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L^p-VARIATIONAL SOLUTIONS OF MULTIVALUED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. We prove the existence and uniqueness of the L^p-variational solution, with $p > 1$, of the following multivalued backward stochastic differential equation with p-integrable data:

$$\begin{cases} -dY_t + \partial_y \Psi(t, Y_t)\mathrm{d}Q_t \ni H(t, Y_t, Z_t)\mathrm{d}Q_t - Z_t \mathrm{d}B_t,\ 0 \le t < \tau, \\ Y_\tau = \eta, \end{cases}$$

where τ is a stopping time, Q is a progressively measurable increasing continuous stochastic process and $\partial_y \Psi$ is the subdifferential of the convex lower semicontinuous function $y \mapsto \Psi(t, y)$. In the framework of [14] (the case $p \ge 2$), the strong solution found it there is the unique variational solution, via the uniqueness property proved in the present article.

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1. Introduction

The study of the standard backward stochastic differential equations (BSDEs) was initiated by E. Pardoux and S. Peng [18]. The authors have proved the existence and the uniqueness of the solution for the BSDE on fixed time interval, under the assumption of Lipschitz continuity of the generator $(t, y, z) \mapsto F(t, y, z)$ with respect to the variable y and z and the square integrability of $F(\cdot, 0, 0)$ on $\Omega \times [0, T]$ and respectively of η on Ω. The case of BSDEs on random time interval have been treated by R.W.R. Darling and E. Pardoux [5], where it is obtained, as application, the existence of a continuous viscosity solution to the elliptic partial differential equations (PDEs) with Dirichlet boundary conditions. The more general case of reflected BSDEs was considered for the first time by N. El Karoui *et al.* [8].

In this paper, we prove the existence and uniqueness of a new type of solution, called L^p-variational solution, in the case $p > 1$, of the generalized backward stochastic variational inequality (BSVI for short) with p-integrable

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data:

$$\begin{cases} Y_t + \int_{t\wedge\tau}^{\tau} \mathrm{d}K_s = \eta + \int_{t\wedge\tau}^{\tau} \left[F\left(s,Y_s,Z_s\right)\mathrm{d}s + G\left(s,Y_s\right)\mathrm{d}A_s\right] - \int_{t\wedge\tau}^{\tau} Z_s \mathrm{d}B_s\,, \quad t\geq 0, \\ \mathrm{d}K_t \in \partial\varphi\left(Y_t\right)\mathrm{d}t + \partial\psi\left(Y_t\right)\mathrm{d}A_t\,, \quad \text{on } \mathbb{R}_+, \end{cases} \tag{1.1}$$

where $\partial\varphi$ and $\partial\psi$ are the subdifferentials of two proper convex lower semicontinuous (l.s.c. for short) functions φ and ψ and $\{A_t : t \geq 0\}$ is a progressively measurable increasing continuous stochastic process.

We prove the uniqueness property of the solution on a random time interval $[0,\tau]$; the existence is obtained also on a random time interval, but first in the case of a deterministic time interval, *i.e.* $\tau = T > 0$, and it is made using the Moreau-Yosida regularization of φ and ψ and a mollifier approximation of the generators F and G.

In fact, we will define and prove the existence and uniqueness of the L^p-variational solution for an equivalent form of (1.1):

$$\begin{cases} Y_t + \int_{t\wedge\tau}^{\tau} \mathrm{d}K_s = \eta + \int_{t\wedge\tau}^{\tau} H\left(s,Y_s,Z_s\right)\mathrm{d}Q_s - \int_{t\wedge\tau}^{\tau} Z_s \mathrm{d}B_s\,,\ t\geq 0 \\ \mathrm{d}K_t \in \partial_y\Psi\left(t,Y_t\right)\mathrm{d}Q_t\,,\ \text{on } \mathbb{R}_+, \end{cases} \tag{1.2}$$

with Q, H and Ψ adequately defined.

The second condition in (1.1) says, among others, that the first component Y of the solution is forced to stay in the set $\mathrm{Dom}\left(\partial\varphi\right)\cap\mathrm{Dom}\left(\partial\psi\right)$. The role of K is to act in the evolution of the process Y and also to keep Y in these domains.

We emphasize that, unlike the case $p \geq 2$, in the case $1 \leq p < 2$ it is not possible to obtain the boundedness of the term $\mathbb{E}\Big(\int_0^T e^{2V_r}|\nabla\Psi_\varepsilon(r,Y_r^\varepsilon)|^2\mathrm{d}Q_r\Big)^{p/2}$, which is essential in order to obtain a strong solution, where Ψ_ε is the Moreau-Yosida's regularization of Ψ. Therefore we propose a generalization of the strong solution and we give the definition of the solution using an inequality (instead of a equality) involving only the function Ψ and not the subdifferential operator $\partial\Psi$. However, under this kind of definition, we were able to prove the uniqueness property (even if the solution (Y,Z) satisfies an inequality).

We mention that the presence of the process A is justified by the possible applications of equation (1.1) in proving probabilistic proofs for the existence of a solution of PDEs with Neumann boundary conditions on a domain from $\mathbb{R}^m$. The stochastic approach of the existence problem for multivalued parabolic PDEs, was considered by L. Maticiuc and A. Răşcanu [13, 15]. We emphasize that if the obstacles are fixed, the reflected BSDEs becomes a particular case of the BSVI of type (1.1), by taking φ as convex indicator of the interval defined by obstacles. In this case the solution of the BSVI belongs to the domain of the multivalued operator $\partial\varphi$ and it is reflected at the boundary of this domain.

The standard work on BSVI in the finite dimensional case is that of E. Pardoux and A. Răşcanu [19], where it is proved the existence and uniqueness of the solution (Y,Z,K) for the BSVI (1.1) with $A\equiv 0$, under the following assumptions on F: continuity with respect to y, monotonicity with respect to y (in the sense that $\langle y'-y, F(t,y',z)-F(t,y,z)\rangle \leq \alpha|y'-y|^2$), Lipschitzianity with respect to z and a sublinear growth for $F\left(t,y,0\right)$. Moreover, it was shown that, unlike the forward case, the process K is absolute continuous with respect to $\mathrm{d}t$. [20] the same authors extend these results to the Hilbert spaces framework.

We mention that assumptions of Lipschitz continuity of the generator F with respect to y and z and the square integrability of the final condition and $F\left(t,0,0\right)$ (as in articles El Karoui *et al.* [8] and E. Pardoux and S. Peng [18]) are sometimes too strong for applications (see, *e.g.*, D. Duffie and L. Epste [7] and El Karoui *et al.* [9] for the applications in mathematical finance and P. Briand *et al.* [3] and A. Rozkosz and L. Słomiński [24] for the applications to PDEs). A possibility is to weaken the integrability conditions imposed on η and F or to weaken the assumption which concerns the Lipschitz continuity of the generators. In P. Briand and R. Carmona [3] or E. Pardoux [17] it is considered the case where the generators are Lipschitz continuous with respect to z, continuous with respect to y and satisfy a monotonicity condition and a growth condition of the type $|F(t,y,z)| \leq |F(t,0,z)| + \phi(|y|)$, where ϕ is a polynomial or even an arbitrary positive increasing continuous function.

We recall that the previous assumption was used [17] in order to prove the existence of a solution in L^2. This result was generalized by P. Briand *et al.* [4], where it is proved the existence and uniqueness of L^p solutions, with $p \in [1,2]$, for BSDEs considered with a random terminal time T : in the case $p \in (1,2]$, if $\eta \in L^p$, $\int_0^T |F(s,0,0)| \, ds \in L^p$ and $\int_0^T \sup_{|y|\leq r} |F(s,y,0) - F(s,0,0)| \, ds \in L^1$, for any $r > 0$, and if F is Lipschitz continuous with respect to z, continuous with respect to y and satisfies a monotonicity condition, then there exists a unique L^p solution. In the case $p = 1$ similar result is proved if T is a fixed deterministic terminal time and under additional assumptions.

We also note that the study of the reflected BSDEs was the subject, *e.g.*, of the papers: J.P. Lepeltier *et al.* [12] (in the case of the general growth condition with respect to y and for $p = 2$), S. Hamadène and A. Popier [10] (in the case of Lipschitz continuity with respect to y and for $p \in (1,2)$). Studies made, roughly speaking, under the assumptions of [4] are, *e.g.*: A. Aman [1] (in the case of a generalized reflected BSDE and for $p \in (1,2)$), A. Rozkosz and L. Słomiński [23] (for $p \in [1,2]$) and T. Klimsiak [11] (in the case of BSDE with two irregular reflecting barriers and for $p \in [1,2]$).

Our paper generalizes the existence and uniqueness results from [19] by considering the L^p solutions in the case $p \in (1,2)$, the Lebesgue-Stieltsjes integral terms, and by assuming a weaker boundedness condition for the generator F (instead of the sublinear growth):

$$\mathbb{E}\Big(\int_0^T F_\rho^{\#}(s) ds\Big)^p < \infty, \quad \text{where } F_\rho^{\#}(t) \overset{\text{def}}{=\!=} \sup_{|y|\leq\rho} |F(t,y,0)| \,. \tag{1.3}$$

We remark that article [14] concerns the same type of backward equation as in our study (and under similar assumptions), considered in the infinite dimensional framework and for $p \geq 2$. [14] the existence and uniqueness of a strong solution is proved. In addition, it is also introduced the variational solution and it is proved the existence of this type of solution.

More precisely, Theorem 5.5 generalizes the results from [13] for $p = 2$ and Theorem 5.8 generalizes (except the Hilbert spaces framework) the results from [14] for $p \geq 2$. We emphasize that the assumptions on F and G are weaker to those adopted [14]; also the form of our hypothesis is more simplified (and therefore more easily to be verified) with respect to those from [14].

Finally, let us notice that our main results (the uniqueness provided by Cor. 4.2 and the existence of a variational solution provided by Thm. 5.5) hold true in the case of infinite dimensional spaces (with minor changes in their proofs and using the suitable framework of [14]).

In this paper we use the following notation: $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, the set $\mathcal{N} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$, $\{\mathcal{F}_t\}_{t\geq 0}$ is a right continuous and complete filtration generated by a standard k-dimensional Brownian motion $(B_t)_{t\geq 0}$.

$S_m^p[0,T]$ is the space of (equivalent classes of) continuous progressively measurable stochastic processes (p.m.s.p. for short) $X : \Omega \times [0,T] \to \mathbb{R}^m$, if $p \geq 0$, and such that $\mathbb{E}\sup_{t\in[0,T]} |X_t|^p < \infty$, if $p > 0$. The notation S_m^p is the space of (equivalent classes of) continuous p.m.s.p. $X : \Omega \times [0,\infty) \to \mathbb{R}^m$ such that, for all $T > 0$, the restriction $X\big|_{[0,T]}$ belongs to $S_m^p[0,T]$. For other details, *e.g.* metrics, see Section 1.1.4 of [21].

$\Lambda_m^p(0,T)$ is the space of p.m.s.p. $X : \Omega \times [0,T] \to \mathbb{R}^m$ such that $\int_0^T |X_t|^2 dt < \infty$, $\mathbb{P}$-a.s. if $p = 0$ and $\mathbb{E}\big(\int_0^T |X_t|^2 dt\big)^{p/2} < \infty$, if $p > 0$. The notation Λ_m^p is the space of p.m.s.p. $X : \Omega \times [0,\infty) \to \mathbb{R}^m$ such that, for all $T > 0$, the restriction $X\big|_{[0,T]}$ belongs to $\Lambda_m^p(0,T)$. For other details, *e.g.* metrics, see Section 2.1 of [21].

The article is organized as follows: next section is dedicated to the presentation of the assumptions needed in our study. In the third section we present an intuitive introduction and the definition of the notion of L^p-variational solution. The next section deals with proof of the uniqueness and continuity properties. The fifth section is devoted to the proof of the existence of our type of solution both in the case of a deterministic and random time interval. The Appendix contains, mainly following [21], some results useful throughout the paper.

2. Assumptions and definitions

At the beginning of this subsection we introduce the assumptions about equation (1.1).

We consider throughout this paper that $p>1$.

(A$_1$) The random variable $\tau:\Omega\rightarrow[0,\infty]$ is a stopping time;

(A$_2$) The random variable $\eta:\Omega\rightarrow\mathbb{R}^m$ is $\mathcal{F}_\tau$-measurable such that $\mathbb{E}\left|\eta\right|^p<\infty$ and $(\xi,\zeta)\in S_m^p\times\Lambda_{m\times k}^p$ is the unique pair associated to η given by the martingale representation formula (see [21], Cor. 2.44)

$$\begin{cases}\xi_t=\eta-\displaystyle\int_t^\infty\zeta_s\mathrm{d}B_s\,, & t\geq0,\ \mathbb{P}-\text{a.s.},\\ \xi_t=\mathbb{E}^{\mathcal{F}_t}\eta\quad\text{and}\quad\zeta_t=\mathbf{1}_{[0,\tau]}\left(t\right)\zeta_t\end{cases}\tag{2.1}$$

(or equivalently, $\xi_t=\eta-\displaystyle\int_{t\wedge\tau}^{\tau}\zeta_s\mathrm{d}B_s\,,\ t\geq0,\ \mathbb{P}$-a.s.);

(A$_3$) The process $\{A_t:t\geq0\}$ is an increasing and continuous p.m.s.p. such that $A_0=0$ and

$$\mathbb{E}\left(e^{\alpha A_T}\right)<\infty,\quad\text{for any }\alpha,T>0;\tag{2.2}$$

(A$_4$) $\varphi,\psi:\mathbb{R}^m\rightarrow[0,+\infty]$ are proper l.s.c. functions, $\partial\varphi$ and $\partial\psi$ denote their subdifferentials and we suppose that $0\in\partial\varphi\left(0\right)\cap\partial\psi\left(0\right)$ (or equivalently $0=\varphi\left(0\right)\leq\varphi\left(y\right)$ and $0=\psi\left(0\right)\leq\psi\left(y\right)$ for all $y\in\mathbb{R}^m$);
In addition, we suppose that

$$\varphi\left(\eta\right)+\psi\left(\eta\right)<\infty,\quad\mathbb{P}-\text{a.s.};$$

(A$_5$) The functions $F:\Omega\times\mathbb{R}_+\times\mathbb{R}^m\times\mathbb{R}^{m\times k}\rightarrow\mathbb{R}^m$ and $G:\Omega\times\mathbb{R}_+\times\mathbb{R}^m\rightarrow\mathbb{R}^m$ are such that $F\left(\cdot,\cdot,y,z\right)$, $G\left(\cdot,\cdot,y\right)$ are p.m.s.p., for all $(y,z)\in\mathbb{R}^m\times\mathbb{R}^{m\times k}$, $F\left(\omega,t,\cdot,\cdot\right)$, $G\left(\omega,t,\cdot\right)$ are continuous functions, $d\mathbb{P}\otimes\mathrm{d}t$-a.e. and, $\mathbb{P}$-a.s.,

$$\int_0^T F_\rho^{\#}\left(s\right)\mathrm{d}s+\int_0^T G_\rho^{\#}\left(s\right)\mathrm{d}A_s<\infty,\quad\text{for all }\rho,T\geq0,\tag{2.3}$$

where

$$F_\rho^{\#}\left(\omega,s\right)\overset{\text{def}}{=}\sup\nolimits_{|y|\leq\rho}\left|F\left(\omega,s,y,0\right)\right|,\quad G_\rho^{\#}\left(\omega,s\right)\overset{\text{def}}{=}\sup\nolimits_{|y|\leq\rho}\left|G\left(\omega,s,y\right)\right|\,;\tag{2.4}$$

(A$_6$) Let

$$n_p\overset{\text{def}}{=}\left(p-1\right)\wedge1\quad\text{and}\quad\Lambda\in\left(0,1\right).\tag{2.5}$$

Assume there exist three p.m.s.p. $\mu,\nu:\Omega\times\mathbb{R}_+\rightarrow\mathbb{R}$, $\ell:\Omega\times\mathbb{R}_+\rightarrow\mathbb{R}_+$, such that

$$\mathbb{E}\exp\left(p\int_0^T\left(\mu_s^++\frac{1}{2n_p\lambda}\,\ell_s^2\right)\mathrm{d}s+p\int_0^T\nu_s^+\mathrm{d}A_s\right)<\infty,\quad\text{for all }\ T>0,\tag{2.6}$$

and for all $t \geq 0$, $y, y' \in \mathbb{R}^m$, $z, z' \in \mathbb{R}^{m\times k}$, $\mathbb{P}$-a.s.

$$\begin{aligned} &\langle y'-y, F(t,y',z)-F(t,y,z)\rangle \leq \mu_t\ |y'-y|^2, \\ &\langle y'-y, G(t,y')-G(t,y)\rangle \leq \nu_t\ |y'-y|^2, \\ &|F(t,y,z')-F(t,y,z)| \leq \ell_t\ |z'-z|. \end{aligned} \tag{2.7}$$

We define

$$Q_t = t + A_t,$$

and let $\{\alpha_t : t \geq 0\}$ be the real positive p.m.s.p. such that $\alpha \in [0,1]$ and

$$\mathrm{d}t = \alpha_t \mathrm{d}Q_t \quad \text{and} \quad \mathrm{d}A_t = (1-\alpha_t)\,\mathrm{d}Q_t.$$

Let us introduce the functions

$$\begin{aligned} &H(t,y,z) \overset{\text{def}}{=\!=} \mathbf{1}_{[0,\tau]}(t)\left[\alpha_t F(t,y,z) + (1-\alpha_t)\,G(t,y)\right], \\ &\Psi(t,y) \overset{\text{def}}{=\!=} \mathbf{1}_{[0,\tau]}(t)\left[\alpha_t \varphi(y) + (1-\alpha_t)\,\psi(y)\right]. \end{aligned} \tag{2.8}$$

Obviously, from (2.7) we see that

$$\begin{aligned} &\langle y'-y, H(t,y',z)-H(t,y,z)\rangle \leq \mathbf{1}_{[0,\tau]}(t)\left[\mu_t\alpha_t + \nu_t(1-\alpha_t)\right]|y'-y|^2, \\ &|H(t,y,z')-H(t,y,z)| \leq \mathbf{1}_{[0,\tau]}(t)\ \alpha_t\,\ell_t\ |z'-z|. \end{aligned} \tag{2.9}$$

Here and subsequently

$$V_t \overset{\text{def}}{=\!=} \int_0^t \mathbf{1}_{[0,\tau]}(r)\left(\mu_r + \frac{1}{2n_p\lambda}\,\ell_r^2\right)\mathrm{d}r + \int_0^t \mathbf{1}_{[0,\tau]}(r)\,\nu_r \mathrm{d}A_r \tag{2.10}$$

and

$$V_t^{(+)} \overset{\text{def}}{=\!=} \int_0^t \mathbf{1}_{[0,\tau]}(r)\left(\mu_r^+ + \frac{1}{2n_p\lambda}\,\ell_r^2\right)\mathrm{d}r + \int_0^t \mathbf{1}_{[0,\tau]}(r)\,\nu_r^+ \mathrm{d}A_r. \tag{2.11}$$

By assumption (2.6) we clearly have, for all $T > 0$,

$$\begin{aligned} &\mathbb{E}\exp(pV_T) \leq \mathbb{E}\left(\sup\nolimits_{r\in[0,T]} e^{pV_r}\right) \leq \mathbb{E}\exp\left(pV_T^{(+)}\right) \\ &\leq \mathbb{E}\exp\left(p\int_0^T\left(\mu_s^+ + \frac{1}{2n_p\lambda}\,\ell_s^2\right)\mathrm{d}s + p\int_0^T \nu_s^+\,\mathrm{d}A_s\right) < \infty. \end{aligned} \tag{2.12}$$

Remark 2.1. Assumption (2.6) is necessary for some estimates throughout the proofs.

Remark 2.2. Usually, the monotonicity coefficients μ_t, ν_t and the Lipschitz coefficient ℓ_t are considered deterministic constants. But in many concrete cases the coefficients are stochastic processes (may depend on ω and

t); for instance, if we take, as a simple example, in one dimensional case, a BSDE with the generators

$$\begin{aligned} F(t,y,z) &= \frac{\tilde{a}\, B_t\, |B_t|^{\bar{a}}}{t^{\alpha}}\,(y - f_1(B_t y)) + \frac{\tilde{b}\, |B_t|^{(\bar{b}+1)/2}}{t^{\beta/2}}\, f_2(z)\,, \\ G(t,y) &= \frac{\tilde{c}\, B_t\, |B_t|^{\bar{c}}}{(t+A_t)^{\gamma}}\,(y - f_3(B_t y))\,, \end{aligned}$$

where $\tilde{a}, \tilde{b}, \tilde{c} > 0$ and $0 < \bar{a} \le 1$, $-1 < \bar{b} \le 1$, $0 < \bar{c} < 1$ and $0 \le \alpha, \beta < 1$, $1/2 < \gamma < 1$ are some suitable constants and f_1, f_2, f_3 are three derivable and nondecreasing functions.

In this case we obtain the monotonicity coefficient functions $\mu_t = \frac{\tilde{a}\, B_t |B_t|^{\bar{a}}}{t^{\alpha}}$, $\nu_t = \frac{\tilde{c}\, B_t |B_t|^{\bar{c}}}{(t+A_t)^{\gamma}}$ and the Lipschitz coefficient function $\ell_t = \frac{\tilde{b} |B_t|^{(\bar{b}+1)/2}}{t^{\beta/2}}$.

We deduce that the condition

$$\mathbb{E}\exp\left(p\int_0^T \left(|\mu_s| + \frac{1}{2n_p\lambda}\,\ell_s^2\right)\mathrm{d}s + p\int_0^T |\nu_s|\,\mathrm{d}A_s\right) < \infty, \quad \text{for all } \ T > 0, \tag{2.13}$$

is satisfied if

$$\frac{3p\tilde{a}}{1-\alpha} \le \frac{1}{2}\,T^{\alpha-2}, \qquad 2\gamma\left(\frac{3p\tilde{c}}{1-\gamma}\right)^{1/\gamma} \le \frac{1}{2T} \qquad \text{and} \qquad \frac{3p\tilde{b}^2}{1-\beta}\,\frac{1}{n_p\lambda} \le T^{\beta-2}, \tag{2.14}$$

since we have (2.2) and

$$\mathbb{E}\Big(\exp\big(\tilde{a}\sup\nolimits_{t\in[0,T]} |B_t|^2\big)\Big) < \infty \quad \text{iff} \quad \tilde{a} < \frac{1}{2T}$$

(for the details see, for instance, [6], Thm. 4.1).

Therefore, under above restrictions, our assumption ($A_6 - 2.6$) is satisfied.

Definition 2.3. The notation $\mathrm{d}K_t \in \partial_y \Psi(t, Y_t)\,\mathrm{d}Q_t$ means that K is an $\mathbb{R}^m$-valued locally bounded variation stochastic process, Q is a real increasing stochastic process, Y is an $\mathbb{R}^m$-valued continuous stochastic process such that $\int_0^T \Psi(t, Y_t)\,\mathrm{d}Q_t < \infty$, $\mathbb{P}$-a.s., for all $T \ge 0$ and, $\mathbb{P}$-a.s., for any $0 \le t \le s$

$$\int_t^s \langle y(r) - Y_r, \mathrm{d}K_r\rangle + \int_t^s \Psi(r, Y_r)\,\mathrm{d}Q_r \le \int_t^s \Psi(r, y(r))\,\mathrm{d}Q_r, \quad \text{for any } y \in C(\mathbb{R}_+; \mathbb{R}^m)\,.$$

Remark 2.4. The condition $0 \in \partial\varphi(0) \cap \partial\psi(0)$ does not restrict the generality of the problem, since from $Dom(\partial\varphi) \cap Dom(\partial\psi) \neq \emptyset$ it follows that there exists $u_0 \in Dom(\partial\varphi) \cap Dom(\partial\psi)$ and $\hat{u}_{01} \in \partial\varphi(u_0)$, $\hat{u}_{02} \in \partial\psi(u_0)$. In this case equation (1.1) is equivalent to

$$\begin{cases} \hat{Y}_t + \displaystyle\int_{t\wedge\tau}^{\tau} \mathrm{d}\hat{K}_s = \eta + \int_{t\wedge\tau}^{\tau} \big[\hat{F}(s,\hat{Y}_s,\hat{Z}_s)\mathrm{d}s + \hat{G}(s,\hat{Y}_s)\mathrm{d}A_s\big] - \int_{t\wedge\tau}^{\tau} \hat{Z}_s \mathrm{d}B_s, \quad \mathbb{P}-\text{a.s.}, \\ \mathrm{d}\hat{K}_t \in \partial\hat{\varphi}(\hat{Y}_t)\mathrm{d}t + \partial\hat{\psi}(\hat{Y}_t)\mathrm{d}A_t, \quad \text{for all } t \ge 0, \end{cases}$$

where

$$\hat{Y}_t := Y_t - u_0\,, \quad \hat{Z}_t := Z_t\,, \quad \hat{\eta} := \eta - u_0$$

and

$$\begin{aligned}
&\hat{F}(s,y,z) = F(t,y+u_0,z) - \hat{u}_{01}\,, \quad \hat{G}(s,y,z) = G(t,y+u_0) - \hat{u}_{02}\,,\\
&\hat{\varphi}(y) = \varphi(y+u_0) - \langle \hat{u}_{01}, y\rangle - \varphi(u_0)\,, \quad \hat{\psi}(y) = \psi(y+u_0) - \langle \hat{u}_{02}, y\rangle - \psi(u_0)\,,\\
&\partial\hat{\varphi}(y) = \partial\varphi(y+u_0) - \hat{u}_{01}\,, \quad \partial\hat{\psi}(y) = \partial\psi(y+u_0) - \hat{u}_{02}\\
&\text{and}\\
&\mathrm{d}\hat{K}_t = \mathrm{d}K_t - \hat{u}_{01}\mathrm{d}t - \hat{u}_{02}\mathrm{d}A_t\,.
\end{aligned}$$

Let $\varepsilon > 0$ and the Moreau-Yosida regularization of φ :

$$\varphi_\varepsilon(y) \overset{\mathrm{def}}{=\!=} \inf\Big\{\frac{1}{2\varepsilon}|y-v|^2 + \varphi(v) : v \in \mathbb{R}^m\Big\}, \tag{2.15}$$

which is a C^1-convex function.

The gradient $\nabla\varphi_\varepsilon(x) = \partial\varphi_\varepsilon(x) \in \partial\varphi(J_\varepsilon(x))\,,$ where $J_\varepsilon(x) \overset{\mathrm{def}}{=\!=} x - \varepsilon\nabla\varphi_\varepsilon(x)$ and the next inequalities are satisfied

$$\begin{aligned}
&(a) \quad |J_\varepsilon(x) - J_\varepsilon(y)| \le |x-y|\,,\\
&(b) \quad |\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y)| \le \tfrac{1}{\varepsilon}\,|x-y|\,,\\
&(c) \quad \varphi_\varepsilon(y) = \tfrac{|y-J_\varepsilon(y)|^2}{2\varepsilon} + \varphi(J_\varepsilon(y))
\end{aligned} \tag{2.16}$$

and

$$-\langle u-v, \nabla\varphi_\varepsilon(u) - \nabla\varphi_\delta(v)\rangle \le (\varepsilon+\delta)\langle\nabla\varphi_\varepsilon(u), \nabla\varphi_\delta(v)\rangle \le \frac{\varepsilon+\delta}{2}\Big[|\nabla\varphi_\varepsilon(u)|^2 + |\nabla\varphi_\delta(v)|^2\Big] \tag{2.17}$$

(for other useful inequalities see, *e.g.*, [14], inequalities (2.8)).

Since $0 \in \partial\varphi(0)$ we deduce that

$$\begin{aligned}
&0 = \varphi(0) \le \varphi(J_\varepsilon(u)) \le \varphi_\varepsilon(u) \le \varphi(u)\,, \quad \text{for any } u \in \mathbb{R}^m,\\
&J_\varepsilon(0) = 0, \quad \nabla\varphi_\varepsilon(0) = 0, \quad \text{and } \varphi_\varepsilon(0) = 0.
\end{aligned} \tag{2.18}$$

We introduce the compatibility conditions between φ, ψ and F, G.

(A_7) For all $\varepsilon > 0$, $t \ge 0$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^{m\times k}$

$$\begin{aligned}
(i) &\quad \langle\nabla\varphi_\varepsilon(y), \nabla\psi_\varepsilon(y)\rangle \ge 0,\\
(ii) &\quad \langle\nabla\varphi_\varepsilon(y), G(t,y)\rangle \le |\nabla\psi_\varepsilon(y)|\,|G(t,y)|\,, \quad \mathbb{P}-\text{a.s.},\\
(iii) &\quad \langle\nabla\psi_\varepsilon(y), F(t,y,z)\rangle \le |\nabla\varphi_\varepsilon(y)|\,|F(t,y,z)|\,, \quad \mathbb{P}-\text{a.s.}.
\end{aligned} \tag{2.19}$$

Example 2.5.
(a) If $\varphi = \psi$ then the compatibility assumptions (2.19) are clearly satisfied.
(b) Let $m = 1$. Since $\nabla\varphi_\varepsilon$ and $\nabla\psi_\varepsilon$ are increasing monotone functions on $\mathbb{R}$, we see that, if $y \cdot G(t,y) \le 0$ and $y \cdot F(t,y,z) \le 0$, for all t, y, z, then the compatibility assumptions (2.19) are satisfied.

(b) Let $m=1$. If $\varphi,\psi:\mathbb{R}\to(-\infty,+\infty]$ are the convexity indicator functions $\varphi(y)=\begin{cases}0, & \text{if } y\in[a,b],\\ +\infty, & \text{if } y\notin[a,b],\end{cases}$ and $\psi(y)=\begin{cases}0, & \text{if } y\in[c,d],\\ +\infty, & \text{if } y\notin[c,d],\end{cases}$ where $-\infty\le a\le b\le\infty$ and $-\infty\le c\le d\le\infty$ are such that $0\in[a,b]\cap[c,d]$ (see (A_6)), then $\nabla\varphi_\varepsilon(y)=\frac{1}{\varepsilon}[(y-b)^+-(a-y)^+]$, and $\nabla\psi_\varepsilon(y)=\frac{1}{\varepsilon}[(y-d)^+-(c-y)^+]$.

Assumption (A_7-i) is clearly fulfilled; the remaining compatibility assumptions are satisfies if, for example, $G(t,y)\ge 0$, for $y\le a$, $G(t,y)\le 0$, for $y\ge b$, and, respectively, $F(t,y,z)\ge 0$, for $y\le c$, $F(t,y,z)\le 0$, for $y\ge d$.

3. Intuitive introduction of L^p-variational solutions

For $a\ge 0$, let us define the space $\mathcal{V}_m^a$ of the m-dimensional local continuous semimartingales M such that for all $T>0$,

$$\mathbb{E}\left(\sup\nolimits_{r\in[0,T]}e^{aV_r}|M_r|^a\right)<\infty,\quad \text{if } a>1 \tag{3.1}$$

and given by

$$\begin{aligned} &M_t=\gamma-\int_0^t N_r\mathrm{d}Q_r+\int_0^t R_r\mathrm{d}B_r\quad\text{or equivalently}\\ &M_t=M_T+\int_t^T N_r\mathrm{d}Q_r-\int_t^T R_r\mathrm{d}B_r\,,\quad M_0=\gamma, \end{aligned} \tag{3.2}$$

where $\gamma\in\mathbb{R}^m$ and $N:\Omega\times\mathbb{R}_+\to\mathbb{R}^m$, $R:\Omega\times\mathbb{R}_+\to\mathbb{R}^{m\times k}$ are p.m.s.p. such that for all $T>0:$

$$\mathbb{E}\Big(\int_0^T e^{V_r}|N_r|\,\mathrm{d}Q_r\Big)^a+\mathbb{E}\Big(\int_0^T e^{2V_r}|R_r|^2\,\mathrm{d}r\Big)^{a/2}<\infty,\quad\text{if } a>0 \tag{3.3}$$

and

$$\int_0^T|N_r|\,\mathrm{d}Q_r+\int_0^T|R_r|^2\,\mathrm{d}r<\infty,\ \mathbb{P}-\text{a.s.},\quad\text{if } a=0.$$

For an intuitive introduction, let (Y,Z,U) be a strong a solution of (1.1) or (1.2), that is $Y,Z,$ and U are p.m.s.p., Y has continuous trajectories,

$$\int_0^T|Z_r|^2\,\mathrm{d}r+\int_0^T|U_r|^2\,\mathrm{d}r<\infty,\quad\mathbb{P}-\text{a.s.},\quad\text{for all } T\ge 0,$$

and the following equation is satisfied, for all $T\ge 0$,

$$\begin{cases} Y_t+\displaystyle\int_t^T\mathrm{d}K_r=Y_T+\int_t^T H(r,Y_r,Z_r)\,\mathrm{d}Q_r-\int_t^T Z_r\mathrm{d}B_r\,,\quad\mathbb{P}-\text{a.s.},\quad\text{for all } t\in[0,T],\\ \mathrm{d}K_r=U_r\mathrm{d}Q_r\in\partial_y\Psi(r,Y_r)\,\mathrm{d}Q_r\,, \end{cases}$$

and

$$e^{V_t}\left|Y_t-\xi_t\right|+\int_t^\infty e^{2V_r}\left|Z_r-\zeta_r\right|^2 \mathrm{d}r \xrightarrow{\ \mathbb{P}\ } 0, \quad \text{as } t\to\infty.$$

For $\delta\in(0,1]$ we define

$$\delta_q \overset{\text{def}}{=\!=} \delta\,\mathbf{1}_{[1,2)}(q)=\begin{cases}\delta, & \text{if } 1\le q<2,\\ 0, & \text{otherwise.}\end{cases} \tag{3.4}$$

Let $q\in[1,2]$, $n_q \overset{\text{def}}{=\!=} (q-1)\wedge 1=q-1$, $M\in\mathcal{V}_m^0$ of form (3.2) and

$$\Gamma_t \overset{\text{def}}{=\!=} \Big(\left|M_t-Y_t\right|^2+\delta_q\Big)^{1/2}. \tag{3.5}$$

By Itô's formula applied to $(\Gamma_t)^q$ we deduce, using inequality (6.4) from Remark 6.2, that, for all $0\le t\le s$ and for all $\delta\in(0,1]$,

$$\begin{aligned}&(\Gamma_t)^q+\frac{q}{2}\int_t^s(\Gamma_r)^{q-4}\Big(n_q\left|M_r-Y_r\right|^2+\delta_q\Big)\left|R_r-Z_r\right|^2\mathrm{d}r-q\int_t^s(\Gamma_r)^{q-2}\left\langle M_r-Y_r,U_r\,\mathrm{d}Q_r\right\rangle\\ &\le(\Gamma_s)^q+q\int_t^s(\Gamma_r)^{q-2}\langle M_r-Y_r,N_r-H(r,Y_r,Z_r)\rangle\mathrm{d}Q_r\\ &\quad-q\int_t^s(\Gamma_r)^{q-2}\left\langle M_r-Y_r,(R_r-Z_r)\,\mathrm{d}B_r\right\rangle,\end{aligned} \tag{3.6}$$

where $U_t\mathrm{d}Q_t\in\partial_y\Psi(t,Y_t)\,\mathrm{d}Q_t$.

Using the subdifferential inequality

$$\langle M_r-Y_r,U_t\mathrm{d}Q_t\rangle+\Psi(r,Y_r)\,\mathrm{d}Q_r\le\Psi(r,M_r)\,\mathrm{d}Q_r$$

we get, from (3.6),

$$\begin{aligned}&(\Gamma_t)^q+\frac{q}{2}\int_t^s(\Gamma_r)^{q-4}\Big(n_q\left|M_r-Y_r\right|^2+\delta_q\Big)\left|R_r-Z_r\right|^2\mathrm{d}r+q\int_t^s(\Gamma_r)^{q-2}\,\Psi(r,Y_r)\,\mathrm{d}Q_r\\ &\le(\Gamma_s)^q+q\int_t^s(\Gamma_r)^{q-2}\,\Psi(r,M_r)\,\mathrm{d}Q_r+q\int_t^s(\Gamma_r)^{q-2}\langle M_r-Y_r,N_r-H(r,Y_r,Z_r)\rangle\mathrm{d}Q_r\\ &\quad-q\int_t^s(\Gamma_r)^{q-2}\left\langle M_r-Y_r,(R_r-Z_r)\,\mathrm{d}B_r\right\rangle.\end{aligned} \tag{3.7}$$

Remark 3.1. Let $\delta>0$, $p>1$ and $q\in\{2,p\wedge 2\}$. We have

$$n_p \overset{\text{def}}{=\!=} (p-1)\wedge 1\le q-1=(q-1)\wedge 1 \overset{\text{def}}{=\!=} n_q\le 1$$

and moreover

– if $q=p\wedge 2$, then $n_q=q-1=(p-1)\wedge 1=n_p$ and $\delta_q=\delta\,\mathbf{1}_{p<2}$;
– if $q=2$, then $n_p\le 1=n_q$ and $\delta_q=0$.

3.1. Definition and preliminary estimates

Following the approach used for the forward stochastic variational inequalities from article [22], we propose, starting from inequality (3.7) and using $n_q(\Gamma_r)^{q-2} \leq (\Gamma_r)^{q-4}\left(n_q\left|M_r - Y_r\right|^2 + \delta_q\right)$, the next variational formulation for a solution of the multivalued BSDE (1.2).

Definition 3.2. We say that $(Y_t, Z_t)_{t\geq 0}$ is an L^p-variational solution of (1.2), where $p > 1$, if:

- $Y : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$ and $Z : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{m\times k}$ are two p.m.s.p., Y has continuous trajectories satisfying
$$\mathbb{E}\left(\sup\nolimits_{r\in[0,\tau]} e^{pV_r}\left|Y_r\right|^p\right) < \infty \tag{3.8}$$
and
$$\mathbb{E}\left(\int_0^\tau e^{2V_r}\left|Z_r\right|^2 \mathrm{d}r\right)^{p/2} + \mathbb{E}\left(\int_0^\tau e^{2V_r}\Psi\left(r, Y_r\right)\mathrm{d}Q_r\right)^{p/2} < \infty, \tag{3.9}$$
where V is defined by (2.10);
- $(Y_t, Z_t) = (\xi_t, \zeta_t) = (\eta, 0)$, for $t > \tau$ and
$$e^{pV_T}\left|Y_T - \xi_T\right|^p + \left(\int_T^\infty e^{2V_s}\left|Z_s - \zeta_s\right|^2 \mathrm{d}s\right)^{p/2} \xrightarrow[T\to\infty]{\mathbb{P}} 0 \tag{3.10}$$
(where (ξ, ζ) are provided by (2.1));
- let Γ_t be defined by (3.5), *i.e.* $\Gamma_t = \left(\left|M_t - Y_t\right|^2 + \delta_q\right)^{1/2}$; then, for every $q \in \{2, p\wedge 2\}$ and $\delta \in (0, 1]$, it holds
$$\begin{aligned}&(\Gamma_t)^q + \frac{q\,(q-1)}{2}\int_t^s (\Gamma_r)^{q-2}\left|R_r - Z_r\right|^2 \mathrm{d}r + q\int_t^s (\Gamma_r)^{q-2}\Psi\left(r, Y_r\right)\mathrm{d}Q_r\\ &\leq (\Gamma_s)^q + q\int_t^s (\Gamma_r)^{q-2}\Psi\left(r, M_r\right)\mathrm{d}Q_r + q\int_t^s (\Gamma_r)^{q-2}\langle M_r - Y_r, N_r - H\left(r, Y_r, Z_r\right)\rangle \mathrm{d}Q_r\\ &\quad -q\int_t^s (\Gamma_r)^{q-2}\langle M_r - Y_r, \left(R_r - Z_r\right)\mathrm{d}B_r\rangle,\end{aligned} \tag{3.11}$$
for any $0 \leq t \leq s < \infty$ and $M \in \mathcal{V}_m^0$ of form (3.2), *i.e.* $M_t = M_T + \int_t^T N_r \mathrm{d}Q_r - \int_t^T R_r \mathrm{d}B_r$.

Remark 3.3. For $q = 2$ inequality (3.11) becomes
$$\begin{aligned}&\left|M_t - Y_t\right|^2 + \int_t^s \left|R_r - Z_r\right|^2 \mathrm{d}r + 2\int_t^s \Psi\left(r, Y_r\right)\mathrm{d}Q_r\\ &\leq \left|M_s - Y_s\right|^2 + 2\int_t^s \Psi\left(r, M_r\right)\mathrm{d}Q_r + 2\int_t^s \langle M_r - Y_r, N_r - H\left(r, Y_r, Z_r\right)\rangle \mathrm{d}Q_r\\ &\quad -2\int_t^s \langle M_r - Y_r, \left(R_r - Z_r\right)\mathrm{d}B_r\rangle, \quad \mathbb{P}-\text{a.s.},\end{aligned} \tag{3.12}$$
which is exactly the definition of the variational solution in the case $p \geq 2$ from article [14] (since, in this case, $q \in \{2, p\wedge 2\}$ means $q = 2$).

Remark 3.4. As we see above, in the case $p \geq 2$, inequality (3.11) from Definition 3.2 should be satisfied only for $q = 2$.

But in the case $1 < p < 2$, we ask inequality (3.11) to be satisfied for $q = p$ and also for $q = 2$ (in order to obtain the existence of a solution $(Y, Z) \in S_m^p \times \Lambda_{m\times k}^p$). This is due to the fact that, without inequality (3.11) accomplished for $q = 2$, we are not able to obtain the estimates for the term $\mathbb{E}\big(\int_0^T e^{2V_r} |Z_r|^2 \mathrm{d}r\big)^{p/2}$ (see inequality (3.17)) and therefore the fact that $t \mapsto \int_0^t e^{pV_r} (\Gamma_r)^{p-2} \langle Y_r, Z_r \mathrm{d}B_r\rangle$ is a martingale (see Prop. 3.7).

Proposition 3.5. *Let $\{L_t : t \geq 0\}$ be an arbitrary continuous bounded variation p.m.s.p.. Then inequality (3.11) from Definition 3.2 is equivalent to*

$$\begin{aligned} & e^{qL_t} (\Gamma_t)^q + q \int_t^s e^{qL_r} (\Gamma_r)^q \,\mathrm{d}L_r + \frac{q}{2} n_q \int_t^s e^{qL_r} (\Gamma_r)^{q-2} |R_r - Z_r|^2 \,\mathrm{d}r \\ & \quad + q \int_t^s e^{qL_r} (\Gamma_r)^{q-2} \Psi(r, Y_r) \,\mathrm{d}Q_r \\ & \leq e^{qL_s} (\Gamma_s)^q + q \int_t^s e^{qL_r} (\Gamma_r)^{q-2} \Psi(r, M_r) \,\mathrm{d}Q_r \\ & \quad + q \int_t^s e^{qL_r} (\Gamma_r)^{q-2} \langle M_r - Y_r, N_r - H(r, Y_r, Z_r)\rangle \mathrm{d}Q_r \\ & \quad - q \int_t^s e^{qL_r} (\Gamma_r)^{q-2} \langle M_r - Y_r, (R_r - Z_r) \,\mathrm{d}B_r\rangle, \end{aligned} \tag{3.13}$$

for any $q \in \{2, p \wedge 2\}$, $\delta \in (0, 1]$, $0 \leq t \leq s < \infty$, and $M \in \mathcal{V}_m^0$ of form (3.2).

