

LINEAR HYPERBOLIC SYSTEMS ON NETWORKS: WELL-POSEDNESS AND QUALITATIVE PROPERTIES*

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Abstract. We study hyperbolic systems of one-dimensional partial differential equations under general, possibly non-local boundary conditions. A large class of evolution equations, either on individual 1-dimensional intervals or on general networks, can be reformulated in our rather flexible formalism, which generalizes the classical technique of first-order reduction. We study forward and backward well-posedness; furthermore, we provide necessary and sufficient conditions on both the boundary conditions and the coefficients arising in the first-order reduction for a given subset of the relevant ambient space to be invariant under the flow that governs the system. Several examples are studied.

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1. INTRODUCTION

This paper is devoted to the study of systems of partial differential equations in 1-dimensional setting, more precisely, on collections of intervals: not only internal couplings are allowed, but also interactions at the endpoints of the intervals. It is then natural to interpret these systems as networks, and in fact, we will dwell on this viewpoint throughout the paper.

Partially motivated by investigations in quantum chemistry since the 1950s, differential operators of second order on networks have been often considered in the mathematical literature since the pioneering investigations by Lumer [50] and Faddeev and Pavlov [59]: in these early examples, either heat or Schrödinger equations were of interest. This has paved the way to a manifold of investigations, see *e.g.* the historical overview in [54].

The equations we are going to study in this paper will, however, be rather hyperbolic; more precisely, the *hyperbolic systems* of partial differential equations of our interest are of the form

$$\dot{u} = Mu' + Nu,$$

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where, here and below, we denote everywhere by \dot{u} and u' the partial derivative of a function u with respect to the time variable t and to the space variable x , respectively.

Each of these equations models a physical system: we consider several of these systems and allow them to interact at their boundaries, thus producing a collection of hyperbolic systems on a network. Hyperbolic evolution equations of different kinds taking place on the edges of a network have been frequently considered in the literature, we refer to [20, 22, 45, 54, 55] for an overview. Let us emphasize that we shall only consider *linear* systems: for a survey on some recent developments of the theory for nonlinear hyperbolic systems and many practical applications see *e.g.* [13].

On each edge of the network we allow for possibly different dynamics (say, Dirac-like, wave-like, beam-like, *etc.*), thus it would be more precise to write

$$\dot{u}_e = M_e u'_e + N_e u_e, \quad e \in E, \quad (1.1)$$

where E is the edge set of the considered network. In particular, in the easiest cases M_e may be a diagonal matrix of coefficients of a transport-like equations, but M_e may well have off-diagonal entries, or even have a symplectic structure: additionally, we allow all these M_e 's to have different size, which of course has to be taken into account by the boundary conditions.

We are *not* going to assume the matrices M_e to be either positive or negative semidefinite – in fact, not even Hermitian; therefore, it is at a first glance not clear at which endpoints the boundary conditions should be imposed at all. Indeed, the choice of appropriate transmission conditions in the vertices of the network is the biggest difficulty one has to overcome.

While Ali Mehmeti began the study of wave equations on networks already in [1], it was to the best of our knowledge only at the end of the 1990s that first order differential operators on networks began to be studied. In [16], Carlson defined on a network the *momentum operator* – *i.e.*, the operator defined edgewise as $v \frac{d}{dx}$ – and gave a sufficient condition – in terms of the boundary conditions satisfied by functions in its domain – for self-adjointness, hence for generation of a unitary group governing a system of equations

$$\dot{u} = \pm u' \quad (1.2)$$

with couplings in the boundary (*i.e.*, in the nodes of the networks). Similar ideas were revived in [23, 30], where different sufficient conditions of combinatorial or algebraic nature were proposed. A characteristic equation and the long-time behavior of the semigroup governing (1.1) as well as further spectral and extension theoretical properties were discussed in [36, 37], respectively, in dependence of the boundary conditions. While all the above mentioned authors – as well as the present manuscript – apply Hilbert space techniques, a semigroup approach to study simple transport equations in Banach spaces (like the space of L^1 -functions along the edges of a network) was presented in [22, 42, 52], see also ([7], Sect. 18) and the references given there.

All these above mentioned papers treat essentially the same parametrization of boundary conditions, namely

$$\underline{u(0)} = T \underline{u(\ell)}$$

for a suitable matrix T (possibly consisting of diagonal blocks that correspond to the network's vertices), where $\underline{u(0)}$ and $\underline{u(\ell)}$ denote the vectors of boundary values of u at the initial and terminal endpoints of all intervals, respectively.

Bolte and Harrison studied in [9] the Dirac equation on networks. The 1D Dirac equation consists of a system of two coupled first order (both in time and space) equations, much like (1.1); the matrix M_e is Hermitian, which allows for simple integration by parts and, in turn, for the emergence of a convenient symplectic structure. Both internal and boundary couplings had to be considered, and the relevant coupling matrix is indefinite. They thus

adopted the parametrization

$$\underline{Au(0)} + \underline{Bu(\ell)} = 0$$

for the boundary conditions, for suitable matrices A, B : mimicking ideas from [39], they were able to characterize those A, B that lead to self-adjoint extensions. Self-adjointness of more general first-order differential operator matrices has been studied in [65].

In this paper, we opt for yet another parametrization of the boundary conditions, inspired by a classical Sturm–Liouville formalism borrowed by Kuchment to discuss self-adjoint extensions of Laplacians on networks in [43] (see also [56] for the “telegrapher’s equation” on networks with similar boundary conditions). More precisely, we impose boundary conditions of the form

$$\begin{pmatrix} \underline{u(0)} \\ \underline{u(\ell)} \end{pmatrix} \in Y$$

for a subspace Y of the space of boundary values; and find sufficient conditions on Y that, in dependence on M and an auxiliary matrix Q , guarantee that the abstract Cauchy problem associated with (1.1) is governed by a (possibly unitary, under stronger assumptions) group, or a (possibly contractive, under stronger assumptions) semigroup.

The auxiliary matrix Q – often called a *Friedrichs symmetrizer* in the literature, see Definition 2.1 of [8] – will play a fundamental role in our approach. Roughly speaking, its role is *not* to diagonalize M , but only to make it Hermitian; this is done by suitably modifying the inner product of the L^2 -space over the network by means of Q , which therefore has in turn to be positive definite; especially for this reason, our whole theory is essentially relying upon the Hilbert space structure. Our approach allows us in particular to prove generation of unitary C_0 -groups and contractive C_0 -semigroups (and, by perturbation, of general C_0 -(semi)groups). This has a long tradition that goes back to Lax and Phillips [47], who already propose the idea of transforming boundary conditions into the requirement that at each boundary point v the boundary values belong to a given subspace Y_v . Indeed, while our well-posedness results are not surprising *once the correct boundary conditions are found*, the actually tricky task – as long as M is not diagonalizable, the standard assumption among others in [6, 8, 24, 35, 64] – is to actually find the right dimension of the space Y_v . In this paper, we pursue this task by a fair amount of linear algebra that eventually allows us to parametrize the boundary conditions leading to *contractive* (semi)groups. This should be compared with the more involved situation in higher dimension, see *e.g.* [63], which allows for less explicit representation of the boundary conditions. Our setting is thus arguably more general than the approaches to hyperbolic systems on networks that have recently emerged, including port-Hamiltonian systems [34, 67, 69] and hyperbolic systems that can be transformed into characteristic forms via Riemann coordinates [6], both based on diagonalization arguments.

A relevant by-product of our approach is the possibility to characterize in terms of Q, M, N positivity and further qualitative properties of the solutions of the initial value problem associated with (1.1). In this context, we regard as particularly relevant Proposition 4.11 and Lemma 4.13, which roughly speaking state that the semigroup governing (1.1) can only be positive if M is diagonal, up to technical assumptions (including that Q is diagonal too; this is not quite restrictive, as *e.g.* all of the examples we will discuss in Sect. 5 will satisfy it); this negative result essentially prevents most evolution equations of non-transport type arising in applications from being governed by a positive semigroup.

The obtained results also form a basis for studies of different stability and control problems related to the presented hyperbolic systems. In Remark 4.4 we give some immediate indications for these studies but leave further problems for possible future considerations.

Let us sketch the structure of our paper. In Section 2 we present our general assumptions and discuss their role by showing that a broad class of examples fits into our scheme. In Sections 3 and 4 we then show that our description of boundary condition allows for easy description of realizations that generate (semi)groups. We

also find necessary and sufficient conditions for qualitative properties of these semigroups, including reality and positivity.

We conclude this paper by reviewing in Section 5 several applications of our method; among other we discuss forward and backward well-posedness of different equations modeling wave phenomena on networks, including 1D Saint–Venant, Maxwell, and Dirac equations. We study different regimes for the Saint-Venant equation and discuss transmission conditions in the vertices that imply forward, but not backward well-posedness of the Dirac equation. We also study in detail an interesting model of mathematical physics for heat propagation in supercold molecules; we extend the results from [62] by providing physically meaningful classes of transmission conditions implying well-posedness and proving nonpositivity of the semigroup governing this system.

Some technical results, which seem to be folklore, are recalled in the appendices.

2. GENERAL SETTING AND MAIN EXAMPLES

Let E be a nonempty finite set, which we will identify with the edges of a network upon associating a length ℓ_e with each $e \in E$. To fix the ideas, take $e \in E$ and $\ell_e > 0$. (We restrict for simplicity to the case of a network consisting of edges of finite length only, although our results can be easily extended to the case of networks consisting of finitely many leads – semi-bounded intervals – attached to a “core” of finitely many edges of finite length.) We will consider evolution equations of the form

$$\dot{u}_e(t, x) = M_e(x)u'_e(t, x) + N_e(x)u_e(t, x), \quad t \geq 0, \quad x \in (0, \ell_e), \quad (2.1)$$

where u_e is a vector-valued function of size $k_e \in \mathbb{N}_1 := \{1, 2, \dots\}$, and M_e and N_e are matrix-valued functions of size $k_e \times k_e$. We will couple equations (2.1) for different $e \in E$ *via* boundary conditions given later on.

