ESAIM: COCV 27 (2021) S22
https://doi.org/10.1051/cocv/2020080

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

WEIGHTED SOBOLEV INEQUALITIES IN CD$(0,N)$ SPACES

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Abstract. In this note, we prove global weighted Sobolev inequalities on non-compact CD$(0,N)$ spaces satisfying a suitable growth condition, extending to possibly non-smooth and non-Riemannian structures a previous result from [V. Minerbe, *G.A.F.A.* **18** (2009) 1696–1749] stated for Riemannian manifolds with non-negative Ricci curvature. We use this result in the context of RCD$(0,N)$ spaces to get a uniform bound of the corresponding weighted heat kernel *via* a weighted Nash inequality.

Mathematics Subject Classification. 46E36, 53C23, 35K08, 51K10.

Received November 8, 2019. Accepted November 18, 2020.

1. Introduction

Riemannian manifolds with non-negative Ricci curvature have strong analytic properties. Indeed, the doubling condition and the local L^2-Poincaré inequality are satisfied on such spaces, and they imply many important results, like the well-known Li-Yau Gaussian estimates for a class of Green functions including the heat kernel [19] or powerful local Sobolev inequalities and parabolic Harnack inequalities (see *e.g.* [28]).

In the recent years, several classes of possibly non-smooth metric measure spaces containing the collection of Riemannian manifolds with non-negative Ricci curvature have been under investigation, both from a geometric and an analytic point of view. For instance, in the context of measure spaces endowed with a suitable Dirichlet form, Sturm proved existence and uniqueness of the fundamental solution of parabolic operators along with Gaussian estimates and parabolic Harnack inequalities [29, 30], provided the doubling and Poincaré properties hold. Afterwards, general doubling spaces with Poincaré-type inequalities were studied at length by Hajlasz and Koskela [15] who proved local Sobolev-type inequalities, a Trudinger inequality, a Rellich-Kondrachov theorem, and many related results.

Approximately a decade ago, Sturm [31] and Lott and Villani [20] independently proposed the curvature-dimension condition CD$(0,N)$, for $N \in [1,+\infty)$, as an extension of non-negativity of the Ricci curvature and bound from above by N of the dimension for possibly non-smooth metric measure spaces. Coupled with the infinitesimal Hilbertiannity introduced later on by Ambrosio, Gigli and Savaré [2] to rule out non-Riemannian structures, the CD$(0,N)$ condition leads to the stronger RCD$(0,N)$ condition, where R stands for Riemannian.

The classes of CD$(0,N)$ and RCD$(0,N)$ spaces have been extensively studied over the past few years, and it is by now well-known that they both contain the measured Gromov-Hausdorff closure of the class of Riemannian manifolds with non-negative Ricci curvature and dimension lower than N, as well as Alexandrov spaces with non-negative generalized sectional curvature and locally finite and non-zero n-dimensional Hausdorff measure,

Keywords and phrases: Sobolev inequalities, metric measure spaces, curvature-dimension conditions, heat kernel.

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n being lower that N. Moreover, $\mathrm{CD}(0,N)$ spaces satisfy the doubling and Poincaré properties, and $\mathrm{RCD}(0,N)$ spaces are, in addition, endowed with a regular and strongly local Dirichlet form called Cheeger energy (see Sect. 2). Therefore, the works of Sturm [29, 30] imply existence and uniqueness of an heat kernel, which by the way satisfies Gaussian estimates, on $\mathrm{RCD}(0,N)$ spaces.

One of the interest of the $\mathrm{CD}(0,N)$ and $\mathrm{RCD}(0,N)$ conditions, and of the more general $\mathrm{CD}(K,N)$ and $\mathrm{RCD}(K,N)$ conditions for arbitrary $K \in \mathbb{R}$, is the possibility of proving classical functional inequalities on spaces with rather loose structure thanks to optimal transport or gradient flow arguments. In this regard, Lott and Villani obtained in ([21], Thm. 5.29) a global Sobolev-type inequality for $\mathrm{CD}(K,N)$ spaces with $K>0$ and $N\in(2,+\infty)$. Later on, in their striking work ([8], Thm. 1.11), Cavaletti and Mondino proved a global Sobolev-type inequality with sharp constant for bounded essentially non-branching $\mathrm{CD}^*(K,N)$ spaces with $K\in\mathbb{R}$ and $N\in(1,+\infty)$; in case $K>0$ and $N>2$, they get the classical Sobolev inequality with sharp constant. This last inequality had been previously justified on $\mathrm{RCD}^*(K,N)$ spaces with $K>0$ and $N>2$ by Profeta [26].

The aim of this note is to provide a new related analytic result, namely a global weighted Sobolev inequality, for certain non-compact $\mathrm{CD}(0,N)$ spaces with $N>2$. It is worth underlying that our result does not require the Riemannian synthetic condition $\mathrm{RCD}(0,N)$. Here and throughout the paper, if $(X,\mathsf{d},\mathfrak{m})$ is a metric measure space, we write $B_r(x)$ for the ball of radius $r>0$ centered at $x\in X$, and $V(x,r)$ for $\mathfrak{m}(B_r(x))$.

Theorem 1.1 (Weighted Sobolev inequalities)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be a $\mathrm{CD}(0,N)$ space with $N>1$. Assume that there exists $1<\eta\le N$ such that*

$$0<\Theta_{inf} := \liminf_{r\to+\infty}\frac{V(o,r)}{r^\eta} \le \Theta_{sup} := \limsup_{r\to+\infty}\frac{V(o,r)}{r^\eta} < +\infty \tag{1.1}$$

for some $o\in X$. Then for any $1\le p<\eta$, there exists a constant $C>0$, depending only on N, η, Θ_{inf}, Θ_{sup} and p, such that for any continuous function $u: X\to\mathbb{R}$ admitting an upper gradient $g\in L^p(X,\mathfrak{m})$,

$$\left(\int_X |u|^{p^*}\mathrm{d}\mu\right)^{\frac{1}{p^*}} \le C\left(\int_X g^p\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}}$$

where $p^*=Np/(N-p)$ and μ is the measure absolutely continuous with respect to $\mathfrak{m}$ with density $w_o = V(o,\mathsf{d}(o,\cdot))^{p/(N-p)}\mathsf{d}(o,\cdot)^{-Np/(N-p)}$.

Theorem 1.1 extends a result by Minerbe stated for $p=2$ on n-dimensional Riemannian manifolds with non-negative Ricci curvature ([23], Thm. 0.1). The motivation there was that the classical L^2-Sobolev inequality does not hold on those manifolds which satisfy (1.1) with $\eta<N=n$, see ([23], Prop. 2.21). This phenomenon also holds on some metric measure spaces including Finsler manifolds, see the forthcoming [32] for related results.

Our proof is an adaptation of Minerbe's proof to the setting of $\mathrm{CD}(0,N)$ spaces and is based upon ideas of Grigor'yan and Saloff-Coste introduced in the smooth category [14] which extend easily to the setting of metric measure spaces. More precisely, we apply an abstract process (Thm. 2.15) which permits to patch local inequalities into a global one by means of an appropriate discrete Poincaré inequality. In the broader context of metric measure spaces with a global doubling condition, a local Poincare inequality, and a reverse doubling condition weaker than (1.1), this method provides "adimensional" weighted Sobolev inequalities, as explained in the recent work [33].

After that, we follow a classical approach (see *e.g.* [4]) which was neither considered in [23] nor in the subsequent related work [17] to deduce a weighted Nash inequality (Thm. 4.1) for $\mathrm{CD}(0,N)$ spaces satisfying the growth assumption (1.1), provided $\eta>2$. Let us mention that in the context of non-reversible Finsler manifolds, Ohta put forward an unweighted Nash inequality [24] and that Bakry, Bolley, Gentil and Maheux introduced weighted Nash inequalities in the study of possibly non-ultracontractive Markov semigroups [3], but these inequalities seem presently unrelated to our.

We conclude this note with a natural consequence in the setting of RCD$(0,N)$ spaces satisfying a uniform local Ahlfors regularity property, namely a uniform bound for the weighted heat kernel associated with a suitable modification of the Cheeger energy. To the best knowledge of the author, this is the first appearance of this weighted heat kernel whose properties would require a deeper investigation.

The paper is organized as follows. In Section 2, we introduce the tools of non-smooth analysis that we shall use throughout the article. We also define the CD$(0,N)$ and RCD$(0,N)$ conditions, and present the aforementioned patching process. Section 3 is devoted to the proof of Theorem 1.1. Section 4 deals with the weighted Nash inequality and the uniform bound on the weighted heat kernel we mentioned earlier. The final Section 5 provides a non-trivial non-smooth space to which our main theorem applies.

2. Preliminaries

Unless otherwise mentioned, in the whole article $(X,\mathsf{d},\mathfrak{m})$ denotes a triple where (X,d) is a proper, complete and separable metric space and $\mathfrak{m}$ is a Borel measure, positive and finite on balls with finite and non-zero radius, such that $\mathrm{supp}(\mathfrak{m})=X$. We use the standard notations for function spaces: $C(X)$ for the space of d-continuous functions, $\mathrm{Lip}(X)$ for the space of d-Lipschitz functions and $L^p(X,\mathfrak{m})$ (respectively $L^p_{loc}(X,\mathfrak{m})$) for the space of p-integrable (respectively locally p-integrable) functions, for any $1\le p\le+\infty$. If U is an open subset of X, we denote by $C_c(U)$ the space of continuous functions on X compactly supported in U. We also write $L^0(X,\mathfrak{m})$ (respectively $L^0_+(X,\mathfrak{m})$) for the space of $\mathfrak{m}$-measurable (respectively non-negative $\mathfrak{m}$-measurable) functions. If A is a subset of X, we denote by $\overline{A}$ its closure. For any $x\in X$ and $r>0$, we write $S_r(x)$ for $\overline{B_r(x)}\backslash B_r(x)$. For any $\lambda>0$, if B denotes a ball of radius $r>0$, we write λB for the ball with same center as B and of radius λr. If A is a bounded Borel subset of X, then for any locally integrable function $u:X\to\mathbb{R}$, we write u_A or $⨍_A u\,\mathrm{d}\mathfrak{m}$ for the mean value $\frac{1}{\mathfrak{m}(A)}\int_A u\,\mathrm{d}\mathfrak{m}$, and $\langle u\rangle_A$ for the mean value $\frac{1}{\mu(A)}\int_A u\,\mathrm{d}\mu$, where μ is as in Theorem 1.1.

Several constants appear in this work. For better readability, if a constant C depends only on parameters $a_1,a_2,\cdots$ we always write $C=C(a_1,a_2,\ldots)$ for its first occurrence, and then write more simply C if there is no ambiguity.

