleman's estimation, and requires additional information on the differential operator L as well as on the geometric condition of the domain (see Sect. 5 for wave equations).

3. ASYMPTOTIC STABILITY

Recall that the operators L and γ satisfy conditions (2.1)–(2.3). In what follows, we assume furthermore that the operator γ is compact from V into V' .

Let A and D be symmetric and positive semi-definite matrices. Consider the following second order evolution equations:

$$U'' + \mathcal{L}U + AU + D\mathcal{G}U' = 0. \quad (3.1)$$

Defining the linear operator \mathcal{A} by

$$\mathcal{A}(U, \widehat{U}) = (\widehat{U}, -\mathcal{L}U - AU - D\mathcal{G}\widehat{U}) \quad (3.2)$$

with the domain

$$D(\mathcal{A}) = \{(U, \widehat{U}) \in \mathcal{V} \times \mathcal{V} : \mathcal{L}U + AU + D\mathcal{G}\widehat{U} \in \mathcal{V}\}, \quad (3.3)$$

we transform (3.1) into an abstract formulation as follows:

$$(U, \widehat{U})' = \mathcal{A}(U, \widehat{U}). \quad (3.4)$$

Proposition 3.1. *The operator \mathcal{A} defined in (3.2)–(3.3) generates a semi-group of contractions in the space $\mathcal{V} \times \mathcal{H}$. Moreover, \mathcal{A}^{-1} is compact in $\mathcal{V} \times \mathcal{H}$.*

Proof. Using (2.1)–(2.3), we first check that \mathcal{A} is dissipative:

$$(\mathcal{A}(U, \widehat{U}), (U, \widehat{U}))_{\mathcal{V} \times \mathcal{H}} = -(D\mathcal{G}\widehat{U}, \widehat{U})_{\mathcal{V}, \mathcal{V}} \leq 0.$$

We next show that $R(\mathcal{A}) = \mathcal{V} \times \mathcal{H}$. Then by Hill–Yosida's Theorem (see [26]), it generates a semi-group of contractions in the space $\mathcal{V} \times \mathcal{H}$.

To this end, for any given $(F, \widehat{F}) \in \mathcal{V} \times \mathcal{H}$, we solve the equation $\mathcal{A}(U, \widehat{U}) = (F, \widehat{F})$, namely,

$$\widehat{U} = F, \quad \mathcal{L}U + AU = -(DGF + \widehat{F}).$$

Noting (2.1) and the convention of pivot space, it follows that

$$(U, \Phi)_{\mathcal{V}} + (AU, \Phi)_{\mathcal{H}} = -\langle DGF + \widehat{F}, \Phi \rangle_{\mathcal{V}', \mathcal{V}}, \quad \forall \Phi \in \mathcal{V},$$

which, due to Lax-Milgram's Lemma, admits a unique solution $U \in \mathcal{V}$ with the continuous dependence:

$$\|U\|_{\mathcal{V}} \leq c\|(DGF + \widehat{F})\|_{\mathcal{V}'}.$$

Since the operator γ is compact from V to V' , so is the operator \mathcal{G} from \mathcal{V} to \mathcal{V}' . On the other hand, noting that the embeddings $V \subset H \subset V'$ are compact, so the map

$$(F, \widehat{F}) \rightarrow U = -(\mathcal{L} + A)^{-1}(DGF + \widehat{F})$$

is compact from $\mathcal{V} \times \mathcal{H}$ to \mathcal{V} . Therefore the map

$$(F, \widehat{F}) \rightarrow (U, \widehat{U}) = (-(\mathcal{L} + A)^{-1}(DGF + \widehat{F}), F)$$

is compact from $\mathcal{V} \times \mathcal{H}$ into $\mathcal{V} \times \mathcal{H}$. Therefore, \mathcal{A}^{-1} is compact in $\mathcal{V} \times \mathcal{H}$. \square

Definition 3.2. System (3.1) is asymptotically (strongly) stable in $\mathcal{V} \times \mathcal{H}$ if for any given initial data $(U_0, U_1) \in \mathcal{V} \times \mathcal{H}$, the corresponding solution U satisfies

$$(U(t), U'(t)) \rightarrow (0, 0) \quad \text{in } \mathcal{V} \times \mathcal{H} \text{ as } t \rightarrow +\infty. \quad (3.5)$$

Proposition 3.3. Assume that system (3.1) is asymptotically stable in $\mathcal{V} \times \mathcal{H}$. Then we necessarily have Kalman's rank condition (1.6).

Proof. If (1.6) fails, then there exist $E \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$, such that

$$AE = \lambda E \quad \text{and} \quad DE = 0.$$

Applying E to (3.1) and setting $\phi = (E, U)$, we get a conservative equation:

$$\phi'' + L\phi + \lambda\phi = 0,$$

which is never asymptotically stable. \square

The following theorem suggests to handle the asymptotic stability of the whole system (3.1) by means of the uniqueness of a scalar problem.

Theorem 3.4. Let the pair of matrices (A, D) satisfy Kalman's rank condition (1.6). Then, system (3.1) is asymptotically stable in any one of the situations described in Theorem 2.6.

Proof. Noting that \mathcal{A}^{-1} is compact in the space $\mathcal{V} \times \mathcal{H}$, by the classic theory of semi-groups (see [2, 3]), the dissipative system (3.1) is asymptotically stable if and only if \mathcal{A} has no pure imaginary eigenvalues.