Proof. Let $T > 0$ be arbitrary and $0 \leq t \leq s \leq T$. Let $M \in \mathcal{V}_m^0$ of form (3.2) be such that

$$\int_0^T \Psi(r, M_r) \,\mathrm{d}Q_r < \infty, \quad \mathbb{P} - \text{a.s.}.$$

(3.11) $\Longrightarrow$ (3.13):

We remark that the stochastic process

$$\begin{aligned} \Lambda_t \overset{\text{def}}{=\!=} & \frac{q(q-1)}{2} \int_0^t (\Gamma_r)^{q-2} |R_r - Z_r|^2 \,\mathrm{d}r + q \int_0^t (\Gamma_r)^{q-2} \Psi(r, Y_r) \,\mathrm{d}Q_r \\ & - q \int_0^t (\Gamma_r)^{q-2} \Psi(r, M_r) \,\mathrm{d}Q_r - q \int_0^t (\Gamma_r)^{q-2} \langle M_r - Y_r, N_r - H(r, Y_r, Z_r)\rangle \mathrm{d}Q_r \\ & + q \int_0^t (\Gamma_r)^{q-2} \langle M_r - Y_r, (R_r - Z_r) \,\mathrm{d}B_r\rangle \end{aligned}$$

is a local semimartingale, and from (3.11) it follows that

$$t \mapsto (\Gamma_t)^q - \Lambda_t$$

is a continuous nondecreasing stochastic process.

Therefore, $\Gamma^q = [\Gamma^q - \Lambda] + \Lambda$ is a local semimartingale and, for all $0 \leq t \leq s \leq T$,

$$\begin{aligned} e^{qL_s} (\Gamma_s)^q - e^{qL_t} (\Gamma_t)^q &= \int_t^s d\left[e^{qL_r} (\Gamma_r)^q\right] \\ &= q \int_t^s e^{qL_r} (\Gamma_r)^q \,\mathrm{d}L_r + \int_t^s e^{qL_r} d\left[(\Gamma_r)^q - \Lambda_r\right] + \int_t^s e^{qL_r} \mathrm{d}\Lambda_r \end{aligned}$$

$$\geq q\int_t^s e^{qL_r}\left(\Gamma_r\right)^q \mathrm{d}L_r+\int_t^s e^{qL_r}\mathrm{d}\Lambda_r\,,$$

which clearly yields (3.13).

The implication (3.13) $\Longrightarrow$ (3.11) is proved in the same manner. Let

$$\begin{aligned}\tilde{\Lambda}_t=&\,q\int_0^t e^{qL_r}\left(\Gamma_r\right)^q \mathrm{d}L_r+\frac{q\left(q-1\right)}{2}\int_0^t e^{qL_r}(\Gamma_r)^{q-2}\left|R_r-Z_r\right|^2\mathrm{d}r\\&+q\int_0^t e^{qL_r}\left(\Gamma_r\right)^{q-2}\Psi\left(r,Y_r\right)\mathrm{d}Q_r-q\int_0^t e^{qL_r}\left(\Gamma_r\right)^{q-2}\Psi\left(r,M_r\right)\mathrm{d}Q_r\\&-q\int_0^t e^{qL_r}(\Gamma_r)^{q-2}\langle M_r-Y_r,N_r-H\left(r,Y_r,Z_r\right)\rangle\mathrm{d}Q_r\\&+q\int_0^t e^{qL_r}(\Gamma_r)^{q-2}\left\langle M_r-Y_r,\left(R_r-Z_r\right)\mathrm{d}B_r\right\rangle.\end{aligned}$$

Then $t\mapsto\left(e^{L_t}\Gamma_t\right)^q-\tilde{\Lambda}_t$ is a continuous nondecreasing stochastic process and

$$\begin{aligned}\left(\Gamma_s\right)^q-\left(\Gamma_t\right)^q=&\int_t^s\mathrm{d}\left[e^{-qL_r}\left(e^{L_r}\Gamma_r\right)^q\right]\\=&-q\int_t^s e^{-qL_r}\left(e^{L_r}\Gamma_r\right)^q\mathrm{d}L_r+\int_t^s e^{-qL_r}\mathrm{d}\left[\left(e^{L_r}\Gamma_r\right)^q-\tilde{\Lambda}_r\right]+\int_t^s e^{-qL_r}\mathrm{d}\tilde{\Lambda}_r\\\geq&-q\int_t^s\left(\Gamma_r\right)^q\mathrm{d}L_r+\int_t^s e^{-qL_r}\mathrm{d}\tilde{\Lambda}_r\,.\end{aligned}$$

□

Remark 3.6. In the following results we will often use the continuous bounded variation p.m.s.p. $\{V_t:t\geq 0\}$ given by (2.10) in the place of $\{L_t:t\geq 0\}$.

Proposition 3.7. *Let $(Y_t,Z_t)_{t\geq 0}$ an L^p-variational solution in the sense of Definition 3.2 and $q=p\wedge 2$, and $M\in\mathcal{V}_m^q$ of form (3.2). Then*

$$t\mapsto\int_0^t e^{qV_r}\left(\Gamma_r\right)^{q-2}\left\langle M_r-Y_r,\left(R_r-Z_r\right)\mathrm{d}B_r\right\rangle$$

is a continuous martingale.

If moreover

$$\mathbb{E}\Big(\int_0^T e^{V_r}\left|H\left(r,Y_r,Z_r\right)\right|\mathrm{d}Q_r\Big)^{p\wedge 2}<\infty,$$

then, for all $T \geq 0$, $M \in \mathcal{V}_m^q$ of form (3.2), and for all stopping times $0 \leq \sigma \leq \theta \leq T$:

$$\begin{aligned}
& e^{qV_\sigma} (\Gamma_\sigma)^q + q\,\mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} (\Gamma_r)^q \,\mathrm{d}V_r + \frac{q(q-1)}{2}\,\mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} (\Gamma_r)^{q-2} |R_r - Z_r|^2 \,\mathrm{d}r \\
& \quad + q\,\mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} (\Gamma_r)^{q-2} \Psi(r, Y_r) \,\mathrm{d}Q_r \\
& \leq \mathbb{E}^{\mathcal{F}_\sigma} e^{qV_\theta} (\Gamma_\theta)^q + q\,\mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} (\Gamma_r)^{q-2} \Psi(r, M_r) \,\mathrm{d}Q_r \\
& \quad + q\,\mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} (\Gamma_r)^{q-2} \langle M_r - Y_r, N_r - H(r, Y_r, Z_r)\rangle \mathrm{d}Q_r, \quad \mathbb{P}-a.s..
\end{aligned} \tag{3.14}$$

Proof. We have

$$\begin{aligned}
& \mathbb{E}\left[\int_0^T e^{2qV_r} (\Gamma_r)^{2q-4} |M_r - Y_r|^2 |R_r - Z_r|^2 \,\mathrm{d}r\right]^{1/2} \leq \mathbb{E}\left[\int_0^T e^{2qV_r} (\Gamma_r)^{2q-2} |R_r - Z_r|^2 \,\mathrm{d}r\right]^{1/2} \\
& \leq \mathbb{E}\left[\sup\nolimits_{r\in[0,T]} \left(e^{(q-1)V_r} \big(|M_r - Y_r|^2 + \delta_q\big)^{(q-1)/2}\right) \cdot \left(\int_0^T e^{2V_r} |R_r - Z_r|^2 \,\mathrm{d}r\right)^{1/2}\right] \\
& \leq \left[\mathbb{E}\left(\sup\nolimits_{r\in[0,T]} e^{qV_r} \left(|M_r - Y_r|^2 + \delta_q\right)^{q/2}\right)\right]^{(q-1)/q} \left[\mathbb{E}\left(\int_0^T e^{2V_r} |R_r - Z_r|^2 \,\mathrm{d}r\right)^{q/2}\right]^{1/q} \\
& < \infty,
\end{aligned}$$

since, from (2.12),

$$(\delta_q)^{q/2}\, \mathbb{E}\left(\sup\nolimits_{r\in[0,T]} e^{qV_r}\right) < \infty, \quad \text{for all } T > 0$$

and inequalities (3.1),(3.3), (3.8) and (3.9) hold.

Consequently, the stochastic integral $t \mapsto \int_0^t e^{qV_r} (\Gamma_r)^{q-2} \langle M_r - Y_r, (R_r - Z_r) \,\mathrm{d}B_r\rangle$ is a continuous martingale. We also have

$$\begin{aligned}
& \mathbb{E}\int_0^T e^{qV_r} (\Gamma_r)^{q-2} |\langle M_r - Y_r, N_r - H(r, Y_r, Z_r)\rangle| \,\mathrm{d}Q_r \\
& \leq \mathbb{E}\int_0^T e^{qV_r} (\Gamma_r)^{q-1} [|N_r| + |H(r, Y_r, Z_r)|] \,\mathrm{d}Q_r \\
& \leq \mathbb{E}\left[\sup\nolimits_{r\in[0,T]} \left(e^{(q-1)V_r} \left(|M_r - Y_r|^2 + \delta_q\right)^{(q-1)/2}\right) \cdot \left(\int_0^T e^{V_r} [|N_r| + |H(r, Y_r, Z_r)|] \,\mathrm{d}Q_r\right)\right] \\
& \leq \left[\mathbb{E}\left(\sup\nolimits_{r\in[0,T]} e^{qV_r} \left(|M_r - Y_r|^2 + \delta_q\right)^{q/2}\right)\right]^{(q-1)/q} \\
& \quad \cdot \left[\mathbb{E}\left(\int_0^T e^{V_r} [|N_r| + |H(r, Y_r, Z_r)|] \,\mathrm{d}Q_r \mathrm{d}r\right)^q\right]^{1/q} \\
& < \infty.
\end{aligned}$$

Hence, using inequality (3.13) with $L = V$, inequality (3.14) follows. ∎

Remark 3.8. From Section 3 (see the proof of inequality (3.7)) we see that a strong solution $(Y,Z,K) \in S_m^0 \times \Lambda_{m\times k}^0 \times \Lambda_m^0$ of BSDE (1.2), such that (3.8), (3.9) and (3.10) are satisfied, is also an L^p-variational solution. Conversely, we have the next result.

Corollary 3.9. *If (Y,Z) is an L^p-variational solution of BSDE (1.2) with $\varphi = \psi = 0$, V is a continuous nondecreasing process and*

$$\mathbb{E}\Big(\int_0^T e^{V_r} \left| H\left(r, Y_r, Z_r\right) \right| \mathrm{d}Q_r \Big)^{p\wedge 2} < \infty, \quad \text{for all } T > 0,$$

then $(Y,Z,0)$ is a strong solution of BSDE (1.2).

Proof. By Corollary 2.45 of [21] there exists a unique $(M,R) \in S_m^q\left[0,T\right] \times \Lambda_{m\times k}^q\left(0,T\right)$ such that

$$M_t = Y_T + \int_t^T H\left(r, Y_r, Z_r\right) \mathrm{d}Q_r - \int_t^T R_r \mathrm{d}B_r$$

and

$$\begin{aligned} &\mathbb{E}\sup\nolimits_{r\in[0,T]} e^{qV_r} \left|M_r\right|^q + \mathbb{E}\Big(\int_0^T e^{2V_r} \left|R_r\right|^2 \mathrm{d}r \Big)^{\frac{q}{2}} \\ &\leq C_q\, \mathbb{E}\left[e^{qV_T} \left|Y_T\right|^q + \Big(\int_0^T e^{V_r} \left|H\left(r,Y_r,Z_r\right)\right| \mathrm{d}Q_r \Big)^q \right] \end{aligned}$$

for $q = p \wedge 2$.

With this M inequality (3.13) becomes (since $\Psi = 0$) $\mathbb{P}$-a.s.

$$\begin{aligned} &e^{qV_t}\left(\Gamma_t\right)^q + q\int_t^s e^{qV_r}\left(\Gamma_r\right)^q \mathrm{d}V_r + \frac{q}{2} n_q \int_t^s e^{qV_r}\left(\Gamma_r\right)^{q-2} \left|R_r - Z_r\right|^2 \mathrm{d}r \\ &\leq e^{qV_s}\left(\Gamma_s\right)^q - q\int_t^s e^{qV_r}\left(\Gamma_r\right)^{q-2} \langle M_r - Y_r, \left(R_r - Z_r\right) \mathrm{d}B_r \rangle, \end{aligned}$$

for any $\delta \in (0,1]$ and $0 \leq t \leq s < \infty$.

By Proposition 3.7, the stochastic integral is a martingale and therefore we obtain, using the last inequality with $s = T$, for all $0 \leq t \leq T$, $\mathbb{P}$-a.s.

$$e^{qV_t}\left(\Gamma_t\right)^q + \frac{q}{2} n_q\, \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{qV_r} \frac{1}{\left(\left|M_r - Y_r\right|^2 + 1\right)^{(2-q)/2}} \left|R_r - Z_r\right|^2 \mathrm{d}r \leq \left(\delta_q\right)^{q/2} \mathbb{E}^{\mathcal{F}_t} e^{qV_T}, \tag{3.15}$$

since $M_T = Y_T$ and V is nondecreasing.

Passing to limit as $\delta \to 0_+$ we obtain, by Fatou's Lemma, for all $0 \leq t \leq T$, $\mathbb{P}$-a.s.

$$e^{qV_t}\left|M_t - Y_t\right|^q + \frac{q\left(q-1\right)}{2} \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{qV_r} \frac{1}{\left(\left|M_r - Y_r\right|^2 + 1\right)^{(2-q)/2}} \left|R_r - Z_r\right|^2 \mathrm{d}r = 0,$$

which clearly yields $(M,R) = (Y,Z)$ in $S_m^q\left[0,T\right] \times \Lambda_{m\times k}^q\left(0,T\right)$, hence

$$Y_t = Y_T + \int_t^T H\left(r, Y_r, Z_r\right) \mathrm{d}Q_r - \int_t^T Z_r \mathrm{d}B_r\,.$$

□

Proposition 3.10. *Let $M \in \mathcal{V}_m^0$ of form (3.2). Let $Y : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$ and $Z : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{m\times k}$ be two p.m.s.p. such that Y has continuous trajectories and $\mathbb{P}$-a.s.,*

$$\begin{array}{rl} (i) & \displaystyle\int_0^T e^{2V_r} |R_r - Z_r|^2 \,\mathrm{d}r + \int_0^T e^{2V_r} \Psi(r, Y_r) \,\mathrm{d}Q_r < \infty, \quad \textit{for all } T > 0, \\ (ii) & \Psi(r, M_r) \le \mathbf{1}_{q\ge 2} \Psi(r, M_r), \\ (iii) & \langle M_r - Y_r, N_r \rangle \,\mathrm{d}Q_r \le |M_r - Y_r| \,\mathrm{d}L_r, \quad \textit{a.e. } r \in [0, T], \end{array}$$

with L an increasing and continuous p.m.s.p. with $L_0 = 0$.
I. *If inequality (3.11) holds for $q = 2$, then, for all $a > 0$ and for any stopping times $0 \le \sigma \le \theta < \infty$,*

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^\theta e^{2V_r} |R_r - Z_r|^2 \,\mathrm{d}r\Big)^{a/2} + \mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^\theta e^{2V_r} \Psi(r, Y_r) \,\mathrm{d}Q_r\Big)^{a/2} \\ &\le C_{a,\lambda}\, \mathbb{E}^{\mathcal{F}_\sigma}\Big[\sup\nolimits_{r\in[\sigma,\theta]} e^{aV_r} |M_r - Y_r|^a + \Big(\int_\sigma^\theta e^{V_r} \Psi(r, M_r) \,\mathrm{d}Q_r\Big)^{a/2} \\ &\quad + \Big(\int_\sigma^\theta e^{V_r} |M_r - Y_r| \left[\mathrm{d}L_r + |H(r, M_r, R_r)| \,\mathrm{d}Q_r\right]\Big)^{a/2}\Big] \\ &\le 2C_{a,\lambda}\, \mathbb{E}^{\mathcal{F}_\sigma}\Big[\sup\nolimits_{r\in[\sigma,\theta]} e^{aV_r} |M_r - Y_r|^a + \Big(\int_\sigma^\theta e^{V_r} \Psi(r, M_r) \,\mathrm{d}Q_r\Big)^{a/2} \\ &\quad + \Big(\int_\sigma^\theta e^{V_r} \left[\mathrm{d}L_r + |H(r, M_r, R_r)| \,\mathrm{d}Q_r\right]\Big)^{a}\Big], \quad \mathbb{P}-a.s.. \end{aligned} \tag{3.16}$$

In particular, for $\gamma = 0, N = 0, R = 0$ (hence $M = 0$) and $L = 0$, it follows

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^\theta e^{2V_r} |Z_r|^2 \,\mathrm{d}r\Big)^{a/2} + \mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^\theta e^{2V_r} \Psi(r, Y_r) \,\mathrm{d}Q_r\Big)^{a/2} \\ &\le C_{a,\lambda}\, \mathbb{E}^{\mathcal{F}_\sigma}\Big[\sup\nolimits_{r\in[\sigma,\theta]} e^{aV_r} |Y_r|^a + \Big(\int_\sigma^\theta e^{V_r} |Y_r| \,|H(r, 0, 0)| \,\mathrm{d}Q_r\Big)^{a/2}\Big] \\ &\le 2C_{a,\lambda}\, \mathbb{E}^{\mathcal{F}_\sigma}\Big[\mathbb{E}^{\mathcal{F}_\sigma} \sup\nolimits_{r\in[\sigma,\theta]} e^{aV_r} |Y_r|^a + \Big(\int_\sigma^\theta e^{V_r} |H(r, 0, 0)| \,\mathrm{d}Q_r\Big)^{a}\Big], \quad \mathbb{P}-a.s.. \end{aligned} \tag{3.17}$$

II. *If inequality (3.11) holds and for some fixed stopping times $0 \le \sigma \le \theta < \infty$, $1 < q \le a$*

$$\mathbb{E}\left(\sup\nolimits_{r\in[\sigma,\theta]} e^{aV_r} |M_r - Y_r|^a\right) < \infty, \tag{3.18}$$

then

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_\sigma} \sup\nolimits_{r\in[\sigma,\theta]} e^{aV_r} |M_r - Y_r|^a \\ &\le C_{\lambda,q,a}\, \mathbb{E}^{\mathcal{F}_\sigma}\Big[e^{aV_\theta} |M_\theta - Y_\theta|^a + \Big(\int_\sigma^\theta e^{V_r} |M_r - Y_r|^{q-2} \,\mathbf{1}_{q\ge 2}\, \Psi(r, M_r) \,\mathrm{d}Q_r\Big)^{a/q} \\ &\quad + \Big(\int_\sigma^\theta e^{qV_r} |M_r - Y_r|^{q-1} \left[\mathrm{d}L_r + |H(r, M_r, R_r)| \,\mathrm{d}Q_r\right]\Big)^{a/q}\Big], \quad \mathbb{P}-a.s. \end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_\sigma}\Big(\sup\nolimits_{r\in[\sigma,\theta]} e^{aV_r}\left|M_r-Y_r\right|^a\Big)+\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^\theta e^{qV_r}\left|M_r-Y_r\right|^{q-2}\mathbf{1}_{M_r\neq Y_r}\left|R_r-Z_r\right|^2\mathrm{d}r\Big)^{a/q}\\
&\quad+\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^\theta e^{qV_r}\left|M_r-Y_r\right|^{q-2}\mathbf{1}_{M_r\neq Y_r}\Psi\left(r,Y_r\right)\mathrm{d}Q_r\Big)^{a/q}\\
&\leq C_{\lambda,q,a}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[e^{aV_\theta}\left|M_\theta-Y_\theta\right|^a+\Big(\int_\sigma^\theta e^{V_r}\mathbf{1}_{q\geq2}\Psi\left(r,M_r\right)\mathrm{d}Q_r\Big)^{a/2}\\
&\quad+\Big(\int_\sigma^\theta e^{V_r}\left[\mathrm{d}L_r+\left|H\left(r,M_r,R_r\right)\right|\mathrm{d}Q_r\right]\Big)^a\Big],\quad\mathbb{P}-a.s..
\end{aligned}\tag{3.20}$$

In particular, for $\gamma=0,N=0,R=0$ *(hence* $M=0$*) and* $L=0$*, it follows*

$$\mathbb{E}^{\mathcal{F}_\sigma}\sup\nolimits_{r\in[\sigma,\theta]} e^{aV_r}\left|Y_r\right|^a\leq C_{\lambda,q,a}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[e^{aV_\theta}\left|Y_\theta\right|^a+\Big(\int_\sigma^\theta e^{qV_r}\left|Y_r\right|^{q-1}\left|H\left(r,0,0\right)\right|\mathrm{d}Q_r\Big)^{a/q}\Big]\tag{3.21}$$

and

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_\sigma}\Big(\sup\nolimits_{r\in[\sigma,\theta]} e^{aV_r}\left|Y_r\right|^a\Big)+\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^\theta e^{qV_r}\left|Y_r\right|^{q-2}\mathbf{1}_{M_r\neq Y_r}\left|Z_r\right|^2\mathrm{d}r\Big)^{a/q}\\
&\quad+\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^\theta e^{qV_r}\left|Y_r\right|^{q-2}\mathbf{1}_{M_r\neq Y_r}\Psi\left(r,Y_r\right)\mathrm{d}Q_r\Big)^{a/q}\\
&\leq C_{\lambda,q,a}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[e^{aV_\theta}\left|Y_\theta\right|^a+\Big(\int_\sigma^\theta e^{V_r}\left|H\left(r,0,0\right)\right|\mathrm{d}Q_r\Big)^a\Big].
\end{aligned}\tag{3.22}$$

Proof. Using the monotonicity of H we have

$$\begin{aligned}
&\langle M_r-Y_r,-H\left(r,Y_r,Z_r\right)\mathrm{d}Q_r\rangle\\
&=\langle M_r-Y_r,-H\left(r,M_r,R_r\right)\mathrm{d}Q_r\rangle+\langle M_r-Y_r,H\left(r,M_r,R_r\right)-H\left(r,Y_r,Z_r\right)\mathrm{d}Q_r\rangle\\
&\leq\left|M_r-Y_r\right|\left|H\left(r,M_r,R_r\right)\right|\mathrm{d}Q_r+\left|M_r-Y_r\right|^2\mathrm{d}V_r+\frac{n_p\lambda}{2}\left|R_r-Z_r\right|^2\mathrm{d}s\\
&=\left|M_r-Y_r\right|\left|H\left(r,M_r,R_r\right)\right|\mathrm{d}Q_r+\left(\Gamma_r\right)^2\mathrm{d}V_r-\delta_q\,\mathrm{d}V_r+\frac{n_p\lambda}{2}\left|R_r-Z_r\right|^2\mathrm{d}s.
\end{aligned}$$

If we suppose that (3.11) is satisfied, then

$$\begin{aligned}
&\left(\Gamma_t\right)^q+\frac{q\left(n_q-n_p\lambda\right)}{2}\int_t^s\left(\Gamma_r\right)^{q-2}\left|R_r-Z_r\right|^2\mathrm{d}r\\
&\quad+q\delta_q\int_t^s\left(\Gamma_r\right)^{q-2}\mathrm{d}V_r+q\int_t^s\left(\Gamma_r\right)^{q-2}\Psi\left(r,Y_r\right)\mathrm{d}Q_r\\
&\leq\left(\Gamma_s\right)^q+q\int_t^s\left(\Gamma_r\right)^q\mathrm{d}V_r+q\int_t^s\left(\Gamma_r\right)^{q-2}\mathbf{1}_{q\geq2}\Psi\left(r,M_r\right)\mathrm{d}Q_r\\
&\quad+q\int_t^s\left(\Gamma_r\right)^{q-2}\left|M_r-Y_r\right|\left[\mathrm{d}L_r+\left|H\left(r,M_r,R_r\right)\right|\mathrm{d}Q_r\right]\\
&\quad-q\int_t^s\left(\Gamma_r\right)^{q-2}\langle M_r-Y_r,\left(R_r-Z_r\right)\mathrm{d}B_r\rangle.
\end{aligned}$$

for all $0 \le t \le s < \infty$.

Since $q \in \{2, p \wedge 2\}$, we have (see Rem. 3.1)

$$n_q - n_p\lambda \ge n_q (1-\lambda) = (q-1)(1-\lambda)$$

and therefore we deduce, using a stochastic Gronwall's type inequality for the previous inequality (see, for instance, [16], Lem. 12),

$$\begin{aligned} & e^{qV_t} (\Gamma_t)^q + \frac{q}{2} n_q (1-\lambda) \int_t^s e^{qV_r} (\Gamma_r)^{q-2} |R_r - Z_r|^2 \mathrm{d}r + q\,\delta_q \int_t^s e^{qV_r} (\Gamma_r)^{q-2} \mathrm{d}V_r \\ & \quad + q \int_t^s e^{qV_r} (\Gamma_r)^{q-2} \Psi(r, Y_r) \mathrm{d}Q_r \\ & \le e^{qV_s} (\Gamma_s)^q + q \int_t^s e^{qV_r} (\Gamma_r)^{q-2} \mathbf{1}_{q\ge 2} \Psi(r, M_r) \mathrm{d}Q_r \\ & \quad + q \int_t^s e^{qV_r} (\Gamma_r)^{q-2} |M_r - Y_r| [\mathrm{d}L_r + |H(r, M_r, R_r)| \mathrm{d}Q_r] \\ & \quad - q \int_t^s e^{qV_r} (\Gamma_r)^{q-2} \langle M_r - Y_r, (R_r - Z_r) \mathrm{d}B_r \rangle. \end{aligned} \tag{3.23}$$

I. Writing (3.23) for $q = 2$ we get

$$\begin{aligned} & e^{2V_t} |M_t - Y_t|^2 + (1-\lambda) \int_t^s e^{2V_r} |R_r - Z_r|^2 \mathrm{d}r + 2 \int_t^s e^{2V_r} \Psi(r, Y_r) \mathrm{d}Q_r \\ & \le e^{2V_s} |M_s - Y_s|^2 + 2 \int_t^s e^{qV_r} \Psi(r, M_r) \mathrm{d}Q_r + 2 \int_t^s e^{2V_r} |M_r - Y_r| [\mathrm{d}L_r + |H(r, M_r, R_r)| \mathrm{d}Q_r] \\ & \quad - 2 \int_t^s e^{2V_r} \langle M_r - Y_r, (R_r - Z_r) \mathrm{d}B_r \rangle, \ \mathbb{P}-\text{a.s.}, \end{aligned}$$

for all $0 \le t \le s < \infty$, which yields (3.16), by Proposition 6.3 from the Appendix.

II. Using Fatou's Lemma, Lebesgue dominated convergence theorem and the continuity in probability of the stochastic integral we clearly deduce from (3.23), as $\delta \to 0_+$, that:

$$\begin{aligned} & e^{qV_t} |M_t - Y_t|^q + \frac{q}{2} (q-1)(1-\lambda) \int_t^s e^{qV_r} |M_r - Y_r|^{q-2} \mathbf{1}_{M_r \ne Y_r} |R_r - Z_r|^2 \mathrm{d}r \\ & \quad + q \int_t^s e^{qV_r} |M_r - Y_r|^{q-2} \mathbf{1}_{M_r \ne Y_r} \Psi(r, Y_r) \mathrm{d}Q_r \\ & \le e^{qV_s} |M_s - Y_s|^q + q \int_t^s e^{qV_r} |M_r - Y_r|^{q-2} \mathbf{1}_{q \ge 2} \Psi(r, M_r) \mathrm{d}Q_r \\ & \quad + q \int_t^s e^{qV_r} |M_r - Y_r|^{q-1} [\mathrm{d}L_r + |H(r, M_r, R_r)| \mathrm{d}Q_r] \\ & \quad - q \int_t^s e^{qV_r} |M_r - Y_r|^{q-2} \langle M_r - Y_r, (R_r - Z_r) \mathrm{d}B_r \rangle, \end{aligned} \tag{3.24}$$

since $\Gamma_r \xrightarrow[\delta \to 0_+]{} |M_r - Y_r| \mathbf{1}_{M_r \ne Y_r}$, $\mathbb{P}$-a.s., for all $r \ge 0$.

Using Proposition 6.4 and Remark 6.5 we get (3.19) and (3.20). □

4. Uniqueness and continuity of L^p-variational solutions

In order to prove the uniqueness of the variational solution it is sufficient that inequality (3.11) from Definition 3.2 is satisfied only for $q = p \wedge 2$. The both cases $q \in \{p \wedge 2, 2\}$ are necessarily for the existence of a solution (the case $q = 2$ is needed for obtaining the estimates for the term $\mathbb{E}\big(\int_0^T e^{2V_r}|Z_r|^2\,\mathrm{d}r\big)^{p/2}$ (see also Rem. 3.4).

Theorem 4.1 (Continuity)**.** *We suppose that assumptions* $(\mathrm{A}_1 - \mathrm{A}_6)$ *are satisfied. Let* $(\hat{Y}, \hat{Z}), (\tilde{Y}, \tilde{Z})$ *be two* L^p*-variational solutions of (1.2) corresponding to* $(\hat{\eta}, \hat{H})$ *and* $(\tilde{\eta}, \tilde{H})$ *respectively, where* $\hat{H}$ *and* $\tilde{H}$ *have the same coefficients* μ, ν, ℓ.

Then, for any stopping time $0 \le \sigma \le \tau$, *it holds,* $\mathbb{P}$*-a.s.,*

$$\begin{aligned}
& e^{qV_\sigma}|\hat{Y}_\sigma - \tilde{Y}_\sigma|^q + c_{q,\lambda}\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\tau e^{qV_r}|\hat{Y}_r - \tilde{Y}_r|^{q-2}\,\mathbf{1}_{\hat{Y}_r \neq \tilde{Y}_r}\,|\hat{Z}_r - \tilde{Z}_r|^2\mathrm{d}r \\
& \le \mathbb{E}^{\mathcal{F}_\sigma} e^{qV_\tau}|\hat{\eta} - \tilde{\eta}|^q + q\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\tau e^{qV_r}|\hat{Y}_r - \tilde{Y}_r|^{q-1}\big|\hat{H}(r, \hat{Y}_r, \hat{Z}_r) - \tilde{H}(r, \hat{Y}_r, \hat{Z}_r)\big|\mathrm{d}Q_r \\
& \le \mathbb{E}^{\mathcal{F}_\sigma} e^{qV_\tau}|\hat{\eta} - \tilde{\eta}|^q \\
& \quad + C_{q,\lambda}\left[\mathbb{E}^{\mathcal{F}_\sigma}\left(\Lambda_{\sigma,\tau}\right)\right]^{(q-1)/q}\cdot\left[\mathbb{E}^{\mathcal{F}_\sigma}\left(\int_\sigma^\tau e^{V_r}\big|\hat{H}(r, \hat{Y}_r, \hat{Z}_r) - \tilde{H}(r, \hat{Y}_r, \hat{Z}_r)\big|\mathrm{d}Q_r\right)^q\right]^{1/q},
\end{aligned} \tag{4.1}$$

where

$$\Lambda_{\sigma,\tau} = e^{qV_\tau}|\hat{\eta}|^q + e^{qV_\tau}|\tilde{\eta}|^q + \Big(\int_\sigma^\tau e^{V_r}\big|\hat{H}(r,0,0)\big|\mathrm{d}Q_r\Big)^q + \Big(\int_\sigma^\tau e^{V_r}\big|\tilde{H}(r,0,0)\big|\mathrm{d}Q_r\Big)^q \tag{4.2}$$

and $c_{q,\lambda}, C_{q,\lambda} > 0$.

Moreover, for all $0 < \alpha < 1$,

$$\begin{aligned}
& \mathbb{E}\sup\nolimits_{t\in[0,\tau]} e^{\alpha q V_t}|\hat{Y}_t - \tilde{Y}_t|^{\alpha q} + (q-1)\,\mathbb{E}\bigg(\int_0^\tau \frac{1}{\big(e^{V_r}|\hat{Y}_r - \tilde{Y}_r| + 1\big)^{2-q}}\,e^{2V_r}|\hat{Z}_r - \tilde{Z}_r|^2\mathrm{d}r\bigg)^\alpha \\
& \le \mathbb{E}\sup\nolimits_{t\in[0,\tau]} e^{\alpha q V_t}|\hat{Y}_t - \tilde{Y}_t|^{\alpha q} + (q-1)\,\mathbb{E}\bigg(\int_0^\tau e^{qV_r}|\hat{Y}_r - \tilde{Y}_r|^{q-2}\,|\hat{Z}_r - \tilde{Z}_r|^2\mathrm{d}r\bigg)^\alpha \\
& \le C_{\alpha,q,\lambda}\left(\mathbb{E}e^{qV_\tau}|\hat{\eta} - \tilde{\eta}|^q\right)^\alpha \\
& \quad + C_{\alpha,q,\lambda}\left[\mathbb{E}\left(\Lambda_{\sigma,\tau}\right)\right]^{\alpha(q-1)/q}\cdot\bigg(\mathbb{E}\bigg(\int_0^\tau e^{V_r}\big|\hat{H}(r, \hat{Y}_r, \hat{Z}_r) - \tilde{H}(r, \hat{Y}_r, \hat{Z}_r)\big|\mathrm{d}Q_r\bigg)^q\bigg)^{\alpha/q},
\end{aligned} \tag{4.3}$$

where $\Lambda_{\sigma,\tau}$ *is given by (4.2) and* $C_{\alpha,q,\lambda} > 0$.

Corollary 4.2. *Under the assumptions of Theorem 4.1 the uniqueness of the* L^p*-variational solution holds.*

Proof. The uniqueness property of the L^p-variational solution follows clearly from continuity property (4.3). Indeed, if $\hat{\eta} = \tilde{\eta}$ and $\hat{H} = \tilde{H}$, we conclude, from (4.3), in the case $q > 1$, that $\hat{Y} = \tilde{Y}$ in S_m^0 and $\hat{Z} = \tilde{Z}$ in $\Lambda_{m\times k}^0$. □

Remark 4.3. We also notice that both results (uniqueness by continuity property (4.3) and existence of a variational solution, by Thm. 5.5) hold true in the case of infinite dimensional spaces without any major change in their proofs; the suitable framework is that from [14].

Proof. **Step I.** *Obtaining of inequality (4.8) for an approximation* M^ε

Let $M \in \mathcal{V}_m^q$ of form (3.2) and, following (3.5), we define

$$\hat{\Gamma}_t = \big(|M_t - \hat{Y}_t|^2 + \delta_q\big)^{1/2} \quad \text{and} \quad \tilde{\Gamma}_t = \big(|M_t - \tilde{Y}_t|^2 + \delta_q\big)^{1/2}.$$

Let $T > 0$ and the stopping times $0 \leq \sigma \leq \theta \leq T \wedge \tau$ such that

$$\mathbb{E}\bigg(\int_\sigma^\theta e^{V_r} \Big[|\hat{H}(r, \hat{Y}_r, \hat{Z}_r)| + |\tilde{H}(r, \tilde{Y}_r, \tilde{Z}_r)| \Big] \, \mathrm{d}Q_r \bigg)^q < \infty. \tag{4.4}$$

Using (3.14) (which is an equivalent form of (3.11)), we deduce that

$$\begin{aligned}
&\big[e^{qV_\sigma}(\hat{\Gamma}_\sigma)^q + e^{qV_\sigma}(\tilde{\Gamma}_\sigma)^q\big] + q\,\mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} \big[(\hat{\Gamma}_r)^q + (\tilde{\Gamma}_r)^q\big] \mathrm{d}V_r \\
&\quad + \frac{q}{2}\, n_q\, \mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} \Big[(\hat{\Gamma}_r)^{q-2} \big|R_r - \hat{Z}_r\big|^2 + (\tilde{\Gamma}_r)^{q-2} \big|R_r - \tilde{Z}_r\big|^2 \Big] \, \mathrm{d}r \\
&\quad + q\,\mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} \big[(\hat{\Gamma}_r)^{q-2} \Psi\big(r, \hat{Y}_r\big) + (\tilde{\Gamma}_r)^{q-2} \Psi\big(r, \tilde{Y}_r\big) \big] \mathrm{d}Q_r \\
&\leq \mathbb{E}^{\mathcal{F}_\sigma} \big[e^{qV_\theta}(\hat{\Gamma}_\theta)^q + e^{qV_\theta}(\tilde{\Gamma}_\theta)^q \big] + q\,\mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} \big[(\hat{\Gamma}_r)^{q-2} + (\tilde{\Gamma}_r)^{q-2} \big] \Psi\left(r, M_r\right) \mathrm{d}Q_r \\
&\quad + q\,\mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} (\hat{\Gamma}_r)^{q-2} \langle M_r - \hat{Y}_r, N_r - \hat{H}(r, \hat{Y}_r, \hat{Z}_r) \rangle \mathrm{d}Q_r \\
&\quad + q\,\mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} (\tilde{\Gamma}_r)^{q-2} \langle M_r - \tilde{Y}_r, N_r - \tilde{H}(r, \tilde{Y}_r, \tilde{Z}_r) \rangle \mathrm{d}Q_r \,, \quad \mathbb{P}-\text{a.s.}.
\end{aligned} \tag{4.5}$$

Let

$$Y_r \overset{\text{def}}{=\!=} \frac{1}{2}\big(\hat{Y}_r + \tilde{Y}_r\big).$$

We have, for all $\beta > 0$, by Young's inequality[1],

$$\begin{aligned}
|M_r - \hat{Y}_r|^2 &\leq \frac{1+\beta}{\beta}\, |M_r - Y_r|^2 + \frac{1+\beta}{4}\, |\hat{Y}_r - \tilde{Y}_r|^2, \\
|M_r - \tilde{Y}_r|^2 &\leq \frac{1+\beta}{\beta}\, |M_r - Y_r|^2 + \frac{1+\beta}{4}\, |\hat{Y}_r - \tilde{Y}_r|^2
\end{aligned}$$

and therefore

$$\begin{aligned}
&(\hat{\Gamma}_r)^{q-2} \big|R_r - \hat{Z}_r\big|^2 + (\tilde{\Gamma}_r)^{q-2} \big|R_r - \tilde{Z}_r\big|^2 \\
&= \big(|M_r - \hat{Y}_r|^2 + \delta_q\big)^{(q-2)/2} \big|R_r - \hat{Z}_r\big|^2 + \big(|M_r - \tilde{Y}_r|^2 + \delta_q\big)^{(q-2)/2} \big|R_r - \tilde{Z}_r\big|^2 \\
&\geq \left[\frac{1+\beta}{\beta}\, |M_r - Y_r|^2 + \frac{1+\beta}{4}\, |\hat{Y}_r - \tilde{Y}_r|^2 + \delta_q \right]^{(q-2)/2} \Big[\big|R_r - \hat{Z}_r\big|^2 + \big|R_r - \tilde{Z}_r\big|^2 \Big],
\end{aligned}$$

[1] Indeed, for any $a, b \in \mathbb{R}$ and any $\beta > 0$, we have, by Young's inequality,

$$(a+b)^2 = a^2 + b^2 + 2ab \leq a^2 + b^2 + \frac{1}{\beta}\, a^2 + \beta\, b^2 = \left(1 + \frac{1}{\beta}\right) a^2 + (1+\beta)\, b^2.$$

since $1<q\leq 2$.

Hence

$$\begin{aligned}&(\hat{\Gamma}_r)^{q-2}\big|R_r-\hat{Z}_r\big|^2+(\tilde{\Gamma}_r)^{q-2}\big|R_r-\tilde{Z}_r\big|^2\\&\geq \frac{1}{2}\left[\frac{1+\beta}{\beta}\,|M_r-Y_r|^2+\frac{1+\beta}{4}\,|\hat{Y}_r-\tilde{Y}_r|^2+\delta_q\right]^{(q-2)/2}\big|\hat{Z}_r-\tilde{Z}_r\big|^2.\end{aligned}\tag{4.6}$$

Let $0<\varepsilon\leq 1$ and

$$M_t^\varepsilon=\frac{1}{Q_\varepsilon}\,\mathbb{E}^{\mathcal{F}_t}\int_{t\vee\varepsilon}^\infty e^{-\frac{Q_r-Q_{t\vee\varepsilon}}{Q_\varepsilon}}\,Y_r\,\mathrm{d}Q_r\,,\quad t\geq 0.$$

Then, by Proposition 6.12, $(M^\varepsilon,R^\varepsilon)\in S_m^p\times\Lambda_{m\times k}^p$ is the unique solution of the BSDE:

$$\begin{cases}M_t^\varepsilon=M_T^\varepsilon+\dfrac{1}{Q_\varepsilon}\displaystyle\int_t^T\mathbf{1}_{[\varepsilon,\infty)}\,(r)\,(Y_r-M_r^\varepsilon)\,\mathrm{d}Q_r-\int_t^T R_r^\varepsilon\,\mathrm{d}B_r\,,\quad\text{for any }T>0,\quad t\in[0,T]\,,\\ \lim_{T\to\infty}\mathbb{E}\,|M_T^\varepsilon-\xi_T|^p=0\end{cases}$$

and

$$\begin{array}{ll}(a) & |M_t^\varepsilon|\leq\mathbb{E}^{\mathcal{F}_t}\sup_{r\geq0}|Y_r|\,,\quad\mathbb{P}-\text{a.s., for all }t\geq0,\\ (b) & \lim_{\varepsilon\to0}M_t^\varepsilon=Y_t\,,\quad\mathbb{P}-\text{a.s., for all }t\geq0,\\ (c) & \lim_{\varepsilon\to0}\mathbb{E}\sup_{t\in[0,T]}|M_t^\varepsilon-Y_t|^p=0,\quad\text{for all }T>0.\end{array}$$

We will replace now in (4.5) M by M^ε, R by R^ε and N by $N^\varepsilon=\frac{1}{Q_\varepsilon}\mathbf{1}_{[\varepsilon,\infty)}\,(r)\,(Y_r-M_r^\varepsilon)$. This is possible since our M^ε is from $\mathcal{V}_m^p$ and of form (3.2).

