

## APPROXIMATE BOUNDARY SYNCHRONIZATION BY GROUPS FOR A COUPLED SYSTEM OF WAVE EQUATIONS WITH COUPLED ROBIN BOUNDARY CONDITIONS

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**Abstract.** In this paper, we first give an algebraic characterization of uniqueness of continuation for a coupled system of wave equations with coupled Robin boundary conditions. Then, the approximate boundary controllability and the approximate boundary synchronization by groups for a coupled system of wave equations with coupled Robin boundary controls are developed around this fundamental characterization.

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### 1. INTRODUCTION

The phenomenon of synchronization was observed by Huygens in 1665 [8]. The first related mathematical research goes back to Wiener [33] in 1950's. The previous studies focused only on systems described by ODE. The synchronization in the PDE case was first studied for a coupled system of wave equations with Dirichlet boundary controls by Li and Rao in [16, 18] for the exact boundary synchronization, and in [19, 20] for the approximate boundary synchronization. Later, the synchronization for a coupled system of wave equations with Neumann boundary controls was carried in [14, 21]. The most part of the results was recently collected in the monograph [22].

In the framework of classical solutions, the exact boundary synchronization for a coupled system of 1-D wave equations with various boundary controls was considered in [6, 7] for linear and quasilinear cases.

Now let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary  $\Gamma = \Gamma_1 \cup \Gamma_0$  such that  $\bar{\Gamma}_1 \cap \bar{\Gamma}_0 = \emptyset$ . Let  $A, B$  be two matrices of order  $N$  and  $D$  a full column-rank matrix of order  $N \times M$  with  $M \leq N$ . Let  $U = (u^{(1)}, \dots, u^{(N)})^T$  and  $H = (h^{(1)}, \dots, h^{(M)})^T$  stand for the state variables and the boundary controls applied on  $\Gamma_1$ , respectively. Consider the following coupled system of wave equations with coupled Robin boundary

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controls:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu U + BU = DH & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (1.1)$$

with the initial condition

$$t = 0: \quad U = \widehat{U}_0, \quad U' = \widehat{U}_1 \quad \text{in } \Omega, \quad (1.2)$$

where  $\partial_\nu$  denotes the outward normal derivative.

In [15], the exact boundary controllability for system (1.1) was established as  $M = N$ . In the case of fewer boundary controls, namely, as  $M < N$ , the non-exact boundary controllability was also established for a parallelepiped domain. Moreover, the exact boundary synchronization by  $p$ -groups was also studied under the condition  $M = N - p$ . The exact boundary controllability as well as the exact boundary synchronization by  $p$ -groups are intrinsically linked with the number of applied boundary controls. In order to reduce the number of boundary controls, we return to consider the approximate boundary controllability and the approximate boundary synchronization by  $p$ -groups.

Consider the following adjoint system for the adjoint variable  $\Phi = (\phi^{(1)}, \dots, \phi^{(N)})^T$ :

$$\begin{cases} \Phi'' - \Delta \Phi + A^T \Phi = 0 & \text{in } (0, +\infty) \times \Omega, \\ \Phi = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \Phi + B^T \Phi = 0 & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (1.3)$$

with the initial data

$$t = 0: \quad \Phi = \widehat{\Phi}_0, \quad \Phi' = \widehat{\Phi}_1 \quad \text{in } \Omega. \quad (1.4)$$

We say (see Defs. 4.1 and 4.3) that system (1.1) is approximately controllable at the time  $T > 0$ , if for any given initial data  $(\widehat{U}_0, \widehat{U}_1)$ , there exists a sequence  $\{H_n\}$  of boundary controls, such that the corresponding sequence  $\{U_n\}$  of solutions goes to zero for  $t \geq T$  as  $n \rightarrow +\infty$ . Accordingly, the adjoint system (1.3) is  $D$ -observable on a finite time interval  $[0, T]$ , if the  $D$ -observation

$$D^T \Phi \equiv 0 \quad \text{on } [0, T] \times \Gamma_1 \quad (1.5)$$

implies that  $\Phi \equiv 0$ .

Similar to Dirichlet boundary controls in [17], the approximate boundary controllability of system (1.1) is still equivalent to the  $D$ -observability of the adjoint system (1.3). The main interest of the approximate boundary controllability consists in the fact that the rank  $M$  of the matrix  $D$  for realizing it may be substantially smaller than the number  $N$  of state variables.

For Dirichlet boundary controls, it was shown in [19] that the following Kalman's criterion

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N \quad (1.6)$$

is necessary for the  $D$ -observability of the corresponding adjoint system. For coupled Robin boundary controls, we want to find a similar characterization on the matrices  $A, B$  and  $D$ , which is necessary for the  $D$ -observability of the adjoint system (1.3). But the situation seems to be more complicated because of the presence of the second coupling matrix  $B$ .

Let  $V$  be a subspace, which is contained in  $\text{Ker}(D^T)$  and invariant for both  $A^T$  and  $B^T$ . We observe that system (1.3) is not  $D$ -observable in the subspace  $V$ .

We will construct a composite matrix  $\mathcal{R}$  (see (2.3)) to characterize the subspace  $V$  of this kind. We will show that  $\text{Ker}(\mathcal{R}^T)$  is the largest subspace of all the subspaces which are contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$  and  $B^T$  (see Lem. 2.1 below). As a direct consequence,  $\text{Ker}(\mathcal{R}^T) = \{0\}$  is a necessary condition for the  $D$ -observability of the adjoint system (1.3). The approximate boundary controllability will be first developed around this fundamental characterization.

Next, when  $\text{Ker}(\mathcal{R}^T) = \text{Span}\{E_1, \dots, E_p\}$ , assume that both  $A$  and  $B$  admit a common invariant subspace  $\text{Span}\{e_1, \dots, e_p\}$  such that  $(E_i, e_j) = \delta_{ij}$  ( $i, j = 1, \dots, p$ ), then both  $\text{Span}\{e_1, \dots, e_p\}$  and  $\text{Span}\{E_1, \dots, E_p\}^\perp$  are invariant for  $A$  and  $B$ . Moreover, the projection of system (1.1) on  $\text{Span}\{e_1, \dots, e_p\}$  is independent of applied boundary controls, therefore uncontrollable, while, the projection of system (1.1) on  $\text{Span}\{E_1, \dots, E_p\}^\perp$  is approximately null controllable. This is the basic idea that we develop in this paper for the approximate boundary synchronization by  $p$ -groups.

However, the approach is quite flexible and can be easily applied to other types of wave equations with memory and variable density [12], outside a star-shaped domain [31], orbital stability [34] and viscoelastic Petrovsky-type equation [11, 13].

The paper is organized as follows. In Section 2, we give an algebraic Lemma, which generalizes Kalman's criterion or Hautus test. In Section 3, we establish the well-posedness of problems. Section 4 is devoted to the  $D$ -observability and the approximate boundary null controllability. Generally speaking, the condition  $\dim \text{Ker}(\mathcal{R}^T) = 0$  is not sufficient for the uniqueness of continuation of solutions to the adjoint system (1.3) with  $D$ -observation (1.4). In fact, this is not a standard type of Holmgren's uniqueness theorem. In Section 5, we outline some known results on the topic. In Section 6, we consider the approximate boundary synchronization of system (1.1) in the case that  $\dim \text{Ker}(\mathcal{R}^T) \neq 0$ . For this purpose, we first show that  $\dim \text{Ker}(\mathcal{R}^T) \leq 1$  is a necessary condition for the approximate boundary synchronization (Thm. 6.4). Then, under the condition that  $\dim \text{Ker}(\mathcal{R}^T) = 1$ , we show the necessity of the condition of  $C_1$ -compatibility for the coupling matrices  $A$  and  $B$  related to the synchronization matrix  $C_1$ , the independence of the approximately synchronizable states with respect to applied boundary controls, and the approximate boundary synchronization under the condition of  $C_1$ -compatibility (Thms. 6.5 and 6.7). In Section 7, we generalize the above consideration to the approximate boundary synchronization by  $p$ -groups and carry on the study from a general point of view. In Section 8, based on the sharp regularity in Lasiecka and Triggiani [9, 10] on the solution to the wave equation with Neumann boundary conditions, we establish the necessity of some algebraic properties on the matrices  $A$  and  $B$  for the existence of the approximately synchronizable state.

Let us comment some related literatures. One of the motivation of studying the synchronization consists of establishing a weak exact boundary controllability in the case of fewer boundary controls. In order to realize the exact boundary controllability, because of its uniform character with respect to the state variables, the number of boundary controls must be equal to the degrees of freedom of the considered system. However, when the components of initial data are allowed to have different levels of energy, the exact boundary controllability by means of only one boundary control for a system of two wave equations was established in Liu and Rao [27], Rosier and de Teresa [32], and for a cascade system of  $N$  wave equations in Alabau-Boussouira [1]. In [4], Dehman *et al* established the controllability of two coupled wave equations on a compact manifold with only one local distributed control. Moreover, both the optimal time of controllability and the controllable spaces are given in the cases with the same or different wave speeds.

The approximate boundary null controllability is more flexible with respect to the number of applied boundary controls. In Li and Rao [19] as well as in the present paper, for a coupled system of wave equations with Dirichlet/Neuman/Robin boundary controls, some fundamental algebraic properties on the coupling matrices are used to characterize the uniqueness of continuation for the solution to the corresponding adjoint systems. Although these criteria are only necessary in general, they open an important way to the research on the uniqueness of continuation for the system of hyperbolic partial differential equations.

In contrast with hyperbolic systems, in Ammar Khodja [4] (also [5] and the reference therein), it was shown that Kalman's criterion is sufficient to the exact boundary null controllability for systems of parabolic equations. Recently, Wang and Yuan [24] have established the minimal time for a control problem related to the exact synchronization for a linear parabolic system.

The average controllability proposed by Zuazua in [29, 35] gives another way to deal with the controllability with fewer controls. The observability inequality is particularly interesting for a trial on the decay rate of approximate controllability.

## 2. AN ALGEBRAIC LEMMA

Let  $A$  be a matrix of order  $N$  and  $D$  a full column-rank matrix of order  $N \times M$  with  $M \leq N$ . We have shown that the following Kalman's criterion (see [19]):

$$\text{rank}(D, AD, \dots, A^{N-1}D) \geq N - d \quad (2.1)$$

holds if and only if the dimension of any given subspace, contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$ , does not exceed  $d$ . In particular, the equality holds if and only if the dimension of the largest subspace, contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$ , is exactly equal to  $d$ .

Let  $A, B$  be two matrices of order  $N$  and  $D$  a full column-rank matrix of order  $N \times M$  with  $M \leq N$ . For any given non-negative integers  $p, q, \dots, r, s \geq 0$ , we define a matrix of order  $N \times M$  by

$$\mathcal{R}_{(p,q,\dots,r,s)} = A^p B^q \dots A^r B^s D. \quad (2.2)$$

We construct an enlarged matrix

$$\mathcal{R} = (\mathcal{R}_{(p,q,\dots,r,s)}, \mathcal{R}_{(p',q',\dots,r',s')}, \dots) \quad (2.3)$$

by the matrices  $\mathcal{R}_{(p,q,\dots,r,s)}$  for all possible  $(p, q, \dots, r, s)$ , which, by Theorem of Cayley-Hamilton, essentially constitute a finite set  $\mathcal{M}$  with  $\dim(\mathcal{M}) \leq MN$ .