2.1. Non-smooth analysis

Let us recall that a continuous function $\gamma:[0,L]\to X$ is called a rectifiable curve if its length

$$L(\gamma):=\sup\left\{\sum_{i=1}^n \mathsf{d}(\gamma(x_i),\gamma(x_{i-1}))\ :\ 0=x_0<\cdots<x_n=L,\ n\in\mathbb{N}\backslash\{0\}\right\}$$

is finite. If $\gamma:[0,L]\to X$ is rectifiable then so is its restriction $\gamma\big|_{[t,s]}$ to any subinterval $[t,s]$ of $[0,L]$; moreover, there exists a continuous function $\bar\gamma:[0,L(\gamma)]\to X$, called arc-length parametrization of γ, such that $L(\bar\gamma\big|_{[t,s]})=|t-s|$ for all $0\le t\le s\le L(\gamma)$, and a non-decreasing continuous map $\varphi:[0,L]\to[0,L(\gamma)]$, such that $\gamma=\bar\gamma\circ\varphi$ (see *e.g.* [6], Prop. 2.5.9). When $\gamma=\bar\gamma$, we say that γ is parametrized by arc-length.

In the context of metric analysis, a weak notion of norm of the gradient of a function is available and due to Heinonen and Koskela [18].

Definition 2.1 (Upper gradients)**.** Let $u:X\to[-\infty,+\infty]$ be an extended real-valued function. A Borel function $g:X\to[0,+\infty]$ is called upper gradient of u if for any rectifiable curve $\gamma:[0,L]\to X$ parametrized by arc-length,

$$|u(\gamma(L))-u(\gamma(0))|\le\int_0^L g(\gamma(s))\,\mathrm{d}s.$$

Building on this, one can introduce the so-called Cheeger energies and the associated Sobolev spaces $H^{1,p}(X,\mathsf{d},\mathfrak{m})$, where $p\in[1,+\infty)$, in the following way:

Definition 2.2 (Cheeger energies and Sobolev spaces)**.** Let $1 \le p < +\infty$. The p-Cheeger energy of a function $u \in L^p(X, \mathfrak{m})$ is set as

$$\mathrm{Ch}_p(u) := \inf \liminf_{i\to\infty} \|g_i\|_{L^p}^p.$$

where the infimum is taken over all the sequences $(u_i)_i \subset L^p(X, \mathfrak{m})$ and $(g_i)_i \subset L^0_+(X, \mathfrak{m})$ such that g_i is an upper gradient of u_i and $\|u_i - u\|_{L^p} \to 0$. The Sobolev space $H^{1,p}(X, \mathsf{d}, \mathfrak{m})$ is then defined as the closure of $\mathrm{Lip}(X) \cap L^p(X, \mathfrak{m})$ with respect to the norm

$$\|u\|_{H^{1,p}} := \left(\|u\|_{L^p}^p + \mathrm{Ch}_p(u)\right)^{1/p}.$$

Remark 2.3. Following a classical convention, we call Cheeger energy the 2-Cheeger energy and write Ch instead of Ch_2.

The above relaxation process can be performed with slopes of bounded Lipschitz functions instead of upper gradients, see Lemma 4.2. Recall that the slope of a Lipschitz function f is defined as

$$|\nabla f|(x) := \begin{cases} \limsup\limits_{y\to x} \frac{|f(x)-f(y)|}{d(x,y)} & \text{if } x \in X \text{ is not isolated,} \\ 0 & \text{otherwise,} \end{cases}$$

and that it satisfies the chain rule, namely $|\nabla(fg)| \le f|\nabla g| + g|\nabla f|$ for any $f, g \in \mathrm{Lip}(X)$.

Let us recall that $(X, \mathsf{d}, \mathfrak{m})$ is called doubling if there exists $C_D \ge 1$ such that

$$V(x, 2r) \le C_D V(x, r) \qquad \forall x \in X, \forall r > 0, \tag{2.1}$$

and that it satisfies a uniform weak local L^p-Poincaré inequality, where $p \in [1, +\infty)$, if there exists $\lambda > 1$ and $C_P > 0$ such that

$$\int_B |u - u_B|^p \,\mathrm{d}\mathfrak{m} \le C_P r^p \int_{\lambda B} g^p \,\mathrm{d}\mathfrak{m} \tag{2.2}$$

holds for any ball B of arbitrary radius $r > 0$, any $u \in L^1_{loc}(X, \mathfrak{m})$ and any upper gradient $g \in L^p(X, \mathfrak{m})$ of u. If (2.2) holds with $\lambda = 1$, we say that a uniform *strong* local L^p-Poincaré inequality holds.

The next notion serves to turn weak inequalities into strong inequalities, see *e.g.* ([15], Sect. 9).

Definition 2.4 (John domain)**.** A bounded open set $\Omega \subset X$ is called a John domain if there exists $x_o \in \Omega$ and $C_J > 0$ such that for every $x \in \Omega$, there exists a Lipschitz curve $\gamma : [0, L] \to \Omega$ parametrized by arc-length such that $\gamma(0) = x$, $\gamma(L) = x_o$ and $t^{-1}\mathsf{d}(\gamma(t), X\backslash\Omega) \ge C_J$ for any $t \in [0, L]$.

Finally let us introduce a technical property taken from [15]. For any $v \in L^0(X, \mathfrak{m})$ and $0 < t_1 < t_2 < +\infty$, we denote by $v_{t_1}^{t_2}$ the truncated function $\min(\max(0, v - t_1), t_2 - t_1) + t_1$. We write χ_A for the characteristic function of a set $A \subset X$.

Definition 2.5 (Truncation property)**.** We say that a pair of $\mathfrak{m}$-measurable functions (u, g) such that for some $p \in [1, +\infty)$, $C_P > 0$ and $\lambda > 1$, the inequality (2.2) holds for any ball B of arbitrary radius $r > 0$, has the truncation property if for any $0 < t_1 < t_2 < +\infty$, $b \in \mathbb{R}$ and $\varepsilon \in \{-1, 1\}$, there exists $C > 0$ such that (2.2) holds for any ball B of arbitrary radius $r > 0$ with u, g and C_P replaced by $(\varepsilon(u - b))_{t_1}^{t_2}$, $g\chi_{\{t_1<u<t_2\}}$ and C respectively.

The next proposition is a particular case of ([15], Thm. 10.3).

Proposition 2.6. *If $(X,\mathsf{d},\mathfrak{m})$ satisfies a uniform weak local L^1-Poincaré inequality, any pair (u,g) where $u\in C(X)$ and $g\in L^1_{loc}(X,\mathfrak{m})$ is an upper gradient of u has the truncation property.*

2.2. The CD$(0,N)$ and RCD$(0,N)$ conditions

Let us give the definition of the curvature-dimension conditions CD$(0,N)$ and RCD$(0,N)$. For the general condition CD(K,N) with $K\in\mathbb{R}$, we refer to ([34], Chap. 29 & 30).

Recall that a curve $\gamma:[0,1]\to X$ is called a geodesic if $\mathsf{d}(\gamma(s),\gamma(t))=|t-s|\mathsf{d}(\gamma(0),\gamma(1))$ for any $s,t\in[0,1]$. The space (X,d) is called geodesic if for any couple of points $(x_0,x_1)\in X^2$ there exists a geodesic γ such that $\gamma(0)=x_0$ and $\gamma(1)=x_1$. We denote by $\mathcal{P}(X)$ the set of probability measures on X and by $\mathcal{P}_2(X)$ the set of probability measures μ on X with finite second moment, i.e. such that there exists $x_o\in X$ for which $\int_X \mathsf{d}^2(x_o,x)\,\mathrm{d}\mu(x)<+\infty$. The Wasserstein distance between two measures $\mu_0,\mu_1\in\mathcal{P}_2(X)$ is by definition

$$W_2(\mu_0,\mu_1):=\inf\left(\int_{X\times X}\mathsf{d}(x_0,x_1)^2\,\mathrm{d}\pi(x_0,x_1)\right)^{1/2}$$

where the infimum is taken among all the probability measures π on $X\times X$ with first marginal equal to μ_0 and second marginal equal to μ_1. A standard result of optimal transport theory states that if the space (X,d) is geodesic, then the metric space $(\mathcal{P}_2,W_2)$ is geodesic too. Let us introduce the Rényi entropies.

Definition 2.7 (Rényi entropies)**.** Given $N\in(1,+\infty)$, the N-Rényi entropy relative to $\mathfrak{m}$, denoted by $S_N(\cdot|\mathfrak{m})$, is defined as follows:

$$S_N(\mu|\mathfrak{m}):=-\int_X \rho^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m}\qquad\forall\mu\in\mathcal{P}(X),$$

where $\mu=\rho\mathfrak{m}+\mu^{sing}$ is the Lebesgue decomposition of μ with respect to $\mathfrak{m}$.

We are now in a position to introduce the CD$(0,N)$ condition, which could be summarized as weak geodesical convexity of all the N'-Rényi entropies with $N'\geq N$.

Definition 2.8 (CD$(0,N)$ condition)**.** Given $N\in(1,+\infty)$, a complete, separable, geodesic metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfies the $CD(0,N)$ condition if for any $N'\geq N$, the N'-Rényi entropy is weakly geodesically convex, meaning that for any couple of measures $(\mu_0,\mu_1)\in\mathcal{P}_2(X)^2$, there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ between μ_0 and μ_1 such that for any $t\in[0,1]$,

$$S_N(\mu_t|\mathfrak{m})\leq(1-t)S_N(\mu_0|\mathfrak{m})+tS_N(\mu_1|\mathfrak{m}).$$

Any space satisfying the CD$(0,N)$ condition is called a CD$(0,N)$ space.

The Bishop-Gromov theorem holds on CD$(0,N)$ spaces ([34], Thm. 30.11), and as a direct consequence, the doubling condition (2.1) holds too, with $C_D=2^N$. Moreover, Rajala proved the following uniform weak local L^1-Poincaré inequality ([27], Thm. 1.1).

Proposition 2.9. *Assume that $(X,\mathsf{d},\mathfrak{m})$ is a* CD$(0,N)$ *space. Then for any function $u\in C(X)$ and any upper gradient $g\in L^1_{loc}(X,\mathfrak{m})$ of u, for any ball $B\subset X$ of arbitrary radius $r>0$,*

$$\int_B|u-u_B|\,\mathrm{d}\mathfrak{m}\leq 4r\int_{2B}g\,\mathrm{d}\mathfrak{m}.$$

The CD$(0,N)$ condition does not distinguish between Riemannian-like and non-Riemannian-like structures: for instance, $\mathbb{R}^n$ equipped with the distance induced by the L^∞-norm and the Lebesgue measure satisfies the

CD$(0,N)$ condition (see the last theorem in [34]), though it is not a Riemannian structure because the L^∞-norm is not induced by any scalar product. To focus on Riemannian-like structures, Ambrosio, Gigli and Savaré added to the theory the notion of infinitesimal Hilbertianity, leading to the so-called RCD condition, R standing for *Riemannian* [2].