We see first that, for any $r\geq\varepsilon$,

$$\begin{aligned}&\langle M_r^\varepsilon-\hat{Y}_r,N_r^\varepsilon\rangle=\langle M_r^\varepsilon-\hat{Y}_r,\frac{1}{Q_\varepsilon}\,(Y_r-M_r^\varepsilon)\rangle=\frac{1}{2Q_\varepsilon}\,\langle M_r^\varepsilon-\hat{Y}_r,(\hat{Y}_r-M_r^\varepsilon)+(\tilde{Y}_r-M_r^\varepsilon)\rangle\\&\leq\frac{1}{2Q_\varepsilon}\,\Big[-|M_r^\varepsilon-\hat{Y}_r|^2+|M_r^\varepsilon-\hat{Y}_r||M_r^\varepsilon-\tilde{Y}_r|\Big]=\frac{1}{2Q_\varepsilon}\,\Big[|M_r^\varepsilon-\tilde{Y}_r|-|M_r^\varepsilon-\hat{Y}_r|\Big]\,|M_r^\varepsilon-\hat{Y}_r|\end{aligned}$$

and similarly

$$\langle M_r^\varepsilon-\tilde{Y}_r,N_r^\varepsilon\rangle\leq\frac{1}{2Q_\varepsilon}\,\Big[|M_r^\varepsilon-\hat{Y}_r|-|M_r^\varepsilon-\tilde{Y}_r|\Big]\,|M_r^\varepsilon-\tilde{Y}_r|.$$

Therefore

$$\begin{aligned}&(\hat{\Gamma}_r^\varepsilon)^{q-2}\langle M_r^\varepsilon-\hat{Y}_r,N_r^\varepsilon\rangle+(\tilde{\Gamma}_r^\varepsilon)^{q-2}\langle M_r^\varepsilon-\tilde{Y}_r,N_r^\varepsilon\rangle\\&=\big(|M_r^\varepsilon-\hat{Y}_r|^2+\delta_q\big)^{(q-2)/2}\langle M_r^\varepsilon-\hat{Y}_r,N_r^\varepsilon\rangle+\big(|M_r^\varepsilon-\tilde{Y}_r|^2+\delta_q\big)^{(q-2)/2}\langle M_r^\varepsilon-\tilde{Y}_r,N_r^\varepsilon\rangle\\&\leq\frac{-1}{2Q_\varepsilon}\,\Big[\big(|M_r^\varepsilon-\hat{Y}_r|^2+\delta_q\big)^{(q-2)/2}|M_r^\varepsilon-\hat{Y}_r|-\big(|M_r^\varepsilon-\tilde{Y}_r|^2+\delta_q\big)^{(q-2)/2}|M_r^\varepsilon-\tilde{Y}_r|\Big]\\&\quad\cdot\Big[|M_r^\varepsilon-\hat{Y}_r|-|M_r^\varepsilon-\tilde{Y}_r|\Big]\leq0,\end{aligned}\tag{4.7}$$

since we have, for all $a, b \geq 0$, $\delta \geq 0$ and $\beta \geq -1/2$,

$$\Big[\big(a^2+\delta\big)^{\beta} a - \big(b^2+\delta\big)^{\beta} b\Big](a-b) \geq 0.$$

We use inequalities (4.6) and (4.7) and inequality (4.5) becomes:

$$\begin{aligned}
&\Big[e^{qV_\sigma}(\hat{\Gamma}^\varepsilon_\sigma)^q + e^{qV_\sigma}(\tilde{\Gamma}^\varepsilon_\sigma)^q\Big] + q\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\theta e^{qV_r}\Big[(\hat{\Gamma}^\varepsilon_r)^q + (\tilde{\Gamma}^\varepsilon_r)^q\Big]\mathrm{d}V_r\\
&\quad+\frac{q}{4}\,n_q\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\theta e^{qV_r}\Big[\frac{1+\beta}{\beta}\,|M^\varepsilon_r - Y_r|^2 + \frac{1+\beta}{4}\,|\hat{Y}_r - \tilde{Y}_r|^2 + \delta_q\Big]^{(q-2)/2}|\hat{Z}_r - \tilde{Z}_r|^2\mathrm{d}r\\
&\quad+q\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\theta e^{qV_r}\Big[(\hat{\Gamma}^\varepsilon_r)^{q-2}\Psi(r,\hat{Y}_r) + (\tilde{\Gamma}^\varepsilon_r)^{q-2}\Psi(r,\tilde{Y}_r)\Big]\mathrm{d}Q_r\\
&\leq \mathbb{E}^{\mathcal{F}_\sigma}\Big[e^{qV_\theta}(\hat{\Gamma}^\varepsilon_\theta)^q + e^{qV_\theta}(\tilde{\Gamma}^\varepsilon_\theta)^q\Big] + q\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\theta e^{qV_r}\Big[(\hat{\Gamma}^\varepsilon_r)^{q-2} + (\tilde{\Gamma}^\varepsilon_r)^{q-2}\Big]\Psi\left(r, M^\varepsilon_r\right)\mathrm{d}Q_r\\
&\quad+q\,\mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^\theta e^{qV_r}\Big[(\hat{\Gamma}^\varepsilon_r)^{q-2}\langle M^\varepsilon_r - \hat{Y}_r, -\hat{H}(r,\hat{Y}_r,\hat{Z}_r)\rangle + (\tilde{\Gamma}^\varepsilon_r)^{q-2}\langle M^\varepsilon_r - \tilde{Y}_r, -\tilde{H}(r,\tilde{Y}_r,\tilde{Z}_r)\rangle\Big]\mathrm{d}Q_r\,.
\end{aligned}\tag{4.8}$$

Step II. *Working with a sequence of stopping times*

Let $0 \leq u \leq v$ and the stopping times $v^* = Q^{-1}_v$, $u^* = Q^{-1}_u$, where $Q^{-1}_\cdot$ is the inverse of the function $r \mapsto Q_r : [0,\infty) \to [0,\infty)$.

Let, for each $k, i \in \mathbb{N}^*$, the stopping times[2]

$$\begin{aligned}
\alpha_k = \inf\Big\{u \geq 0 : \updownarrow V \updownarrow_u &+ \sup\nolimits_{r\in[0,u]}|e^{V_r}\hat{Y}_r - \hat{Y}_0| + \sup\nolimits_{r\in[0,u]}|e^{V_r}\tilde{Y}_r - \tilde{Y}_0| + \int_0^u e^{2V_r}|\hat{Z}_r|^2\mathrm{d}r\\
&+\int_0^u e^{2V_r}|\tilde{Z}_r|^2\mathrm{d}r + \int_0^u e^{V_r}|\hat{H}(r,\hat{Y}_r,\hat{Z}_r)|\mathrm{d}Q_r + \int_0^u e^{V_r}|\tilde{H}(r,\tilde{Y}_r,\tilde{Z}_r)|\mathrm{d}Q_r\\
&+\int_0^u e^{2V_r}\Psi(r,\hat{Y}_r)\mathrm{d}Q_r + \int_0^u e^{2V_r}\Psi(r,\tilde{Y}_r)\mathrm{d}Q_r \geq k\Big\}.
\end{aligned}$$

and define

$$u^*_k = \sigma \wedge u^* \wedge \alpha_k \quad \text{and} \quad v^*_{k+i} = \theta \wedge v^* \wedge \alpha_{k+i}$$

(in which case (4.4) is satisfied).

We consider in (4.8)

$$\sigma = u^*_k \quad \text{and} \quad \theta = v^*_{k+i}$$

[2] The notation $\updownarrow V \updownarrow_t$ means the total variation of the function $V \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$ on the interval $[0,t]$. For details see Section 6.3.5 of [21].

and passing to $\liminf_{\varepsilon\searrow 0}$ we obtain (using Prop. 6.12, Fatou's Lemma and Lebesgue dominated convergence theorem):

$$\begin{aligned}
&2\,e^{qV_{u_k^*}}\left(\frac{1}{4}\left|\hat{Y}_{u_k^*}-\tilde{Y}_{u_k^*}\right|^2+\delta_q\right)^{q/2}+2q\,\mathbb{E}^{\mathcal{F}_{u_k^*}}\int_{u_k^*}^{v_{k+i}^*}e^{qV_r}\left(\frac{1}{4}\left|\hat{Y}_r-\tilde{Y}_r\right|^2+\delta_q\right)^{q/2}\mathrm{d}V_r\\
&\quad+\frac{q}{4}\,n_q\,\mathbb{E}^{\mathcal{F}_{u_k^*}}\int_{u_k^*}^{v_{k+i}^*}e^{qV_r}\left(\frac{1+\beta}{4}\,|\hat{Y}_r-\tilde{Y}_r|^2+\delta_q\right)^{(q-2)/2}|\hat{Z}_r-\tilde{Z}_r|^2\mathrm{d}r\\
&\quad+q\,\mathbb{E}^{\mathcal{F}_{u_k^*}}\int_{u_k^*}^{v_{k+i}^*}e^{qV_r}\left(\frac{1}{4}\,|\hat{Y}_r-\tilde{Y}_r|^2+\delta_q\right)^{(q-2)/2}\left[\Psi(r,\hat{Y}_r)+\Psi(r,\tilde{Y}_r)\right]\mathrm{d}Q_r\\
&\le 2\,\mathbb{E}^{\mathcal{F}_{u_k^*}}e^{qV_{v_{k+i}^*}}\left(\frac{1}{4}\left|\hat{Y}_{v_{k+i}^*}-\tilde{Y}_{v_{k+i}^*}\right|^2+\delta_q\right)^{q/2}\\
&\quad+q\,\mathbb{E}^{\mathcal{F}_{u_k^*}}\int_{u_k^*}^{v_{k+i}^*}e^{qV_r}\left(\frac{1}{4}\,|\hat{Y}_r-\tilde{Y}_r|^2+\delta_q\right)^{(q-2)/2}\cdot 2\Psi\left(r,Y_r\right)\mathrm{d}Q_r\\
&\quad+\frac{q}{2}\,\mathbb{E}^{\mathcal{F}_{u_k^*}}\int_{u_k^*}^{v_{k+i}^*}e^{qV_r}\left(\frac{1}{4}\,|\hat{Y}_r-\tilde{Y}_r|^2+\delta_q\right)^{(q-2)/2}\langle\hat{Y}_r-\tilde{Y}_r,\hat{H}(r,\hat{Y}_r,\hat{Z}_r)-\tilde{H}(r,\tilde{Y}_r,\tilde{Z}_r)\rangle\mathrm{d}Q_r\,.
\end{aligned}$$

By Fatou's Lemma, as as $\beta\to 0$, we deduce that the previous inequality holds true also for $\beta=0$.

We remark now that

$$2\Psi\left(r,Y_r\right)=2\Psi\Big(r,\frac{1}{2}\hat{Y}_r+\frac{1}{2}\tilde{Y}_r\Big)\le\Psi(r,\hat{Y}_r)+\Psi(r,\tilde{Y}_r)$$

and

$$\begin{aligned}
&\langle\hat{Y}_r-\tilde{Y}_r,\hat{H}(r,\hat{Y}_r,\hat{Z}_r)-\tilde{H}(r,\hat{Y}_r,\hat{Z}_r)+\tilde{H}(r,\hat{Y}_r,\hat{Z}_r)-\tilde{H}(r,\tilde{Y}_r,\tilde{Z}_r)\rangle\mathrm{d}Q_r\\
&\le\langle\hat{Y}_r-\tilde{Y}_r,\hat{H}(r,\hat{Y}_r,\hat{Z}_r)-\tilde{H}(r,\hat{Y}_r,\hat{Z}_r)\rangle\mathrm{d}Q_r+|\hat{Y}_r-\tilde{Y}_r|^2\mathrm{d}V_r+\frac{n_p\lambda}{2}\,|\hat{Z}_r-\tilde{Z}_r|^2\mathrm{d}r.
\end{aligned}$$

Hence

$$\begin{aligned}
&2\,e^{qV_{u_k^*}}\left(\frac{1}{4}\left|\hat{Y}_{u_k^*}-\tilde{Y}_{u_k^*}\right|^2+\delta_q\right)^{q/2}+2q\delta_q\,\mathbb{E}^{\mathcal{F}_{u_k^*}}\int_{u_k^*}^{v_{k+i}^*}e^{qV_r}\left(\frac{1}{4}\,|\hat{Y}_r-\tilde{Y}_r|^2+\delta_q\right)^{(q-2)/2}\mathrm{d}V_r\\
&\quad+\frac{q}{4}\left(n_q-n_p\lambda\right)\,\mathbb{E}^{\mathcal{F}_{u_k^*}}\int_{u_k^*}^{v_{k+i}^*}e^{qV_r}\left(\frac{1}{4}\,|\hat{Y}_r-\tilde{Y}_r|^2+\delta_q\right)^{(q-2)/2}|\hat{Z}_r-\tilde{Z}_r|^2\mathrm{d}r\\
&\le 2\,\mathbb{E}^{\mathcal{F}_{u_k^*}}e^{qV_{v_{k+i}^*}}\left(\frac{1}{4}\left|\hat{Y}_{v_{k+i}^*}-\tilde{Y}_{v_{k+i}^*}\right|^2+\delta_q\right)^{q/2}\\
&\quad+\frac{q}{2}\,\mathbb{E}^{\mathcal{F}_{u_k^*}}\int_{u_k^*}^{v_{k+i}^*}e^{qV_r}\left(\frac{1}{4}\,|\hat{Y}_r-\tilde{Y}_r|^2+\delta_q\right)^{(q-2)/2}\langle\hat{Y}_r-\tilde{Y}_r,\hat{H}(r,\hat{Y}_r,\hat{Z}_r)-\tilde{H}(r,\hat{Y}_r,\hat{Z}_r)\rangle\mathrm{d}Q_r\,.
\end{aligned}$$

Step III. *Passing to the limit*

Since $q\in\{2,p\wedge 2\}$, we have (see Rem. 3.1)

$$n_q-n_p\lambda\ge n_q\left(1-\lambda\right)=\left(q-1\right)\left(1-\lambda\right),$$

passing to $\lim_{u,v\to\infty}$, by Fatou's Lemma and Lebesgue's dominated convergence theorem and using the continuity of the natural filtration $\{\mathcal{F}_r : r \geq 0\}$, it follows that

$$\begin{aligned}
&2\, e^{qV_{\sigma\wedge\alpha_k}} \left(\frac{1}{4}\left|\hat{Y}_{\sigma\wedge\alpha_k} - \tilde{Y}_{\sigma\wedge\alpha_k}\right|^2 + \delta_q\right)^{q/2} \\
&\quad +2q\delta_q\, \mathbb{E}^{\mathcal{F}_{\sigma\wedge\alpha_k}} \int_{\sigma\wedge\alpha_k}^{\theta\wedge\alpha_{k+i}} e^{qV_r} \left(\frac{1}{4}|\hat{Y}_r - \tilde{Y}_r|^2 + \delta_q\right)^{(q-2)/2} \mathrm{d}V_r \\
&\quad +\frac{q}{4}(q-1)(1-\lambda)\, \mathbb{E}^{\mathcal{F}_{\sigma\wedge\alpha_k}} \int_{\sigma\wedge\alpha_k}^{\theta\wedge\alpha_{k+i}} e^{qV_r} \left(\frac{1}{4}|\hat{Y}_r - \tilde{Y}_r|^2 + \delta_q\right)^{(q-2)/2} |\hat{Z}_r - \tilde{Z}_r|^2 \mathrm{d}r \\
&\leq 2\, \mathbb{E}^{\mathcal{F}_{\sigma\wedge\alpha_k}} e^{qV_{\theta\wedge\alpha_{k+i}}} \left(\frac{1}{4}\left|\hat{Y}_{\theta\wedge\alpha_{k+i}} - \tilde{Y}_{\theta\wedge\alpha_{k+i}}\right|^2 + \delta_q\right)^{q/2} \\
&\quad +q\, \mathbb{E}^{\mathcal{F}_{\sigma\wedge\alpha_k}} \int_{\sigma\wedge\alpha_k}^{\theta\wedge\alpha_{k+i}} e^{qV_r} \left(\frac{1}{4}|\hat{Y}_r - \tilde{Y}_r|^2 + \delta_q\right)^{(q-1)/2} \left|\hat{H}(r,\hat{Y}_r,\hat{Z}_r) - \tilde{H}(r,\hat{Y}_r,\hat{Z}_r)\right| \mathrm{d}Q_r\,.
\end{aligned} \tag{4.9}$$

Passing to the limit, as $\delta \to 0_+$, by Fatou's Lemma for the left-hand side and inequality

$$\delta_q \left(\frac{1}{4}|\hat{Y}_r - \tilde{Y}_r|^2 + \delta_q\right)^{(q-2)/2} \leq {\delta_q}^{q/2},$$

and by Lebesgue's dominated convergence theorem for the right-hand side we get

$$\begin{aligned}
&e^{qV_{\sigma\wedge\alpha_k}} \left|\hat{Y}_{\sigma\wedge\alpha_k} - \tilde{Y}_{\sigma\wedge\alpha_k}\right|^q \\
&\quad +\frac{q}{2}(q-1)(1-\lambda)\, \mathbb{E}^{\mathcal{F}_{\sigma\wedge\alpha_k}} \int_{\sigma\wedge\alpha_k}^{\theta\wedge\alpha_{k+i}} e^{qV_r} |\hat{Y}_r - \tilde{Y}_r|^{q-2} |\hat{Z}_r - \tilde{Z}_r|^2 \mathrm{d}r \\
&\leq \mathbb{E}^{\mathcal{F}_{\sigma\wedge\alpha_k}} e^{qV_{\theta\wedge\alpha_{k+i}}} \left|\hat{Y}_{\theta\wedge\alpha_{k+i}} - \tilde{Y}_{\theta\wedge\alpha_{k+i}}\right|^q \\
&\quad +q\, \mathbb{E}^{\mathcal{F}_{\sigma\wedge\alpha_k}} \int_{\sigma\wedge\alpha_k}^{\theta\wedge\alpha_{k+i}} e^{qV_r} |\hat{Y}_r - \tilde{Y}_r|^{q-1} \left|\hat{H}(r,\hat{Y}_r,\hat{Z}_r) - \tilde{H}(r,\hat{Y}_r,\hat{Z}_r)\right| \mathrm{d}Q_r\,.
\end{aligned} \tag{4.10}$$

Passing to the limit successively, $\lim_{i\to\infty}$, $\lim_{k\to\infty}$ and $\lim_{T\to\infty}$, in (4.10), we obtain (using Fatou's Lemma and Lebesgue dominated convergence theorem via condition (3.8)), for any stopping times $0 \leq \sigma \leq \theta \leq \tau$ and $\mathbb{P}$-a.s.,

$$\begin{aligned}
&e^{qV_\sigma} \left|\hat{Y}_\sigma - \tilde{Y}_\sigma\right|^q + \frac{q}{2}(q-1)(1-\lambda)\, \mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} |\hat{Y}_r - \tilde{Y}_r|^{q-2} |\hat{Z}_r - \tilde{Z}_r|^2 \mathrm{d}r \\
&\leq \mathbb{E}^{\mathcal{F}_\sigma} e^{qV_\theta} \left|\hat{Y}_\theta - \tilde{Y}_\theta\right|^q + q\, \mathbb{E}^{\mathcal{F}_\sigma} \int_\sigma^\theta e^{qV_r} |\hat{Y}_r - \tilde{Y}_r|^{q-1} \left|\hat{H}(r,\hat{Y}_r,\hat{Z}_r) - \tilde{H}(r,\hat{Y}_r,\hat{Z}_r)\right| \mathrm{d}Q_r
\end{aligned} \tag{4.11}$$

(we also used the continuity of the natural filtration $\{\mathcal{F}_r : r \geq 0\}$).

From (4.11) we obtain conclusion (4.1) if we use Holder's inequality and boundedness property (3.22) from Proposition 3.10 and inequality $\left|\hat{Y}_r - \tilde{Y}_r\right|^q \leq 2^{q-1}\left(|\hat{Y}_r|^q + |\tilde{Y}_r|^q\right)$.

Applying now a stochastic Gronwall's type inequality provided by Proposition 6.6, we infer from (4.11) that, for all $0<\alpha<1$,

$$\begin{aligned} &\mathbb{E}\sup\nolimits_{r\in[\sigma,\theta]} e^{\alpha q V_r}\big|\hat{Y}_r-\tilde{Y}_r\big|^{\alpha q}+(q-1)^{\alpha}\,\mathbb{E}\Big(\int_\sigma^\theta e^{qV_r}|\hat{Y}_r-\tilde{Y}_r|^{q-2}\,|\hat{Z}_r-\tilde{Z}_r|^2\mathrm{d}r\Big)^{\alpha}\\ &\leq C_{q,\alpha,\lambda}\left[\Big(\mathbb{E}e^{qV_\theta}\big|\hat{Y}_\theta-\tilde{Y}_\theta\big|^q\Big)^{\alpha}+L^{\frac{\alpha(q-1)}{q}}\left(\mathbb{E}\Big(\int_\sigma^\theta e^{V_r}\big|\hat{H}(r,\hat{Y}_r,\hat{Z}_r)-\tilde{H}(r,\hat{Y}_r,\hat{Z}_r)\big|\mathrm{d}Q_r\Big)^q\right)^{\frac{\alpha}{q}}\right], \end{aligned}\tag{4.12}$$

where L is a constant such that, see inequality (3.22),

$$\mathbb{E}\sup\nolimits_{r\in[0,T]} e^{qV_r}\big|\hat{Y}_r-\tilde{Y}_r\big|^q\leq 2^{q-1}\,\mathbb{E}\sup\nolimits_{r\in[0,T]} e^{qV_r}\big(|\hat{Y}_r|^q+|\tilde{Y}_r|^q\big)\leq L.$$

Hence conclusion (4.3) follows. □

Remark 4.4. We notice that if we have the uniqueness of Y and if we multiply (4.9) with $\delta^{1-q/2}$, then the uniqueness of Z follows also from (4.9) by taking $\delta\to 0_+$.

5. Existence of the solution

5.1. Existence on a deterministic interval time $[0,T]$

The existence of a L^p-variational solution will be proved firstly in the case of a deterministic time interval, *i.e.* $\tau=T>0$.

The proof of the main result of this section (see Thm. 5.5) will be splitted into several steps. The main idea is to use firstly more restrictive assumptions (more precisely, (5.1), (5.2), (5.3) and (5.4)) in order to obtain the existence of a strong solution (see Lem. 5.1); this is only an intermediate step and we will renounce to these assumptions.

Lemma 5.1 (Strong solution)**.** *We suppose that assumptions* $(\mathrm{A}_2-\mathrm{A}_7)$ *are satisfied. Let* $V^{(+)}$ *be given by definition (2.11). In addition we assume that:*

(i) *there exists* $L>0$ *such that*

$$|\eta|+\varphi(\eta)+\psi(\eta)+\ell_t+F_1^{\#}(t)+G_1^{\#}(t)\leq L,\quad a.e.\ t\in[0,T],\ \mathbb{P}-a.s.;\tag{5.1}$$

(ii) *there exists* $\delta>0$ *such that*

$$\mathbb{E}\exp\Big[(2+\delta)\,V_T^{(+)}\Big]<\infty;\tag{5.2}$$

(iii) *there exists* $\tilde{L}>0$ *such that*

$$\big|e^{V_T^{(+)}}\eta\big|^2+\Big(\int_0^T e^{V_r^{(+)}}\big(F_1^{\#}(r)\,\mathrm{d}r+G_1^{\#}(r)\,\mathrm{d}A_r\big)\Big)^2\leq\tilde{L},\quad\mathbb{P}-a.s.;\tag{5.3}$$

(iv) *for*[3] $\tilde{\rho}_0\overset{\mathrm{def}}{=\!=}(C_\lambda\tilde{L})^{1/2}>0$ *it holds*

$$\mathbb{E}\int_0^T e^{2V_r^{(+)}}\Big[\big(F_{1+\tilde{\rho}_0}^{\#}(r)\big)^2\mathrm{d}r+\big(G_{1+\tilde{\rho}_0}^{\#}(r)\big)^2\mathrm{d}A_r\Big]<\infty.\tag{5.4}$$

[3]The constant $C_\lambda:=C_{2,\lambda}$, where $C_{2,\lambda}$ is given by (6.35).

Then the multivalued BSDE

$$\begin{cases} Y_t + \int_t^T \mathrm{d}K_r = \eta + \int_t^T H(r, Y_r, Z_r)\,\mathrm{d}Q_r - \int_t^T Z_r \mathrm{d}B_r\,, \\ \mathrm{d}K_t = U_t^{(1)}\mathrm{d}t + U_t^{(2)}\mathrm{d}A_t\,, \\ U_t^{(1)}\mathrm{d}t \in \partial\varphi(Y_t)\,\mathrm{d}t \quad \textit{and} \quad U_t^{(2)}\mathrm{d}A_t \in \partial\psi(Y_t)\,\mathrm{d}A_t\,, \quad t \in [0,T]\,, \end{cases}$$

has a strong a solution $\left(Y, Z, U^{(1)}, U^{(2)}\right) \in S_m^0 \times \Lambda_{m\times k}^0 \times \Lambda_m^0 \times \Lambda_m^0$ *such that*

$$\mathbb{E}\sup_{r\in[0,T]} e^{2V_r^{(+)}}|Y_r|^2 + \mathbb{E}\int_0^T e^{2V_r^{(+)}}|Z_r|^2\,\mathrm{d}r + \mathbb{E}\int_0^T e^{2V_r^{(+)}}\left|U_r^{(1)}\right|^2\mathrm{d}r + \mathbb{E}\int_0^T e^{2V_r^{(+)}}\left|U_r^{(2)}\right|^2\mathrm{d}A_r < \infty.$$

Proof. **Step I.** *Approximating equation*

Let $0 < \varepsilon \leq 1$. We consider the approximating BSDE

$$Y_t^\varepsilon + \int_t^T \nabla_y \Psi^\varepsilon(r, Y_r^\varepsilon)\mathrm{d}Q_r = \eta + \int_t^T H_\varepsilon(r, Y_r^\varepsilon, Z_r^\varepsilon)\mathrm{d}Q_r - \int_t^T Z_r^\varepsilon\,\mathrm{d}B_r\,, \quad \mathbb{P}-\text{a.s.},\ t\in[0,T]\,, \tag{5.5}$$

where

$$\begin{aligned} &\Psi^\varepsilon(r,y) \overset{\text{def}}{=\!=} \alpha_r\,\varphi_\varepsilon(y) + (1-\alpha_r)\ \psi_\varepsilon(y)\ \mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r)\,, \\ &\nabla_y\Psi^\varepsilon(r,y) = \left[\alpha_r\,\nabla\varphi_\varepsilon(y) + (1-\alpha_r)\ \nabla\psi_\varepsilon(y)\right]\mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r)\,, \\ &H_\varepsilon(r,y,z) \overset{\text{def}}{=\!=} \left[\alpha_r\,F_\varepsilon(r,y,z) + (1-\alpha_r)\ G_\varepsilon(r,y)\right]\mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r)\,, \end{aligned} \tag{5.6}$$

where φ_ε and ψ_ε are the Moreau-Yosida's regularization given by (2.15) and $F_\varepsilon, G_\varepsilon$ are the mollifier approximations defined by Section 6.4.

By properties provided by (6.50), Remark 6.13 and (2.16), and by assumption (5.1) we see that the function

$$\Phi_\varepsilon(r,y,z) \overset{\text{def}}{=\!=} H_\varepsilon(r,y,z) - \nabla_y\Psi^\varepsilon(r,y)$$

is a Lipschitz function:

$$\begin{aligned} &|\Phi_\varepsilon(r,y,z) - \Phi_\varepsilon(r,\hat{y},\hat{z})| \\ &\leq \left[\alpha_r\left(\ell_r\ |z-\hat{z}| + \frac{\kappa(1+\ell_r)}{\varepsilon^2}\ |y-\hat{y}|\right) + (1-\alpha_r)\ \frac{\kappa}{\varepsilon^2}\ |y-\hat{y}|\right. \\ &\quad \left. + \frac{1}{\varepsilon}\,\alpha_r\ |y-\hat{y}| + \frac{1}{\varepsilon}(1-\alpha_r)\ |y-\hat{y}|\right]\mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r) \\ &\leq \left[\alpha_r\,\frac{\kappa L+\kappa+1}{\varepsilon^2}\ |y-\hat{y}| + (1-\alpha_r)\ \frac{\kappa+1}{\varepsilon^2}\ |y-\hat{y}| + L\,\alpha_r\,|z-\hat{z}|\right]\mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r) \\ &\leq \left[\frac{\kappa L+\kappa+1}{\varepsilon^2}\ |y-\hat{y}| + L\,\alpha_r\,|z-\hat{z}|\right]\mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r)\,. \end{aligned}$$

If we use the above properties of Φ_ε , the presence of the indicator $\mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r)$ in the definition of Φ_ε , assumption (5.1) and properties (6.49) and (6.50$-a$) of the mollifier approximations of F and G, we see that the assumptions of Lemma 5.20 in [21] are satisfied for any $p' \geq 2$ arbitrary fixed.

Therefore equation (5.5) has a unique solution $(Y^{\varepsilon},Z^{\varepsilon})\in S_m^{p'}[0,T]\times\Lambda_{m\times k}^{p'}(0,T)$ and consequently, for any $p'\geq 2$,

$$\mathbb{E}\sup\nolimits_{t\in[0,T]}|Y_t^{\varepsilon}|^{p'}<\infty. \tag{5.7}$$

Step II. *Boundedness of the approximating solution*
Remark that, by property (6.51), since $n_p\leq 1$,

$$\begin{aligned}
&\left\langle Y_t^{\varepsilon},\Phi_{\varepsilon}(t,Y_t^{\varepsilon},Z_t^{\varepsilon})\right\rangle \mathrm{d}Q_t\\
&=\left\langle Y_t^{\varepsilon},F_{\varepsilon}(t,Y_t^{\varepsilon},Z_t^{\varepsilon})\right\rangle \mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_t)\,\mathrm{d}t+\left\langle Y_t^{\varepsilon},G_{\varepsilon}(t,Y_t^{\varepsilon})\right\rangle \mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_t)\,\mathrm{d}A_t\\
&\quad-\left\langle Y_t^{\varepsilon},\nabla\varphi_{\varepsilon}(Y_t^{\varepsilon})\right\rangle \mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_t)\,\mathrm{d}t-\left\langle Y_t^{\varepsilon},\nabla\psi_{\varepsilon}(Y_t^{\varepsilon})\right\rangle \mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_t)\,\mathrm{d}A_t\\
&\leq\Big[|Y_t^{\varepsilon}|\,F_1^{\#}(t)+\Big(\mu_t+\frac{1}{2n_p\lambda}\ell_t^2\Big)^+|Y_t^{\varepsilon}|^2+\frac{n_p\lambda}{2}|Z_t^{\varepsilon}|^2\Big]\mathrm{d}t+\Big[|Y_t^{\varepsilon}|\,G_1^{\#}(t)+\nu_t^+|Y_t^{\varepsilon}|^2\Big]\mathrm{d}A_t\\
&\leq|Y_t^{\varepsilon}|\,\bar{H}_1^{\#}(t)\,\mathrm{d}Q_t+|Y_t^{\varepsilon}|^2\,\mathrm{d}V_t^{(+)}+\frac{\lambda}{2}|Z_t^{\varepsilon}|^2\,\mathrm{d}t,
\end{aligned}$$

where

$$\bar{H}_1^{\#}(t)\overset{\mathrm{def}}{=\!=}\alpha_tF_1^{\#}(t)+(1-\alpha_t)\,G_1^{\#}(t).$$

By Young's inequality and assumption (5.2) and (5.7) we have

$$\begin{aligned}
\mathbb{E}\sup\nolimits_{t\in[0,T]}e^{2V_t^{(+)}}|Y_t^{\varepsilon}|^2&\leq\mathbb{E}\Big[\big(\exp 2V_T^{(+)}\big)\sup\nolimits_{t\in[0,T]}|Y_t^{\varepsilon}|^2\Big]\\
&\leq\Big[\frac{2}{2+\delta}\mathbb{E}\exp(2+\delta)V_T^{(+)}+\frac{\delta}{2+\delta}\mathbb{E}\sup\nolimits_{t\in[0,T]}|Y_t^{\varepsilon}|^{(4+2\delta)/\delta}\Big]\\
&<\infty,
\end{aligned}\tag{5.8}$$

therefore, by Proposition 6.8 (our main stochastic Gronwall's type inequality), we have

$$\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{r\in[t,T]}\big|e^{V_r^{(+)}}Y_r^{\varepsilon}\big|^2+\int_t^Te^{2V_r^{(+)}}|Z_r^{\varepsilon}|^2\,\mathrm{d}r\Big]\leq C_{\lambda}\,\mathbb{E}^{\mathcal{F}_t}\Big[\big|e^{V_T^{(+)}}\eta\big|^2+\Big(\int_t^Te^{V_r^{(+)}}\bar{H}_1^{\#}(r)\,\mathrm{d}Q_r\Big)^2\Big]$$

($C_{\lambda}=C_{2,\lambda}$, where $C_{2,\lambda}$ is given by (6.35)).
From the above inequality we get, using (5.3) and property (6.49), that, $\mathbb{P}$-a.s., for all $t\in[0,T]$,

$$\begin{aligned}
&(a)\quad|Y_t^{\varepsilon}|\leq e^{V_t^{(+)}}|Y_t^{\varepsilon}|\leq\Big[\mathbb{E}^{\mathcal{F}_t}\Big(\sup\nolimits_{r\in[t,T]}\big|e^{V_r^{(+)}}Y_r^{\varepsilon}\big|^2\Big)\Big]^{1/2}\leq(C_{\lambda}\tilde{L})^{1/2}=\tilde{\rho}_0,\\
&(b)\quad\mathbb{E}\Big(\int_0^Te^{2V_r^{(+)}}|Z_r^{\varepsilon}|^2\,\mathrm{d}r\Big)\leq\tilde{\rho}_0^2,\\
&(c)\quad|F_{\varepsilon}(t,Y_t^{\varepsilon},Z_t^{\varepsilon})|\leq\ell_t|Z_t^{\varepsilon}|+F_{1+\tilde{\rho}_0}^{\#}(t),\quad|G_{\varepsilon}(t,Y_t^{\varepsilon})|\leq G_{1+\tilde{\rho}_0}^{\#}(t),\\
&(d)\quad|H_{\varepsilon}(r,Y_r^{\varepsilon},Z_r^{\varepsilon})|\leq\Big[\alpha_r\Big(\ell_r|Z_r^{\varepsilon}|+F_{1+\tilde{\rho}_0}^{\#}(r)\Big)+(1-\alpha_r)\,G_{1+\tilde{\rho}_0}^{\#}(r)\Big]\mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r).
\end{aligned}\tag{5.9}$$

Step III. *Boundedness in L^2 of $\nabla\varphi^{\varepsilon}$ and $\nabla\psi^{\varepsilon}$*

Using the stochastic subdifferential inequality (see [21], Lem. 2.38, Rem. 2.39)

$$\begin{aligned} e^{2V_t^{(+)}}\varphi_\varepsilon(Y_t^\varepsilon) \leq e^{2V_s^{(+)}}\varphi_\varepsilon(Y_s^\varepsilon) + \int_t^s e^{2V_r^{(+)}}\left\langle \nabla\varphi_\varepsilon(Y_r^\varepsilon), \Phi_\varepsilon(r, Y_r^\varepsilon, Z_r^\varepsilon)\right\rangle \mathrm{d}Q_r \\ - \int_t^s e^{2V_r^{(+)}}\left\langle \nabla\varphi_\varepsilon(Y_r^\varepsilon), Z_r^\varepsilon \mathrm{d}B_r\right\rangle, \quad 0 \leq t \leq s \leq T \end{aligned}$$

(and similar inequality for ψ_ε) and following the ideas from [13], [14] and Section 5.6.2 of [21], we deduce that:

$$\begin{aligned} & e^{2V_t^{(+)}}\left[\varphi_\varepsilon(Y_t^\varepsilon) + \psi_\varepsilon(Y_t^\varepsilon)\right] \\ & \quad + \int_t^s e^{2V_r^{(+)}}\Big[\alpha_r|\nabla\varphi_\varepsilon(Y_r^\varepsilon)|^2 + \langle\nabla\varphi_\varepsilon(Y_r^\varepsilon), \nabla\psi_\varepsilon(Y_r^\varepsilon)\rangle + (1-\alpha_r)\,|\nabla\psi_\varepsilon(Y_r^\varepsilon)|^2\Big]\mathrm{d}Q_r \\ & \leq e^{2V_s^{(+)}}\left[\varphi_\varepsilon(Y_s^\varepsilon) + \psi_\varepsilon(Y_s^\varepsilon)\right] + \int_t^s e^{2V_r^{(+)}}\left\langle \nabla\varphi_\varepsilon(Y_r^\varepsilon) + \nabla\psi_\varepsilon(Y_r^\varepsilon), H_\varepsilon(r, Y_r^\varepsilon, Z_r^\varepsilon)\right\rangle \mathrm{d}Q_r \\ & \quad - \int_t^s e^{2V_r^{(+)}}\left\langle \nabla\varphi_\varepsilon(Y_r^\varepsilon) + \nabla\psi_\varepsilon(Y_r^\varepsilon), Z_r^\varepsilon \mathrm{d}B_r\right\rangle. \end{aligned} \tag{5.10}$$

The compatibility assumptions (2.19) and inequality (6.49) yield for $|y| \leq \tilde{\rho}_0$:

$$\begin{aligned} & \langle\nabla\psi_\varepsilon(y), F_\varepsilon(t,y,z)\rangle \\ & = \int_{\overline{B(0,1)}} \langle\nabla\psi_\varepsilon(y) - \nabla\psi_\varepsilon(y-\varepsilon u), F\left(t, y-\varepsilon u, \beta_\varepsilon\left(z\right)\right)\rangle\, \mathbf{1}_{[0,1]}\left(\varepsilon\left|F\left(t, y-\varepsilon u, 0\right)\right|\right)\rho\left(u\right)\mathrm{d}u \\ & \quad + \int_{\overline{B(0,1)}} \langle\nabla\psi_\varepsilon(y-\varepsilon u), F\left(t, y-\varepsilon u, \beta_\varepsilon\left(z\right)\right)\rangle\, \mathbf{1}_{[0,1]}\left(\varepsilon\left|F\left(t, y-\varepsilon u, 0\right)\right|\right)\rho\left(u\right)\mathrm{d}u \\ & \leq \frac{1}{\varepsilon}\int_{\overline{B(0,1)}} |\varepsilon u|\, \left|F\left(t, y-\varepsilon u, \beta_\varepsilon\left(z\right)\right)\right|\, \mathbf{1}_{[0,1]}\left(\varepsilon\left|F\left(t, y-\varepsilon u, 0\right)\right|\right)\rho\left(u\right)\mathrm{d}u \\ & \quad + \int_{\overline{B(0,1)}} |\nabla\psi_\varepsilon(y-\varepsilon u)|\left|F\left(t, y-\varepsilon u, \beta_\varepsilon\left(z\right)\right)\right|\, \mathbf{1}_{[0,1]}\left(\varepsilon\left|F\left(t, y-\varepsilon u, 0\right)\right|\right)\rho\left(u\right)\mathrm{d}u \\ & \leq \left|F\right|_\varepsilon(t,y,z) + \int_{\overline{B(0,1)}} \left[\left|\nabla\varphi_\varepsilon(y-\varepsilon u) - \nabla\varphi_\varepsilon(y)\right| + \left|\nabla\varphi_\varepsilon(y)\right|\right] \\ & \qquad\qquad \cdot \left|F\left(t, y-\varepsilon u, \beta_\varepsilon\left(z\right)\right)\right|\, \mathbf{1}_{[0,1]}\left(\varepsilon\left|F\left(t, y-\varepsilon u, 0\right)\right|\right)\rho\left(u\right)\mathrm{d}u \\ & \leq \left|F\right|_\varepsilon(t,y,z) + \left(1 + \left|\nabla\varphi_\varepsilon(y)\right|\right)\, \left|F\right|_\varepsilon(t,y,z) = \left(2 + \left|\nabla\varphi_\varepsilon(y)\right|\right)\, \left|F\right|_\varepsilon(t,y,z) \\ & \leq 2L\left|z\right| + 2F^{\#}_{1+\tilde{\rho}_0}(t) + \left|\nabla\varphi_\varepsilon(y)\right|\Big(L\left|z\right| + F^{\#}_{1+\tilde{\rho}_0}(t)\Big) \end{aligned}$$

and similarly

$$\langle\nabla\varphi_\varepsilon(y), G_\varepsilon(t,y)\rangle = \left(2 + \left|\nabla\psi_\varepsilon(y)\right|\right)\, \left|G\right|_\varepsilon(t,y) \leq 2G^{\#}_{1+\tilde{\rho}_0}(t) + \left|\nabla\psi_\varepsilon(y)\right| G^{\#}_{1+\tilde{\rho}_0}(t)\,.$$

Hence, using the above inequalities and the definition of $H_\varepsilon(t,y,z)$, we have, for any $|y|\leq\tilde{\rho}_0$,

$$\begin{aligned}&\langle\nabla\varphi_\varepsilon(y)+\nabla\psi_\varepsilon(y),H_\varepsilon(s,y,z)\rangle\\&=\langle\nabla\varphi_\varepsilon(y)+\nabla\psi_\varepsilon(y),\alpha_s F_\varepsilon(s,y,z)+(1-\alpha_s)\,G_\varepsilon(s,y)\rangle\,\mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r)\\&\leq\alpha_t\;(2+2\,|\nabla\varphi_\varepsilon(y)|)\;|F|_\varepsilon(t,y,z)+(1-\alpha_t)\;(2+2\,|\nabla\psi_\varepsilon(y)|)\;|G|_\varepsilon(t,y)\\&\leq\alpha_t\Big[\frac{1}{2}\,|\nabla\varphi_\varepsilon(y)|^2+1+3\big(\,|F|_\varepsilon(t,y,z)\,\big)^2\Big]+(1-\alpha_t)\Big[\frac{1}{2}\,|\nabla\psi_\varepsilon(y)|^2+1+3\big(\,|G|_\varepsilon(t,y)\,\big)^2\Big].\end{aligned}$$

Using (2.18) we deduce inequality

$$\mathbb{E}\Big(e^{2V_T^{(+)}}\,(\varphi_\varepsilon(Y_T^\varepsilon)+\psi_\varepsilon(Y_T^\varepsilon))\Big)\leq\mathbb{E}\Big(e^{2V_T^{(+)}}\,(\varphi(\eta)+\psi(\eta))\Big).\tag{5.11}$$

On the other hand,

$$M_t^\varepsilon=\int_0^t e^{2V_r^{(+)}}\,\langle\nabla\varphi_\varepsilon(Y_r^\varepsilon)+\nabla\psi_\varepsilon(Y_r^\varepsilon),Z_r^\varepsilon\mathrm{d}B_r\rangle\quad\text{is a martingale.}\tag{5.12}$$

By Young's inequality and assumption (5.2) we have

$$\mathbb{E}\int_0^T e^{2V_r^{(+)}}\mathrm{d}Q_r\leq\mathbb{E}\big(e^{2V_T^{(+)}}Q_T\big)\leq\Big[\frac{2}{2+\delta}\,\mathbb{E}\exp\left(2+\delta\right)V_T^{(+)}+\frac{\delta}{2+\delta}\,\mathbb{E}Q_T^{(2+\delta)/\delta}\Big]<\infty\tag{5.13}$$

and

$$\mathbb{E}\big(e^{2V_T^{(+)}}\,(\varphi(\eta)+\psi(\eta))\,\big)\leq 2L\,\mathbb{E}\big(e^{2V_T^{(+)}}\big)<\infty.\tag{5.14}$$

Therefore, using inequalities (6.49), (5.9$-a$) and (5.11)–(5.14), we deduce from (5.10) that, for all $0\leq t\leq s\leq T$,

$$\begin{aligned}&\mathbb{E}e^{2V_t^{(+)}}\varphi_\varepsilon(Y_t^\varepsilon)+\mathbb{E}e^{2V_t^{(+)}}\psi_\varepsilon(Y_t^\varepsilon)+\frac{1}{2}\,\mathbb{E}\int_t^T e^{2V_r^{(+)}}\Big[|\nabla\varphi_\varepsilon(Y_r^\varepsilon)|^2\mathrm{d}r+|\nabla\psi_\varepsilon(Y_r^\varepsilon)|^2\mathrm{d}A_r\Big]\\&\leq\mathbb{E}\Big[e^{2V_T^{(+)}}\,(\varphi(\eta)+\psi(\eta))\Big]\\&\quad+\mathbb{E}\int_t^T e^{2V_r^{(+)}}\Big(1+3\big(\,|F|_\varepsilon(r,Y_r^\varepsilon,Z_r^\varepsilon)\,\big)^2\Big)\,\mathrm{d}r+\mathbb{E}\int_t^T e^{2V_r^{(+)}}\Big(1+3\big(\,|G|_\varepsilon(r,Y_r^\varepsilon)\,\big)^2\Big)\,\mathrm{d}A_r\\&\leq\mathbb{E}\Big[e^{2V_T^{(+)}}\,(\varphi(\eta)+\psi(\eta))\Big]\\&\quad+\mathbb{E}\int_t^T e^{2V_r^{(+)}}\Big[1+6L^2\,|Z_r^\varepsilon|^2+6\big(F_{1+\tilde{\rho}_0}^{\#}(r)\,\big)^2\Big]\,\mathrm{d}r+\mathbb{E}\int_t^T e^{2V_r^{(+)}}\Big[1+3\big(G_{1+\tilde{\rho}_0}^{\#}(r)\,\big)^2\Big]\,\mathrm{d}A_r\,.\end{aligned}\tag{5.15}$$

Therefore, by assumption (5.4) and (5.9$-b$),

$$\begin{aligned}&(a)\quad\sup_{t\in[0,T]}\Big[\mathbb{E}e^{2V_t^{(+)}}\varphi_\varepsilon(Y_t^\varepsilon)+\mathbb{E}e^{2V_t^{(+)}}\psi_\varepsilon(Y_t^\varepsilon)\Big]\leq C_{\tilde{\rho}_0,L,T,\lambda}\\&(b)\quad\mathbb{E}\int_0^T e^{2V_r^{(+)}}\Big[|\nabla\varphi_\varepsilon(Y_r^\varepsilon)|^2\mathrm{d}r+|\nabla\psi_\varepsilon(Y_r^\varepsilon)|^2\mathrm{d}A_r\Big]\leq C_{\tilde{\rho}_0,L,T,\lambda}\end{aligned}\tag{5.16}$$

($C_{\tilde{\rho}_0,L,T,\lambda}$ is independent of ε).