If $M_e(x)$ is Hermitian for all x , then integrating by parts we obtain for all $u \in \bigoplus_{e \in E} H^1(0, \ell_e)^{k_e}$ ¹

$$2\Re \sum_{e \in E} \int_0^{\ell_e} M_e u'_e \cdot \bar{u}_e \, dx = - \sum_{e \in E} \int_0^{\ell_e} u_e \cdot M'_e \bar{u}_e \, dx + \sum_{e \in E} [M_e u_e \cdot \bar{u}_e] \Big|_0^{\ell_e}, \quad (2.2)$$

which – provided M'_e is essentially bounded – allows for an elementary dissipativity analysis of the operator that governs the abstract Cauchy problem associated with (2.1) in a natural Hilbert space. Also the case of diagonalizable matrices M_e is benign enough, see *e.g.* Section 7.3 [29]. In the case of general M_e , however, it is not easy to control all terms that arise when integrating against test functions and we have to resort to different ideas.

Assumption 2.1. For each $e \in E$ the following holds.

- (1) The matrix $M_e(x)$ is invertible for each $x \in [0, \ell_e]$ and the mapping $[0, \ell_e] \ni x \mapsto M_e(x) \in M_{k_e}(\mathbb{C})$ is Lipschitz continuous, in other words, $M_e \in W^{1, \infty}(0, \ell_e)$.
- (2) The mapping $[0, \ell_e] \ni x \mapsto N_e(x) \in M_{k_e}(\mathbb{C})$ is of class L^∞ .
- (3) There exists a Lipschitz continuous function $[0, \ell_e] \ni x \mapsto Q_e(x) \in M_{k_e}(\mathbb{C})$ such that
 - (i) $Q_e(x)$ and $Q_e(x)M_e(x)$ are Hermitian for all $x \in [0, \ell_e]$,
 - (ii) $Q_e(\cdot)$ is *uniformly positive definite*, *i.e.*, there exists $q > 0$ such that

$$Q_e(x)\xi \cdot \bar{\xi} \geq q\|\xi\|^2 \text{ for all } \xi \in \mathbb{C}^{k_e} \text{ and } x \in [0, \ell_e].^2$$

If $M_e(x)$ is Hermitian for all x , then Assumptions 2.1.(3i) are trivially satisfied by taking Q_e to be the $k_e \times k_e$ identity matrix, although this is not the only possible choice and, in fact, it is sometimes actually possible and

¹Throughout this paper $\bigoplus_{e \in E} H_e$ denotes the direct sum of the Hilbert spaces H_e , $e \in E$, *i.e.*, $\bigoplus_{e \in E} H_e := \{(h_e)_{e \in E} : h_e \in H_e\}$.

²Throughout this paper, \mathbb{C}^{k_e} is equipped with its Euclidean norm $\|\cdot\|$ and $M_{k_e}(\mathbb{C})$ with the induced operator norm, also denoted by $\|\cdot\|$, since no confusion is possible.

convenient to take non-diagonal Q_e . Assumptions 2.1.(3) holds if and only if the system (2.1) is hyperbolic in the sense of [6], see Lemma A.1 below. But we prefer this formulation because the matrices Q_e will be involved in the boundary conditions.

The fact that $Q_e(x)M_e(x)$ is Hermitian for all x greatly simplifies our analysis. At the same time, many examples from physics, chemistry, biology, *etc.*, fit in this framework.

The most trivial examples are obtained by taking M_e as a diagonal matrix with spatially constant entries: this choice leads to classical (vector-valued) transport problems on networks. For $k_e \equiv 1$ they were considered in [42] and subsequent papers, *cf.* the literature quoted in Section 18 of [7].

Example 2.2. The 2×2 hyperbolic system

$$\begin{cases} \dot{p} + Lq' + Gp + Hq = 0 & \text{in } (0, \ell) \times (0, +\infty), \\ \dot{q} + Pp' + Kq + Jp = 0 & \text{in } (0, \ell) \times (0, +\infty), \end{cases} \quad (2.3)$$

on a real interval $(0, \ell)$ generalizes the first order reduction of the wave equation and offers a general framework to treat models that appear in several applications. The analysis of this system on networks with different boundary conditions has been performed in [56].

In electrical engineering [33, 51], p (resp. q) represents the voltage V (resp. the electrical current I) at $(\ell - x, t)$, $H = J = 0$, $L = \frac{1}{C}$, $P = \frac{1}{L}$, $G = \frac{\hat{G}}{C}$, $K = \frac{R}{L}$, where $C > 0$ is the capacitance, $L > 0$ the inductance, $\hat{G} \geq 0$ the conductance, and $R \geq 0$ the resistance: (2.3) is then referred to as “telegrapher’s equation”.

This system also models arterial blood flow [12, 18] for which p is the pressure and q the flow rate at (x, t) , $L = \frac{1}{C}$, $P = A$, $K = -\frac{2\alpha}{\alpha-1} \frac{\nu}{A}$, $G = 0$, where $A > 0$ is the vessel cross-sectional area, $C > 0$ is the vessel compliance, $\nu \geq 0$ is the kinematic viscosity coefficient ($\nu \approx 3.2 \times 10^{-6}$ m²/s for blood) and $\alpha > 1$ is the Coriolis coefficient or correction coefficient ($\alpha = 4/3$ for Newtonian fluids, while $\alpha = 1.1$ for non-Newtonian fluids, like blood).

Given $L, P \in \mathbb{C}$, the Assumptions 2.1. (3) hold for system (2.3) with

$$M_e = -\begin{pmatrix} 0 & L \\ P & 0 \end{pmatrix}, \quad N_e = -\begin{pmatrix} G & H \\ K & J \end{pmatrix}, \quad \text{and} \quad Q_e = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2.4)$$

if and only if ³

$$a, d \in \mathbb{R}, \quad b = \bar{c}, \quad a > 0, \quad ad > |b|^2, \quad aL = d\bar{P}, \quad bP, b\bar{L} \in \mathbb{R}, \quad LP \neq 0. \quad (2.5)$$

Example 2.3. The momentum operator $\nu \frac{d}{dx}$ [17, 30] does *not* satisfy Assumptions 2.1. More generally, if M_e is skew-Hermitian, then Assumptions 2.1 imply that there exists a Hermitian, positive definite matrix Q_e that anti-commutes with M_e . But then $\text{Tr } Q_e$, the trace of the matrix Q_e , satisfies

$$\text{Tr } Q_e = \text{Tr}(M_e^{-1}Q_eM_e) = -\text{Tr } Q_e,$$

hence $\text{Tr } Q_e = 0$ which is in contradiction with positive definiteness of Q_e .

Example 2.4. The linearized Saint-Venant equation gives rise to a case where $N_e \neq 0$, see equation (1.27) of [6]. Indeed, it corresponds to the 2×2 system

$$\begin{cases} \dot{h} &= -Vh' - Hu' - V'h - H'u & \text{in } (0, \ell) \times (0, +\infty), \\ \dot{u} &= -Vu' - gh' + C_f \frac{V^2}{H^2} h - (V' + 2C_f \frac{V}{H})u & \text{in } (0, \ell) \times (0, +\infty), \end{cases} \quad (2.6)$$

³ For a complex number z , \bar{z} denotes its complex conjugate.

where h is the water depth and u the water velocity, and corresponds to the linearization around a steady state (H, V) of the Saint-Venant model, that in particular satisfies

$$H \neq 0, (HV)' = 0 \text{ and } gH - V^2 \neq 0. \quad (2.7)$$

Here, g is the constant of gravity and C_f is a (positive) constant friction coefficient.

Note that Assumptions 2.1 holds for system (2.6) with

$$M_e := \begin{pmatrix} -V & -H \\ -g & -V \end{pmatrix} \quad \text{and} \quad Q_e := \begin{pmatrix} g & 0 \\ 0 & H \end{pmatrix} \quad (2.8)$$

whenever H and V are of class H^1 and $H > 0$, which holds as soon as we consider a non trivial and smooth enough steady state (H, V) , see (2.7).

Finally N_e is clearly given by

$$N_e := \begin{pmatrix} -V' & -H' \\ C_f \frac{V^2}{H^2} & -(V' + 2C_f \frac{V}{H}) \end{pmatrix}. \quad (2.9)$$

3. PARAMETRIZATION OF THE REALIZATIONS: THE ISOMETRIC CASE

The catchiest application of our general theory arises whenever we discuss hyperbolic equations (or even systems thereof) on *networks* (also known as *metric graphs* in the literature); in this case, it is natural to interpret \mathbf{E} as a set of intervals, each with length ℓ_e ; and boundary conditions in the endpoints $0, \ell_e$ turn into transmission conditions in the ramification nodes. Indeed, for each edge e , k_e boundary conditions are required. In general, they are expressed in Riemann (characteristic) coordinates, see for instance [6]. This means that system (2.1) is transformed into an equivalent system with a diagonal matrix \tilde{M}_e with k_e^+ (resp. k_e^-) positive (resp. negative) eigenvalues with $k_e^+ + k_e^- = k_e$ and k_e^+ (resp. k_e^-) boundary conditions are imposed at 0 (resp. ℓ_e), which allows to fix the incoming information. Here, we prefer to write them in the original unknowns. Furthermore it is *a priori* not clear how these conditions should be adapted to the case of a network, so we will conversely try to parametrize all those transmission conditions in the network's vertices that lead to an evolution governed by a semigroup (of isometries).

In particular, we are going to look for dissipativity, hence m -dissipativity of the operator $\pm \mathcal{A}$ whose restriction to the edge e is given by

$$(\mathcal{A}u)_e := M_e u'_e + N_e u_e \quad (3.1)$$

in a natural Hilbert space, see (2.1).

To begin with, let us impose the following assumption.

Assumption 3.1. Let \mathcal{G} be a finite network (or metric graph) with underlying (discrete) graph \mathbf{G} , *i.e.*, $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is a finite, directed graph with node set \mathbf{V} and edge set \mathbf{E} and each $e \in \mathbf{E}$ is identified with an interval $(0, \ell_e)$ whereby the parametrization of the interval agrees with the orientation of the edge.