Definition 2.10 (RCD$(0,N)$ condition)**.** $(X,\mathsf{d},\mathfrak{m})$ is called infinitesimally Hilbertian if Ch is a quadratic form. If in addition $(X,\mathsf{d},\mathfrak{m})$ is a CD$(0,N)$ space, it is said to satisfy the RCD$(0,N)$ condition, or more simply it is called a RCD$(0,N)$ space.

Let us provide some standard facts taken from [2, 12]. First, note that $(X,\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian if and only if $H^{1,2}(X,\mathsf{d},\mathfrak{m})$ is a Hilbert space, whence the terminology. Moreover, for infinitesimally Hilbertian spaces, a suitable diagonal argument justifies for any $f\in H^{1,2}(X,\mathsf{d},\mathfrak{m})$ the existence of a function $|\nabla f|_*\in L^2(X,\mathfrak{m})$, called *minimal relaxed slope* or *minimal generalized upper gradient* of f, which gives integral representation of Ch, meaning:

$$\mathsf{Ch}(f)=\int_X|\nabla f|_*^2\,\mathrm{d}\mathfrak{m}\qquad\forall f\in H^{1,2}(X,\mathsf{d},\mathfrak{m}).$$

The minimal relaxed slope is a local object, meaning that $|\nabla f|_*=|\nabla g|_*$ $\mathfrak{m}$-a.e. on $\{f=g\}$ for any $f,g\in H^{1,2}(X,\mathsf{d},\mathfrak{m})$, and it satisfies the chain rule, namely $|\nabla(fg)|_*\le f|\nabla g|_*+g|\nabla f|_*$ $\mathfrak{m}$-a.e. on X for all $f,g\in H^{1,2}(X,\mathsf{d},\mathfrak{m})$. In addition, the function

$$\langle\nabla f_1,\nabla f_2\rangle:=\lim_{\epsilon\to0}\frac{|\nabla(f_1+\epsilon f_2)|_*^2-|\nabla f_1|_*^2}{2\epsilon}$$

provides a symmetric bilinear form on $H^{1,2}(X,\mathsf{d},\mathfrak{m})\times H^{1,2}(X,\mathsf{d},\mathfrak{m})$ with values in $L^1(X,\mathfrak{m})$, and

$$\mathsf{Ch}(f_1,f_2):=\int_X\langle\nabla f_1,\nabla f_2\rangle\,\mathrm{d}\mathfrak{m}\qquad\forall f_1,f_2\in H^{1,2}(X,\mathsf{d},\mathfrak{m}),$$

defines a strongly local, regular and symmetric Dirichlet form. Finally, the infinitesimally Hilbertian condition allows to apply the general theory of gradient flows on Hilbert spaces, ensuring the existence of the L^2-gradient flow $(h_t)_{t\ge0}$ of the convex and lower semicontinuous functional Ch, called *heat flow* of $(X,\mathsf{d},\mathfrak{m})$. This heat flow is a linear, continuous, self-adjoint and Markovian contraction semigroup in $L^2(X,\mathfrak{m})$. The terminology 'heat flow' comes from the characterization of $(h_t)_{t\ge0}$ as the only semigroup of operators such that $t\mapsto h_tf$ is locally absolutely continuous in $(0,+\infty)$ with values in $L^2(X,\mathfrak{m})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}h_tf=\Delta h_tf\quad\text{for }\mathcal{L}^1\text{-a.e. }t\in(0,+\infty)$$

holds for any $f\in L^2(X,\mathfrak{m})$, the Laplace operator Δ being defined in this context by:

$$f\in D(\Delta)\iff\exists h:=\Delta f\in L^2(X,\mathfrak{m})\text{ s.t. }\mathsf{Ch}(f,g)=-\int_X hg\,\mathrm{d}\mathfrak{m}\ \ \forall g\in H^{1,2}(X,\mathsf{d},\mathfrak{m}).$$

2.3. Patching process

Let us present now the patching process [14, 23] that we shall apply to get Theorem 1.1. In the whole paragraph, (X,d) is a metric space equipped with two Borel measures $\mathfrak{m}_1$ and $\mathfrak{m}_2$ both finite and nonzero on balls with finite and nonzero radius and such that $\mathrm{supp}(\mathfrak{m}_1)=\mathrm{supp}(\mathfrak{m}_2)=X$. For any bounded Borel set $A\subset X$ and any locally $\mathfrak{m}_2$-integrable function $u:X\to\mathbb{R}$, we denote by $\{u\}_A$ the mean value $\frac{1}{\mathfrak{m}_2(A)}\int_A u\,\mathrm{d}\mathfrak{m}_2$. For any given set S, we denote by $Card(S)$ its cardinality.

Definition 2.11 (Good covering)**.** Let $A \subset A^{\#} \subset X$ be two Borel sets. A countable family $(U_i, U_i^*, U_i^{\#})_{i\in I}$ of triples of Borel subsets of X with finite $\mathfrak{m}_j$-measure for any $j \in \{1,2\}$ is called a good covering of $(A, A^{\#})$ with respect to $(\mathfrak{m}_1, \mathfrak{m}_2)$ if:

1. for every $i \in I$, $U_i \subset U_i^* \subset U_i^{\#}$;
2. there exists a Borel set $E \subset A$ such that $A\backslash E \subset \bigcup_i U_i \subset \bigcup_i U_i^{\#} \subset A^{\#}$ and $\mathfrak{m}_1(E) = \mathfrak{m}_2(E) = 0$;
3. there exists $Q_1 > 0$ such that $Card(\{i \in I : U_{i_0}^{\#} \cap U_i^{\#} \neq \emptyset\}) \leq Q_1$ for any $i_0 \in I$;
4. for any $(i,j) \in I \times I$ such that $\overline{U_i} \cap \overline{U_j} \neq \emptyset$, there exists $k(i,j) \in I$ such that $U_i \cup U_j \subset U^*_{k(i,j)}$;
5. there exists $Q_2 > 0$ such that for any $(i,j) \in I \times I$ satisfying $\overline{U_i} \cap \overline{U_j} \neq \emptyset$,
$$\mathfrak{m}_2(U^*_{k(i,j)}) \leq Q_2 \min(\mathfrak{m}_2(U_i), \mathfrak{m}_2(U_j)).$$

When $A = A^{\#} = X$, we say that $(U_i, U_i^*, U_i^{\#})_{i\in I}$ is a good covering of (X, d) with respect to $(\mathfrak{m}_1, \mathfrak{m}_2)$.

For the sake of clarity, we call condition 3. the *overlapping condition*, condition 4. the *embracing condition* and condition 5. the *measure control condition* of the good covering. Note that in [23] the measure control condition was required also for $\mathfrak{m}_1$ though never used in the proofs.

From now on, we consider two numbers $p, q \in [1, +\infty)$ and two Borel sets $A \subset A^{\#} \subset X$. We assume that a good covering $(U_i, U_i^*, U_i^{\#})_{i\in I}$ of $(A, A^{\#})$ with respect to $(\mathfrak{m}_1, \mathfrak{m}_2)$ exists.

Let us explain how to define from $(U_i, U_i^*, U_i^{\#})_{i\in I}$ a canonical weighted graph $(\mathcal{V}, \mathcal{E}, \nu)$, where $\mathcal{V}$ is the set of vertices of the graph, $\mathcal{E}$ is the set of edges, and ν is a weight on the graph (i.e. a function $\nu : \mathcal{V} \sqcup \mathcal{E} \to \mathbb{R}$). We define $\mathcal{V}$ by associating to each U_i a vertex i (informally, we put a point i on each U_i). Then we set $\mathcal{E} := \{(i,j) \in \mathcal{V} \times \mathcal{V} : i \neq j \text{ and } \overline{U_i} \cap \overline{U_j} \neq \emptyset\}$. Finally we weight the vertices of the graph by setting $\nu(i) := \mathfrak{m}_2(U_i)$ for every $i \in \mathcal{V}$ and the edges by setting $\nu(i,j) := \max(\nu(i), \nu(j))$ for every $(i,j) \in \mathcal{E}$.

The patching theorem (Thm. 2.15) states that if some local inequalities are true on the pieces of the good covering and if a discrete inequality holds on the associated canonical weighted graph, then the local inequalities can be patched into a global one. Let us give the precise definitions.

Definition 2.12 (Local continuous $L^{q,p}$-Sobolev-Neumann inequalities)**.** We say that the good covering $(U_i, U_i^*, U_i^{\#})_{i\in I}$ satisfies local continuous $L^{q,p}$-Sobolev-Neumann inequalities if there exists a constant $S_c > 0$ such that for all $i \in I$,

$$\left(\int_{U_i} |u - \{u\}_{U_i}|^q \, \mathrm{d}\mathfrak{m}_2\right)^{\frac{1}{q}} \leq S_c \left(\int_{U_i^*} g^p \, \mathrm{d}\mathfrak{m}_1\right)^{\frac{1}{p}} \tag{2.3}$$

for all $u \in L^1(U_i, \mathfrak{m}_2)$ and all upper gradients $g \in L^p(U_i^*, \mathfrak{m}_1)$, and

$$\left(\int_{U_i^*} |u - \{u\}_{U_i^*}|^q \, \mathrm{d}\mathfrak{m}_2\right)^{\frac{1}{q}} \leq S_c \left(\int_{U_i^{\#}} g^p \, \mathrm{d}\mathfrak{m}_1\right)^{\frac{1}{p}} \tag{2.4}$$

for all $u \in L^1(U_i^*, \mathfrak{m}_2)$ and all upper gradients $g \in L^p(U_i^{\#}, \mathfrak{m}_1)$.

