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ON TANGENT CONE TO SYSTEMS OF INEQUALITIES AND EQUATIONS IN BANACH SPACES UNDER RELAXED CONSTANT RANK CONDITION

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Abstract. Under the relaxed constant rank condition, introduced by Minchenko and Stakhovski, we prove that the linearized cone is contained in the tangent cone (Abadie condition) for sets represented as solution sets to systems of finite numbers of inequalities and equations given by continuously differentiable functions defined on Banach spaces.

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1. Introduction

Conditions ensuring the equality between the tangent and the linearized cones to the constraint set are at the core of optimality conditions in constrained optimization. Let E be a Banach space and

$$\mathcal{F} := \{x \in E \mid h_i(x) = 0,\ i \in I_0,\ h_i(x) \leq 0,\ i \in I\}, \tag{1.1}$$

where $h_i: E \to \mathbb{R}$, $i \in I_0 \cup I$ are C^1 functions in a neighbourhood of $x_0 \in \mathcal{F}$. Sets I_0, I are finite, $I_0 \cup I = \{1, 2, \dots, n\}$, we admit either I_0 or I to be empty.

Abadie condition has been introduced in [1]. It says that the tangent and linearized cone coincide (see Sect. 6). In [17] it was shown that CRCQ implies Abadie constraint qualification. In the case $I_0 = \emptyset$ and $h_i: \mathbb{R}^n \to \mathbb{R}$, $i \in I$ are convex it was shown in Theorem 3.5 of [20] that Abadie condition is equivalent to the metric regularity of the respective set-valued mapping.

In finite-dimensional settings relationships between constant rank constraint qualification and Abadie condition were investigated in [6, 7], and for relationships between relaxed Mangasarian-Fromovitz and Abadie condition see *e.g.*, [18]. When h_i, $i \in I_0 \cup I$ are of class C^2, the Abadie condition has been investigated in [3]. The survey of the existing finite-dimensional constraint qualification ensuring the Abadie conditions can be found in [27].

Keywords and phrases: Tangent cone, relaxed constant rank condition, Abadie condition, rank theorem, Ljusternik theorem, Lagrange multipliers.

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In infinite-dimensional case, the most commonly used constraint qualification is Robinson's condition (see [5, 19, 26]) and the relationship to Lagrange multipliers (see [10, 25, 30]).

Due to limitations of applicability of Robinson's condition there exist a number of recent attempts to define other constraint qualification without referring to active index set (for example various forms of cone-continuity properties in [12] and the references therein).

In Proposition 1 of [15], in metric spaces, it was shown that calmness implies Abadie condition. Recently, the Abadie condition on manifolds was investigated in [9].

In the present investigation we consider constrained optimization problems defined on Banach spaces with finite number of constraints. A natural example of such an optimization problem is the projection in Hilbert space onto set of the form (1.1).

The aim of the present paper is to investigate the relationship between the relaxed constant rank constraint qualification (RCRCQ, Def. 6.1), the form of the tangent cone to $\mathcal{F}$ at a given $x_0 \in \mathcal{F}$ (the Abadie condition) and the existence of Lagrange multipliers to the problem

$$\begin{array}{ll} \text{minimize} & h_0(x) \\ \text{s.t.} & x \in \mathcal{F}, \end{array} \tag{P}$$

$h_0 : \ E \to \mathbb{R}$, in Banach space setting.

In the finite-dimensional setting, when $E = \mathbb{R}^n$, this question has been discussed in Theorem 1 of [23] (see also [22]). When dealing with the infinite-dimensional case our main tools are Banach space versions of local representation theorem (Thm. 2.5, [2], Thm. 2.1.15), rank theorem (Thm. 4.3, [2], Thm. 2.5.15) and Ljusternik theorem (see *e.g.*, [16], Sect. 0.2.4).

The main results are Theorem 6.5 and Proposition 8.1. The proof of Theorem 6.5 relies mainly on Proposition 5.2. This proposition can be viewed as a variant of the Implicit Function type theorem and relates constant rank condition with functional independence and functional dependence of a finite number of C^1 functions defined on a Banach space. In Proposition 5.2 we use the concept of functional dependence/independence which generalizes to Banach spaces the concept of Example 2.5.16 in [2] and the proof of Proposition 5.2 is based on rank theorem (Thm. 4.3).

The organization of the paper is as follows. Section 3 is devoted to constant rank condition (CRC). With the help of CRC, in Section 4, we reformulate the classical rank theorem in the case considered (finite number of functions defined on a Banach space). In Section 5 we prove Proposition 5.2 which is the main tool in deriving our main results of Section 6. In Section 6, in Definition 6.1, we define the relaxed constant rank condition (RCRCQ) and we use it to prove the Abadie condition (Thm. 6.5). In Section 7 we discuss other concepts of functional dependence and in Section 8 in Proposition 8.1 we prove the nonemptiness of the Lagrange multipliers set under RCRCQ.

2. Preliminary facts

We start with some preliminary facts and definitions which will be useful in the sequel.

Let E be a real Banach space and denote E^* its dual. Let $\langle\cdot,\cdot\rangle : \ E^* \times E \to \mathbb{R}$ denote the duality mapping between spaces E, E^*. We have $\varphi(x) = \langle \varphi\,,\, x\rangle$ for all $\varphi \in E^*$, $x \in E$. Let $\|\cdot\|$ denote norm on E and $\|\cdot\|_*$ denote norm on E^*.

Definition 2.1. The closed subspace H of a Banach space E splits or is complemented, if there is a closed subspace $G \subset E$ such that $E = H \oplus G$ (*i.e.* $E = H + G$ and $H \cap G = \{0\}$), where $\oplus$ denotes the direct sum of H and G.

Let $f:\ U \to \mathbb{R}^n$, $U \subset E$ open set and $x_0 \in U$. Fréchet derivative $Df(x_0):\ U \to \mathbb{R}^n$ is a linear operator defined as

$$\lim_{\Delta x \to 0} \frac{\|f(x_0+\Delta x) - f(x_0) - Df(x_0)\Delta x\|}{\|\Delta x\|} = 0.$$

Moreover, assuming $f = [f_1, \dots, f_n]$, where $f_i:\ U \to \mathbb{R}$, we write $Df(x_0) = [Df_1(x_0), \dots, Df_n(x_0)]^T$, where $Df_i:\ U \to \mathbb{R}$.

In the sequel we will use representations of the space E as a direct sum of closed subspaces.

Fact 2.2. Let V be a finite-dimensional subspace of a Banach space E. Let $\mathcal{B} = \{e_i, i = 1, \dots, n\}$ be a basis for V. Then there exists a basis $\{e_i^*, i = 1, \dots, n\} \subset E^*$ of V^* s.t.

$$e_i^*(e_j) = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.1}$$

Furthermore, for all $x \in V$, $e_i^*(x) = 0$ for $i = 1, 2, \dots, n$ implies that $x = 0$ (*i.e.* $\{e_i^*\}_{j=1}^n$ is a total family of functionals on V).

Proof. Let us put $w_i(e_j) = \delta_{ij}$ for all $j = 1, 2, \dots, n$ then, by Hahn-Banach Theorem, there exists a unique linear functionals $e_i^* \in E^*$, such that $e_i^*|_V = w_i$, $i = 1, 2, \dots, n$. Let us take any $x \in V$ *i.e.* $x = \sum_{i=1}^n \alpha_i e_i$ and let us observe that for all $j = 1, \dots, n$ we have

$$e_j^*(\sum_{i=1}^n \alpha_i e_i) = w_j(\sum_{i=1}^n \alpha_i e_i) = \alpha_j$$

and hence, equality $e_j^*(x) = 0$ for all $j = 1, 2, \dots, n$ implies that $\alpha_j = 0$ for all $j = 1, 2, \dots, n$. So $\{e_j^*\}_{j=1}^n$ is a total family of functionals on V. □

Proposition 2.3. *Let E be a Banach space, $f : E \to \mathbb{R}^\kappa$, $f = [f_1, \dots, f_\kappa]$, $f_i : E \to \mathbb{R}$, $i = 1, \dots, \kappa$, $x_0 \in E$. Assume that functionals $Df_i(x_0)$, $i = 1, \dots, \kappa$, are linearly independent. Then $E = E_1 \oplus E_2$, where $E_2 := \ker Df(x_0)$, $E_1 := span\{Df_1(x_0)^*, \dots, Df_\kappa(x_0)^*\}$ with $Df_i(x_0)^*$, $i = 1, \dots, \kappa$, defined as in Fact 2.2.*

Proof. Taking $e_i := Df_i(x_0) \in E^*$, $i = 1, \dots, \kappa$, we get $X_1 := \operatorname{span}\{e_1, \dots, e_\kappa\} = \operatorname{span}\{Df_1(x_0), \dots, Df_\kappa(x_0)\}$, $\dim X_1 = \kappa$. By Fact 2.2, there exist vectors $e_i^* := Df_i(x_0)^* \in E^{**}$, $i = 1, \dots, \kappa$ satisfying (2.1), *i.e.*

$$e_i^*(e_j) = Df_i(x_0)^*(Df_j(x_0)) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i = j \end{cases}, \tag{2.2}$$

which are linearly independent and $X_1^* = \operatorname{span}\{Df_i(x_0)^*,\ 1 = 1, \dots, \kappa\} \subset E^{**}$ is a κ-dimensional subspace of E^{**}.

Since X_1 is the finite-dimensional space there exists canonical isomorphism (see *e.g.* By Proposition 10.4 of [14]) $eval_{X_1}:\ X_1 \to X_1^{**}$ defined as

$$eval_{X_1}(v)(u^*) = u^*(v) \quad \text{for every } v \in X_1,\ u^* \in X_1^*$$

and we have $e_i^*(e_j) = e_j(e_i^*) = Df_i(x_0)Df_j(x_0)^*$ for all $i, j \in \{1, \dots, \kappa\}$. By (2.2),

$$Df(x_0)(Df_i(x_0)^*) = v_i, \quad v_i = [0, \dots, \underbrace{1}_{i}, \dots, 0] \in \mathbb{R}^\kappa. \tag{2.3}$$

Now we show that $E = X_1^* \oplus \ker Df(x_0)$. For any $x \in E$, $Df(x_0)(x) \in \mathbb{R}^\kappa$, there exist $\alpha_i(x) \in \mathbb{R}$, $i = 1, \dots, \kappa$ such that $Df(x_0)(x) = \sum_{i=1}^{\kappa} \alpha_i(x) v_i$. Take $m \in X_1^*$, $m := \sum_{i=1}^{\kappa} \alpha_i(x) Df_i(x_0)^*$. By (2.3), we have

$$Df(x_0)(x-m) = Df(x_0)(x) - Df(x_0)(m) = \sum_{i=1}^{\kappa} \alpha_i(x) v_i - \sum_{i=1}^{\kappa} \alpha_i(x) Df(x_0)(Df_i(x_0)^*) = 0.$$

This shows that $x - m \in \ker Df(x_0)$ which proves the assertion with $E_1 := X_1^*$ and $E_2 := \ker Df(x_0)$. □

In the sequel we will make a frequent use of the following observation.