Step IV. *Cauchy sequence*

Let $\varepsilon, \delta \in (0,1]$. We have

$$Y_t^\varepsilon - Y_t^\delta = \int_t^T \mathrm{d}K_r^{\varepsilon,\delta} - \int_t^T (Z_r^\varepsilon - Z_r^\delta)\mathrm{d}B_r\,, \quad \mathbb{P}-\text{a.s.},$$

with

$$\begin{aligned}
\mathrm{d}K_r^{\varepsilon,\delta} &= \Big[H_\varepsilon(r,Y_r^\varepsilon,Z_r^\varepsilon) - H_\delta(r,Y_r^\delta,Z_r^\delta) - \big[\nabla_y\Psi^\varepsilon(r,Y_r^\varepsilon) - \nabla_y\Psi^\delta(r,Y_r^\delta)\big]\Big]\mathrm{d}Q_r \\
&= \alpha_r\left[F_\varepsilon(r,Y_r^\varepsilon,Z_r^\varepsilon) - F_\delta(r,Y_r^\delta,Z_r^\delta)\right]\,\mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r)\,\mathrm{d}r \\
&\quad + (1-\alpha_r)\left[G_\varepsilon(r,Y_r^\varepsilon) - G_\delta(r,Y_r^\delta)\right]\,\mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r)\,\mathrm{d}A_r \\
&\quad + \alpha_r\,F_\delta(r,Y_r^\delta,Z_r^\delta)\Big(\mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r) - \mathbf{1}_{[0,\frac{1}{\delta}]}(A_r)\Big)\mathrm{d}r \\
&\quad + (1-\alpha_r)\,G_\delta(r,Y_r^\delta)\Big(\mathbf{1}_{[0,\frac{1}{\varepsilon}]}(A_r) - \mathbf{1}_{[0,\frac{1}{\delta}]}(A_r)\Big)\mathrm{d}A_r \\
&\quad - \alpha_r\left[\nabla\varphi_\varepsilon(Y_r^\varepsilon) - \nabla\varphi_\delta(Y_r^\delta)\right]\mathrm{d}r - (1-\alpha_r)\left[\nabla\psi_\varepsilon(Y_r^\varepsilon) - \nabla\psi_\delta(Y_r^\delta)\right]\mathrm{d}A_r\,.
\end{aligned}$$

By inequalities (2.17) and property (6.52$-c$) and (5.9$-a$) we have (since $n_p \le 1$)

$$\begin{aligned}
\left\langle Y_r^\varepsilon - Y_r^\delta, \mathrm{d}K_r^{\varepsilon,\delta}\right\rangle &\le \mathrm{d}R_r^{\varepsilon,\delta} + |Y_r^\varepsilon - Y_r^\delta| dN_r^{\varepsilon,\delta} + |Y_r^\varepsilon - Y_r^\delta|^2\mathrm{d}V_r^{(+)} + \frac{\lambda}{2}\,|Z_r^\varepsilon - Z_r^\delta|^2\mathrm{d}r \\
&\le (1+2\tilde\rho_0)\,d\left(R_r^{\varepsilon,\delta} + N_r^{\varepsilon,\delta}\right) + |Y_r^\varepsilon - Y_r^\delta|^2\mathrm{d}V_r^{(+)} + \frac{\lambda}{2}\,|Z_r^\varepsilon - Z_r^\delta|^2\mathrm{d}r,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{d}R_r^{\varepsilon,\delta} &= |\varepsilon-\delta|\,\Big[\mu_r^+\,|\varepsilon-\delta| + 2F_{1+\tilde\rho_0}^\#(r) + 2\ell_r\,|Z_r^\varepsilon|\Big]\,\mathrm{d}r + |\varepsilon-\delta|\,\Big[\nu_r^+\,|\varepsilon-\delta| + 2G_{1+\tilde\rho_0}^\#(r)\Big]\,\mathrm{d}A_r \\
&\quad + \frac{\varepsilon+\delta}{2}\big(|\nabla\varphi_\varepsilon(Y_r^\varepsilon)|^2 + |\nabla\varphi_\delta(Y_r^\delta)|^2\big)\mathrm{d}r + \frac{\varepsilon+\delta}{2}\big(|\nabla\psi_\varepsilon(Y_r^\varepsilon)|^2 + |\nabla\psi_\delta(Y_r^\delta)|^2\big)\mathrm{d}A_r
\end{aligned}$$

and

$$\begin{aligned}
dN_r^{\varepsilon,\delta} &= \Big[2\mu_r^+\,|\varepsilon-\delta| + \ell_r\,\big|Z_r^\delta\big|\,\mathbf{1}_{[\frac{1}{\varepsilon}\wedge\frac{1}{\delta},\infty)}\,\big(\big|Z_r^\delta\big| + A_r\big)\,\mathbf{1}_{\varepsilon\neq\delta} \\
&\quad + \big(F_{1+\tilde\rho_0}^\#(r) + \ell_r\,\big|Z_r^\delta\big|\,\big)\,\mathbf{1}_{[\frac{1}{\varepsilon}\wedge\frac{1}{\delta},\infty)}\big(F_{1+\tilde\rho_0}^\#(r) + A_r\big)\Big]\mathrm{d}r \\
&\quad + \Big[2\nu_r^+\,|\varepsilon-\delta| + G_{1+\tilde\rho_0}^\#(r)\,\mathbf{1}_{[\frac{1}{\varepsilon}\wedge\frac{1}{\delta},\infty)}\big(G_{1+\tilde\rho_0}^\#(r) + A_r\big)\Big]\,\mathrm{d}A_r\,.
\end{aligned}$$

By (5.8) and Proposition 6.8 we get

$$\mathbb{E}\sup\nolimits_{r\in[0,T]} e^{2V_r^{(+)}}|Y_r^\varepsilon - Y_r^\delta|^2 + \mathbb{E}\int_0^T e^{2V_r^{(+)}}|Z_r^\varepsilon - Z_r^\delta|^2\mathrm{d}r \le C_\lambda\,\mathbb{E}\int_0^T e^{2V_r^{(+)}}d\left(R_r^{\varepsilon,\delta} + N_r^{\varepsilon,\delta}\right).$$

Boundedness assumptions (2.2), (2.6), (5.1), (5.2), (5.4) and (5.9$-b$), (5.16$-b$) yield

$$\lim\nolimits_{\varepsilon,\delta\to0}\mathbb{E}\int_0^T e^{2V_r^{(+)}}d\left(R_r^{\varepsilon,\delta} + N_r^{\varepsilon,\delta}\right) = 0 \tag{5.17}$$

(also the calculus for obtaining (5.8) is useful).

For instance, if we denote

$$\bar{H}^{\#}_{1+\tilde{\rho}_0}(t) \overset{\text{def}}{=\!=} \alpha_t F^{\#}_{1+\tilde{\rho}_0}(t) + (1-\alpha_t)\, G^{\#}_{1+\tilde{\rho}_0}(t),$$

we deduce

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{2V_r^{(+)}}\Big[F^{\#}_{1+\tilde{\rho}_0}(r)\,\mathrm{d}r + G^{\#}_{1+\tilde{\rho}_0}(r)\,\mathrm{d}A_r\Big] = \mathbb{E}\int_0^T e^{2V_r^{(+)}}\bar{H}^{\#}_{1+\tilde{\rho}_0}(r)\,\mathrm{d}Q_r\\
&\leq \Big(\mathbb{E}\int_0^T e^{2V_r^{(+)}}\big(\bar{H}^{\#}_{1+\tilde{\rho}_0}(r)\big)^2\mathrm{d}Q_r\Big)^{1/2}\Big(\mathbb{E}\int_0^T e^{2V_r^{(+)}}\mathrm{d}Q_r\Big)^{1/2}\\
&\leq \sqrt{2}\Big(\mathbb{E}\int_0^T e^{2V_r^{(+)}}\Big[\big(F^{\#}_{1+\tilde{\rho}_0}(r)\big)^2\mathrm{d}r + \big(G^{\#}_{1+\tilde{\rho}_0}(r)\big)^2\mathrm{d}A_r\Big]\Big)^{1/2}\Big(\mathbb{E}e^{2V_T^{(+)}}Q_T\Big)^{1/2}
\end{aligned}$$

which is finite, using assumption (5.4) and inequality (5.13).

For instance, for any $a > 0$,

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{2V_r^{(+)}}\left(\mu_r^+\,\mathrm{d}r + \nu_r^+\,\mathrm{d}A_r\right) \leq \mathbb{E}\Big(e^{2V_T^{(+)}}\int_0^T\left(\mu_r^+\,\mathrm{d}r + \nu_r^+\,\mathrm{d}A_r\right)\Big)\\
&\leq \frac{2}{2+a}\,\mathbb{E}\exp(2+a)\,V_T^{(+)} + \frac{a}{2+a}\,\mathbb{E}\Big(\int_0^T\left(\mu_r^+\,\mathrm{d}r + \nu_r^+\,\mathrm{d}A_r\right)\Big)^{(2+a)/a}.
\end{aligned}$$

On the other hand, by Holder's inequality,

$$\begin{aligned}
\mathbb{E}\Big(\int_0^T\left(\mu_r^+\,\mathrm{d}r + \nu_r^+\,\mathrm{d}A_r\right)\Big)^{(2+a)/a} &\leq \Big(\mathbb{E}\Big(\int_0^T\left(\mu_r^+\,\mathrm{d}r + \nu_r^+\,\mathrm{d}A_r\right)\Big)^k\Big)^{\frac{2+a}{ak}}\\
&\leq \Big(\frac{k!}{p^k}\Big)^{\frac{2+a}{ak}}\Big(\mathbb{E}\Big(e^{p\int_0^T\left(\mu_r^+\,\mathrm{d}r + \nu_r^+\,\mathrm{d}A_r\right)}\Big)\Big)^{\frac{2+a}{ak}},
\end{aligned}$$

where $\mathbb{N}^* \ni k = \left[\frac{2+a}{a}\right] + 1 > \frac{2+a}{a}$, since

$$\alpha^k \leq \frac{k!}{p^k}\,e^{p\alpha}, \quad \text{for any } \alpha > 0, p > 1.$$

Hence $\mathbb{E}\int_0^T e^{2V_r^{(+)}}\left(\mu_r^+\,\mathrm{d}r + \nu_r^+\,\mathrm{d}A_r\right) < \infty$.

For instance,

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{2V_r^{(+)}}\ell_r\,\left|Z_r^\delta\right|\,\mathbf{1}_{[\frac{1}{\varepsilon}\wedge\frac{1}{\delta},\infty)}\left(\left|Z_r^\delta\right| + A_r\right)\mathrm{d}r \leq L(\varepsilon+\delta)\,\mathbb{E}\int_0^T e^{2V_r^{(+)}}\left|Z_r^\delta\right|\left(\left|Z_r^\delta\right| + A_r\right)\mathrm{d}r\\
&= L(\varepsilon+\delta)\Big(\mathbb{E}\int_0^T e^{2V_r^{(+)}}\left|Z_r^\delta\right|^2\mathrm{d}r + \mathbb{E}\int_0^T e^{2V_r^{(+)}}\left|Z_r^\delta\right|\,A_r\,\mathrm{d}r\Big),
\end{aligned}$$

since

$$\mathbf{1}_{[\frac{1}{\varepsilon}\wedge\frac{1}{\delta},\infty)}\left(\left|Z_r^\delta\right| + A_r\right) \leq \frac{\left|Z_r^\delta\right| + A_r}{\frac{1}{\varepsilon}\wedge\frac{1}{\delta}} \leq (\varepsilon+\delta)\left(\left|Z_r^\delta\right| + A_r\right), \quad \text{for any } r.$$

On the other hand,

$$\begin{aligned}\mathbb{E}\int_0^T e^{2V_r^{(+)}}\left|Z_r^\delta\right| A_r\,\mathrm{d}r &\leq \frac{1}{2}\mathbb{E}\int_0^T e^{2V_r^{(+)}}\left|Z_r^\delta\right|^2\mathrm{d}r+\frac{1}{2}\mathbb{E}\int_0^T e^{2V_r^{(+)}}A_r^2\,\mathrm{d}r\\
&\leq \frac{1}{2}\mathbb{E}\int_0^T e^{2V_r^{(+)}}\left|Z_r^\delta\right|^2\mathrm{d}r+\frac{T}{2}\,\mathbb{E}\left(e^{2V_T^{(+)}}A_T^2\right)\\
&\leq \frac{1}{2}\mathbb{E}\int_0^T e^{2V_r^{(+)}}\left|Z_r^\delta\right|^2\mathrm{d}r+\frac{2}{2+\delta}\,\mathbb{E}\exp\left(2+\delta\right)V_T^{(+)}+\frac{\delta}{2+\delta}\,\mathbb{E}A_T^{(4+2\delta)/\delta}\\
&\leq \frac{1}{2}\mathbb{E}\int_0^T e^{2V_r^{(+)}}\left|Z_r^\delta\right|^2\mathrm{d}r+\frac{2}{2+\delta}\,\mathbb{E}\exp\left(2+\delta\right)V_T^{(+)}+\frac{\delta}{2+\delta}\,\mathbb{E}e^{\frac{4+2\delta}{\delta}A_T}.\end{aligned}$$

Hence $\mathbb{E}\int_0^T e^{2V_r^{(+)}}\ell_r\left|Z_r^\delta\right|\mathbf{1}_{[\frac{1}{\varepsilon}\wedge\frac{1}{\delta},\infty)}\left(\left|Z_r^\delta\right|+A_r\right)\mathrm{d}r\to 0$, as $\varepsilon,\delta\to 0$.

Using the similar calculus for the other quantities, conclusion (5.17) is completely proved.

Step V. *Passing to the limit*

Consequently there exists $(Y,Z)\in S_m^0\times\Lambda_{m\times k}^0$ such that

$$\mathbb{E}\sup\nolimits_{r\in[0,T]}e^{2V_r^{(+)}}|Y_r^\varepsilon-Y_r|^2+\mathbb{E}\int_0^T e^{2V_r^{(+)}}|Z_r^\varepsilon-Z_r|^2\mathrm{d}r\to 0,\quad \text{as } \varepsilon\to 0.$$

From (5.16) there exist two p.m.s.p. $U^{(1)}$ and $U^{(2)}$, such that along a sequence $\varepsilon_n\to 0$, we have

$$\begin{aligned}&e^{V^{(+)}}\nabla\varphi_{\varepsilon_n}(Y^{\varepsilon_n})\longrightarrow e^{V^{(+)}}U^{(1)},\quad \text{weakly in } L^2\left(\Omega\times[0,T],d\mathbb{P}\otimes\mathrm{d}t;\mathbb{R}^m\right),\\
&e^{V^{(+)}}\nabla\psi_{\varepsilon_n}(Y^{\varepsilon_n})\longrightarrow e^{V^{(+)}}U^{(2)},\quad \text{weakly in } L^2\left(\Omega\times[0,T],d\mathbb{P}\otimes\mathrm{d}A_t;\mathbb{R}^m\right).\end{aligned}$$

Passing to limit in the approximating equation (5.5), for $\varepsilon=\varepsilon_n\to 0$, we infer

$$Y_t+\int_t^T U_r\mathrm{d}Q_r=\eta+\int_t^T H(r,Y_r,Z_r)\mathrm{d}Q_r-\int_t^T Z_r\mathrm{d}B_r\,,\ \mathbb{P}-\text{a.s.},$$

where

$$U_r\overset{\text{def}}{=\!=}\left[\alpha_rU_r^1+(1-\alpha_r)\,U_r^2\right],\ \text{for } r\in[0,T]\,.$$

Since $\nabla\varphi_\varepsilon(y)\in\partial\varphi\left(y-\varepsilon\nabla\varphi_\varepsilon(y)\right)$ then for all $E\in\mathcal{F}$, $0\leq t\leq s\leq T$ and $X\in S_m^2\left[0,T\right]$

$$\begin{aligned}\mathbb{E}\int_t^s\left\langle e^{2V_r^{(+)}}\nabla\varphi_{\varepsilon_n}(Y_r^{\varepsilon_n}),X_r-Y_r^{\varepsilon_n}\right\rangle\mathbf{1}_E\,\mathrm{d}r+\mathbb{E}\int_t^s e^{2V_r^{(+)}}\varphi(Y_r^{\varepsilon_n}-\varepsilon\nabla\varphi_\varepsilon(Y_r^{\varepsilon_n}))\,\mathbf{1}_E\,\mathrm{d}r&\\
\leq\mathbb{E}\int_t^s e^{2V_r^{(+)}}\varphi(X_r)\,\mathbf{1}_E\,\mathrm{d}r.&\end{aligned}$$

Passing to $\liminf_{n\to\infty}$ in the above inequality we obtain $U_s^{(1)}\in\partial\varphi(Y_s)$, $\quad d\mathbb{P}\otimes\mathrm{d}s$-a.e. and, with similar arguments, $U_s^{(2)}\in\partial\psi(Y_s)$, $\quad d\mathbb{P}\otimes\mathrm{d}A_s$-a.e..

We conclude that $(Y,Z,U)\in S_m^0\left[0,T\right]\times\Lambda_{m\times k}^0\left[0,T\right]\times\Lambda_m^0\left[0,T\right]$ is a strong solution of

$$\left\{\begin{array}{l}Y_t+\displaystyle\int_t^T\big(U_s^{(1)}\mathrm{d}s+U_s^{(2)}\mathrm{d}A_s\big)=\eta+\int_t^T\left[F\left(s,Y_s,Z_s\right)\mathrm{d}s+G\left(s,Y_s\right)\mathrm{d}A_s\right]-\int_t^T Z_s\mathrm{d}B_s\,,\\ \hfill\text{for any } t\in\left[0,T\right],\\ U_s^{(1)}\in\partial\varphi(Y_s),\quad d\mathbb{P}\otimes\mathrm{d}s-\text{a.e.}\quad\text{and}\quad U_s^{(2)}\in\partial\psi(Y_s),\quad d\mathbb{P}\otimes\mathrm{d}A_s-\text{a.e.},\quad\text{on }\left[0,T\right].\end{array}\right.\tag{5.18}$$

Moreover, from (5.9),

$$\begin{array}{ll}(a) & \left|Y_t\right|\leq e^{V_t^{(+)}}\left|Y_t\right|\leq\left[\mathbb{E}^{\mathcal{F}_t}\left(\sup_{r\in[t,T]}\left|e^{V_r^{(+)}}Y_r\right|^2\right)\right]^{1/2}\leq\tilde{\rho}_0\,,\\ (b) & \mathbb{E}\bigg(\displaystyle\int_0^T e^{2V_r^{(+)}}\left|Z_r\right|^2\mathrm{d}r\bigg)\leq\tilde{\rho}_0^2\,.\end{array}\tag{5.19}$$

From inequalities (6.49) and (5.9$-a$) we have

$$\begin{array}{l}\big(\left|F\right|_\varepsilon\left(r,Y_r^\varepsilon,Z_r^\varepsilon\right)\big)^2\leq 2L^2\left|Z_r^\varepsilon\right|^2+2\big(F_{1+\tilde{\rho}_0}^{\#}\left(r\right)\big)^2\quad\text{and}\\ \big(\left|G\right|_\varepsilon\left(r,Y_r^\varepsilon\right)\big)^2\leq\big(G_{1+\tilde{\rho}_0}^{\#}\left(r\right)\big)^2\end{array}$$

and therefore, passing to $\liminf_{\varepsilon\to0}$ in the first inequality of (5.15) and using (6.50$-b$) and Fatou's Lemma and Lebesgue's dominated convergence theorem, we have

$$\begin{array}{l}\dfrac{1}{2}\,\mathbb{E}\displaystyle\int_0^T e^{2V_r^{(+)}}\Big[\big|U_r^{(1)}\big|^2\mathrm{d}r+\big|U_r^{(2)}\big|^2\mathrm{d}A_r\Big]\\ \leq\mathbb{E}\Big[e^{2V_T^{(+)}}\left(\varphi(\eta)+\psi(\eta)\right)\Big]+\mathbb{E}\displaystyle\int_0^T e^{2V_r^{(+)}}\Big(1+6L^2\left|Z_r\right|^2+6\left|F\left(r,Y_r,0\right)\right|^2\Big)\mathrm{d}r\\ \quad+\mathbb{E}\displaystyle\int_0^T e^{2V_r^{(+)}}\Big(1+3\left|G\left(r,Y_r\right)\right|^2\Big)\mathrm{d}A_r\,.\end{array}\tag{5.20}$$

□

The next result, Proposition 5.2, provides the proof for the existence of a variational solution but under some still strong hypotheses (more precisely, assumption (5.3) of Lemma 5.1 is relaxed and becomes (5.21) and we have added (5.22) and (5.23)).

Proposition 5.2. *We suppose that assumptions* $(\mathrm{A}_2-\mathrm{A}_7)$ *are satisfied. Let* $V^{(+)}$ *be given by definition (2.11). In addition we assume that:*

(i) (see (5.3)) there exists $\hat{L}>0$ *such that*

$$\big|e^{V_T^{(+)}}\eta\big|^2+\Big(\int_0^T e^{V_r^{(+)}}\left(\left|F\left(r,0,0\right)\right|\mathrm{d}r+\left|G\left(r,0\right)\right|\mathrm{d}A_r\right)\Big)^2\leq\hat{L};\tag{5.21}$$

(ii) there exists $a\in(1+n_p\lambda,p\wedge2)$ *such that*

$$\begin{array}{ll}(a) & \mathbb{E}\bigg(\displaystyle\int_0^T\ell_s^2\mathrm{d}s\bigg)^{\frac{a}{2-a}}<\infty,\\ (b) & \mathbb{E}\bigg[\displaystyle\int_0^T e^{V_s^{(+)}}\Big(F_{1+\hat{\rho}_0}^{\#}\left(s\right)\mathrm{d}s+G_{1+\hat{\rho}_0}^{\#}\left(s\right)\mathrm{d}A_s\Big)\bigg]^a<\infty,\end{array}\tag{5.22}$$

where[4] $\hat{\rho}_0 \overset{\text{def}}{=\!=} (C_\lambda \hat{L})^{1/2}$;

(iii) there exists a p.m.s.p. $(\Theta_t)_{t\geq 0}$ *and, for each* $\rho \geq 0$, *there exist an non-decreasing function* $K_\rho : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$F_\rho^{\#}(t) + G_\rho^{\#}(t) \leq K_\rho(\Theta_t), \quad a.e.\ t \in [0,T], \ \mathbb{P}-a.s.. \tag{5.23}$$

Then the multivalued BSDE

$$\begin{cases} Y_t + \int_t^T \mathrm{d}K_r = \eta + \int_t^T H(r, Y_r, Z_r)\,\mathrm{d}Q_r - \int_t^T Z_r \mathrm{d}B_r, & \mathbb{P}-a.s., \text{ for all } t \in [0,T], \\ \mathrm{d}K_r = U_r \mathrm{d}Q_r \in \partial_y \Psi(r, Y_r)\,\mathrm{d}Q_r \end{cases}$$

has a unique L^p*-variational solution, in the sense of Definition 3.2.*

Moreover, for all $t \in [0,T]$, $\mathbb{P}$*-a.s.,*

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_t}\Big(\sup\nolimits_{s\in[t,T]} \big|e^{V_s^{(+)}} Y_s\big|^p\Big) + \mathbb{E}^{\mathcal{F}_t}\Big(\int_t^T e^{2V_s^{(+)}} (\varphi(Y_s)\,\mathrm{d}s + \psi(Y_s)\,\mathrm{d}A_s)\Big)^{p/2} \\ &\qquad + \mathbb{E}^{\mathcal{F}_t}\Big(\int_0^T e^{2V_s^{(+)}} |Z_s|^2\,\mathrm{d}s\Big)^{p/2} \\ &\leq C_{p,\lambda}\, \mathbb{E}^{\mathcal{F}_t}\Big[e^{pV_T^{(+)}} |\eta|^p + \Big(\int_t^T e^{V_s^{(+)}} (|F(r,0,0)|\,\mathrm{d}r + |G(t,0)|\,\mathrm{d}A_r)\Big)^p\Big]. \end{aligned} \tag{5.24}$$

Proof. **Step I.** *Approximating equation*

Let $t \in [0,T]$, $n \in \mathbb{N}^*$ and

$$\beta_t = t + A_t + |\mu_t| + |\nu_t| + \ell_t + V_t^{(+)} + F_{1+\hat{\rho}_0}^{\#}(t) + G_{1+\hat{\rho}_0}^{\#}(t) + \Theta_t.$$

Consider the BSDE

$$\begin{cases} Y_t^{(n)} + \int_t^T U_s^{(n)} \mathrm{d}Q_s = \eta^{(n)} + \int_t^T H^{(n)}\big(s, Y_s^{(n)}, Z_s^{(n)}\big)\mathrm{d}Q_s - \int_t^T Z_s^{(n)} \mathrm{d}B_s, \quad t \in [0,T], \\ U_s^{(n)} = \alpha_r U_r^{(1,n)} + (1-\alpha_r)\, U_r^{(2,n)} \\ U_s^{(1,n)} \in \partial\varphi(Y_s^{(n)}), \quad d\mathbb{P}\otimes \mathrm{d}s - \text{a.e.} \quad \text{and} \quad U_s^{(2,n)} \in \partial\psi(Y_s^{(n)}), \quad d\mathbb{P}\otimes \mathrm{d}A_s - \text{a.e.,} \end{cases} \tag{5.25}$$

where

$$\begin{aligned} &\eta^{(n)} \overset{\text{def}}{=\!=} \eta\, \mathbf{1}_{[0,n]}\big(|\eta| + \varphi(\eta) + \psi(\eta) + V_T^{(+)}\big), \\ &F^{(n)}(t,y,z) \overset{\text{def}}{=\!=} F(t,y,z)\ \mathbf{1}_{[0,n]}(\beta_t) \quad \text{and} \quad G^{(n)}(t,y) \overset{\text{def}}{=\!=} G(t,y)\ \mathbf{1}_{[0,n]}(\beta_t), \\ &H^{(n)}(s,y,z) \overset{\text{def}}{=\!=} \alpha_s F^{(n)}(s,y,z) + (1-\alpha_s)\, G^{(n)}(s,y). \end{aligned}$$

If we denote

$$\mu_t^{(n)} \overset{\text{def}}{=\!=} \mathbf{1}_{[0,n]}(\beta_t)\ \mu_t, \qquad \nu_t^{(n)} \overset{\text{def}}{=\!=} \mathbf{1}_{[0,n]}(\beta_t)\ \nu_t, \qquad \ell_t^{(n)} \overset{\text{def}}{=\!=} \mathbf{1}_{[0,n]}(\beta_t)\ \ell_t,$$

[4] The constant $C_\lambda := C_{2,\lambda}$, where $C_{2,\lambda}$ is given by (6.35).

then we have

$$\begin{aligned}&\left\langle y-\hat{y},H^{(n)}(t,y,z)-H^{(n)}(t,\hat{y},z)\right\rangle \leq \left(\alpha_t\mu_t^{(n)}+(1-\alpha_t)\,\nu_t^{(n)}\right)|y-\hat{y}|^2\,,\\&\left|H^{(n)}(t,y,z)-H^{(n)}(t,y,\hat{z})\right|\leq \alpha_t\ell_t^{(n)}\ |z-\hat{z}|\,.\end{aligned}$$

Of course,

$$\begin{aligned}&\left|\eta^{(n)}\right|\leq n\,\mathbf{1}_{[0,n]}\big(\,|\eta|+V_T^{(+)}\big),\\&\left|\mu_t^{(n)}\right|\leq n\,\mathbf{1}_{[0,n]}\left(\beta_t\right), \quad \left|\nu_t^{(n)}\right|\leq n\,\mathbf{1}_{[0,n]}\left(\beta_t\right), \quad \left|\ell_t^{(n)}\right|\leq n\,\mathbf{1}_{[0,n]}\left(\beta_t\right),\\&F_1^{(n)\#}\left(t\right)=\sup\nolimits_{|u|\leq 1}\left|F^{(n)}\left(t,u,0\right)\right|\leq n\,\mathbf{1}_{[0,n]}\left(\beta_t\right),\\&G_1^{(n)\#}\left(t\right)=\sup\nolimits_{|u|\leq 1}\left|G^{(n)}\left(t,u\right)\right|\leq n\,\mathbf{1}_{[0,n]}\left(\beta_t\right).\end{aligned}$$

Let

$$\theta_n:=\inf\big\{r\geq 0:r+A_r+V_r^{(+)}>n\big\}.$$

We have $\mathbf{1}_{[0,n]}\left(\beta_r\right)\leq \mathbf{1}_{[0,\theta_n]}\left(r\right)$ and therefore

$$\begin{aligned}V_t^{(n,+)}&\overset{\text{def}}{=\!=}\int_0^t\left[\left(\mu_r^{(n)}+\frac{1}{2n_p\lambda}\left(\ell_r^{(n)}\right)^2\right)^+\mathrm{d}r+\left(\nu_r^{(n)}\right)^+\mathrm{d}A_r\right]\\&=\int_0^t\mathbf{1}_{[0,n]}\left(\beta_r\right)\left[\left(\mu_r+\frac{1}{2n_p\lambda}\left(\ell_r\right)^2\right)^+\mathrm{d}r+\nu_r^+\mathrm{d}A_r\right]\\&\leq\int_0^t\mathbf{1}_{[0,\theta_n]}\left(\beta_r\right)\left[\left(\mu_r+\frac{1}{2n_p\lambda}\left(\ell_r\right)^2\right)^+\mathrm{d}r+\nu_r^+\mathrm{d}A_r\right]\\&=V_{t\wedge\theta_n}^{(+)}\leq V_{\theta_n}^{(+)}\leq n\end{aligned}\tag{5.26}$$

and

$$\begin{aligned}&\left|e^{V_T^{(n,+)}}\eta^{(n)}\right|^2+\Big(\int_0^T e^{V_r^{(n,+)}}\big(F_1^{(n)\#}\left(r\right)\mathrm{d}r+G_1^{(n)\#}\left(r\right)\mathrm{d}A_r\big)\Big)^2\\&\leq n^2e^{2n}+n^2e^{2n}\bigg(\int_0^{T\wedge\theta_n}\left(\mathrm{d}r+\mathrm{d}A_r\right)\bigg)^2\leq n^2e^{2n}\left(1+n^2\right)=\tilde{L}^{(n)}\end{aligned}$$

and, for every $\rho\geq 0$,

$$F_\rho^{(n)\#}\left(t\right)+G_\rho^{(n)\#}\left(t\right)\leq\left[F_\rho^{\#}\left(t\right)+G_\rho^{\#}\left(t\right)\right]\,\mathbf{1}_{[0,n]}\left(\beta_t\right)\leq K_\rho\left(\Theta_t\right)\,\mathbf{1}_{[0,n]}\left(\beta_t\right)\leq K_\rho\left(n\right).$$

Therefore assumptions (5.1)–(5.4) are satisfied.

Hence, by Lemma 5.1, there exists a unique (strong) solution $\left(Y^{(n)},Z^{(n)},U^{(n)}\right)\in S_m^0\left[0,T\right]\times\Lambda_{m\times k}^0\left(0,T\right)\times\Lambda_m^0\left(0,T\right)$ of BSDE (5.25).

Step II. *Boundedness of the approximating solution*

We have

$$\begin{aligned}
&\langle Y_t^{(n)}, H^{(n)}(t, Y_t^{(n)}, Z_t^{(n)}) - U_t^{(n)}\rangle \mathrm{d}Q_t \\
&\leq \Big[\Big(\alpha_t \mu_t + (1-\alpha_t)\,\nu_t + \alpha_t\,\frac{1}{2n_p\lambda}\,\ell_t^2\Big)\,\mathbf{1}_{[0,n]}\,(\beta_t)\,\big|Y_t^{(n)}\big|^2 \\
&\quad +\alpha_t\,\mathbf{1}_{[0,n]}\,(\beta_t)\,\frac{n_p\lambda}{2}\,\big|Z_t^{(n)}\big|^2 + |H^{(n)}\,(t,0,0)\,|\big|Y_t^{(n)}\big|\Big]\mathrm{d}Q_t \\
&\leq \big|Y_t^{(n)}\big|\mathrm{d}\bar{N}_t + \big|Y_t^{(n)}\big|^2 \mathrm{d}V_t^{(n,+)} + \frac{\lambda}{2}\,\big|Z_t^{(n)}\big|^2 \mathrm{d}t,
\end{aligned} \tag{5.27}$$

where

$$\bar{N}_t := \int_0^t \left[|F\,(r,0,0)|\,\mathrm{d}r + |G\,(r,0)|\,\mathrm{d}A_r\right]. \tag{5.28}$$

Since by (5.19), for all $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$\big|Y_t^{(n)}\big|^2 \leq \big|e^{V_t^{(n,+)}} Y_t^{(n)}\big|^2 \leq \mathbb{E}^{\mathcal{F}_t}\Big(\sup\nolimits_{r\in[t,T]}\big|e^{V_r^{(n,+)}} Y_r^{(n)}\big|^2\Big) \leq C_\lambda \tilde{L}^{(n)} =: (\tilde{\rho}_0^n)^2$$

and $\big|\eta^{(n)}\big| \leq |\eta|$, we deduce, using the stochastic Gronwall's type inequality provided by Proposition 6.8, applied to (5.27), that, for all $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\Big(\sup\nolimits_{r\in[t,T]}\big|e^{V_r^{(n,+)}} Y_r^{(n)}\big|^2\Big) + \mathbb{E}^{\mathcal{F}_t}\Big(\int_t^T e^{2V_r^{(n,+)}}\big|Z_r^{(n)}\big|^2 \mathrm{d}r\Big) \\
&\leq C_\lambda\,\mathbb{E}^{\mathcal{F}_t}\Big[\big|e^{V_T^{(n,+)}}\eta\big|^2 + \Big(\int_t^T e^{V_r^{(n,+)}}\mathrm{d}\bar{N}_r\Big)^2\Big] \\
&\leq C_\lambda\,\mathbb{E}^{\mathcal{F}_t}\Big[\big|e^{V_T^{(+)}}\eta\big|^2 + \Big(\int_t^T e^{V_r^{(+)}}\mathrm{d}\bar{N}_r\Big)^2\Big]
\end{aligned}$$

(the constant $C_\lambda := C_{2,\lambda}$, where $C_{2,\lambda}$ is given by (6.35)).

Therefore, by assumption (5.21) we have, for all $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&(a)\quad \big|Y_t^{(n)}\big| \leq \big|e^{V_t^{(n,+)}} Y_t^{(n)}\big| \leq \Big[\mathbb{E}^{\mathcal{F}_t}\Big(\sup\nolimits_{r\in[t,T]}\big|e^{V_r^{(n,+)}} Y_r^{(n)}\big|^2\Big)\Big]^{1/2} \leq (C_\lambda \hat{L})^{1/2} = \hat{\rho}_0\,, \\
&(b)\quad \mathbb{E}\int_0^T e^{2V_r^{(n,+)}}\big|Z_r^{(n)}\big|^2 \mathrm{d}r \leq \hat{\rho}_0^2\,.
\end{aligned} \tag{5.29}$$

Step III. *Cauchy sequence*

Let $n, i \in \mathbb{N}^*$ arbitrary fixed. We have

$$\begin{aligned}
&Y_t^{(n+i)} - Y_t^{(n)} + \int_t^T \big(U_s^{(n+i)} - U_s^{(n)}\big)\mathrm{d}Q_s \\
&= \eta^{(n+i)} - \eta^{(n)} + \int_t^T \Big(H^{(n+i)}\big(s, Y_s^{(n+i)}, Z_s^{(n+i)}\big) - H^{(n)}\big(s, Y_s^{(n)}, Z_s^{(n)}\big)\Big)\mathrm{d}Q_s \\
&\qquad\qquad - \int_t^T \big(Z_s^{(n+i)} - Z_s^{(n)}\big)\mathrm{d}B_s\,.
\end{aligned}$$

We recall the property

$$\left\langle Y_s^{(n+i)}-Y_s^{(n)},U_s^{(n+i)}-U_s^{(n)}\right\rangle \mathrm{d}Q_s\geq 0.$$

On the other hand

$$\begin{aligned}
&\left\langle Y_s^{(n+i)}-Y_s^{(n)},H^{(n+i)}\big(s,Y_s^{(n+i)},Z_s^{(n+i)}\big)-H^{(n)}\big(s,Y_s^{(n)},Z_s^{(n)}\big)\right\rangle \mathrm{d}Q_s\\
&=\left\langle Y_s^{(n+i)}-Y_s^{(n)},H^{(n+i)}\big(s,Y_s^{(n+i)},Z_s^{(n+i)}\big)-H^{(n+i)}\big(s,Y_s^{(n)},Z_s^{(n)}\big)\right\rangle \mathrm{d}Q_s\\
&\quad+\left\langle Y_s^{(n+i)}-Y_s^{(n)},H^{(n+i)}\big(s,Y_s^{(n)},Z_s^{(n)}\big)-H^{(n)}\big(s,Y_s^{(n)},Z_s^{(n)}\big)\right\rangle \mathrm{d}Q_s\\
&\leq \mathbf{1}_{[0,n+i]}(\beta_s)\Big(\mu_s\mathrm{d}s+\nu_s\mathrm{d}A_s+\frac{1}{2n_p\lambda}\,\ell_s^2\mathrm{d}s\Big)\big|Y_s^{(n+i)}-Y_s^{(n)}\big|^2+\frac{n_p\lambda}{2}\,\big|Z_s^{(n+i)}-Z_s^{(n)}\big|^2\mathrm{d}s\\
&\quad+\big|Y_s^{(n+i)}-Y_s^{(n)}\big|\;\big|\mathbf{1}_{[0,n+i]}(\beta_s)-\mathbf{1}_{[0,n]}(\beta_s)\big|\;\big(\ell_s\big|Z_s^{(n)}\big|+F_{\hat\rho_0}^{\#}(s)\,\mathrm{d}s+G_{\hat\rho_0}^{\#}(s)\,\mathrm{d}A_s\big)\\
&\leq \big|Y_s^{(n+i)}-Y_s^{(n)}\big|\,\mathbf{1}_{(n,\infty)}(\beta_s)\;\big(\ell_s\big|Z_s^{(n)}\big|+F_{\hat\rho_0}^{\#}(s)\,\mathrm{d}s+G_{\hat\rho_0}^{\#}(s)\,\mathrm{d}A_s\big)\\
&\quad+\big|Y_s^{(n+i)}-Y_s^{(n)}\big|^2\mathrm{d}V_s^{(+)}+\frac{n_a\lambda'}{2}\big|Z_s^{(n+i)}-Z_s^{(n)}\big|^2\mathrm{d}s,
\end{aligned}$$

since

$$\big|\mathbf{1}_{[0,n+i]}(\beta_s)-\mathbf{1}_{[0,n]}(\beta_s)\big|\leq \mathbf{1}_{(n,\infty)}(\beta_s) \tag{5.30}$$

and

$$n_p\lambda\leq n_a\lambda',\quad \text{with }\lambda':=\frac{n_p\lambda+a-1}{2\,(a-1)}\in(0,1)$$

(where a is given by assumption (5.22) and $n_a:=(a-1)\wedge 1=a-1$).