Definition 2.13 (Discrete L^q-Poincaré inequality)**.** We say that the weighted graph $(\mathcal{V}, \mathcal{E}, \nu)$ satisfies a discrete L^q-Poincaré inequality if there exists $S_d > 0$ such that:

$$\left(\sum_{i\in\mathcal{V}} |f(i)|^q \nu(i)\right)^{\frac{1}{q}} \leq S_d \left(\sum_{\{i,j\}\in\mathcal{E}} |f(i) - f(j)|^q \nu(i,j)\right)^{\frac{1}{q}} \qquad \forall f \in L^q(\mathcal{V}, \nu). \tag{2.5}$$

Remark 2.14. Here we differ a bit from Minerbe's terminology. Indeed, in [23], the following discrete L^q Sobolev-Dirichlet inequalities of order k were introduced for any $k \in (1,+\infty]$ and any $q \in [1,k)$:

$$\left(\sum_{i\in\mathcal{V}} |f(i)|^{\frac{qk}{k-q}}\nu(i)\right)^{\frac{k-q}{qk}} \leq S_d \left(\sum_{\{i,j\}\in\mathcal{E}} |f(i)-f(j)|^q \nu(i,j)\right)^{\frac{1}{q}} \qquad \forall f \in L^q(\mathcal{V},\nu).$$

In the present paper we only need the case $k=+\infty$, in which we recover (2.5): here is why we have chosen the terminology "Poincaré" which seems, in our setting, more appropriate.

We are now in a position to state the patching theorem.

Theorem 2.15 (Patching theorem)**.** *Let (X,d) be a metric space equipped with two Borel measures $\mathfrak{m}_1$ and $\mathfrak{m}_2$, both finite and nonzero on balls with finite and nonzero radius, such that $\operatorname{supp}(\mathfrak{m}_1)=\operatorname{supp}(\mathfrak{m}_2)=X$. Let $A\subset A^{\#}\subset X$ be two Borel sets, and $p,q\in[1,+\infty)$ be such that $q\geq p$. Assume that $(A,A^{\#})$ admits a good covering $(U_i,U_i^*,U_i^{\#})$ with respect to $(\mathfrak{m}_1,\mathfrak{m}_2)$ which satisfies the local $L^{q,p}$-Sobolev-Neumann inequalities (2.3) and (2.4) and whose associated weighted graph $(\mathcal{V},\mathcal{E},\nu)$ satisfies the discrete L^q-Poincaré inequality (2.5). Then there exists a constant $C=C(p,q,Q_1,Q_2,S_c,S_d)>0$ such that for any function $u\in C_c(A^{\#})$ and any upper gradient $g\in L^p(A^{\#},\mathfrak{m}_1)$ of u,*

$$\left(\int_A |u|^q \,\mathrm{d}\mathfrak{m}_2\right)^{\frac{1}{q}} \leq C\left(\int_{A^{\#}} g^p \,\mathrm{d}\mathfrak{m}_1\right)^{\frac{1}{p}}.$$

Although the proof of Theorem 2.15 is a straightforward adaptation of ([23], Thm. 1.8), we provide it for the reader's convenience.

Proof. Let us consider $u\in C_c(A^{\#})$. Then

$$\int_A |u|^q \,\mathrm{d}\mathfrak{m}_2 \leq \sum_{i\in\mathcal{V}} \int_{U_i} |u|^q \,\mathrm{d}\mathfrak{m}_2.$$

From convexity of the function $t\mapsto |t|^q$, we deduce $|u|^q \leq 2^{q-1}(|u-\{u\}_{U_i}|^q + |\{u\}_{U_i}|^q)$ $\mathfrak{m}_2$-a.e. on each U_i, and then

$$\int_A |u|^q \,\mathrm{d}\mathfrak{m}_2 \leq 2^{q-1}\sum_{i\in\mathcal{V}}\int_{U_i}|u-\{u\}_{U_i}|^q \,\mathrm{d}\mathfrak{m}_2 + 2^{q-1}\sum_{i\in\mathcal{V}}|\{u\}_{U_i}|^q\nu(i). \tag{2.6}$$

From (2.3) and the fact that $\sum_j x_j^{q/p} \leq (\sum_j x_j)^{q/p}$ for any finite family of non-negative numbers $\{x_j\}$ (since $q\geq p$), we get

$$\begin{aligned}\sum_{i\in\mathcal{V}}\int_{U_i}|u-\{u\}_{U_i}|^q \,\mathrm{d}\mathfrak{m}_2 &\leq S_c^{q/p}\left(\sum_{i\in\mathcal{V}}\int_{U_i^*} g^p \,\mathrm{d}\mathfrak{m}_1\right)^{q/p}\\ &\leq S_c^{q/p}Q_1^{q/p}\left(\int_{A^{\#}} g^p \,\mathrm{d}\mathfrak{m}_1\right)^{q/p},\end{aligned} \tag{2.7}$$

this last inequality being a direct consequence of the overlapping condition 3. Now the discrete L^q-Poincaré inequality (2.5) implies

$$\sum_{i\in\mathcal{V}} |\{u\}_{U_i}|^q \nu(i) \le S_d \sum_{(i,j)\in\mathcal{E}} |\{u\}_{U_i} - \{u\}_{U_j}|^q \nu(i,j). \tag{2.8}$$

For any $(i,j) \in \mathcal{E}$, a double application of Hölder's inequality yields to

$$|\{u\}_{U_i} - \{u\}_{U_j}|^q \nu(i,j) \le \frac{\nu(i,j)}{\mathfrak{m}_2(U_i)\mathfrak{m}_2(U_j)} \int_{U_i}\int_{U_j} |u(x)-u(y)|^q \,\mathrm{d}\mathfrak{m}_2(x)\,\mathrm{d}\mathfrak{m}_2(y),$$

and as the measure control condition 5. ensures $\nu(i,j) = \max(\mathfrak{m}_2(U_i), \mathfrak{m}_2(U_j)) \le Q_2 \mathfrak{m}_2(U^*_{k(i,j)})$, the embracing condition 4. implies

$$|\{u\}_{U_i} - \{u\}_{U_j}|^q \nu(i,j) \le \frac{Q_2}{\mathfrak{m}_2(U^*_{k(i,j)})} \int_{U^*_{k(i,j)}}\int_{U^*_{k(i,j)}} |u(x)-u(y)|^q \,\mathrm{d}\mathfrak{m}_2(x)\,\mathrm{d}\mathfrak{m}_2(y)$$

and then

$$|\{u\}_{U_i} - \{u\}_{U_j}|^q \nu(i,j) \le Q_2 2^q \int_{U^*_{k(i,j)}} |u - \{u\}_{U^*_{k(i,j)}}|^q \,\mathrm{d}\mathfrak{m}_2$$

where we have used again the convexity of $t \mapsto |t|^q$. Summing over $(i,j) \in \mathcal{E}$, we get

$$\sum_{(i,j)\in\mathcal{E}} |\{u\}_{U_i} - \{u\}_{U_j}|^q \nu(i,j) \le Q_2 2^q \sum_{(i,j)\in\mathcal{E}} \int_{U^*_{k(i,j)}} |u - \{u\}_{U^*_{k(i,j)}}|^q \,\mathrm{d}\mathfrak{m}_2. \tag{2.9}$$

Then (2.4) yields to

$$\sum_{(i,j)\in\mathcal{E}} |\{u\}_{U_i} - \{u\}_{U_j}|^q \nu(i,j) \le Q_2 2^q S_c^{q/p} \left(\sum_{(i,j)\in\mathcal{E}} \int_{U^\#_{k(i,j)}} g^p \,\mathrm{d}\mathfrak{m}_1 \right)^{q/p}. \tag{2.10}$$

Finally, a simple counting argument shows that

$$\sum_{(i,j)\in\mathcal{E}} \int_{U^\#_{k(i,j)}} g^p \,\mathrm{d}\mathfrak{m}_1 \le Q_1^3 \int_{A^\#} g^p \,\mathrm{d}\mathfrak{m}. \tag{2.11}$$

The result follows from combining (2.6), (2.7), (2.8), (2.9), (2.10) and (2.11). □

A similar statement holds if we replace the discrete L^q-Poincaré inequality by a discrete "L^q-Poincaré-Neumann" version:

$$\left(\sum_{i\in\mathcal{V}} |f(i) - \nu(f)|^q \nu(i)\right)^{\frac{1}{q}} \le S_d \left(\sum_{\{i,j\}\in\mathcal{E}} |f(i)-f(j)|^q \nu(i,j)\right)^{\frac{1}{q}} \tag{2.12}$$

for all compactly supported $f : \mathcal{V} \to \mathbb{R}$, where $\nu(f) = \left(\sum_{i\,:\,f(i)\neq 0} \nu(i)\right)^{-1} \sum_i f(i)\nu(i)$. The terminology "Poincaré-Neumann" comes from the mean-value in the left-hand side of (2.12) and the analogy with the local Poincare inequality used in the study of the Laplacian on bounded Euclidean domains with Neumann boundary conditions, see ([28], Sect. 1.5.2).

Theorem 2.16 (Patching theorem - Neumann version)**.** *Let* (X, d) *be a metric space equipped with two Borel measures* $\mathfrak{m}_1$ *and* $\mathfrak{m}_2$*, both finite and nonzero on balls with finite and nonzero radius, such that* $\mathrm{supp}(\mathfrak{m}_1) = \mathrm{supp}(\mathfrak{m}_2) = X$*. Let* $A \subset A^\# \subset X$ *be two Borel sets such that* $0 < \mathfrak{m}(A) < +\infty$ *and* $p, q \in [1, +\infty)$ *such that* $q \geq p$*. Assume that* $(A, A^\#)$ *admits a good covering* $(U_i, U_i^*, U_i^\#)$ *with respect to* $(\mathfrak{m}_1, \mathfrak{m}_2)$ *which satisfies the local* $L^{q,p}$*-Sobolev-Neumann inequalities* (2.3) *and* (2.4) *and whose associated weighted graph* $(\mathcal{V}, \mathcal{E}, \nu)$ *satisfies the discrete* L^q*-Poincaré-Neumann inequality* (2.12)*. Then there exists a constant* $C = C(p, q, Q_1, Q_2, S_c, S_d) > 0$ *such that for any* $u \in C_c(A^\#)$ *and any upper gradient* $g \in L^p(A^\#, \mathfrak{m}_1)$*,*

$$\left(\int_A |u - \{u\}_A|^q \, \mathrm{d}\mathfrak{m}_2 \right)^{\frac{1}{q}} \leq C \left(\int_{A^\#} g^p \, \mathrm{d}\mathfrak{m}_1 \right)^{\frac{1}{p}}.$$

The proof of Theorem 2.16 is similar to the proof of Theorem 2.15 and writes exactly as ([23], Thm. 1.10) with upper gradients instead of norms of gradients, so we skip it.

3. Proof of the main result

In this section, we prove Theorem 1.1 after a few preliminary results.

As already pointed out in [23], the local continuous $L^{2^*,2}$-Sobolev-Neumann inequalities on Riemannian manifolds (where $2^* = 2n/(n-2)$ and n is the dimension of the manifold) can be derived from the doubling condition and the uniform strong local L^2-Poincaré inequality which are both implied by non-negativity of the Ricci curvature. However, the discrete L^{2^*}-Poincaré inequality requires an additional reverse doubling condition which is an immediate consequence of the growth condition (1.1), as shown in the next lemma.