Lemma 2.4. *Let E be a Banach space and $f : E \to \mathbb{R}^p$, $f = [f_1, \dots, f_p]$, $f_i : E \to \mathbb{R}$, $i = 1, \dots, p$ be C^1 functions in some neighbourhood of $x_0 \in E$. If $Df_j(x_0)$, $j = 1, \dots, p$ are linearly independent, then there exists a neighbourhood $U_0(x_0)$ such that elements $Df_j(x)$, $j = 1, \dots, p$ are linearly independent for all $x \in U_0(x_0)$.*

Proof. By assumption, $Df_i(x_0)$, $i = 1, \dots, p$ is a basis of $\operatorname{span}\{Df_i(x_0),\ i = 1, \dots, p\}$. By Fact 2.2, there exist $(e_1(x_0))^*, \dots, (e_p(x_0))^*$, such that

$$(e_j(x_0))^*(Df_i(x_0)) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \quad i, j \in \{1, \dots, p\}. \tag{2.4}$$

Put $(e(x_0))^* := [(e_1(x_0))^*, \dots, (e_p(x_0))^*]^T$ and $G(x) := [(e(x_0))^* Df_1(x), \dots, (e(x_0))^* Df_p(x)]$, $x \in E$. By (2.4),

$$\det G(x_0) = \det[(e(x_0))^* Df_1(x_0), \dots, (e(x_0))^* Df_p(x_0)] = 1$$

and by the continuity of Df_i, there exists a neighbourhood $U_0(x)$ such that $\det G(x) > 0$ for $x \in U_0(x_0)$. Consequently, for any $x \in U_0(x_0)$,

$$\begin{aligned} &\alpha_1 Df_1(x) + \dots + \alpha_p Df_p(x) = 0 \Rightarrow (e(x_0))^*(\alpha_1 Df_1(x) + \dots \alpha_p Df_p(x)) = 0 \\ &\iff \alpha_1 (e(x_0))^* Df_1(x) + \dots + \alpha_p (e(x_0))^* Df_p(x) = 0 \iff \alpha_1, \dots, \alpha_p = 0. \end{aligned}$$

Therefore, for any $x \in U_0(x_0)$, $Df_i(x)$, $i = 1, \dots, p$ are linearly independent. □

Theorem 2.5. *([2], Thm. 2.5.14. Local Representation Theorem) Let $f :\ U \to \mathbb{R}^n$ be of class C^r, $r \geq 1$ in a neighbourhood of $x_0 \in U$, $U \subset E$ open set. Let F_1 be closed split image of $Df(x_0)$ with closed complement F_2. Suppose that $Df(x_0)$ has split kernel $E_2 = \ker Df(x_0)$ with closed complement E_1. Then there are open sets $U_1 \subset U \subset E_1 \oplus E_2$ and $U_2 \subset F_1 \oplus E_2$, $x_0 \in U_2$ and a C^r diffeomorphism $\psi :\ U_2 \to U_1$ such that $(f \circ \psi)(u, v) = (u, \eta(u, v))$ for any $(u, v) \in U_1$, where $u \in E_1$, $v \in E_2$ and $\eta :\ U_2 \to E_2$ is a C^r map satisfying $D\eta(\psi^{-1}(x_0)) = 0$.*

Remark 2.6. If $\dim \operatorname{range} Df(x) = k$ for all x in some neighbourhood $U'(x_0)$, then, by Inverse Function Theorem (see *e.g.* [2], Thm. 2.5.7), there exists an invertible function $\Psi' :\ U'(x_0) \to U$ such that $f \circ \Psi'$ depends on k variables.

3. Constant rank condition (CRC)

In the present section we recall basic facts related to constant rank condition for C^1 functions. By rank of A (where A is a finite set of elements of vector space) we will understand the cardinality of maximally linearly independent subset of elements of A.

Definition 3.1. Let $f_i :\ U \to \mathbb{R}$, $i = 1, \dots, \kappa$ be C^1 functions in a neighbourhood of $x_0 \in U$, $U \subset E$ open set. We say that *constant rank condition* (CRC) holds at x_0 if there exists a neighbourhood $V(x_0)$ such that

$$\operatorname{rank}\{Df_i(x_0),\ i = 1, \dots, \kappa\} = \mathit{const} = \operatorname{rank}\{Df_i(x),\ i = 1, \dots, \kappa\}$$

for all $x \in V(x_0)$. We also admit *const* $= 0$, which corresponds to the case $Df_i(x_0) = 0$, $i = 1, \dots, \kappa$.

The constant rank condition appears in the literature ([2], p. 127, [11], p. 47 and [29], p. 503 (under the name *same rank*).[1]

Let us note that, when $f = [f_1, \dots, f_n]$, $f_i : U \to \mathbb{R}$, $i = 1, \dots, n$ are of class C^1, $U \subset E$ open, and $\operatorname{rank}\{Df_i(x_0),\ i = 1, \dots, n\} = k$, then $\dim F_1 = \dim Df(x_0)(E) = k$, where

$$Df(x_0)y = \begin{bmatrix} \langle Df_1(x_0)\,,\, y\rangle \\ \vdots \\ \langle Df_n(x_0)\,,\, y\rangle \end{bmatrix}, \quad y \in E.$$

Consequently, F_1, $\dim F_1 = k$, splits $\mathbb{R}^n$ and F_2 is a closed complement of F_1, $\dim F_2 = n - k$.

Moreover, for any $e \in B(0, \delta) \subset E$,

$$\|Df(x_0)e\| := \left\| \begin{matrix} \langle Df_1(x_0)\,,\, e\rangle \\ \vdots \\ \langle Df_n(x_0)\,,\, e\rangle \end{matrix} \right\|_1 = \sum_{i=1}^{n} |\langle Df_i(x_0)\,,\, e\rangle| \le \|e\| \sum_{i=1}^{n} \|Df_i(x_0)\|_* < \varepsilon,$$

whenever $\delta < \frac{\varepsilon}{\|Df_i(x_0)\|_*}$, *i.e.*, $Df(x_0)$ is continuous, and $E_2 := \ker Df(x_0) = \{h \in E \mid Df(x_0)h = 0\}$ is a closed subspace of E. By Proposition 2.3, its complement E_1 is a closed subspace of E, moreover, $\dim E_1 = k$.

4. Rank theorem under CRC

In this section we reformulate the classical rank theorem, Theorem 4.3, under CRC condition.

Let $x_0 \in E$ and $E_2 = \ker Df(x_0)$. By E_1 we denote its closed complement (see Prop. 2.3).

Lemma 4.1. *Let E be a Banach space. Let $f_i :\ U \to \mathbb{R}$, $i = 1, \dots, \kappa$, $U \subset E$ open, be C^1 functions in a neighbourhood of $x_0 \in U$ and let $f = [f_1, \dots, f_\kappa]$. Assume that functionals $Df_i(x)$, $i = 1, \dots, \kappa$, are linearly independent for x from a neighbourhood $U_0(x_0)$. Then for any $x \in U_0(x_0)$, the vectors*

$$e^i(x) := Df(x)Df_i(x_0)^*, \quad i = 1, \dots \kappa,$$

form a basis in $\mathbb{R}^\kappa$, where $Df_i(x_0)^* \in E$, $i = 1, \dots \kappa$, (see Fact 2.2) are such that

$$Df_i(x_0)^*(Df_j(x_0)) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases}$$

Proof. The existence of $Df_i(x_0)^*$, $i = 1, \dots, \kappa$ is ensured by Fact 2.2. Since $Df_1(x_0), \dots, Df_\kappa(x_0)$ are linearly independent, by Fact 2.2, $Df_1(x_0)^*, \dots, Df_\kappa(x_0)^*$ are linearly independent.

Let $E_1 := \operatorname{span}\{Df_1(x_0)^*, \dots, Df_\kappa(x_0)^*\}$. First, let us show that for all $x \in U_0(x_0)$ the mapping $Df(x)|_{E_1} : E_1 \to Df(x)(E)$ is an injection.

We start this by showing that for $x = x_0$ this mapping is an injection. Indeed, suppose that there exists $e_1, e_2 \in E_1$, $e_1 \ne e_2$ such that $Df(x_0)(e_1) = Df(x_0)(e_2)$. Then $e_1 - e_2 \in E_1$ since E_1 is a linear space and at the same time $e_1 - e_2 \in E_2$. This contradicts the fact that $e_1 \ne e_2$.

By assumption, for any $x \in U_0(x_0)$ there exists a linear isomorphism

$$L_x :\ E_1 \to \operatorname{span}\{Df_1(x)^*, \dots, Df_\kappa(x)^*\}$$

[1]Let us note that the same terminology (constant rank condition) has been already used in [4] (Def. 1) and is stronger than that proposed in Definition 3.1.

defined as $L_x(Df_j(x_0)^*) = Df_j(x)^*$, $j = 1, \dots, \kappa$.

Hence, for any $x \in U_0(x_0)$, $E_1 = L_x^{-1}(\text{span}\{Df_1(x)^*, \dots, Df_\kappa(x)^*\})$ and $Df(x)|_{E_1} :\ E_1 \to Df(x)(E)$ is an injection (by injectivity of composition of two injective mappings).

Take any $x \in U_0(x_0)$ and $\alpha_i(x)$, $i = 1, \dots, \kappa$ such that

$$\alpha_1(x)e^1(x) + \dots + \alpha_\kappa(x)e^\kappa(x) = 0.$$

By injectivity of $DF(x)|_{E_1}$, $k \in \mathbb{N}$ and by linear independence of $Df_i(x_0)^*$, $i = 1, \dots, \kappa$, for all k we are getting the lineary independece of $e^i(x), i = 1, 2, \dots, \kappa$ for $x \in U_0(x_0)$.

Thus, $e^j(x)$, $j = 1, \dots, \kappa$, form a basis in $\mathbb{R}^\kappa$ for any $x \in U_0(x_0)$.

□

Proposition 4.2. *Let E be a Banach space. Let $f_i :\ U \to \mathbb{R}$, $i = 1, \dots, \kappa$, $U \subset E$ open, be C^1 functions in a neighbourhood of $x_0 \in U$. Then the following statements are equivalent:*

(i) CRC holds at x_0
(ii) the mapping $Df(x)|_{E_1} :\ E_1 \to Df(x)(E)$ is an isomorphism for x in some neighbourhood of x_0.