We are going to study the problem (2.1) in the vector space

$$\mathbf{L}^2(\mathcal{G}) := \bigoplus_{e \in \mathbf{E}} L^2(0, \ell_e)^{k_e}.$$

Clearly, $\mathbf{L}^2(\mathcal{G})$ becomes a Hilbert space once equipped with the inner product

$$(u, v) := \sum_{e \in E} \int_0^{\ell_e} Q_e(x) u_e(x) \cdot \bar{v}_e(x) dx, \quad u, v \in \mathbf{L}^2(\mathcal{G}), \quad (3.2)$$

(where, here and below, $z \cdot \bar{z}_1$ means the Euclidean inner product in \mathbb{C}^{k_e} between z and z_1), which is equivalent to the canonical one. The associated norm will be denoted by $\|\cdot\|$, because no confusion is possible with the Euclidean and matrix norms introduced before.

Of course, if M_e is diagonal, then Assumptions 2.1.(3i) is satisfied *e.g.* whenever $Q_e(x)$ is the identity for all x ; however, Example 2.2 shows that $Q_e M_e$ may be Hermitian even when M_e is not. It thus turns out that such an alternative inner product is tailor-made for the class of hyperbolic systems we are considering. The main reason for restricting to the Hilbert space setting is that checking dissipativity in L^p -spaces is less immediate.

3.1. Transmission conditions in the vertices

In order to tackle the problem of determining the correct transmission conditions on \mathcal{A} , let us first introduce the maximal domain

$$D_{\max} := \bigoplus_{e \in E} \{u \in L^2(0, \ell_e)^{k_e} : M_e u' \in L^2(0, \ell_e)^{k_e}\}. \quad (3.3)$$

We want to explicitly state the following, whose easy proof we leave to the reader. Recall that invertibility of $M_e(x)$ is assumed for all $x \in [0, \ell_e]$.

Lemma 3.2. *It holds*

$$D_{\max} = \bigoplus_{e \in E} H^1(0, \ell_e)^{k_e}, \quad (3.4)$$

and therefore D_{\max} is densely and compactly embedded in $\mathbf{L}^2(\mathcal{G})$.

We stress that compactness of the embedding can actually fail if 0 is an eigenvalue of $M_e(\cdot)$ at the endpoints of $(0, \ell_e)$: to see this, take over – with obvious changes – the proof of Lemma 4.2 of [44].

Let us see why we have chosen to define an alternative inner product on $\mathbf{L}^2(\mathcal{G})$. Under our standing assumption $Q_e(x)M_e(x)$ is for all x a Hermitian matrix, so it can be diagonalized – although these matrices need not commute, so they will in general not be simultaneously diagonalizable. If however there exists a diagonal matrix D_e such that

$$D_e = Q_e(x)M_e(x) \quad \text{for all } x, \quad (3.5)$$

then the semigroup generated by $M_e \frac{d}{dx}$ on $\mathbf{L}^2(\mathcal{G})$ with respect to the inner product in (3.2) agrees with the semigroup generated by $Q_e(\cdot)M_e(\cdot) \frac{d}{dx} = D_e \frac{d}{dx}$ on $\mathbf{L}^2(\mathcal{G})$ with respect to the canonical inner product: the latter one is simply the shift semigroup, up to taking into account the boundary conditions, *cf.* Proposition 3.3 of [21] or Proposition 18.7 of [7] for a special case where $D_e = I$. So, the complete operator \mathcal{A} will generate a semigroup $(e^{t\mathcal{A}})_{t \geq 0}$ that can be semi-explicitly written down by means of the Dyson–Phillips Series, since the perturbation $Q_e N_e$ is bounded, see Proposition 3.1.2, page 77 of [60] or Theorem 1.10 [28]; and also by means of Trotter’s Product Formula, see Exercise III.5.11 of [28].

Even when (3.5) does not hold we are still in a commendable situation: indeed,

$$(\mathcal{A}u, v) = \sum_{e \in E} \int_0^{\ell_e} Q_e (M_e u'_e + N_e u_e) \cdot \bar{v}_e dx \quad \text{for all } u, v \in D_{\max}. \quad (3.6)$$

Due to our standing assumptions, $Q_e M_e$ and hence its space derivative are hermitian: integrating by parts we hence find

$$\begin{aligned} \sum_{e \in E} \int_0^{\ell_e} Q_e M_e u'_e \cdot \bar{v}_e \, dx &= - \sum_{e \in E} \int_0^{\ell_e} u_e \cdot (\overline{Q_e M_e v_e})' \, dx + \sum_{e \in E} [Q_e M_e u_e \cdot \bar{v}_e] \Big|_0^{\ell_e} \\ &= - \sum_{e \in E} \int_0^{\ell_e} u_e \cdot \overline{Q_e M_e v'_e} \, dx - \sum_{e \in E} \int_0^{\ell_e} u_e \cdot \overline{(Q_e M_e)' v_e} \, dx \\ &\quad + \sum_{e \in E} [Q_e M_e u_e \cdot \bar{v}_e] \Big|_0^{\ell_e} \end{aligned} \quad (3.7)$$

Now, for $u = v$ (3.7) can be equivalently written as

$$2\Re \sum_{e \in E} \int_0^{\ell_e} Q_e M_e u'_e \cdot \bar{u}_e \, dx = - \sum_{e \in E} \int_0^{\ell_e} (Q_e M_e)' u_e \cdot \bar{u}_e \, dx + \sum_{e \in E} [Q_e M_e u_e \cdot \bar{u}_e] \Big|_0^{\ell_e}; \quad (3.8)$$

both addends on the right hand side are real, in view of our standing assumptions on Q_e, M_e .

For all $v \in V$, let us denote by E_v , the set of all edges incident in v . We introduce for each $v \in V$ the trace operator $\gamma_v : \bigoplus_{e \in E} H^1(0, \ell_e)^{k_e} \rightarrow \mathbb{C}^{k_v}$ defined by

$$\gamma_v(u) = (u_e(v))_{e \in E_v}, \quad v \in V,$$

where $k_v := \sum_{e \in E_v} k_e$, and the $k_v \times k_v$ block-diagonal matrix T_v with $k_e \times k_e$ diagonal blocks

$$T_v := \text{diag} (Q_e(v) M_e(v) \iota_{ve})_{e \in E_v}, \quad v \in V, \quad (3.9)$$

where we recall that the $|V| \times |E|$ *incidence matrix* $\mathcal{I} = (\iota_{ve})$ of the graph G is defined by

$$\iota_{ve} := \begin{cases} -1 & \text{if } v \text{ is initial endpoint of } e, \\ +1 & \text{if } v \text{ is terminal endpoint of } e, \\ 0 & \text{otherwise.} \end{cases} \quad (3.10)$$

With these notation, we see that the identity (3.8) is equivalent to

$$\begin{aligned} 2\Re \sum_{e \in E} \int_0^{\ell_e} Q_e M_e u'_e \cdot \bar{u}_e \, dx &= - \sum_{e \in E} \int_0^{\ell_e} (Q_e M_e)' u_e \cdot \bar{u}_e \, dx \\ &\quad + \sum_{v \in V} T_v \gamma_v(u) \cdot \gamma_v(\bar{u}). \end{aligned} \quad (3.11)$$

Taking the real part of (3.6) and using the last identity we find

$$\begin{aligned} \Re(\mathcal{A}u, u) &= \Re \sum_{e \in E} \int_0^{\ell_e} Q_e N_e u_e \cdot \bar{u}_e \, dx \\ &\quad - \frac{1}{2} \sum_{e \in E} \int_0^{\ell_e} (Q_e M_e)' u_e \cdot \bar{u}_e \, dx + \frac{1}{2} \sum_{v \in V} T_v \gamma_v(u) \cdot \gamma_v(\bar{u}). \end{aligned} \quad (3.12)$$

The boundary terms vanish if so does $\gamma_v(u)$ for all $v \in V$; however, upon introducing the quadratic form

$$q_v(\xi) := T_v \xi \cdot \bar{\xi}, \quad \xi \in \mathbb{C}^{k_v}, \quad (3.13)$$

it is more generally sufficient to impose that $\gamma_v(u)$ belongs to a *totally isotropic subspace* Y_v associated with q_v , i.e., to a vector space Y_v such that the restriction of q_v to Y_v vanishes identically, see Definition C.1. (Observe that $q_v(\xi) \in \mathbb{R}$, due to our standing assumptions on Q_e, M_e .) This means that it suffices to assume that

$$\gamma_v(u) \in Y_v \quad \text{for all } v \in V. \quad (3.14)$$

Remark 3.3. Introducing

$$\omega(f, g) := \sum_{e \in E} \int_0^{\ell_e} Q_e M_e u'_e \cdot \bar{v}_e \, dx, \quad f, g \in D_{\min},$$

defines a skew-symmetric form in the sense of Definition 2.2 in [65], where $D_{\min} := \bigoplus_{e \in E} H_0^1(0, \ell_e)^{k_e}$. All skew-adjoint extensions of $\mathcal{A}|_{D_{\min}}$ can be then parametrized by Theorem 3.6 of [65]. We are however rather interested in the general case of possibly variable coefficients and therefore prefer to pursue an approach based on the classical Lumer–Phillips Theorem.

Example 3.4. For the system (2.3) with the choice of Q_e from (2.4) in which we take $a = P$, $b = c = 0$, $d = L$, the matrix $Q_e M_e$ is given by

$$Q_e M_e = - \begin{pmatrix} 0 & PL \\ LP & 0 \end{pmatrix}$$

and therefore, with $u_e = (p_e, q_e)^\top$, the expression $Q_e M_e u_e \cdot \bar{u}_e$ takes the form

$$Q_e M_e u_e \cdot \bar{u}_e = -2 \Re(PLq_e \bar{p}_e).$$

One may *e.g.* consider the vertex transmission conditions (see [56], p. 56)

- p is continuous across the vertices and $\sum_{e \in E_v} q_e(v) \iota_{ve} = 0$ for all $v \in V$; or
- q is continuous across the vertices and $\sum_{e \in E_v} p_e(v) \iota_{ve} = 0$ for all $v \in V$.