Now, let $\beta \in \mathbb{R}$ and $(\Phi, \Psi) \in \mathcal{V} \times \mathcal{H}$, such that

$$\mathcal{A}(\Phi, \Psi) = i\beta(\Phi, \Psi).$$

It follows that

$$\mathcal{L}\Phi + A\Phi + i\beta D\mathcal{G}\Phi = \beta^2\Phi. \quad (3.6)$$

Since $\mathcal{L} + A$ is coercive, we have $\beta \neq 0$. Then, noting that \mathcal{L} and $D\mathcal{G}$ are symmetric and positive semi-definite, we deduce that (3.6) is equivalent to system (2.4)–(2.5), which has only the trivial solution $\Phi \equiv 0$ by virtue of Theorem 2.6. \square

Thanks to Theorem 2.6, the above theorem confirms well the asymptotic stability of a large class of linear dissipative systems (see examples in Sect. 5).

On the other hand, due to the absence of a general uniqueness theorem, although the equivalence between the controllability and the D -observability for wave systems has been established in [16, 17, 19], however, except few specific systems, we don't know if a system of wave equations is actually controllable or not.

4. ASYMPTOTIC SYNCHRONIZATION BY GROUPS

By Proposition 3.3, when the pair of matrices (A, D) does not satisfy Kalman's rank condition (1.6), the dissipative system (3.1) is not asymptotically stable. Instead of asymptotic stability, we have introduced the asymptotic synchronization by p -groups in Definition 1.2.

Now let S_r be the full row-rank matrix of order $(n_r - n_{r-1} - 1) \times (n_r - n_{r-1})$:

$$S_r = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}, \quad 1 \leq r \leq p. \quad (4.1)$$

Define the $(N - p) \times N$ matrix C_p of synchronization by p -groups as

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}. \quad (4.2)$$

Let $\epsilon_1, \dots, \epsilon_N$ be the vectors of the canonical basis of \mathbb{R}^N . Defining

$$e_r = \sum_{i=n_{r-1}+1}^{n_r} \epsilon_i, \quad 1 \leq r \leq p, \quad (4.3)$$

we have

$$\text{Ker}(C_p) = \text{Span}\{e_1, \dots, e_p\}. \quad (4.4)$$

Then the asymptotic synchronization by p -groups (1.12) can be equivalently written as

$$C_p(U(t), U'(t)) \rightarrow (0, 0) \quad \text{in } (V \times H)^{N-p} \text{ as } t \rightarrow +\infty. \quad (4.5)$$

Before starting the study on the asymptotic synchronization by p -groups, we first give some algebraic preliminaries.

Definition 4.1. Matrix A satisfies the condition of C_p -compatibility if

$$AKer(C_p) \subseteq Ker(C_p). \quad (4.6)$$

Proposition 4.2. *The condition of C_p -compatibility (4.6) is equivalent to the existence of a symmetric and positive semi-definite matrix \bar{A}_p of order $(N - p)$, such that*

$$(C_p C_p^T)^{-1/2} C_p A = \bar{A}_p (C_p C_p^T)^{-1/2} C_p. \quad (4.7)$$

Proof. It is sufficient to note that $Ker((C_p C_p^T)^{-1/2} C_p) = Ker(C_p)$. Then, we conclude by virtue of Lemma 3.3 in [23].

Moreover, the reduced matrix \bar{A}_p can be explicitly given by

$$\bar{A}_p = (C_p C_p^T)^{-1/2} C_p A C_p^T (C_p C_p^T)^{-1/2}, \quad (4.8)$$

which shows that the reduced matrix \bar{A}_p is also symmetric and positive semi-definite. \square

Definition 4.3. Matrix D satisfies the condition of strong C_p -compatibility if

$$Ker(C_p) \subseteq Ker(D). \quad (4.9)$$

Proposition 4.4. *The condition of strong C_p -compatibility (4.9) is equivalent to the existence of a symmetric and positive semi-definite matrix R of order $(N - p)$, such that*

$$D = C_p^T R C_p. \quad (4.10)$$

Proof. Noting that $Ker(\sqrt{D}) = Ker(D)$, it follows from (4.9) that

$$Im(\sqrt{D}) = (Ker(\sqrt{D}))^\perp = (Ker(D))^\perp \subseteq Im(C_p^T).$$

Thus, there exists a matrix \hat{R} of order $(N - p) \times N$, such that $\sqrt{D} = C_p^T \hat{R}$, which gives the expression in (4.10) with $R = \hat{R} \hat{R}^T$.

Moreover, setting

$$\bar{D}_p = (C_p C_p^T)^{1/2} R (C_p C_p^T)^{1/2}, \quad (4.11)$$

we easily check that \bar{D}_p satisfies

$$(C_p C_p^T)^{-1/2} C_p D = \bar{D}_p (C_p C_p^T)^{-1/2} C_p. \quad (4.12)$$

This justifies well the terminology “the condition of strong C_p -compatibility”. \square

Remark 4.5. Noting the special form of (4.4), the conditions of compatibility (4.6) and (4.9) are equivalent to the row-sum conditions by blocks (1.15).