Proof. (i) $\implies$ (ii). Let CRC hold at x_0 with a neighbourhood $V(x_0)$. Let rank $\{Df_i(x_0), i = 1, \dots, \kappa\} = k$. By Lemma 2.4, there exist indices $i_1, \dots, i_k \subset \{1, \dots, \kappa\}$, such that $i_j \neq i_l$ for $j \neq l$, and a neighbourhood $U_0(x_0)$ such that

$$Df_{i_1}(x), \dots, Df_{i_k}(x) \tag{4.1}$$

form a maximally linearly independent subset of $\{Df_j(x),\ j = 1, \dots, \kappa\}$, $x \in U_0(x_0)$.

Let $f^1(x) := [f_{i_1}(x), \dots, f_{i_k}(x)]$. Clearly, $\ker Df(x_0) = \ker Df^1(x_0)$, where $\ker Df^1(x_0) = \{h \in E \mid Df^1(x_0)h = 0\}$. By Proposition 2.3,

$$E_1 = \text{span}\{Df_{i_1}(x_0)^*, \dots, Df_{i_k}(x_0)^*\}$$

and $\dim E_1 = k$. By CRC, $\dim(Df(x)(E)) = k$ for all $x \in U_0(x_0)$.

Since $Df_{i_1}(x_0), \dots, Df_{i_k}(x_0)$ are linearly independent, the mapping $Df^1(x)|_{E_1} :\ E_1 \to Df^1(x_0)(E)$ is an injection for all $x \in U(x_0)$ (see proof of Lem. 4.1).

Now we discuss the surjectivity of $Df(x)|_{E_1} :\ E_1 \to Df(x)(E)$ in a neighbourhood of x_0. To this aim we note that it is enough to investigate the surjectivity of $Df^1(x)|_{E_1} :\ E_1 \to Df^1(x)(E)$.

Let us note that $Df(x)e$, $e \in E_1$ is fully determined by $Df^1(x)e$. To see this take $e \in E_1$. Then $e = \sum_{j=1}^k \lambda_j (Df_{i_j}(x_0))^*$, where $\lambda_j \in \mathbb{R}$, $j = 1, \dots, k$. For any $x \in U_0(x_0)$ we have

$$Df^1(x)e = [Df_{i_l}(x)e]_{l=1}^k = \left[\sum_{j=1}^k \lambda_j Df_{i_l}(x) Df_{i_j}(x_0)^*\right]_{l=1}^k.$$

Again, by Lemma 2.4, $Df_l(x)^*$, $l \in \{1, \dots, \kappa\} \setminus \{i_1, \dots, i_k\}$ depend linearly on $Df_{i_1}(x)^*, \dots, Df_{i_k}(x)^*$, $x \in U_0(x_0) \cap V(x_0)$. We have

$$Df_l(x)e = \sum_{j=1}^k \alpha_j^l(x) Df_{i_j}(x)e, \tag{4.2}$$

where $\alpha_j^l(x) \in \mathbb{R}$, $j \in 1, \dots, k$, $l \in \{1, \dots, \kappa\} \setminus \{i_1, \dots, i_k\}$, $x \in U_0(x_0) \cap V(x_0)$ and

$$Df_l(x) = \sum_{j=1}^{k} \alpha_j^l(x) Df_{i_j}(x).$$

Now we show the surjectivity of $Df(x)|_{E_1} : \ E_1 \to Df(x)(E)$ for x in some neighbourhood of x_0. By Lemma 4.1, there exists a neighbourhood $U_1(x_0)$ such that the vectors

$$e^j(x) := Df^1(x) Df_{i_j}(x_0)^*, \quad j = 1, \dots, k$$

form a basis in $\mathbb{R}^k$. Let $x \in U_0(x_0) \cap U_1(x_0) \cap V(x_0)$, $g \in Df(x)(E)$ and for $l \in \{1, \dots, \kappa\}$ let us denote by g_l its l-th component. By (4.2), we have $g_l = \sum_{j=1}^{k} \alpha_j^l(x) g_{i_h}$, $l \in \{1, \dots, \kappa\} \setminus \{i_1, \dots, i_k\}$ and, moreover

$$\begin{bmatrix} g_{i_1} \\ \vdots \\ g_{i_k} \end{bmatrix} = \sum_{j=1}^{k} \beta_j(x) e^j(x),$$

for some $\beta_j(x) \in \mathbb{R}$, $j = 1, \dots, k$. Hence,

$$\begin{bmatrix} g_{i_1} \\ \vdots \\ g_{i_k} \end{bmatrix} = \sum_{j=1}^{k} \beta_j(x) Df^1(x) Df_{i_j}(x_0)^* = Df^1(x)(\sum_{j=1}^{k} \beta_j(x) Df_{i_j}(x_0)^*).$$

And, for $l \in \{1, \dots, \kappa\} \setminus \{i_1, \dots, i_k\}$,

$$g_l = \sum_{h=1}^{k} \alpha_h^l(x) Df_{i_h}(x)(\sum_{j=1}^{k} \beta_j(x) Df_{i_j}(x_0)^*).$$

Observe that $\sum_{j=1}^{k} \beta_j(x) Df_{i_j}(x_0)^* \in E_1$, and hence $Df(x)|_{E_1} : \ E_1 \to Df(x)(E)$ is surjective for $x \in U_0(x_0) \cap U_1(x_0) \cap V(x_0)$. Since $Df(x)|_{E_1} : \ E_1 \to Df(x)(E)$ is surjection and injection between finite-dimensional spaces, it is a (linear) isomorphism.

(ii) $\Longrightarrow$ (i) We have $E_2 = \ker Df(x_0) = \{h \in E \mid Df(x_0)h = 0\}$ and let $k = \operatorname{rank}\{Df_1(x_0), \dots, Df_\kappa(x_0)\}$. There exists $i_1, \dots, i_k \subset \{1, \dots, \kappa\}$ such that the system $Df_{i_1}(x_0), \dots, Df_{i_k}(x_0)$ forms a maximally linearly independent subset of set $\{Df_1(x_0), \dots, Df_\kappa(x_0)\}$. Moreover,

$$E_1 = E_2^\perp = \operatorname{span}\{Df_1(x_0)^*, \dots, D(f_\kappa(x_0))^*\} = \operatorname{span}\{Df_{i_1}(x_0)^*, \dots, Df_{i_k}(x_0)^*\}.$$

Since $f_{i_1}, \dots, f_{i_k}$ are of class C^1 there exists a neighbourhood $U_0(x_0)$ such that $Df_{i_1}(x), \dots, Df_{i_k}(x)$ are linearly independent.

By assumption $\dim Df(x)(E) = k$ for all x in a neighbourhood $V_1(x_0) \subset U_0(x_0)$. Thus $\{Df_{i_1}(x)^*, \dots, Df_{i_k}(x)^*\}$ forms a maximally linearly independent subset of set $\{Df_1(x)^*, \dots, Df_\kappa(x)^*\}$, $x \in V_1(x_0)$. Hence CRC holds for functions $f_i : \ E \to \mathbb{R}$, $i = 1, \dots, \kappa$ at x_0 with neighbourhood $V_1(x_0)$. □

In view of Proposition 4.2, Theorem 2.5.15 of [2] (for the finite-dimensional case see [21, 29]) takes the following form in the case considered in the present paper.

Theorem 4.3 (Rank theorem under CRC)**.** *Let E be a Banach space. Let $x_0 \in U$, where U is an open subset of E and $f : \ U \to \mathbb{R}^\kappa$, $f = [f_1, \dots, f_\kappa]$, $f_i : \ E \to \mathbb{R}$, $i = 1, \dots, n$ be C^1 functions in a neighbourhood of x_0. Assume* *that CRC holds at x_0 with a neighbourhood $V(x_0)$ and the constant rank k. As previously, let $E_2 = \ker Df(x_0)$, and let E_1 be its closed complement. Then there exist open sets $U_1 \subset \mathbb{R}^k \oplus E_2$, $U_2 \subset E$, $V_1 \subset \mathbb{R}^\kappa$, $V_2 \subset \mathbb{R}^k \oplus E_2$ and diffeomorphisms of class C^1, $\varphi : V_1 \to V_2$ and $\psi : U_1 \to U_2$, $x_0 = (x_{01}, x_{02}) \in U_2 \subset U \subset E_1 \oplus E_2$, i.e. $x_{01} \in E_1$, $x_{02} \in E_2$, $f(x_0) \in V_1$ satisfying*

$$(\varphi \circ f \circ \psi)(w, e) = (w, 0), \quad \textit{where } w \in E_1,\ e \in E_2$$

for all $(w, e) \in U_1$.

5. Functional dependence

In this section, by exploiting Theorem 4.3, we prove Proposition 5.2 which is an important tool in the proof of Theorem 6.5 and can be viewed as a variant of the classical Implicit Function Theorem.

To this aim we extend to Banach spaces the definition of functional dependence of functions $f_i : U \to \mathbb{R}$, $i = 1, \ldots, \kappa$, $U \subset E$ open, at some $x_0 \in U$ given in Example 2.5.16 of [2].

Definition 5.1. Let $U \subset E$ be an open set and let functions $f_i : U \to \mathbb{R}$, $i = 1, \ldots, \kappa$ be of class C^1 in a neighbourhood of $x_0 \in U$. Functions $f_1 \ldots, f_\kappa$ are *functionally dependent* at x_0 if there exist neighbourhoods $U(x_0)$, $V(y_0)$, where $y_0 := (f_1(x_0), \ldots, f_\kappa(x_0)) \in \mathbb{R}^\kappa$ and a function $F : V(y_0) \to \mathbb{R}$ of class C^1 such that

1. $F(f_1(x), \ldots, f_\kappa(x)) = 0$ for all $x \in U(x_0)$,
2. $DF(y_0) \neq 0$.

Functions $f_i : U \to \mathbb{R}$, $i = 1, \ldots, \kappa$, are *functionally independent* at x_0 if $f_i : U \to \mathbb{R}$, $i = 1, \ldots, \kappa$ are not functionally dependent at x_0, *i.e.* for any neighbourhoods $V(y_0)$, $U(x_0)$, and for any $F : V(y_0) \to \mathbb{R}$ of class C^1, if $F(f_1(x), \ldots, f_\kappa(x)) = 0$ for all $x \in U(x_0)$, then $DF(y_0) = 0$.

Now we discuss conditions ensuring functional dependence/independence.