**Lemma 2.1.**  *$\text{Ker}(\mathcal{R}^T)$  is the largest subspace of all the subspaces which are contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$  and  $B^T$ .*

*Proof.* First, noting that  $\text{Im}(D) \subseteq \text{Im}(\mathcal{R})$ , we have  $\text{Ker}(\mathcal{R}^T) \subseteq \text{Ker}(D^T)$ . We now show that  $\text{Ker}(\mathcal{R}^T)$  is invariant for  $A^T$  and  $B^T$ . Let  $x \in \text{Ker}(\mathcal{R}^T)$ . We have

$$D^T (B^T)^s (A^T)^r \dots (B^T)^q (A^T)^p x = 0 \quad (2.4)$$

for any given integers  $p, q, \dots, r, s \geq 0$ . Then, it follows that  $A^T x \in \text{Ker}(\mathcal{R}^T)$ , namely,  $\text{Ker}(\mathcal{R}^T)$  is invariant for  $A^T$ . Similarly,  $\text{Ker}(\mathcal{R}^T)$  is invariant for  $B^T$ . Thus, the subspace  $\text{Ker}(\mathcal{R}^T)$  is contained in  $\text{Ker}(D^T)$  and invariant for both  $A^T$  and  $B^T$ .

Now let  $V$  be another subspace, contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$  and  $B^T$ . For any given  $y \in V$ , we have

$$D^T y = 0, \quad A^T y \in V, \quad B^T y \in V. \quad (2.5)$$

Then, it is easy to see that

$$(B^T)^s (A^T)^r \dots (B^T)^q (A^T)^p y \in V \quad (2.6)$$

for any given integers  $p, q, \dots, r, s \geq 0$ . Thus, by the first formula of (2.5) we have

$$D^T(B^T)^s(A^T)^r \dots (B^T)^q(A^T)^p y = 0 \quad (2.7)$$

for any given integers  $p, q, \dots, r, s \geq 0$ , namely, we have

$$V \subseteq \text{Ker}(\mathcal{R}^T). \quad (2.8)$$

The proof is then complete.  $\square$

By the rank-nullity theorem, we have  $\text{rank}(\mathcal{R}) + \dim \text{Ker}(\mathcal{R}^T) = N$ . The following lemma is a dual version of Lemma 2.1.

**Lemma 2.2.** *Let  $d \geq 0$  be an integer. Then*  
 (i) *the rank condition*

$$\text{rank}(\mathcal{R}) \geq N - d \quad (2.9)$$

*holds true if and only if the dimension of any given subspace, contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$  and  $B^T$ , does not exceed  $d$ ;*

*(ii) the rank condition*

$$\text{rank}(\mathcal{R}) = N - d \quad (2.10)$$

*holds true if and only if the dimension of the largest subspace, contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$  and  $B^T$ , is exactly equal to  $d$ .*

*Proof.* (i) Let  $V$  be a subspace which is contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$  and  $B^T$ . By Lemma 2.1, we have

$$V \subseteq \text{Ker}(\mathcal{R}^T). \quad (2.11)$$

Assume that (2.9) holds, it follows from (2.11) that

$$N - d \leq \text{rank}(\mathcal{R}) = N - \dim \text{Ker}(\mathcal{R}^T) \leq N - \dim(V), \quad (2.12)$$

namely,

$$\dim(V) \leq d. \quad (2.13)$$

Conversely, assume that (2.13) holds for any given subspace  $V$  which is contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$  and  $B^T$ . In particular, by Lemma 2.1, we have  $\dim \text{Ker}(\mathcal{R}^T) \leq d$ . Then it follows that

$$\text{rank}(\mathcal{R}) = N - \dim \text{Ker}(\mathcal{R}^T) \geq N - d, \quad (2.14)$$

The proof is then complete.

(ii) Noting that (2.10) can be written as

$$\text{rank}(\mathcal{R}) \geq N - d \quad (2.15)$$

and

$$\text{rank}(\mathcal{R}) \leq N - d. \quad (2.16)$$

By (i), the rank condition (2.15) means that  $\dim(V) \leq d$  for any given invariant subspace  $V$  of  $A^T$  and  $B^T$ , contained in  $\text{Ker}(D^T)$ . We claim that there exists a subspace  $V_0$ , which is contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$  and  $B^T$ , such that  $\dim(V_0) = d$ . Otherwise, all the subspaces of this kind have a dimension less than or equal to  $(d - 1)$ . By (i), we get

$$\text{rank}(\mathcal{R}) \geq N - d + 1, \quad (2.17)$$

which contradicts (2.16). It proves (ii).  $\square$

**Remark 2.3.** In the special case that  $B = I$ , it is easy to see that

$$\mathcal{R} = (D, AD, \dots, A^{N-1}D). \quad (2.18)$$

Then, by Lemma 2.2, we find again (see [19]) that Kalman's criterion (2.1) holds if and only if the dimension of any given subspace, contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$ , does not exceed  $d$ . In particular, the equality holds if and only if the dimension of the largest subspace, contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$ , is exactly equal to  $d$ .

### 3. WELL-POSEDNESS

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary  $\Gamma = \Gamma_1 \cup \Gamma_0$  such that  $\bar{\Gamma}_1 \cap \bar{\Gamma}_0 = \emptyset$ . Let

$$U = (u^{(1)}, \dots, u^{(N)})^T \text{ and } H = (h^{(1)}, \dots, h^{(M)})^T$$

stands for the state variables and the boundary controls applied on  $\Gamma_1$ , respectively. Consider the following coupled system of wave equations with coupled Robin boundary controls:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu U + BU = DH & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (3.1)$$

with the initial condition

$$t = 0: \quad U = \widehat{U}_0, \quad U' = \widehat{U}_1 \quad \text{in } \Omega, \quad (3.2)$$

where  $\partial_\nu$  denotes the outward normal derivative.

Accordingly, let

$$\Phi = (\phi^{(1)}, \dots, \phi^{(N)})^T.$$

Consider the following adjoint system

$$\begin{cases} \Phi'' - \Delta \Phi + A^T \Phi = 0 & \text{in } (0, +\infty) \times \Omega, \\ \Phi = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \Phi + B^T \Phi = 0 & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (3.3)$$

with the initial data

$$t = 0 : \quad \Phi = \Phi_0, \quad \Phi' = \Phi_1 \quad \text{in } \Omega. \quad (3.4)$$

Denote

$$\mathcal{H}_0 = L^2(\Omega), \quad \mathcal{H}_1 = H_{\Gamma_0}^1(\Omega), \quad \mathcal{L} = L_{loc}^2(0, +\infty; L^2(\Gamma_1))$$

and by  $\mathcal{H}_{-1}$  the dual space of  $\mathcal{H}_1$  with respect to the pivot space  $\mathcal{H}_0$ , here  $H_{\Gamma_0}^1(\Omega)$  denotes the subspace of  $H^1(\Omega)$ , composed of functions with null trace on the boundary  $\Gamma_0$ .

We first consider the adjoint system (3.3) with the homogeneous boundary conditions by a direct method given in [25], which has the advantage of applying the semi-group approach in [30] in the present situation.

**Proposition 3.1.** *Assume that the matrix  $B$  is symmetric. Then for any given initial data  $(\Phi_0, \Phi_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , the adjoint problem (3.3)–(3.4) admits a unique solution:*

$$\Phi \in C_{loc}^0([0, +\infty); (\mathcal{H}_1)^N) \cap C_{loc}^1([0, +\infty); (\mathcal{H}_0)^N). \quad (3.5)$$

*Proof.* We first formulate system (3.3) into the following variational form:

$$\int_{\Omega} (\Phi'', \widehat{\Phi}) dx + \int_{\Omega} \langle \nabla \Phi, \nabla \widehat{\Phi} \rangle dx + \int_{\Gamma_1} (B\Phi, \widehat{\Phi}) d\Gamma + \int_{\Omega} (A\Phi, \widehat{\Phi}) dx = 0 \quad (3.6)$$

for any given test function  $\widehat{\Phi} \in (\mathcal{H}_1)^N$ , where  $(\cdot, \cdot)$  denotes the inner product of  $\mathbb{R}^N$ , while  $\langle \cdot, \cdot \rangle$  denotes the inner product of  $\mathbb{M}^{N \times N}(\mathbb{R})$ . Recalling the following interpolation inequality

$$\int_{\Gamma} |\phi|^2 d\Gamma \leq c \|\phi\|_{H^1(\Omega)} \|\phi\|_{L^2(\Omega)}, \quad \forall \phi \in H^1(\Omega),$$

we have

$$\int_{\Gamma_1} (B\Phi, \Phi) d\Gamma \leq \|B\| \int_{\Gamma_1} |\Phi|^2 d\Gamma \leq c \|B\| \|\Phi\|_{(\mathcal{H}_1)^N} \|\Phi\|_{(\mathcal{H}_0)^N},$$

then it follows that

$$\int_{\Omega} \langle \nabla \Phi, \nabla \Phi \rangle dx + \int_{\Gamma_1} (B\Phi, \Phi) d\Gamma + \lambda \|\Phi\|_{(\mathcal{H}_0)^N}^2 \geq c' \|\Phi\|_{(\mathcal{H}_1)^N}^2$$

for some suitable constants  $\lambda > 0$  and  $c' > 0$ . Therefore, the symmetric bilinear form

$$\int_{\Omega} \langle \nabla \Phi, \nabla \widehat{\Phi} \rangle dx + \int_{\Gamma_1} (B\Phi, \widehat{\Phi}) d\Gamma$$

is coercive. Moreover, the non-symmetric part in (3.6) satisfies

$$\int_{\Omega} (A\Phi, \widehat{\Phi}) dx \leq \|A\| \|\Phi\|_{(\mathcal{H}_0)^N} \|\widehat{\Phi}\|_{(\mathcal{H}_0)^N}.$$

By Theorem 1.1 ([25], p. 151), the variational problem (3.6) with the initial data (3.4) admits a unique solution  $\Phi$  with (3.5). The proof is complete.  $\square$

Now we consider problem (3.1)–(3.2) with inhomogeneous boundary conditions by the duality method given in [26].

**Definition 3.2.**  $U$  is a weak solution to problem (3.1)–(3.2), if

$$U \in C_{loc}^0([0, +\infty); (\mathcal{H}_0)^N) \cap C_{loc}^1([0, +\infty); (\mathcal{H}_{-1})^N) \quad (3.7)$$

such that

$$\begin{cases} \langle (U'(t), -U(t)), (\Phi(t), \Phi'(t)) \rangle = \langle (U_1, -U_0), (\Phi_0, \Phi_1) \rangle \\ + \int_0^t \int_{\Gamma_1} (DH(\tau), \Phi(\tau)) d\Gamma d\tau, \quad \forall t \geq 0 \end{cases} \quad (3.8)$$

holds for the solution  $\Phi$  to problem (3.3) with any given initial data  $(\Phi_0, \Phi_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , here and hereafter  $\langle \cdot, \cdot \rangle$  denotes the dual product between  $(\mathcal{H}_{-1})^N \times (\mathcal{H}_0)^N$  and  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ .

**Proposition 3.3.** *Assume that the matrix  $B$  is symmetric. Then for any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$  and for any given boundary function  $H \in \mathcal{L}^M$  with compact support in  $[0, T]$ , problem (3.1)–(3.2) admits a unique weak solution  $U$ . Moreover, the linear mapping*

$$(\widehat{U}_0, \widehat{U}_1, H) \rightarrow (U, U') \quad (3.9)$$

is continuous with respect to the corresponding topologies.