Proposition 4.6. *Assume that A satisfies the condition of C_p -compatibility (4.7) and D is given by (4.10). Then the asymptotic synchronization by p -groups of system (3.1) is equivalent to the asymptotic stability of the following reduced system with the variable $W = (w^{(1)}, \dots, w^{(N-p)})^T$:*

$$W'' + \mathcal{L}W + \bar{A}_p W + \bar{D}_p G W' = 0. \quad (4.13)$$

Proof. Applying the matrix $(C_p C_p^T)^{-1/2} C_p$ to system (3.1), noting the conditions of C_p -compatibility (4.7) and (4.12) and setting

$$W = (C_p C_p^T)^{-1/2} C_p U, \quad (4.14)$$

we get the self-closed reduced system (4.13). Moreover, since the reduced matrices \bar{A}_p and \bar{D}_p are still symmetric and positive semi-definite, the asymptotic stability of the reduced system (4.13) can be obtained by applying Theorem 3.4. \square

The necessity of the conditions of C_p -compatibility is a delicate question. Intuitively, we think that this question is intrinsically linked with the rank of Kalman's matrix. In what follows, we will give a lower bound on the rank of Kalman's matrix in Theorem 4.7. Next, in Theorem 4.8, we show that the necessity of the conditions of C_p -compatibility is a consequence of the minimality of the rank of Kalman's matrix. We establish the asymptotic synchronization by p -groups in Theorem 4.10. We clarify the two kinds of asymptotic synchronizations by p -groups in Theorem 4.11 at the end of the section.

Theorem 4.7. *Assume that system (3.1) is asymptotically synchronizable by p -groups. Then we necessarily have*

$$\text{rank}(D, AD, \dots, A^{N-1}D) \geq N - p. \quad (4.15)$$

Proof. Assume that

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N - q \quad \text{with } q > p.$$

Noting that A is symmetric, by Proposition 2.1, without loss of generality, we may assume that there exists a subspace $\text{Span}\{E_1, \dots, E_q\}$ such that

$$AE_r = \lambda_r E_r \quad \text{and} \quad DE_r = 0, \quad r = 1, \dots, q.$$

Since

$$\dim \text{Im}(C_p^T) + \dim \text{Span}\{E_1, \dots, E_q\} = N - p + q > N,$$

there exists a unit vector $E \in \text{Span}\{E_1, \dots, E_q\} \cap \text{Im}(C_p^T)$. Let $x \in \mathbb{R}^{N-p}$ such that $E = C_p^T x$. Setting $\phi = (E, U) = (x, C_p U)$, by (4.5) we have

$$(\phi, \phi') \rightarrow (0, 0) \quad \text{in } V \times H \quad \text{as } t \rightarrow +\infty. \quad (4.16)$$

On the other hand, since $E \in \text{Span}\{E_1, \dots, E_q\}$, we can write $E = \sum_{r=1}^q a_r E_r$. Then, setting $\phi_r = (E_r, U)$ for $r = 1, \dots, q$, we get

$$\phi = \sum_{r=1}^q a_r (E_r, U) = \sum_{r=1}^q a_r \phi_r.$$

Applying E_r to system (3.1), we get

$$\phi_r'' + L\phi_r + \lambda_r \phi_r = 0, \quad r = 1, \dots, q \quad (4.17)$$

associated with the initial data:

$$t = 0 : \quad \phi_r = v_r, \quad \phi_r' = 0, \quad r = 1, \dots, q, \quad (4.18)$$

where the functions v_r are given by

$$Lv_r = \beta_r^2 v_r, \quad r = 1, \dots, q. \quad (4.19)$$

Since the eigenfunctions v_1, \dots, v_q are orthonormal, so are the solutions ϕ_1, \dots, ϕ_q . Then we have

$$\|\phi\|_H^2 = \left\| \sum_{r=1}^q a_r \phi_r \right\|_H^2 = \sum_{r=1}^q |a_r|^2 \|\phi_r\|_H^2 = \sum_{r=1}^q |a_r|^2 \equiv 1.$$

This contradicts the convergence (4.16). The proof is then complete. \square

Now we show that the necessity of the conditions of C_p -compatibility is in fact the consequence of the minimality of the rank of Kalman's matrix.

Theorem 4.8. *Assume that system (3.1) is asymptotically synchronizable by p -groups under the minimal rank condition*

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N - p. \quad (4.20)$$

Then, A satisfies the condition of C_p -compatibility (4.7) and D satisfies the condition of strong C_p -compatibility (4.10).

Proof. By Proposition 2.1, the rank condition (4.20) implies the existence of p orthonormal vectors E_1, \dots, E_p such that

$$AE_r = \lambda_r E_r \quad \text{and} \quad DE_r = 0, \quad r = 1, \dots, p. \quad (4.21)$$

Let β_r^2 be the eigenvalue of L associated with the eigenvector v_r , defined by (4.19). Setting

$$\beta_r^2 = \mu_r^2 + \lambda_r > 0,$$

we check easily that for each r with $1 \leq r \leq p$, the function $e^{i\beta_r t} v_r E_r$ is a solution to system (3.1). Then, by (4.5) we have

$$e^{i\beta_r t} v_r C_p E_r \rightarrow 0 \quad \text{in } V^{N-p} \quad \text{as } t \rightarrow +\infty.$$

It follows thus that

$$C_p E_r = 0, \quad r = 1, \dots, p,$$

namely,

$$\text{Ker}(C_p) = \text{Span}\{E_1, \dots, E_p\},$$

which together with (4.21) imply conditions (4.6) and (4.9). \square

Proposition 4.9. *Assume that A satisfies the condition of C_p -compatibility (4.7) and D is given by (4.10). Then,*

$$\text{rank}(C_p(D, AD, \dots, A^{N-1}D)) = \text{rank}(D, AD, \dots, A^{N-1}D). \quad (4.22)$$