They both fit to our framework. Indeed, if for simplicity we write $\gamma_v(u) = ((p_e(v))_{e \in E_v}, (q_e(v))_{e \in E_v})^\top$, then in the first case it suffices to take

$$Y_v = \text{span}\{\mathbf{1}_{E_v}\} \oplus \text{span}\{\iota_{E_v}\}^\perp,$$

where $\mathbf{1}_{E_v} := (1, \dots, 1) \in \mathbb{C}^{|E_v|}$ and ι_{E_v} is the vector in $\mathbb{C}^{|E_v|}$ whose e -th entry equals ι_{ve} , the appropriate nonzero entry of the incidence matrix defined in (3.10). Observe that (3.14) now yields that the values $p_e(v)$ coincide for all $e \in E_v$ while the vectors $(q_e(v))_{e \in E_v}$ and ι_{E_v} are orthogonal. To cover the second set of transmission conditions we on the contrary let

$$Y_v = \text{span}\{\iota_{E_v}\}^\perp \oplus \text{span}\{\mathbf{1}_{E_v}\}.$$

Before proving our first well-posedness result, we reformulate the condition (3.14) for constant vector fields u_e . Namely, if we assume that $u_e \equiv K_e \in \mathbb{C}^{k_e}$ for all edges e , (3.14) is equivalent to

$$(K_e)_{e \in E_v} \in Y_v \quad \text{for all } v \in V. \quad (3.15)$$

Denoting $I_v := \{1, 2, \dots, \dim Y_v^\perp\}$ and fixing a basis $\{\mathbf{w}^{(v,i)}\}_{i \in I_v}$ of $Y_v^\perp \subset \mathbb{C}^{k_v}$, (3.15) is equivalent to

$$(K_{\mathbf{e}})_{\mathbf{e} \in E_v} \cdot \overline{\mathbf{w}^{(v,i)}} = 0 \quad \text{for all } i \in I_v, v \in V. \quad (3.16)$$

To write this in a global way, we first let $k := \sum_{\mathbf{e} \in E} k_{\mathbf{e}}$. Now recall that each $\mathbf{w}^{(v,i)}$ is an element of \mathbb{C}^{k_v} , hence it can be identified with the vector $(\mathbf{w}_{\mathbf{e}}^{(v,i)})_{\mathbf{e} \in E_v}$. We denote by $\widetilde{\mathbf{w}}^{(v,i)} \in \mathbb{C}^k$ its extension to the whole set of edges, namely,

$$\widetilde{\mathbf{w}}_{\mathbf{e}}^{(v,i)} := \begin{cases} \mathbf{w}_{\mathbf{e}}^{(v,i)}, & \text{if } \mathbf{e} \in E_v, \\ 0 & \text{else.} \end{cases} \quad (3.17)$$

With this notation we see that (3.16), hence also (3.14) in this case, is equivalent to,

$$(K_{\mathbf{e}})_{\mathbf{e} \in E} \cdot \overline{\widetilde{\mathbf{w}}^{(v,i)}} = 0 \quad \text{for all } i \in I_v, v \in V. \quad (3.18)$$

In the same way each coordinate of an element of $Y_v, Y_v^\perp \subset \mathbb{C}^{k_v}$ corresponds to some $\mathbf{e} \in E_v$ and as above we can extend these spaces to $\widetilde{\mathbb{C}^k}$ by putting a 0 to the coordinate corresponding to \mathbf{e} whenever $\mathbf{e} \notin E_v$. Denote these extensions by \widetilde{Y}_v , and \widetilde{Y}_v^\perp , respectively.

Lemma 3.5. *The set $\{\widetilde{\mathbf{w}}^{(v,i)}\}_{i \in I_v, v \in V}$ is a basis of \mathbb{C}^k if and only if*

$$\dim \sum_{v \in V} \widetilde{Y}_v^\perp = k = \sum_{v \in V} \dim Y_v. \quad (3.19)$$

Proof. By construction, $\{\widetilde{\mathbf{w}}^{(v,i)}\}_{i \in I_v}$ is a basis of \widetilde{Y}_v^\perp . Therefore, $\{\widetilde{\mathbf{w}}^{(v,i)}\}_{i \in I_v, v \in V}$ is a basis of \mathbb{C}^k if and only if

$$\sum_{v \in V} \widetilde{Y}_v^\perp = \bigoplus_{v \in V} \widetilde{Y}_v^\perp = \mathbb{C}^k$$

which is further equivalent to the dimensions condition

$$\dim \sum_{v \in V} \widetilde{Y}_v^\perp = \sum_{v \in V} \dim \widetilde{Y}_v^\perp = k. \quad (3.20)$$

Now, observe that $\dim \widetilde{Y}_v^\perp = \dim Y_v^\perp = k_v - \dim Y_v$. Moreover, by the hand-shaking lemma, $\sum_{v \in V} k_v = 2k$, hence

$$\sum_{v \in V} \dim \widetilde{Y}_v^\perp = 2k - \sum_{v \in V} \dim Y_v.$$

Plugging this into (3.20) yields (3.19). □

Remark 3.6. The equivalent assertions in Lemma 3.5 mean that the number of boundary conditions in (3.14) (that is equivalent to (3.18) in the special case) is exactly equal to k and that these boundary conditions are linearly independent. Furthermore, as the support of the vector $\widetilde{\mathbf{w}}^{(v,i)}$ corresponds to the set of the edges incident to v , the vectors $\widetilde{\mathbf{w}}^{(v,i)}$ and $\widetilde{\mathbf{w}}^{(v',i')}$, and hence also the subspaces \widetilde{Y}_v^\perp and $\widetilde{Y}_{v'}^\perp$, are linearly independent if v and

v' are not adjacent. However, the first equality in (3.19) is equivalent to the mutual linear independence of all \widetilde{Y}_v^\perp , that is,

$$\widetilde{Y}_v^\perp \cap \sum_{v' \neq v} \widetilde{Y}_{v'}^\perp = \{0\} \quad \text{for all } v \in \mathbf{V}. \quad (3.21)$$

We are finally in the position to formulate a well-posedness result in terms of the transmission conditions in (3.14).

Theorem 3.7. *For all $v \in \mathbf{V}$, let Y_v be a totally isotropic subspace associated with the quadratic form q_v defined by (3.13) and assume that (3.19) holds. Then both $\pm \mathcal{A}$, defined as (3.1) on the domain*

$$D(\mathcal{A}) := \{u \in D_{\max} : \gamma_v(u) \in Y_v \text{ for all } v \in \mathbf{V}\}, \quad (3.22)$$

are quasi- m -dissipative operators. In particular, both $\pm \mathcal{A}$ generate a strongly continuous semigroup and hence a strongly continuous group in $\mathbf{L}^2(\mathcal{G})$. The operator \mathcal{A} has compact resolvent, hence pure point spectrum.

Proof. Under Assumptions 2.1.(2), $u \mapsto Nu$ is a bounded perturbation of \mathcal{A} . By the Bounded Perturbation Theorem (cf. [28], Thm. III.1.3), we may without loss of generality in this proof assume that $N = 0$. Under this assumption and in view of (3.12) and the definition of $D(\mathcal{A})$, we see that

$$\Re(\pm \mathcal{A}u, u) = \mp \frac{1}{2} \sum_{e \in \mathbf{E}} \int_0^{\ell_e} (Q_e M_e)' u_e \cdot \bar{u}_e \, dx \quad (3.23)$$

for all $u \in D(\mathcal{A})$. By the assumptions made on the matrices Q_e, M_e , we deduce that there exists a positive constant C such that

$$|\Re(\mathcal{A}u, u)| \leq \frac{1}{2} \left(\max_{e \in \mathbf{E}} \sup_{x \in (0, \ell_e)} \|(Q_e M_e)'(x)\|_2 \right) \sum_{e \in \mathbf{E}} \int_0^{\ell_e} \|u_e(x)\|_2^2 \, dx \leq C \|u\|^2, \quad (3.24)$$

for all $u \in D(\mathcal{A})$ which means that $\pm \mathcal{A}$ with domain $D(\mathcal{A})$ are both quasi-dissipative. By the Lumer–Phillips Theorem, it remains to check their maximality. To this aim, for any $f \in \mathbf{L}^2(\mathcal{G})$, we first look for a solution $u \in D(\mathcal{A})$ of

$$M_e(x)u'_e(x) = f_e(x) \quad \text{for } x \in (0, \ell_e) \text{ and all } e \in \mathbf{E}. \quad (3.25)$$

Such a solution is given by

$$u_e(x) = K_e + \int_0^x M_e^{-1}(y) f_e(y) \, dy, \text{ for all } x \in [0, \ell_e], \text{ for all } e \in \mathbf{E}, \quad (3.26)$$

with $K_e \in \mathbb{C}^{k_e}$. It then remains to fix the vectors K_e in order to enforce the condition $u_e \in D(\mathcal{A})$. Since (3.14) is in our situation a $k_e \times k_e$ linear system in $(K_e)_{e \in \mathbf{E}}$, the existence of this vector is equivalent to its uniqueness. By the previous considerations, this means that it suffices to show that system (3.18) has the sole solution $K_e = 0$, which holds due to our assumption (3.19) in Lemma 3.5. This shows that the operator \mathcal{A} is an isomorphism from $D(\mathcal{A})$ into $\mathbf{L}^2(\mathcal{G})$ and proves that $\pm \mathcal{A}$ is maximal.

To conclude, we observe that Lemma 3.2 directly implies that \mathcal{A} has compact resolvent, since $D(\mathcal{A})$ is continuously embedded in D_{\max} . \square

Example 3.8. Imposing Dirichlet conditions on *all* endpoints is a possibility allowed for by our formalism, taking

$$\gamma_{\mathbf{v}}(u) \in \{(0, \dots, 0)^{\top}\} =: Y_{\mathbf{v}}, \quad \mathbf{v} \in \mathbf{V}.$$

However, our dimension condition rules it out, as in this case $\dim Y_{\mathbf{v}} = 0$ for all \mathbf{v} , hence (3.19) is not satisfied.