By (5.29−a) and Proposition 6.8 and Hölder's inequality we obtain:

$$\begin{aligned}
&\mathbb{E}\sup\nolimits_{s\in[0,T]}e^{aV_s^{(+)}}\big|Y_s^{(n+i)}-Y_s^{(n)}\big|^a+\mathbb{E}\Big(\int_0^T e^{aV_s^{(+)}}\big|Z_s^{(n+i)}-Z_s^{(n)}\big|^2\mathrm{d}s\Big)^{a/2}\\
&\leq C_{a,\lambda}\,\mathbb{E}\Big[e^{aV_T^{(+)}}\,|\eta|^a\;\mathbf{1}_{(n,\infty)}\big(|\eta|+V_T^{(+)}\big)\Big]\\
&\quad+C_{a,\lambda}\,\mathbb{E}\bigg(\int_0^T e^{V_s^{(+)}}\,\mathbf{1}_{(n,\infty)}(\beta_s)\,\Big[\ell_s\big|Z_s^{(n)}\big|\mathrm{d}s+F_{\hat\rho_0}^{\#}(s)\,\mathrm{d}s+G_{\hat\rho_0}^{\#}(s)\,\mathrm{d}A_s\Big]\bigg)^a\\
&\leq C_{a,\lambda}\,\mathbb{E}\Big[e^{aV_T^{(+)}}\,|\eta|^a\;\mathbf{1}_{(n,\infty)}\big(|\eta|+V_T^{(+)}\big)\Big]+2^{a-1}C_{a,\lambda}\,\mathbb{E}\Big(\int_0^T e^{V_s^{(+)}}\,\mathbf{1}_{(n,\infty)}(\beta_s)\;\ell_s\,\big|Z_s^{(n)}\big|\mathrm{d}s\Big)^a\\
&\quad+2^{a-1}C_{a,\lambda}\,\mathbb{E}\bigg(\int_0^T e^{V_s^{(+)}}\,\mathbf{1}_{(n,\infty)}(\beta_s)\,\Big(F_{\hat\rho_0}^{\#}(s)\,\mathrm{d}s+G_{\hat\rho_0}^{\#}(s)\,\mathrm{d}A_s\Big)\bigg)^a\\
&\leq C_{a,\lambda}\,\mathbb{E}\Big[e^{aV_T^{(+)}}\,|\eta|^a\;\mathbf{1}_{(n,\infty)}\big(|\eta|+V_T^{(+)}\big)\Big]\\
&\quad+C'_{a,\lambda}\mathbb{E}\bigg[\Big(\int_0^T e^{2\left(V_s^{(+)}-V_s^{(n,+)}\right)}\,\ell_s^2\,\mathbf{1}_{(n,\infty)}(\beta_s)\,\mathrm{d}s\Big)^{\frac{a}{2}}\Big(\int_0^T e^{2V_s^{(n,+)}}\,|Z_s^n|^2\,\mathrm{d}s\Big)^{\frac{a}{2}}\bigg]\\
&\quad+C'_{a,\lambda}\,\mathbb{E}\bigg(\int_0^T e^{V_s^{(+)}}\,\mathbf{1}_{(n,\infty)}(\beta_s)\,\Big(F_{\hat\rho_0}^{\#}(s)\,\mathrm{d}s+G_{\hat\rho_0}^{\#}(s)\,\mathrm{d}A_s\Big)\bigg)^a\\
&\leq C_{a,\lambda}\,\hat L^{\frac{a}{2}}\,\mathbb{E}\Big[\mathbf{1}_{(n,\infty)}\big(|\eta|+V_T^{(+)}\big)\Big]
\end{aligned}$$

$$+ C'_{a,\lambda}\left[\mathbb{E}\Big(\int_0^T e^{2\left(V_s^{(+)}-V_s^{(n,+)}\right)}\,\ell_s^2\,\mathbf{1}_{(n,\infty)}\left(\beta_s\right)\mathrm{d}s\Big)^{\frac{a}{2-a}}\right]^{\frac{2-a}{2}}\left[\mathbb{E}\Big(\int_0^T e^{2V_s^{(n,+)}}\left|Z_s^n\right|^2\mathrm{d}s\Big)\right]^{\frac{a}{2}}$$
$$+ C'_{a,\lambda}\,\mathbb{E}\Big(\int_0^T e^{V_s^{(+)}}\,\mathbf{1}_{(n,\infty)}\left(\beta_s\right)\Big(F_{\hat\rho_0}^{\#}\left(s\right)\mathrm{d}s+G_{\hat\rho_0}^{\#}\left(s\right)\mathrm{d}A_s\Big)\Big)^{a}.$$

Step IV. *Passing to the limit*
Hence there exists $(Y,Z)\in S_m^0\left[0,T\right]\times\Lambda_{m\times k}^0\left(0,T\right)$ such that

$$\begin{aligned}
&(j)\quad |Y_t|\le e^{V_t^{(+)}}\,|Y_t|\le (C_\lambda\hat L)^{1/2}=\hat\rho_0\,,\quad \text{for all } t\in\left[0,T\right],\ \mathbb{P}-\text{a.s.},\\
&(jj)\quad \mathbb{E}\int_0^T e^{2V_r^{(+)}}\,|Z_r|^2\,\mathrm{d}r\le\hat\rho_0^2\,,\\
&(jjj)\quad \lim_{n\to\infty}\Big[\mathbb{E}\sup\nolimits_{s\in[0,T]}e^{aV_s^{(+)}}\,|Y_s^n-Y_s|^a+\mathbb{E}\Big(\int_0^T e^{aV_s^{(+)}}\,|Z_s^n-Z_s|^2\,\mathrm{d}s\Big)^{a/2}\Big]=0,\\
&(jv)\quad (Y_t,Z_t)=(\eta,0)\,,\quad \text{for all } t>T.
\end{aligned}\tag{5.31}$$

Using (5.25) and assumption (A_4) we deduce

$$\varphi\big(Y_t^{(n)}\big)\mathrm{d}t+\psi\big(Y_t^{(n)}\big)\mathrm{d}A_t\le\langle Y_t^{(n)},U_t^{(1,n)}\rangle\mathrm{d}t+\langle Y_t^{(n)},U_t^{(2,n)}\rangle\mathrm{d}A_t$$

and therefore

$$\begin{aligned}
&\varphi\big(Y_t^{(n)}\big)\mathrm{d}t+\psi\big(Y_t^{(n)}\big)\mathrm{d}A_t+\langle Y_t^{(n)},H^{(n)}(t,Y_t^{(n)},Z_t^{(n)})-U_t^{(n)}\rangle\mathrm{d}Q_t\\
&\le\langle Y_t^{(n)},H^{(n)}(t,Y_t^{(n)},Z_t^{(n)})\rangle\mathrm{d}Q_t\\
&\le\Big[\Big(\alpha_t\mu_t+(1-\alpha_t)\,\nu_t+\alpha_t\,\frac{1}{2n_p\lambda}\,\ell_t^2\Big)\,\mathbf{1}_{[0,n]}\left(\beta_t\right)\big|Y_t^{(n)}\big|^2\\
&\quad+\alpha_t\,\mathbf{1}_{[0,n]}\left(\beta_t\right)\,\frac{n_p\lambda}{2}\,\big|Z_t^{(n)}\big|^2+\big|H^{(n)}\left(t,0,0\right)\big|\big|Y_t^{(n)}\big|\Big]\mathrm{d}Q_t\\
&\le\big|Y_t^{(n)}\big|\mathrm{d}\bar N_t+\big|Y_t^{(n)}\big|^2\mathrm{d}V_t^{(+)}+\frac{n_p\lambda}{2}\,\big|Z_t^{(n)}\big|^2\mathrm{d}r,
\end{aligned}$$

where $\bar N$ is defined by (5.28).
Also by (5.29) and assumption (2.6) we have

$$\mathbb{E}\sup\nolimits_{t\in[0,T]}e^{pV_t^{(+)}}\big|Y_t^{(n)}\big|^p\le\hat\rho_0^p\,\mathbb{E}\exp\Big(p\int_0^T\Big(|\mu_s|+\frac{1}{2n_p\lambda}\,\ell_s^2\Big)\,\mathrm{d}s+p\int_0^T|\nu_s|\,\mathrm{d}A_s\Big)<\infty.$$

Hence, by Proposition 6.8, we deduce that, for all $t\in\left[0,T\right]$,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\Big(\sup\nolimits_{s\in[t,T]}\big|e^{V_s^{(+)}}Y_s^{(n)}\big|^p\Big)+\mathbb{E}^{\mathcal{F}_t}\Big(\int_t^T e^{2V_s^{(+)}}\Big(\varphi\big(Y_s^{(n)}\big)\mathrm{d}s+\psi\big(Y_s^{(n)}\big)\mathrm{d}A_s\Big)\Big)^{p/2}\\
&\quad+\mathbb{E}^{\mathcal{F}_t}\Big(\int_t^T e^{2V_s^{(+)}}\big|Z_s^{(n)}\big|^2\mathrm{d}s\Big)^{p/2}\\
&\le C_{p,\lambda}\,\mathbb{E}^{\mathcal{F}_t}\Big[e^{pV_T^{(+)}}\,|\eta|^p+\Big(\int_t^T e^{V_s^{(+)}}\,\mathrm{d}\bar N_s\Big)^p\Big],\quad \mathbb{P}-\text{a.s.}.
\end{aligned}\tag{5.32}$$

By Remark 3.8 and inequality $V_t \leq V_t^{(+)}$, we see that $\left(Y^{(n)}, Z^{(n)}\right)$, as a strong solution of (5.25), is also an L^p-variational solution on $[0,T]$ for (5.25).

Hence, for

$$q \in \{2, p\wedge 2\}, \quad \delta_q = \delta\, \mathbf{1}_{[1,2)}\left(q\right) \quad \text{and} \quad \Gamma_t^{(n)} = \left(\left|M_t - Y_t^{(n)}\right|^2 + \delta_q\right)^{1/2},$$

it holds

$$\begin{aligned}
&\left(\Gamma_t^{(n)}\right)^q + \frac{q\left(q-1\right)}{2}\int_t^s \left(\Gamma_r^{(n)}\right)^{q-2}\left|R_r - Z_r^{(n)}\right|^2 \mathrm{d}r + q\int_t^s \left(\Gamma_r^{(n)}\right)^{q-2}\Psi\left(r, Y_r^{(n)}\right)\mathrm{d}Q_r \\
&\leq \left(\Gamma_s^{(n)}\right)^q + q\int_t^s \left(\Gamma_r^{(n)}\right)^{q-2}\Psi\left(r, M_r\right)\mathrm{d}Q_r \\
&\quad + q\int_t^s \left(\Gamma_r^{(n)}\right)^{q-2}\langle M_r - Y_r^{(n)}, N_r - H\left(r, Y_r^{(n)}, Z_r^{(n)}\right)\rangle \mathrm{d}Q_r \\
&\quad - q\int_t^s \left(\Gamma_r^{(n)}\right)^{q-2}\langle M_r - Y_r^{(n)}, \left(R_r - Z_r^{(n)}\right)\mathrm{d}B_r\rangle;
\end{aligned} \tag{5.33}$$

for any $0 \leq t \leq s < \infty$ and $M \in \mathcal{V}_m^0$ of form (3.2), *i.e.* $M_t = M_T + \int_t^T N_r \mathrm{d}Q_r - \int_t^T R_r \mathrm{d}B_r$.

By convergence result $(5.31-jjj)$ and assumptions $(\mathrm{A}_4 - \mathrm{A}_6)$ we can pass to $\liminf_{n\to\infty}$ (on a subsequence) in (5.32) and (5.33) to conclude that (Y,Z) is also an L^p-variational solution on $[0,T]$ and inequality (5.24) holds. ◻

Corollary 5.3. *Let the assumptions of Proposition 5.2 be satisfied. If we consider the particular case $\varphi = \psi = 0$, then the BSDE*

$$Y_t = \eta + \int_t^T H\left(r, Y_r, Z_r\right)\mathrm{d}Q_r - \int_t^T Z_r \mathrm{d}B_r\,, \quad \mathbb{P} - a.s.,\ t \in [0,T]\,, \tag{5.34}$$

has a unique strong solution $(Y,Z) \in S_m^p\left[0,T\right] \times \Lambda_{m\times k}^p\left(0,T\right)$.

Proof. Based on the results from (5.31) and assumptions $(\mathrm{A}_4 - \mathrm{A}_6)$ we can pass to limit $\lim_{n\to\infty}$ in the approximating equation (5.25) with $\varphi = \psi = 0$ and $U^{(1)} = U^{(2)} = 0$ to infer that (Y,Z) satisfies (5.34). From (5.24) and assumption (5.21) we get $(Y,Z) \in S_m^p\left[0,T\right] \times \Lambda_{m\times k}^p\left(0,T\right)$. Moreover by $(5.31-j)$

$$|Y_t| \leq \hat{\rho}_0\,, \quad \text{for all } t \in [0,T]\,,\ \mathbb{P} - \text{a.s.}.$$

◻

Another consequence of the proof of Proposition 5.2 is the existence of a strong solution under weaker conditions than those of Lemma 5.1.

Corollary 5.4. *Let the assumptions of Proposition 5.2 be satisfied. In addition we assume that:*

$$\begin{aligned}
(i) &\quad \mathbb{E}\left[e^{2V_T^{(+)}}\left(\varphi(\eta) + \psi(\eta)\right)\right] < \infty, \\
(ii) &\quad \mathbb{E}\int_0^T e^{2V_r^{(+)}}\mathrm{d}Q_r < \infty, \\
(iii) &\quad \mathbb{E}\int_0^T e^{2V_r^{(+)}}\left(\left|F_{\hat{\rho}_0}^{\#}\left(r\right)\right|^2 \mathrm{d}r + \left|G_{\hat{\rho}_0}^{\#}\left(r\right)\right|^2 \mathrm{d}A_r\right) < \infty.
\end{aligned} \tag{5.35}$$

Then the BSDE

$$\begin{cases} Y_t + \int_t^T \mathrm{d}K_r = Y_T + \int_t^T H(r, Y_r, Z_r)\, \mathrm{d}Q_r - \int_t^T Z_r \mathrm{d}B_r, \quad \mathbb{P}-a.s., \text{ for all } t \in [0,T], \\ \mathrm{d}K_r = U_r^{(1)}\mathrm{d}r + U_r^{(2)}\mathrm{d}A_r\,, \\ U^{(1)}\mathrm{d}r \in \partial\varphi(Y_r)\,\mathrm{d}r \quad \text{and} \quad U^{(2)}\mathrm{d}A_r \in \partial\psi(Y_r)\,\mathrm{d}A_r \end{cases}$$

has a unique strong a solution $\left(Y, Z, U^{(1)}, U^{(2)}\right) \in S_m^0 \times \Lambda_{m\times k}^0 \times \Lambda_m^0 \times \Lambda_m^0$ *such that*

$$\begin{aligned} &\mathbb{E}\sup\nolimits_{t\in[0,T]} e^{2V_t}|Y_t|^2 + \mathbb{E}\Big(\int_0^T e^{2V_r}|Z_r|^2\,\mathrm{d}r\Big) + \mathbb{E}\Big(\int_0^T e^{2V_r}\big|U_r^{(1)}\big|^2\mathrm{d}r\Big) \\ &\qquad\qquad + \mathbb{E}\Big(\int_0^T e^{2V_r}\big|U_r^{(2)}\big|^2\mathrm{d}A_r\Big) < \infty. \end{aligned} \tag{5.36}$$

Moreover

$$|Y_t| \le e^{V_t^{(+)}}|Y_t| \le \hat\rho_0\,, \quad \text{for all } t\in[0,T],\ \mathbb{P}-a.s..$$

Proof. We are in the framework of the proof of Proposition 5.2. We can deduce again inequality (5.15) and therefore, using (5.29),

$$\begin{aligned} &\frac{1}{2}\,\mathbb{E}\int_0^T e^{2V_r^{(n,+)}}\Big[\big|U_r^{(1,n)}\big|^2\mathrm{d}r + \big|U_r^{(2,n)}\big|^2\mathrm{d}A_r\Big] \\ &\le \mathbb{E}\Big[e^{2V_T^{(n,+)}}\big(\varphi(\eta^{(n)}) + \psi(\eta^{(n)})\big)\Big] + \mathbb{E}\int_0^T e^{2V_r^{(n,+)}}\Big(1 + 6L^2\big|Z_r^{(n)}\big|^2 + 6\big|F^{(n)}(r, Y_r^{(n)}, 0)\big|^2\Big)\,\mathrm{d}r \\ &\quad + \mathbb{E}\int_0^T e^{2V_r^{(n,+)}}\Big(1 + 3\big|G^{(n)}(r, Y_r^{(n)})\big|^2\Big)\,\mathrm{d}A_r \\ &\le \mathbb{E}\Big[e^{2V_T^{(+)}}\big(\varphi(\eta) + \psi(\eta)\big)\Big] + \mathbb{E}\int_0^T e^{2V_r^{(+)}}(\mathrm{d}r + \mathrm{d}A_r) + 6L^2\hat\rho_0^2 \\ &\quad + 6\,\mathbb{E}\int_0^T e^{2V_r^{(+)}}\big|F_{\hat\rho_0}^{\#}(r)\big|^2\mathrm{d}r + 3\,\mathbb{E}\int_0^T e^{2V_r^{(+)}}\big|G_{\hat\rho_0}^{\#}(r)\big|^2\mathrm{d}A_r\,, \end{aligned}$$

where $V^{(n,+)}$ is defined by (5.26).

Hence, using assumptions (5.35), there exists $\big(\hat U^{(1)}, \hat U^{(2)}\big) \in \Lambda_m^0(0,T) \times \Lambda_m^0(0,T)$ such that, on a subsequence still denoted by $\left\{U^{(1,n)}, U^{(2,n)};\ n\in\mathbb{N}^*\right\}$, we have, if we denote $\big(U^{(1)}, U^{(2)}\big) \overset{\text{def}}{=\!=} \big(e^{-V^{(+)}}\hat U^{(1)}, e^{-V^{(+)}}\hat U^{(2)}\big) \in \Lambda_m^0(0,T)\times\Lambda_m^0(0,T)$:

$$\begin{aligned} &e^{V_r^{(n,+)}}U^{(1,n)} \rightharpoonup e^{V^{(+)}}U^{(1)}, \quad \text{weakly in } L^2(\Omega\times[0,T], d\mathbb{P}\otimes \mathrm{d}t; \mathbb{R}^m), \\ &e^{V^{(n,+)}}U^{(2,n)} \rightharpoonup e^{V^{(+)}}U^{(2)}, \quad \text{weakly in } L^2(\Omega\times[0,T], d\mathbb{P}\otimes \mathrm{d}A_t; \mathbb{R}^m). \end{aligned}$$

and, passing to the limit,

$$\begin{aligned}
&\frac{1}{2}\,\mathbb{E}\int_0^T e^{2V_r^{(+)}}\left[\left|U_r^{(1)}\right|^2\mathrm{d}r+\left|U_r^{(2)}\right|^2\mathrm{d}A_r\right]\\
&\leq \mathbb{E}\left[e^{2V_T^{(+)}}\left(\varphi(\eta)+\psi(\eta)\right)\right]+\mathbb{E}\int_0^T e^{2V_r^{(+)}}\left(1+6L^2\left|Z_r\right|^2+6\left|F\left(r,Y_r,0\right)\right|^2\right)\mathrm{d}r\\
&\quad+\mathbb{E}\int_0^T e^{2V_r^{(+)}}\left(1+3\left|G\left(r,Y_r\right)\right|^2\right)\mathrm{d}A_r\,.
\end{aligned}$$

Passing to $\lim_{n\to\infty}$ in the approximating equation (5.25) and using the results from the proof of Proposition 5.2 we infer

$$\begin{cases}
Y_t+\int_t^T U_s\mathrm{d}Q_s=\eta+\int_t^T H\left(s,Y_s,Z_s\right)\mathrm{d}Q_s-\int_t^T Z_s\mathrm{d}B_s\,,\ t\in[0,T]\,,\\
U_s=\alpha_r U_r^{(1)}+\left(1-\alpha_r\right)U_r^{(2)}\\
U_s^{(1)}\in\partial\varphi(Y_s)\ ,\quad d\mathbb{P}\otimes\mathrm{d}s-a.e.\quad\text{and}\quad U_s^{(2)}\in\partial\psi(Y_s),\quad d\mathbb{P}\otimes\mathrm{d}A_s-a.e.\quad\text{on}\ \ [0,T]\,,
\end{cases}\tag{5.37}$$

and the conclusion follows. □

The main result of this section (and one of the main goal of our article, among the uniqueness property provided by Theorem 4.1) is the next Theorem 5.5. We remark that, with respect to Proposition 5.2, the restrictive assumption (5.21) is relaxed and becomes (5.38); the assumptions (5.22) and (5.23) are kept.

Theorem 5.5 (L^p-variational solution)**.** *We suppose that assumptions* $(\mathrm{A}_2-\mathrm{A}_7)$ *are satisfied. In addition we assume that:*

(i) (see (5.21))

$$\mathbb{E}\left[e^{pV_T}\left|\eta\right|^p+\left(\int_0^T e^{V_r}\left(\left|F\left(r,0,0\right)\right|\mathrm{d}r+\left|G\left(r,0\right)\right|\mathrm{d}A_r\right)\right)^p\right]<\infty,\tag{5.38}$$

where V *is defined by (2.10);*

(ii) (see (5.22)) there exists $a\in(1+n_p\lambda,p\wedge 2)$ *such that*

$$\begin{aligned}
&(a)\quad \mathbb{E}\Big(\int_0^T \ell_s^2\mathrm{d}s\Big)^{\frac{a}{2-a}}<\infty,\\
&(b)\quad \mathbb{E}\left[\int_0^T e^{V_r^{(+)}}\left(F_\rho^{\#}\left(r\right)\mathrm{d}r+G_\rho^{\#}\left(r\right)\mathrm{d}A_r\right)\right]^a<\infty,\quad \textit{for all } \rho>0,
\end{aligned}\tag{5.39}$$

where $V^{(+)}$ *is defined by (2.11) and* $F_\rho^{\#}$, $G_\rho^{\#}$ *are defined by (2.4);*

(iii) (we recall (5.23)) there exists a positive p.m.s.p. $\left(\Theta_t\right)_{t\geq0}$ *and for each* $\rho\geq0$ *there exist an non-decreasing function* $K_\rho:\mathbb{R}_+\to\mathbb{R}_+$ *such that*

$$F_\rho^{\#}\left(t\right)+G_\rho^{\#}\left(t\right)\leq K_\rho\left(\Theta_t\right),\quad a.e.\ t\in[0,T]\,.\tag{5.40}$$

Then the multivalued BSDE

$$\left\{\begin{array}{l} Y_t + \int_t^T \mathrm{d}K_r = \eta + \int_t^T H(r,Y_r,Z_r)\,\mathrm{d}Q_r - \int_t^T Z_r \mathrm{d}B_r\,, \quad \mathbb{P}-a.s., \text{ for all } t\in[0,T]\,,\\ \mathrm{d}K_r = U_r \mathrm{d}Q_r \in \partial_y \Psi(r,Y_r)\,\mathrm{d}Q_r \end{array}\right.$$

has a unique L^p-variational solution, in the sense of Definition 3.2.

Moreover this solution satisfies

$$\begin{array}{l} \mathbb{E}\Big(\sup_{t\in[0,T]} e^{pV_t}\,|Y_t|^p\Big) + \mathbb{E}\Big(\int_0^T e^{2V_r}\,|Z_r|^2\,\mathrm{d}r\Big)^{p/2} + \mathbb{E}\Big(\int_0^T e^{2V_r}\Psi(r,Y_r)\,\mathrm{d}Q_r\Big)^{p/2}\\ \quad + \mathbb{E}\Big(\int_0^T e^{qV_r}\,|Y_r|^{q-2}\,|Z_r|^2\,\mathrm{d}r\Big)^{p/q} + \mathbb{E}\Big(\int_0^T e^{qV_r}\,|Y_r|^{q-2}\,\Psi(r,Y_r)\,\mathrm{d}Q_r\Big)^{p/q}\\ \leq C_{p,\lambda}\,\mathbb{E}\Big[e^{pV_T}\,|\eta|^p + \Big(\int_0^T e^{V_r}\,|H(r,0,0)|\,\mathrm{d}Q_r\Big)^p\Big]\,, \end{array} \tag{5.41}$$

where $q\in\{2,p\wedge 2\}$.

Remark 5.6. If we consider the particular coefficients given by Remark 2.2, we mention that assumptions (5.38)–(5.40) are satisfied (under some supplementary restrictions).

Indeed, by (2.13), we have

$$\mathbb{E}\big(e^{p\bar{V}_T}\big) < \infty, \quad \text{where } \bar{V}_t \stackrel{\text{def}}{=\!=} \int_0^t \Big(|\mu_r| + \frac{1}{2n_p\lambda}\,\ell_r^2\Big)\mathrm{d}r + \int_0^t |\nu_r|\,\mathrm{d}A_r\,.$$

Hence

$$\begin{array}{l} \mathbb{E}\Big(\int_0^T e^{V_r}\,(|F(r,0,0)|\,\mathrm{d}r + |G(r,0)|\,\mathrm{d}A_r)\Big)^p \leq c_p\,\mathbb{E}\Big(\int_0^T e^{\bar{V}_r}\,(|\mu_r|\,\mathrm{d}r + \ell_r\,\mathrm{d}r + |\nu_r|\,\mathrm{d}A_r)\Big)^p\\ \leq c_p\,\mathbb{E}\Big(\int_0^T e^{\bar{V}_r}\Big(|\mu_r|\,\mathrm{d}r + \Big(\frac{1}{2n_p\lambda}\,\ell_r^2 + \frac{n_p\lambda}{2}\Big)\mathrm{d}r + |\nu_r|\,\mathrm{d}A_r\Big)\Big)^p\\ \leq c_p\,2^{p-1}\,\mathbb{E}\Big(\int_0^T e^{\bar{V}_s}\mathrm{d}\bar{V}_s\Big)^p + c_p\,2^{p-1}\,\Big(\frac{n_p\lambda}{2}\Big)^p\,\mathbb{E}\Big(\int_0^T e^{\bar{V}_s}\mathrm{d}s\Big)^p\\ \leq c_p\,2^{p-1}\,\Big(1+\Big(\frac{n_p\lambda T}{2}\Big)^p\Big)\,\mathbb{E}\big(e^{p\bar{V}_T}\big) < \infty \end{array}$$

and therefore assumption (5.38) is satisfied.

Assumption ($5.39-a$) is clearly satisfied by the Burkholder-Davis-Gundy inequality;

In order to fulfill assumption ($5.39-b$) we should impose, in addition, that there exist $m,n>0$ such that $|f_i(y)| \leq e^{m|y|+n}, \quad y\in\mathbb{R}, i=\overline{1,3}$. Hence, for $a<p$,

$$\begin{array}{l} \mathbb{E}\Big[\int_0^T e^{V_r^{(+)}}\,\big(F_\rho^{\#}(r)\,\mathrm{d}r + G_\rho^{\#}(r)\,\mathrm{d}A_r\big)\Big]^a\\ \leq \mathbb{E}\Big[\int_0^T e^{\bar{V}_r}\,\Big(\rho + e^{m\rho|B_r|+n}\Big)\,(|\mu_r|\,\mathrm{d}r + \ell_r\,\mathrm{d}r + |\nu_r|\,\mathrm{d}A_r)\Big]^a \end{array}$$

$$\begin{aligned}
&\leq \mathbb{E}\Big[\Big(\rho+e^{m\rho\sup_{r\in[0,T]}|B_r|+n}\Big)\int_0^T e^{\bar V_r}\Big(\mathrm{d}\bar V_r+\frac{n_p\lambda}{2}\,\mathrm{d}r\Big)\Big]^a\\
&\leq 2^{2(a-1)}\,\mathbb{E}\Big[\Big(\rho^a+e^{am\rho\sup_{r\in[0,T]}|B_r|+an}\Big)\Big(\Big(\int_0^T e^{\bar V_r}\mathrm{d}\bar V_r\Big)^a+\Big(\frac{n_p\lambda}{2}\Big)^a\Big(\int_0^T e^{\bar V_r}\mathrm{d}r\Big)^a\Big)\Big]\\
&\leq 2^{2(a-1)}\Big(1+\Big(\frac{n_p\lambda T}{2}\Big)^a\Big)\Big[\rho^a\mathbb{E}\big(e^{a\bar V_T}\big)+e^{an}\,\mathbb{E}\big(e^{am\rho\sup_{r\in[0,T]}|B_r|}\,e^{a\bar V_T}\big)\Big]\\
&\leq 2^{2(a-1)}\Big(1+\Big(\frac{n_p\lambda T}{2}\Big)^a\Big)\Big[\rho^a\mathbb{E}\big(e^{a\bar V_T}\big)+e^{an}\,\mathbb{E}\big(e^{\frac{am}{2}\sup_{r\in[0,T]}|B_r|^2+\frac{am\rho^2}{2}}\,e^{a\bar V_T}\big)\Big]
\end{aligned}$$

which is finite if, for instance, m is such that $\frac{amp}{2(p-a)}<\frac{1}{2T}$.

Therefore we deduce that assumption (5.39$-b$) is satisfied.

Assumptions (5.40) is satisfied with $K_\rho(r)=r\,(\rho+e^{m\rho\,r+n})$, $r\in\mathbb{R}_+$ and $\Theta_t=|B_t|+|\mu_t|+|\nu_t|+\ell_t$.

Proof. **Step I.** *Approximating equation*

Let $t\in[0,T]$ and

$$\beta_t=t+A_t+|\mu_t|+|\nu_t|+\ell_t+V_t^{(+)}+|F(t,0,0)|+|G(t,0)|+\Theta_t,$$

Define, for $n\in\mathbb{N}^*$,

$$\begin{aligned}
&\eta^{(n)}\overset{\mathrm{def}}{=\!=}\eta\,\mathbf{1}_{[0,n]}\big(|\eta|+V_T^{(+)}\big),\\
&F^{(n)}(t,y,z)\overset{\mathrm{def}}{=\!=}F(t,y,z)-F(t,0,0)\;\mathbf{1}_{(n,\infty)}(\beta_t),\\
&G^{(n)}(t,y)\overset{\mathrm{def}}{=\!=}G(t,y)-G(t,0)\;\mathbf{1}_{(n,\infty)}(\beta_t),\\
&H^{(n)}(t,y,z)\overset{\mathrm{def}}{=\!=}\alpha_tF^{(n)}(t,y,z)+(1-\alpha_t)\,G^{(n)}(t,y).
\end{aligned}$$

We highlight the following properties of the function $H^{(n)}$:

$$\begin{aligned}
&\Big\langle y'-y,H^{(n)}(t,y',z)-H^{(n)}(t,y,z)\Big\rangle\leq[\mu_t\alpha_t+\nu_t(1-\alpha_t)]\,|y'-y|^2,\\
&\Big|H^{(n)}(t,y,z')-H^{(n)}(t,y,z)\Big|\leq\alpha_t\,\ell_t\,|z'-z|,\\
&\Big|H^{(n+i)}(t,y,z)-H^{(n)}(t,y,z)\Big|\leq[\alpha_t\,|F(t,0,0)|+(1-\alpha_t)\,|G(t,0)|]\;\mathbf{1}_{(n,\infty)}(\beta_t)
\end{aligned}\tag{5.42}$$

and, using the previous, the monotonicity properties

$$\begin{aligned}
&\big\langle y,H^{(n)}(t,y,z)\big\rangle\\
&\leq|y|\,[\alpha_t\,|F(t,0,0)|+(1-\alpha_t)\,|G(t,0)|]\;\mathbf{1}_{[0,n]}(\beta_t)+|y|^2\mathrm{d}V_s+\alpha_t\,\frac{n_p\lambda}{2}\,|z|^2\\
&\leq|y|\,[\alpha_t\,|F(t,0,0)|+(1-\alpha_t)\,|G(t,0)|]\;\mathbf{1}_{[0,n]}(\beta_t)+|y|^2\mathrm{d}V_s^{(+)}+\alpha_t\,\frac{n_p\lambda}{2}\,|z|^2
\end{aligned}$$

and

$$\begin{aligned}
\big\langle Y_t'-Y_t,H^{(n)}(t,Y_t',Z_t')-H^{(n)}(t,Y_t,Z_t)\big\rangle\mathrm{d}Q_t&\leq|Y_t'-Y_t|^2\,\mathrm{d}V_t+\frac{n_p\lambda}{2}\,|Z_t'-Z_t|^2\,\mathrm{d}t\\
&\leq|Y_t'-Y_t|^2\,\mathrm{d}V_t^{(+)}+\frac{n_p\lambda}{2}\,|Z_t'-Z_t|^2\,\mathrm{d}t.
\end{aligned}$$

Clearly, the assumptions of Proposition 5.2 are satisfied for the approximating BSDE

$$\left\{\begin{array}{l} Y_t^{(n)}+\int_t^T \mathrm{d}K_s^{(n)}=\eta^{(n)}+\int_t^T H^{(n)}\big(s,Y_s^{(n)},Z_s^{(n)}\big)\mathrm{d}Q_s-\int_t^T Z_s^{(n)}\mathrm{d}B_s\,, \quad t\in[0,T]\,,\\ \mathrm{d}K_r^{(n)}\in\partial_y\Psi\big(r,Y_r^{(n)}\big)\mathrm{d}Q_r=\alpha_r\partial\varphi\big(Y_r^{(n)}\big)\mathrm{d}r+(1-\alpha_r)\,\partial\psi\big(Y_r^{(n)}\big)\mathrm{d}A_r \end{array}\right. \tag{5.43}$$

and therefore there exists a unique L^p-variational solution $\big(Y^{(n)},Z^{(n)}\big)$ of (5.43).

Since $V\leq V^{(+)}$, we obtain from (5.24)

$$\mathbb{E}\Big(\sup\nolimits_{r\in[0,T]}e^{pV_r}\big|Y_r^{(n)}\big|^p\Big)+\mathbb{E}\Big(\int_0^T e^{2V_r}\big|Z_r^{(n)}\big|^2\mathrm{d}r\Big)^{p/2}+\mathbb{E}\Big(\int_0^T e^{2V_r}\Psi\big(r,Y_r^{(n)}\big)\mathrm{d}Q_r\Big)^{p/2}<\infty$$

and for any $q\in\{2,p\wedge 2\}$, $\delta_q=\delta\,\mathbf{1}_{[1,2)}(q)$ and $\Gamma_t^{(n)}\overset{\mathrm{def}}{=\!=}\Big(\big|M_t-Y_t^{(n)}\big|^2+\delta_q\Big)^{1/2}$ it holds

$$\begin{aligned} &\big(\Gamma_t^{(n)}\big)^q+\frac{q\,(q-1)}{2}\int_t^s\big(\Gamma_r^{(n)}\big)^{q-2}\big|R_r-Z_r^{(n)}\big|^2\mathrm{d}r+q\int_t^s\big(\Gamma_r^{(n)}\big)^{q-2}\Psi\big(r,Y_r^{(n)}\big)\mathrm{d}Q_r\\ &\leq\big(\Gamma_s^{(n)}\big)^q+q\int_t^s\big(\Gamma_r^{(n)}\big)^{q-2}\Psi\left(r,M_r\right)\mathrm{d}Q_r\\ &\quad+q\int_t^s\big(\Gamma_r^{(n)}\big)^{q-2}\langle M_r-Y_r^{(n)},N_r-H^{(n)}\big(r,Y_r^{(n)},Z_r^{(n)}\big)\rangle\mathrm{d}Q_r\\ &\quad-q\int_t^s\big(\Gamma_r^{(n)}\big)^{q-2}\langle M_r-Y_r^{(n)},\big(R_r-Z_r^{(n)}\big)\mathrm{d}B_r\rangle \end{aligned} \tag{5.44}$$

for any $0\leq t\leq s<\infty$ and for any $M\in\mathcal{V}_m^0$ of form (3.2), *i.e.* $M_t=M_T+\int_t^T N_r\mathrm{d}Q_r-\int_t^T R_r\mathrm{d}B_r$.

Since $\mathbb{E}\big(\sup_{r\in[0,T]}e^{pV_r}\big|Y_r^{(n)}\big|^p\big)<\infty$ and inequality (5.44) holds for $1<q=p\wedge 2\leq p$, boundedness properties (3.22) and (3.17) from Proposition 3.10 yield

$$\begin{aligned} &\mathbb{E}\Big(\sup\nolimits_{t\in[0,T]}e^{pV_t}\big|Y_t^{(n)}\big|^p\Big)+\mathbb{E}\Big(\int_0^T e^{2V_r}\big|Z_r^{(n)}\big|^2\mathrm{d}r\Big)^{p/2}+\mathbb{E}\Big(\int_0^T e^{2V_r}\Psi\big(r,Y_r^{(n)}\big)\mathrm{d}Q_r\Big)^{p/2}\\ &\quad+\mathbb{E}\Big(\int_0^T e^{qV_r}\big|Y_r^{(n)}\big|^{q-2}\big|Z_r^{(n)}\big|^2\mathrm{d}r\Big)^{p/q}+\mathbb{E}\Big(\int_0^T e^{qV_r}\big|Y_r^{(n)}\big|^{q-2}\Psi\big(r,Y_r^{(n)}\big)\mathrm{d}Q_r\Big)^{p/q}\\ &\leq C_{p,\lambda}\,\mathbb{E}\left[e^{pV_T}\left|\eta\right|^p+\left(\int_0^T e^{V_r}\left|H\left(r,0,0\right)\right|\mathrm{d}Q_r\right)^p\right]. \end{aligned} \tag{5.45}$$

Step II. *Cauchy sequence*

From the continuity property (4.3), for $q=p\wedge 2$, we have for all $0<\alpha<1:$

$$\begin{aligned} &\mathbb{E}\sup\nolimits_{t\in[0,T]}e^{\alpha qV_t}\big|Y_t^{(n+i)}-Y_t^{(n)}\big|^{\alpha q}+\left(\mathbb{E}\int_0^T e^{2V_r}\frac{\big|Z_r^{(n+i)}-Z_r^{(n)}\big|^2}{\big(e^{V_r}\big|Y_t^{(n+i)}-Y_t^{(n)}\big|+1\big)^{2-q}}\,\mathrm{d}r\right)^\alpha\\ &\leq C_{\alpha,q,\lambda}\Bigg[\mathbb{E}e^{qV_T}\left|\eta^{(n+i)}-\eta^{(n)}\right|^q\\ &\quad+K\left(\mathbb{E}\Big(\int_0^T e^{V_r}\big|H^{(n+i)}(t,Y_t^{(n)},Z_t^{(n)})-H^{(n)}(t,Y_t^{(n)},Z_t^{(n)})\big|\mathrm{d}Q_r\Big)^q\right)^{1/q}\Bigg]^\alpha, \end{aligned} \tag{5.46}$$

where

$$K=\Big[\mathbb{E}e^{qV_T}\big|\eta^{(n+i)}\big|^q+\mathbb{E}\Big(\int_0^T e^{V_r}\big|H^{(n+i)}\left(r,0,0\right)\big|\mathrm{d}Q_r\Big)^q+\mathbb{E}e^{qV_T}\big|\eta^{(n)}\big|^q$$
$$+\mathbb{E}\Big(\int_0^T e^{V_r}\big|H^{(n)}\left(r,0,0\right)\big|\mathrm{d}Q_r\Big)^q\Big]^{(q-1)/q}$$
$$\leq 2^{(q-1)/q}\Big[\mathbb{E}\Big(e^{qV_T}\left|\eta\right|^q+\Big(\int_0^T e^{V_r}\left|F\left(r,0,0\right)\right|\mathrm{d}r+\left|G\left(r,0\right)\right|\mathrm{d}A_r\Big)^q\Big)\Big]^{(q-1)/q}$$

and $C_{\alpha,q,\lambda}$ is a positive constant depending only α,q and λ.

First we remark, see also (5.30),

$$\mathbb{E}e^{qV_T}\big|\eta^{(n+i)}-\eta^{(n)}\big|^q\leq\mathbb{E}e^{qV_T}\left|\eta\right|^q\mathbf{1}_{(n,\infty)}\big(\left|\eta\right|+V_T^{(+)}\big)\to 0,\quad\mathbb{P}-\text{a.s., for }n\to\infty,$$

since, by $1<q\leq p$ and assumption (5.38), we have

$$\mathbb{E}e^{qV_T}\left|\eta\right|^q\leq\big(\mathbb{E}\,e^{pV_T}\left|\eta\right|^p\big)^{q/p}<\infty.$$

Again by (5.30) and assumption (5.38) we deduce:

$$\mathbb{E}\Big(\int_0^T e^{V_r}\big|H^{(n+i)}(t,Y_t^{(n)},Z_t^{(n)})-H^{(n)}(t,Y_t^{(n)},Z_t^{(n)})\big|\mathrm{d}Q_r\Big)^q$$
$$\leq\mathbb{E}\Big(\int_0^T e^{V_r}\left[\left|F\left(r,0,0\right)\right|\mathbf{1}_{(n,\infty)}\left(\beta_r\right)\mathrm{d}r+\left|G\left(r,0\right)\right|\mathbf{1}_{(n,\infty)}\left(\beta_r\right)\mathrm{d}A_r\right]\Big)^q$$
$$\leq 2^{q-1}\Big[\mathbb{E}\Big(\int_0^T e^{V_r}\left|F\left(r,0,0\right)\right|\mathbf{1}_{(n,\infty)}\left(\beta_r\right)\mathrm{d}r\Big)^q+\mathbb{E}\Big(\int_0^T e^{V_r}\left|G\left(r,0\right)\right|\mathbf{1}_{(n,\infty)}\left(\beta_r\right)\mathrm{d}A_r\Big)^q\Big]$$
$$\to 0,\quad\mathbb{P}-\text{a.s., for }n\to\infty.$$

From (5.46) we conclude that there exists $(Y,Z)\in S_m^0\times\Lambda_{m\times k}^0$ such that (on a subsequence)

$$\sup\nolimits_{t\in[0,T]}\big|Y_t^{(n)}-Y_t\big|+\int_0^T\big|Z_r^{(n)}-Z_r\big|^2\mathrm{d}r\to 0,\quad\mathbb{P}-\text{a.s., for }n\to\infty.$$

Passing to $\liminf_{n\to\infty}$ in (5.44) and (5.45) we infer that (Y,Z) is an L^p-variational solution. □

Remark 5.7. If we consider BSDE (1.1) in the particular case $G\equiv 0$, $\psi\equiv 0$, F and η deterministic and F independent of Y and Z, we obtain the deterministic differential equation

$$\begin{cases}Y_t'\in\partial\varphi\left(Y_t\right)-F\left(t\right), & t\in\left[0,T\right],\\ Y_T=\eta,\end{cases}\tag{5.47}$$

since the solution $(Y_t,Z_t)=(Y_t,0)$ is deterministic.

Let us remark that a variational solution in the sense of definition (3.11) implies that Y satisfies the inequality

$$\left|M-Y_t\right|^2+2\int_t^s\varphi\left(Y_r\right)\mathrm{d}r\leq\left|M-Y_s\right|^2+2\int_t^s\varphi\left(M\right)\mathrm{d}r+2\int_t^s\left\langle M-Y_r,-F\left(r\right)\right\rangle\mathrm{d}r,$$

for any $M \in \mathbb{R}^m$ and any $0 \leq t \leq s \leq T$.

On the other hand, see ([2], Chap. III, Def. 2.1), the integral solution of (5.47) means that Y satisfies the inequality (let us remark that our equation (5.47) is with final condition)

$$|M - Y_t|^2 \leq |M - Y_s|^2 + 2\int_t^s \langle M - Y_r, M^* - F(r)\rangle \, \mathrm{d}r, \quad \text{for any } M^* \in \partial\varphi(M)$$

and any $0 \leq t \leq s \leq T$.

It is easy to see that any variational solution of (5.47) is an integral solution of the same problem. Using now Chapter III, Corollary 2.2 of [2], we can prove the converse, *i.e.* that any integral solution of (5.47) is a limit of strong solutions and therefore (see Rem. 3.8) is a variational solution of (5.47) in the sense of definition (3.11).

Therefore, the definition of the variational solution for (5.47) coincides with the definition of integral solution for the same problem.

Hence the question about the difference between the variational solution (our definition) and the strong solution is equivalent with the problem of the difference between the integral solution and the strong solution of the same problem. We also remark that, in this framework, our assumptions about the generator F reduces to the assumption that $F \in L^1(0, T, \mathbb{R}^m)$. Finally, we remark that, accordingly with Chapter III, Theorem 2.1 of [2], under the assumption $F \in L^1(0, T, \mathbb{R}^m)$ the solution is only integral (in order to have a strong solution it is necessary to impose that $F \in L^2(0, T, \mathbb{R}^m)$, see [2], Chap. IV, Thm. 2.1).