Proof. By the condition of C_p -compatibility (4.6) and the condition of strong C_p -compatibility (4.9), we have

$$\text{Ker}(C_p) \subseteq \text{Ker} \begin{pmatrix} D \\ DA \\ \vdots \\ DA^{N-1} \end{pmatrix},$$

namely,

$$\text{Im}(D, AD, \dots, A^{N-1}D) \subseteq \text{Im}(C_p^T).$$

Then it follows that

$$\text{Ker}(C_p) \cap \text{Im}(D, AD, \dots, A^{N-1}D) \subseteq \text{Ker}(C_p) \cap \text{Im}(C_p^T) = \{0\},$$

which is equivalent to (4.22). \square

As for the asymptotic stability, the following theorem indicates that the asymptotic synchronization by p -groups of system (3.1) can be reduced to the uniqueness of solution to an over-determined scalar problem.

Theorem 4.10. *Let the pair (A, D) satisfy Kalman's rank condition (4.20). Assume that A satisfies the condition of C_p -compatibility (4.7) and D is given by (4.10). Then, system (3.1) is asymptotically synchronizable by p -groups in any one of the cases described in Theorem 2.6.*

Proof. By Proposition 4.6, it suffices to show the asymptotic stability of the reduced system (4.13). By Theorem 3.4, system (4.13) is asymptotically stable under the condition

$$\text{rank}(\overline{D}_p, \overline{A}_p \overline{D}_p, \dots, \overline{A}_p^{N-p-1} \overline{D}_p) = N - p. \quad (4.23)$$

In order to complete the proof, we will show that the rank condition (4.20) is equivalent to (4.23).

In fact, noting (4.7) and (4.12), we have

$$(\overline{D}_p, \overline{A}_p \overline{D}_p, \dots, \overline{A}_p^{N-p-1} \overline{D}_p) = (C_p C_p^T)^{-\frac{1}{2}} C_p(D, AD, \dots, A^{N-1}D),$$

then it follows that

$$\text{rank}(\overline{D}_p, \overline{A}_p \overline{D}_p, \dots, \overline{A}_p^{N-p-1} \overline{D}_p) = \text{rank}(C_p(D, AD, \dots, A^{N-1}D)).$$

By Proposition 4.9, we obtain (4.23). \square

Theorem 4.11. *Assume that A satisfies the condition of C_p -compatibility (4.7) and D is given by (4.10). If system (3.1) is asymptotically synchronizable by p -groups, then for any given initial data $(U_0, U_1) \in \mathcal{V} \times \mathcal{H}$, there exist some scalar functions u_1, \dots, u_p such that*

$$(u^{(k)}(t) - u_r(t), (u^{(k)})'(t) - u_r'(t)) \rightarrow (0, 0) \quad \text{in } V \times H \quad (4.24)$$

as $t \rightarrow +\infty$ for all $n_{r-1} + 1 \leq k \leq n_r$ and $1 \leq r \leq p$.

Proof. Noting (4.4), there exist some real numbers α_{rs} with $\alpha_{rs} = \alpha_{sr}$, such that

$$Ae_r = \sum_{s=1}^p \alpha_{rs} e_s, \quad r = 1, \dots, p.$$

Then, applying e_r to (3.1) and setting $v_r = (e_r, U)$ for $r = 1, \dots, p$, we get

$$v_r'' + Lv_r + \sum_{s=1}^p \alpha_{rs} v_s = 0 \quad (4.25)$$

associated with the initial data

$$t = 0: \quad v_r = (U_0, e_r), \quad v_r' = (U_1, e_r). \quad (4.26)$$

Setting

$$S = \begin{pmatrix} e_1^T \\ \vdots \\ e_p^T \\ C_p^T \end{pmatrix},$$

by (4.5) we have

$$SU = \begin{pmatrix} (e_1, U) \\ \vdots \\ (e_p, U) \\ C_p U \end{pmatrix} \rightarrow \begin{pmatrix} v_1 \\ \vdots \\ v_p \\ 0 \end{pmatrix} \quad \text{in } \mathcal{V} \times \mathcal{H} \text{ as } t \rightarrow +\infty.$$

On the other hand, since

$$S e_r = \|e_r\|^2 \epsilon_r, \quad r = 1, \dots, p,$$

setting $u_r = v_r / \|e_r\|^2$, it follows that

$$U \rightarrow S^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_p \\ 0 \end{pmatrix} = \sum_{r=1}^p v_r S^{-1} \epsilon_r = \sum_{r=1}^p \frac{v_r}{\|e_r\|^2} e_r = \sum_{r=1}^p u_r e_r$$

in $\mathcal{V} \times \mathcal{H}$ as $t \rightarrow +\infty$. The proof is thus complete. \square

Remark 4.12. The convergence in (1.12) will be called the asymptotic synchronization by p -groups in the consensus sense, and the convergence in (4.24) will be called in the pinning sense. Theorem 4.11 clarifies that the two notions are simply the same. However, the functions u_1, \dots, u_p given by Theorem 4.11 are not unique. In fact, any functions $\hat{u}_1, \dots, \hat{u}_p$ such that for $r = 1, \dots, p$,

$$(\hat{u}_r - u_r, \hat{u}_r' - u_r') \rightarrow (0, 0) \quad \text{in } \mathcal{V} \times \mathcal{H} \quad \text{as } t \rightarrow +\infty \quad (4.27)$$

satisfy also (4.24).