In the proposition below we generalize Proposition 1 of Section 8.6.3 of [29] to the case, where the argument space is a Banach space and the functional dependence is understood in the sense of Definition 5.1. Assertion 2 of the proposition below establishes a connection of CRC at x_0 and Implicit Function Theorem.

Proposition 5.2. *Let E be a Banach space. Let $x_0 \in U$, $U \subset E$, U - open, $f = [f_1, \ldots, f_\kappa]$, $f_i : U \to \mathbb{R}$, $i = 1, \ldots, \kappa$ be C^1 functions in a neighbourhood of x_0. Assume that CRC holds at x_0 with a neighbourhood $V(x_0)$, i.e.*

$$rank\{Df_i(x),\ i = 1, \ldots, \kappa\} = rank\{Df_i(x_0),\ i = 1, \ldots, \kappa\} = k \quad \forall x \in V(x_0).$$

Let $E_2 = \ker Df(x_0)$ and E_1 be its closed complement. Let $i_1, \ldots, i_k \subset \{1, \ldots, \kappa\}$ be such that $i_j \neq i_l$ for $j \neq l$ and $Df_{i_1}(x_0), \ldots, Df_{i_k}(x_0)$ are linearly independent.

(1) If $k = \kappa$, then functions $f_1, \ldots, f_\kappa$ are functionally independent at x_0.
(2) If $k < \kappa$, then for any $l \in \{1, \ldots, \kappa\} \setminus \{i_1, \ldots, i_k\}$ functions $f_{i_1} \ldots, f_{i_k}, f_l$ are functionally dependent at x_0 and there exists a function $g_l : \mathbb{R}^k \to \mathbb{R}$ of class C^1 such that for any x in some neighbourhood of x_0

$$f_l(x) = g_l(f_{i_1}(x), \ldots, f_{i_k}(x)).$$

Proof.

(1) The proof follows the lines of the proof of Proposition 1 of Section 8.6.3 of [29].
Let $f = [f_1, \ldots, f_\kappa]$. By Theorem 4.3, there exist diffeomorphisms of class C^1, $\varphi : V_1 \to V_2$ and $\psi : U_1 \to U_2$ such that

$$(\varphi \circ f \circ \psi)(w, e) = (w, 0) \quad \text{for all } (w, e) \in U_1 \subset E_1 \oplus E_2.$$

Since φ, ψ are diffeomorphisms we have

$$f = \varphi^{-1} \circ (\varphi \circ f \circ \psi) \circ \psi^{-1},$$

and hence, $y_0 := f(x_0)$ is an interior point (in space $\mathbb{R}^\kappa$) of the image of a neighbourhood of $x_0 \in E$ (note that $E_2 = \{0\}$). Thus, for any function F, the relation

$$F(f_1(x), \dots, f_\kappa(x)) \equiv 0$$

holds in a neighbourhood of x_0 only if

$$F(y_1, \dots, y_\kappa) \equiv 0$$

in an neighbourhood of y_0. Hence, $DF(y_0) = 0$.

(2) The proof follows the lines of the proof of Theorem 2.5.12 in [2].
If $\{1, \dots, \kappa\} \setminus \{i_1, \dots, i_k\} = \emptyset$, the assertion is automatically satisfied. Suppose that $\{1, \dots, \kappa\} \setminus \{i_1, \dots, i_k\} \neq \emptyset$. Without loss of generality we assume that $i_j = j$, $j = 1, \dots, k$.
By Theorem 4.3, there exist open sets $U_1 \subset \mathbb{R}^k \oplus E_2$, $U_2 \subset E$, $V_1 \subset \mathbb{R}^\kappa$, $V_2 \subset \mathbb{R}^k \oplus E_2$ and diffeomorphisms of class C^1, $\varphi :\ V_1 \to V_2$ and $\psi :\ U_1 \to U_2$, $x_0 = (x_{01}, x_{02}) \in U_2 \subset U \subset E_1 \oplus E_2$, *i.e.* $x_{01} \in E_1$, $x_{02} \in E_2$, $f(x_0) \in V_1$ satisfying

$$(\varphi \circ f \circ \psi)(w, e) = (w, 0), \quad \text{where } w \in \mathbb{R}^k,\ e \in E_2 \tag{5.1}$$

for all $(w, e) \in U_1$. Note that the diffeomorphism ψ is the same as in Theorem 2.5. Hence,

$$\bar{f}(w, e) := (f \circ \psi)(w, e) = \varphi^{-1}(w, 0) = (w, \eta(w, e)), \tag{5.2}$$

where $\eta :\ \mathbb{R}^k \times E_2 \to E_2$ is the same as in Theorem 2.5. Thus, $\bar{f}$ does not depend on $e \in E_2$.
Let $x \in U_2$ and denote

$$y_i = f_i(x), \quad i = 1, \dots, \kappa. \tag{5.3}$$

There exists $u = (w, e) \in U_1 \in \mathbb{R}^k \oplus E_2$ such that $x = \psi(u)$. Hence,

$$y_j = f_j(\psi(w, e)) = w_j, \quad j = 1, \dots, k.$$

For $l \in \{k+1, \dots, \kappa\}$ we have

$$y_l = f_l(x) = f_l(\psi(w, e)) = \bar{f}_l(w, e) = \bar{f}_l(y_1, \dots, y_k, e).$$

In consequence, by (5.2), $y_l = \bar{f}_l(y_1, \dots, y_k)$, $l \in \{k+1, \dots, \kappa\}$. Hence, for any $x \in U_2$, $f_l(x) = \bar{f}_l(f_1(x), \dots, f_k(x))$, $l \in \{k+1, \dots, \kappa\}$.

□

The following example illustrates functional independence of functions at x_0 under CRC.

Example 5.3. Let ℓ_2 be the Hilbert space of square summable series. Let $f_1, f_2 :\ \ell_2 \to \mathbb{R}$ be given as $f_1(x) = x_1$, $f_2(x) = x_2$, where $x = (x_1, x_2, \dots) \in \ell_2$. We will show that f_1, f_2 are functionally independent at $x_0 = 0 \in \ell_2$. Suppose, by contrary, that f_1, f_2 are functionally dependent, i.e there exists a function $F :\ \mathbb{R}^2 \to \mathbb{R}$ of class C^1 such that $F(f_1(x), f_2(x)) = 0$ for all x in some neighbourhood of $0 \in \ell_2$ and $DF \neq 0$ in some neighbourhood of $(f_1(0), f_2(0)) = (0,0) \in \mathbb{R}^2$. Indeed, by Implicit Function Theorem (see *e.g.* [2], Thm. 2.5.7), there exists $g:\ \mathbb{R} \to \mathbb{R}$ such that $f_2(x) = g(f_1(x))$ for all x in some neighbourhood of x_0 and

$$Df_2(x) = \begin{bmatrix} \frac{dg}{df_1}(f_1(x)) \cdot \frac{\partial f_1}{\partial x_1} \\ \frac{dg}{df_1}(f_1(x)) \cdot \frac{\partial f_1}{dx_2} \\ 0 \\ \vdots \end{bmatrix}^T = \frac{dg}{df_1}(f_1(x)) Df_1(x),$$

i.e. $Df_1(x), Df_2(x)$ are linearly dependent for all x in some neighbourhood of $0 \in \ell_2$, which is not true.

6. Tangent and linearized cones

In the present section we prove our main result, namely the Abadie condition in Banach space for the set $\mathcal{F}$ given by (1.1) under RCRCQ. The finite-dimensional case has been proved by Minchenko and Stakhovski in [23].

Let E be a Banach space. Let C be a subset of E and $x_0 \in \operatorname{cl} C$. We use the classical definition of tangent cone of C at x_0,

$$\begin{aligned} T_C(x_0) := \{d \in E \mid \exists \varepsilon > 0 \\ &\exists \text{ a vector function } o(t) \text{ such that } \|o(t)\| t^{-1} \to 0, \text{ as } t \downarrow 0 \\ &\text{and } x_0 + td + o(t) \in C \ \forall\ 0 \le t \le \varepsilon\}. \end{aligned}$$

For the set $\mathcal{F}$ given by (1.1) and $x_0 \in \mathcal{F}$, the linearized cone is given as

$$\Gamma_{\mathcal{F}}(x_0) := \{d \in E \mid \langle Dh_i(x_0) \mid d\rangle \le 0,\ i \in I(x_0),\ \langle Dh_i(x_0) \mid d\rangle = 0,\ i \in I_0\},$$

where $I(x_0) := \{i \in I \mid h_i(x_0) = 0\}$ is the active index set of $\mathcal{F}$ at x_0.

Definition 6.1 (Relaxed Constant Rank Constraint Qualification)**.** The *relaxed constant rank constraint qualification* (RCRCQ) holds for set $\mathcal{F}$, given by (1.1) at $\bar{x} \in \mathcal{F}$, if there exists a neighbourhood $U(\bar{x})$ of $\bar{x}$ such that, for any index set J, $I_0 \subset J \subset I_0 \cup I(\bar{x})$, for every $x \in U(\bar{x})$, the system of vectors $\{Dh_i(x), i \in J\}$ has constant rank. Precisely, for any J, $I_0 \subset J \subset I_0 \cup I(\bar{x})$,

$$\operatorname{rank}(Dh_i(x), i \in J) = \operatorname{rank}(Dh_i(\bar{x}), i \in J) \quad \text{for all } x \in U(\bar{x}).$$

Remark 6.2. Note that RCRCQ holds for $\mathcal{F}$ at $x_0 \in \mathcal{F}$ if and only if for any index set J, $I_0 \subset J \subset I_0 \cup I(x_0)$, CRC holds at x_0 for functions h_i, $i \in J$.

In Theorem 6.5 we will use Ljusternik theorem (see [16], Sect. 0.2.4).

Theorem 6.3. *(Ljusternik Theorem) Let X and Y be Banach spaces, let U be a neighborhood of a point $x_0 \in X$, and let $F:\ U \to Y$ be a Fréchet differentiable mapping. Assume that F is regular at x_0, i.e., that $\operatorname{Im} DF(x_0) = Y$, and that its derivative is continuous at this point (in the uniform operator topology of the space $\mathcal{L}(X,Y)$). Then the tangent space $T_M(x_0)$ to the set*

$$M = \{x \in U \mid F(x) = F(x_0)\}$$

at the point x_0 coincides with the kernel of the operator $DF(x_0)$,

$$T_M(x_0) = Ker\, DF(x_0).$$

Moreover, if the assumptions of the theorem are satisfied, then there exist a neighborhood $U' \subset U$ of the point x_0, a number $K > 0$, and a mapping $\xi \to x(\xi)$ of the set U' into X such that

$$F(\xi + x(\xi)) = F(x_0),$$
$$\|x(\xi)\| \le K\|F(\xi) - F(x_0)\|$$

for all $\xi \in U'$.