*Proof.* Define the linear form

$$L_t(\Phi_0, \Phi_1) = \langle (\widehat{U}_1, -\widehat{U}_0), (\Phi_0, \Phi_1) \rangle + \int_0^t \int_{\Gamma_1} (DH(\tau), \Phi(\tau)) d\Gamma d\tau.$$

Clearly,  $L_t$  is bounded in  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ . Let  $S_t$  be the semi-group associated to the problem (3.1)–(3.2) with the homogeneous boundary conditions on the Hilbert space  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ . The composed linear form  $L_t \circ S_t^{-1}$  is bounded in  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ . Then, by Riesz-Fr chet's representation theorem, there exists a unique element  $(U'(t), -U(t)) \in (\mathcal{H}_{-1})^N \times (\mathcal{H}_0)^N$ , such that

$$L_t \circ S_t^{-1}(\Phi(t), \Phi'(t)) = \langle (U'(t), -U(t)), (\Phi(t), \Phi'(t)) \rangle$$

for any given  $(\Phi_0, \Phi_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ . Noting

$$L_t \circ S_t^{-1}(\Phi(t), \Phi'(t)) = L_t(\Phi_0, \Phi_1),$$

we get (3.8) for any given  $(\Phi_0, \Phi_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ . Moreover, for any given  $T > 0$ , we have

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|(U'(t), -U(t))\|_{(\mathcal{H}_{-1})^N \times (\mathcal{H}_0)^N} \\ & \leq c_T (\|(\widehat{U}_1, \widehat{U}_0)\|_{(\mathcal{H}_{-1})^N \times (\mathcal{H}_0)^N} + \|H\|_{\mathcal{L}^M}), \end{aligned}$$

where  $c_T > 0$  is a positive constant depending on  $T$ . This gives the continuous dependence.

Finally, by a classic argument of density, we get the regularity (3.7) for all initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ . The proof is then complete.  $\square$



**Remark 3.4.** Suppose that  $B$  is similar to a symmetric matrix. Let  $P$  be an invertible matrix such that  $PBP^{-1}$  is symmetric. The new variable  $\tilde{U} = PU$  satisfies the same system (3.1) with the coupling matrix  $\tilde{A} = PAP^{-1}$  and the symmetric matrix  $\tilde{B} = PBP^{-1}$ . Hence, in order to guarantee the well-posedness of problem (3.1)–(3.2), in what follows, we always assume that  $B$  is similar to a symmetric matrix.

#### 4. APPROXIMATE BOUNDARY NULL CONTROLLABILITY

**Definition 4.1.** For  $(\Phi_0, \Phi_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ , the adjoint system (3.3) is  $D$ -observable on a finite interval  $[0, T]$ , if the observation

$$D^T \Phi \equiv 0 \quad \text{on } [0, T] \times \Gamma_1 \quad (4.1)$$

implies that  $\Phi_0 = \Phi_1 \equiv 0$ , then  $\Phi \equiv 0$ .

**Proposition 4.2.** *If the adjoint system (3.3) is  $D$ -observable, then we necessarily have  $\text{rank}(\mathcal{R}) = N$ . Conversely, if  $\text{rank}(D) = N$ , then system (3.3) is  $D$ -observable.*

*Proof.* Otherwise, we have  $\dim \text{Ker}(\mathcal{R}^T) = d \geq 1$ . Let  $\text{Ker}(\mathcal{R}^T) = \text{Span}\{E_1, \dots, E_d\}$ . By Lemma 2.1,  $\text{Ker}(\mathcal{R}^T)$  is contained in  $\text{Ker}(D^T)$  and invariant for  $A^T$  and  $B^T$ , namely, we have

$$D^T E_r = 0, \quad 1 \leq r \leq d \quad (4.2)$$

and there exist coefficients  $\alpha_{rs}$  and  $\beta_{rs}$  such that

$$A^T E_r = \sum_{s=1}^d \alpha_{rs} E_s, \quad B^T E_r = \sum_{s=1}^d \beta_{rs} E_s, \quad 1 \leq r \leq d. \quad (4.3)$$

In what follows, we restrict system (3.3) on the subspace  $\text{Ker}(\mathcal{R}^T)$  and look for a solution of the form

$$\Phi = \sum_{r=1}^d \phi_r E_r, \quad (4.4)$$

which, because of (4.2), obviously satisfies the  $D$ -observation (4.1).

Inserting the function (4.4) into system (3.3) and noting (4.3), it is easy to see that for  $1 \leq s \leq d$ , we have

$$\begin{cases} \phi_s'' - \Delta \phi_s + \sum_{r=1}^d \alpha_{rs} \phi_r = 0 & \text{in } (0, +\infty) \times \Omega, \\ \phi_s = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \phi_s + \sum_{r=1}^d \beta_{rs} \phi_r = 0 & \text{on } (0, +\infty) \times \Gamma_1. \end{cases} \quad (4.5)$$

For any non-trivial initial data:

$$t = 0: \quad \phi_s = \phi_{0s}, \quad \phi_s' = \phi_{1s}, \quad (1 \leq s \leq d), \quad (4.6)$$

we have  $\Phi \not\equiv 0$ . This contradicts the  $D$ -observability of system (3.3).

Conversely, when  $\text{rank}(D) = N$ , the  $D$ -observation (4.1) implies that

$$\partial_\nu \Phi \equiv \Phi \equiv 0 \quad \text{on } (0, T) \times \Gamma_1. \quad (4.7)$$

Then, Holmgren's uniqueness theorem implies well  $\Phi \equiv 0$ , provided that  $T > 0$  is large enough.  $\square$

**Definition 4.3.** System (3.1) is approximately null controllable at the time  $T > 0$ , if for any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ , there exists a sequence  $\{H_n\}$  of boundary controls in  $\mathcal{L}^M$  with compact support in  $[0, T]$ , such that the sequence  $\{U_n\}$  of solutions to problem (3.1)–(3.2) satisfies

$$u_n^{(k)} \longrightarrow 0 \quad \text{in } C_{loc}^0([T, +\infty); \mathcal{H}_0) \cap C_{loc}^1([T, +\infty); \mathcal{H}_{-1}) \quad (4.8)$$

for all  $1 \leq k \leq N$  as  $n \rightarrow +\infty$ .

By a similar argument as in [17], we can prove the following

**Proposition 4.4.** System (3.1) is approximately null controllable at the time  $T > 0$ , if and only if its adjoint system (3.3) is  $D$ -observable on the interval  $[0, T]$ .

**Corollary 4.5.** If system (3.1) is approximately controllable, then we necessarily have  $\text{rank}(\mathcal{R}) = N$ . In particular, as  $M = N$ , namely,  $D$  is invertible, system (3.1) is approximately null controllable.

*Proof.* This Corollary follows immediately from Proposition 4.2 and Proposition 4.4. However, here we prefer to give a direct proof from the point of view of control.

Suppose that  $\dim \text{Ker}(\mathcal{R}^T) = d \geq 1$ . Let  $\text{Ker}(\mathcal{R}^T) = \text{Span}\{E_1, \dots, E_d\}$ . By Lemma 2.1,  $\text{Ker}(\mathcal{R}^T)$  is contained in  $\text{Ker}(D^T)$  and invariant for both  $A^T$  and  $B^T$ , then we still have (4.2) and (4.3). Applying  $E_r$  to problem (3.1)–(3.2) and setting  $u_r = (E_r, U)$  for  $1 \leq r \leq d$ , it follows that for  $1 \leq r \leq d$ , we have

$$\begin{cases} u_r'' - \Delta u_r + \sum_{s=1}^d \alpha_{rs} u_s = 0 & \text{in } (0, +\infty) \times \Omega, \\ u_r = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu u_r + \sum_{s=1}^d \beta_{rs} u_s = 0 & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (4.9)$$

with the initial condition

$$t = 0: \quad u_r = (E_r, \widehat{U}_0), \quad u_r' = (E_r, \widehat{U}_1) \quad \text{in } \Omega. \quad (4.10)$$

Thus, the projections  $u_1, \dots, u_d$  of  $U$  on the subspace  $\text{Ker}(\mathcal{R}^T)$  are independent of applied boundary controls  $H$ , therefore, uncontrollable. This contradicts the approximate boundary null controllability of system (3.1). The proof is then complete.  $\square$

## 5. UNIQUENESS OF CONTINUATION

By Proposition 4.2,  $\text{rank}(\mathcal{R}) = N$  is a necessary condition for the  $D$ -observability.

**Proposition 5.1.** Let

$$\mu = \sup_{\alpha, \beta \in \mathbb{C}} \dim \text{Ker} \begin{pmatrix} A^T - \alpha I \\ B^T - \beta I \end{pmatrix}. \quad (5.1)$$

Assume that

$$\text{Ker}(\mathcal{R}^T) = \{0\}. \quad (5.2)$$

Then we have the following lower bound estimate:

$$\text{rank}(D) \geq \mu. \quad (5.3)$$

*Proof.* Let  $\alpha, \beta \in \mathbb{C}$ , such that

$$V = \text{Ker} \begin{pmatrix} A^T - \alpha I \\ B^T - \beta I \end{pmatrix} \quad (5.4)$$

is of dimension  $\mu$ . It is easy to see that any given subspace  $W$  of  $V$  is still invariant for  $A^T$  and  $B^T$ , then by Lemma 2.1, condition (5.2) implies that  $\text{Ker}(D^T) \cap V = \{0\}$ . Then, it follows that

$$\dim \text{Ker}(D^T) + \dim(V) \leq N, \quad (5.5)$$

namely,

$$\mu = \dim(V) \leq N - \dim \text{Ker}(D^T) = \text{rank}(D). \quad (5.6)$$

The proof is complete.  $\square$

In general, the condition  $\dim \text{Ker}(\mathcal{R}^T) = 0$  does not imply  $\text{rank}(D) = N$ , so, the  $D$ -observation (4.1) does not imply

$$\Phi = 0 \quad \text{on } (0, T) \times \Gamma_1. \quad (5.7)$$

Therefore, the uniqueness of continuation for the solution to the adjoint system (3.3) with  $D$ -observatiouon (4.1) is not a standard type of Holmgren's uniqueness theorem. Up to now, we only know fewer results on it, which we outline as follows.

Consider the following Robin type mixed problem of a system of two equations

$$\begin{cases} u'' - \Delta u + au + bv = 0 & \text{in } (0, +\infty) \times \Omega, \\ v'' - \Delta v + cu + dv = 0 & \text{in } (0, +\infty) \times \Omega, \\ u = v = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu u + \alpha u = 0 & \text{on } (0, +\infty) \times \Gamma_1, \\ \partial_\nu v + \beta v = 0 & \text{on } (0, +\infty) \times \Gamma_1. \end{cases} \quad (5.8)$$

Here, since the boundary coupling matrix  $B$  is assumed to be similar to a symmetric matrix, without loss of generality, we suppose that  $B = \text{diag}(\alpha, \beta)$  is a diagonal matrix. The following result can be easily checked.

**Proposition 5.2.** *We have  $\text{Ker}(\mathcal{R}^T) = \{0\}$  in the following cases.*

- (i) Case  $\alpha \neq \beta$ . Let  $D = (d_1, d_2)^T$ .
  - (a)  $d_1 \neq 0$ , if  $(1, 0)^T$  is the only common eigenvector of  $A^T$  and  $B^T$ ,
  - (b)  $d_2 \neq 0$ , if  $(0, 1)^T$  is the only common eigenvector of  $A^T$  and  $B^T$ ,
  - (c)  $d_1 d_2 \neq 0$ , if both  $(1, 0)^T$  and  $(0, 1)^T$  are eigenvectors of  $A^T$  and  $B^T$ ,
  - (d)  $d_1^2 + d_2^2 \neq 0$ , if there is no common eigenvector for  $A^T$  and  $B^T$ .
- (ii) Case  $\alpha = \beta$ .
  - (a)  $D = \mu_1 x_1 + \mu_2 x_2$  with  $\mu_1 \mu_2 \neq 0$ , if  $A$  possesses two different eigenvalues, associated to two eigenvectors  $x_1, x_2$ .
  - (b)  $D = \mu_1 x_1 + \mu_2 x_2$  with  $\mu_1 \neq 0$ , if  $A$  possesses only one eigenvalue associated to an eigenvector  $x_1$  and a root vector  $x_2$ .