5. APPLICATIONS

5.1. Wave equations with boundary damping

We will give some classical examples to illustrate possible applications of the developed abstract theory. However, the approach is quite flexible and can be easily applied to other types of wave equations with memory and variable density [15], outside a star-shaped domain [28] and orbital stability [30].

Let Ω be a bounded domain in \mathbb{R}^n with the boundary $\Gamma = \Gamma_1 \cup \Gamma_0$ such that $mes(\Gamma_1) > 0$. Consider the following system of wave equations:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ U = 0 & \text{on } \mathbb{R}^+ \times \Gamma_0, \\ \partial_\nu U + DU' = 0 & \text{on } \mathbb{R}^+ \times \Gamma_1, \end{cases} \quad (5.1)$$

where ∂_ν denotes the outward normal derivative on the boundary.

Multiplying system (5.1) by a test function $\Phi \in (H_{\Gamma_0}^1(\Omega))^N$ and integrating by parts, we get the following variational formulation:

$$\int_{\Omega} (U'', \Phi) dx + \int_{\Omega} (\nabla U, \nabla \Phi) dx + \int_{\Omega} (AU, \Phi) dx + \int_{\Gamma_1} (DU', \Phi) d\Gamma = 0. \quad (5.2)$$

Let L and γ be defined by

$$\langle Lu, \phi \rangle = \int_{\Omega} \nabla u \cdot \nabla \phi dx \quad \text{and} \quad \langle \gamma v, \phi \rangle = \int_{\Gamma_1} v \phi d\Gamma, \quad (5.3)$$

respectively. Setting \mathcal{L} and \mathcal{G} as in (1.2), the variational equation (5.2) can be rewritten as

$$U'' + \mathcal{L}U + AU + D\mathcal{G}U' = 0. \quad (5.4)$$

Obviously, the operators L and γ defined by (5.3) satisfy conditions (2.1)–(2.3). Then by Proposition 3.1, system (5.4) generates a semi-group of contractions.

5.1.1. Case with Gap Condition

We first consider a specific situation of (5.1) on a rectangular domain

$$\Omega = (0, \pi) \times (0, a\pi), \quad \Gamma_0 = \{(0, y) \cup (\pi, y), \quad 0 < y < a\pi\}, \quad (5.5)$$

where $a > 0$ is a parameter.

Theorem 5.1. *Let a^2 be a rational. Assume that the pair (A, D) satisfies the rank condition (1.6). Then system (5.1) is asymptotically stable in $(H_{\Gamma_0}^1(\Omega) \times L^2(\Omega))^N$, provided that A satisfies the ϵ -closing condition with $\epsilon > 0$ small enough.*

Proof. Consider the following eigen-problem:

$$\begin{cases} -\Delta \phi = \alpha \phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma_0, \\ \partial_\nu \phi = 0 & \text{on } \Gamma_1 \end{cases} \quad (5.6)$$

with the additional observation

$$\phi = 0, \quad \text{on } \Gamma_1. \quad (5.7)$$

A straightforward computation gives the eigenvalues and the associated eigenvectors as follows:

$$\alpha_{k,l} = k^2 + \frac{l^2}{a^2}, \quad \phi_{k,l} = \sin(kx) \cos\left(\frac{ly}{a}\right).$$

We check easily, or directly by Carleman's unique continuation theorem [5, 10], that the over-determined system (5.6)–(5.7) has only the trivial solution.

Moreover, when a^2 is a rational, the eigenvalues satisfy the c -gap condition:

$$|\alpha_{k,l} - \alpha_{p,q}| = \left| k^2 - p^2 + \frac{l^2 - q^2}{a^2} \right| \geq c > 0. \quad (5.8)$$

Then by Theorem 3.4, system (5.1) is asymptotically stable, provided that A satisfies the ϵ -closing condition with $\epsilon > 0$ small enough. \square

5.1.2. Case with Observability Inequality

In this section, we assume that there exists $x_0 \in R^n$, such that setting $m = x - x_0$, we have $(m \cdot \nu) \leq 0$ on Γ_0 , where ν denotes the outward normal vector on the boundary.

Proposition 5.2. *The pair (L, γ) defined by (5.3) is observable.*

Proof. By definition, we have to show that

$$\int_{\Omega} |\phi|^2 dx \leq c \int_{\Omega} |f|^2 dx \quad (5.9)$$

for any given solution ϕ to the following over-determined scalar problem:

$$\begin{cases} \beta^2 \phi + \Delta \phi = f & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma_0, \\ \partial_{\nu} \phi = 0 & \text{on } \Gamma_1 \end{cases} \quad (5.10)$$

with the observation

$$\phi = 0 \quad \text{on } \Gamma_1. \quad (5.11)$$

Let us first recall Rellich's identity

$$2 \int_{\Omega} \Delta \phi (m \cdot \nabla \phi) dx = (n-2) \int_{\Omega} |\nabla \phi|^2 dx + \int_{\Gamma} (2 \partial_{\nu} \phi (m \cdot \nabla \phi) - (m \cdot \nu) |\nabla \phi|^2) d\Gamma$$

for all $\phi \in H^2(\Omega)$, where $m = x - x_0$.