We start with the following technical lemma (see also [23] for finite-dimensional case).

Lemma 6.4. *Let $x_0 \in \mathcal{F}$, where $\mathcal{F}$ is given by (1.1) and $d \in \Gamma_{\mathcal{F}}(x_0)$. For any vector function $r:\ (0,1) \to E$ such that $\|r(t)\|t^{-1} \to 0$, as $t \downarrow 0$, there exists a number $\varepsilon_0 > 0$ such that*

$$h_i(x_0 + td + r(t)) < 0 \text{ for all } i \in I \setminus I(x_0, d) \text{ and for all } t \in (0, \varepsilon_0), \tag{6.1}$$

where $I(x_0, d) := \{i \in I(x_0) \mid \langle Dh_i(x_0)\,,\, d\rangle = 0\}$.

Proof. Let $d \in \Gamma_{\mathcal{F}}(x_0)$. If $i \in I \setminus I(x_0)$, then $h_i(x_0) < 0$ and, therefore,

$$\begin{aligned} h_i(x_0 + td + r(t)) &= h_i(x_0) + \langle Dh_i(x_0 + \theta(td + r(t)))\,,\, td + r(t)\rangle \\ &= h_i(x_0) + t\langle Dh_i(x_0 + \theta(td + r(t))\,,\, d\rangle + t\langle Dh_i(x_0 + \theta(td + r(t)))\,,\, \frac{r(t)}{t}\rangle < 0, \end{aligned}$$

where $0 \le \theta \le 1$ for all sufficiently small $t > 0$.

If $i \in I(x_0) \setminus I(x_0, d)$, then

$$h_i(x_0 + td + r(t)) = h_i(x_0) + t\langle Dh_i(x_0)\,,\, d\rangle + o_i(t) = t\langle Dh_i(x_0)\,,\, d\rangle + o_i(t),$$

where

$$\begin{aligned} o_i(t) := {} & t\langle Dh_i(x_0 + \theta(td + r(t))\,,\, d\rangle + t\langle Dh_i(x_0 + \theta(td + r(t)))\,,\, \frac{r(t)}{t}\rangle \\ & - t\langle Dh_i(x_0)\,,\, d\rangle. \end{aligned}$$

In this case $h_i(x_0 + td + r(t)) < 0$ for sufficiently small $t > 0$ since

$$\langle Dh_i(x_0)\,,\, d\rangle < 0 \text{ and } o_i(t)t^{-1} \to 0.$$

Consequently, $h_i(x_0 + td + r(t)) < 0$, for all $i \in I \setminus I(x_0, d)$ and for all $t \in (0, \varepsilon_0)$, which proves (6.1). ☐

Let us note that Lemma 6.4 is valid also in the case $I(x_0, d) = \emptyset$. Now we are ready to prove our main result.

Theorem 6.5. *Let E be a Banach space and $\mathcal{F} \subset E$ be given as in (1.1). Assume that RCRCQ holds for $\mathcal{F}$ at $x_0 \in \mathcal{F}$. Then Abadie condition holds, i.e. $\Gamma_{\mathcal{F}}(x_0) = T_{\mathcal{F}}(x_0)$.*

Moreover, for each $d \in T_{\mathcal{F}}(x_0)$ there is a vector function $r:\ (0,1) \to E$, $\|r(t)\|/t \to 0$ when $t \downarrow 0$, such that for all t sufficiently small

$$\begin{aligned} h_i(x_0 + td + r(t)) = 0,\ & i \in J(d), \\ h_\ell(x_0 + td + r(t)) \le 0,\ & \ell \in I \setminus J(d), \end{aligned} \qquad J(d) := I_0 \cup I(x_0, d). \tag{6.2}$$

Additionally, whenever $J(d) \neq \emptyset$, $d \in ker\, Dh(x_0)$, where $h := [h_{i_j}]$, $j = 1, \ldots, k$,

$$rank\{Dh_i(x_0 + td + r(t)), i \in J(d)\} = rank\{Dh_i(x_0), i \in J(d)\} = k$$

for all t sufficiently small.

Proof. The inclusion $T_{\mathcal{F}}(x_0) \subset \Gamma_{\mathcal{F}}(x_0)$ is immediate. To see the converse, take any $d \in \Gamma_{\mathcal{F}}(x_0)$. We start by considering the case $J := J(d) \neq \emptyset$. By RCRCQ of $\mathcal{F}$ at x_0, we have

$$\text{rank}\,\{Dh_i(x_0 + td + r), i \in J\} = \text{rank}\,\{Dh_i(x_0), i \in J\} = k,$$

for (t, r) in some neighbourhood of $(0, 0) \in \mathbb{R} \times E$. By Lemma 2.4, there exist indices $i_1, \ldots, i_k$, such that $Dh_{i_1}(x_0 + td + r), \ldots, Dh_{i_k}(x_0 + td + r)$ are linearly independent for (t, r) in some neighbourhood of $(0, 0)$. Without loss of generality, we can assume that $i_j = j$, $j = 1, \ldots, k$.

If $k = |J|$, *i.e.* $J \setminus \{1, \ldots, k\} = \emptyset$, then, by applying Ljusternik Theorem 6.3 to

$$M := \{x \in E \mid h_i(x) = h_i(x_0) = 0,\ i \in J = \{1, 2, \ldots, k\}\},$$

the conclusion holds.

If $k < |J|$ then, by (2) of Proposition 5.2, applied to h_i, $i \in 1 \ldots, k$, there exist functions g_l, $l \in J \setminus \{1, \ldots, k\}$ of class C^1, such that

$$h_l(x_0 + td + r) = g_l(h_1(x_0 + td + r), \ldots, h_k(x_0 + td + r)), \tag{6.3}$$

for (t, r) in some neighbourhood of $(0, 0)$.

Consider the system

$$h_i(x_0 + td + r) = 0, \quad i \in J \tag{6.4}$$

with respect to variables t, r. Let us note that system (6.4) is satisfied for $(t, r) = (0, 0)$.

Obviously, in some neighbourhood of $(0, 0)$, system (6.4) is equivalent to

$$\begin{cases} h_1(x_0 + td + r) = 0 \\ \quad\cdots \\ h_k(x_0 + td + r) = 0 \end{cases} \tag{6.5}$$

with additional condition

$$h_l(x_0 + td + r) = g_l(h_1(x_0 + td + r), \ldots, h_k(x_0 + td + r)) = 0,\ l \in J \setminus \{1, \ldots, k\}. \tag{6.6}$$

Note that $g_l(h_1(x_0), \ldots, h_k(x_0)) = 0$, $l \in J \setminus \{1, \ldots, k\}$ and therefore $g_l(0, \ldots, 0) = 0$, $l \in J \setminus \{1, \ldots, k\}$.

We have

$$\langle Dh_i(x_0)\,,\, d\rangle = 0, \quad i \in J = I_0 \cup I(x_0, d).$$

Hence, $d \in \ker Dh(x_0)$, where $h(x) = [h_1(x) \ldots, h_k(x)]$. By applying Ljusternik Theorem 6.3 with $F = h$ at x_0, we obtain that $d \in T_M(x_0)$, where

$$M := \{x \in E \mid h(x) = 0\}.$$

This means that there exist $\varepsilon > 0$ and a function $r: [0,\varepsilon) \to E$, $\|r(t)\|t^{-1} \to 0$, $t \downarrow 0$, such that

$$\begin{cases} h_1(x_0 + td + r(t)) = 0 \\ \dots \\ h_k(x_0 + td + r(t)) = 0. \end{cases}$$

By (6.6), $h_i(x_0 + td + r(t)) = 0$, $i \in J$ for $t \in [0,\varepsilon]$. By Lemma 6.4, there exists $\varepsilon_0 > 0$ such that

$$x_0 + td + r(t) \in \mathcal{F} \quad t \in [0, \min\{\varepsilon_0, \varepsilon\}]. \tag{6.7}$$

Thus, $d \in T_{\mathcal{F}}(x_0)$.

Now, let us consider the case $J = \emptyset$ (*i.e.* the case when both $I_0 = \emptyset$ and $I(x_0, d) = \emptyset$). Then, by Lemma 5.1, for any vector function $r: (0,1) \to E$, $\|r(t)\|/t \to 0$ when $t \downarrow 0$ there exists $\varepsilon > 0$ such that

$$x_0 + td + r(t) \in \mathcal{F} \quad t \in [0,\varepsilon], \tag{6.8}$$

i.e., $d \in T_{\mathcal{F}}(x_0)$. □

The following corollary refers to the special case, where there is no inequality constraints in the definition of the set $\mathcal{F}$.

Corollary 6.6. *Suppose that $I = \emptyset$, i.e. there is no inequalities in the representation (1.1) of the set $\mathcal{F}$ i.e.*

$$\mathcal{F} = \{x \in E \mid h_i(x) = 0 \quad i = 1,\dots,n\}$$

and CRC holds at $x_0 \in \mathcal{F}$, i.e. there exists a neigbourhood $U(x_0)$ s.t.

$$rank\{Dh_i(x_0), i = 1,2,\dots,n\} = rank\{Dh_i(x), i = 1,2,\dots,n\} = k$$

for all $x \in U(x_0)$. Then

$$T_{\mathcal{F}}(x_0) = \{d : \langle Dh_i(x_0)\,,\, d\rangle = 0,\ i = 1,2,\dots,n\}.$$

Moreover, if $I_k = \{i_1, i_2, \dots, i_k\}$ is such that $Dh_{i_j}(x_0)$, $i_j \in I_k$ are linearly independent, then

$$T_{\mathcal{F}}(x_0) = ker \begin{bmatrix} Dh_{i_1}(x_0) \\ \vdots \\ Dh_{i_k}(x_0) \end{bmatrix}.$$

Proof. By assumption, for any $\ell \notin I_k$

$$Dh_\ell(x_0) = \sum_{i \in I_k} \lambda_i^\ell Dh_i(x_0).$$

This shows that $T_{\mathcal{F}}(x_0)$ does not depend upon the choice of the set I_k. □

7. Functional dependence/independence without CRC

In our main theorem (Thm. 6.5) we used constant rank condition (and RCRCQ) to be able to apply Proposition 5.2, where we used the concept of functional dependence/independence according to Definition 5.1.

In this additional section we investigate functional dependence/independence with respect to Definition 5.1 without CRC. Moreover, in Subsection 7.1 we review other most common concepts of functional dependence/independence (Defs. 7.5, 7.9, 7.13).