**Theorem 5.3.** ([2], Thm. 2.6) Let  $(u, v)$  be a solution to the following system of two equations:

$$\begin{cases} u'' - \Delta u = 0 & \text{in } (0, +\infty) \times \Omega, \\ v'' - \Delta v + u = 0 & \text{in } (0, +\infty) \times \Omega, \\ u = v = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (5.9)$$

with initial data in  $H_{\Gamma_0}^1(\Omega) \times H_{\Gamma_0}^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ . Then, the observation

$$d_1 u + d_2 v \equiv 0 \quad \text{on } [0, T] \times \Gamma_1 \quad (5.10)$$

implies that  $u \equiv v \equiv 0$ , provided that  $d_2 \neq 0$  and  $T > 0$  is large enough.

**Theorem 5.4.** ([28]) Let  $(u, v)$  be a solution to the following system of two equations:

$$\begin{cases} u'' - \Delta u = 0 & \text{in } (0, +\infty) \times \Omega, \\ v'' - \Delta v = 0 & \text{in } (0, +\infty) \times \Omega, \\ u = v = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu u + \alpha u = 0 & \text{on } (0, +\infty) \times \Gamma_1, \\ \partial_\nu v + \beta v = 0 & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (5.11)$$

with initial data in  $H_{\Gamma_0}^1(\Omega) \times H_{\Gamma_0}^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ . Assume that  $\alpha \neq \beta$  and  $d_1 d_2 \neq 0$ . Then

(i) In higher dimensional case, the observation in the infinite horizon:

$$d_1 u + d_2 v \equiv 0 \quad \text{on } (0, +\infty) \times \Gamma_1 \quad (5.12)$$

implies that  $u \equiv v \equiv 0$ .

(ii) In one-space-dimensional case, the observation in a finite horizon:

$$d_1 u(1) + d_2 v(1) \equiv 0 \quad \text{for } 0 \leq t \leq T \quad (5.13)$$

implies that  $u \equiv v \equiv 0$ , provided that  $T > 0$  is large enough.

Let us consider the following slightly modified system:

$$\begin{cases} u'' - \Delta u = 0 & \text{in } (0, +\infty) \times \Omega, \\ v'' - \Delta v + u = 0 & \text{in } (0, +\infty) \times \Omega, \\ u = v = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu u + \alpha u = 0 & \text{on } (0, +\infty) \times \Gamma_1, \\ \partial_\nu v + \beta v = 0 & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (5.14)$$

with the partial observation (5.10) corresponding to  $D = (d_1, d_2)^T$ . By Lemma 2.2 (ii),  $\text{Ker}(\mathcal{R}^T) = \{0\}$  if and only if  $\text{Ker}(D^T)$  does not contain any common eigenvector of  $A^T$  and  $B^T$ . Since  $(0, 1)^T$  is the only common eigenvector of  $A^T$  and  $B^T$ ,  $\text{Ker}(\mathcal{R}^T) = \{0\}$  if and only if  $(0, 1)^T \notin \text{Ker}(D^T)$ , namely, if and only if  $d_2 \neq 0$ . Unfortunately, the multiplier approach used in [2] is quite technically delicate, we don't know up to now if it can be adapted to get the uniqueness of continuation for system (5.14) with the partial observation (5.10).

## 6. APPROXIMATE BOUNDARY SYNCHRONIZATION

**Definition 6.1.** System (3.1) is approximately synchronizable at the time  $T > 0$ , if for any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ , there exists a sequence  $\{H_n\}$  of boundary controls in  $\mathcal{L}^M$  with compact support in  $[0, T]$ , such that the corresponding sequence  $\{U_n\}$  of solutions to problem (3.1)–(3.2) satisfies

$$u_n^{(k)} - u_n^{(l)} \rightarrow 0 \quad \text{in } C_{loc}^0([T, +\infty); \mathcal{H}_0) \cap C_{loc}^1([T, +\infty); \mathcal{H}_{-1}) \quad (6.1)$$

for all  $k, l$  with  $1 \leq k, l \leq N$  as  $n \rightarrow +\infty$ .

Define the synchronization matrix of order  $(N-1) \times N$  by

$$C_1 = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}. \quad (6.2)$$

Clearly,

$$\text{Ker}(C_1) = \text{Span}\{e_1\} \text{ with } e_1 = (1, \dots, 1)^T. \quad (6.3)$$

Then, the approximate boundary synchronization (6.1) can be equivalently rewritten as

$$C_1 U_n \rightarrow 0 \quad \text{in } (C_{loc}^0([T, +\infty); \mathcal{H}_0))^{N-1} \cap (C_{loc}^1([T, +\infty); (\mathcal{H}_{-1}))^{N-1} \quad (6.4)$$

as  $n \rightarrow +\infty$ .

**Definition 6.2.** The matrix  $A$  satisfies the condition of  $C_1$ -compatibility, if there exists a unique matrix  $\overline{A}_1$  of order  $(N-1)$ , such that

$$C_1 A = \overline{A}_1 C_1. \quad (6.5)$$

The matrix  $\overline{A}_1$  is called the reduced matrix of  $A$  by  $C_1$ .

**Remark 6.3.** It was shown in [23] that the condition of  $C_1$ -compatibility (6.5) is equivalent to

$$A \text{Ker}(C_1) \subseteq \text{Ker}(C_1). \quad (6.6)$$

Then, noting (6.3), the vector  $e_1 = (1, \dots, 1)^T$  is an eigenvector of  $A$ , corresponding to the eigenvalue  $a$  given by

$$a = \sum_{j=1}^N a_{ij}, \quad i = 1, \dots, N. \quad (6.7)$$

In (6.7),  $\sum_{j=1}^N a_{ij}$  is independent of  $i = 1, \dots, N$ , called the row-sum condition, which is also equivalent to the condition of  $C_1$ -compatibility (6.5) or (6.6).

Similarly, the matrix  $B$  satisfies the condition of  $C_1$ -compatibility, if there exists a unique matrix  $\overline{B}_1$  of order  $(N-1)$ , such that

$$C_1 B = \overline{B}_1 C_1, \quad (6.8)$$

which is equivalent to the fact that

$$BKer(C_1) \subseteq Ker(C_1). \quad (6.9)$$

Moreover, the vector  $e_1 = (1, \dots, 1)^T$  is also an eigenvector of  $B$ , corresponding to the eigenvalue  $b$  given by

$$b = \sum_{j=1}^N b_{ij}, \quad i = 1, \dots, N, \quad (6.10)$$

where the sum  $\sum_{j=1}^N b_{ij}$  is independent of  $i = 1, \dots, N$ .

**Theorem 6.4.** *Assume that system (3.1) is approximately synchronizable. Then we necessarily have  $\text{rank}(\mathcal{R}) \geq N - 1$ .*

*Proof.* Otherwise, we have  $\dim \text{Ker}(\mathcal{R}^T) > 1$ . Let  $\text{Ker}(\mathcal{R}^T) = \text{Span}\{E_1, \dots, E_d\}$  with  $d > 1$ . Noting that

$$\dim \text{Im}(C_1^T) + \dim \text{Ker}(\mathcal{R}^T) = N - 1 + d > N, \quad (6.11)$$

there exists a unit vector  $E \in \text{Im}(C_1^T) \cap \text{Ker}(\mathcal{R}^T)$ . Let  $E = C_1^T x$  with  $x \in \mathbb{R}^{N-1}$ . The approximate boundary synchronization (6.4) implies that

$$(E, U_n) = (x, C_1 U_n) \rightarrow 0 \quad \text{in } C_{loc}^0([T, +\infty); \mathcal{H}_0) \cap C_{loc}^1([T, +\infty); \mathcal{H}_{-1}) \quad (6.12)$$

as  $n \rightarrow +\infty$ .

On the other hand, since  $E \in \text{Ker}(\mathcal{R}^T)$ , we have

$$E = \sum_{r=1}^d \alpha_r E_r, \quad (6.13)$$

where the coefficients  $\alpha_1, \dots, \alpha_d$  are not all zero. By Lemma 2.1,  $\text{Ker}(\mathcal{R}^T)$  is contained in  $\text{Ker}(D^T)$  and invariant for both  $A^T$  and  $B^T$ , therefore we still have (4.2) and (4.3). Thus, applying  $E_r$  to problem (3.1)–(3.2) and setting  $u_r = (E_r, U_n)$  for  $1 \leq r \leq d$ , we find again problem (4.9)–(4.10) with homogeneous boundary conditions. Noting that problem (4.9)–(4.10) is independent of  $n$ , it follows from (6.12) and (6.13) that

$$\sum_{r=1}^d \alpha_r u_r(T) \equiv \sum_{r=1}^d \alpha_r u_r'(T) \equiv 0. \quad (6.14)$$

Then, by well-posedness, it is easy to see that

$$\sum_{r=1}^d \alpha_r (E_r, \widehat{U}_0) \equiv \sum_{r=1}^d \alpha_r (E_r, \widehat{U}_1) \equiv 0 \quad (6.15)$$

for any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ . This yields

$$\sum_{r=1}^d \alpha_r E_r = 0. \quad (6.16)$$

Because of the linear independence of the vectors  $E_1, \dots, E_d$ , we get a contradiction  $\alpha_1 = \dots = \alpha_d = 0$ .  $\square$

**Theorem 6.5.** *Assume that system (3.1) is approximately synchronizable under the minimum  $\text{rank}(\mathcal{R}) = N - 1$ . Then, we have the following assertions:*

(i) *There exists a vector  $E_1 \in \text{Ker}(\mathcal{R}^T)$ , such that  $(E_1, e_1) = 1$  with  $e_1 = (1, 1, \dots, 1)^T$ .*

(ii) *For any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ , there exists a unique scalar function  $u$  such that*

$$u_n^{(k)} \rightarrow u \quad \text{in } C_{loc}^0([T, +\infty); \mathcal{H}_0) \cap C_{loc}^1([T, +\infty); \mathcal{H}_{-1}) \quad (6.17)$$

for all  $1 \leq k \leq N$  as  $n \rightarrow +\infty$ .

(iii) *The matrices  $A$  and  $B$  satisfy the conditions of  $C_1$ -compatibility (6.5) and (6.8), respectively.*

*Proof.* (i) Noting that  $\dim \text{Ker}(\mathcal{R}^T) = 1$ , by Lemma 2.1, there exists a non-zero vector  $E_1 \in \text{Ker}(\mathcal{R}^T)$ , such that

$$D^T E_1 = 0, \quad A^T E_1 = \alpha E_1, \quad B^T E_1 = \beta E_1. \quad (6.18)$$

We claim that  $E_1 \notin \text{Im}(C_1^T)$ . Otherwise, applying  $E_1$  to problem (3.1)–(3.2) with  $U = U_n$  and  $H = H_n$ , and setting  $u = (E_1, U_n)$ , it follows that

$$\begin{cases} u'' - \Delta u + \alpha u = 0 & \text{in } (0, +\infty) \times \Omega, \\ u = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu u + \beta u = 0 & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (6.19)$$

with the following initial data

$$t = 0: \quad u = (E_1, \widehat{U}_0), \quad u' = (E_1, \widehat{U}_1) \quad \text{in } \Omega. \quad (6.20)$$

Suppose that  $E_1 \in \text{Im}(C_1^T)$ , there exists a vector  $x \in \mathbb{R}^{N-1}$ , such that  $E_1 = C_1^T x$ . Then, the approximate boundary synchronization (6.4) implies

$$(u(T), u'(T)) = ((x, C_1 U_n(T)), (x, C_1 U_n'(T))) \rightarrow (0, 0) \quad (6.21)$$

in the space  $\mathcal{H}_0 \times \mathcal{H}_{-1}$  as  $n \rightarrow +\infty$ . Since problem (6.19)–(6.20) is independent of  $n$ , so is the solution  $u$ . We get thus

$$u(T) \equiv u'(T) \equiv 0. \quad (6.22)$$

Thus, because of the well-posedness of problem (6.19)–(6.20), it follows that

$$(E_1, \widehat{U}_0) = (E_1, \widehat{U}_1) = 0 \quad (6.23)$$

for any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ . This yields a contradiction  $E_1 = 0$ .