Then, multiplying the equation in (5.10) by $2m \cdot \nabla \phi + (n-1)\phi$ and integrating by parts, we have

$$\int_{\Omega} (|\beta \phi|^2 + |\nabla \phi|^2) dx - \int_{\Gamma_0} (m \cdot \nu) |\partial_{\nu} \phi|^2 d\Gamma = - \int_{\Omega} f (2m \cdot \nabla \phi + (n-1)\phi) dx.$$

Since $(m \cdot \nu) \leq 0$ on Γ_0 , we get a stronger version than (5.9):

$$\int_{\Omega} (|\beta\phi|^2 + |\nabla\phi|^2) dx \leq c \int_{\Omega} |f|^2 dx. \quad (5.12)$$

The proof is complete. \square

Theorem 5.3. *Assume that the pair (A, D) satisfies the rank condition (4.20). Let A satisfy the condition of C_p -compatibility (4.7) and D be given by (4.10). Assume furthermore that A satisfies the ϵ -closing condition (2.7) with $\epsilon > 0$ small enough. Then system (5.1) is asymptotically synchronizable by p -groups in $(H_{\Gamma_0}^1(\Omega) \times L^2(\Omega))^N$.*

Proof. Noting that the trace operator γ is compact from $H_{\Gamma_0}^1(\Omega)$ into $L^2(\Gamma_1)$. Furthermore, by Proposition 5.2, the pair (L, γ) is observable. Then, by Theorem 4.10 we get the conclusion. \square

Remark 5.4. The inequality (5.12) is much stronger than (5.9). It can be served to establish the decay rate, for example, the uniform decay rate as in [22]:

$$\|C_p(U(t), U'(t))\|_{(\mathcal{V} \times \mathcal{H})^{N-p}} \leq M e^{-\omega t} \|C_p(U_0, U_1)\|_{(\mathcal{V} \times \mathcal{H})^{N-p}}, \quad (5.13)$$

where M and ω are positive constants, or the polynomial decay rate for the smooth initial data:

$$\|C_p(U(t), U'(t))\|_{(\mathcal{V} \times \mathcal{H})^{N-p}} = O(t^{-\delta}), \quad (5.14)$$

where the constant $\delta > 0$ is independent of the initial data. We refer to [7, 27] and the references therein for the recent progress on the polynomial stability of indirectly damped wave equations.

5.2. Wave equations with distributed damping

Let Ω be a bounded domain in \mathbb{R}^n with smooth boundary Γ . Consider the following coupled system of wave equations:

$$\begin{cases} U'' - \Delta U + AU + DU' = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ U = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \quad (5.15)$$

Multiplying system (5.15) by a test function $\Phi \in (H_0^1(\Omega))^N$ and integrating by parts, we get the following variational formulation:

$$\int_{\Omega} (U'', \Phi) dx + \int_{\Omega} (\nabla U, \nabla \Phi) dx + \int_{\Omega} (AU, \Phi) dx + \int_{\omega} (DU', \Phi) d\Gamma = 0. \quad (5.16)$$

Let L and γ be defined by

$$\langle Lu, \phi \rangle = \int_{\Omega} \nabla u \cdot \nabla \phi dx \quad \text{and} \quad \langle \gamma v, \phi \rangle = \int_{\Omega} v \phi dx, \quad (5.17)$$

respectively. The variational problem (5.16) can be rewritten as

$$U'' + \mathcal{L}U + AU + D\mathcal{G}U' = 0. \quad (5.18)$$

Theorem 5.5. *Assume that the pair (A, D) satisfies the rank condition (4.20). Assume furthermore that A satisfies the condition of C_p -compatibility (4.7) and D is given by (4.10). Then system (5.15) is asymptotically synchronizable by p -groups in $(H_0^1(\Omega) \times L^2(\Omega))^N$.*

Proof. It is a direct consequence of Theorem 4.10. \square

5.3. Kirchhoff plate equations with shear force damping

In this section, Ω is a star-shaped domain in \mathbb{R}^2 , occupied by an elastic thin plate. Let μ with $0 < \mu < 1/2$ be the Poisson ratio. Let the shear force operator B_1 and the bending moment operator B_2 be defined by

$$\begin{cases} B_1\phi = 2\nu_1\nu_2\partial_{xy}\phi - \nu_1^2\partial_{yy}\phi - \nu_2^2\partial_{xx}\phi, \\ B_2\phi = (\nu_1^2 - \nu_2^2)\partial_{xy}\phi + \nu_1\nu_2(\partial_{xx}\phi - \partial_{yy}\phi). \end{cases} \quad (5.19)$$

Consider the following system (see [11, 12] for more precise description):

$$\begin{cases} U'' + \Delta^2 U + AU = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \Delta U + (1 - \mu)B_1U = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \\ \partial_\nu \Delta U + (1 - \mu)\partial_\tau B_2U = DU' & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases} \quad (5.20)$$

where ∂_τ denotes the tangential derivative on the boundary. For the sake of simplicity, the shear force damping is acted on the whole boundary Γ .