Lemma 3.1. *Let* $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ *be a metric measure space such that*

$$0 < \Theta_{inf} := \liminf_{r \to +\infty} \frac{\mathfrak{m}_Y(B_r(y_o))}{r^\alpha} \leq \Theta_{sup} := \limsup_{r \to +\infty} \frac{\mathfrak{m}_Y(B_r(y_o))}{r^\alpha} < +\infty \tag{3.1}$$

for some $y_o \in Y$ *and* $\alpha > 0$*. Then there exists* $A > 0$ *and* $C_{RD} = C_{RD}(\Theta_{inf}, \Theta_{sup}) > 0$ *such that*

$$\frac{\mathfrak{m}_Y(B_R(y_o))}{\mathfrak{m}_Y(B_r(y_o))} \geq C_{RD} \left(\frac{R}{r} \right)^\alpha \qquad \forall\, A < r \leq R. \tag{3.2}$$

Proof. The growth condition (3.1) implies the existence of $A > 0$ such that for any $R \geq r > A$, $\Theta_{inf}/2 \leq r^{-\alpha}\mathfrak{m}_Y(B_r(y_o)) \leq 2\Theta_{sup}$ and $R^{-\alpha}\mathfrak{m}_Y(B_R(y_o)) \geq \Theta_{inf}/2$, whence (3.2) with $C_{RD} = \Theta_{inf}/(4\Theta_{sup})$. □

Remark 3.2. Note that the doubling condition (2.1) easily implies (3.2): see for instance ([13], p. 9) for a proof giving $C_{RD} = (1 + C_D^{-4})^{-1}$ and $\alpha = \log_2(1 + C_D^{-4})$. But in this case, $\alpha > 1$ if and only if $C_D < 1$ which is impossible. So we emphasize that in our context, in which we want the segment $(1, \alpha)$ to be non-empty, doubling and reverse doubling must be thought as complementary hypotheses.

The next result, a strong local L^p-Sobolev inequality for $\mathrm{CD}(0, N)$ spaces, is an important technical tool for our purposes. In the context of Riemannian manifolds, it was proved by Maheux and Saloff-Coste [22].

Lemma 3.3. *Let $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ be a* CD$(0, N)$ *space. Then for any $p \in [1, N)$ there exists $C = C(N, p) > 0$ such that for any $u \in C(Y)$, any upper gradient $g \in L^1_{loc}(Y, \mathfrak{m}_Y)$, and any ball B with arbitrary radius $r > 0$,*

$$\left(\int_B |u - u_B|^{p^*} \, \mathrm{d}\mathfrak{m}_Y\right)^{\frac{1}{p^*}} \leq C \frac{r}{\mathfrak{m}_Y(B)^{1/N}} \left(\int_B g^p \, \mathrm{d}\mathfrak{m}_Y\right)^{\frac{1}{p}}, \tag{3.3}$$

where $p^* = Np/(N - p)$.

Proof. Let u be a continuous function on Y, $g \in L^1_{loc}(Y, \mathfrak{m}_Y)$ be an upper gradient of u, B be a ball with arbitrary radius $r > 0$, and $p \in [1, N)$. In this proof u_B stands for $\mathfrak{m}_Y(B)^{-1} \int_B u \, \mathrm{d}\mathfrak{m}_Y$. Thanks to Hölder's inequality and the doubling property, Proposition 2.9 implies

$$\fint_B |u - u_B| \, \mathrm{d}\mathfrak{m}_Y \leq 2^{N+2} r \left(\fint_{2B} g^p \, \mathrm{d}\mathfrak{m}_Y\right)^{1/p}.$$

Let $x_0, x_1 \in Y$ and $r_0, r_1 > 0$ be such that $x_1 \in B_{r_0}(x_0)$ and $r_1 \leq r_0$. Then

$$\frac{\mathfrak{m}_Y(B_{r_1}(x_1))}{\mathfrak{m}_Y(B_{r_0}(x_0))} \geq \frac{\mathfrak{m}_Y(B_{r_1}(x_1))}{\mathfrak{m}_Y(B_{r_0 + \mathsf{d}_Y(x_0,x_1)}(x_1))} \geq 2^{-N} \left(\frac{r_1}{r_0 + \mathsf{d}_Y(x_0, x_1)}\right)^N \geq 2^{-2N} \left(\frac{r_1}{r_0}\right)^N$$

by the doubling condition. Moreover, we know from Proposition 2.6 that (u, g) satisfies the truncation property, so that ([15], Thm. 5.1, 1) applies and gives

$$\left(\fint_B |u - u_B|^{p^*} \, \mathrm{d}\mathfrak{m}_Y\right)^{1/p^*} \leq \tilde{C} r \left(\fint_{10B} g^p \, \mathrm{d}\mathfrak{m}_Y\right)^{1/p}$$

where $\tilde{C}$ depends only on p and the doubling and Poincaré constants of $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ which depend only on N. As $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ is a $CD(0, N)$ space, the metric structure (Y, d_Y) is proper and geodesic, so it follows from ([15], Cor. 9.5) that all the balls in Y are John domains with a universal constant $C_J > 0$. Then ([15], Thm. 9.7) applies and yields to the result since $1/p^* - 1/p = 1/N$. □

Finally, let us state a result whose proof - omitted here - can be deduced from ([23], Prop. 2.8) by using Proposition 2.9. Note that even if Proposition 2.9 provides only a weak inequality, one can harmlessly substitute it to the strong one used in the proof of ([23], Prop. 2.8), because it is applied there to a function f which is Lipschitz on a ball B and extended by 0 outside of B. Note also that Proposition 2.9 being a L^1-Poincaré inequality, we can assume $\alpha > 1$ (a L^2-Poincaré inequality would have only permit $\alpha > 2$).

Proposition 3.4. *Let $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ be a* CD$(0, N)$ *space satisfying the growth condition (3.1) with $\alpha > 1$. Then there exists $\kappa_0 = \kappa_0(N, \alpha) > 1$ such that for any $R > 0$ such that $S_R(y_o)$ is non-empty, for any couple of points $(x, x') \in S_R(y_o)^2$, there exists a rectifiable curve from x to x' that remains inside $B_R(y_o) \backslash B_{\kappa_0^{-1} R}(y_o)$.*

Let us prove now Theorem 1.1. Let $(X, \mathsf{d}, \mathfrak{m})$ be a non-compact CD$(0, N)$ space with $N \geq 3$ satisfying the growth condition (1.1) with parameter $\eta \in (1, N]$, and $p \in [1, \eta)$. We recall that μ is the measure absolutely continuous with respect to $\mathfrak{m}$ with density $w_o = V(o, \mathsf{d}(o, \cdot))^{p/(N-p)} \mathsf{d}(o, \cdot)^{-Np/(N-p)}$, and that $p^* = Np/(N - p)$. Note that Lemma 3.1 applied to $(X, \mathsf{d}, \mathfrak{m})$, assuming with no loss of generality that $A = 1$, implies:

$$\frac{V(o, R)}{V(o, r)} \geq C_{RD} \left(\frac{R}{r}\right)^\eta \qquad \forall\, 1 < r < R. \tag{3.4}$$

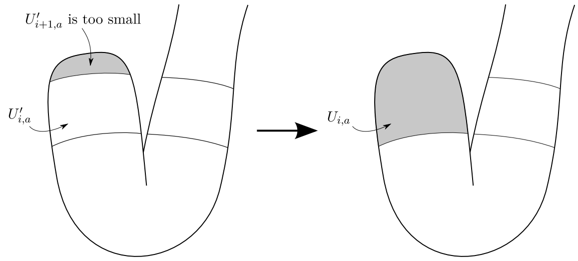

FIGURE 1. for simplicity assume $a' = a$; if $U'_{i+1,a} \cap S_{\kappa^{i+1}}(o) = \emptyset$, then we glue the small piece $U'_{i+1,a}$ to the adjacent piece $U'_{i,a}$ to form $U_{i,a}$.

Step 1: The good covering.

Let us briefly explain how to construct a good covering on $(X, \mathsf{d}, \mathfrak{m})$, referring to ([23], Sect. 2.3.1) for additional details. Define κ as the square-root of the constant κ_0 given by Proposition 3.4. Then for any $R > 0$, two connected components X_1 and X_2 of $B_{\kappa R}(o)\backslash \overline{B_R}(o)$ are always contained in one component of $B_{\kappa R}(o)\backslash B_{\kappa^{-1}R}(o)$: otherwise, linking $x \in \overline{X_1} \cap S_{\kappa R}(o)$ and $x' \in \overline{X_2} \cap S_{\kappa R}(o)$ by a curve remaining inside $B_{\kappa R}(o)\backslash B_{\kappa^{-1}R}(o)$ would not be possible.

Every point in a complete geodesic metric space of infinite diameter is the origin of some geodesic ray: see *e.g.* ([25], Prop. 10.1.1). Therefore, there exists a geodesic ray γ starting from o. For any $i \in \mathbb{N}$, let us write $A_i = B_{\kappa^i}(o)\backslash B_{\kappa^{i-1}}(o)$ and denote by $(U'_{i,a})_{0\le a\le h'_i}$ the connected components of A_i, $U'_{i,0}$ being set as the one intersecting γ. The next simple result was used without a proof in [23].

Claim **1.** There exists a constant $h = h(N, \kappa) < \infty$ such that $\sup_i h'_i \le h$.