Since  $E_1 \notin \text{Im}(C_1^T)$ , noting that  $\text{Im}(C_1^T) = \text{Span}\{e_1\}^\perp$ , we have  $(E_1, e_1) \neq 0$ . Without loss of generality, we can take  $E_1$  such that  $(E_1, e_1) = 1$ .

(ii) Since  $E_1 \notin \text{Im}(C_1^T)$ , the matrix  $\begin{pmatrix} C_1 \\ E_1^T \end{pmatrix}$  is invertible. Moreover, we have

$$\begin{pmatrix} C_1 \\ E_1^T \end{pmatrix} e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (6.24)$$

Noting (6.4), we have

$$\begin{pmatrix} C_1 \\ E_1^T \end{pmatrix} U_n = \begin{pmatrix} C_1 U_n \\ (E_1, U_n) \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ u \end{pmatrix} = u \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (6.25)$$

as  $n \rightarrow +\infty$  in the space

$$(C_{loc}^0([T, +\infty); \mathcal{H}_0))^N \cap (C_{loc}^1([T, +\infty); \mathcal{H}_{-1}))^N. \quad (6.26)$$

Then, noting (6.24), it follows that

$$U_n = \begin{pmatrix} C_1 \\ E_1^T \end{pmatrix}^{-1} \begin{pmatrix} C_1 U_n \\ E_1^T U_n \end{pmatrix} \rightarrow u \begin{pmatrix} C_1 \\ E_1^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = ue_1 \quad (6.27)$$

in the the space (6.26), namely, (6.17) holds.

(iii) Applying  $C_1$  to system (3.1) with  $U = U_n$  and  $H = H_n$ , and passing to the limit as  $n \rightarrow +\infty$ , it follows from (6.4) and (6.27) that

$$C_1 A e_1 u = 0 \quad \text{in } [T, +\infty) \times \Omega \quad (6.28)$$

and

$$C_1 B e_1 u = 0 \quad \text{on } [T, +\infty) \times \Gamma_1. \quad (6.29)$$

We claim that at least for an initial data  $(\widehat{U}_0, \widehat{U}_1)$ , we have

$$u \neq 0 \quad \text{on } [T, +\infty) \times \Gamma_1. \quad (6.30)$$

Otherwise, it follows from system (6.19) that

$$\partial_\nu u \equiv u \equiv 0 \quad \text{on } [T, +\infty) \times \Gamma_1, \quad (6.31)$$

then, by Holmgren's uniqueness theorem, we get  $u \equiv 0$  for all the initial data  $(\widehat{U}_0, \widehat{U}_1)$ , namely, system (3.1) is approximately null controllable under the condition  $\dim \text{Ker}(\mathcal{R}^T) = 1$ . This contradicts Corollary 4.5. Then, it follows from (6.28) and (6.29) that  $C_1 A e_1 = 0$  and  $C_1 B e_1 = 0$ , which give the conditions of  $C_1$ -compatibility for  $A$  and  $B$ , respectively. The proof is complete.  $\square$

Assume that both  $A$  and  $B$  satisfy the corresponding conditions of  $C_1$ -compatibility, namely, there exist two matrices  $\overline{A}_1$  and  $\overline{B}_1$  such that  $C_1 A = \overline{A}_1 C_1$  and  $C_1 B = \overline{B}_1 C_1$ , respectively. Setting  $W = C_1 U$  in problem (3.1)–(3.2), we get the following reduced system

$$\begin{cases} W'' - \Delta W + \overline{A}_1 W = 0 & \text{in } (0, +\infty) \times \Omega, \\ W = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu W + \overline{B}_1 W = C_1 D H & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (6.32)$$

with the initial condition

$$t = 0: \quad W = C_1 \widehat{U}_0, \quad W' = C_1 \widehat{U}_1 \quad \text{in } \Omega. \quad (6.33)$$



Since  $B$  is similar to a symmetric matrix, so is its reduced matrix  $\bar{B}_1$  (cf. Prop. 7.4). Then, by Proposition 3.3, the reduced problem (6.32)–(6.33) is well-posed in the space  $(\mathcal{H}_0)^{N-1} \times (\mathcal{H}_{-1})^{N-1}$ .

Accordingly, consider the reduced adjoint system

$$\begin{cases} \Psi'' - \Delta\Psi + \bar{A}_1^T \Psi = 0 & \text{in } (0, T) \times \Omega, \\ \Psi = 0 & \text{on } (0, T) \times \Gamma_0, \\ \partial_\nu \Psi + \bar{B}_1^T \Psi = 0 & \text{on } (0, T) \times \Gamma_1 \end{cases} \quad (6.34)$$

with the  $C_1D$ -observation

$$(C_1D)^T \Psi \equiv 0 \quad \text{on } (0, T) \times \Gamma_1. \quad (6.35)$$

Obviously, we have

**Proposition 6.6.** *Under the conditions of  $C_1$ -compatibility for  $A$  and  $B$ , system (3.1) is approximately synchronizable if and only if the reduced system (6.32) is approximately null controllable, or equivalently, if and only if the reduced adjoint system (6.34) is  $C_1D$ -observable.*

**Theorem 6.7.** *Assume that  $A$  and  $B$  satisfy the conditions of  $C_1$ -compatibility (6.5) and (6.8), respectively. Assume furthermore that  $A^T$  and  $B^T$  admit a common eigenvector  $E_1$  such that  $(E_1, e_1) = 1$  with  $e_1 = (1, \dots, 1)^T$ . Let  $D$  be defined by*

$$\text{Im}(D) = \text{Span}\{E_1\}^\perp. \quad (6.36)$$

Then system (3.1) is approximate synchronizable. Moreover, we have  $\text{rank}(\mathcal{R}) = N - 1$ .

*Proof.* Since  $(E_1, e_1) = 1$ , noting (6.36), we have  $e_1 \notin \text{Im}(D)$  and  $\text{Ker}(C_1) \cap \text{Im}(D) = \{0\}$ . Therefore, by Lemma 2.2 in [20], we have

$$\text{rank}(C_1D) = \text{rank}(D) = N - 1. \quad (6.37)$$

Thus, the adjoint system (6.34) is  $C_1D$ -observable because of Holmgren's uniqueness theorem. By Proposition 6.6, system (3.1) is approximate synchronizable.

Noting (6.36), we have  $E_1 \in \text{Ker}(D^T)$ . Moreover, since  $E_1$  is a common eigenvector of  $A^T$  and  $B^T$ , we have  $E_1 \in \text{Ker}(\mathcal{R}^T)$ , hence  $\dim \text{Ker}(\mathcal{R}^T) \geq 1$ , namely,  $\text{rank}(\mathcal{R}) \leq N - 1$ . On the other hand, since  $\text{rank}(\mathcal{R}) \geq \text{rank}(D) = N - 1$ , we get  $\text{rank}(\mathcal{R}) = N - 1$ . The proof is complete.  $\square$

## 7. APPROXIMATE BOUNDARY SYNCHRONIZATION BY $p$ -GROUPS

In this section, let  $p \geq 1$  be an integer and

$$0 = n_0 < n_1 < n_2 < \dots < n_p = N. \quad (7.1)$$

We rearrange the components of the state variable  $U$  into  $p$  groups:

$$(u^{(1)}, \dots, u^{(n_1)}), (u^{(n_1+1)}, \dots, u^{(n_2)}), \dots, (u^{(n_{p-1}+1)}, \dots, u^{(n_p)}). \quad (7.2)$$

**Definition 7.1.** System (3.1) is approximately synchronizable by  $p$ -groups at the time  $T > 0$ , if for any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ , there exists a sequence  $\{H_n\}$  of boundary controls in  $\mathcal{L}^M$  with compact

support in  $[0, T]$ , such that the corresponding sequence  $\{U_n\}$  of solutions to problem (3.1)–(3.2) satisfies

$$u_n^{(k)} - u_n^{(l)} \rightarrow 0 \quad \text{in } C_{loc}^0([T, +\infty); \mathcal{H}_0) \cap C_{loc}^1([T, +\infty); \mathcal{H}_{-1}) \quad (7.3)$$

for  $n_{r-1} + 1 \leq k, l \leq n_r$  and  $1 \leq r \leq p$  as  $n \rightarrow +\infty$ .

Let  $S_r$  be the following  $(n_r - n_{r-1} - 1) \times (n_r - n_{r-1})$  matrix

$$S_r = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}. \quad (7.4)$$

Let  $C_p$  be the following  $(N - p) \times N$  full row-rank matrix of synchronization by  $p$ -groups:

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}. \quad (7.5)$$

For  $1 \leq r \leq p$ , setting

$$(e_r)_j = \begin{cases} 1, & n_{r-1} + 1 \leq j \leq n_r, \\ 0, & \text{otherwise,} \end{cases} \quad (7.6)$$

it is clear that

$$\text{Ker}(C_p) = \text{Span}\{e_1, e_2, \dots, e_p\}. \quad (7.7)$$

Moreover, the approximate boundary synchronization by  $p$ -groups (7.3) can be equivalently rewritten as

$$C_p U_n \rightarrow 0 \quad \text{in } (C_{loc}^0([T, +\infty); \mathcal{H}_0))^{N-p} \cap (C_{loc}^1([T, +\infty); (\mathcal{H}_{-1}))^{N-p} \quad (7.8)$$

as  $n \rightarrow +\infty$ .

**Definition 7.2.** The matrix  $A$  satisfies the condition of  $C_p$ -compatibility, if there exists a unique matrix  $\bar{A}_p$  of order  $(N - p)$ , such that

$$C_p A = \bar{A}_p C_p. \quad (7.9)$$

The matrix  $\bar{A}_p$  is called the reduced matrix of  $A$  by  $C_p$ .

**Remark 7.3.** The condition of  $C_p$ -compatibility (7.9) is equivalent to

$$A \text{Ker}(C_p) \subseteq \text{Ker}(C_p). \quad (7.10)$$

Moreover, the reduced matrix  $\bar{A}_p$  is given by

$$\bar{A}_p = C_p A C_p^T (C_p C_p^T)^{-1} \quad (7.11)$$

(see [23], Lem. 3.3). Similarly, the matrix  $B$  satisfies the condition of  $C_p$ -compatibility, if there exists a unique matrix  $\bar{B}_p$  of order  $(N - p)$ , such that

$$C_p B = \bar{B}_p C_p, \quad (7.12)$$

which is equivalent to

$$B \text{Ker}(C_p) \subseteq \text{Ker}(C_p). \quad (7.13)$$

**Proposition 7.4.** *Assume that the matrix  $A$  satisfies the condition of  $C_p$ -compatibility (7.9). Let  $\{x_l^{(k)}\}_{1 \leq k \leq d, 1 \leq l \leq r_k}$  be a system of root vectors of the matrix  $A$ , corresponding to the eigenvalues  $\lambda_k$  ( $1 \leq k \leq d$ ), such that for each  $k$  ( $1 \leq k \leq d$ ) we have*

$$A x_l^{(k)} = \lambda_k x_l^{(k)} + x_{l+1}^{(k)}, \quad 1 \leq l \leq r_k \text{ with } x_{r_k+1}^{(k)} = 0. \quad (7.14)$$

Define the following projected vectors by

$$\bar{x}_l^{(k)} = C_p x_l^{(k)}, \quad 1 \leq k \leq \bar{d}, \quad 1 \leq l \leq \bar{r}_k, \quad (7.15)$$

where  $\bar{d}$  ( $1 \leq \bar{d} \leq d$ ) and  $\bar{r}_k$  ( $1 \leq \bar{r}_k \leq r_k$ ) are given by (7.16) below. Then  $\{\bar{x}_l^{(k)}\}_{1 \leq k \leq \bar{d}, 1 \leq l \leq \bar{r}_k}$  forms a system of root vectors of the reduced matrix  $\bar{A}_p$ . In particular, if  $A$  is similar to a symmetric matrix, then so is  $\bar{A}_p$ .