Define the symmetric bilinear form by

$$a(\phi, \psi) = \phi_{xx}\psi_{xx} + \phi_{yy}\psi_{yy} + \mu(\phi_{xx}\psi_{yy} + \phi_{yy}\psi_{xx}) + 2(1 - \mu)\phi_{xy}\psi_{xy}. \quad (5.21)$$

Recall the Green's formula (see (4.3.20) in [11]):

$$\begin{aligned} \int_\Omega \Delta^2 \phi \psi \, dx dy &= \int_\Omega a(\phi, \psi) \, dx dy \\ &+ \int_\Gamma (\partial_\nu \Delta \phi + (1 - \mu)\partial_\tau B_2 \phi) \psi \, d\Gamma - \int_\Gamma (\Delta \phi + (1 - \mu)B_1 \phi) \partial_\nu \psi \, d\Gamma. \end{aligned} \quad (5.22)$$

Multiplying system (5.20) by a test function $\Phi \in (H^2(\Omega))^N$ and integrating by parts, we get the following variational formulation:

$$\int_\Omega (U'', \Phi) \, dx dy + \int_\Omega a(U, \Phi) \, dx dy + \int_\Omega (AU, \Phi) \, dx dy + \int_\Gamma (DU', \Phi) \, d\Gamma = 0. \quad (5.23)$$

Let L and γ be defined by

$$\langle Lu, \phi \rangle = \int_\Omega a(u, \phi) \, dx dy \quad \text{and} \quad \langle \gamma v, \phi \rangle = \int_\Gamma v \phi \, d\Gamma. \quad (5.24)$$

(5.23) can be interpreted as

$$U'' + \mathcal{L}U + AU + D\mathcal{G}U' = 0, \quad (5.25)$$

which, by Proposition 3.1, is well-posed in the sense of semi-group of contractions in the space $(H^2(\Omega) \times L^2(\Omega))^N$.

Proposition 5.6. *The pair (L, γ) defined by (5.24) is observable.*

Proof. We have to show

$$\int_\Omega |\phi|^2 \, dx dy \leq c \int_\Omega |f|^2 \, dx dy \quad (5.26)$$

for any given solution ϕ to the following over-determined scalar problem:

$$\begin{cases} \beta^2\phi - \Delta^2\phi = f & \text{in } \Omega, \\ \Delta\phi + (1-\mu)B_1\phi = 0 & \text{on } \Gamma, \\ \partial_\nu\Delta\phi + (1-\mu)\partial_\tau B_2\phi = 0 & \text{on } \Gamma \end{cases} \quad (5.27)$$

with the observation

$$\phi = 0 \quad \text{on } \Gamma. \quad (5.28)$$

Recall the following identity (see Lem. 4.5.1 in [11]):

$$\begin{aligned} \int_{\Omega} \Delta^2\phi(m \cdot \nabla\phi) dx dy &= \int_{\Omega} a(\phi, \phi) dx dy + \frac{1}{2} \int_{\Gamma} (m \cdot \nu) a(\phi, \phi) d\Gamma \\ &\quad + \int_{\Gamma} (\partial_\nu\Delta\phi + (1-\mu)\partial_\tau B_2\phi)(m \cdot \nabla\phi) d\Gamma - \int_{\Gamma} (\Delta\phi + (1-\mu)B_1\phi)\partial_\nu(m \cdot \nabla\phi) d\Gamma. \end{aligned}$$

Multiplying the equation in (5.27) by $m \cdot \nabla\phi$ and integrating by parts, we have

$$\int_{\Omega} |\beta\phi|^2 dx dy + \int_{\Omega} a(\phi, \phi) dx dy + \frac{1}{2} \int_{\Gamma} (m \cdot \nu) a(\phi, \phi) d\Gamma = - \int_{\Omega} f(m \cdot \nabla\phi) dx dy,$$

which implies (5.26). The proof is complete. \square

Theorem 5.7. *Assume that the pair (A, D) satisfies the rank condition (4.20). Let A satisfy the condition of C_p -compatibility (4.7), and D be given by (4.10). Assume furthermore that A satisfies the ϵ -closing condition (2.7) with $\epsilon > 0$ small enough. Then system (5.20) is asymptotically synchronizable by p -groups in $(H^2(\Omega) \times L^2(\Omega))^N$.*

Proof. Since the trace operator γ is compact from $H^2(\Omega)$ into $L^2(\Gamma)$, and the pair (L, γ) is observable, then, applying Theorem 4.10 we get the conclusion. \square

6. COMMENTS

In this section, we want to mention several possibilities to enlarge the research field on this topic.

6.1. Kirchhoff plate with distributed damping

Let A, D_1, D_2 be symmetric and semi-positive matrices. Consider the following system

$$\begin{cases} U'' + \Delta^2 U + AU + D_1 U' - D_2 \Delta U' = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ U = \partial_\nu U = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \quad (6.1)$$

We will investigate the asymptotic behavior of system (6.1) under the joint action of the damping given by the shear force $D_1 U'$ and the bending moment $D_2 \Delta U'$.

Theorem 6.1. *System (6.1) is asymptotically stable in $(H_0^2(\Omega) \times L^2(\Omega))^N$ if and only if (A, D) satisfies Kalman's rank condition (1.6) with $D = (D_1, D_2)$.*

Proof. The asymptotic stability of system (6.1) is equivalent to the uniqueness of solution to the eigen-problem

$$\begin{cases} \Delta^2 \Phi + A\Phi = \beta^2 \Phi & \text{in } \Omega, \\ \Phi = \partial_\nu \Phi = 0 & \text{on } \Gamma \end{cases} \quad (6.2)$$

associated with the mixed observations

$$D_1\Phi = 0 \quad \text{and} \quad D_2\Delta\Phi = 0 \quad \text{in } \Omega. \quad (6.3)$$