Proof. Take $i \in \mathbb{N}$. For every $0 \le a \le h'_i$, pick x_a in $U_{i,a} \cap S_{(\kappa^i+\kappa^{i-1})/2}(o)$. As the balls $(B_a := B_{(\kappa^i-\kappa^{i-1})/4}(x_a))_{0\le a\le h'_i}$ are disjoint and all included in $B_{\kappa^i}(o)$, we have

$$h'_i \min_{0\le a\le h'_i} \mathfrak{m}(B_a) \le \sum_{0\le a\le h'_i} \mathfrak{m}(B_a) \le V(o, \kappa^i).$$

With no loss of generality, we can assume that $\min_{0\le a\le h'_i} \mathfrak{m}(B_a) = \mathfrak{m}(B_0)$. Notice that $d(o, x_0) \le \kappa^i$. Then

$$h'_i \le \frac{V(o,\kappa^i)}{\mathfrak{m}(B_0)} \le \frac{V(x_0, \kappa^i + \mathsf{d}(o, x_0))}{\mathfrak{m}(B_0)} \le \left(\frac{8\kappa^i}{\kappa^i - \kappa^{i-1}}\right)^N$$

by the doubling condition. This yields to the result with $h := \left(\frac{8\kappa}{\kappa-1}\right)^N$. □

Define then the covering $(U'_{i,a}, U'^{*}_{i,a}, U'^{\#}_{i,a})_{i\in\mathbb{N},0\le a\le h'_i}$ where $U'^{*}_{i,a}$ is by definition the union of the sets $U'_{j,b}$ such that $\overline{U'_{j,b}} \cap \overline{U'_{i,a}} \neq \emptyset$, and $U'^{\#}_{i,a}$ is by definition the union of the sets $U'^{*}_{j,b}$ such that $\overline{U'^{*}_{j,b}} \cap \overline{U'^{*}_{i,a}} \neq \emptyset$. Note that $(U'_{i,a}, U'^{*}_{i,a}, U'^{\#}_{i,a})_{i\in\mathbb{N},0\le a\le h'_i}$ is not necessarily a good covering, as pieces $U'_{i,a}$ might be arbitrary small compared to their neighbors: in this case, the measure control condition 5. would not be true. So whenever $\overline{U'_{i+1,a}} \cap S_{\kappa^{i+1}}(o) = \emptyset$ (this condition being satisfied by all "small" pieces), we set $U_{i,a} := U'_{i+1,a} \cup U'_{i,a'}$ where a' is the integer such that $\overline{U'_{i+1,a}} \cap \overline{U_{i,a'}} \neq \emptyset$; otherwise we set $U_{i+1,a} := U'_{i+1,a}$.

We define $U^*_{i,a}$ and $U^{\#}_{i,a}$ in a similar way from $U'^*_{i,a}$ and $U'^{\#}_{i,a}$ respectively. Using the doubling condition, one can easily show that $(U_{i,a}, U^*_{i,a}, U^{\#}_{i,a})_{i\in\mathbb{N},0\leq a\leq h_i}$ is a good covering of (X, d) with respect to $(\mu, \mathfrak{m})$, with constants Q_1 and Q_2 depending only on N.

Step 2: The discrete L^{p^*}-Poincaré inequality.

Let $(\mathcal{V}, \mathcal{E}, \nu)$ be the weighted graph obtained from $(U_{i,a}, U^*_{i,a}, U^{\#}_{i,a})_{i\in\mathbb{N},0\leq a\leq h_i}$. Define the degree $\deg(i, a)$ of a vertex (i, a) as the number of vertices (j, b) such that $\overline{U_{i,a}} \cap \overline{U_{j,b}} = \emptyset$. As a consequence of Claim 1, $\sup\{\deg(i, a) : (i, a) \in \mathcal{V}\} \leq 2h$. Moreover:

Claim **2.** There exists $C \geq 1$ such that $C^{-1} \leq \nu(j, b)/\nu(i, a) \leq C$ for any $(i, a), (j, b) \in \mathcal{E}$.

Proof. Take $(i, a), (j, b) \in \mathcal{E}$. With no loss of generality we can assume $j = i + 1$. Take $x \in U_{i,a} \cap S_{(\kappa^i+\kappa^{i-1})/2}(o)$ and set $r = (\kappa^i - \kappa^{i-1})/4$, $R = 2\kappa^{i+1}$, so that $B_r(x) \subset U_{i,a}$ and $U_{i+1,b} \subset B_R(x)$. Then the doubling condition implies

$$\nu(i + 1, b) \leq \mathfrak{m}_2(B_R(x)) \leq C_D(R/r)^{\log_2 C_D}\mathfrak{m}_2(B_r(x)) \leq \bar{C}\nu(i, a)$$

where $\bar{C} = C_D(8\kappa^2/(\kappa - 1))^{\log_2(C_D)} \geq 1$. A similar reasoning starting from $x \in U_{i+1,b} \cap S_{(\kappa^{i+1}+\kappa^i)/2}(o)$ provides the existence of $C' \geq 1$ such that $\nu(i, a) \leq C'\nu(i + 1, b)$. Set $C = \max(\bar{C}, C')$ to conclude. □

We are now in a position to apply ([23], Prop. 1.12) which ensures that the discrete L^1-Poincaré inequality implies the L^q one for any given $q \geq 1$. But the discrete L^1-Poincaré inequality is equivalent to the isoperimetric inequality ([23], Prop. 1.14): there exists a constant $\mathcal{I} > 0$ such that for any $\Omega \subset \mathcal{V}$ with finite measure,

$$\frac{\nu(\Omega)}{\nu(\partial\Omega)} \leq \mathcal{I}$$

where $\partial\Omega := \{(i, a), (j, b) \in \mathcal{E} : (i, a) \in \Omega, (j, b) \notin \Omega\}$. The only ingredients to prove this isoperimetric inequality are the doubling and reverse doubling conditions, see Section 2.3.3 in [23]. Then the discrete L^q-Poincaré inequality holds for any $q \geq 1$, with a constant S_d depending only on q, η, Θ_{inf}, Θ_{sup} and on the doubling and Poincaré constants of $(X, \mathsf{d}, \mathfrak{m})$, i.e. on N. In case $q = p^*$, we have $S_d = S_d(N, \eta, p, \Theta_{inf}, \Theta_{sup})$.

Step 3: The local continuous $L^{p^*,p}$-Sobolev-Neumann inequalities.

Let us explain how to get the local continuous $L^{p^*,p}$-Sobolev-Neumann inequalities. We start by deriving from the strong local L^p-Sobolev inequality (3.3) a L^p-Sobolev-type inequality on connected Borel subsets of annuli.

Claim **3.** Let $R > 0$ and $\alpha > 1$. Let A be a connected Borel subset of $B_{\alpha R}(o)\backslash B_R(o)$. For $0 < \delta < 1$, denote by $[A]_\delta$ the δ-neighborhood of A, i.e. $[A]_\delta = \bigcup_{x\in A} B_\delta(x)$. Then there exists a constant $C = C(N, \delta, \alpha, p) > 0$ such that for any function $u \in C(X)$ and any upper gradient $g \in L^p([A]_\delta, \mathfrak{m})$ of u,

$$\left(\int_A |u - u_A|^{p^*} \,\mathrm{d}\mathfrak{m}\right)^{1/p^*} \leq C\frac{R^p}{V(o, R)^{p/N}} \left(\int_{[A]_\delta} g^p \,\mathrm{d}\mathfrak{m}\right)^{1/p}.$$

Proof. Define $s = \delta R$ and choose an s-lattice of A (i.e. a maximal set of points whose distance between two of them is at least s) $(x_j)_{j\in J}$. Set $V_i = B(x_i, s)$ and $V^*_i = V^{\#}_i = B(x_i, 3s)$. Using the doubling condition, there is no difficulty in proving that $(V_i, V^*_i, V^{\#}_i)$ is a good covering of $(A, [A]_\delta)$ with respect to $(\mathfrak{m}, \mathfrak{m})$. A discrete L^{p^*}-Poincaré inequality holds on the associated weighted graph, as one can easily check following the lines of ([23], Lem. 2.10). The local continuous $L^{p^*,p}$-Sobolev-Neumann inequalities stem from the proof of ([23], Lem. 2.11), where we replace (14) there by (3.3). Then Theorem 2.16 gives the result. □

Let us prove that Claim 3 implies the local continuous $L^{p^*,p}$-Sobolev-Neumann inequalities with a constant S_c depending only on N, η and p. Take a piece of the good covering $U_{i,a}$. Choose $\delta = (1-\kappa^{-1})/2$ so that $[U_{i,a}]_\delta \subset U^*_{i,a}$. Take a function $u \in C(X)$ and an upper gradient $g \in L^p([U_{i,a}]_\delta, \mathfrak{m})$ of u. Since $|u - \langle u\rangle_{U_{i,a}}| \leq |u-c| + |c - \langle u\rangle_{U_{i,a}}|$ for any $c \in \mathbb{R}$, convexity of $t \mapsto |t|^{p^*}$ and Hölder's inequality imply

$$\begin{aligned}\int_{U_{i,a}} |u - \langle u\rangle_{U_{i,a}}|^{p^*}\,\mathrm{d}\mu &\leq 2^{p^*-1}\int_{U_{i,a}} (|u-c|^{p^*} + |c-\langle u\rangle_{U_{i,a}}|^{p^*})\,\mathrm{d}\mu \\ &\leq 2^{p^*}\inf_{c\in\mathbb{R}}\int_{U_{i,a}} |u-c|^{p^*}\,\mathrm{d}\mu \leq 2^{p^*}\int_{U_{i,a}} |u-u_{U_{i,a}}|^{p^*} w_o\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

As w_o is a radial function, we can set $\bar{w}_o(r) := w_o(x)$ for any $r > 0$ and any $x \in X$ such that $\mathsf{d}(o,x) = r$. Note that by the Bishop-Gromov theorem, $\bar{w}_o$ is a decreasing function, so

$$\int_{U_{i,a}} |u - \langle u\rangle_{U_{i,a}}|^{p^*}\,\mathrm{d}\mu \leq 2^{p^*}\bar{w}_o(\kappa^{i-1})\int_{U_{i,a}} |u-u_{U_{i,a}}|^{p^*}\,\mathrm{d}\mathfrak{m}.$$

Applying Claim 3 with $A = U_{i,a}$, $R = \kappa^{i-1}$ and $\alpha = \kappa^2$ yields to

$$\begin{aligned}\int_{U_{i,a}} |u - \langle u\rangle_{U_{i,a}}|^{p^*}\,\mathrm{d}\mu &\leq 2^{p^*}\bar{w}_o(\kappa^{i-1})\,C\,\frac{\kappa^{pp^*(i-1)}}{V(o,\kappa^{i-1})^{pp^*/N}}\left(\int_{U^*_{i,a}} g^p\,\mathrm{d}\mathfrak{m}\right)^{p^*/p} \\ &\leq C\left(\int_{U^*_{i,a}} g^p\,\mathrm{d}\mathfrak{m}\right)^{p^*/p}\end{aligned}$$

where we used the same letter C to denote different constants depending only on N, p, and κ. As κ depends only on N, η and p, we get the result.

An analogous argument implies the inequalities between levels 2 and 3.

Step 4: Conclusion.

Apply Theorem 2.15 to get the result.

4. Weighted nash inequality and bound of the corresponding heat kernel

In this section, we deduce from Theorem 1.1 a weighted Nash inequality. We use this result in the context of $\mathrm{RCD}(0,N)$ spaces to get a uniform bound on a corresponding weighted heat kernel.