*Proof.* Since  $\text{Ker}(C_p)$  is an invariant subspace of  $A$ , without loss of generality, we may assume that there exist some integers  $\bar{d}$  ( $1 \leq \bar{d} \leq d$ ) and  $\bar{r}_k$  ( $1 \leq \bar{r}_k \leq r_k$ ), such that the  $\{x_l^{(k)}\}_{1 \leq k \leq \bar{d}, 1 \leq l \leq \bar{r}_k}$  forms a root system for the restriction of  $A$  on the invariant subspace  $\text{Ker}(C_p)$ . Then,

$$\text{Ker}(C_p) = \text{Span}\{x_l^{(k)} : 1 \leq k \leq \bar{d}, 1 \leq l \leq \bar{r}_k\}. \quad (7.16)$$

In particular, we have

$$\sum_{k=1}^{\bar{d}} (r_k - \bar{r}_k) = p. \quad (7.17)$$

Noting that  $C_p^T (C_p C_p^T)^{-1} C_p$  is a projection from  $\mathbb{R}^N$  onto  $\text{Im}(C_p^T)$ , we have

$$C_p^T (C_p C_p^T)^{-1} C_p x = x, \quad \forall x \in \text{Im}(C_p^T). \quad (7.18)$$

On the other hand, by  $\mathbb{R}^N = \text{Im}(C_p^T) \oplus \text{Ker}(C_p)$  we can write

$$x_l^{(k)} = \hat{x}_l^{(k)} + \tilde{x}_l^{(k)} \quad \text{with} \quad \hat{x}_l^{(k)} \in \text{Im}(C_p^T), \quad \tilde{x}_l^{(k)} \in \text{Ker}(C_p), \quad (7.19)$$

then it follows from (7.15) that

$$\bar{x}_l^{(k)} = C_p \hat{x}_l^{(k)}, \quad 1 \leq k \leq \bar{d}, \quad 1 \leq l \leq \bar{r}_k. \quad (7.20)$$

Thus, noting (7.11) and (7.18), we have

$$\bar{A}_p \bar{x}_l^{(k)} = C_p A C_p^T (C_p C_p^T)^{-1} C_p \hat{x}_l^{(k)} = C_p A \hat{x}_l^{(k)}. \quad (7.21)$$

Since  $\text{Ker}(C_p)$  is invariant for  $A$ ,  $A\tilde{x}_l^{(k)} \in \text{Ker}(C_p)$ , then  $C_p A\tilde{x}_l^{(k)} = 0$ . It follows that

$$\overline{A}_p \overline{x}_l^{(k)} = C_p A(\widehat{x}_l^{(k)} + \tilde{x}_l^{(k)}) = C_p A\widehat{x}_l^{(k)}. \quad (7.22)$$

Then, using (7.14) and (7.15), it is easy to see that

$$\overline{A}_p \overline{x}_l^{(k)} = C_p(\lambda_k x_l^{(k)} + x_{l+1}^{(k)}) = \lambda_k \overline{x}_l^{(k)} + \overline{x}_{l+1}^{(k)}. \quad (7.23)$$

Therefore,  $\overline{x}_1^{(k)}, \overline{x}_2^{(k)}, \dots, \overline{x}_{\bar{r}_k}^{(k)}$  is a Jordan chain with length  $\bar{r}_k$  of the reduced matrix  $\overline{A}_p$ , corresponding to the eigenvalue  $\lambda_k$ .

Since  $\dim \text{Ker}(C_p) = p$ , the projected system  $\{\overline{x}_l^{(k)}\}_{1 \leq k \leq \bar{d}, 1 \leq l \leq \bar{r}_k}$  is of rank  $(N - p)$ . On the other hand, by (7.17), system  $\{\overline{x}_l^{(k)}\}_{1 \leq k \leq \bar{d}, 1 \leq l \leq \bar{r}_k}$  contains  $(N - p)$  vectors, therefore, forms a system of root vectors of the reduced matrix  $\overline{A}_p$ . The proof is complete.  $\square$

Assume that  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (7.9) and (7.12), respectively. Setting  $W = C_p U$  in problem (3.1)–(3.2), we get the following reduced system:

$$\begin{cases} W'' - \Delta W + \overline{A}_p W = 0 & \text{in } (0, +\infty) \times \Omega, \\ W = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu W + \overline{B}_p W = C_p D H & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (7.24)$$

with the initial condition

$$t = 0: \quad W = C_p \widehat{U}_0, \quad W' = C_p \widehat{U}_1 \quad \text{in } \Omega. \quad (7.25)$$

Since  $B$  is similar to a symmetric matrix, by Proposition 7.4, the reduced matrix  $\overline{B}_p$  is also similar to a symmetric matrix. Then by Proposition 3.3 and Remark 3.4, the reduced problem (7.24)–(7.25) is well-posed in the space  $(\mathcal{H}_0)^{N-p} \times (\mathcal{H}_{-1})^{N-p}$ .

Accordingly, consider the reduced adjoint system

$$\begin{cases} \Psi'' - \Delta \Psi + \overline{A}_p^T \Psi = 0 & \text{in } (0, +\infty) \times \Omega, \\ \Psi = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \Psi + \overline{B}_p^T \Psi = 0 & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (7.26)$$

together with the  $C_p D$ -observation

$$(C_p D)^T \Psi \equiv 0 \quad \text{on } (0, T) \times \Gamma_1. \quad (7.27)$$

We have

**Proposition 7.5.** *Assume that  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (7.9) and (7.12), respectively. Then system (3.1) is approximately synchronizable by  $p$ -groups if and only if the reduced system (7.24) is approximately null controllable, or equivalently, if and only if the reduced adjoint system (7.26) is  $C_p D$ -observable.*

**Corollary 7.6.** *Under the conditions of  $C_p$ -compatibility (7.9) and (7.12), if system (3.1) is approximately synchronizable by  $p$ -groups, we necessarily have the following rank condition:*

$$\text{rank}(C_p \mathcal{R}) = N - p. \quad (7.28)$$

*Proof.* Let  $\bar{\mathcal{R}}$  be the matrix defined by (2.2)–(2.3) corresponding to the reduced matrices  $\bar{A}_p, \bar{B}_p$  and  $\bar{D} = C_p D$ . Noting (7.9) and (7.12), we have

$$\bar{A}_p^r \bar{B}_p^s \bar{D} = \bar{A}_p^r \bar{B}_p^s C_p D = C_p A^r B^s D, \quad (7.29)$$

then

$$\bar{\mathcal{R}} = C_p \mathcal{R}. \quad (7.30)$$

Under the assumption that system (3.1) is approximately synchronizable by  $p$ -groups, by Proposition 7.5, the reduced system (7.24) is approximately null controllable, then by Corollary 4.5, we have  $\text{rank}(\bar{\mathcal{R}}) = N - p$  which together with (7.30) implies (7.28).  $\square$

**Proposition 7.7.** *Assume that system (3.1) is approximately synchronizable by  $p$ -groups. Then, we necessarily have  $\text{rank}(\mathcal{R}) \geq N - p$ .*

*Proof.* Assume that  $\dim \text{Ker}(\mathcal{R}^T) = d$  with  $d > p$ . Let  $\text{Ker}(\mathcal{R}^T) = \text{Span}\{E_1, \dots, E_d\}$ . Since

$$\dim \text{Ker}(\mathcal{R}^T) + \dim \text{Im}(C_p^T) = d + N - p > N,$$

we have  $\text{Ker}(\mathcal{R}^T) \cap \text{Im}(C_p^T) \neq \{0\}$ . Hence, there exists a non-zero vector  $x \in \mathbb{R}^{N-d}$  and coefficients  $\beta_1, \dots, \beta_d$  not all zero, such that

$$\sum_{r=1}^d \beta_r E_r = C_p^T x. \quad (7.31)$$

Moreover, by Lemma 2.1, we still have (4.2) and (4.3). Then, applying  $E_r$  to problem (3.1)–(3.2) with  $U = U_n$  and  $H = H_n$  and setting  $u_r = (E_r, U_n)$  for  $1 \leq r \leq d$ , it follows that

$$\begin{cases} u_r'' - \Delta u_r + \sum_{s=1}^d \alpha_{rs} u_s = 0 & \text{in } (0, +\infty) \times \Omega, \\ u_r = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu u_r + \sum_{s=1}^d \beta_{rs} u_s = 0 & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (7.32)$$

with the initial condition

$$t = 0: \quad u_r = (E_r, \widehat{U}_0), \quad u_r' = (E_r, \widehat{U}_1) \quad \text{in } \Omega. \quad (7.33)$$

Noting (7.8), it follows from (7.31) that

$$\sum_{r=1}^d \beta_r u_r = (x, C_p U_n) \rightarrow 0 \quad \text{in } C_{loc}^0([T, +\infty); \mathcal{H}_0) \cap C_{loc}^1([T, +\infty); \mathcal{H}_{-1}) \quad (7.34)$$

as  $n \rightarrow +\infty$ . Since problem (7.32)–(7.33) is independent of  $n$ , so is the solution  $(u_1, \dots, u_d)$ . It follows that

$$\sum_{r=1}^d \beta_r u_r(T) = \sum_{r=1}^d \beta_r u'_r(T) = 0 \quad \text{in } \Omega. \quad (7.35)$$

Then, it follows from the well-posedness of problem (7.32)–(7.33) that

$$\sum_{r=1}^d \beta_r (E_r, \widehat{U}_0) = \sum_{r=1}^d \beta_r (E_r, \widehat{U}_1) = 0 \quad (7.36)$$

for any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ . In particular, we get

$$\sum_{r=1}^d \beta_r E_r = 0, \quad (7.37)$$

then a contradiction:  $\beta_1 = \dots = \beta_d = 0$ , because of the linear independence of the vectors  $E_1, \dots, E_d$ . The proof is achieved.  $\square$

**Theorem 7.8.** *Let  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (7.9) and (7.12), respectively. Assume that  $A^T$  and  $B^T$  admit a common invariant subspace  $V$ , which is bi-orthonormal to  $\text{Ker}(C_p)$ . Then, setting the boundary control matrix  $D$  by*

$$\text{Im}(D) = V^\perp, \quad (7.38)$$

system (3.1) is approximately synchronizable by  $p$ -groups. Moreover, we have  $\text{rank}(\mathcal{R}) = N - p$ .