5.2. Existence on a random interval time $[0, \tau]$

Theorem 5.8. *We suppose that assumptions* $(\mathrm{A}_1 - \mathrm{A}_7)$ *are satisfied. Let* V *and* $V^{(+)}$ *be given by definitions (2.10) and (2.11) respectively. In addition we assume that:*

(i)

$$0 \leq \Psi(r, \eta) \leq \mathbf{1}_{q \geq 2} \Psi(r, \eta), \quad a.e.\ r \geq 0;$$

(ii)

$$\begin{aligned} &\mathbb{E}\big(\sup\nolimits_{t \geq 0} e^{pV_t} |\xi_t|^p\big) + \mathbb{E}\bigg(\int_0^\tau e^{2V_r} |\zeta_r|^2 \, \mathrm{d}r\bigg)^{p/2} + \mathbb{E}\bigg(\int_0^\tau e^{2V_r} \Psi(r, \xi_r) \, \mathrm{d}Q_r\bigg)^{p/2} \\ &\quad + \mathbb{E}\bigg(\int_0^\tau e^{V_r} (|F(r, 0, 0)| \, \mathrm{d}r + |G(r, 0)| \, \mathrm{d}A_r)\bigg)^p \overset{\mathrm{def}}{=} L < \infty; \end{aligned} \tag{5.48}$$

(iii)

$$\lim\nolimits_{t \to \infty} \mathbb{E}(\Lambda_t) = 0, \tag{5.49}$$

where $\Lambda_t \overset{\mathrm{def}}{=} \bigg(\int_t^\infty e^{V_r} \mathbf{1}_{q \geq 2} \Psi(r, \xi_r) \, \mathrm{d}Q_r\bigg)^{p/2} + \bigg(\int_t^\infty e^{V_r} |H(r, \xi_r, \zeta_r)| \, \mathrm{d}Q_r\bigg)^p$;

(iv) there exists $a \in (1 + n_p\lambda, p \wedge 2)$ *such that for every* $T \geq 0$:

$$\begin{aligned} &(a) \quad \mathbb{E}\bigg(\int_0^T \ell_s^2 \mathrm{d}s\bigg)^{\frac{a}{2-a}} < \infty, \\ &(b) \quad \mathbb{E}\bigg(\int_0^T e^{V_s^{(+)}} \big(F_\rho^\#(s) \, \mathrm{d}s + G_\rho^\#(s) \, \mathrm{d}A_s\big)\bigg)^a < \infty, \quad \textit{for all } \rho > 0; \end{aligned} \tag{5.50}$$

(v) there exists a positive p.m.s.p. $(\Theta_t)_{t\geq 0}$ *and, for each* $\rho \geq 0$*, there exist an non-decreasing function* $K_\rho : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$F_\rho^{\#}(t) + G_\rho^{\#}(t) \leq K_\rho(\Theta_t), \quad a.e.\ t \geq 0,\ \mathbb{P}-a.s.. \tag{5.51}$$

Then the multivalued BSDE

$$\begin{cases} Y_t + \int_{t\wedge\tau}^{\tau} \mathrm{d}K_r = \eta + \int_{t\wedge\tau}^{\tau} H(r, Y_r, Z_r)\,\mathrm{d}Q_r - \int_{t\wedge\tau}^{\tau} Z_r \mathrm{d}B_r\,, \quad \mathbb{P}-a.s., \text{ for all } t \geq 0, \\ \mathrm{d}K_r = U_r \mathrm{d}Q_r \in \partial_y \Psi(r, Y_r)\,\mathrm{d}Q_r \end{cases}$$

has a unique L^p*-variational solution, in the sense of Definition 3.2.*

Moreover this solution satisfies

$$\begin{aligned} &\mathbb{E}\Big(\sup\nolimits_{t\in[0,\tau]} e^{pV_t}|Y_t|^p\Big) + \mathbb{E}\left(\int_0^\tau e^{2V_r}|Z_r|^2\,\mathrm{d}r\right)^{p/2} + \mathbb{E}\left(\int_0^\tau e^{2V_r}\Psi(r, Y_r)\,\mathrm{d}Q_r\right)^{p/2} \\ &\quad + \mathbb{E}\left(\int_0^\tau e^{qV_r}|Y_r|^{q-2}|Z_r|^2\,\mathrm{d}r\right)^{p/q} + \mathbb{E}\left(\int_0^\tau e^{qV_r}|Y_r|^{q-2}\,\Psi(r, Y_r)\,\mathrm{d}Q_r\right)^{p/q} \\ &\leq C_{p,\lambda}\,\mathbb{E}\left[e^{pV_\tau}|\eta|^p + \left(\int_0^\tau e^{V_r}|H(r,0,0)|\,\mathrm{d}Q_r\right)^p\right]. \end{aligned} \tag{5.52}$$

Remark 5.9. Our initial assumptions are about η but the first three terms from (5.48) involves the processes ξ and ζ (associated to η). However, we remark that in order to obtain that the first three terms in (5.48) are bounded it is sufficient to impose

$$\mathbb{E}\big(\sup\nolimits_{t\geq 0} e^{pV_t}|\eta|^p\big) < \infty$$

(for the proof we can apply ([21], Cor. 2.45) for the process $\sup_{t\geq 0} V_t$) and respectively

$$\mathbb{E}\left(\int_0^\tau e^{2V_r}\Psi(r,\eta)\,\mathrm{d}Q_r\right)^{p/2} < \infty.$$

Proof. **Step I.** *Approximating equation and estimates*

By Theorem 5.5 there exists a unique pair $\big(Y^{(n)}, Z^{(n)}\big)$ as L^p-variational solution on $[0,n]$ of the BSDE, with the final data $\xi_n = \mathbb{E}^{\mathcal{F}_n}\eta$,

$$\begin{cases} Y_t^{(n)} + \int_t^n \mathrm{d}K_s^{(n)} = \xi_n + \int_t^n H\big(s, Y_s^{(n)}, Z_s^{(n)}\big)\mathrm{d}Q_s - \int_t^n Z_s^{(n)}\mathrm{d}B_s\,, \quad t\in[0,n]\,, \\ \mathrm{d}K_s^{(n)} \in \partial_y\Psi(s, Y_s^{(n)})\mathrm{d}Q_s\,. \end{cases}$$

Hence

$$\big(Y_t^{(n)}, Z_t^{(n)}\big) = (\xi_t, \zeta_t)\,, \quad \text{for all } t\geq n \quad \text{and} \quad \big(Y_t^{(n)}, Z_t^{(n)}\big) = (\eta, 0)\,, \quad \text{for all } t \geq \tau. \tag{5.53}$$

By (5.41) we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup\nolimits_{t\geq 0} e^{pV_t}\big|Y_t^{(n)}\big|^p\Big)+\mathbb{E}\left(\int_0^\infty e^{2V_r}\big|Z_r^{(n)}\big|^2\mathrm{d}r\right)^{p/2}+\mathbb{E}\left(\int_0^\infty e^{2V_r}\Psi\big(r,Y_r^{(n)}\big)\mathrm{d}Q_r\right)^{p/2}\\
&\leq \mathbb{E}\Big(\sup\nolimits_{t\in[0,n]} e^{pV_t}\big|Y_t^{(n)}\big|^p\Big)+\mathbb{E}\left(\int_0^n e^{2V_r}\big|Z_r^{(n)}\big|^2\mathrm{d}r\right)^{p/2}+\mathbb{E}\left(\int_0^n e^{2V_r}\Psi\big(r,Y_r^{(n)}\big)\mathrm{d}Q_r\right)^{p/2}\\
&\quad+\mathbb{E}\Big(\sup\nolimits_{t\geq 0} e^{pV_t}\left|\xi_t\right|^p\Big)+\mathbb{E}\left(\int_0^\tau e^{2V_r}\left|\zeta_r\right|^2\mathrm{d}r\right)^{p/2}+\mathbb{E}\left(\int_0^\tau e^{2V_r}\Psi\left(r,\xi_r\right)\mathrm{d}Q_r\right)^{p/2}\\
&\leq C_{p,\lambda}\,\mathbb{E}\left[e^{pV_n}\left|\xi_n\right|^p+\Big(\int_0^n e^{V_r}\left|H\left(r,0,0\right)\right|\mathrm{d}Q_r\Big)^p\right]+L\\
&\leq L\cdot C_{p,\lambda}+L\overset{\mathrm{def}}{=}\tilde{L}.
\end{aligned}\tag{5.54}$$

On the other hand, let us take in Proposition 3.10

$$M_t=\xi_t=\mathbb{E}^{\mathcal{F}_t}\eta,\quad R_t=\zeta_t\,,\quad N_t=0,\quad \text{and } L_t=0$$

and therefore

$$M_t=M_T+\int_t^T 0\,\mathrm{d}r-\int_t^T R_r\mathrm{d}B_r\,,\quad \text{for all } 0\leq t\leq T<\infty.$$

Since, for all $k\in\mathbb{N}^*$,

$$\mathbb{E}\Big(\sup\nolimits_{t\in[0,k]} e^{pV_t}\big|M_t-Y_t^{(k)}\big|^p\Big)\leq 2^{p-1}\,\mathbb{E}\sup\nolimits_{t\in[0,k]} e^{pV_t}\left|\xi_t\right|^p+2^{p-1}\,\mathbb{E}\sup\nolimits_{t\in[0,k]} e^{pV_t}\big|Y_t^{(k)}\big|^p<\infty,$$

we can use boundedness properties (3.20) and (3.16) from Proposition 3.10 in order to deduce that, for all $0\leq t\leq s\leq k$,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\Big(\sup\nolimits_{r\in[t,s]} e^{pV_r}\big|\xi_r-Y_r^{(k)}\big|^p\Big)+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^s e^{2V_r}\big|\zeta_r-Z_r^{(k)}\big|^2\mathrm{d}r\right)^{p/2}\\
&\leq C_{p,\lambda}\,\mathbb{E}^{\mathcal{F}_t}\Big[e^{pV_s}\big|\xi_s-Y_s^{(k)}\big|^p\\
&\quad+\left(\int_t^s e^{V_r}\,\mathbf{1}_{q\geq 2}\Psi\left(r,\xi_r\right)\mathrm{d}Q_r\right)^{p/2}+\Big(\int_t^s e^{V_r}\left|H\left(r,\xi_r,\zeta_r\right)\right|\mathrm{d}Q_r\Big)^p\Big].
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}e^{pV_s}\big|\xi_s-Y_s^{(k)}\big|^p\leq \mathbb{E}^{\mathcal{F}_t}\Big(\sup\nolimits_{s\in[t,k]} e^{pV_s}\big|\xi_s-Y_s^{(k)}\big|^p\Big)\\
&\leq C_{p,\lambda}\,\mathbb{E}^{\mathcal{F}_t}\Big[e^{pV_k}\big|\xi_k-Y_k^{(k)}\big|^p+\Big(\int_t^k e^{V_r}\,\mathbf{1}_{q\geq 2}\Psi\left(r,\xi_r\right)\mathrm{d}Q_r\Big)^{p/2}\\
&\quad+\Big(\int_t^k e^{V_r}\left|H\left(r,\xi_r,\zeta_r\right)\right|\mathrm{d}Q_r\Big)^p\Big]\\
&=C_{p,\lambda}\,\mathbb{E}^{\mathcal{F}_t}\Big[\left(\int_t^k e^{V_r}\,\mathbf{1}_{q\geq 2}\Psi\left(r,\xi_r\right)\mathrm{d}Q_r\right)^{p/2}+\Big(\int_t^k e^{V_r}\left|H\left(r,\xi_r,\zeta_r\right)\right|\mathrm{d}Q_r\Big)^p\Big].
\end{aligned}$$

Using the above two inequalities we deduce

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\Big(\sup\nolimits_{r\in[t,s]} e^{pV_r}\big|\xi_r - Y_r^{(k)}\big|^p\Big) + \mathbb{E}^{\mathcal{F}_t}\Big(\int_t^s e^{2V_r}\big|\zeta_r - Z_r^{(k)}\big|^2 \mathrm{d}r\Big)^{p/2}\\
&\leq C_{p,\lambda}\,\mathbb{E}^{\mathcal{F}_t}\Big[\Big(\int_t^k e^{V_r}\,\mathbf{1}_{q\geq 2}\Psi(r,\xi_r)\,\mathrm{d}Q_r\Big)^{p/2} + \Big(\int_t^k e^{V_r}\,|H(r,\xi_r,\zeta_r)|\,\mathrm{d}Q_r\Big)^p\Big]\\
&\leq C_{p,\lambda}\,\mathbb{E}^{\mathcal{F}_t}\Big[\Big(\int_t^\infty e^{V_r}\,\mathbf{1}_{q\geq 2}\Psi(r,\xi_r)\,\mathrm{d}Q_r\Big)^{p/2} + \Big(\int_t^\infty e^{V_r}\,|H(r,\xi_r,\zeta_r)|\,\mathrm{d}Q_r\Big)^p\Big]\\
&= C_{p,\lambda}\cdot\mathbb{E}^{\mathcal{F}_t}(\Lambda_t)\,.
\end{aligned}$$

Step II. *Cauchy sequence*

In particular, since

$$\big(Y_r^{(n)}, Z_r^{(n)}\big) = (\xi_r,\zeta_r)\,,\quad \text{for all } r\geq n \quad\text{and}\quad \big(Y_r^{(n+i)}, Z_r^{(n+i)}\big) = (\xi_r,\zeta_r)\,,\quad \text{for all } r\geq n+i,$$

we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_n}\Big(\sup\nolimits_{r\geq n} e^{pV_r}\big|Y_r^{(n)} - Y_r^{(n+i)}\big|^p\Big) + \mathbb{E}^{\mathcal{F}_n}\Big(\int_n^\infty e^{2V_r}\big|Z_r^{(n)} - Z_r^{(n+i)}\big|^2 \mathrm{d}r\Big)^{p/2}\\
&= \mathbb{E}^{\mathcal{F}_n}\Big(\sup\nolimits_{r\in[n,n+i]} e^{pV_r}\big|Y_r^{(n)} - Y_r^{(n+i)}\big|^p\Big) + \mathbb{E}^{\mathcal{F}_n}\Big(\int_n^{n+i} e^{2V_r}\big|Z_r^{(n)} - Z_r^{(n+i)}\big|^2 \mathrm{d}r\Big)^{p/2}\\
&= \mathbb{E}^{\mathcal{F}_n}\Big(\sup\nolimits_{r\in[n,n+i]} e^{pV_r}\big|\xi_r - Y_r^{(n+i)}\big|^p\Big) + \mathbb{E}^{\mathcal{F}_n}\Big(\int_n^{n+i} e^{2V_r}\big|\zeta_r - Z_r^{(n+i)}\big|^2 \mathrm{d}r\Big)^{p/2}\\
&\leq C_{p,\lambda}\,\mathbb{E}^{\mathcal{F}_n}\Big[\Big(\int_n^\infty e^{V_r}\,\mathbf{1}_{q\geq 2}\Psi(r,\xi_r)\,\mathrm{d}Q_r\Big)^{p/2} + \Big(\int_n^\infty e^{V_r}\,|H(r,\xi_r,\zeta_r)|\,\mathrm{d}Q_r\Big)^p\Big]\\
&= C_{p,\lambda}\cdot\mathbb{E}^{\mathcal{F}_n}(\Lambda_n)
\end{aligned}$$

and therefore, using (5.49),

$$\lim\nolimits_{n\to\infty}\mathbb{E}\Big(\sup\nolimits_{r\geq n} e^{pV_r}\big|\xi_r - Y_r^{(n)}\big|^p\Big) + \mathbb{E}\Big(\int_n^\infty e^{2V_r}\big|\zeta_r - Z_r^{(n)}\big|^2 \mathrm{d}r\Big)^{p/2} = 0.$$

By the continuity property (4.3) on $[0,n]$, with $0<\alpha<1$,

$$\begin{aligned}
&\mathbb{E}\sup\nolimits_{t\in[0,n]} e^{\alpha qV_t}|Y_t^{(n+i)} - Y_t^{(n)}|^{\alpha q} + \Big(\mathbb{E}\int_0^n e^{2V_r}\frac{\big|Z_r^{(n+i)} - Z_r^{(n)}\big|^2}{\big(e^{V_r}\big|Y_r^{(n+i)} - Y_r^{(n)}\big| + 1\big)^{2-q}}\mathrm{d}r\Big)^\alpha\\
&\leq C_{\alpha,q,\lambda}\Big[\mathbb{E}e^{qV_n}|Y_n^{(n+i)} - Y_n^{(n)}|^q\Big]^\alpha \leq C_{\alpha,q,\lambda}\Big[\mathbb{E}e^{pV_n}|Y_n^{(n+i)} - \xi_n|^p\Big]^{\alpha q/p}\\
&\leq C_{\alpha,p,\lambda}\cdot[\mathbb{E}(\Lambda_n)]^{\alpha q/p}\to 0,\quad \text{as } n\to\infty.
\end{aligned}\tag{5.55}$$

Hence, using again Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\sup\nolimits_{t\geq 0} e^{\alpha q V_t}|Y_t^{(n+i)}-Y_t^{(n)}|^{\alpha q}\\
&\leq \mathbb{E}\sup\nolimits_{t\in[0,n]} e^{\alpha q V_t}|Y_t^{(n+i)}-Y_t^{(n)}|^{\alpha q}+\Big(\mathbb{E}\Big(\sup\nolimits_{r\geq n} e^{pV_r}\big|Y_r^{(n)}-Y_r^{(n+i)}\big|^p\Big)\Big)^{\alpha q/p}\\
&\leq C_{\alpha,p,\lambda}\cdot[\mathbb{E}(\Lambda_n)]^{\alpha q/p}\rightarrow 0,\quad \text{as } n\rightarrow\infty.
\end{aligned}$$

Step III. *Passing to the limit*

Hence there exists $Y\in S_m^0$ such that

$$\mathbb{E}\sup\nolimits_{t\geq 0} e^{\alpha q V_t}|Y_t-Y_t^{(n)}|^{\alpha q}\rightarrow 0,\quad \text{as } n\rightarrow\infty$$

and, on a subsequence denoted also by $Y^{(n)}$,

$$\sup\nolimits_{t\geq 0} e^{\alpha q V_t}|Y_t-Y_t^{(n)}|^{\alpha q}\rightarrow 0,\quad \mathbb{P}-\text{a.s., as } n\rightarrow\infty. \tag{5.56}$$

Since $\big(Y_t^{(n)},Z_t^{(n)}\big)=(\eta,0)$, for all $t\geq\tau$ and $n\in\mathbb{N}^*$, it clearly follows that $Y_t=\eta$, for all $t\geq\tau$.

By Fatou's Lemma applied to inequality (5.54) we deduce

$$\mathbb{E}\Big(\sup\nolimits_{t\geq 0} e^{pV_t}\,|Y_t|^p\Big)+\mathbb{E}\left(\int_0^\infty e^{2V_r}\Psi(r,Y_r)\,\mathrm{d}Q_r\right)^{p/2}\leq\tilde{L}.$$

From (5.54) we also infer that there exists $Z\in\Lambda_{m\times k}^0$ such that

$$Z^{(n)}\longrightarrow Z,\quad \text{weakly in } L^p\left(\Omega;L^2\left(\mathbb{R}_+;\mathbb{R}^{m\times k}\right)\right),\ \text{as } n\rightarrow\infty$$

and

$$\mathbb{E}\left(\int_0^\infty e^{2V_r}\,|Z_r|^2\,\mathrm{d}r\right)^{p/2}\leq\liminf\nolimits_{n\rightarrow\infty}\mathbb{E}\left(\int_0^\infty e^{2V_r}\big|Z_r^{(n)}\big|^2\mathrm{d}r\right)^{p/2}\leq\tilde{L}.$$

Now, by the continuity property (4.1) on $[0,n]$, we have, $\mathbb{P}$-a.s., for all $t\in[0,n]$,

$$\begin{aligned}
&e^{qV_t}\big|Y_t^{(n+i)}-Y_t^{(n)}\big|^q+c_{q,\lambda}\,\mathbb{E}^{\mathcal{F}_t}\int_t^n e^{2V_r}\frac{\big|Z_r^{(n+i)}-Z_r^{(n)}\big|^2}{\big(e^{V_r}\big|Y_r^{(n+i)}-Y_r^{(n)}\big|+1\big)^{2-q}}\,\mathrm{d}r\\
&\quad+c_{q,\lambda}\,\mathbb{E}^{\mathcal{F}_t}\int_t^n e^{qV_r}\big|Y_r^{(n+i)}-Y_r^{(n)}\big|^{q-2}\big|Z_r^{(n+i)}-Z_r^{(n)}\big|^2\mathrm{d}r\\
&\leq\mathbb{E}^{\mathcal{F}_t}e^{qV_n}\big|Y_n^{(n+i)}-Y_n^{(n)}\big|^q\leq\Big(\mathbb{E}^{\mathcal{F}_t}e^{pV_n}\big|Y_n^{(n+i)}-\xi_n\big|^p\Big)^{q/p}\\
&\leq\big[\mathbb{E}^{\mathcal{F}_t}\big(C_{\lambda,\alpha,p}\,\mathbb{E}^{\mathcal{F}_n}(\Lambda_n)\big)\big]^{q/p}=C_{\lambda,\alpha,p}\left[\mathbb{E}^{\mathcal{F}_t}(\Lambda_n)\right]^{q/p}.
\end{aligned}\tag{5.57}$$

Therefore, if we denote $\Delta_t^{(n)}\overset{\text{def}}{=\!=}\sup_{i\in\mathbb{N}^*}e^{V_t}\big|Y_t^{(n+i)}-Y_t^{(n)}\big|$, then

$$\big(\Delta_t^{(n)}\big)^p\leq C_{\lambda,\alpha,p}\,\mathbb{E}^{\mathcal{F}_t}(\Lambda_n),\quad \mathbb{P}-\text{a.s.},\quad \text{for all } t\in[0,n].$$

From Proposition 1.56 of [21] we infer

$$\mathbb{E}\sup\nolimits_{t\in[0,T]}\big(\Delta_t^{(n)}\big)^{\alpha p}\leq\frac{1}{1-\alpha}\,\left(C_{\lambda,\alpha,p}\,\mathbb{E}\left(\Lambda_n\right)\right)^{\alpha},\quad\text{for all }0<\alpha<1.$$

Consequently, by Beppo Levi monotone convergence theorem for $T\to\infty$, it follows

$$\mathbb{E}\sup\nolimits_{t\geq 0}\big(\Delta_t^{(n)}\big)^{\alpha p}\leq\frac{1}{1-\alpha}\,\left(C_{\lambda,\alpha,p}\,\mathbb{E}\left(\Lambda_n\right)\right)^{\alpha},\to 0,\quad\text{as }n\to\infty.$$

Let $T>0$ be arbitrary and $T\leq n$. Then, from (5.57),

$$\begin{aligned}&c_{q,\lambda}\,\mathbb{E}\bigg(\frac{1}{\big(\sup_{t\geq0}\Delta_t^{(n)}+1\big)^{2-q}}\int_0^T e^{2V_r}\big|Z_r^{(n+i)}-Z_r^{(n)}\big|^2\mathrm{d}r\bigg)\\&\leq c_{q,\lambda}\,\mathbb{E}\int_0^n e^{2V_r}\frac{\big|Z_r^{(n+i)}-Z_r^{(n)}\big|^2}{\big(e^{V_r}\big|Y_r^{(n+i)}-Y_r^{(n)}\big|+1\big)^{2-q}}\,\mathrm{d}r\leq\left[C_{\lambda,\alpha,p}\,\mathbb{E}\left(\Lambda_n\right)\right]^{q/p}.\end{aligned}$$

In this inequality we pass to $\liminf_{i\to\infty}$ and it follows

$$c_{q,\lambda}\,\mathbb{E}\bigg(\frac{1}{\big(\sup_{t\geq0}\Delta_t^{(n)}+1\big)^{2-q}}\int_0^T e^{2V_r}\big|Z_r-Z_r^{(n)}\big|^2\mathrm{d}r\bigg)\leq\left[C_{\lambda,\alpha,p}\,\mathbb{E}\left(\Lambda_n\right)\right]^{q/p}.$$

Now, by Beppo Levi monotone convergence theorem for $T\to\infty$, we obtain

$$c_{q,\lambda}\,\mathbb{E}\bigg(\frac{1}{\big(\sup_{t\geq0}\Delta_t^{(n)}+1\big)^{2-q}}\int_0^\infty e^{2V_r}\big|Z_r-Z_r^{(n)}\big|^2\mathrm{d}r\bigg)\leq\left[C_{\lambda,\alpha,p}\,\mathbb{E}\left(\Lambda_n\right)\right]^{q/p}\to 0,\quad\text{as }n\to\infty.$$

Hence, on a subsequence denoted also by $Z^{(n)}$, we get

$$\int_0^\infty e^{2V_r}\big|Z_r-Z_r^{(n)}\big|^2\mathrm{d}r\to 0,\quad\mathbb{P}-\text{a.s., as }n\to\infty.\tag{5.58}$$

Since $Z_r^{(n)}=0$ for all $r>\tau$, we clearly deduce $Z_r=0$, for all $r>\tau$.

We shall verify that (3.10) is satisfied. For $1\leq n<T$ we have

$$\begin{aligned}&\limsup\nolimits_{T\to\infty}\bigg[e^{pV_T}\left|Y_T-\xi_T\right|^p+\bigg(\int_T^\infty e^{2V_s}\left|Z_s-\zeta_s\right|^2\mathrm{d}s\bigg)^{p/2}\bigg]\\&=\limsup\nolimits_{T\to\infty}\bigg[e^{pV_T}\big|Y_T-Y_T^{(n)}\big|^p+\bigg(\int_T^\infty e^{2V_s}\big|Z_s-Z_s^{(n)}\big|^2\mathrm{d}s\bigg)^{p/2}\bigg]\\&\leq\big(\sup\nolimits_{t\geq0}e^{pV_t}\big|Y_t-Y_t^{(n)}\big|^p\big)^{\alpha q/p}+\bigg(\int_0^\infty e^{2V_s}\big|Z_s-Z_s^{(n)}\big|^2\mathrm{d}s\bigg)^{p/2}.\end{aligned}$$

From (5.56) and (5.58) we get

$$e^{pV_T}\left|Y_T-\xi_T\right|^p+\bigg(\int_T^\infty e^{2V_s}\left|Z_s-\zeta_s\right|^2\mathrm{d}s\bigg)^{p/2}\to 0,\quad\mathbb{P}-\text{a.s.,}\ \ \text{as }T\to\infty,$$

and consequently the convergence holds true in $L^0(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)$ (which is the convergence in probability).

In order to verify (3.11), let $0\le t\le s<\infty$ be arbitrarily chosen and let $n>s$.

Then, for $q\in\{2,p\wedge 2\}$, $\delta_q=\delta\,\mathbf{1}_{[1,2)}(q)$ and $\Gamma_t^{(n)}=\big(\big|M_t-Y_t^{(n)}\big|^2+\delta_q\big)^{1/2}$, it holds

$$\begin{aligned}&\big(\Gamma_t^{(n)}\big)^q+\frac{q(q-1)}{2}\int_t^s\big(\Gamma_r^{(n)}\big)^{q-2}\big|R_r-Z_r^{(n)}\big|^2\mathrm{d}r+q\int_t^s\big(\Gamma_r^{(n)}\big)^{q-2}\Psi\big(r,Y_r^{(n)}\big)\mathrm{d}Q_r\\&\le\big(\Gamma_s^{(n)}\big)^q+q\int_t^s\big(\Gamma_r^{(n)}\big)^{q-2}\Psi(r,M_r)\,\mathrm{d}Q_r-q\int_t^s\big(\Gamma_r^{(n)}\big)^{q-2}\langle M_r-Y_r^{(n)},\big(R_r-Z_r^{(n)}\big)\mathrm{d}B_r\rangle\\&\quad+q\int_t^s\big(\Gamma_r^{(n)}\big)^{q-2}\langle M_r-Y_r^{(n)},N_r-H^{(n)}\big(r,Y_r^{(n)},Z_r^{(n)}\big)\rangle\mathrm{d}Q_r\,,\end{aligned}\tag{5.59}$$

for all $M\in\mathcal{V}_m^0$ of the form

$$M_t=M_T+\int_t^T N_r\mathrm{d}Q_r-\int_t^T R_r\mathrm{d}B_r\,.$$

Passing to the limit, for $n\to\infty$, in (5.59) we infer (using for the left-hand side the Fatou's Lemma and, for the right-hand side, the Lebesgue's dominated convergence theorem and the continuity of the stochastic integral with respect to the convergence in probability) that the pair (Y,Z) satisfies inequality (3.11). □

6. Appendix

In this section, mainly based on some results from [21] and their proofs, we recall and we obtain new inequalities and properties useful in our framework and frequently used in our paper. These results concern mainly inequalities for BSDEs and are interesting by themselves. For more details the interested readers are referred to the monograph of Pardoux and Rășcanu [21].

Let $\{B_t:t\ge 0\}$ be a k-dimensional Brownian motion with respect to a given stochastic basis $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge 0})$, where $(\mathcal{F}_t)_{t\ge 0}$ is the natural filtration associated to $\{B_t:t\ge 0\}$ augmented with $\mathcal{N}$ (the set of $\mathbb{P}$-null events of $\mathcal{F}$).

Notation 6.1. *If $p\ge 1$, we denote $n_p\overset{\text{def}}{=\!=}(p-1)\wedge 1$.*

6.1. An Itô's formula and some backward stochastic inequalities

For the proof of the next result see equality (2.24) from the proof of Proposition 2.26 in [21] and Corollary 2.29 in [21].

Proposition 6.1. *Let $p\in\mathbb{R}$, $\rho\ge 0$ and δ such that $\delta\ge 0$, if $p\ge 2$ and $\delta>0$, if $p<2$.*
Let $Y\in S_d^0$ be a local semimartingale of the form

$$Y_t=Y_0-\int_0^t\mathrm{d}K_r+\int_0^t R_r\mathrm{d}B_r,\quad t\ge 0,\tag{6.1}$$

where $R\in\Lambda^0_{m\times k}$, $K\in S_m^0$, $K_\cdot\in\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^m)$, $\mathbb{P}$-a.s..
Let $\varphi_{\rho,\delta}:\mathbb{R}^d\to(0,\infty)$,

$$\varphi_{\rho,\delta}(x)=\left(\frac{|x|^2}{1+\rho|x|^2}+\delta\right)^{1/2}.$$

By Itô's formula, applied to $\varphi_{\rho,\delta}^{p}(Y_t)$, *with* $p\in\mathbb{R}$, *we have, for all* $0\le t\le s\le T$,

$$\begin{aligned}&\varphi_{\rho,\delta}^{p}(Y_t)+\frac{p}{2}\int_t^s R_r^{(p,\rho,\delta)}\mathrm{d}r+\frac{p}{2}\left[L_s^{(p,\rho,\delta)}-L_t^{(p,\rho,\delta)}\right]\\&=\varphi_{\rho,\delta}^{p}(Y_s)+\frac{p}{2}\int_t^s Q_r^{(p,\rho,\delta)}\mathrm{d}r+p\int_t^s\left\langle U_r^{(p,\rho,\delta)},\mathrm{d}K_r\right\rangle-p\int_t^s\left\langle U_r^{(p,\rho,\delta)},R_r\mathrm{d}B_r\right\rangle,\quad \mathbb{P}-a.s.,\end{aligned}\tag{6.2}$$

where

$U_r^{(p,\rho,\delta)}=\varphi_{\rho,\delta}^{p-2}(Y_r)\ \frac{1}{(1+\rho|Y_r|^2)^2}\,Y_r$,

$R_r^{(p,\rho,\delta)}=\varphi_{\rho,\delta}^{p-4}(Y_r)\ \frac{1}{(1+\rho|Y_r|^2)^3}\left[\frac{p-1}{1+\rho|Y_r|^2}\ |R_r^*Y_r|^2+\left(|R_r|^2|Y_r|^2-|R_r^*Y_r|^2\right)\right]$,

$L_t^{(p,\rho,\delta)}=\delta\int_0^t\varphi_{\rho,\delta}^{p-4}(Y_r)\ \frac{1}{(1+\rho|Y_r|^2)^3}\ \left[|R_r|^2+\rho\left(|R_r|^2|Y_r|^2-|R_r^*Y_r|^2\right)\right]\mathrm{d}r$,

and

$Q_r^{(p,\rho,\delta)}=\varphi_{\rho,\delta}^{p-2}(Y_r)\ \frac{3\rho}{(1+\rho|Y_r|^2)^3}\ |R_r^*Y_r|^2$.

In the case $\rho=0$ *we have*

$$\begin{aligned}&\left(|Y_t|^2+\delta\right)^{p/2}+\frac{p}{2}\int_t^s\left(|Y_r|^2+\delta\right)^{(p-4)/2}\left[(p-1)|R_r^*Y_r|^2+\left(|R_r|^2|Y_r|^2-|R_r^*Y_r|^2\right)\right]\mathrm{d}r\\&\quad+\frac{p}{2}\int_t^s\delta\left(|Y_r|^2+\delta\right)^{(p-4)/2}|R_r|^2\mathrm{d}r\\&=\left(|Y_s|^2+\delta\right)^{p/2}+p\int_t^s\left(|Y_r|^2+\delta\right)^{(p-2)/2}\langle Y_r,\mathrm{d}K_r\rangle\\&\quad-p\int_t^s\left(|Y_r|^2+\delta\right)^{(p-2)/2}\langle Y_r,R_r\mathrm{d}B_r\rangle.\end{aligned}\tag{6.3}$$

Remark 6.2. If $p\ge 1$, then

$$\begin{aligned}&(p-1)|R_r^*Y_r|^2+\left(|R_r|^2|Y_r|^2-|R_r^*Y_r|^2\right)+\delta|R_r|^2\\&\ge n_p\left[|R_r^*Y_r|^2+\left(|R_r|^2|Y_r|^2-|R_r^*Y_r|^2\right)\right]+\delta|R_r|^2\\&=\left(n_p|Y_r|^2+\delta\right)|R_r|^2\ge n_p\left(|Y_r|^2+\delta\right)|R_r|^2\end{aligned}$$

and from (6.3) we infer

$$\begin{aligned}&\left(|Y_t|^2+\delta\right)^{p/2}+\frac{p}{2}\int_t^s\left(|Y_r|^2+\delta\right)^{(p-4)/2}\left(n_p|Y_r|^2+\delta\right)|R_r|^2\mathrm{d}r\\&\le\left(|Y_s|^2+\delta\right)^{p/2}+p\int_t^s\left(|Y_r|^2+\delta\right)^{(p-2)/2}\langle Y_r,\mathrm{d}K_r\rangle\\&\quad-p\int_t^s\left(|Y_r|^2+\delta\right)^{(p-2)/2}\langle Y_r,R_r\mathrm{d}B_r\rangle,\quad\mathbb{P}-\text{a.s.},\end{aligned}\tag{6.4}$$

for all $0\le t\le s\le T$.

6.2. Backward stochastic inequalities

Based on Proposition 6.80 in [21] and its proof we adapt here the Pardoux-Răşcanu's inequalities (6.92) and (6.94) from [21] to the case of our framework (namely the fact that the solutions are defined using not an equality but a stochastic inequality, see Definition 3.2).

Proposition 6.3. *Let $(Y,Z)\in S_m^0\times\Lambda^0_{m\times k}$ and $a\geq 0$, $\gamma\in\mathbb{R}$ such that, for all $0\leq t\leq s<\infty$,*

$$\int_t^s |Z_r|^2\,\mathrm{d}r+\int_t^s \mathrm{d}D_r\leq a\,|Y_s|^2+a\int_t^s(\mathrm{d}R_r+|Y_r|\,dN_r)+\gamma\int_t^s\langle Y_r,Z_r\mathrm{d}B_r\rangle,\quad \mathbb{P}-a.s.,$$

where R,N and D are increasing and continuous p.m.s.p. with $R_0=N_0=D_0=0$.

Then, for all $q>0$ and for all stopping times $0\leq\sigma\leq\theta<\infty$, the following inequality hold:

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^\theta|Z_r|^2\,\mathrm{d}r\Big)^{q/2}+\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^\theta \mathrm{d}D_r\Big)^{q/2}\\&\leq C_{a,\gamma,q}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[\sup\nolimits_{r\in[\sigma,\theta]}|Y_r|^q+\Big(\int_\sigma^\theta\mathrm{d}R_r\Big)^{q/2}+\Big(\int_\sigma^\theta|Y_r|\,dN_r\Big)^{q/2}\Big]\\&\leq 2\,C_{a,\gamma,q}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[\sup\nolimits_{r\in[\sigma,\theta]}|Y_r|^q+\Big(\int_\sigma^\theta\mathrm{d}R_r\Big)^{q/2}+\Big(\int_\sigma^\theta dN_r\Big)^{q}\Big],\quad\mathbb{P}-a.s.,\end{aligned}\tag{6.5}$$

where $C_{a,\gamma,q}$ is a positive constant depending only on a,γ and q.

Proof. We follow the first part of the proof of Proposition 6.80 in [21]. Let the sequence of stopping times

$$\theta_n=\theta\wedge\inf\Big\{s\geq\sigma:\sup\nolimits_{r\in[\sigma,\sigma\vee s]}|Y_r-Y_\sigma|+\int_\sigma^{\sigma\vee s}|Z_r|^2\,\mathrm{d}r+\int_\sigma^{\sigma\vee s}d\,(D_r+R_r+N_r)\geq n\Big\}.\tag{6.6}$$

We have, for $q>0$,

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n}|Z_r|^2\,\mathrm{d}r\Big)^{q/2}+\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n}\mathrm{d}D_r\Big)^{q/2}\\&\leq 2\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n}|Z_r|^2\,\mathrm{d}r+\int_\sigma^{\theta_n}\mathrm{d}D_r\Big)^{q/2}\\&\leq C'_{a,\gamma,q}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[|Y_{\theta_n}|^q+\Big(\int_\sigma^{\theta_n}\mathrm{d}R_r\Big)^{q/2}+\Big(\int_\sigma^{\theta_n}|Y_r|\,dN_r\Big)^{q/2}+\Big|\int_\sigma^{\theta_n}\langle Y_r,Z_r\mathrm{d}B_r\rangle\Big|^{q/2}\Big].\end{aligned}\tag{6.7}$$

By Burkholder-Davis-Gundy inequality we get

$$\begin{aligned}C'_{a,\gamma,q}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big|\int_\sigma^{\theta_n}\langle Y_r,Z_r\mathrm{d}B_r\rangle\Big|^{q/2}&\leq C''_{a,\gamma,q}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n}|Y_r|^2\,|Z_r|^2\,\mathrm{d}r\Big)^{q/4}\\&\leq C''_{a,\gamma,q}\,\mathbb{E}^{\mathcal{F}_\sigma}\sup\nolimits_{r\in[\sigma,\theta_n]}|Y_r|^{q/2}\Big(\int_\sigma^{\theta_n}|Z_r|^2\,\mathrm{d}r\Big)^{q/4}\\&\leq\frac12\,\big(C''_{a,\gamma,q}\big)^2\,\mathbb{E}^{\mathcal{F}_\sigma}\sup\nolimits_{r\in[\sigma,\theta_n]}|Y_r|^q+\frac12\,\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n}|Z_r|^2\,\mathrm{d}r\Big)^{q/2}\end{aligned}$$

and consequently from (6.7) the following inequality holds

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n}|Z_r|^2\,\mathrm{d}r\Big)^{q/2}+\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n}\mathrm{d}D_r\Big)^{q/2}\\&\leq C_{a,\gamma,q}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[\sup\nolimits_{r\in[\sigma,\theta]}|Y_r|^q+\Big(\int_\sigma^{\theta}\mathrm{d}R_r\Big)^{q/2}+\Big(\int_\sigma^{\theta}|Y_r|\,dN_r\Big)^{q/2}\Big].\end{aligned}\tag{6.8}$$

Since

$$\Big(\int_\sigma^{\theta}|Y_r|\,dN_r\Big)^{q/2}\leq\frac{1}{2}\,\sup\nolimits_{r\in[\sigma,\theta]}|Y_r|^q+\frac{1}{2}\Big(\int_\sigma^{\theta}dN_r\Big)^{q},$$

from (6.8) we infer

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n}|Z_r|^2\,\mathrm{d}r\Big)^{q/2}+\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n}\mathrm{d}D_r\Big)^{q/2}\\&\leq C_{a,\gamma,q}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[\sup\nolimits_{r\in[\sigma,\theta]}|Y_r|^q+\Big(\int_\sigma^{\theta}\mathrm{d}R_r\Big)^{q/2}+\Big(\int_\sigma^{\theta}dN_r\Big)^{q}\Big].\end{aligned}\tag{6.9}$$

Consequently, by Fatou's Lemma, as $n\to\infty$, inequality (6.5) follows. □

Proposition 6.4. *Let* $(Y,Z)\in S_m^0\times\Lambda_{m\times k}^0$ *,* $a\geq 0$*,* $\gamma\in\mathbb{R}$ *and* $1<q\leq p$ *satisfying for all* $0\leq t\leq s<\infty$ *and* $\mathbb{P}$*-a.s.,*

$$\begin{aligned}&|Y_t|^q+\int_t^s|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,|Z_r|^2\,\mathrm{d}r+\int_t^s|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,\mathrm{d}D_r\\&\leq a\,|Y_s|^q+a\int_t^s\Big[|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r+|Y_r|^{q-1}\,dN_r\Big]+\gamma\int_t^s|Y_r|^{q-2}\,\langle Y_r,Z_r\mathrm{d}B_r\rangle,\end{aligned}$$

where[5] R,N *and* D *are increasing and continuous p.m.s.p. with* $R_0=N_0=D_0=0$*.*

If σ *and* θ *are two stopping times such that* $0\leq\sigma\leq\theta<\infty$ *and*

$$\mathbb{E}\sup\nolimits_{r\in[\sigma,\theta]}|Y_r|^p<\infty,\tag{6.10}$$

then, $\mathbb{P}$*-a.s.,*

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_\sigma}\Big[\sup\nolimits_{r\in[\sigma,\theta]}|Y_r|^p+\Big(\int_\sigma^{\theta}|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,|Z_r|^2\,\mathrm{d}r\Big)^{p/q}+\Big(\int_\sigma^{\theta}|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,dD_r\Big)^{p/q}\Big]\\&\leq C_{p,q,a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[|Y_\theta|^p+\Big(\int_\sigma^{\theta}\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{p/2}+\Big(\int_\sigma^{\theta}dN_r\Big)^{p}\Big].\end{aligned}\tag{6.11}$$

where $C_{p,q,a,\gamma}$ *is a positive constant depending only on* p,q,a *and* γ*.*

Proof. We follow the proof of Proposition 6.80 in [21]. Let the stopping time θ_n be defined by

$$\theta_n=\theta\wedge\inf\Big\{s\geq\sigma:\sup\nolimits_{r\in[\sigma,\sigma\vee s]}|Y_r-Y_\sigma|+\int_\sigma^{\sigma\vee s}|Z_r|^2\,\mathrm{d}r+\int_\sigma^{\sigma\vee s}d\,(D_r+R_r+N_r)\geq n\Big\}.$$

[5] We use the convention: $|Y_r|^{q-2}\,Y_r=|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,Y_r$, for any $q\geq 1$.