Proposition 7.1. *Let E be a Banach space. Let $x_0 \in U$, $U \subset E$ open and $f_1, \ldots, f_\kappa : U \to \mathbb{R}$. Suppose that in every neighbourhood $U(x_0)$ there exists $x \in U(x_0)$ such that $Df_1(x), \ldots, Df_n(x)$ are linearly independent. Then functions $f_1, \ldots, f_\kappa$ are functionally independent at x_0.*

Proof. Let $F : V \to \mathbb{R}$ be a function of class C^1 defined on a neighbourhood $V(y_0)$ such that $F(f_1(x), \ldots, f_\kappa(x)) = 0$ for any x in some neighbourhood $U(x_0)$.

We show that it must be $DF(y_0) = 0$, where $y_0 = (f_1(x_0), \ldots, f_\kappa(x_0))$. By assumption, let $U(x_0)$ be a neighbourhood of x_0 and $x' \in U(x_0)$ be such that $Df_1(x'), \ldots, Df_\kappa(x')$ are linearly independent.

There exists a neighbourhood $U(x') \subset U(x_0)$ and $Df_1(z'), \ldots, Df_\kappa(z')$ are linearly independent for all $z \in U(x')$. By (1) of Proposition 5.2, it must be $DF(f(x')) = 0$. By smoothness of function F and f, the latter equality implies $DF(f(x_0)) = 0$. □

Proposition 7.2. *(Local stability of functional dependence) Let E be a Banach space. If $f_1, \ldots, f_\kappa : U \to \mathbb{R}$, $U \subset E$ open, are functionally dependent at $x_0 \in U$, then there exists a neighbourhood $U(x_0)$ such that $f_1, \ldots, f_\kappa$ are functionally dependent at any $x \in U(x_0)$.*

Proof. Let $f = [f_1, \ldots, f_\kappa]$. Assume that $f_1, \ldots, f_\kappa$ are functionally dependent at x_0, *i.e.* there exist neighbourhoods $U(x_0)$, $V(y_0)$, $y_0 = f(x_0)$, and a function $F : V(y_0) \to \mathbb{R}$ such that $DF(y_0) \neq 0$ and $F(f_1(x), \ldots, f_\kappa(x)) = 0$ for all $x \in U(x_0)$.

Since $DF(y_0) \neq 0$ and F is of class C^1 there exists a neighbourhood $V_1(y_0)$ such that $DF(y) \neq 0$ for all $y \in V_1(y_0)$. Let $x' \in U(x_0) \cap f^{-1}(V_1(y_0))$, $U(x') := U(x_0) \cap f^{-1}(V_1(y_0))$ and $y' := f(x')$. Then function $F : V_1(y_0) \cap V(y_0) \to \mathbb{R}$ satisfies $DF(y') \neq 0$ and $F(f_1(x), \ldots, f_\kappa(x)) = 0$ for all $x \in U(x')$. Hence $f_1, \ldots, f_\kappa$ are functionally dependent at any $x' \in U(x_0) \cap f^{-1}(V_1(y_0))$. □

The fact below relates functional dependence with linear dependence of gradients.

Fact 7.3. Let E be a Banach space. Let $x_0 \in U$, $U \subset E$ open, and $f_1, \ldots, f_\kappa : U \to \mathbb{R}$. Suppose that $f_1, \ldots, f_\kappa$ are functionally dependent at x_0. Then there exists a neighbourhood $U(x_0)$ such that $Df_1(x), \ldots, Df_\kappa(x)$, $x \in U(x_0)$, are linearly dependent.

Proof. The proof follows immediately from Lemma 2.4, (1) of Proposition 5.2 and Proposition 7.2. □

The following proposition provides sufficient conditions for functional independence.

Proposition 7.4. *Let $x_0 \in U$, $U \subset E$ open, $f_1, \ldots, f_\kappa : U \to \mathbb{R}$. If, for any neighbourhood $U(x_0)$, $\operatorname{int} f(U(x_0)) \neq \emptyset$, then $f_1, \ldots, f_\kappa$ are functionally independent at x_0.*

Proof. Let E be a Banach space. Let $U(x_0)$ be a neighbourhood of x_0 and $V(y_0)$ be a neighbourhood of $y_0 = f(x_0)$ and $F : V \to \mathbb{R}$ be of class C^1 such that $F(f_1(x), \ldots, f_\kappa(x)) = 0$ for all $x \in U(x_0)$.

By the continuity of f, for any $m \in \mathbb{N}$ there exists $U'_m(x_0)$ such that $f(U'_m(x_0)) \subset B(y_0, \frac{1}{m})$. Let $U''_m(x_0) = U(x_0) \cap U'_m(x_0)$. Then, by assumption, $\operatorname{int}(f(U''_m(x_0))) \neq \emptyset$ and, moreover, $A_m := [\operatorname{int} f(U''_m(x_0))] \cap B(y_0, \frac{1}{m}) = \operatorname{int} f(U''_m(x_0))$ is a nonempty open set. Since $F(y) = 0$ for all $y \in A_m$ we have $DF(y) = 0$ for all $y \in A_m$.

Since $F : V \to \mathbb{R}$ is of class C^1, there exists a sequence $y_m \to y_0$, $y_m \in A_n$, such that $DF(y_m) = 0$. By the smoothness of F, it must be $DF(y_0) = 0$. In consequence, functions $f_1, \ldots, f_\kappa$ are functionally independent at x_0. □

7.1. Functional dependence/independence, other definitions

Here we compare the concept of functional dependence given in Definition 5.1 with other concepts of functional dependence appearing in the literature.

Let us note that in the proof of Theorem 6.5 and Proposition 7.4 we use the concept of functional dependence as defined in Definition 5.1. In general, without condition 2. of Definition 5.1 we are not able to deduce formula (6.3).

The definition of functional dependence at x_0 given in Chapter II.7 of [11] can be rewritten in Banach spaces as follows.

Definition 7.5. Let E be a Banach space. Let $U \subset E$ be an open set and let functions $f_i : U \to \mathbb{R}$, $i = 1, \dots, \kappa$ be of class C^1 in a neighbourhood of $x \in U$. Functions $f_1 \dots, f_\kappa$ are said to be *functionally dependent* at x_0 if there exists a neighbourhood $U(x_0)$ and a neighbourhood $V(y_0)$, where $y_0 := (f_1(x_0), \dots, f_\kappa(x_0)) \in \mathbb{R}^\kappa$ and a function $F : V \to \mathbb{R}$ of class C^1 such that

1. $F(f_1(x), \dots, f_\kappa(x)) = 0$ for all $x \in U(x_0)$,
2. $F \not\equiv 0$ on $V(y_0)$.

Definition 7.5 for continuous functions $f_1 \dots, f_\kappa$, F in finite-dimensional settings was given in Paragraph 8.6.3 of [29].

Remark 7.6. Clearly, if $f_1, \dots, f_\kappa$ are functionally dependent at x_0 in the sense of Definition 5.1, then $f_1, \dots, f_\kappa$ are functionally dependent at x_0 in the sense of Definition 7.5.

The example below illustrates the difference between of definitions of functional dependence given in Definition 5.1 and Definition 7.5. Let us note that the functions f_1, f_2 from the example below do not satisfy the CRC condition at $x_0 = (0, 0)$.

Example 7.7. Let $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f_1(x_1, x_2) = x_1^2$, $f_2(x_1, x_2) = x_2^2$ and $x_0 = (0, 0)$. We will show that f_1, f_2 are functionally dependent at x_0 in the sense of Definition 7.5 and are functionally independent at x_0 in the sense of Definition 5.1.

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows

$$F(y_1, y_2) := \begin{cases} 0 & \text{if} \quad y_1 \geq 0 \ \wedge \ y_2 \geq 0, \\ y_1^2 & \text{if} \quad y_1 < 0 \ \wedge \ y_2 \geq 0, \\ y_2^2 & \text{if} \quad y_1 \geq 0 \ \wedge \ y_2 < 0, \\ y_1^2 + y_2^2 & \text{if} \quad y_1 < 0 \ \wedge y_2 < 0 \end{cases} .$$

Then F is of class C^1 and

$$F(f_1(x_1, x_2), f_2(x_1, x_2)) = F(x_1^2, x_2^2) = 0$$

for all $(x_1, x_2) \in \mathbb{R}^2$. Moreover, in any neighbourhood of $y_0 := (f_1(x_0), f_2(x_0)) = (0, 0)$ there exists $y = (y_1, y_2)$ such that $y_1 < 0$ or $y_2 < 0$, *i.e.* $F(y) \neq 0$. Hence, f_1, f_2 are functionally dependent at x_0 in the sense of Definition 7.5

Now we show that f_1, f_2 are functionally independent at x_0 in the sense of Definition 5.1. By contrary, suppose, that f_1, f_2 are functionally dependent at x_0 in the sense of Definition 5.1. Then there exists a function $F : \mathbb{R}^2 \to \mathbb{R}$ of class C^1 such that $DF(y_0) \neq 0$ at $y_0 := (f_1(x_0), f_2(x_0)) = (0, 0)$ and $F(x_1^2, x_2^2) = 0$ for all (x_1, x_2) in some neighbourhood of x_0.

Let $U(x_0)$ be any neighbourhood of x_0 and $V(y_0)$ be any neighbourhood of y_0. Let $U'(x_0) = \{(x_1, x_2) \in \mathbb{R}^2 \mid (\text{sgn}(x_1)x_1^2, \text{sgn}(x_2)x_2^2) \in U(x_0)\}$, where sgn is the signum function. Let us take any $y' \in V(y_0) \cap U'(x_0)$ such that $y' = (y_1', y_2')$, where $y_1' > 0$ and $y_2' > 0$. Let $V(y') = V(y_0) \cap U'(x_0) \cap \mathbb{R}^2_{++}$, where $R^2_{++} := \{(x_1, x_2) \mid x_1 > 0 \ \wedge x_2 > 0\}$. Let us note that $y_0 \in \text{cl}(V(y'))$. Then for all $y = (y_1, y_2) \in V(y')$ we have $(\sqrt{y_1}, \sqrt{y_2}) \in U'(x_0)$ and, moreover, $F(y) = F(\sqrt{y_1}^2, \sqrt{y_2}^2) = 0$. Thus, $DF(y) = 0$ for all $y \in V(y') \subset V(y_0)$, which is a contradiction with the assumption $DF(y_0) \neq 0$.

Definition 7.8. Let $\Omega \subset E$ be a nonempty set. Subset $A \subset \Omega$ is *nowhere dense* in Ω if for all $U \subset \Omega$, U open in Ω, $U \neq \emptyset$ there exists $V \subset U$, V open in Ω, $V \neq \emptyset$ such that $A \cap V = \emptyset$, *i.e.* $A \subset \Omega \setminus V$.