*Proof.* Since  $V$  is bi-orthonormal to  $\text{Ker}(C_p)$ , we have

$$\text{Ker}(C_p) \cap V^\perp = \text{Ker}(C_p) \cap \text{Im}(D) = \{0\}, \quad (7.39)$$

therefore, by Lemma 2.2 in [20], we have

$$\text{rank}(C_p D) = \text{rank}(D) = N - p. \quad (7.40)$$

Thus, the  $C_p D$ -observation (6.35) becomes the full observation

$$\Psi \equiv 0 \quad \text{on } (0, T) \times \Gamma_1. \quad (7.41)$$

By Holmgren's uniqueness theorem, the reduced adjoint system (7.26) is observable and the reduced system (7.24) is approximately null controllable. Then, by Proposition 7.5, the original system (3.1) is approximately synchronizable by  $p$ -groups. Noting that  $\text{Ker}(D^T) = V$ , by Lemma 2.1, it is easy to see that  $\text{rank}(\mathcal{R}) = N - p$ . The proof is then complete.  $\square$

**Theorem 7.9.** *Assume that system (3.1) is approximately synchronizable by  $p$ -groups. Assume furthermore that  $\text{rank}(\mathcal{R}) = N - p$ . Then, we have the following assertions:*

- (i)  $\text{Ker}(\mathcal{R}^T)$  is bi-orthonormal to  $\text{Ker}(C_p)$ .
- (ii) For any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ , there exist unique scalar functions  $u_1, u_2, \dots, u_p$  such that

$$u_n^{(k)} \rightarrow u_r \quad \text{in } C_{loc}^0([T, +\infty); \mathcal{H}_0) \cap C_{loc}^1([T, +\infty); H^{-1}(\Omega)) \quad (7.42)$$

for  $n_{r-1} + 1 \leq k \leq n_r$  and  $1 \leq r \leq p$  as  $n \rightarrow +\infty$ .

(iii) The coupling matrices  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (7.9) and (7.12), respectively.

*Proof.* (i) We claim that  $\text{Ker}(\mathcal{R}^T) \cap \text{Im}(C_p^T) = \{0\}$ . Then, noting that  $\text{Ker}(\mathcal{R}^T)$  and  $\text{Ker}(C_p)$  have the same dimension  $p$  and

$$\text{Ker}(\mathcal{R}^T) \cap \{\text{Ker}(C_p)\}^\perp = \text{Ker}(\mathcal{R}^T) \cap \text{Im}(C_p^T) = \{0\}, \quad (7.43)$$

by Proposition 4.1 in [18],  $\text{Ker}(\mathcal{R}^T)$  and  $\text{Ker}(C_p)$  are bi-orthonormal. Let  $\text{Ker}(\mathcal{R}^T) = \text{Span}\{E_1, \dots, E_p\}$  and  $\text{Ker}(C_p) = \text{Span}\{e_1, \dots, e_p\}$  such that

$$(E_r, e_s) = \delta_{rs}, \quad r, s = 1, \dots, p. \quad (7.44)$$

Now we return to check that  $\text{Ker}(\mathcal{R}^T) \cap \text{Im}(C_p^T) = \{0\}$ . If  $\text{Ker}(\mathcal{R}^T) \cap \text{Im}(C_p^T) \neq \{0\}$ , there exist a non-zero vector  $x \in \mathbb{R}^{N-p}$  and some coefficients  $\beta_1, \dots, \beta_p$  not all zero, such that

$$\sum_{r=1}^p \beta_r E_r = C_p^T x. \quad (7.45)$$

By Lemma 2.1, we still have (4.2) and (4.3) with  $d = p$ . For  $1 \leq r \leq p$ , applying  $E_r$  to problem (3.1)–(3.2) with  $U = U_n$  and  $H = H_n$ , and setting

$$u_r = (E_r, U), \quad (7.46)$$

it follows that

$$\begin{cases} u_r'' - \Delta u_r + \sum_{s=1}^p \alpha_{rs} u_s = 0 & \text{in } (0, +\infty) \times \Omega, \\ u_r = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu u_r + \sum_{s=1}^p \beta_{rs} u_s = 0 & \text{on } (0, +\infty) \times \Gamma_1 \end{cases} \quad (7.47)$$

with the initial condition

$$t = 0: \quad u_r = (E_r, \widehat{U}_0), \quad u_r' = (E_r, \widehat{U}_1). \quad (7.48)$$

Noting (7.8), we have

$$\sum_{r=1}^p \beta_r u_r = (x, C_p U_n) \rightarrow 0 \text{ in } C_{loc}^0([T, +\infty); \mathcal{H}_0) \cap C_{loc}^1([T, +\infty); \mathcal{H}_{-1}) \quad (7.49)$$

as  $n \rightarrow +\infty$ .

Since the functions  $u_1, \dots, u_p$  are independent of  $n$  and of applied boundary controls, we have

$$\sum_{r=1}^p \beta_r u_r(T) \equiv \sum_{r=1}^p \beta_r u_r'(T) \equiv 0 \quad \text{in } \Omega. \quad (7.50)$$

Then, it follows from the well-posedness of problem (7.47)–(7.48) that

$$\sum_{r=1}^p \beta_r (E_r, \widehat{U}_0) = \sum_{r=1}^p \beta_r (E_r, \widehat{U}_1) = 0 \quad \text{in } \Omega \quad (7.51)$$

for any given initial data  $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_{-1})^N$ . In particular, we get

$$\sum_{r=1}^p \beta_r E_r = 0, \quad (7.52)$$

then, a contradiction:  $\beta_1 = \dots = \beta_p = 0$ , because of the linear independence of the vectors  $E_1, \dots, E_p$ .

(ii) Noting (7.8), we have

$$\begin{pmatrix} C_p \\ E_1^T \\ \cdot \\ E_p^T \end{pmatrix} U_n = \begin{pmatrix} C_p U_n \\ E_1^T U_n \\ \cdot \\ E_p^T U_n \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ u_1 \\ \cdot \\ u_p \end{pmatrix} \quad (7.53)$$

as  $n \rightarrow +\infty$  in the space

$$(C_{loc}^0([T, +\infty); \mathcal{H}_0))^N \cap (C_{loc}^1([T, +\infty); \mathcal{H}_{-1}))^N, \quad (7.54)$$

where  $u_1, \dots, u_p$  are given by (7.47). Since  $\text{Ker}(\mathcal{R}^T) \cap \text{Im}(C_p^T) = \{0\}$ , the matrix  $\begin{pmatrix} C_p \\ E_1^T \\ \cdot \\ E_p^T \end{pmatrix}$  is invertible. Thus it follows from (7.53) that there exists  $U$  such that

$$U_n \rightarrow \begin{pmatrix} C_p \\ E_1^T \\ \cdot \\ E_p^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ u_1 \\ \cdot \\ u_p \end{pmatrix} =: U \quad (7.55)$$

as  $n \rightarrow +\infty$  in the space (7.54). Moreover, (7.8) implies that

$$t \geq T : \quad C_p U \equiv 0 \quad \text{in } \Omega.$$

Noting (7.7), (7.44) and (7.46), it follows that

$$t \geq T : \quad U = \sum_{r=1}^p (E_r, U) e_r = \sum_{r=1}^p u_r e_r \quad \text{in } \Omega. \quad (7.56)$$

Noting (7.6), we get then (7.42).

(iii) Applying  $C_p$  to system (3.1) with  $U = U_n$  and  $H = H_n$ , and passing to the limit as  $n \rightarrow +\infty$ , by (7.8), (7.55) and (7.56), it is easy to get that

$$\sum_{r=1}^p C_p A e_r u_r(T) \equiv 0 \quad \text{in } \Omega \quad (7.57)$$

and

$$\sum_{r=1}^p C_p B e_r u_r(T) \equiv 0 \quad \text{on } \Gamma_1. \quad (7.58)$$



Since system (7.47) is well-posed in  $(\mathcal{H}_1)^p \times (\mathcal{H}_0)^p$  and time-invertible, so it defines an isomorphism from  $(\mathcal{H}_1)^p \times (\mathcal{H}_0)^p$  onto  $(\mathcal{H}_1)^p \times (\mathcal{H}_0)^p$ . On the other hand, the mapping

$$(U_0, U_1) \rightarrow ((E_r, \widehat{U}_0), (E_r, \widehat{U}_1))_{1 \leq r \leq p} \quad (7.59)$$

is surjective from  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$  onto  $(\mathcal{H}_1)^p \times (\mathcal{H}_0)^p$ . Then,  $(u_1, \dots, u_p)$  will fulfil the space  $(\mathcal{H}_1)^p \times (\mathcal{H}_0)^p$  as the initial data  $(U_0, U_1)$  runs through the space  $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ . There exists thus an initial date  $(U_0, U_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$  such that the corresponding  $(u_1(T), \dots, u_p(T))$  are linearly independent. Then, it follows from (7.57) and (7.58) that

$$C_p A e_r = 0 \quad \text{and} \quad C_p B e_r = 0 \quad \text{for } 1 \leq r \leq p. \quad (7.60)$$

We get thus the conditions of  $C_p$ -compatibility for  $A$  and  $B$ , respectively. The proof is complete.  $\square$

**Remark 7.10.** The convergence (7.42) will be called the approximate boundary synchronization by  $p$ -groups in the pinning sense, and  $(u_1, \dots, u_p)^T$  will be called the approximately synchronizable state by  $p$ -groups. While the convergence (7.8) given by Definition 7.1 will be called the approximate boundary synchronization by  $p$ -groups in the consensus sense.

In general, the convergence (7.8) does not imply the convergence (7.42). In fact, we even don't know if the sequence  $\{U_n\}$  is bounded. However, under the rank condition  $\text{rank}(\mathcal{R}) = N - p$ , the convergence (7.8) actually implies the convergence (7.42). Moreover, the fonctions  $u_1, \dots, u_p$  are independent of applied boundary controls.

Let  $\mathbb{D}_p$  be the set of all the boundary control matrices  $D$  which realize the approximate boundary synchronization by  $p$ -groups for system (3.1). In order to show the dependence on  $D$ , we prefer to write  $\mathcal{R}_D$  instead of  $\mathcal{R}$  in (2.3). Then, we may define the minimal rank as

$$N_p = \inf_{D \in \mathbb{D}_p} \text{rank}(\mathcal{R}_D). \quad (7.61)$$

Noting that  $\text{rank}(\mathcal{R}_D) = N - \dim \text{Ker}(\mathcal{R}_D^T)$ , because of Proposition 7.7, we have

$$N_p \geq N - p. \quad (7.62)$$

Moreover, we have the following

**Corollary 7.11.** *The equality*

$$N_p = N - p \quad (7.63)$$

*holds if and only if the coupling matrices  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (7.9) and (7.12), respectively and  $A^T, B^T$  possess a common invariant subspace, which is bi-orthonormal to  $\text{Ker}(C_p)$ . Moreover, the approximate synchronization is in the pinning sense.*

*Proof.* Assume that (7.63) holds. Then there exists a matrix  $D \in \mathbb{D}_p$ , such that  $\dim \text{Ker}(\mathcal{R}_D^T) = p$ . By Theorem 7.9, the coupling matrices  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (7.9) and (7.12), respectively, and  $\text{Ker}(\mathcal{R}_D^T)$  which, by Lemma 2.1, is bi-orthonormal to  $\text{Ker}(C_p)$ , is invariant for both  $A^T$  and  $B^T$ . Moreover, the approximate synchronization is in the pinning sense.

Conversely, let  $V$  be a subspace, which is invariant for both  $A^T$  and  $B^T$ , and bi-orthonormal to  $\text{Ker}(C_p)$ . Noting that  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (7.9) and (7.12), respectively, by Theorem 7.8, there exists a matrix  $D \in \mathbb{D}_p$ , such that  $\dim \text{Ker}(\mathcal{R}_D^T) = p$ , which together with (7.62) implies (7.63).  $\square$

**Remark 7.12.** If  $N_p > N - p$ , then the situation is more complicated. We don't know if the conditions of  $C_p$ -compatibility (7.9) and (7.12) are necessary, either if the approximate boundary synchronization by  $p$ -groups is in the pinning sense.

## 8. APPROXIMATELY SYNCHRONIZABLE STATE BY $p$ -GROUPS

In Theorem 7.9, we have shown that if system (3.1) is approximately synchronizable by  $p$ -groups under the condition  $\dim \text{Ker}(\mathcal{R}^T) = p$ , then  $A$  and  $B$  satisfy the corresponding conditions of  $C_p$ -compatibility, and  $\text{Ker}(\mathcal{R}^T)$  is bi-orthonormal to  $\text{Ker}(C_p)$ , moreover, the approximately synchronizable state by  $p$ -groups is independent of applied boundary controls. The following is the counterpart.