For any stopping time $\tau \in [\sigma, \theta_n]$ we have

$$\begin{aligned}&|Y_\tau|^q + \int_\tau^{\theta_n} |Y_r|^{q-2}\ \mathbf{1}_{Y_r\neq 0}\ |Z_r|^2\,\mathrm{d}r + \int_\tau^{\theta_n} |Y_r|^{q-2}\ \mathbf{1}_{Y_r\neq 0}\, dD_r \\ &\leq a\,|Y_{\theta_n}|^q + a\int_\tau^{\theta_n} \Big(|Y_r|^{q-2}\ \mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r + |Y_r|^{q-1}\,dN_r\Big) + \gamma\int_\tau^{\theta_n} |Y_r|^{q-2}\ \langle Y_r, Z_r\mathrm{d}B_r\rangle.\end{aligned} \tag{6.12}$$

Remark that

$$M_s = \int_0^s \mathbf{1}_{[\sigma,\theta_n]}(r)\ |Y_r|^{q-2}\ \langle Y_r, Z_r\mathrm{d}B_r\rangle, \quad s\geq 0$$

is a martingale, since for any $T > 0$,

$$\begin{aligned}\mathbb{E}\Big(\int_0^T \mathbf{1}_{[\sigma,\theta_n]}(r)\ |Y_r|^{2q-2}\,|Z_r|^2\,\mathrm{d}r\Big)^{1/2} &\leq \mathbb{E}\Big[\sup\nolimits_{r\in[\sigma,\theta_n]} |Y_r|^{q-1}\Big(\int_\sigma^{\theta_n} |Z_r|^2\,\mathrm{d}r\Big)^{1/2}\Big] \\ &\leq \frac{q-1}{q}\,\mathbb{E}\sup\nolimits_{r\in[\sigma,\theta_n]} |Y_r|^q + \frac{1}{q}\,\mathbb{E}\Big(\int_\sigma^{\theta_n} |Z_r|^2\,\mathrm{d}r\Big)^{q/2} \\ &\leq \frac{q-1}{q}\,\mathbb{E}\,(|Y_\sigma| + n)^q + \frac{1}{q}\,n^{q/2} < \infty.\end{aligned}$$

Therefore, from (6.12),

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_\sigma}\Big[\Big(\int_\sigma^{\theta_n} |Y_r|^{q-2}\ \mathbf{1}_{Y_r\neq 0}\ |Z_r|^2\,\mathrm{d}r\Big)^{p/q} + \Big(\int_\sigma^{\theta_n} |Y_r|^{q-2}\ \mathbf{1}_{Y_r\neq 0}\,dD_r\Big)^{p/q}\Big] \\ &\leq C_{p,q,a}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[|Y_{\theta_n}|^p + \Big(\int_\sigma^{\theta_n}\Big(|Y_r|^{q-2}\ \mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)\Big)^{p/q} + \Big(\int_\sigma^{\theta_n} |Y_r|^{q-1}\,dN_r\Big)^{p/q}\Big]\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_\sigma}\sup\nolimits_{\tau\in[\sigma,\theta_n]} |Y_\tau|^p &\leq C'_{p,q,a,\gamma}\Big[\mathbb{E}^{\mathcal{F}_\sigma}\,|Y_{\theta_n}|^p + \mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n} |Y_r|^{q-2}\ \mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{p/q} \\ &\quad + \mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n} |Y_r|^{q-1}\,dN_r\Big)^{p/q} + \mathbb{E}^{\mathcal{F}_\sigma}\sup\nolimits_{\tau\in[\sigma,\theta_n]} |M_{\theta_n} - M_\tau|^{p/q}\Big] \\ &\leq C''_{p,q,a,\gamma}\Big[\mathbb{E}^{\mathcal{F}_\sigma}\,|Y_{\theta_n}|^p + \mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n} |Y_r|^{q-2}\ \mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{p/q} \\ &\quad + \mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n} |Y_r|^{q-1}\,dN_r\Big)^{p/q} + \mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n} |Y_r|^{2q-2}\,|Z_r|^2\,\mathrm{d}r\Big)^{p/(2q)}\Big].\end{aligned} \tag{6.14}$$

On the other hand, from (6.13),

$$\begin{aligned}&C''_{p,q,a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n} |Y_r|^{2q-2}\,|Z_r|^2\,\mathrm{d}r\Big)^{p/(2q)} \\ &\leq C''_{p,q,a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[\sup\nolimits_{r\in[\sigma,\theta_n]} |Y_r|^{p/2}\Big(\int_\sigma^{\theta_n} \mathbf{1}_{Y_r\neq 0}\ |Y_r|^{q-2}\,|Z_r|^2\,\mathrm{d}r\Big)^{p/(2q)}\Big] \\ &\leq \frac{1}{2}\,\mathbb{E}^{\mathcal{F}_\sigma}\sup\nolimits_{r\in[\sigma,\theta_n]} |Y_r|^p + \frac{\big(C''_{p,q,a,\gamma}\big)^2}{2}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n} \mathbf{1}_{Y_r\neq 0}\ |Y_r|^{q-2}\,|Z_r|^2\,\mathrm{d}r\Big)^{p/q}\end{aligned}$$

$$\leq \frac{1}{2}\,\mathbb{E}^{\mathcal{F}_\sigma} \sup\nolimits_{r\in[\sigma,\theta_n]} |Y_r|^p + C'''_{p,q,a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[|Y_{\theta_n}|^p + \Big(\int_\sigma^{\theta_n}\Big(|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)\Big)^{p/q}\Big]$$
$$+ C'''_{p,q,a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n} |Y_r|^{q-1}\,dN_r\Big)^{p/q}.$$

Using this last inequality in (6.14) we obtain

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_\sigma} \sup\nolimits_{\tau\in[\sigma,\theta_n]} |Y_\tau|^p\\ &\leq C_{p,q,a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[|Y_{\theta_n}|^p + \Big(\int_\sigma^{\theta_n}|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{p/q} + \Big(\int_\sigma^{\theta_n}|Y_r|^{q-1}\,dN_r\Big)^{p/q}\Big].\end{aligned}\tag{6.15}$$

Now, by Young's inequality,

$$\begin{aligned}&C_{p,q,a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[\Big(\int_\sigma^{\theta_n}|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{p/q} + \Big(\int_\sigma^{\theta_n}|Y_r|^{q-1}\,dN_r\Big)^{p/q}\Big]\\ &\leq C_{a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[\Big(\sup\nolimits_{r\in[\sigma,\theta_n]}\Big(|Y_r\,\mathbf{1}_{Y_r\neq 0}|^{q-2}\,\mathbf{1}_{q\geq 2}\Big)\int_\sigma^{\theta_n}\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{p/q}\\ &\quad+\Big(\sup\nolimits_{r\in[\sigma,\theta_n]}|Y_r|^{q-1}\int_\sigma^{\theta_n}dN_r\Big)^{p/q}\Big]\\ &\leq \frac{1}{2}\,\mathbb{E}^{\mathcal{F}_\sigma}\sup\nolimits_{\tau\in[\sigma,\theta_n]}|Y_\tau|^p + \hat{C}_{p,q,a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n}\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{p/2} + \hat{C}_{p,q,a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^{\theta_n}dN_r\Big)^p,\end{aligned}$$

which yields, via (6.15),

$$\mathbb{E}^{\mathcal{F}_\sigma}\sup\nolimits_{\tau\in[\sigma,\theta_n]}|Y_\tau|^p \leq C_{p,q,a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[|Y_{\theta_n}|^p + \Big(\int_\sigma^{\theta_n}\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{p/2} + \Big(\int_\sigma^{\theta_n}dN_r\Big)^p\Big].\tag{6.16}$$

Hence, from the last two inequalities, we deduce

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_\sigma}\Big[\Big(\int_\sigma^{\theta_n}|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{p/q} + \Big(\int_\sigma^{\theta_n}|Y_r|^{q-1}\,dN_r\Big)^{p/q}\Big]\\ &\leq \tilde{C}_{p,q,a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[|Y_{\theta_n}|^p + \Big(\int_\sigma^{\theta_n}\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{p/2} + \Big(\int_\sigma^{\theta_n}dN_r\Big)^p\Big].\end{aligned}\tag{6.17}$$

By Beppo Levi monotone convergence theorem and by Lebesgue dominated convergence theorem we deduce, from (6.13), (6.16) and (6.17), inequalities (6.18) and (6.11). □

Remark 6.5. Passing to the limit in (6.13) and (6.15), as $n\to\infty$, we deduce (using Beppo Levi's monotone convergence theorem and condition (6.10) and Lebesgue dominated convergence theorem) that the following inequality holds

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_\sigma}\Big[\sup\nolimits_{r\in[\sigma,\theta]}|Y_r|^p + \Big(\int_\sigma^\theta |Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,|Z_r|^2\,\mathrm{d}r\Big)^{p/q} + \Big(\int_\sigma^\theta |Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,dD_r\Big)^{p/q}\Big]\\ &\leq C_{p,q,a,\gamma}\,\mathbb{E}^{\mathcal{F}_\sigma}\Big[|Y_\theta|^p + \Big(\int_\sigma^\theta |Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{p/q} + \Big(\int_\sigma^\theta |Y_r|^{q-1}\,dN_r\Big)^{p/q}\Big],\quad \mathbb{P}-\text{a.s.}.\end{aligned}\tag{6.18}$$

Moreover (using again the same Beppo Levi theorem and Lebesgue theorem) we can see that inequalities (6.11) and (6.18) are also true in the case $0 \leq \sigma \leq \theta \leq \infty$.

Proposition 6.6. *Let Y be a continuous stochastic process. Let $q \geq 1$ and $b, L \geq 0$ such that*

$$\mathbb{E}\sup_{r\in[0,T]} |Y_r|^q \leq L < \infty$$

and, for all $0 \leq t \leq T < \infty$,

$$\mathbb{E}^{\mathcal{F}_t}\Big(|Y_t|^q + \int_t^T dD_r\Big) \leq b\,\mathbb{E}^{\mathcal{F}_t}\Big[|Y_T|^q + \int_0^T \Big(|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r + |Y_r|^{q-1}\,dN_r\Big)\Big],$$

$\mathbb{P}$-a.s., where R, N and D are increasing and continuous p.m.s.p. $R_0 = N_0 = D_0 = 0$.

Then, for any $0 < \alpha < 1$,

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T]} |Y_t|^{\alpha q} + \mathbb{E}\Big(\int_0^T dD_r\Big)^{\alpha} \\ &\leq \frac{2b^{\alpha}}{1-\alpha}\Big[(\mathbb{E}\,|Y_T|^q)^{\alpha} + L^{\frac{\alpha(q-2)}{q}}\Big(\mathbb{E}\Big(\int_0^T \mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{\frac{q}{2}}\Big)^{\frac{2\alpha}{q}} + L^{\frac{\alpha(q-1)}{q}}\Big(\mathbb{E}\Big(\int_0^T dN_r\Big)^{q}\Big)^{\frac{\alpha}{q}}\Big] \end{aligned} \tag{6.19}$$

and, for any $\varepsilon, \delta > 0$, there exists $C_{\alpha,q,b,\varepsilon,\delta} > 0$ such that

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T]} |Y_t|^{\alpha q} + \mathbb{E}\Big(\int_0^T dD_r\Big)^{\alpha} \\ &\leq C_{\alpha,q,b,\varepsilon,\delta}\Big[(\mathbb{E}\,|Y_T|^q)^{\alpha} + \Big(\mathbb{E}\Big(\int_0^T \mathbf{1}_{q\geq 2}\,\mathrm{d}R_r\Big)^{\frac{q}{2}+\varepsilon}\Big)^{\frac{\alpha q}{q+2\varepsilon}} + \Big(\mathbb{E}\Big(\int_0^T dN_r\Big)^{q+\delta}\Big)^{\frac{\alpha q}{q+\delta}}\Big]. \end{aligned} \tag{6.20}$$

Proof. By Proposition 1.56, (A_3) of [21], the conclusions clearly hold true in the case $q = 1$.

Let $q > 1$ and $0 < \alpha < 1$.

We remark first that

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T]} |Y_t|^{\alpha q} \leq \mathbb{E}\sup_{t\in[0,T]}\Big(|Y_t|^q + \int_t^T dD_r\Big)^{\alpha}, \\ &\mathbb{E}\Big(\int_0^T dD_r\Big)^{\alpha} \leq \mathbb{E}\sup_{t\in[0,T]}\Big(|Y_t|^q + \int_t^T dD_r\Big)^{\alpha}, \end{aligned}$$

hence

$$\mathbb{E}\sup_{t\in[0,T]} |Y_t|^{\alpha q} + \mathbb{E}\Big(\int_0^T dD_r\Big)^{\alpha} \leq 2\,\mathbb{E}\sup_{t\in[0,T]}\Big(|Y_t|^q + \int_t^T dD_r\Big)^{\alpha}. \tag{6.21}$$

By Proposition 1.56, (A_3) of [21] we have

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T]}\Big(|Y_t|^q + \int_t^T dD_r\Big)^{\alpha} \\ &\leq \frac{b^{\alpha}}{1-\alpha}\Big[\mathbb{E}\,|Y_T|^q + \mathbb{E}\int_0^T |Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\geq 2}\,\mathrm{d}R_r + \mathbb{E}\int_0^T |Y_r|^{q-1}\,dN_r\Big]^{\alpha} \\ &\leq \frac{b^{\alpha}}{1-\alpha}\big(\mathbb{E}\,|Y_T|^q\big)^{\alpha} + A^{\alpha} + B^{\alpha}, \end{aligned} \tag{6.22}$$

where

$$A^{\alpha}=\frac{b^{\alpha}}{1-\alpha}\Big(\mathbb{E}\Big(\sup\nolimits_{r\in[0,T]}\Big(|Y_r\,\mathbf{1}_{Y_r\neq0}|^{q-2}\;\mathbf{1}_{q\geq2}\Big)\cdot\int_0^T\mathbf{1}_{q\geq2}\,\mathrm{d}R_r\Big)\Big)^{\alpha},$$
$$B^{\alpha}=\frac{b^{\alpha}}{1-\alpha}\Big(\mathbb{E}\Big(\sup\nolimits_{r\in[0,T]}|Y_r|^{q-1}\int_0^T dN_r\Big)\Big)^{\alpha}.$$

By Hölder's inequality with $\beta=\frac{q}{q-1}>1$ and $\beta'=q$ we have

$$\begin{aligned}B^{\alpha}&\leq\frac{b^{\alpha}}{1-\alpha}\Big(\mathbb{E}\sup\nolimits_{r\in[0,T]}|Y_r|^{\beta(q-1)}\Big)^{\alpha/\beta}\cdot\Big(\mathbb{E}\Big(\int_0^T dN_r\Big)^{\beta'}\Big)^{\alpha/\beta'}\\&\leq\frac{b^{\alpha}}{1-\alpha}\,L^{\alpha(q-1)/q}\Big(\mathbb{E}\Big(\int_0^T dN_r\Big)^{q}\Big)^{\alpha/q}.\end{aligned}\tag{6.23}$$

In addition, again by Hölder's inequality and Young's inequality, with $\beta=\frac{\alpha q}{q-1}>1$, $\beta'=\frac{\alpha}{\alpha-(q-1)/q}$, $\gamma=\frac{\beta}{\alpha}=\frac{q}{q-1}>1$ and $\gamma'=q$, we obtain the next inequality for any α such that $\frac{q-1}{q}<\alpha<1$:

$$\begin{aligned}B^{\alpha}&\leq\frac{1}{8}\Big(\mathbb{E}\sup\nolimits_{r\in[0,T]}|Y_r|^{\beta(q-1)}\Big)^{\alpha\gamma/\beta}+C_{\alpha,\beta,\gamma,b}\Big(\mathbb{E}\Big(\int_0^T dN_r\Big)^{\beta'}\Big)^{\alpha\gamma'/\beta'}\\&=\frac{1}{8}\,\mathbb{E}\sup\nolimits_{r\in[0,T]}|Y_r|^{\alpha q}+C_{\alpha,q,b}\Big(\mathbb{E}\Big(\int_0^T dN_r\Big)^{\alpha q/(\alpha q-q+1)}\Big)^{\alpha q-q+1}.\end{aligned}\tag{6.24}$$

Of course,

$$A^{\alpha}=\begin{cases}0, & \text{if } 1<q<2,\\ \dfrac{b^{\alpha}}{1-\alpha}\Big(\mathbb{E}\displaystyle\int_0^T\mathrm{d}R_r\Big)^{\alpha}, & \text{if } q=2.\end{cases}\tag{6.25}$$

If $q>2$, by Hölder's inequality with $\beta=\frac{q}{q-2}>1$ and $\beta'=\frac{q}{2}$, we have

$$\begin{aligned}A^{\alpha}&\leq\frac{b^{\alpha}}{1-\alpha}\Big(\mathbb{E}\sup\nolimits_{r\in[0,T]}|Y_r|^{\beta(q-2)}\Big)^{\alpha/\beta}\cdot\Big(\mathbb{E}\Big(\int_0^T\mathrm{d}R_r\Big)^{\beta'}\Big)^{\alpha/\beta'}\\&\leq\frac{b^{\alpha}}{1-\alpha}\,L^{\alpha(q-2)/q}\Big(\mathbb{E}\Big(\int_0^T\mathrm{d}R_r\Big)^{q/2}\Big)^{2\alpha/q}.\end{aligned}\tag{6.26}$$

We see that inequality (6.26) is satisfied also in the case $1<q\leq2$.

In addition, again by Hölder's inequality and Young's inequality, with $\beta=\frac{\alpha q}{q-2}>1$, $\beta'=\frac{\alpha}{\alpha-(q-2)/q}$, $\gamma=\frac{\beta}{\alpha}=\frac{q}{q-2}>1$ and $\gamma'=\frac{q}{2}$, we obtain the next inequality for any α such that $\frac{q-2}{q}<\alpha<1$:

$$\begin{aligned}A^{\alpha}&=\frac{1}{8}\Big(\mathbb{E}\sup\nolimits_{r\in[0,T]}|Y_r|^{\beta(q-2)}\Big)^{\alpha\gamma/\beta}+C_{\alpha,\beta,\gamma,b}\Big(\mathbb{E}\Big(\int_0^T\mathrm{d}R_r\Big)^{\beta'}\Big)^{\alpha\gamma'/\beta'}\\&=\frac{1}{8}\,\mathbb{E}\sup\nolimits_{r\in[0,T]}\big(|Y_r|^{\alpha q}\big)+C_{\alpha,q,b}\Big(\mathbb{E}\Big(\int_0^T\mathrm{d}R_r\Big)^{\alpha q/(\alpha q-q+2)}\Big)^{(\alpha q-q+2)/2}.\end{aligned}\tag{6.27}$$

We see that inequality (6.27) is satisfied also in the case $1 < q \le 2$ and for any α such that $0 < \alpha < 1$.

Now it is clear that inequality (6.19) follows from inequalities (6.21), (6.22), (6.23), (6.25) and (6.26).

On the other hand, from inequalities (6.21), (6.22), (6.24), (6.25) and (6.27) we deduce that, for any α such that $\frac{q-2}{q} < \frac{q-1}{q} < \alpha < 1$,

$$\begin{aligned}&\mathbb{E}\sup\nolimits_{t\in[0,T]}|Y_t|^{\alpha q}+\mathbb{E}\Big(\int_0^T dD_r\Big)^{\alpha}\\&\le C_{\alpha,q,b}\left[\left(\mathbb{E}\,|Y_T|^q\right)^{\alpha}+\left(\mathbb{E}\Big(\int_0^T \mathbf{1}_{q\ge 2}\mathrm{d}R_r\Big)^{\frac{\alpha q}{\alpha q-q+2}}\right)^{\frac{\alpha q-q+2}{2}}+\left(\mathbb{E}\Big(\int_0^T dN_r\Big)^{\frac{\alpha q}{\alpha q-q+1}}\right)^{\alpha q-q+1}\right].\end{aligned}$$

If $0 < \alpha < 1$ is arbitrary fixed, then the last inequality hold also for α replaced by any $\bar{\alpha}$ such that

$$\alpha\vee\frac{\mathbf{1}_{q\ge 2}\ (q-2)\,(q+2\varepsilon)}{q\,(q+2\varepsilon-2)}\vee\frac{(q-1)\,(q+\delta)}{q\,(q+\delta-1)}<\bar{\alpha}<1. \tag{6.28}$$

By Hölder's inequality we have

$$\begin{aligned}&\mathbb{E}\sup\nolimits_{t\in[0,T]}|Y_t|^{\alpha q}+\mathbb{E}\Big(\int_0^T dD_r\Big)^{\alpha}\le\Big(\mathbb{E}\sup\nolimits_{t\in[0,T]}|Y_t|^{\bar{\alpha} q}\Big)^{\alpha/\bar{\alpha}}+\left(\mathbb{E}\Big(\int_0^T dD_r\Big)^{\bar{\alpha}}\right)^{\frac{\alpha}{\bar{\alpha}}}\\&\le 2\left[\mathbb{E}\sup\nolimits_{t\in[0,T]}|Y_t|^{\bar{\alpha} q}+\mathbb{E}\Big(\int_0^T dD_r\Big)^{\bar{\alpha}}\right]^{\frac{\alpha}{\bar{\alpha}}}\\&\le C_{\bar{\alpha},q,b}\left[\left(\mathbb{E}\,|Y_T|^q\right)^{\bar{\alpha}}+\left(\mathbb{E}\Big(\textstyle\int_0^T \mathbf{1}_{q\ge 2}\mathrm{d}R_r\Big)^{\frac{\bar{\alpha} q}{\bar{\alpha} q-q+2}}\right)^{\frac{\bar{\alpha} q-q+2}{2}}\right.\\&\qquad\left.+\left(\mathbb{E}\Big(\int_0^T dN_r\Big)^{\frac{\bar{\alpha} q}{\bar{\alpha} q-q+1}}\right)^{\bar{\alpha} q-q+1}\right]^{\frac{\alpha}{\bar{\alpha}}}\\&\le C_{\bar{\alpha},q,b}\left[\left(\mathbb{E}\,|Y_T|^q\right)^{\alpha}+\left(\mathbb{E}\Big(\textstyle\int_0^T \mathbf{1}_{q\ge 2}\mathrm{d}R_r\Big)^{\frac{\bar{\alpha} q}{\bar{\alpha} q-q+2}}\right)^{\frac{\bar{\alpha} q-q+2}{\bar{\alpha} q}\cdot\frac{\alpha q}{2}}\right.\\&\qquad\left.+\left(\mathbb{E}\Big(\int_0^T dN_r\Big)^{\frac{\bar{\alpha} q}{\bar{\alpha} q-q+1}}\right)^{\frac{\bar{\alpha} q-q+1}{\bar{\alpha} q}\cdot\alpha q}\right]\end{aligned}$$

Using (6.28) we obtain

$$\frac{\bar{\alpha} q}{\bar{\alpha} q-q+2}\le\frac{q}{2}+\varepsilon\quad\text{and}\quad\frac{\bar{\alpha} q}{\bar{\alpha} q-q+1}\le q+\delta$$

and, by Hölder's inequality,

$$\begin{aligned}&\left(\mathbb{E}\Big(\int_0^T \mathbf{1}_{q\ge 2}\,\mathrm{d}R_r\Big)^{\frac{\bar{\alpha} q}{\bar{\alpha} q-q+2}}\right)^{\frac{\bar{\alpha} q-q+2}{\bar{\alpha} q}\ \frac{\alpha q}{2}}\le\left(\mathbb{E}\Big(\int_0^T \mathbf{1}_{q\ge 2}\,\mathrm{d}R_r\Big)^{\frac{q}{2}+\varepsilon}\right)^{\frac{\alpha q}{q+2\varepsilon}}\\&\left(\mathbb{E}\Big(\int_0^T dN_r\Big)^{\frac{\bar{\alpha} q}{\bar{\alpha} q-q+1}}\right)^{\frac{\bar{\alpha} q-q+1}{\bar{\alpha} q}\ \alpha q}\le\left(\mathbb{E}\Big(\int_0^T dN_r\Big)^{q+\delta}\right)^{\frac{\alpha q}{q+\delta}}.\end{aligned}$$

Consequently inequality (6.20) holds for any $0 < \alpha < 1$. ☐

Proposition 6.7. *Let:*

– $(Y, Z) \in S^0_m \times \Lambda^0_{m\times k}$;

– $K \in S_m^0$ *and* $K_\cdot \in \mathrm{BV}_{\mathrm{loc}}\left(\mathbb{R}_+;\mathbb{R}^m\right)$, $\mathbb{P}$*-a.s.;*
– $D, R, N, \tilde{R}$ *be some increasing continuous p.m.s.p. with* $D_0 = R_0 = N_0 = 0$*;*
– V *be a bounded variation p.m.s.p. with* $V_0 = 0$;
– σ *and* θ *be two stopping times such that* $0 \le \sigma \le \theta < \infty$.

I. *If for all* $0 \le t \le s < \infty$, $\mathbb{P}$*-a.s.*

$$|Y_t|^2 + \int_t^s |Z_r|^2 \,\mathrm{d}r + \int_t^s \mathrm{d}D_r \le |Y_s|^2 + 2\int_t^s \langle Y_r, \mathrm{d}K_r\rangle - 2\int_t^s \langle Y_r, Z_r \mathrm{d}B_r\rangle, \tag{6.29}$$

and for some $\lambda < 1$

$$\int_t^s \langle Y_r, \mathrm{d}K_r\rangle \le \int_t^s \left(\mathrm{d}R_r + |Y_r| dN_r + |Y_r|^2 \mathrm{d}V_r\right) + \frac{\lambda}{2}\int_t^s |Z_r|^2 \,\mathrm{d}r, \tag{6.30}$$

then, for any $q > 0$*, there exists a positive constant* $C_{q,\lambda}$ *such that* $\mathbb{P}$*-a.s.*

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^\theta e^{2V_r} |Z_r|^2 \,\mathrm{d}s\Big)^{q/2} + \mathbb{E}^{\mathcal{F}_\sigma}\Big(\int_\sigma^\theta e^{2V_r} \mathrm{d}D_r\Big)^{q/2} \\ &\le C_{q,\lambda}\mathbb{E}^{\mathcal{F}_\sigma}\left[\sup\nolimits_{r\in[\sigma,\theta]} \left|e^{V_r} Y_r\right|^q + \Big(\int_\sigma^\theta e^{2V_r}\mathrm{d}R_r\Big)^{q/2} + \Big(\int_\sigma^\theta e^{2V_r}|Y_r|\, dN_r\Big)^{q/2}\right] \\ &\le 2C_{q,\lambda}\mathbb{E}^{\mathcal{F}_\sigma}\left[\sup\nolimits_{r\in[\sigma,\theta]} \left|e^{V_r} Y_r\right|^q + \Big(\int_\sigma^\theta e^{2V_r}\mathrm{d}R_r\Big)^{q/2} + \Big(\int_\sigma^\theta e^{V_r} dN_r\Big)^{q}\right]. \end{aligned} \tag{6.31}$$

II. *If* $q > 1$,

$$\begin{aligned} (i)\quad & |Y_t|^q + \frac{q}{2} n_q \int_t^s |Y_r|^{q-2}\, \mathbf{1}_{Y_r\neq 0}\, |Z_r|^2 \,\mathrm{d}r + \int_t^s |Y_r|^{q-2}\, \mathbf{1}_{Y_r\neq 0}\, \mathrm{d}D_r \\ & \le |Y_s|^q + q\int_t^s |Y_r|^{q-2}\, \mathbf{1}_{Y_r\neq 0}\left[d\tilde{R}_r + \langle Y_r, \mathrm{d}K_r\rangle\right] - q\int_t^s |Y_r|^{q-2}\, \mathbf{1}_{Y_r\neq 0}\, \langle Y_r, Z_r\mathrm{d}B_r\rangle, \\ (ii)\quad & \mathbb{E}\sup\nolimits_{r\in[\sigma,\theta]} e^{qV_r}|Y_r|^q < \infty \end{aligned} \tag{6.32}$$

and for some $\lambda < 1$

$$d\tilde{R}_r + \langle Y_r, \mathrm{d}K_r\rangle \le \left(\mathbf{1}_{q\ge 2}\,\mathrm{d}R_r + |Y_r| dN_r + |Y_r|^2\mathrm{d}V_r\right) + \frac{n_q}{2}\lambda\, |Z_r|^2 \,\mathrm{d}t, \tag{6.33}$$

then there exists some positive constants $C_{q,\lambda}$, $C'_{q,\lambda}$ *such that,* $\mathbb{P}$*-a.s.,*

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_\sigma}\left[\sup\nolimits_{r\in[\sigma,\theta]}\left|e^{V_r}Y_r\right|^q + \int_\sigma^\theta e^{qV_r}|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,|Z_r|^2\,\mathrm{d}s + \int_\sigma^\theta e^{qV_r}|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,\mathrm{d}D_r\right] \\ &\le C_{q,\lambda}\,\mathbb{E}^{\mathcal{F}_\sigma}\left[\left|e^{V_\theta}Y_\theta\right|^q + \Big(\int_\sigma^\theta e^{qV_r}|Y_r|^{q-2}\,\mathbf{1}_{Y_r\neq 0}\,\mathbf{1}_{q\ge 2}\,\mathrm{d}R_r\Big) + \Big(\int_\sigma^\theta e^{qV_r}|Y_r|^{q-1}\,dN_r\Big)\right] \\ &\le C'_{q,\lambda}\,\mathbb{E}^{\mathcal{F}_\sigma}\left[\left|e^{V_\theta}Y_\theta\right|^q + \Big(\int_\sigma^\theta e^{2V_r}\,\mathbf{1}_{q\ge 2}\,\mathrm{d}R_r\Big)^{q/2} + \Big(\int_\sigma^\theta e^{V_r}dN_r\Big)^q\right]. \end{aligned} \tag{6.34}$$

Proof. Using inequalities (6.29) and (6.30) we obtain, for all $0 \le t \le s < \infty$,

$$\begin{aligned} &|Y_t|^2 + (1-\lambda) \int_t^s |Z_r|^2 \mathrm{d}r + \int_t^s \mathrm{d}D_r \\ &\le |Y_s|^2 + \int_t^s \left[(2\mathrm{d}R_r + 2|Y_r|dN_r) + |Y_r|^2 d\,(2V_r)\right] - 2\int_t^s \langle Y_r, Z_r \mathrm{d}B_r\rangle, \end{aligned}$$

which yields, applying ([21], Prop. 6.69) (or [16], Lem. 12),

$$\begin{aligned} &\left|e^{V_t} Y_t\right|^2 + (1-\lambda) \int_t^s \left|e^{V_r} Z_r\right|^2 \mathrm{d}r + \int_t^s e^{2V_r} \mathrm{d}D_r \\ &\le \left|e^{V_s} Y_s\right|^2 + 2\int_t^s \left[e^{2V_r}\mathrm{d}R_r + |e^{V_r}Y_r| e^{V_r} dN_r\right] - 2\int_t^s \langle e^{V_r} Y_r, e^{V_r} Z_r \mathrm{d}B_r\rangle. \end{aligned}$$

Inequality (6.31) follows now by Proposition 6.3.

In the same manner, using (6.32), (6.33) and Proposition 6.69 of [21], we infer

$$\begin{aligned} &\left|e^{V_t} Y_t\right|^q + \frac{q}{2} n_q\ (1-\lambda) \int_t^s \left|e^{V_r} Y_r\right|^{q-2} \mathbf{1}_{Y_r \neq 0} \left|e^{V_r} Z_r\right|^2 \mathrm{d}r + \int_t^s \left|e^{V_r} Y_r\right|^{q-2} \mathbf{1}_{Y_r \neq 0}\, e^{2V_r} \mathrm{d}D_r \\ &\le \left|e^{V_s} Y_s\right|^q + q\int_t^s \left[\left|e^{V_r} Y_r\right|^{q-2} \mathbf{1}_{Y_r \neq 0}\, \mathbf{1}_{q\ge 2}\, e^{2V_r}\mathrm{d}R_r + |e^{V_r}Y_r|^{q-1} e^{V_r} dN_r\right] \\ &\quad -q\int_t^s \left|e^{V_r} Y_r\right|^{q-2} \mathbf{1}_{Y_r \neq 0} \left\langle e^{V_r} Y_r, e^{V_r} Z_r \mathrm{d}B_r\right\rangle. \end{aligned}$$

Inequalities from (6.34) follow now by Proposition 6.4 and Remark 6.5. □

With a similar approach we deduce the next result; for its proof see Proposition 6.80 of [21].

Proposition 6.8 (Pardoux–Rășcanu)**.** *Let $(Y,Z) \in S_m^0 \times \Lambda_{m\times k}^0$ satisfying*

$$Y_t = Y_T + \int_t^T \mathrm{d}K_s - \int_t^T Z_s \mathrm{d}B_s,\ 0 \le t \le T, \quad \mathbb{P}-a.s.,$$

where $K \in S_m^0$ and $K_\cdot \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^m)$, $\mathbb{P}$-a.s..

Let τ and σ be two stopping times such that $0 \le \tau \le \sigma < \infty$. Assume that there exists three increasing and continuous p.m.s.p. D, R, N with $D_0 = R_0 = N_0 = 0$ and a bounded variation p.m.s.p. V with $V_0 = 0$ such that for, $\lambda < 1$,

$$\mathrm{d}D_t + \langle Y_t, \mathrm{d}K_t\rangle \le \mathrm{d}R_t + |Y_t|dN_t + |Y_t|^2 \mathrm{d}V_t + \frac{\lambda}{2}\,|Z_t|^2\,\mathrm{d}t.$$

Then, for any $q > 0$, there exists a positive constant $C_{q,\lambda}$ such that, $\mathbb{P}$-a.s.,

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_\tau}\Big(\int_\tau^\sigma e^{2V_s}\mathrm{d}D_s\Big)^{q/2} + \mathbb{E}^{\mathcal{F}_\tau}\Big(\int_\tau^\sigma e^{2V_s}\,|Z_s|^2\,\mathrm{d}s\Big)^{q/2} \\ &\le C_{q,\lambda}\,\mathbb{E}^{\mathcal{F}_\tau}\left[\sup\nolimits_{s\in[\tau,\sigma]} \left|e^{V_s} Y_s\right|^q + \Big(\int_\tau^\sigma e^{2V_s}\mathrm{d}R_s\Big)^{q/2} + \Big(\int_\tau^\sigma e^{V_s} dN_s\Big)^q\right]. \end{aligned}$$

Moreover, if $p>1$ and

$$\begin{array}{l}\mathrm{d}D_t+\langle Y_t,\mathrm{d}K_t\rangle\leq\big(\mathbf{1}_{p\geq 2}\,\mathrm{d}R_t+|Y_t|dN_t+|Y_t|^2\mathrm{d}V_t\big)+\dfrac{n_p}{2}\lambda\,|Z_t|^2\,\mathrm{d}t,\\ \mathbb{E}\sup_{s\in[\tau,\sigma]}e^{pV_s}\,|Y_s|^p<\infty,\end{array}$$

then there exists a positive constant $C_{p,\lambda}$ such that, $\mathbb{P}$-a.s.,

$$\begin{array}{l}\mathbb{E}^{\mathcal{F}_\tau}\Big(\sup_{s\in[\tau,\sigma]}\left|e^{V_s}Y_s\right|^p\Big)+\mathbb{E}^{\mathcal{F}_\tau}\Big(\displaystyle\int_\tau^\sigma e^{2V_s}\mathrm{d}D_s\Big)^{p/2}+\mathbb{E}^{\mathcal{F}_\tau}\Big(\displaystyle\int_\tau^\sigma e^{2V_s}\,|Z_s|^2\,\mathrm{d}s\Big)^{p/2}\\ \leq C_{p,\lambda}\,\mathbb{E}^{\mathcal{F}_\tau}\Big[\left|e^{V_\sigma}Y_\sigma\right|^p+\Big(\displaystyle\int_\tau^\sigma e^{2V_s}\,\mathbf{1}_{p\geq 2}\,\mathrm{d}R_s\Big)^{p/2}+\Big(\displaystyle\int_\tau^\sigma e^{V_s}dN_s\Big)^p\Big].\end{array}\tag{6.35}$$

6.3. Smoothing approximations

Lemma 6.9. *Let $\varepsilon>0$ and let $Q:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ be a strictly increasing continuous stochastic process such that $Q_0=0$ and $\lim_{t\to\infty}Q_t=\infty$, and let $G:\Omega\times\mathbb{R}_+\to\mathbb{R}^m$ be a measurable stochastic process such that $\sup_{t\in\mathbb{R}_+}|G_t|<\infty$, $\mathbb{P}$-a.s..*

Define

$$G_t^\varepsilon=\frac{1}{Q_\varepsilon}\int_{t\vee\varepsilon}^\infty e^{-\frac{Q_r-Q_{t\vee\varepsilon}}{Q_\varepsilon}}G_r\,\mathrm{d}Q_r\,.\tag{6.36}$$

Then $G^\varepsilon:\Omega\times\mathbb{R}_+\to\mathbb{R}^m$ are continuous stochastic processes and, $\mathbb{P}$-a.s.,

$$\begin{array}{ll}(a)&|G_t^\varepsilon|\leq\sup_{r\geq 0}|G_r|\,,\quad\text{for all } t\geq 0;\\ (b)&\lim_{\varepsilon\to 0}G_t^\varepsilon=G_t\,,\quad\text{a.e. } t\geq 0;\\ (c)&|G_t^\varepsilon-G_t|\leq\exp\big(2-1/\sqrt{Q_\varepsilon}\big)\sup_{r\geq 0}|G_r|\\ &\quad+\sup_{r\geq 0}\left\{|G_r-G_t|:0\leq Q_r-Q_t\leq\sqrt{Q_\varepsilon}\vee Q_\varepsilon\right\},\quad\text{for all } t\geq 0.\end{array}\tag{6.37}$$

Moreover, if G is a continuous stochastic process, then, for all $T>0$,

$$\lim_{\varepsilon\to 0}\Big(\sup_{s\in[0,T]}|G_s^\varepsilon-G_s|\Big)=0,\quad\mathbb{P}-a.s..\tag{6.38}$$

Proof. (b) Let $n\in\mathbb{N}^*$. We can assume that $0<\varepsilon<t$.

$$\begin{aligned}|G_t^\varepsilon-G_t|&\leq\frac{1}{Q_\varepsilon}\int_t^\infty e^{-\frac{Q_r-Q_t}{Q_\varepsilon}}|G_r-G_t|\mathrm{d}Q_r\\ &=\int_0^\infty e^{-s}\left|G_{Q^{-1}(Q_t+sQ_\varepsilon)}-G_{Q^{-1}(Q_t)}\right|\mathrm{d}s\\ &\leq\int_0^n\left|G_{Q^{-1}(Q_t+sQ_\varepsilon)}-G_{Q^{-1}(Q_t)}\right|\mathrm{d}s+2\sup_{r\geq 0}|G_r|\int_n^\infty e^{-s}\mathrm{d}s.\end{aligned}$$

Since almost all points $t\in[0,n]$ are Lebesgue points for the measurable and bounded function $G_{Q^{-1}(\cdot)}$, we have

$$\lim_{\varepsilon\to 0}\int_0^n\left|G_{Q^{-1}(Q_t+sQ_\varepsilon)}-G_{Q^{-1}(Q_t)}\right|\mathrm{d}s=0,\quad\text{a.e. } t\in[0,n]\,,$$

and therefore, for all $n \in \mathbb{N}^*$,

$$\limsup_{\varepsilon\to 0} |G_t^\varepsilon - G_t| \le 2e^{-n} \sup_{r\ge 0} |G_r|, \quad \text{a.e. } t\in(0,T),$$

which yields (b).

(c) Let $t_\varepsilon = Q^{-1}\left(Q_t + \sqrt{Q_\varepsilon}\right)$. We have

$$\begin{aligned}
|G_t^\varepsilon - G_t| &\le \frac{1}{Q_\varepsilon}\int_{t\vee\varepsilon}^{\infty} e^{-\frac{Q_r - Q_{t\vee\varepsilon}}{Q_\varepsilon}} |G_r - G_t| \mathrm{d}Q_r \\
&\le \sup_{r\in[t\vee\varepsilon, t_\varepsilon\vee\varepsilon]} |G_r - G_t| \frac{1}{Q_\varepsilon}\int_{t\vee\varepsilon}^{t_\varepsilon\vee\varepsilon} e^{-\frac{Q_r - Q_{t\vee\varepsilon}}{Q_\varepsilon}} \mathrm{d}Q_r + 2\sup_{s\ge 0}|G_s| \frac{1}{Q_\varepsilon}\int_{t_\varepsilon\vee\varepsilon}^{\infty} e^{-\frac{Q_r - Q_{t\vee\varepsilon}}{Q_\varepsilon}} \mathrm{d}Q_r \\
&\le \sup_{r\in[t\vee\varepsilon, t_\varepsilon\vee\varepsilon]} |G_r - G_t| \int_0^{\frac{Q_{t_\varepsilon\vee\varepsilon} - Q_{t\vee\varepsilon}}{Q_\varepsilon}} e^{-s}\mathrm{d}s + 2\sup_{s\ge 0}|G_s| \frac{1}{Q_\varepsilon}\int_{t_\varepsilon}^{\infty} e^{-\frac{Q_r - Q_{t\vee\varepsilon}}{Q_\varepsilon}} \mathrm{d}Q_r \\
&\le \sup_{r\in[t\vee\varepsilon, t_\varepsilon\vee\varepsilon]} |G_r - G_t| + 2e^{-\frac{Q_{t_\varepsilon} - Q_{t\vee\varepsilon}}{Q_\varepsilon}} \sup_{s\ge 0}|G_s|.
\end{aligned}$$

Since

$$\frac{Q_{t_\varepsilon} - Q_{t\vee\varepsilon}}{Q_\varepsilon} = \frac{\sqrt{Q_\varepsilon}}{Q_\varepsilon} + \frac{Q_t - Q_{t\vee\varepsilon}}{Q_\varepsilon} \ge \frac{1}{\sqrt{Q_\varepsilon}} - 1,$$

inequality (6.37$-c$) follows.

Clearly, (6.38) follows from (6.37$-c$). □

Remark 6.10. Let $\varepsilon > 0$ and let $Q : \Omega\times\mathbb{R}\to\mathbb{R}$ be a strictly increasing continuous stochastic process such that $Q_0 = 0$ and $\lim_{t\to\infty} Q_t = \infty$, $\lim_{t\to-\infty} Q_t = -\infty$, and let $G:\Omega\times\mathbb{R}\to\mathbb{R}^m$ be a measurable stochastic process such that $\sup_{t\in\mathbb{R}_+}|G_t| < \infty$, $\mathbb{P}$-a.s..

Then similar boundedness and convergence results as in the previous Lemma 6.9 hold true for $G^{i,\varepsilon}:\Omega\times\mathbb{R}\to\mathbb{R}^m$, $i=\overline{1,4}$, defined by

$$\begin{aligned}
G_t^{1,\varepsilon} &= \frac{1}{Q_\varepsilon}\int_t^{\infty} G_r\, e^{-\frac{Q_r - Q_t}{Q_\varepsilon}} \mathrm{d}Q_r, \quad t\in\mathbb{R}, \\
G_t^{2,\varepsilon} &= \frac{1}{Q_\varepsilon}\int_{-\infty}^{t} G_r\, e^{-\frac{Q_t - Q_r}{Q_\varepsilon}} \mathrm{d}Q_r, \quad t\in\mathbb{R}, \\
G_t^{3,\varepsilon} &= e^{-\frac{Q_t}{Q_\varepsilon}} G_0 + \frac{1}{Q_\varepsilon}\int_0^{t} G_r\, e^{-\frac{Q_t - Q_r}{Q_\varepsilon}} \mathrm{d}Q_r \\
&= \frac{1}{Q_\varepsilon}\int_{-\infty}^{t} \left[\mathbf{1}_{(-\infty,0)}(r)\, G_0 + \mathbf{1}_{[0,\infty)}(r)\, G_r\right] e^{-\frac{Q_t - Q_r}{Q_\varepsilon}} \mathrm{d}Q_r, \quad t\ge 0, \\
G_t^{4,\varepsilon} &= \mathbf{1}_{[0,\varepsilon)}(t)\, G_0 + \mathbf{1}_{[\varepsilon,\infty)}(t)\, \frac{1}{Q_\varepsilon}\int_0^{t} G_r\, e^{-\frac{Q_t - Q_{r\vee\varepsilon}}{Q_\varepsilon}} \mathrm{d}Q_r, \quad t\ge 0.
\end{aligned}$$

Corollary 6.11. *Let the assumptions of Lemma 6.9 be satisfied and* $\varphi:\mathbb{R}^m\to(-\infty,+\infty]$ *be a proper convex lower semicontinuous function such that* $\int_0^\infty |\varphi(G_u)|\,\mathrm{d}Q_u < \infty$, $\mathbb{P}$*-a.s..*

Then, for any $0\le\alpha\le\beta$,

$$\lim_{\varepsilon\to 0}\int_\alpha^\beta \varphi(G_r^\varepsilon)\,\mathrm{d}Q_r = \int_\alpha^\beta \varphi(G_r)\,\mathrm{d}Q_r,$$

where G^ε is given by (6.36).