By the Lumer–Phillips Theorem and (3.11), the semigroup generated by \mathcal{A} is isometric if and only if

$$\Re \sum_{\mathbf{e} \in \mathbf{E}} \int_0^{\ell_{\mathbf{e}}} \left(Q_{\mathbf{e}} N_{\mathbf{e}} - \frac{1}{2} (Q_{\mathbf{e}} M_{\mathbf{e}})' \right) u_{\mathbf{e}} \cdot \bar{u}_{\mathbf{e}} dx = 0 \quad \text{for all } u \in D(\mathcal{A}). \quad (3.27)$$

As the next result shows, this condition is easy to characterize using Lemma B.3.

Corollary 3.9. *Under the assumptions of Theorem 3.7,*

$$Q_{\mathbf{e}} N_{\mathbf{e}} + (Q_{\mathbf{e}} N_{\mathbf{e}})^* = (Q_{\mathbf{e}} M_{\mathbf{e}})' \quad (3.28)$$

if and only if the system (2.1) on \mathcal{G} with transmission conditions (3.14) is governed by a unitary group on $\mathbf{L}^2(\mathcal{G})$; in particular, the energy

$$\mathcal{E}(t) := \frac{1}{2} \sum_{\mathbf{e} \in \mathbf{E}} \int_0^{\ell_{\mathbf{e}}} Q_{\mathbf{e}} u(t) \cdot \bar{u}(t) dx, \quad t \in \mathbb{R}, \quad (3.29)$$

is conserved.

Proof. Under the assumptions of Theorem 3.7, the identity (3.12) guarantees that

$$\Re(\mathcal{A}u, u) = 0 \quad \text{for all } u \in D(\mathcal{A})$$

if and only if (3.27) holds. But simple calculations show that (3.27) is equivalent to

$$\sum_{\mathbf{e} \in \mathbf{E}} \int_0^{\ell_{\mathbf{e}}} (Q_{\mathbf{e}} N_{\mathbf{e}} + (Q_{\mathbf{e}} N_{\mathbf{e}})^* - (Q_{\mathbf{e}} M_{\mathbf{e}})') u_{\mathbf{e}} \cdot \bar{u}_{\mathbf{e}} dx = 0 \quad \text{for all } u \in D(\mathcal{A}). \quad (3.30)$$

This obviously shows that (3.28) is a sufficient condition for the unitarity property of the semigroup generated by \mathcal{A} . For the necessity, let us observe that $Q_{\mathbf{e}} N_{\mathbf{e}} + (Q_{\mathbf{e}} N_{\mathbf{e}})^* - (Q_{\mathbf{e}} M_{\mathbf{e}})'$ is hermitian. Since the test functions vanishing at each endpoint satisfy all boundary conditions, $\bigoplus_{\mathbf{e} \in \mathbf{E}} \mathcal{D}(0, \ell_{\mathbf{e}})^{k_{\mathbf{e}}}$ is included in $D(\mathcal{A})$ and we deduce that (3.30) implies that

$$\int_0^{\ell_{\mathbf{e}}} (Q_{\mathbf{e}} N_{\mathbf{e}} + (Q_{\mathbf{e}} N_{\mathbf{e}})^* - (Q_{\mathbf{e}} M_{\mathbf{e}})') u_{\mathbf{e}} \cdot \bar{u}_{\mathbf{e}} dx = 0 \quad \text{for all } u_{\mathbf{e}} \in \mathcal{D}(0, \ell_{\mathbf{e}})^{k_{\mathbf{e}}} \text{ and all } \mathbf{e} \in \mathbf{E}.$$

By Lemma B.3 we conclude that (3.28) holds. Finally, because

$$\frac{d}{dt} \mathcal{E}(t) = \Re(\mathcal{A}u(t), u(t))$$

holds along classical solutions u of (2.1), the second assertion is valid as well. \square

Note that condition (3.28) is satisfied in the special case when $Q_e M_e$ is spatially constant and $Q_e N_e$ has zero or purely imaginary entries.

We will see, however, that the condition (3.19) is not satisfied in some relevant applications (in Sect. 5.5, for example). Therefore, let us present an alternative approach to prove well-posedness based on the dissipativity of \mathcal{A} and its adjoint \mathcal{A}^* that is used in Appendix A of [6] for diagonal systems. The first step is to characterize the adjoint operator.

Lemma 3.10. *The adjoint of the operator \mathcal{A} defined in (3.1)–(3.22) is given by*

$$\begin{aligned} D(\mathcal{A}^*) &= \{v \in D_{\max} : \gamma_v(v) \in T_v^{-1} Y_v^\perp \text{ for all } v \in \mathbf{V}\}, \\ (\mathcal{A}^* v)_e &= -M_e v'_e - Q_e^{-1} (Q_e M_e)' v_e + Q_e^{-1} N_e^* Q_e v_e, \quad e \in \mathbf{E}. \end{aligned} \quad (3.31)$$

Proof. The identity (3.7) is equivalent to

$$\begin{aligned} \sum_{e \in \mathbf{E}} \int_0^{\ell_e} Q_e M_e u'_e \cdot \bar{v}_e \, dx &= - \sum_{e \in \mathbf{E}} \int_0^{\ell_e} u_e \cdot \overline{Q_e M_e v'_e} \, dx - \sum_{e \in \mathbf{E}} \int_0^{\ell_e} u_e \cdot \overline{(Q_e M_e)' v_e} \, dx \\ &\quad + \sum_{v \in \mathbf{V}} T_v \gamma_v(u) \cdot \gamma_v(\bar{v}). \end{aligned} \quad (3.32)$$

Since

$$(\mathcal{A}u, v) = \sum_{e \in \mathbf{E}} \int_0^{\ell_e} Q_e (M_e u'_e + N_e u_e) \cdot \bar{v}_e \, dx \quad \text{for all } u, v \in D_{\max},$$

we get

$$\begin{aligned} (\mathcal{A}u, v) &= \sum_{e \in \mathbf{E}} \int_0^{\ell_e} u_e \cdot \overline{(-Q_e M_e v'_e - (Q_e M_e)' v_e + N_e^* Q_e v_e)} \, dx \\ &\quad + \sum_{v \in \mathbf{V}} T_v \gamma_v(u) \cdot \gamma_v(\bar{v}) \quad \text{for all } u, v \in D_{\max}. \end{aligned} \quad (3.33)$$

Now, $v \in \mathbf{L}^2(\mathcal{G})$ belongs to $D(\mathcal{A}^*)$ if and only if there exists $g \in \mathbf{L}^2(\mathcal{G})$ such that

$$(\mathcal{A}u, v) = (u, g) \quad \text{for all } u \in D(\mathcal{A})$$

and in this case $\mathcal{A}^* v = g$.

First, by taking u such that $u_e \in \mathcal{D}(0, \ell_e)$ in this identity we find

$$-Q_e M_e v'_e - (Q_e M_e)' v_e + N_e^* Q_e v_e = Q_e g_e \quad (3.34)$$

in the distributional sense. Since, under our Assumptions 2.1.(1), $Q_e M_e$ is invertible (with a bounded inverse), we find that $v_e \in H^1(0, \ell_e)$, hence v belongs to D_{\max} .

Now we can apply the identity (3.33) and get

$$(u, g) = \sum_{e \in E} \int_0^{\ell_e} u_e \cdot \overline{(-Q_e M_e v'_e - (Q_e M_e)' v_e + N_e^* Q_e v_e)} dx + \sum_{v \in V} T_v \gamma_v(u) \cdot \gamma_v(\bar{v}) \quad \text{for all } u \in D(\mathcal{A}) \quad (3.35)$$

and, by (3.34), we find

$$\sum_{v \in V} T_v \gamma_v(u) \cdot \gamma_v(\bar{v}) = 0 \quad \text{for all } u \in D(\mathcal{A}).$$

Since the trace operator $\gamma_v : D_{\max} \rightarrow \mathbb{C}^{k_v}$ is surjective, so is its restriction from $D(\mathcal{A})$ to Y_v . Hence we get

$$y \cdot T_v \gamma_v(\bar{v}) = 0 \quad \text{for all } y \in Y_v,$$

and therefore

$$T_v \gamma_v(v) \in Y_v^\perp. \quad (3.36)$$

(Here, Y_v^\perp denotes the orthogonal complement of Y_v in \mathbb{C}^{k_v} for the Euclidean inner product.) Since T_v is invertible, (3.36) is equivalent to

$$\gamma_v(v) \in T_v^{-1} Y_v^\perp.$$

This concludes the proof. \square

Theorem 3.11. *For all $v \in V$, let both Y_v and $T_v^{-1} Y_v$ be totally isotropic subspaces associated with the quadratic form q_v defined by (3.13). Then both $\pm \mathcal{A}$, defined on the domain*

$$D(\mathcal{A}) := \{u \in D_{\max} : \gamma_v(u) \in Y_v \text{ for all } v \in V\},$$

are quasi- m -dissipative operators. In particular, both $\pm \mathcal{A}$ generate a strongly continuous semigroup and hence a strongly continuous group in $\mathbf{L}^2(\mathcal{G})$. If, additionally, (3.28) holds, then the group is unitary.