**Theorem 8.1.** *Let  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (7.9) and (7.12), respectively. Assume that system (3.1) is approximately synchronizable by  $p$ -groups. If the projection of any solution  $U$  to problem (3.1)–(3.2) on a subspace  $V$  of dimension  $p$  is independent of applied boundary controls, then  $V = \text{Ker}(\mathcal{R}^T)$ . Moreover,  $\text{Ker}(\mathcal{R}^T)$  is bi-orthonormal to  $\text{Ker}(C_p)$ .*

*Proof.* Fixing  $\widehat{U}_0 = \widehat{U}_1 = 0$ , by Proposition 3.3, the linear map

$$F : H \rightarrow U$$

is continuous, therefore, infinitely differential from the control space  $\mathcal{L}^M$  to the space  $C_{loc}^0([0, +\infty); (\mathcal{H}_0)^N) \cap C_{loc}^1([0, +\infty); (\mathcal{H}_{-1})^N)$ .

Let  $\widehat{U}$  be defined by

$$\widehat{U} = F'(0)\widehat{H},$$

where  $F'(0)$  is the Fréchet differential of  $F$ , and  $\widehat{H} \in \mathcal{L}^M$  is any given boundary control.

Then, by linearity we have

$$\begin{cases} \widehat{U}'' - \Delta \widehat{U} + A\widehat{U} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \widehat{U} = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \widehat{U} + B\widehat{U} = D\widehat{H} & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0 : \widehat{U} = \widehat{U}' = 0 & \text{in } \Omega. \end{cases} \quad (8.1)$$

Let  $V = \text{Span}\{E_1, \dots, E_p\}$ . Then, the independence of the projection of  $U$  on the subspace  $V$ , with respect to the boundary controls, implies that

$$(E_i, \widehat{U}) \equiv 0 \quad \text{in } (0, +\infty) \times \Omega \quad \text{for } 1 \leq i \leq p. \quad (8.2)$$

We first show that  $E_i \notin \text{Im}(C_p^T)$  for any given  $i$  with  $1 \leq i \leq p$ . Otherwise, there exist an  $i$  with  $1 \leq i \leq p$  and a vector  $x_i \in \mathbb{R}^{N-p}$  such that  $E_i = C_p^T x_i$ . Then, it follows from (8.2) that

$$0 = (E_i, \widehat{U}) = (x_i, C_p \widehat{U}).$$

Since  $W = C_p \widehat{U}$  is the solution to the reduced system (7.24) with  $H = \widehat{H}$ , which is approximately controllable, we get thus  $x_i = 0$ , which contradicts  $E_i \neq 0$ . Thus, since  $\dim \text{Im}(C_p^T) = N - p$  and  $\dim(V) = p$ , we have

$V \oplus \text{Im}(C_p^T) = \mathbb{R}^N$ . Then, for any given  $i$  with  $1 \leq i \leq p$ , there exists a vector  $y_i \in \mathbb{R}^{N-p}$ , such that

$$A^T E_i = \sum_{j=1}^p \alpha_{ij} E_j + C_p^T y_i.$$

Noting (8.2) and applying  $E_i$  to system (8.1), it follows that

$$0 = (A\widehat{U}, E_i) = (\widehat{U}, A^T E_i) = (\widehat{U}, C_p^T y_i) = (C_p \widehat{U}, y_i).$$

Once again, the approximate controllability of the reduced system (7.24) implies that  $y_i = 0$  for  $1 \leq i \leq p$ . Then, it follows that

$$A^T E_i = \sum_{j=1}^p \alpha_{ij} E_j, \quad 1 \leq i \leq p.$$

So, the subspace  $V$  is invariant for  $A^T$ .

In [15], by the sharp regularity given in [9, 10] on Neumann type mixed problem, we improved the regularity (3.7) of the solution to problem (8.1). In fact, setting

$$\alpha = \begin{cases} 3/5 - \epsilon, & \Omega \text{ is a bounded smooth domain,} \\ 3/4 - \epsilon, & \Omega \text{ is a parallelepiped,} \end{cases} \quad (8.3)$$

where  $\epsilon > 0$  is a sufficiently small number, the trace

$$\widehat{U}|_{\Gamma_1} \in (H_{loc}^{2\alpha-1}((0, +\infty) \times \Gamma_1))^N \quad (8.4)$$

with the corresponding continuous dependence with respect to  $\widehat{H}$ .

Next, noting (8.2) and applying  $E_i$  ( $1 \leq i \leq p$ ) to the boundary condition on  $\Gamma_1$  in (8.1), we get

$$(D^T E_i, \widehat{H}) = (E_i, B\widehat{U}). \quad (8.5)$$

Then, it follows that

$$\|(D^T E_i, \widehat{H})\|_{H^{2\alpha-1}((0,T) \times \Gamma_1)} \leq c \|\widehat{U}\|_{H^{2\alpha-1}((0,T) \times \Gamma_1)}. \quad (8.6)$$

On the other hand, by the continuous dependence (8.4), we have

$$\|\widehat{U}\|_{H^{2\alpha-1}((0,T) \times \Gamma_1)} \leq c \|\widehat{H}\|_{L^2((0,T) \times \Gamma_1)}. \quad (8.7)$$

Then inserting (8.7) into (8.6), we get

$$\|(D^T E_i, \widehat{H})\|_{H^{2\alpha-1}((0,T) \times \Gamma_1)} \leq c \|\widehat{H}\|_{L^2((0,T) \times \Gamma_1)}. \quad (8.8)$$

Taking  $\widehat{H} = D^T E_i h$  in (8.8), we get

$$\|D^T E_i\| \|h\|_{H^{2\alpha-1}((0,T) \times \Gamma_1)} \leq c \|h\|_{L^2((0,T) \times \Gamma_1)}, \quad \forall h \in L^2((0,T) \times \Gamma_1). \quad (8.9)$$

Because of the compactness of the embedding  $H^{2\alpha-1}((0, T) \times \Gamma_1)$  to  $L^2((0, T) \times \Gamma_1)$  for  $2\alpha - 1 > 0$ , we deduce that

$$D^T E_i = 0, \quad 1 \leq i \leq p. \quad (8.10)$$

Then it follows from (8.10) that

$$V \subseteq \text{Ker}(D^T). \quad (8.11)$$

Moreover, for  $1 \leq i \leq p$  we have

$$(E_i, B\widehat{U}) = 0 \quad \text{on } (0, +\infty) \times \Gamma_1. \quad (8.12)$$

Now, let  $x_i \in \mathbb{R}^{N-p}$ , such that

$$B^T E_i = \sum_{j=1}^p \beta_{ij} E_j + C_p^T x_i. \quad (8.13)$$

Noting (8.2) and inserting the expression (8.13) into (8.12), it follows that

$$(x_i, C_p \widehat{U}) = 0 \quad \text{on } (0, +\infty) \times \Gamma_1.$$

Once again, because of the approximate boundary controllability of the reduced system (7.24), we deduce that  $x_i = 0$  for  $1 \leq i \leq p$ . Then, we get

$$B^T E_i = \sum_{j=1}^p \beta_{ij} E_j, \quad 1 \leq i \leq p.$$

So, the subspace  $V$  is also invariant for  $B^T$ .

Finally, since  $\dim(V) = p$ , by Lemma 2.1 and Proposition 7.7, we have  $\text{Ker}(\mathcal{R}^T) = V$ . Then, by assertion (i) of Theorem 7.9,  $\text{Ker}(\mathcal{R}^T)$  is bi-orthonormal to  $\text{Ker}(C_p)$ . This achieves the proof.  $\square$

Let  $d$  be a column vector of  $D$  and be contained in  $\text{Ker}(C_p)$ . Then it will be canceled in the product matrix  $C_p D$ , therefore it can not give any effect to the reduced system (7.24). However, the vectors in  $\text{Ker}(C_p)$  may play an important role for the approximate boundary controllability. More precisely, we have the following

**Theorem 8.2.** *Let  $A$  and  $B$  satisfy the conditions of  $C_p$ -compatibility (7.9) and (7.12), respectively. Assume that system (3.1) is approximately synchronizable by  $p$ -groups under the action of a boundary control matrix  $D$ . Assume furthermore that*

$$e_1, \dots, e_p \in \text{Im}(D), \quad (8.14)$$

where  $e_1, \dots, e_p$  are given by (7.6). Then system (3.1) is actually approximately null controllable.

*Proof.* By Proposition 4.4, it is sufficient to show that the adjoint system (3.3) is  $D$ -observable. For  $1 \leq r \leq p$ , applying  $e_r$  to the adjoint system (3.3) and noting  $\phi_r = (e_r, \Phi)$ , it follows that

$$\begin{cases} \phi_r'' - \Delta \phi_r + \sum_{s=1}^p \widehat{\alpha}_{rs} \phi_s = 0 & \text{in } (0, +\infty) \times \Omega, \\ \phi_r = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \phi_r + \sum_{s=1}^p \widehat{\beta}_{rs} \phi_s = 0 & \text{on } (0, +\infty) \times \Gamma_1, \end{cases} \quad (8.15)$$

where the constant coefficients  $\widehat{\alpha}_{rs}$  and  $\widehat{\beta}_{rs}$  are given by

$$Ae_r = \sum_{s=1}^p \widehat{\alpha}_{rs} e_s, \quad Be_r = \sum_{s=1}^p \widehat{\beta}_{rs} e_s, \quad 1 \leq r \leq p. \quad (8.16)$$

On the other hand, noting (8.14), the  $D$ -observation (4.1) implies that

$$\phi_r \equiv 0 \quad \text{on } (0, T) \times \Gamma_1 \quad (8.17)$$

for  $1 \leq r \leq p$ . Then, by Holmgren's uniqueness theorem, we get

$$\phi_r \equiv 0 \quad \text{in } (0, +\infty) \times \Omega \quad (8.18)$$

for  $1 \leq r \leq p$ . Thus,  $\Phi \in \text{Im}(C_p^T)$ , then we can write  $\Phi = C_p^T \Psi$  and the adjoint system (3.3) becomes

$$\begin{cases} C_p^T \Psi'' - C_p^T \Delta \Psi + A^T C_p^T \Psi = 0 & \text{in } (0, +\infty) \times \Omega, \\ C_p^T \Psi = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ C_p^T \partial_\nu \Psi + B^T C_p^T \Psi = 0 & \text{on } (0, +\infty) \times \Gamma_1. \end{cases} \quad (8.19)$$

Noting the conditions of  $C_p$ -compatibility (7.9) and (7.12), it follows that

$$\begin{cases} C_p^T (\Psi'' - \Delta \Psi + \overline{A}_p^T \Psi) = 0 & \text{in } (0, +\infty) \times \Omega, \\ C_p^T \Psi = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ C_p^T (\partial_\nu \Psi + \overline{B}_p^T \Psi) = 0 & \text{on } (0, +\infty) \times \Gamma_1. \end{cases} \quad (8.20)$$

Since the map  $C_p^T$  is injective, we find again the reduced adjoint system (7.26). Accordingly, the  $D$ -observation (4.1) implies that

$$D^T \Phi \equiv D^T C_p^T \Psi \equiv 0. \quad (8.21)$$

Since system (3.1) is approximately synchronizable by  $p$ -groups under the action of the boundary control matrix  $D$ , by Proposition 7.5, the reduced adjoint system (7.26) for  $\Psi$  is  $C_p D$ -observable, therefore,  $\Psi \equiv 0$ , then  $\Phi \equiv 0$ . So, the adjoint system (3.3) is  $D$ -observable, then by Proposition 4.4, system (3.1) is approximately null controllable.  $\square$

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