Theorem 4.1 (Weighted Nash inequality)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be a $\mathrm{CD}(0,N)$ space with $N > 2$ satisfying (1.1) with $\eta > 2$. Then there exists a constant $C_{Na} = C_{Na}(N,\eta,\Theta_{inf},\Theta_{sup}) > 0$ such that:*

$$\|u\|^{2+\frac{4}{N}}_{L^2(X,\mu)} \leq C_{Na}\|u\|^{\frac{4}{N}}_{L^1(X,\mu)}\mathsf{Ch}(u) \qquad \forall\, u \in L^1(X,\mu)\cap H^{1,2}(X,\mathsf{d},\mathfrak{m}),$$

where $\mu \ll \mathfrak{m}$ has density $w_o = V(o,\mathsf{d}(o,\cdot))^{2/(N-2)}\mathsf{d}(o,\cdot)^{-2N/(N-2)}$.

To prove this theorem, we need a standard lemma which states that the relaxation procedure defining Ch can be performed with slopes of Lipschitz functions with bounded support (we write $\mathrm{Lip}_{bs}(X)$ in the sequel for the space of such functions) instead of upper gradients of L^2-functions. We omit the proof for brevity and refer to the paragraph after Propositon 4.2 in [1] for a discussion on this result. Note that here and until the end of this section we write $L^p(\mathfrak{m})$, $L^p(\mu)$ instead of $L^p(X,\mathfrak{m})$, $L^p(X,\mu)$ respectively for any $1 \le p \le +\infty$.

Lemma 4.2. *Let $(X,\mathsf{d},\mathfrak{m})$ be a complete and separable metric measure space, and $u \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$. Then*

$$\mathsf{Ch}(u) = \inf\left\{ \liminf_{n\to\infty} \int_X |\nabla u_n|^2 \,\mathrm{d}\mathfrak{m} : (u_n)_n \subset \mathrm{Lip}_{bs}(X), \|u_n - u\|_{L^2(\mathfrak{m})} \to 0 \right\}.$$

In particular, for any $u \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$, there exists a sequence $(u_n)_n \subset \mathrm{Lip}_{bs}(X)$ such that $\|u-u_n\|_{L^2(\mathfrak{m})} \to 0$ and $\||\nabla u_n|\|^2_{L^2(\mathfrak{m})} \to \mathsf{Ch}(u)$ when $n \to +\infty$.

We are now in a position to prove Theorem 4.1.

Proof. By the previous lemma it is sufficient to prove the result for $u \in \mathrm{Lip}_{bs}(X)$. By Hölder's inequality,

$$\|u\|_{L^2(\mu)} \le \|u\|^{\theta}_{L^1(\mu)} \|u\|^{1-\theta}_{L^{2^*}(\mu)}$$

where $\frac{1}{2} = \frac{\theta}{1} + \frac{1-\theta}{2^*}$ i.e. $\theta = \frac{2}{N+2}$. Then by Theorem 1.1 applied in the case $p = 2 < \eta$,

$$\|u\|_{L^2(\mu)} \le C \|u\|^{\frac{2}{N+2}}_{L^1(\mu)} \||\nabla u|\|^{\frac{N}{N+2}}_{L^2(\mathfrak{m})}.$$

It follows from the identification between slopes and minimal relaxed gradients established in ([9], Thm. 5.1) that $\mathsf{Ch}(u) = \||\nabla u|\|^2_{L^2(\mathfrak{m})}$, so the result follows by raising the previous inequality to the power $2(N+2)/N$. □

Let us consider now a RCD$(0,N)$ space $(X,\mathsf{d},\mathfrak{m})$ satisfying the growth condition (1.1) for some $\eta > 2$ and the uniform local N-Ahlfors regularity property:

$$C_o^{-1} \le \frac{V(x,r)}{r^N} \le C_o \qquad \forall x \in X,\ \ \forall\, 0 < r < r_o \tag{4.1}$$

for some $C_o > 1$ and $r_o > 0$. Such spaces are called *weakly non-collapsed* according to the terminology introduced by Gigli and De Philippis in [11]. Note that it follows from [5] that N is an integer which coincides with the essential dimension of $(X,\mathsf{d},\mathfrak{m})$.

We take the weight $w_o = V(o,\mathsf{d}(o,\cdot))^{2/(N-2)}\mathsf{d}(o,\cdot)^{-2N/(N-2)}$ which corresponds to the case $p = 2$ in Theorem 1.1. Note that (4.1) together with Bishop-Gromov's theorem implies that w_o is bounded from above by $C_o^{2/(N-2)}$, thus $L^2(\mathfrak{m}) \subset L^2(\mu)$.

Set $H^{1,2}_{loc}(X,\mathsf{d},\mathfrak{m}) = \{f \in L^2_{loc}(\mathfrak{m}) : \varphi f \in H^{1,2}(X,\mathsf{d},\mathfrak{m}) \ \ \forall \varphi \in \mathrm{Lip}_{bs}(X)\}$ and note that as an immediate consequence of (4.1) combined with Bishop-Gromov's theorem, w_o is bounded from above and below by positive constants on any compact subsets of X, thus $f \in L^2_{loc}(\mathfrak{m})$ if and only if $f \in L^2_{loc}(\mu)$.

Define a Dirichlet form Q on $L^2(\mu)$ as follows. Set

$$\mathcal{D}(Q) := \{f \in L^2(\mu) \cap H^{1,2}_{loc}(X,\mathsf{d},\mathfrak{m}) : |\nabla f|_* \in L^2(\mathfrak{m})\}$$

and

$$Q(f) = \begin{cases} \int_X |\nabla f|_*^2 \,\mathrm{d}\mathfrak{m} & \text{if } f \in \mathcal{D}(Q), \\ +\infty & \text{otherwise.} \end{cases}$$

Q is easily seen to be convex. Let us show that it is a $L^2(\mu)$-lower semicontinuous functional on $L^2(\mu)$. Let $\{f_n\}_n \subset \mathcal{D}(Q)$ and $f \in L^2(\mu)$ be such that $\|f_n - f\|_{L^2(\mu)} \to 0$. Let $K \subset X$ be a compact set. For any $i \in \mathbb{N}\backslash\{0\}$, set

$$\varphi_i(\cdot) = \max(0, 1 - (1/i)\mathsf{d}(\cdot, K))$$

and note that $\varphi_i \in \mathrm{Lip}_{bs}(X)$, $0 \le \varphi_i \le 1$, $\varphi_i \equiv 1$ on K and $|\nabla \varphi_i|_* \le (1/i)$. Then for any i, the sequence $\{\varphi_i f_n\}_n$ converges to $\varphi_i f$ in $L^2(\mathfrak{m})$. The $L^2(\mathfrak{m})$-lower semicontinuity of the Cheeger energy and the chain rule for the slope imply

$$\begin{aligned}\int_K |\nabla f|_*^2 \,\mathrm{d}\mathfrak{m} &\le \int_X |\nabla(\varphi_i f)|_*^2 \,\mathrm{d}\mathfrak{m} \le \liminf_n \int_X |\nabla(\varphi_i f_n)|_*^2 \,\mathrm{d}\mathfrak{m}\\ &\le \liminf_n \int_X |\nabla f_n|_*^2 \,\mathrm{d}\mathfrak{m} + \frac{2}{i}\liminf_n \int_X f_n |\nabla f_n|_* \,\mathrm{d}\mathfrak{m} + \frac{1}{i^2}\liminf_n \int_X f_n^2 \,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Letting i tend to $+\infty$, then letting K tend to X, yields the result.

Then we can apply the general theory of gradient flows to define the semigroup $(h_t^\mu)_{t>0}$ associated to Q which is characterized by the property that for any $f \in L^2(X,\mu)$, $t \to h_t^\mu f$ is locally absolutely continuous on $(0,+\infty)$ with values in $L^2(X,\mu)$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t^\mu f = -A h_t^\mu f \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,+\infty),$$

where the self-adjoint operator $-A$ associated to Q is defined on a dense subset $\mathcal{D}(A)$ of $\mathcal{D}(Q) = \{Q < +\infty\}$ and characterized by:

$$Q(f,g) = \int_X (Af) g \,\mathrm{d}\mu \qquad \forall f \in \mathcal{D}(A),\, \forall g \in \mathcal{D}(Q).$$

Be aware that although Q is defined by integration with respect to $\mathfrak{m}$, it is a Dirichlet form on $L^2(\mu)$, whence the involvement of μ in the above characterization.

Note that by the Markov property, each h_t^μ can be uniquely extended from $L^2(X,\mu) \cap L^1(X,\mu)$ to a contraction from $L^1(X,\mu)$ to itself.

We start with a preliminary lemma stating that a weighted Nash inequality also holds on the appropriate functional space when Ch is replaced by Q.

Lemma 4.3. *Let $(X,\mathsf{d},\mathfrak{m})$ be a $\mathrm{RCD}(0,N)$ space with $N > 3$ satisfying (1.1) and (4.1) for some $\eta > 2$, $C_o > 1$ and $r_o > 0$. Then there exists a constant $C = C(N,\eta,\Theta_{inf},\Theta_{sup}) > 0$ such that:*

$$\|u\|_{L^2(\mu)}^{2+\frac{4}{N}} \le C \|u\|_{L^1(\mu)}^{\frac{4}{N}} Q(u) \qquad \forall u \in L^1(\mu) \cap \mathcal{D}(Q).$$

Proof. Let $u \in L^1(\mu) \cap \mathcal{D}(Q)$. Then $u \in L^2_{loc}(\mathfrak{m})$, $\varphi u \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$ for any $\varphi \in \mathrm{Lip}_{bs}(X)$ and $|\nabla u|_* \in L^2(\mu)$. In particular, if we take $(\chi_n)_n$ as in the proof of Lemma 4.2, for any $n \in \mathbb{N}$ we get that $\chi_n u \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$ and consequently there exists a sequence $(u_{n,k})_k \subset \mathrm{Lip}_{bs}(X)$ such that $u_{n,k} \to \chi_n u$ in $L^2(\mathfrak{m})$ and $\int_X |\nabla u_{n,k}|^2 \,\mathrm{d}\mathfrak{m} \to \int_X |\nabla(\chi_n u)|_*^2 \,\mathrm{d}\mathfrak{m}$. Apply Theorem 4.1 to the functions $u_{n,k}$ to get

$$\|u_{n,k}\|_{L^2(\mu)}^{2+\frac{4}{N}} \le C \|u_{n,k}\|_{L^1(\mu)}^{\frac{4}{N}} \int_X |\nabla u_{n,k}|^2 \,\mathrm{d}\mathfrak{m} \tag{4.2}$$

for any $k \in \mathbb{N}$. As the $u_{n,k}$ and $\chi_n u$ have bounded support, and thanks to (4.1) which ensures boundedness of w_o, the $L^2(\mathfrak{m})$ convergence $u_{n,k} \to \chi_n u$ is equivalent to the $L^2_{loc}(\mathfrak{m})$, $L^2_{loc}(\mu)$, $L^2(\mu)$ and $L^1(\mu)$ convergences. Therefore, passing to the limit $k \to +\infty$ in (4.2), we get

$$\|\chi_n u\|_{L^2(\mu)}^{2+\frac{4}{N}} \leq C\|\chi_n u\|_{L^1(\mu)}^{\frac{4}{N}} \int_X |\nabla(\chi_n u)|_*^2 \,\mathrm{d}\mathfrak{m}.$$

By an argument similar to the proof of Lemma 4.2, we can show that

$$\limsup_{n\to+\infty} \int_X |\nabla(\chi_n u)|_*^2 \,\mathrm{d}\mathfrak{m} \leq \int_X |\nabla u|_*^2 \,\mathrm{d}\mathfrak{m}.$$

And monotone convergence ensures that $\|\chi_n u\|_{L^2(\mu)} \to \|u\|_{L^2(\mu)}$ and $\|\chi_n u\|_{L^1(\mu)} \to \|u\|_{L^1(\mu)}$, whence the result. □

Let us apply Lemma 4.3 to get a bound on the heat kernel of Q.