Proof. We already know that \mathcal{A} is densely defined. By the first part of the proof of Theorem 3.7 (that does not depend on condition (3.19)), we see that both $\pm \mathcal{A}$ are quasi-dissipative. It is easy to verify that \mathcal{A} is closed, see, e.g., the proof of (A.24) in [6]. In view of Corollary II.3.17 in [28], it thus suffices to show that $\pm \mathcal{A}^*$ is quasi-dissipative, too. Observe that (3.11), which is known to hold for all $u \in D_{\max}$, implies that

$$\Re(\mathcal{A}^* u, u) = \Re \sum_{e \in E} \int_0^{\ell_e} (- (Q_e M_e)' u_e + N_e^* Q_e u_e) \cdot \bar{u}_e dx + \frac{1}{2} \sum_{e \in E} \int_0^{\ell_e} (Q_e M_e)' u_e \cdot \bar{u}_e dx - \frac{1}{2} \sum_{v \in V} T_v \gamma_v(u) \cdot \gamma_v(\bar{u}) \quad (3.37)$$

for all $u \in D(\mathcal{A}^*)$. Hence quasi-dissipativity of both $\pm \mathcal{A}^*$ holds if

$$q_v(\xi) = 0, \quad \text{for all } \xi \in T_v^{-1} Y_v^\perp, \quad (3.38)$$

i.e., if $T_V^{-1}Y_V^\perp$ is a totally isotropic subspace associated with the quadratic form q_V . \square

In the remainder of this section, we are going to comment on the possibility of an alternative way of formulating transmission conditions. This is especially relevant in applications to models of theoretical physics when the stress is on unitary well-posedness, see *e.g.* Section 5.5, rather than accurate description of the network structure.

3.2. Global boundary conditions

It is natural to choose transmission conditions that reflect the connectivity of the network, that is the reason of the local boundary condition (3.14). However, nonlocal boundary conditions can be imposed as well by re-writing the term $\sum_{V \in \mathcal{V}} T_V \gamma_V(u) \cdot \gamma_V(\bar{u})$ in a global way as

$$T\gamma(u) \cdot \gamma(\bar{u}),$$

where

$$\gamma(u) := \left(\underline{u(0)}, \underline{u(\ell)} \right)^\top := \left((u_e(0))_{e \in \mathbf{E}}, (u_e(\ell_e))_{e \in \mathbf{E}} \right)^\top$$

and the $2k \times 2k$ matrix T is given by

$$T := \begin{pmatrix} -\text{diag} (Q_e(0)M_e(0))_{e \in \mathbf{E}} & 0 \\ 0 & \text{diag} (Q_e(\ell_e)M_e(\ell_e))_{e \in \mathbf{E}} \end{pmatrix}, \quad (3.39)$$

without any reference to the structure of the network. With this notation,

$$\Re(\mathcal{A}u, u) = \Re \sum_{e \in \mathbf{E}} \int_0^{\ell_e} \left(\left(Q_e N_e - \frac{1}{2} (Q_e M_e)' \right) u_e \cdot \bar{u}_e \right) dx + \frac{1}{2} T\gamma(u) \cdot \gamma(\bar{u}),$$

and this suggests to replace (3.14) by

$$\gamma(u) \in Y, \quad (3.40)$$

i.e., to consider \mathcal{A} with domain

$$D(\mathcal{A}) := \{u \in D_{\max} : \gamma(u) \in Y\},$$

where $Y \subset \mathbb{C}^{2k}$ is a subspace of the null isotropic cone associated with the quadratic form

$$T\xi \cdot \bar{\xi}, \quad \text{for all } \xi \in \mathbb{C}^{2k}. \quad (3.41)$$

This corresponds to glue all vertices together, thus forming a so-called flower graph, and to impose general transmission conditions in the only vertex of such a flower (Fig. 1).

In this setting the well-posedness conditions is different from (3.19) and sums up to

$$\dim Y^\perp = k = \dim P_K Y^\perp, \quad (3.42)$$

where K is the k -dimensional subspace of \mathbb{C}^{2k} defined by

$$K = \left\{ \left((K_e)_{e \in \mathbf{E}}, (K_e)_{e \in \mathbf{E}} \right)^\top : K_e \in \mathbb{C} \text{ for all } e \in \mathbf{E} \right\}$$



FIGURE 1. Gluing all the vertices of a graph: from a complete graph on four vertices (*left*) to a flower graph on six edges (*right*).

and P_K is the orthogonal projection on K with respect to the Euclidean inner product of \mathbb{C}^{2k} . To see this, we elaborate on the proof of Theorem 3.7 and write $I := \{1, 2, \dots, \dim Y^\perp\}$. Once fixed a basis $\{\mathbb{W}^{(i)}\}_{i \in I}$ of $Y^\perp \subset \mathbb{C}^{2k}$, using (3.26) the condition (3.40) is equivalent to

$$((K_e)_{e \in E}, (K_e)_{e \in E})^\top \cdot \overline{\mathbb{W}^{(i)}} = \left((0)_{e \in E}, \left(- \int_0^{\ell_e} M_e^{-1}(y) f_e(y) dy \right)_{e \in E} \right)^\top \cdot \overline{\mathbb{W}^{(i)}} \quad \text{for all } i \in I.$$

Since the vector $((K_e)_{e \in E}, (K_e)_{e \in E})^\top$ belongs to K , we have

$$((K_e)_{e \in E}, (K_e)_{e \in E})^\top \cdot \overline{\mathbb{W}^{(i)}} = ((K_e)_{e \in E}, (K_e)_{e \in E})^\top \cdot \overline{P_K \mathbb{W}^{(i)}},$$

and therefore this last condition is equivalent to

$$((K_e)_{e \in E}, (K_e)_{e \in E})^\top \cdot \overline{P_K \mathbb{W}^{(i)}} = \left((0)_{e \in E}, \left(- \int_0^{\ell_e} M_e^{-1}(y) f_e(y) dy \right)_{e \in E} \right)^\top \cdot \overline{\mathbb{W}^{(i)}} \quad \text{for all } i \in I.$$

Our assumptions (3.42) guarantee that we are dealing with a $k \times k$ linear system for which its associated matrix is invertible. This then proves the maximality condition.

Note that the second condition in (3.42) admits different formulations stated in the next Lemma.

Lemma 3.12. *We have*

$$\dim P_K Y^\perp = k \Leftrightarrow P_K Y^\perp = K \Leftrightarrow Y \cap K = \{0\}. \quad (3.43)$$

Proof. The first equivalence directly follows from the fact that K is of dimension k .

Let us now prove the implication $P_K Y^\perp = K \Rightarrow Y \cap K = \{0\}$. Indeed, each $y \in Y \cap K$ satisfies

$$y \cdot \bar{z} = 0 \quad \text{for all } z \in Y^\perp.$$

Since y is also in K , we have $y \cdot \bar{z} = y \cdot \overline{P_K z}$, thus the last property is equivalent to

$$y \cdot \overline{P_K z} = 0 \quad \text{for all } z \in Y^\perp.$$

By our assumption $P_K Y^\perp = K$ and we deduce that

$$y \cdot \bar{k} = 0 \quad \text{for all } k \in K,$$

which yields $y = 0$ since y is in K .

Let us show the converse implication. Take $k \in K$ orthogonal to $P_K Y^\perp$, *i.e.*,

$$k \cdot \overline{P_K z} = 0 \quad \text{for all } z \in Y^\perp.$$

As k is in K , we get equivalently

$$k \cdot \bar{z} = 0 \quad \text{for all } z \in Y^\perp.$$

Therefore k belongs to Y and hence to $Y \cap K$. Thus $k = 0$, due to the assumption $Y \cap K = \{0\}$, which proves that $P_K Y^\perp = K$. \square

Using $\dim Y = 2k - \dim Y^\perp$ we can find different equivalent conditions to (3.42). One of them is

$$\dim Y = \dim P_K Y^\perp = k. \quad (3.44)$$

This formalism also makes possible to compare solutions of the same system under different transmission conditions in the vertices. We will come back to this in the next section.

Remark 3.13. (Local boundary conditions versus global ones). Assume that \mathcal{A} is defined by the local boundary conditions (3.14), then they can be viewed as global boundary conditions with Y defined here below. Namely for all $\mathbf{v} \in \mathbf{V}$, introduce the matrix operator $\mathcal{P}_{\mathbf{v}}$ (from \mathbb{C}^{2k} to $\mathbb{C}^{k_{\mathbf{v}}}$) defined by

$$\gamma_{\mathbf{v}}(u) = \mathcal{P}_{\mathbf{v}} \gamma(u).$$

Then, given the basis $\{\mathbf{w}^{(\mathbf{v},i)}\}_{i \in I_{\mathbf{v}}}$ of $Y_{\mathbf{v}}^\perp \subset \mathbb{C}^{k_{\mathbf{v}}}$, we define

$$\mathbf{W}^{(\mathbf{v},i)} = \mathcal{P}_{\mathbf{v}}^\top \mathbf{w}^{(\mathbf{v},i)}.$$

We may notice that

$$\gamma(u) \cdot \overline{\mathbf{W}^{(\mathbf{v},i)}} = \gamma_{\mathbf{v}}(u) \cdot \overline{\mathbf{w}^{(\mathbf{v},i)}}, \quad (3.45)$$

hence Y^\perp is simply the space spanned by $\mathbf{W}^{(\mathbf{v},i)}$, for $i \in I_{\mathbf{v}}$ and $\mathbf{v} \in \mathbf{V}$. Let us also remark that

$$\mathbf{W}^{(\mathbf{v},i)} \cdot \overline{\mathbf{W}^{(\mathbf{v}',i')}} = 0, \quad \text{if } \mathbf{v} \neq \mathbf{v}'. \quad (3.46)$$

These two properties imply that (3.42) is equivalent to (3.19). Indeed, by (3.46), and as $\mathcal{P}_{\mathbf{v}}$ being surjective, $\mathcal{P}_{\mathbf{v}}^\top$ is injective, we deduce that

$$\dim Y^\perp = \sum_{\mathbf{v} \in \mathbf{V}} \dim Y_{\mathbf{v}}^\perp, \quad (3.47)$$

while the first one guarantees that

$$\left((K_{\mathbf{e}})_{\mathbf{e} \in \mathbf{E}}, (K_{\mathbf{e}})_{\mathbf{e} \in \mathbf{E}} \right)^\top \cdot \overline{\mathbf{W}^{(\mathbf{v},i)}} = (K_{\mathbf{e}})_{\mathbf{e} \in \mathbf{E}_v} \cdot \overline{\mathbf{w}^{(\mathbf{v},i)}} = (K_{\mathbf{e}})_{\mathbf{e} \in \mathbf{E}} \cdot \overline{\widetilde{\mathbf{w}}^{(\mathbf{v},i)}}. \quad (3.48)$$

Hence if (3.19), then (3.47) directly implies that $\dim Y^\perp = k$. Further by (3.48), one has

$$\left((K_{\mathbf{e}})_{\mathbf{e} \in \mathbf{E}}, (K_{\mathbf{e}})_{\mathbf{e} \in \mathbf{E}} \right)^\top \cdot \overline{P_K \mathbf{W}^{(\mathbf{v},i)}} = (K_{\mathbf{e}})_{\mathbf{e} \in \mathbf{E}} \cdot \overline{\widetilde{\mathbf{w}}^{(\mathbf{v},i)}}, \quad (3.49)$$

and since the vectors $\tilde{w}^{(v,i)}$ span the whole \mathbb{C}^k , if the left-hand side is zero for all i, v , $(K_e)_{e \in E}$ will be zero, hence the vectors $P_K W^{(v,i)}$ span the whole K .