Moreover, if $\mathbb{E}\int_0^\infty |\varphi(G_u)|\,\mathrm{d}Q_u < \infty$, *then, for any stopping times* $0 \le \sigma \le \theta$,

$$\lim_{\varepsilon\to 0} \mathbb{E}\int_\sigma^\theta \varphi(G_r^\varepsilon)\,\mathrm{d}Q_r = \mathbb{E}\int_\sigma^\theta \varphi(G_r)\,\mathrm{d}Q_r\,.$$

Proof. We have

$$\begin{aligned}\int_\alpha^\beta \varphi(G_r^\varepsilon)\,\mathrm{d}Q_r &\le \int_\alpha^\beta \left(\frac{1}{Q_\varepsilon}\int_{r\vee\varepsilon}^\infty e^{-\frac{Q_u-Q_{r\vee\varepsilon}}{Q_\varepsilon}}\varphi(G_u)\,\mathrm{d}Q_u\right)\mathrm{d}Q_r\\ &= \int_0^\infty \varphi(G_u)\left(\int_0^\infty \mathbf{1}_{[\alpha,\beta]}(r)\,\mathbf{1}_{[r\vee\varepsilon,\infty)}(u)\,\frac{1}{Q_\varepsilon}\,e^{-\frac{Q_u-Q_{r\vee\varepsilon}}{Q_\varepsilon}}\mathrm{d}Q_r\right)\mathrm{d}Q_u\\ &= \int_0^\infty \varphi(G_u)\,\mathbf{1}_{[\varepsilon,\infty)}(u)\left(\frac{1}{Q_\varepsilon}\int_0^u \mathbf{1}_{[\alpha,\beta]}(r)\,e^{-\frac{Q_u-Q_{r\vee\varepsilon}}{Q_\varepsilon}}\mathrm{d}Q_r\right)\mathrm{d}Q_u\,,\end{aligned}$$

since $\mathbf{1}_{[r\vee\varepsilon,\infty)}(u) = \mathbf{1}_{[0,u]}(r)\,\mathbf{1}_{[\varepsilon,\infty)}(u)\,.$

We obtain

$$\begin{aligned}&\frac{1}{Q_\varepsilon}\int_0^u \mathbf{1}_{[\alpha,\beta]}(r)\,e^{-\frac{Q_u-Q_r}{Q_\varepsilon}}\mathrm{d}Q_r \le \frac{1}{Q_\varepsilon}\int_0^u \mathbf{1}_{[\alpha,\beta]}(r)\,e^{-\frac{Q_u-Q_{r\vee\varepsilon}}{Q_\varepsilon}}\mathrm{d}Q_r\\ &= \frac{1}{Q_\varepsilon}\int_0^u \mathbf{1}_{[\alpha,\beta]}(r)\left[\mathbf{1}_{[0,\varepsilon)}(r)\,e^{-\frac{Q_u-Q_\varepsilon}{Q_\varepsilon}} + \mathbf{1}_{[\varepsilon,\infty)}(r)\,e^{-\frac{Q_u-Q_r}{Q_\varepsilon}}\right]\mathrm{d}Q_r\\ &\le \frac{Q_{u\wedge\varepsilon}}{Q_\varepsilon}\,e^{1-\frac{Q_u}{Q_\varepsilon}} + \frac{1}{Q_\varepsilon}\int_0^u \mathbf{1}_{[\alpha,\beta]}(r)\,e^{-\frac{Q_u-Q_r}{Q_\varepsilon}}\mathrm{d}Q_r\,,\end{aligned} \tag{6.39}$$

since

$$\int_0^u \mathbf{1}_{[\alpha,\beta]}(r)\,\mathbf{1}_{[0,\varepsilon)}(r)\,\mathrm{d}Q_r \le Q_{u\wedge\varepsilon}\,.$$

By Remark 6.10 (with the extension $Q_r = r$, for $r < 0$),

$$\lim_{\varepsilon\to 0}\frac{1}{Q_\varepsilon}\int_0^u \mathbf{1}_{[\alpha,\beta]}(r)\,e^{-\frac{Q_u-Q_r}{Q_\varepsilon}}\mathrm{d}Q_r = \mathbf{1}_{[\alpha,\beta]}(u)\,,\quad \text{a.e. } u \ge 0,$$

hence, by (6.39),

$$\lim_{\varepsilon\to 0}\frac{1}{Q_\varepsilon}\int_0^u \mathbf{1}_{[\alpha,\beta]}(r)\,e^{-\frac{Q_u-Q_{r\vee\varepsilon}}{Q_\varepsilon}}\mathrm{d}Q_r = \mathbf{1}_{[\alpha,\beta]}(u)\,,\quad \text{a.e. } u \ge 0.$$

On the other hand, since

$$0 \le \frac{1}{Q_\varepsilon}\int_0^u \mathbf{1}_{[\alpha,\beta]}(r)\,e^{-\frac{Q_u-Q_{r\vee\varepsilon}}{Q_\varepsilon}}\mathrm{d}Q_r \le e+1,$$

by Fatou's Lemma and by the Lebesgue dominated convergence theorem, we infer

$$\int_\alpha^\beta \varphi(G_r)\,\mathrm{d}Q_r \le \liminf_{\varepsilon\to 0}\int_\alpha^\beta \varphi(G_r^\varepsilon)\,\mathrm{d}Q_r \le \limsup_{\varepsilon\to 0}\int_\alpha^\beta \varphi(G_r^\varepsilon)\,\mathrm{d}Q_r \le \int_\alpha^\beta \varphi(G_r)\,\mathrm{d}Q_r\,.$$

The second assertion of this corollary follows in the same manner. □

Proposition 6.12. *Let $Q:\Omega\times[0,T]\to\mathbb{R}_+$ be a strictly increasing continuous stochastic process such that $Q_0=0$.*

Let $\tau:\Omega\to[0,\infty]$ be a stopping time and $\eta:\Omega\to\mathbb{R}^m$ a $\mathcal{F}_\tau$-measurable random variable such that $\mathbb{E}\,|\eta|^p<\infty$, if $p>1$, and $(\xi,\zeta)\in S_m^p\times\Lambda_{m\times k}^p$ the unique pair associated to η given by the martingale representation formula (see [21], Cor. 2.44):

$$\begin{cases} \xi_t=\eta-\displaystyle\int_t^\infty\zeta_s\mathrm{d}B_s,\ t\geq0,\quad \mathbb{P}-a.s.,\\ \xi_t=\mathbb{E}^{\mathcal{F}_t}\eta=\mathbb{E}^{\mathcal{F}_{t\wedge\tau}}\eta\quad and\quad \zeta_t=\mathbf{1}_{[0,\tau]}(t)\,\zeta_t\end{cases}$$

(or equivalently, $\xi_t=\eta-\int_{t\wedge\tau}^{\tau}\zeta_s\mathrm{d}B_s,\ t\geq0,\ \mathbb{P}$-a.s.).

Let $U\in S_m^p$, with $p>1$, be such that

$$\begin{array}{ll}(a) & \mathbb{E}\sup_{t\geq0}|U_t|^p<\infty,\\ (b) & \lim_{t\to\infty}\mathbb{E}\,|U_t-\xi_t|^p=0.\end{array}$$

Define

$$U_t^\varepsilon=\frac{1}{Q_\varepsilon}\int_{t\vee\varepsilon}^\infty e^{-\frac{Q_r-Q_{t\vee\varepsilon}}{Q_\varepsilon}}\,U_r\,\mathrm{d}Q_r\quad and\quad M_t^\varepsilon=\mathbb{E}^{\mathcal{F}_t}\big(U_t^\varepsilon\big)\,,\quad t\geq0. \tag{6.40}$$

Then:

I.

$$\begin{array}{ll}(j) & |M_t^\varepsilon|\leq\mathbb{E}^{\mathcal{F}_t}\sup_{r\geq0}|U_r|\,,\quad \mathbb{P}-a.s.,\ for\ all\ t\geq0,\\ (jj) & \mathbb{E}\sup_{t\geq0}|M_t^\varepsilon|^p\leq C_p\,\mathbb{E}\sup_{r\geq0}|U_r|^p\,.\end{array} \tag{6.41}$$

Also, for any $t\geq0$,

$$\begin{aligned}|M_t^\varepsilon-U_t|\leq\mathbb{E}^{\mathcal{F}_t}\Big[&\exp\big(2-1/\sqrt{Q_\varepsilon}\big)\,\sup_{r\geq0}|U_r|\\ &+\sup_{r\geq0}\Big\{|U_r-U_t|:0\leq Q_r-Q_t\leq\sqrt{Q_\varepsilon}\vee Q_\varepsilon\Big\}\Big]\end{aligned} \tag{6.42}$$

which yields

$$\begin{array}{ll}(jjj) & \lim_{\varepsilon\to0}M_t^\varepsilon=U_t\,,\quad \mathbb{P}-a.s.,\quad for\ all\ t\geq0;\\ (jv) & \lim_{\varepsilon\to0}\mathbb{E}\sup_{t\in[0,T]}|M_t^\varepsilon-U_t|^p=0,\quad for\ all\ T>0.\end{array} \tag{6.43}$$

II. *M^ε is the unique solution of the BSDE:*

$$\begin{cases} M_t^\varepsilon=M_T^\varepsilon+\dfrac{1}{Q_\varepsilon}\displaystyle\int_t^T 1_{[\varepsilon,\infty)}(r)\,(U_r-M_r^\varepsilon)\,\mathrm{d}Q_r-\int_t^T R_r^\varepsilon\,\mathrm{d}B_r\,,\quad for\ all\ T>0,\ t\in[0,T]\,,\\ \lim_{t\to\infty}\mathbb{E}\,|M_t^\varepsilon-\xi_t|^p=0.\end{cases} \tag{6.44}$$

Moreover,

$$\lim_{t\to\infty}\mathbb{E}\sup_{s\geq t}|U_t-\xi_t|^p=0\quad\Longrightarrow\quad\lim_{t\to\infty}\mathbb{E}\Big(\sup_{s\geq t}|M_s^\varepsilon-\xi_s|^p\Big)=0. \tag{6.45}$$

III. *Let $\varphi : \mathbb{R}^m \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function such that*

$$\mathbb{E}\int_0^\infty |\varphi(U_r)|\, \mathrm{d}Q_r < \infty.$$

Let $0 \le s \le t$ and the stopping times $s^* = Q_s^{-1}$, $t^* = Q_t^{-1}$, where $Q_\cdot^{-1}$ is the inverse of the function $r \mapsto Q_r : [0,\infty) \to [0,\infty)$.

Then

$$\lim_{\varepsilon\to 0} \mathbb{E}\int_{s^*}^{t^*} \varphi(M_r^\varepsilon)\, \mathrm{d}Q_r = \mathbb{E}\int_{s^*}^{t^*} \varphi(U_r)\, \mathrm{d}Q_r\,.$$

Moreover, if $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}_+$ is a continuous function, $D : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ is a continuous stochastic process such that for all $R > 0$

$$\mathbb{E}\int_0^R |\varphi(U_r)| \sup_{\theta\in[0,r]} g(U_\theta, D_\theta)\, \mathrm{d}Q_r + \mathbb{E}\int_0^R |\varphi(U_r)| \sup_{\theta\in[0,r]} \sup_{0<\varepsilon\le 1} g(M_\theta^\varepsilon, D_\theta)\, \mathrm{d}Q_r < \infty,$$

then, for all $0 \le T \le \infty$,

$$\begin{aligned} (v) \quad & \mathbb{E}\int_{s^*\wedge T}^{t^*\wedge T} g(M_r^\varepsilon, D_r)\, \varphi(M_r^\varepsilon)\, \mathrm{d}Q_r \le \mathbb{E}\int_{s^*\wedge T}^{t^*\wedge T} g(M_r^\varepsilon, D_r)\, \varphi(U_r^\varepsilon)\, \mathrm{d}Q_r\,, \\ (vj) \quad & \lim_{\varepsilon\to 0} \mathbb{E}\int_{s^*\wedge T}^{t^*\wedge T} g(M_r^\varepsilon, D_r)\, \varphi(M_r^\varepsilon)\, \mathrm{d}Q_r = \mathbb{E}\int_{s^*\wedge T}^{t^*\wedge T} g(U_r, D_r)\, \varphi(U_r)\, \mathrm{d}Q_r\,. \end{aligned} \tag{6.46}$$

Proof. By Doob's inequality (see [21], Thm. 1.60) and ($6.41-j$) we get estimate ($6.41-jj$).

Clearly

$$|M_t^\varepsilon - U_t| \le \mathbb{E}^{\mathcal{F}_t} |U_t^\varepsilon - U_t| \le \mathbb{E}^{\mathcal{F}_t} \sup_{r\in[0,T]} |U_t^\varepsilon - U_t|$$

and conclusions (6.42) and (6.43) hold by Lemma 6.9 and Doob's inequality.

Let us to prove (6.44). By the martingale representation theorem we have

$$\begin{aligned} \frac{1}{Q_\varepsilon}\int_\varepsilon^\infty e^{-\frac{Q_r}{Q_\varepsilon}}\, U_r\, \mathrm{d}Q_r &= \mathbb{E}^{\mathcal{F}_t} \frac{1}{Q_\varepsilon}\int_\varepsilon^\infty e^{-\frac{Q_r}{Q_\varepsilon}}\, U_r\, \mathrm{d}Q_r + \int_t^\infty \tilde{R}_r^\varepsilon\, \mathrm{d}B_r \\ &= e^{-\frac{Q_{t\vee\varepsilon}}{Q_\varepsilon}} M_t^\varepsilon + \mathbb{E}^{\mathcal{F}_t} \frac{1}{Q_\varepsilon}\int_\varepsilon^{t\vee\varepsilon} e^{-\frac{Q_r}{Q_\varepsilon}}\, U_r\, \mathrm{d}Q_r + \int_t^\infty \tilde{R}_r^\varepsilon\, \mathrm{d}B_r\,, \end{aligned}$$

which yields

$$e^{-\frac{Q_{t\vee\varepsilon}}{Q_\varepsilon}} M_t^\varepsilon = \frac{1}{Q_\varepsilon}\int_t^\infty 1_{[\varepsilon,\infty)}(r)\, e^{-\frac{Q_r}{Q_\varepsilon}}\, U_r\, \mathrm{d}Q_r - \int_t^\infty \tilde{R}_r^\varepsilon\, \mathrm{d}B_r\,. \tag{6.47}$$

Now by Itô's formula

$$\begin{aligned} M_t^\varepsilon &= M_T^\varepsilon - \int_t^T d\left[e^{\frac{Q_{r\vee\varepsilon}}{Q_\varepsilon}} \left(e^{-\frac{Q_{r\vee\varepsilon}}{Q_\varepsilon}} M_r^\varepsilon\right)\right] \\ &= M_T^\varepsilon - \frac{1}{Q_\varepsilon}\int_t^T 1_{[\varepsilon,\infty)}(r)\, e^{\frac{Q_{r\vee\varepsilon}}{Q_\varepsilon}} \left(e^{-\frac{Q_{r\vee\varepsilon}}{Q_\varepsilon}} M_r^\varepsilon\right) \mathrm{d}Q_r - \int_t^T e^{\frac{Q_{r\vee\varepsilon}}{Q_\varepsilon}}\, d\left(e^{-\frac{Q_{r\vee\varepsilon}}{Q_\varepsilon}} M_r^\varepsilon\right) \end{aligned}$$

$$= M_T^\varepsilon - \frac{1}{Q_\varepsilon}\int_t^T 1_{[\varepsilon,\infty)}(r)\; M_r^\varepsilon \,\mathrm{d}Q_r + \frac{1}{Q_\varepsilon}\int_t^T e^{\frac{Q_{r\vee\varepsilon}}{Q_\varepsilon}} 1_{[\varepsilon,\infty)}(r)\; e^{-\frac{Q_{r\vee\varepsilon}}{Q_\varepsilon}} U_r \,\mathrm{d}Q_r$$
$$- \int_t^T e^{\frac{Q_{r\vee\varepsilon}}{Q_\varepsilon}} \tilde{R}_r^\varepsilon \,\mathrm{d}B_r$$
$$= M_T^\varepsilon + \frac{1}{Q_\varepsilon}\int_t^T 1_{[\varepsilon,\infty)}(r)\; (U_r - M_r^\varepsilon)\,\mathrm{d}Q_r - \int_t^T R_r^\varepsilon \,\mathrm{d}B_r\,,$$

where $R_r^\varepsilon \overset{\text{def}}{=\!=} e^{\frac{Q_{r\vee\varepsilon}}{Q_\varepsilon}} \tilde{R}_r^\varepsilon$.

The convergence result from (6.44) is obtained as follows:

$$|M_t^\varepsilon - \xi_t| = \Big|\mathbb{E}^{\mathcal{F}_t}\frac{1}{Q_\varepsilon}\int_{t\vee\varepsilon}^\infty e^{-\frac{Q_r - Q_{t\vee\varepsilon}}{Q_\varepsilon}}\,(U_r - \xi_r)\,\mathrm{d}Q_r + \mathbb{E}^{\mathcal{F}_t}\frac{1}{Q_\varepsilon}\int_{t\vee\varepsilon}^\infty e^{-\frac{Q_r - Q_{t\vee\varepsilon}}{Q_\varepsilon}}\,(\xi_r - \xi_t)\,\mathrm{d}Q_r\Big|$$
$$\leq \Big|\mathbb{E}^{\mathcal{F}_t}\int_0^\infty e^{-s}\left(U_{Q^{-1}(sQ_\varepsilon + Q_{t\vee\varepsilon})} - \xi_{Q^{-1}(sQ_\varepsilon + Q_{t\vee\varepsilon})}\right)\mathrm{d}s\Big| + \mathbb{E}^{\mathcal{F}_t}\sup\nolimits_{r\geq t}|\xi_r - \xi_t|$$

By Jensen's inequality and, after that, by Burkholder-Davis-Gundy inequality (see [21], Cor. 2.9) we have

$$\left(\mathbb{E}^{\mathcal{F}_t}\sup\nolimits_{r\geq t}|\xi_r - \xi_t|\right)^p \leq \left(\mathbb{E}^{\mathcal{F}_t}\sup\nolimits_{r\geq t}\left|\int_t^r \zeta_s\,\mathrm{d}B_s\right|\right)^p \leq C_p\,\mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty |\zeta_s|^2\,\mathrm{d}s\right)^{p/2}.$$

Hence (by Jensen's and Holder's inequalities)

$$\mathbb{E}\,|M_t^\varepsilon - \xi_t|^p \leq 2^{p-1}\mathbb{E}\left(\mathbb{E}^{\mathcal{F}_t}\int_0^\infty e^{-s}\left|U_{Q^{-1}(sQ_\varepsilon + Q_{t\vee\varepsilon})} - \xi_{Q^{-1}(sQ_\varepsilon + Q_{t\vee\varepsilon})}\right|\mathrm{d}s\right)^p$$
$$+ 2^{p-1}\,\mathbb{E}\left(\mathbb{E}^{\mathcal{F}_t}\sup\nolimits_{r\geq t}|\xi_r - \xi_t|\right)^p$$
$$\leq 2^{p-1}\int_0^\infty e^{-s}\,\mathbb{E}\left|U_{Q^{-1}(sQ_\varepsilon + Q_{t\vee\varepsilon})} - \xi_{Q^{-1}(sQ_\varepsilon + Q_{t\vee\varepsilon})}\right|^p\mathrm{d}s + C_p\,\mathbb{E}\left(\int_t^\infty |\zeta_s|^2\,\mathrm{d}s\right)^{p/2}$$

and, using the Lebesgue dominated convergence theorem, we get

$$\lim\nolimits_{t\to\infty}\mathbb{E}\,|M_t^\varepsilon - \xi_t|^p = 0.$$

In order to prove (6.45), we see that, for $\varepsilon < T \leq t$ and $1 < q < p$,

$$|M_t^\varepsilon - \xi_t|^p \leq 2^{p-1}\,\mathbb{E}^{\mathcal{F}_t}\sup\nolimits_{r\geq T}|U_r - \xi_r|^p + 2^{p-1}\left(\mathbb{E}^{\mathcal{F}_t}\sup\nolimits_{r\geq t}|\xi_r - \xi_t|^q\right)^{p/q}$$

and consequently (by Burkholder-Davis-Gundy and Doob's inequality)

$$\mathbb{E}\sup_{t\geq T}|M_t^\varepsilon - \xi_t|^p \leq 2^{p-1}\,\mathbb{E}\sup\nolimits_{r\geq T}|U_r - \xi_r|^p + 2^{p-1}\,\mathbb{E}\sup\nolimits_{t\geq T}\left[\mathbb{E}^{\mathcal{F}_t}\sup\nolimits_{r\geq t}\left|\int_t^r \zeta_s\,\mathrm{d}B_s\right|^q\right]^{p/q}$$
$$\leq 2^{p-1}\,\mathbb{E}\sup\nolimits_{r\geq T}|U_r - \xi_r|^p + C_{p,q}\,\mathbb{E}\sup\nolimits_{t\geq T}\left[\mathbb{E}^{\mathcal{F}_t}\left(\int_T^\infty |\zeta_s|^2\,\mathrm{d}s\right)^{q/2}\right]^{p/q}$$
$$\leq C_p\,\mathbb{E}\sup\nolimits_{r\geq T}|U_r - \xi_r|^p + C'_{p,q}\,\mathbb{E}\left(\int_T^\infty |\zeta_s|^2\,\mathrm{d}s\right)^{p/2}$$

and (6.45) follows.

Inequality (6.46$-v$) follows since, using the notation $r^* = Q_r^{-1}$,

$$\begin{aligned}
&\mathbb{E}\int_{s^*\wedge T}^{t^*\wedge T} g\left(M_r^\varepsilon, D_r\right)\varphi\left(M_r^\varepsilon\right)\mathrm{d}Q_r = \mathbb{E}\int_{s^*}^{t^*}\mathbf{1}_{[0,T]}\left(r\right)g\left(M_r^\varepsilon, D_r\right)\varphi\left(\mathbb{E}^{\mathcal{F}_r}\left(U_r^\varepsilon\right)\right)\mathrm{d}Q_r\\
&\leq \mathbb{E}\int_{s^*}^{t^*}\mathbb{E}^{\mathcal{F}_r}\left[\mathbf{1}_{[0,T]}\left(r\right)g\left(M_r^\varepsilon, D_r\right)\varphi\left(U_r^\varepsilon\right)\right]\mathrm{d}Q_r\\
&= \mathbb{E}\int_s^t \mathbb{E}^{\mathcal{F}_{r^*}}\left[\mathbf{1}_{[0,T]}\left(r^*\right)g\left(M_{r^*}^\varepsilon, D_{r^*}\right)\varphi\left(U_{r^*}^\varepsilon\right)\right]\mathrm{d}r\\
&= \int_s^t \mathbb{E}\left[\mathbf{1}_{[0,T]}\left(r^*\right)g\left(M_{r^*}^\varepsilon, D_{r^*}\right)\varphi\left(U_{r^*}^\varepsilon\right)\right]\mathrm{d}r = \mathbb{E}\int_s^t\mathbf{1}_{[0,T]}\left(r^*\right)g\left(M_{r^*}^\varepsilon, D_{r^*}\right)\varphi\left(U_{r^*}^\varepsilon\right)\mathrm{d}r\\
&= \mathbb{E}\int_{s^*}^{t^*}\mathbf{1}_{[0,T]}\left(r\right)g\left(M_r^\varepsilon, D_r\right)\varphi\left(U_r^\varepsilon\right)\mathrm{d}Q_r = \mathbb{E}\int_{s^*\wedge T}^{t^*\wedge T} g\left(M_r^\varepsilon, D_r\right)\varphi\left(U_r^\varepsilon\right)\mathrm{d}Q_r\,.
\end{aligned}$$

As in the proof of Corollary 6.11 we have

$$\begin{aligned}
&\mathbb{E}\int_{s^*\wedge T}^{t^*\wedge T} g\left(M_r^\varepsilon, D_r\right)\varphi\left(U_r^\varepsilon\right)\mathrm{d}Q_r\\
&\leq \mathbb{E}\int_0^\infty \varphi\left(U_\theta\right)\,\mathbf{1}_{[\varepsilon,\infty)}\left(\theta\right)\left(\frac{1}{Q_\varepsilon}\int_0^\theta \mathbf{1}_{[s^*\wedge T, t^*\wedge T]}\left(r\right)\,g\left(M_r^\varepsilon, D_r\right)\,e^{-\frac{Q_\theta - Q_{r\vee\varepsilon}}{Q_\varepsilon}}\,\mathrm{d}Q_r\right)\mathrm{d}Q_\theta\\
&\leq \mathbb{E}\int_0^\infty \varphi\left(U_\theta\right)\,\mathbf{1}_{[\varepsilon,\infty)}\left(\theta\right)\cdot\sup\nolimits_{r\in[0,\theta]}\left|g\left(M_r^\varepsilon, D_r\right) - g\left(U_r, D_r\right)\right|\mathrm{d}Q_\theta\\
&\quad+\mathbb{E}\int_0^\infty \varphi\left(U_\theta\right)\,\mathbf{1}_{[\varepsilon,\infty)}\left(\theta\right)\left(\frac{1}{Q_\varepsilon}\int_0^\theta \mathbf{1}_{[s^*\wedge T, t^*\wedge T]}\left(r\right)\,g\left(U_r, D_r\right)\,e^{-\frac{Q_\theta - Q_{r\vee\varepsilon}}{Q_\varepsilon}}\,\mathrm{d}Q_r\right)\mathrm{d}Q_\theta\,.
\end{aligned}$$

Now, by Fatou's Lemma and Remark 6.10 (with the extension $Q_r = r$, for $r < 0$), we have

$$\begin{aligned}
&\mathbb{E}\int_{s^*\wedge T}^{t^*\wedge T} g\left(U_r, D_r\right)\varphi\left(U_r\right)\mathrm{d}Q_r \leq \liminf_{\varepsilon\to 0_+}\mathbb{E}\int_{s^*\wedge T}^{t^*\wedge T} g\left(M_r^\varepsilon, D_r\right)\varphi\left(M_r^\varepsilon\right)\mathrm{d}Q_r\\
&\leq \liminf\nolimits_{\varepsilon\to 0_+}\mathbb{E}\int_{s^*\wedge T}^{t^*\wedge T} g\left(M_r^\varepsilon, D_r\right)\varphi\left(U_r^\varepsilon\right)\mathrm{d}Q_r \leq \limsup\nolimits_{\varepsilon\to 0_+}\mathbb{E}\int_{s^*\wedge T}^{t^*\wedge T} g\left(M_r^\varepsilon, D_r\right)\varphi\left(U_r^\varepsilon\right)\mathrm{d}Q_r\\
&leq\limsup_{\varepsilon\to 0_+}\mathbb{E}\int_0^\infty \varphi\left(U_\theta\right)\Big(\mathbf{1}_{[\varepsilon,\infty)}\left(\theta\right)\,\frac{1}{Q_\varepsilon}\,\int_0^\theta \mathbf{1}_{[s^*\wedge T, t^*\wedge T]}\left(r\right)\,g\left(U_r, D_r\right)\,e^{-\frac{Q_\theta - Q_{r\vee\varepsilon}}{Q_\varepsilon}}\mathrm{d}Q_r\Big)\mathrm{d}Q_\theta\\
&= \mathbb{E}\int_{s^*\wedge T}^{t^*\wedge T} g\left(U_\theta, D_\theta\right)\varphi\left(U_\theta\right)\mathrm{d}Q_\theta
\end{aligned}$$

and convergence (6.46$-vj$) follows. □

6.4. Mollifier approximation

Let $F : \Omega\times\mathbb{R}_+\times\mathbb{R}^m\times\mathbb{R}^{m\times k}\to\mathbb{R}^m$ and $G : \Omega\times\mathbb{R}_+\times\mathbb{R}^m\to\mathbb{R}^m$ be such that assumptions (A_5) and (A_6) are satisfied.

Let $\rho\in C_0^\infty\left(\mathbb{R}^m;\mathbb{R}_+\right)$ such that $\rho\left(y\right) = 0$ if $|y|\geq 1$ and $\int_{\mathbb{R}^m}\rho\left(y\right)dy = 1$.

Let $\kappa > 0$ be such that

$$\int_{\overline{B(0,1)}} |\nabla \rho(v)|\, dv \leq \kappa \quad \text{and} \quad |\nabla_y \rho(y)| \leq \kappa\, \mathbf{1}_{\overline{B(0,1)}}(y), \quad \text{for all } y \in \mathbb{R}^m.$$

Define, for $0 < \varepsilon \leq 1$,

$$\begin{aligned} F_\varepsilon(t, y, z) &= \int_{\overline{B(0,1)}} F(t, y - \varepsilon u, \beta_\varepsilon(z))\, \mathbf{1}_{[0,1]}(\varepsilon |F(t, y - \varepsilon u, 0)|)\, \rho(u)\, \mathrm{d}u \\ &= \int_{\mathbb{R}^m} F(t, y - \varepsilon u, \beta_\varepsilon(z))\, \mathbf{1}_{[0,1]}(\varepsilon |F(t, y - \varepsilon u, 0)|)\, \rho(u)\, \mathrm{d}u \\ &= \frac{1}{\varepsilon^m} \int_{\mathbb{R}^m} F(t, u, \beta_\varepsilon(z))\, \mathbf{1}_{[0,1]}(\varepsilon |F(t, u, 0)|)\, \rho\left(\frac{y-u}{\varepsilon}\right) \mathrm{d}u\,, \end{aligned} \tag{6.48}$$

where

$$\beta_\varepsilon(z) = \frac{z}{1 \vee (\varepsilon |z|)} = \mathrm{Pr}_{\overline{B(0,1/\varepsilon)}}(z)\,.$$

For all $z, \hat{z} \in \mathbb{R}^{m \times k}$ and all $\varepsilon, \delta > 0$ we have

$$\begin{aligned} &|\beta_\varepsilon(z)| \leq |z| \wedge \frac{1}{\varepsilon}\,, \\ &|\beta_\varepsilon(z) - \beta_\varepsilon(\hat{z})| \leq |z - \hat{z}|\,, \\ &|\beta_\varepsilon(z) - \beta_\delta(z)| \leq \mathbf{1}_{[\frac{1}{\varepsilon} \wedge \frac{1}{\delta}, \infty)}(|z|)\, \mathbf{1}_{\varepsilon \neq \delta} \cdot |z|\,. \end{aligned}$$

Clearly, from the assumptions satisfied by F, we have, for all $y, u \in \mathbb{R}^m$ with $|u| \leq 1$ and for all $z \in \mathbb{R}^{m \times k}$,

$$|F(t, y - \varepsilon u, \beta_\varepsilon(z))| \leq \ell_t\, |z| + F^{\#}_{|y|+1}(t)$$

and consequently

$$|F_\varepsilon(t, y, z)| \leq \ell_t\, |z| + F^{\#}_{|y|+1}(t) \quad \text{and} \quad |F_\varepsilon(t, 0, 0)| \leq F^{\#}_1(t)\,. \tag{6.49}$$

The mollifier approximation F_ε of F satisfies the following properties:

$$\begin{aligned} &(a) \quad |F_\varepsilon(t, y, z)| \leq \ell_t\, \beta_\varepsilon(z) + \frac{1}{\varepsilon} \leq \frac{1}{\varepsilon}\,(1 + \ell_t)\,, \\ &(b) \quad |F_\varepsilon(t, y, z) - F_\varepsilon(t, y, \hat{z})| \leq \ell_t\, |z - \hat{z}|\,, \\ &(c) \quad |F_\varepsilon(t, y, z) - F_\varepsilon(t, \hat{y}, z)| \leq \frac{\kappa}{\varepsilon}\left[\ell_t\, |\beta_\varepsilon(z)| + \frac{1}{\varepsilon}\right] |y - \hat{y}| \leq \frac{\kappa\,(1 + \ell_t)}{\varepsilon^2}\, |y - \hat{y}|\,. \end{aligned} \tag{6.50}$$

Indeed,

$$
\begin{aligned}
&|F_\varepsilon(t,y,z)-F_\varepsilon(t,\hat{y},z)|\\
&\le \frac{1}{\varepsilon^m}\int_{\mathbb{R}^m}|F(t,u,\beta_\varepsilon(z))|\ \mathbf{1}_{[0,1]}(\varepsilon|F(t,u,0)|)\ \Big|\rho\Big(\frac{y-u}{\varepsilon}\Big)-\rho\Big(\frac{\hat{y}-u}{\varepsilon}\Big)\Big|\mathrm{d}u\\
&\le \frac{1}{\varepsilon^{m+1}}\ |y-\hat{y}|\int_{\mathbb{R}^m}[\ell_t\ |\beta_\varepsilon(z)|+|F(t,u,0)|]\ \mathbf{1}_{[0,1]}(\varepsilon|F(t,u,0)|)\\
&\qquad\qquad\qquad\qquad\cdot\Big(\int_0^1\Big|\nabla\rho\Big(\frac{y-u}{\varepsilon}+\theta\,\frac{\hat{y}-y}{\varepsilon}\Big)\Big|d\theta\Big)\mathrm{d}u\\
&=\frac{1}{\varepsilon}\ |y-\hat{y}|\int_0^1\Big(\int_{\mathbb{R}^m}[\ell_t\ |\beta_\varepsilon(z)|+|F(t,y+\theta(\hat{y}-y)-\varepsilon v,0)|]\\
&\qquad\qquad\qquad\cdot\mathbf{1}_{[0,1]}(\varepsilon|F(t,y+\theta(\hat{y}-y)-\varepsilon v,0)|)\cdot|\nabla\rho(v)|\,dv\Big)d\theta\\
&\le\frac{1}{\varepsilon}\ |y-\hat{y}|\Big[\ell_t\ |\beta_\varepsilon(z)|+\frac{1}{\varepsilon}\Big]\int_0^1\Big(\int_{\overline{B(0,1)}}|\nabla\rho(v)|\,dv\Big)d\theta\le\frac{\kappa}{\varepsilon}\Big[\ell_t\ |\beta_\varepsilon(z)|+\frac{1}{\varepsilon}\Big]\ |y-\hat{y}|
\end{aligned}
$$

since $0\le(a+b)\ \mathbf{1}_{[0,1]}(\varepsilon b)\le a+\frac{1}{\varepsilon}$, for all $a,b\ge0$.

We also have, for all $y,\hat{y}\in\mathbb{R}^m$ with $|\hat{y}|\le\rho$ and for all $z\in\mathbb{R}^{m\times k}$,

$$
\begin{aligned}
&\langle y-\hat{y},F_\varepsilon(t,y,z)\rangle\le\mu_t\ |y-\hat{y}|^2+|y-\hat{y}|\ \Big[F^{\#}_{\rho+1}(t)+\ell_t\ |z|\Big]\\
&\le|y-\hat{y}|\ F^{\#}_{\rho+1}(t)+\Big(\mu_t+\tfrac{1}{2n_p\lambda}\,\ell_t^2\,\mathbf{1}_{z\neq0}\Big)^+|y-\hat{y}|^2+\tfrac{n_p\lambda}{2}\ |z|^2\,,\quad\text{for all }\lambda>0,
\end{aligned}
\tag{6.51}
$$

where $p>1$, $n_p=(p-1)\wedge1$.

Indeed, by taking

$$
\alpha_\varepsilon(t,y)=\int_{\overline{B(0,1)}}\mathbf{1}_{[0,1]}(\varepsilon|F(t,y-\varepsilon u,0)|)\ \rho(u)\,\mathrm{d}u,
$$

we have $0\le\alpha_\varepsilon(t,y)\le1$ and

$$
\begin{aligned}
&\langle y-\hat{y},F_\varepsilon(t,y,z)\rangle\\
&=\int_{\overline{B(0,1)}}\langle y-\hat{y},F(t,y-\varepsilon u,\beta_\varepsilon(z))-F(t,\hat{y}-\varepsilon u,\beta_\varepsilon(z))\rangle\ \mathbf{1}_{[0,1]}(\varepsilon|F(t,y-\varepsilon u,0)|)\,\rho(u)\,\mathrm{d}u\\
&\quad+\int_{\overline{B(0,1)}}\langle y-\hat{y},F(t,\hat{y}-\varepsilon u,\beta_\varepsilon(z))-F(t,\hat{y}-\varepsilon u,0)\rangle\ \mathbf{1}_{[0,1]}(\varepsilon|F(t,y-\varepsilon u,0)|)\,\rho(u)\,\mathrm{d}u\\
&\quad+\int_{\overline{B(0,1)}}\langle y-\hat{y},F(t,\hat{y}-\varepsilon u,0)\rangle\ \mathbf{1}_{[0,1]}(\varepsilon|F(t,y-\varepsilon u,0)|)\,\rho(u)\,\mathrm{d}u\\
&\le\Big[\mu_t\,|y-\hat{y}|^2+|y-\hat{y}|\,\ell_t\,|\beta_\varepsilon(z)|\Big]\,\alpha_\varepsilon(t,y)+|y-\hat{y}|\,F^{\#}_{\rho+1}(t)\,.
\end{aligned}
$$

Moreover, for all $y,\hat{y}\in\mathbb{R}^m,\lambda\in\mathbb{R}_+^*$ such that $|y|\le\rho,|\hat{y}|\le\rho$:

$$\begin{array}{ll}
(a) & \langle y-\hat{y},F_\varepsilon(t,y,z)-F_\varepsilon(t,\hat{y},z)\rangle \\
& \le \mu_t^+|y-\hat{y}|^2+|y-\hat{y}|\left[F_{\rho+1}^{\#}(t)+\ell_t|z|\right]\mathbf{1}_{[\frac{1}{\varepsilon},\infty)}(F_{\rho+1}^{\#}(t)) \\
(b) & \langle y-\hat{y},F_\varepsilon(t,y,z)-F_\varepsilon(t,\hat{y},\hat{z})\rangle \\
& \le |y-\hat{y}|\left[F_{\rho+1}^{\#}(t)+\ell_t|\hat{z}|\right]\mathbf{1}_{[\frac{1}{\varepsilon},\infty)}(F_{\rho+1}^{\#}(t)) \\
& \quad +\Big(\mu_t^++\dfrac{1}{2n_p\lambda}\ell_t^2\mathbf{1}_{z\ne\hat{z}}\Big)|y-\hat{y}|^2+\dfrac{n_p\lambda}{2}|z-\hat{z}|^2 \\
(c) & \langle y-\hat{y},F_\varepsilon(t,y,z)-F_\delta(t,\hat{y},\hat{z})\rangle \\
& \le |\varepsilon-\delta|\left[\mu_t^+|\varepsilon-\delta|+2F_{\rho+1}^{\#}(t)+2\ell_t|z|\right] \\
& \quad +|y-\hat{y}|\Big[2\mu_t^+|\varepsilon-\delta|+\ell_t|\hat{z}|\mathbf{1}_{[\frac{1}{\varepsilon}\wedge\frac{1}{\delta},\infty)}(|\hat{z}|)\mathbf{1}_{\varepsilon\ne\delta} \\
& \qquad\qquad +(F_{\rho+1}^{\#}(t)+\ell_t|\hat{z}|)\mathbf{1}_{[\frac{1}{\varepsilon}\wedge\frac{1}{\delta},\infty)}(F_{\rho+1}^{\#}(t))\Big] \\
& \quad +\Big(\mu_t^++\dfrac{1}{2n_p\lambda}\ell_t^2\mathbf{1}_{z\ne\hat{z}}\Big)|y-\hat{y}|^2+\dfrac{n_p\lambda}{2}|z-\hat{z}|^2.
\end{array}\tag{6.52}$$

Obviously, it is sufficient to prove (6.52$-c$).

We have

$$\begin{aligned}
&\langle y-\hat{y},F_\varepsilon(t,y,z)-F_\delta(t,\hat{y},\hat{z})\rangle \\
&\le \int_{\overline{B(0,1)}}\langle y-\varepsilon u-(\hat{y}-\delta u)+(\varepsilon-\delta)u,F(t,y-\varepsilon u,\beta_\varepsilon(z))-F(t,\hat{y}-\delta u,\beta_\varepsilon(z))\rangle \\
&\qquad\qquad\qquad\qquad\cdot\mathbf{1}_{[0,1]}(\varepsilon|F(t,y-\varepsilon u,0)|)\rho(u)\,\mathrm{d}u \\
&\quad+\int_{\overline{B(0,1)}}\langle y-\hat{y},F(t,\hat{y}-\delta u,\beta_\varepsilon(z))-F(t,\hat{y}-\delta u,\beta_\delta(\hat{z}))\rangle\mathbf{1}_{[0,1]}(\varepsilon|F(t,y-\varepsilon u,0)|)\rho(u)\,\mathrm{d}u \\
&\quad+\int_{\overline{B(0,1)}}\langle y-\hat{y},F(t,\hat{y}-\delta u,\beta_\delta(\hat{z}))\rangle \\
&\qquad\qquad\cdot\left[\mathbf{1}_{[0,1]}(\varepsilon|F(t,y-\varepsilon u,0)|)-\mathbf{1}_{[0,1]}(\delta|F(t,\hat{y}-\delta u,0)|)\right]\rho(u)\,\mathrm{d}u \\
&\le \mu_t\int_{\overline{B(0,1)}}|y-\varepsilon u-(\hat{y}-\delta u)|^2\mathbf{1}_{[0,1]}(\varepsilon|F(t,y-\varepsilon u,0)|)\rho(u)\,\mathrm{d}u \\
&\quad+2|\varepsilon-\delta|\left[F_{\rho+1}^{\#}(t)+\ell_t|\beta_\varepsilon(z)|\right]+|y-\hat{y}|\,\ell_t|\beta_\varepsilon(z)-\beta_\delta(\hat{z})|\,\alpha_\varepsilon(t,y) \\
&\quad+|y-\hat{y}|\left[F_{\rho+1}^{\#}(t)+\ell_t|\beta_\delta(\hat{z})|\right]\mathbf{1}_{[\frac{1}{\varepsilon}\wedge\frac{1}{\delta},\infty)}(F_{\rho+1}^{\#}(t))
\end{aligned}$$

But

$$\begin{aligned}
\mu_t|y-\varepsilon u-(\hat{y}-\delta u)|^2&\le\mu_t^+|y-\varepsilon u-(\hat{y}-\delta u)|^2 \\
&\le\mu_t^+|y-\hat{y}|^2+2\mu_t^+|y-\hat{y}||\varepsilon-\delta|+\mu_t^+|\varepsilon-\delta|^2
\end{aligned}$$

and, using the properties of β_ε, inequality (6.52$-c$) follows.

Remark 6.13. The function G will be approximate in the same manner. For $0 < \varepsilon \leq 1$:

$$G_\varepsilon(t,y) = \int_{\overline{B(0,1)}} G(t, y - \varepsilon u)\ \mathbf{1}_{[0,1]}(\varepsilon |G(t,y-\varepsilon u)|)\ \rho(u)\,\mathrm{d}u. \tag{6.53}$$

Inequalities (6.50)–(6.52) are similarly obtained for G, with $z = \hat{z} = 0$ and $\ell = 0$.

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