Theorem 4.4 (Bound of the weighted heat kernel)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a* RCD$(0, N)$ *space with $N > 3$ satisfying the growth condition* (1.1) *for some $\eta > 2$ and the uniform local N-Ahlfors regular property* (4.1) *for some $C_o > 1$ and $r_o > 0$. Then there exists $C = C(N, \eta, \Theta_{inf}, \Theta_{sup}) > 0$ such that*

$$\|h_t^\mu\|_{L^1(\mu)\to L^\infty(\mu)} \leq \frac{C}{t^{N/2}}, \qquad \forall t > 0. \tag{4.3}$$

Moreover, for any $t > 0$, h_t^μ admits a kernel p_t^μ with respect to μ such that for some $C = C(N, \eta, \Theta_{inf}, \Theta_{sup}) > 0$,

$$p_t^\mu(x, y) \leq \frac{C}{t^{N/2}} \qquad \forall x, y \in X. \tag{4.4}$$

To prove this theorem we follow closely the lines of ([28], Thm. 4.1.1). The constant C may differ from line to line, note however that it will always depend only on η, N, Θ_{inf} and Θ_{sup}.

Proof. Let $u \in L^1(\mu)$ be such that $\|u\|_{L^1(\mu)} = 1$. Let us show that $\|h_t^\mu u\|_{L^2(\mu)} \leq Ct^{-N/4}$ for any $t > 0$. First of all, by density of $\mathrm{Lip}_{bs}(X)$ in $L^1(\mu)$, we can assume $u \in \mathrm{Lip}_{bs}(X)$ with $\|u\|_{L^1(\mu)} = 1$. Furthermore, since for any $t > 0$, the Markov property ensures that the operator $h_t^\mu : L^1(\mu) \cap L^2(\mu) \to \mathcal{D}(Q)$ extends uniquely to a contraction oerator from $L^1(\mu)$ to itself, we have $h_t^\mu u \in L^1(\mu)\,(\cap \mathcal{D}(Q))$ and $\|h_t^\mu u\|_{L^1(\mu)} \leq 1$. Therefore, we can apply Lemma 4.3 to get:

$$\|h_t^\mu u\|_{L^2(\mu)}^{2+\frac{4}{N}} \leq CQ(h_t^\mu u) \qquad \forall t > 0.$$

As $\displaystyle\int_X |\nabla h_t^\mu u|_*^2 \,\mathrm{d}\mathfrak{m} = \int_X (Ah_t^\mu u) h_t^\mu u \,\mathrm{d}\mu = -\int_X \left(\frac{\mathrm{d}}{\mathrm{d}t} h_t^\mu u\right) h_t^\mu u \,\mathrm{d}\mu = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h_t^\mu u\|_{L^2(\mu)}^2$, we finally end up with the following differential inequality:

$$\|h_t^\mu u\|_{L^2(\mu)}^{2+4/N} \leq -\frac{C}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h_t^\mu u\|_{L^2(\mu)}^2 \qquad \forall t > 0.$$

Writing $\varphi(t) = \|h_t^\mu u\|_{L^2(\mu)}^2$ and $\psi(t) = \frac{N}{2}\varphi(t)^{-2/N}$ for any $t > 0$, we get $\frac{2}{C} \leq \psi'(t)$ and thus $\frac{2}{C}t \leq \psi(t) - \psi(0)$. As $\psi(0) = \frac{N}{2}\|u\|_{L^2(\mu)}^{-4/N} \geq 0$, we obtain $\frac{2}{C}t \leq \psi(t)$, leading to

$$\|h_t^\mu u\|_{L^2(\mu)} \leq \frac{C}{t^{N/4}}.$$

We have consequently $\|h_t^\mu\|_{L^1(\mu)\to L^2(\mu)} \le \frac{C}{t^{N/4}}$. Using the self-adjointness of h_t^μ, we deduce $\|h_t^\mu\|_{L^2(\mu)\to L^\infty(\mu)} \le \frac{C}{t^{N/4}}$ by duality. Finally the semigroup property

$$\|h_t^\mu\|_{L^1(\mu)\to L^\infty(\mu)} \le \|h_{t/2}^\mu\|_{L^1(\mu)\to L^2(\mu)}\|h_{t/2}^\mu\|_{L^2(\mu)\to L^\infty(\mu)}$$

implies (4.3). Then the existence of a measurable kernel p_t^μ of h_t^μ for any $t>0$ together with the bound (4.4) is a direct consequence of Lemma 4.3, thanks to ([7], Thm. (3.25)). ☐

5. A non-smooth example

To conclude, let us provide an example beyond the scope of smooth Riemannian manifolds to which Theorem 1.1 applies. For any positive integer n, let 0_n be the origin of $\mathbb{R}^n$.

In [16], Hattori built a complete four dimensional Ricci-flat manifold (M,g) satisfying (1.1) for some $\eta \in (3,4)$ and whose set of isometry classes of tangent cones at infinity $\mathcal{T}(M,g)$ is homeomorphic to $\mathbb{S}^1$. Of particular interest to us is one specific element of $\mathcal{T}(M,g)$, namely $(\mathbb{R}^3, \mathsf{d}_0^\infty, 0_3)$, where d_0^∞ is the completion of the Riemannian metric fg_e defined on $\mathbb{R}^3\backslash\{0_3\}$ as follows: g_e is the Euclidean metric on $\mathbb{R}^3$ and for any $x=(x_1,x_2,x_3)\in\mathbb{R}^3\backslash\{0_3\}$,

$$f(x)=\int_0^\infty b_x(t)\,\mathrm{d}t \quad \text{with} \quad b_x(t)=\frac{1}{\sqrt{(x_1-t^\alpha)^2+x_2^2+x_3^2}}\,,$$

for some $\alpha>1$. Since $b_x(t)\sim t^{-\alpha}$ when $t\to+\infty$ and $b_x(t)\sim |x|^{-1}$ when $t\to 0$ for any $x\neq 0_3$, then $f(x)$ has no singularity on $\mathbb{R}^3\backslash\{0\}$; however, $b_{0_3}(t)=t^{-\alpha}$ is not integrable on any neighborhood of 0, so $f(x)$ has a singularity at $x=0_3$. In particular, $(\mathbb{R}^3,\mathsf{d}_0^\infty,0_3)$ is a singular space with a unique singularity at 0_3. Hattori proved that this space is neither a metric cone nor a polar metric space.

Let d_g, v_g be the Riemannian distance and Riemannian volume measure associated to g, and $o\in M$ such that $(\mathbb{R}^3,\mathsf{d}_0^\infty,0_3)$ is a tangent cone at infinity of (M,d_g,o). Following a classical method (see *e.g.* [10]), one can equip $(\mathbb{R}^3,\mathsf{d}_0^\infty,0_3)$ with a limit measure μ such that for some infinitesimal sequence $(\varepsilon_i)_i\in(0,+\infty)$ the rescalings $(M,\mathsf{d}_{g_i},\underline{v}_{g_i},o)$, where $g_i=\varepsilon_i^2 g$ and $\underline{v}_{g_i}=v_{g_i}(B_{1/\varepsilon_i}(o))^{-1}v_{g_i}$, converge in the pointed measured Gromov-Hausdorff sense to $(\mathbb{R}^3,\mathsf{d}_0^\infty,\mu,0_3)$. As (M,g) is Ricci-flat, so are any of its rescalings, in particular they are all $\mathrm{RCD}(0,4)$ spaces. The stability of the $\mathrm{RCD}(0,4)$ condition with respect to pointed measured Gromov-Hausdorff convergence implies that $(\mathbb{R}^3,\mathsf{d}_0^\infty,\mu,0_3)$ is $\mathrm{RCD}(0,4)$ too. Let us prove that $(\mathbb{R}^3,\mathsf{d}_0^\infty,\mu,0_3)$ also satisfies (1.1). Set

$$\Theta_{inf}(M,g):=\liminf_{r\to+\infty}\frac{v_g(B_r(o))}{r^\eta} \quad \text{and} \quad \Theta_{sup}(M,g):=\limsup_{r\to+\infty}\frac{v_g(B_r(o))}{r^\eta}\,.$$

Then for any $r>0$,

$$\begin{aligned}\frac{\mu(B_r(0_3))}{r^\eta} &= \lim_{i\to+\infty}\frac{\underline{v}_{g_i}(B_r^i(o))}{r^\eta}=\lim_{i\to+\infty}\frac{v_{g_i}(B_r^i(o))}{v_{g_i}(B_1^i(o))r^\eta}=\lim_{i\to+\infty}\frac{v_g(B_{r/\varepsilon_i}(o))}{v_g(B_{1/\varepsilon_i}(o))r^\eta}\\ &=\lim_{i\to+\infty}\frac{v_g(B_{r/\varepsilon_i}(o))}{(r/\varepsilon_i)^\eta}\frac{(1/\varepsilon_i)^\eta}{v_g(B_{1/\varepsilon_i}(o))}\,,\end{aligned}$$

so

$$\Lambda:=\frac{\Theta_{inf}(M,g)}{\Theta_{sup}(M,g)}\le\frac{\mu(B_r(0_3))}{r_j^\eta}\le\Lambda^{-1}$$

from which (1.1) follows with $\Theta_{inf}\ge\Lambda$ and $\Theta_{sup}\le\Lambda^{-1}$.

Acknowledgements. I warmly thank T. Coulhon who gave the initial impetus to this work. I am also greatly indebted towards L. Ambrosio for many relevant remarks at different stages of the work. Finally, I would like to thank V. Minerbe for useful comments, G. Carron and N. Gigli for helpful final conversations, and the anonymous referees for precious suggestions.

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