Conversely if (3.42) holds, then by (3.47), we have $\sum_{v \in V} \dim Y_v^\perp = k$, that is by the hand-shaking lemma equivalent to $\sum_{v \in V} \dim Y_v = k$. As before, due to (3.49), if the vectors $P_K W^{(v,i)}$ span the whole K , then the vectors $\tilde{w}^{(v,i)}$ span the whole \mathbb{C}^k .

Example 3.14. The arguably easiest application of our theory is the model of flows on networks discussed *e.g.* in [42]. It consists of a system of $k = |E|$ scalar equations

$$\dot{u}_e = c_e u'_e, \quad e \in E,$$

where c_e are positive constants. Hence, $M_e = c_e$, and we can take $Q_e = 1$, for all $e \in E$. The associated matrix T_v is diagonal and takes the form

$$T_v = \text{diag} (c_e(v) \iota_{ve})_{e \in E_v}, \quad v \in V,$$

where ι_{ve} are the entries of the incidence matrix of the graph, see (3.9).

However, in order to treat more general boundary conditions, we switch to the global setting and take

$$T = \begin{pmatrix} -\text{diag}(c_e)_{e \in E} & 0 \\ 0 & \text{diag}(c_e)_{e \in E} \end{pmatrix}.$$

A rather general way of writing the transmission conditions in the vertices is $V_0 u(0) = V_\ell u(\ell)$, where V_0, V_ℓ are $k \times k$ matrices, see [24]. They can be equivalently expressed in our formalism by

$$\gamma(u) \in \ker \begin{pmatrix} V_0 & -V_\ell \end{pmatrix} =: Y.$$

The relevant conditions in Theorem 3.7 are hence whether

- (i) the space $\ker \begin{pmatrix} V_0 & -V_\ell \end{pmatrix} = \{(\psi, \theta) \in \mathbb{C}^{2k} : V_0 \psi = V_\ell \theta\}$ has dimension k and
- (ii) $V_0 \psi = V_\ell \theta$ implies $\sum_{e \in E} c_e |\psi_e|^2 = \sum_{e \in E} c_e |\theta_e|^2$ for all $\psi, \theta \in \mathbb{C}^k$.

Both conditions are *e.g.* satisfied if $V_0, V_\ell = \text{Id}$, which corresponds to transport on k disjoint loops, in which case Theorem 3.7 confirms one's intuition that \mathcal{A} with

$$D(\mathcal{A}) := \{u \in D_{\max} : u_e(0) = u_e(\ell_e), \quad e \in E\} \tag{3.50}$$

generates a strongly continuous group on $\mathbf{L}^2(\mathcal{G})$. With $k = 2$, condition (i) is also fulfilled for $V_\ell = \text{Id}$ if *e.g.* $V_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, or $V_0 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; condition (ii) is satisfied in the former case if $c_{e_1} = c_{e_2}$ (hence we have by Theorem 3.7 a group generator on a network which can be regarded as a loop of length $\ell_{e_1} + \ell_{e_2}$), but not in the latter: it will follow from the results in the next section that this operator, which is a prototype of those considered in [42], still generates a strongly continuous *semigroup* on $\mathbf{L}^2(\mathcal{G})$. This example shows that our conditions on Y_v are tailored for *unitary* group generation, as Corollary 3.9 shows. Some remedies to avoid such a problem will be discussed below.

We stress that the second above condition implies the invertibility of both V_0, V_ℓ , which is proved in Corollary 3.8 of [24] to be equivalent to the assertion that \mathcal{A} with domain

$$D(\mathcal{A}) := \{u \in D_{\max} : \gamma(u) \in \ker \begin{pmatrix} V_0 & -V_\ell \end{pmatrix}\}$$

is a group generator.

4. CONTRACTIVE WELL-POSEDNESS AND QUALITATIVE PROPERTIES

Let us now discuss the more general situation in which the solutions to (2.1) are given by semigroups that are merely contractive. In this case, the above computations show that much more general boundary conditions can be studied. Furthermore, we are also able to describe qualitative behavior of these solutions. We refer to Section 5 for several illustrative examples.

Recall that the *nonpositive* (resp. *nonnegative*) *isotropic cone* associated with a quadratic form $q : \mathbb{C}^k \rightarrow \mathbb{R}$ is the set of vectors $\xi \in \mathbb{C}^k$ such that $q(\xi) \leq 0$ (resp. $q(\xi) \geq 0$), see Definition C.1.

Theorem 4.1. *For all $\mathbf{v} \in \mathbf{V}$, let $Y_{\mathbf{v}}$ be a subspace of the nonpositive isotropic cone associated with the quadratic form $q_{\mathbf{v}}$ given in (3.13) and assume that one of the following conditions is satisfied:*

- (3.19) holds or
- for all $\mathbf{v} \in \mathbf{V}$, the space $T_{\mathbf{v}}^{-1}Y_{\mathbf{v}}^{\perp}$ is a subspace of the nonnegative isotropic cone associated with $q_{\mathbf{v}}$.

Then the following assertions hold.

(1) \mathcal{A} with domain

$$D(\mathcal{A}) := \{u \in D_{\max} : \gamma_{\mathbf{v}}(u) \in Y_{\mathbf{v}} \text{ for all } \mathbf{v} \in \mathbf{V}\}$$

generates a strongly continuous quasi-contractive semigroup $(e^{t\mathcal{A}})_{t \geq 0}$ in $\mathbf{L}^2(\mathcal{G})$ and the system (2.1) on \mathcal{G} with transmission conditions

$$\gamma_{\mathbf{v}}(u) \in Y_{\mathbf{v}}, \quad \mathbf{v} \in \mathbf{V}, \quad (4.1)$$

is well-posed.

- (2) This semigroup is contractive if, additionally, $(Q_{\mathbf{e}}N_{\mathbf{e}})(x) + (Q_{\mathbf{e}}N_{\mathbf{e}})^*(x) - (Q_{\mathbf{e}}M_{\mathbf{e}})'(x)$ is negative semi-definite for all $\mathbf{e} \in \mathbf{E}$ and a.e. $x \in [0, \ell_{\mathbf{e}}]$.
- (3) Under the assumptions of (2), the energy in (3.29) is monotonically decreasing; in this case, there exists a projector commuting with $(e^{t\mathcal{A}})_{t \geq 0}$ whose null space is the set of strong stability of the semigroup and whose range is the closure of the set of periodic vectors under $(e^{t\mathcal{A}})_{t \geq 0}$.

Proof. The proof of the assertion leading to well-posedness is exactly the same as the ones of Theorem 3.7, Corollary 3.9, and Theorem 3.11 for operator \mathcal{A} ; the sufficient condition for contractivity of the semigroup can be read off (3.12). The third assertion is a direct consequence of Corollary V.2.15 in [28] and Lemma 3.2. \square

Remark 4.2. In the previous Theorem we may replace local boundary condition by global ones and then replacing either (3.19) by (3.42) or the second local condition by a global version. Hence Theorem 4.1 especially applies to the setting of Appendix A in [6], since their boundary condition (A.8) can be written as $\gamma(u) \in \ker \mathcal{K}$ for some $k \times 2k$ -matrix \mathcal{K} . The proof of Theorem A.1 in [6] shows that $\ker \mathcal{K}$ is contained in the nonpositive isotropic cone of the quadratic form (3.41). Also the boundary condition (2.2) in [24] can be written as $\gamma(u) \in \ker \mathcal{K}$, and (3.19) is easily seen to be satisfied. Hence, the generation result in Corollary 2.2 of [24] follows from Theorem 4.1; this shows in particular that the well-posedness results for the transport equations in [42, 52] are special cases of our general theory. Let us further stress that, as in Corollary 3.8 of [24], we can chose different $Q_{\mathbf{e}}$ for $M_{\mathbf{e}}$ and $-M_{\mathbf{e}}$ to obtain generation of a group. We will review further, more advanced examples in Section 5.