The definition of functional dependence on a set Ω given in Chapter 1 of [24] is formulated in Banach spaces for C^∞ functions. Here we reformulate it in Banach space settings for C^1 functions in the following way.

Definition 7.9. Let E be a Banach space. Let $\Omega \subset E$ be an open set and let functions $f_i : \ \Omega \to \mathbb{R}$, $i = 1, \dots, \kappa$ be of class C^1 in a neighbourhood of $x_0 \in \Omega$. Functions $f_1 \dots, f_\kappa$ are said to be *functionally dependent* at x_0 if there exist a neighbourhood $U(x_0) \subset \Omega$ and a neighbourhood $V(y_0)$, $y_0 := (f_1(x_0), \dots, f_\kappa(x_0)) \in \mathbb{R}^\kappa$, and a function $F : \ V(y_0) \to \mathbb{R}$ of class C^1 such that

1. $F(f_1(x), \dots, f_\kappa(x)) = 0$ for all $x \in U(x_0)$,
2. $F^{-1}(0)$ is nowhere dense in $V(y_0)$.

The following example shows that the functional dependence in the sense of Definition 7.9 does not imply functional dependence in the sense of Definition 5.1.

Example 7.10. Let $f_1(t) = t^3$, $f_2(t) = t^2$, $t \in \mathbb{R}$. Then functions f_1, f_2 are functionally dependent at $t = 0$ in the sense of Definition 7.9, since by taking $F(x, y) = x^3 - y^2$ we get:

1. $F(f_1(t), f_2(t)) = 0$ for all $t \in \mathbb{R}$
2. $F^{-1}(0)$ is nowhere dense in any neighbourhood of $(0, 0)$.

On the other hand for any neighbourhood of $U(0)$ and $V((0, 0))$ and for any function $F : \ V((0, 0)) \to \mathbb{R}$ of class C^1 if $F(f_1(t), f_2(t)) = 0$ for all $t \in U(0)$, then

$$DF(0,0) = \frac{\partial F(f_1(0), f_2(0))}{\partial f_1} \frac{df_1(0)}{dt} + \frac{\partial F(f_1(0), f_2(0))}{\partial f_2} \frac{df_2(0)}{dt} = 0.$$

Therefore functions f_1, f_2 are functionally independent at $t = 0$ in the sense of Definition 5.1.

Proposition 7.11. *Let E be a Banach space. If functions $f_i : \ \Omega \to \mathbb{R}$, $i = 1, \dots, \kappa$ of class C^1 are functionally dependent at $x_0 \in \Omega$ in the sense of Definition 7.9 then they are functionally dependent in the sense of Definition 7.5.*

Proof. Assume that that $f_1, \dots, f_\kappa$ are functionally dependent at $x_0 \in \Omega$ in the sense of Definition 7.9. Then there are neighbourhood $U(x_0)$ and a neighbourhood $V(y_0)$, $V(y_0) \subset f(U(x_0))$, where $y_0 := (f_1(x_0), \dots, f_\kappa(x_0)) \in \mathbb{R}^\kappa$ and a function $F : \ V \to \mathbb{R}$ of class C^1 such that

1. $F(f_1(x), \dots, f_\kappa(x)) = 0$ for all $x \in U(x_0)$,
2. $F^{-1}(0)$ is nowhere dense in $V(y_0)$.

According to the definition of nowhere dense set, for every nonempty open set $U \subset V(y_0)$, there exists an open set a nonempty set $V \subset U$, such that $V \cap F^{-1}(0) = \emptyset$, *i.e.* $F(x) \neq 0$ for every $x \in V$. In conclusion, F satisfies condition 2. of Definition 7.5. □

Proposition 7.12. *Let E be a Banach space. If functions $f_i : \ \Omega \to \mathbb{R}$, $i = 1, \dots, \kappa$ of class C^1 are functionally dependent at $x_0 \in \Omega$ in the sense of Definition 7.9, then there exists a neighbourhood $U(x_0)$ such that $Df_1(x), \dots, Df_\kappa(x)$, $x \in U(x_0)$ are linearly dependent.*

Proof. Suppose, by contrary that for any neighbourhood $U(x_0)$ there exists $x' \in U(x_0)$ such that $Df_1(x'), \dots, Df_\kappa(x')$ are linearly independent. Then there exists a neighbourhood $U(x')$ such that $Df_1(x), \dots, Df_\kappa(x)$, $x \in U(x')$ are linearly independent. By Theorem 4.3, $f(U(x'))$ has a nonempty interior (see *e.g.* proof of (1) of Prop. 5.2) and $f(U(x')) \subset f(U(x_0)) \subset F^{-1}(0)$, hence $F^{-1}(0)$ would not be nowhere dense. □

The definition of functional dependence at x_0 given in Chapter 4 of [28] can be rewritten in Banach spaces as follows.

Definition 7.13. Let E be a Banach space and let functions $f_i: U \to \mathbb{R}$, $U \subset E$ open, $i = 1, \ldots, \kappa$ be of class C^1 in a neighbourhood of $x_0 \in U$. Functions $f_1 \ldots, f_\kappa$ are *functionally dependent* at x_0, if

$$\operatorname{rank}\{Df_i(x_0),\ i = 1, \ldots, \kappa\} < \kappa. \tag{7.1}$$

Otherwise, we say that $f_1 \ldots, f_\kappa$ are *functionally independent* at x_0.

Let $\Omega \subset U$ be an open set. We say that functions $f_i: U \to \mathbb{R}$, of class C^1, $i = 1, \ldots, \kappa$, are *functionally dependent on* Ω if (7.1) holds for all $x_0 \in \Omega$. Functions $f_i: U \to \mathbb{R}$, of class C^1, $i = 1, \ldots, \kappa$, are *functionally independent on* Ω if

$$\operatorname{rank}\{Df_i(x),\ i = 1, \ldots, \kappa\} = \kappa \quad \text{for all } x \in \Omega. \tag{7.2}$$

In Theorem 4.1.3 of [28] it was shown that for $f_i: \mathbb{R}^\kappa \to \mathbb{R}$, $i = 1, \ldots, \kappa$, Definition 7.5 and Definition 7.13 are equivalent.

Remark 7.14. Let E be a Banach space. By Fact 7.3, if functions $f_i: U \to \mathbb{R}$, of class C^1, $i = 1, \ldots, \kappa$, are functionally dependent at $x_0 \in U$, $U \subset E$ open, in the sense of Definition 5.1, then there exists a neighbourhood $U(x_0)$ such that they are functionally dependent on $U(x_0)$ in the sense of Definition 7.13.

The following example illustrates the fact that the functional dependence in the sense of Definition 7.13 in a neighbourhood of x_0 does not imply the functional dependence at x_0 in the sense of Definition 5.1.

Example 7.15. Let us consider the *geometric tornado* function $f = [f_1, f_2, f_3]$, $f_1, f_2, f_3: \mathbb{R} \to \mathbb{R}$ defined as follows (see Fig. 1)

$$f_1(x) = \begin{cases} x^3 \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$
$$f_2(x) = \begin{cases} x^3 \cos(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$
$$f_3(x) = x^3.$$

We will show, that f_1, f_2, f_3 are functionally dependent in the sense of Definition 7.13 on any open set Ω which contains $x_0 = 0$ and are functionally independent at x_0 in the sense of Definition 5.1.

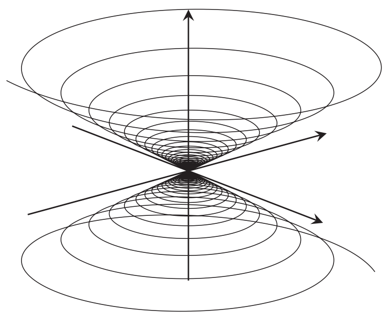

FIGURE 1. The image of $\mathbb{R}$ under $f = [f_1, f_2, f_3]$.

Derivatives of functions f_1, f_2, f_3 are as follows

$$f_1'(x) = \begin{cases} 3x^2\sin(\frac{1}{x}) - x\cos(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{otherwise} \end{cases}$$
$$f_2'(x) = \begin{cases} 3x^2\cos(\frac{1}{x}) + x\sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{otherwise} \end{cases}$$
$$f_3'(x) = 3x^2.$$

For any open set U_0 containing $x_0 = 0$ we have rank $\{f_1'(x), f_2'(x), f_3'(x)\} < 3$, $x \in U_0$. Hence, functions f_1, f_2, f_3 are functionally dependent on U_0 in the sense of Definition 7.13.

Let $U(x_0)$ be any neighbourhood of x_0 and let $F: \mathbb{R}^3 \to \mathbb{R}$ be any C^1 function such that $F(f(U(x_0))) = 0$, where $f(x) = [f_1(x), f_2(x), f_3(x)]$. Let $y_0 = f(x_0) = (0,0,0)$ and $t_n = \frac{1}{\frac{\pi}{2}+2n\pi}$. Then

$$F(t_n^3[1,0,1]) = F(t_n^3, 0, t_n^3) = F(f_1(t_n), f_2(t_n), f_3(t_n)) = 0$$

for sufficiently large n and

$$F'(y_0, [1,0,1]) = \lim_{t\to 0}\frac{F(t[1,0,1])}{t} = \lim_{n\to+\infty}\frac{F(t_n^3[1,0,1])}{t_n^3} = \lim_{n\to+\infty}\frac{0}{t_n^3} = 0.$$

Analogously one can show that

$$F'(y_0, [0,1,1]) = 0, \quad F'(y_0, [-1,0,1]) = 0.$$

Hence $DF(y_0) = 0$, i.e functions f_1, f_2, f_3 cannot be functionally dependent at x_0 in the sense of Definition 5.1.

Remark 7.16. Let $f = [f_1, \dots, f_\kappa]: \mathbb{R}^k \to \mathbb{R}^\kappa$ be of class C^1 in a neighbourhood of $x_0 \in \mathbb{R}^k$. If for any neighbourhood $U(x_0)$ there exist $v_1, \dots, v_\kappa \in \mathbb{R}^\kappa$ linearly independent and a sequence $t_m > 0$, $t_m \to 0$ such that $f(U(x_0)) \cap (f(x_0) + t_m v_i) \neq \emptyset$ for all $m \in \mathbb{N}$, then functions $f_1, \dots, f_\kappa$ are functionally independent at x_0 in the sense of Definition 5.1.

It is clear that Proposition 7.4 implies Remark 7.16 and the converse does not hold (as in Exam. 7.15).