Obviously, Kalman's rank condition (1.6) with $D = (D_1, D_2)$ is still necessary for the uniqueness of solution to (6.2)–(6.3). We then examine the sufficiency. Applying D_1 to (6.2), it follows that

$$\Delta^2 D_1\Phi + D_1 A\Phi = \beta^2 D_1\Phi \quad \text{in } \Omega. \quad (6.4)$$

The first observation in (6.3) implies that $D_1 A\Phi = 0$ in Ω . A simple recurrence successively gives

$$D_1\Phi = D_1 A\Phi = \dots = D_1 A^{N-1}\Phi = 0. \quad (6.5)$$

On the other hand, the second observation in (6.3) implies that $D_2\Phi = 0$ in Ω . Then, by the same procedure as for D_1 , we get

$$D_2\Phi = D_2 A\Phi = \dots = D_2 A^{N-1}\Phi = 0. \quad (6.6)$$

By (6.5) and (6.6), it follows that

$$\Phi \in \text{Ker} \begin{pmatrix} D^T \\ D^T A \\ \vdots \\ D^T A^{N-1} \end{pmatrix} = \{\text{Im}(D, AD, \dots, A^{N-1}D)\}^\perp = \{0\},$$

then $\Phi \equiv 0$. The proof is complete. \square

Remark 6.2. In this example, the feedbacks are global. This case is easy to be treated, but it gives the idea of generalizing Theorem 2.6 to the case of several observations. Once it is realized, we can consider the problems with mixed damping, in particular, plate equations with distributed and boundary damping, wave equations with memory and thermal source, etc. We are waiting for a wide field of applications and discover new challenge in the forthcoming works.

6.2. System of elasticity

Let $u = (u_1, u_2, u_3)^T$ denote the displacement of an elastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary Γ .

Define

$$\epsilon_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{and} \quad \sigma_{ij}(u) = \lambda \sum_{k=1}^3 \epsilon_{kk}(u) \delta_{ij} + 2E \epsilon_{ij}(u),$$

where $\lambda > 0$ is Lamé's coefficient and $E > 0$ Young's module. The evolution of the elastic body obeys the following system:

$$\begin{cases} u'' - \text{div}(\sigma(u)) + Au = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \sigma(u)\nu = Du' & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases} \quad (6.7)$$

where $\sigma(u) = (\sigma_{ij}(u))$, and ν denotes the outside normal vector on the boundary Γ .

Let

$$u^{(l)} = \begin{pmatrix} u_1^{(l)} \\ u_2^{(l)} \\ u_3^{(l)} \end{pmatrix}, \quad U = \begin{pmatrix} u^{(1)} \\ \vdots \\ u^{(N)} \end{pmatrix}, \quad \Sigma(U) = \begin{pmatrix} \sigma(u^{(1)}) \\ \vdots \\ \sigma(u^{(N)}) \end{pmatrix}$$

and

$$\text{Div}(\Sigma(U)) = \begin{pmatrix} \text{div}(\sigma(u^{(1)})) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \text{div}(\sigma(u^{(N)})) \end{pmatrix}.$$

Then, the motion of N elastic bodies is governed by the following evolution equations

$$\begin{cases} U'' - \text{Div}(\Sigma(U)) + \mathcal{A}U = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \Sigma(U)\nu = \mathcal{D}U' & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases} \quad (6.8)$$

where \mathcal{A} and \mathcal{D} are some matrices of order $3N$ with real constant coefficients.

Let us examine a specific situation in which

$$\mathcal{A} = (a_{ij}I_3)_{1 \leq i, j \leq N} \quad \text{and} \quad \mathcal{D} = (d_{ij}I_3)_{1 \leq i, j \leq N}. \quad (6.9)$$

Let

$$A = (a_{ij})_{1 \leq i, j \leq N} \quad \text{and} \quad D = (d_{ij})_{1 \leq i, j \leq N}. \quad (6.10)$$

Clearly,

$$\text{rank}(\mathcal{D}, \mathcal{A}\mathcal{D}, \dots, \mathcal{A}^{3N-1}\mathcal{D}) = 3N \quad (6.11)$$

holds if and only if

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N. \quad (6.12)$$

If (6.12) fails, then there exist $\lambda \in \mathbb{R}$ and $x = (x_i) \in \mathbb{R}^N$, such that

$$Dx = 0 \quad \text{and} \quad Ax = \lambda x.$$

Setting

$$\mathcal{E} = \begin{pmatrix} x_1 I_3 \\ \vdots \\ x_N I_3 \end{pmatrix},$$

we easily check that

$$\mathcal{A}\mathcal{E} = \lambda\mathcal{E} \quad \text{and} \quad \mathcal{D}\mathcal{E} = 0.$$

Then, applying \mathcal{E} to system (6.8) and setting

$$\phi = \sum_{l=1} x_l u^{(l)},$$

we find that ϕ satisfies the following system:

$$\begin{cases} \phi'' - \operatorname{div}(\sigma(\phi)) + \lambda\phi = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \sigma(\phi)\nu = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases}$$

which is conservative, therefore never asymptotically stable.

Thus, the rank condition (6.11) is indeed necessary for the stability. However, if the matrices \mathcal{A} and \mathcal{D} are not of the form (6.9), the stability of system (6.8) is a largely open problem.

Of course, the above discussion can also be carried out for many other situations, such as Timoshenko beam [1, 9], Bresse beam [25], thermo-elastic plate [4, 6], viscoelastic Petrovsky-type equation [14, 24] and Maxwell equations [8].

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