In conclusion, the established relationships between above concepts of functional dependence at x_0 are illustrated in Figure 2.

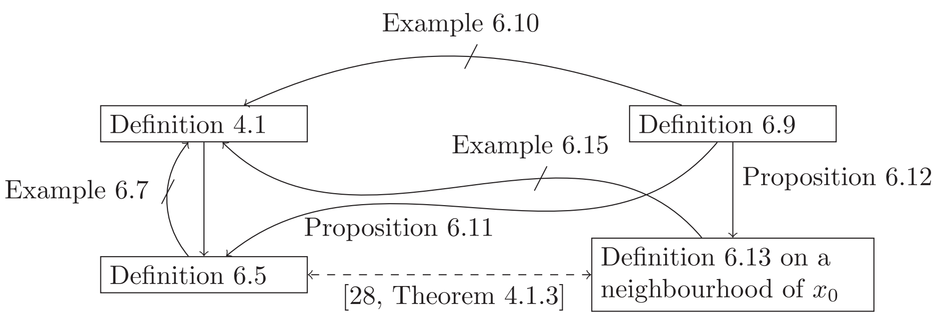

FIGURE 2. Relationships between functional dependence concepts. Theorem 4.1.3 of [28] is given in the case $f_1, \dots, f_\kappa: \mathbb{R}^\kappa \to \mathbb{R}$.

8. Relaxed constant rank constraint qualification and Lagrange multipliers

In this section we prove the non-emptiness of Lagrange multipliers set at a local minimum under RCRCQ and as an application we show that the linearized problem (P) at a local minimum is solvable.

Let us consider the problem (P),

$$\begin{array}{ll} \text{minimize} & h_0(x) \\ \text{s.t.} & x \in \mathcal{F}, \end{array} \tag{P}$$

where $\mathcal{F}$ is of the form (1.1), *i.e.*

$$\mathcal{F} := \{x \in E \mid h_i(x) = 0, i \in I_0,\ h_i(x) \leq 0,\ i \in I\}, \tag{8.1}$$

where E is a Banach space, $h_0, h_i:\ E \to \mathbb{R}$, $i \in I_0 \cup I$ are of class C^1 in a neighbourhood of a given local minimum $x_0 \in \mathcal{F}$ of h_0 and sets I_0, I are finite ($I_0 \cup I = \{1, \ldots, n\}$).

Let $\Lambda(x)$ be the set of Lagrange multipliers at $x \in \mathcal{F}$, *i.e.*

$$\Lambda(x) := \{\lambda \in \mathbb{R}^n \mid D_x L(\lambda, x) = 0,\ \lambda_i \geq 0 \text{ and } \lambda_i h_i(x) = 0,\ i \in I\},$$

where

$$L(\lambda, x) := h_0(x) + \sum_{i=1}^{n} \lambda_i h_i(x), \quad \lambda = (\lambda_1, \ldots, \lambda_n)$$

is the Lagrangean to problem (P). The following proposition relates RCRCQ to the existence of Lagrange multipliers. More on this topic see [8].

Proposition 8.1. *Let $x_0 \in \mathcal{F}$ be a local minimum of problem (P) and let RCRCQ hold for $\mathcal{F}$ at x_0. Then $\Lambda(x_0) \neq \emptyset$.*

Proof. By Theorem 6.5, for any $d \in \Gamma_{\mathcal{F}}(x_0)$ there exists a vector function $o(t)$ such that $\lim_{t\to 0} \|o(t)\| t^{-1} = 0$ and $x_0 + td + o(t) \in \mathcal{F}$. Since x_0 is a local minimum of h_0 on $\mathcal{F}$ we have $h_0(x_0 + td + o(t)) - h_0(x_0) \geq 0$ for all t sufficiently small. By Taylor expansion, we have

$$\begin{aligned} 0 &\leq h_0(x_0 + td + o(t)) - h_0(x_0) \\ &= h_0(x_0) + \langle Dh_0(x_0 + \theta(td + o(t)))\,,\, td + o(t)\rangle - h_0(x_0) \\ &= t\langle Dh_0(x_0 + \theta(td + o(t)))\,,\, d\rangle + \langle Dh_0(x_0 + \theta(td + o(t)))\,,\, o(t)\rangle, \end{aligned}$$

where $\theta \in [0, 1]$ and θ depends on t, d. Hence,

$$\langle Dh_0(x_0 + \theta(td + o(t)))\,,\, d\rangle \geq -\langle Dh_0(x_0 + \theta(td + o(t)))\,,\, o(t)t^{-1}\rangle. \tag{8.2}$$

By passing to the limit with $t \to 0$ in (8.2) we obtain $\langle Dh_0(x_0)\,,\, d\rangle \geq 0$. Hence

$$-\,Dh_0(x_0) \in (\Gamma_{\mathcal{F}}(x_0))^{\circ}, \tag{8.3}$$

where $(\Gamma_{\mathcal{F}}(x_0))^{\circ}$ is dual cone defined as

$$(\Gamma_{\mathcal{F}}(x_0))^{\circ} := \{d^* \in E^* \mid \langle d^*, d\rangle \leq 0, \forall d \in \Gamma_{\mathcal{F}}(x_0)\}.$$

Since

$$\Gamma_{\mathcal{F}}(x_0)=\left\{ d\in E \;\middle|\; \begin{array}{ll} \langle Dh_i(x_0)\,,\,d\rangle\le 0, & i\in I(x_0),\\ \langle Dh_i(x_0)\,,\,d\rangle\le 0, & i\in I_0,\\ \langle -Dh_i(x_0)\,,\,d\rangle\le 0, & i\in I_0 \end{array}\right\},$$

by Theorem 6.40 of [13], the dual cone to $\Gamma_{\mathcal{F}}(x_0)$ is given as follows

$$\begin{aligned}(\Gamma_{\mathcal{F}}(x_0))^\circ=\{d^*\in E^*\mid d^*=&\sum_{i\in I_0\cup I(x_0)}\lambda_i Dh_i(x_0),\\ &\lambda_i\ge 0,\ i\in I(x_0),\ \lambda_i\in\mathbb{R},\ i\in I_0\}.\end{aligned}\tag{8.4}$$

By (8.3) and (8.4), there exist $\lambda_i\ge 0,\ i\in I(x_0),\ \lambda_i\in\mathbb{R},\ i\in I_0$ such that

$$-Dh_0(x_0)=\sum_{i\in I_0\cup I(x_0)}\lambda_i Dh_i(x_0).$$

By putting $\lambda_i=0$ for $i\in\{1,2,\ldots,n\}\setminus(I_0\cup I(x_0))$, we have $\Lambda(x_0)\neq\emptyset$. □

As an application of Proposition 8.1 we show that the linearized problem, at a local minimum to (P), $x_0\in\mathcal{F}$,

$$\begin{array}{lll}\text{minimize} & \langle Dh_0(x_0)\,,\,d\rangle & \\ \text{s.t.} & \langle Dh_i(x_0)\,,\,d\rangle\le 0, & i\in I(x_0),\\ & \langle Dh_i(x_0)\,,\,d\rangle=0, & i\in I_0\end{array}\tag{8.5}$$

is solvable with a solution $d=0$. Problem (8.5) can be equivalently rewritten as

$$\begin{array}{ll}\text{minimize} & \langle Dh_0(x_0)\,,\,d\rangle\\ \text{s.t.} & \langle Dh(x_0)\,,\,d\rangle\in K,\end{array}\tag{8.6}$$

where $h=[h_i]_{i\in I(x_0)\cup I_0}$, $K=\{k=(k_i)\in\mathbb{R}^{|I_0\cup I(x_0)|}\mid k_i\le 0, i\in I(x_0),\ k_i=0,\ i\in I_0\}$. Lagrangian to (8.6) is defined as follows

$$L(d,\lambda)=\langle Dh_0(x_0)\,,\,d\rangle+\langle\lambda\,,\,Dh(x_0)d\rangle,\tag{8.7}$$

where $\lambda\in\mathbb{R}^{|I_0\cup I(x_0)|}$. Let $K^*:=\{k^*=(k_i^*)\in\mathbb{R}^{|I_0\cup I(x_0)|}\mid\langle k\,,\,k^*\rangle\le 0 \text{ for all } k\in K\}=\{k^*\in\mathbb{R}^{|I_0\cup I(x_0)|}\mid k_i^*\ge 0,\ i\in I(x_0),\ k_i^*\in\mathbb{R},\ i\in I_0\}$. The dual to (8.6) takes the form

$$\begin{array}{ll}\text{maximize} & \inf_{d\in E} L(d,\lambda)\\ \text{s.t} & \lambda\in K^*.\end{array}\tag{8.8}$$

Let us consider the objective of the dual. We have

$$\begin{aligned}\inf_{d\in E} L(d,\lambda)&=\inf_{d\in E}\{\langle Dh_0(x_0)\,,\,d\rangle+\langle\lambda\,,\,Dh(x_0)d\rangle\}\\ &=\inf_{d\in E}\left\{\langle Dh_0(x_0)\,,\,d\rangle+\sum_{i\in I_0\cup I(x_0)}\lambda_i\langle Dh_i(x_0)\,,\,d\rangle\right\}\end{aligned}$$

$$= \inf_{d\in E} \left\{ \langle Dh_0(x_0) + \sum_{i\in I_0\cup I(x_0)} \lambda_i Dh_i(x_0)\, ,\, d\rangle \right\}.$$

Hence,

$$\inf_{d\in E} \langle Dh_0(x_0) + \sum_{i\in I_0\cup I(x_0)} \lambda_i Dh_i(x_0)\, ,\, d\rangle$$
$$= \begin{cases} -\infty & \text{if } Dh_0(x_0) + \sum_{i\in I_0\cup I(x_0)} \lambda_i Dh_i(x_0) \neq 0, \\ 0 & \text{if } Dh_0(x_0) + \sum_{i\in I_0\cup I(x_0)} \lambda_i Dh_i(x_0) = 0. \end{cases}$$

Thus (8.8) is equivalent to the following

$$\begin{array}{ll} \text{maximize} & 0 \\ \text{s.t} & Dh_0(x_0) + \sum_{i\in I_0\cup I(x_0)} \lambda_i Dh_i(x_0) = 0, \\ & \lambda \in K^*. \end{array} \tag{8.9}$$

In conclusion, we obtain the following corollary.

Corollary 8.2. *Under assumption of RCRCQ at $x_0 \in \mathcal{F}$, where x_0 is a local minimum of* (P)*, the element $d = 0$ is a solution of* (8.5)*, since, by Proposition 8.1, feasible set of* (8.9) *is nonempty.*

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