Remark 4.3. (Global boundary conditions revisited) It turns out that the global condition (3.42) may fail, see Example 4.16 below. But if all matrices $M_{\mathbf{e}}$ are diagonal or spatially constant and symmetric positive definite, we can replace the surjectivity of \mathcal{A} by the surjectivity of $\lambda \mathbb{1} - \mathcal{A}$ for $\lambda > 0$ large enough. Indeed, using the same approach as in Theorem 3.7, for any $\mathbf{f} \in \mathbf{L}^2(\mathcal{G})$, we look for a solution $u \in D(\mathcal{A})$ of

$$\lambda u'_{\mathbf{e}} - M_{\mathbf{e}}(x)u'_{\mathbf{e}}(x) = \mathbf{f}_{\mathbf{e}}(x) \quad \text{for } x \in (0, \ell_{\mathbf{e}}) \text{ and all } \mathbf{e} \in \mathbf{E}. \quad (4.2)$$

By our assumptions, such a solution is given by

$$u_{\mathbf{e}}(x) = e^{\lambda P_{\ell}(x)} K_{\mathbf{e}} - \int_0^x e^{\lambda(P_{\ell}(x)-P_{\ell}(y))} M_{\mathbf{e}}^{-1}(y) f_{\mathbf{e}}(y) dy, \text{ for all } x \in [0, \ell_{\mathbf{e}}], \text{ for all } \mathbf{e} \in \mathbf{E},$$

with $K_{\mathbf{e}} \in \mathbb{C}^{k_{\mathbf{e}}}$ to be fixed, where

$$P_{\ell}(x) := \int_0^x M_{\mathbf{e}}^{-1}(y) dy, \quad x \in [0, \ell_{\mathbf{e}}], \quad \mathbf{e} \in \mathbf{E}.$$

Obviously, $P_{\ell}(0) = 0$ and $P_{\ell}(\ell_{\mathbf{e}}) = \int_0^{\ell_{\mathbf{e}}} M_{\mathbf{e}}^{-1}(y) dy$ is symmetric positive definite. Hence, using the notation from Section 3.2, the boundary condition (3.40) is equivalent to

$$(K, E_{\lambda} K)^{\top} \cdot \overline{W^{(i)}} = \left((0)_{\mathbf{e} \in \mathbf{E}}, \left(\int_0^{\ell_{\mathbf{e}}} e^{\lambda(P_{\ell}(\ell_{\mathbf{e}})-P_{\ell}(y))} M_{\mathbf{e}}^{-1}(y) f_{\mathbf{e}}(y) dy \right)_{\mathbf{e} \in \mathbf{E}} \right)^{\top} \cdot \overline{W^{(i)}} \quad \text{for all } i \in I. \quad (4.3)$$

where $K = (K_{\mathbf{e}})_{\mathbf{e} \in \mathbf{E}}$ and $E_{\lambda} = \text{diag}(e^{\lambda P_{\ell}(\ell_{\mathbf{e}})})_{\mathbf{e} \in \mathbf{E}}$. Setting $L = E_{\lambda} K$, we have equivalently

$$(E_{-\lambda} L, L)^{\top} \cdot \overline{W^{(i)}} = \left((0)_{\mathbf{e} \in \mathbf{E}}, \left(\int_0^{\ell_{\mathbf{e}}} e^{\lambda(P_{\ell}(\ell_{\mathbf{e}})-P_{\ell}(y))} M_{\mathbf{e}}^{-1}(y) f_{\mathbf{e}}(y) dy \right)_{\mathbf{e} \in \mathbf{E}} \right)^{\top} \cdot \overline{W^{(i)}} \quad \text{for all } i \in I. \quad (4.4)$$

Hence if $\dim Y^{\perp} = k$, we find a $k \times k$ linear system and a unique solution L and then K exists if and only if the determinant of the associated matrix A_{λ} is different from zero. But if for any vectors $(y, z)^{\top} \in \mathbb{C}^{2k}$, with $y, z \in \mathbb{C}^k$ we set

$$\Pi_0(y, z)^{\top} = y \quad \text{and} \quad \Pi_{\ell}(y, z)^{\top} = z,$$

we see that

$$(E_{-\lambda} L, L)^{\top} \cdot \overline{W^{(i)}} = E_{-\lambda} L \cdot \overline{\Pi_0 W^{(i)}} + L \cdot \overline{\Pi_{\ell} W^{(i)}} = L \cdot \overline{E_{-\lambda}^* \Pi_0 W^{(i)} + \Pi_{\ell} W^{(i)}},$$

where $E_{-\lambda}^* = \text{diag}(e^{-\lambda P_{\ell}(\ell_{\mathbf{e}})^*})_{\mathbf{e} \in \mathbf{E}}$. From this expression we see that A_{λ} can be split up as

$$A_{\lambda} = R_{\lambda} + A_{\infty},$$

where A_{∞} is the limit of A_{λ} as λ goes to infinity and is simply the matrix whose i th row is $(\Pi_{\ell} W^{(i)})^{\top}$. Further, since $P_{\ell}(\ell_{\mathbf{e}})^*$ are symmetric positive definite, one can easily see that

$$\lim_{\lambda \rightarrow \infty} \|R_{\lambda}\| = 0.$$

Hence, if we assume that A_{∞} is invertible, A_{λ} will be invertible as well for λ large enough, which allows to conclude the surjectivity of $\lambda I - \mathcal{A}$ for $\lambda > 0$ large enough. In conclusion, as A_{∞} is invertible if and only if $\dim \Pi_{\ell} Y^{\perp} = k$, maximality holds if

$$\dim Y^{\perp} = k = \dim \Pi_{\ell} Y^{\perp}. \quad (4.5)$$

This actually means that maximality holds as soon as maximality holds if we replace the boundary condition (3.40) by the one

$$\underline{u}(\ell) \in \Pi_\ell Y^\perp,$$

at the endpoint ℓ only; in particular, dropping any condition at 0. A local version of this approach can be stated, we let the details to the reader.

Remark 4.4. Our well-posedness results can be also regarded as a first step towards applications to control theory. For example, Proposition VI.8.5 of [28] prevents the control system

$$\begin{cases} \dot{u}_e(t) = M_e u'_e(t) + N_e u_e(t) + B_e y_e(t), & t \geq 0, e \in E, \\ u_e(0) = f_e \end{cases} \quad (\Sigma(\mathcal{A}, \mathcal{B}))$$

from being exactly 2-controllable for any control operator $\mathcal{B} = (B_e)_{e \in E}$ mapping the vertex space $\mathbb{C}^{|V|}$ to $L^2(\mathcal{G})$, for any finite time horizon $t < \infty$. However, Corollary VI.8.11 of [28] can be invoked to show that, provided the assumptions of Theorem 4.1.(2) are satisfied and hence \mathcal{A} generates a contractive semigroup, the control system $(\Sigma(\mathcal{A}, \mathcal{B}))$ will be approximately 2-controllable if and only if $\bigcap_{n \in \mathbb{N}} \ker \mathcal{B}^*(\text{Id} - \mathcal{A}^*)^n = 0$, where \mathcal{A}^* is the adjoint of \mathcal{A} given in Lemma 3.10 and $\mathcal{B} : \mathbb{C}^{|V|} \rightarrow L^2(\mathcal{G})$ is any control operator.

Studying processes on a metric graph, boundary control problems with control imposed in the vertices are of special interest. A semigroup approach to such problems that enables an explicit description of the associated (even exact or positive) reachability space as presented in [25–27] could be adapted for the present situation.

Further, as in [56] in case of generation of a contraction semigroup, some stabilization results can be considered using different approaches, like the use of observability estimates [4, 20], frequency domain approach [11, 61], or port-Hamiltonian techniques [35].

Let us continue by studying qualitative properties of the semigroup generated by \mathcal{A} . In particular, let C be a closed and convex subset of \mathbb{C} and write

$$\mathbf{L}^2(\mathcal{G}; C) := \{u \in \mathbf{L}^2(\mathcal{G}) : u_e(x) \in C^{k_e} \text{ for a.e. } x \in (0, \ell_e) \text{ and all } e \in E\} \quad (4.6)$$

A semigroup $(T(t))_{t \geq 0}$ on $L^2(\mathcal{G})$ is called *real* (resp., *positive*) if each operator $T(t)$ leaves $\mathbf{L}^2(\mathcal{G}; \mathbb{R})$ (resp., $\mathbf{L}^2(\mathcal{G}; \mathbb{R}_+)$) invariant. Moreover, for a closed and convex subset K of a Hilbert space H with inner product $\langle \cdot, \cdot \rangle_H$ and associated norm $\|\cdot\|_H$, the *minimizing projector* P_K onto K assigns to each $u \in H$ the unique element $P_K u \in K$ satisfying

$$\|u - P_K u\|_H = \min\{\|u - w\|_H : w \in K\}$$

or, equivalently (see [14], Thm. 5.2), $P_K u = z \in K$ is the unique element in K such that

$$\Re \langle w - z, u - z \rangle_H \leq 0 \quad \text{for all } w \in K. \quad (4.7)$$

We will use a generalization of Brezis' classical result for invariance under the semigroup generated by a subdifferential. In the linear case, Theorem 2.4 of [68] can be formulated as follows.

Lemma 4.5. *Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and associated norm $\|\cdot\|_H$, K a closed and convex subset of H and P_K the minimizing projector onto K . Let A be an ω -quasi- m -dissipative operator on H for some $\omega \in \mathbb{R}$ and $(T(t))_{t \geq 0}$ the strongly continuous semigroup generated by A . Then K is invariant under $(T(t))_{t \geq 0}$ if and only if*

$$\Re \langle Au, u - P_K u \rangle_H \leq \omega \|u - P_K u\|_H^2 \quad \text{for all } u \in D(A). \quad (4.8)$$

If in particular $\omega = 0$, i.e., A is dissipative, and $P_K u \in D(A)$ for all $u \in D(A)$, then the invariance of K under $(T(t))_{t \geq 0}$ is equivalent to

$$\Re \langle AP_K u, u - P_K u \rangle_H \leq 0 \quad \text{for all } u \in D(A).$$

We can thus describe further properties of the semigroup generated by \mathcal{A} in terms of the matrices Q_e, M_e, N_e , and the boundary conditions. We are interested in the convex subsets of the form $K = \mathbf{L}^2(\mathcal{G}; C)$ where $C \subset \mathbb{C}$ is a closed interval, e.g., $C = \mathbb{R}$ or $C = \mathbb{R}_+$. Let us first relate the minimizing projector P_K^Q with respect to the inner product (3.2) on $\mathbf{L}^2(\mathcal{G})$ to the minimizing projector P_K with respect to the standard inner product

$$\langle u, v \rangle := \sum_{e \in \mathbf{E}} \int_0^{\ell_e} u_e(x) \cdot \bar{v}_e(x) dx, \quad u, v \in \mathbf{L}^2(\mathcal{G}). \quad (4.9)$$

Lemma 4.6. Assume $Q_e^{\frac{1}{2}}(x)$ to be bijective for all $e \in \mathbf{E}$ and all $x \in [0, \ell_e]$ as a map on C , i.e., $Q_e^{\frac{1}{2}}(x)(C) = C$. Then the minimizing projector P_K^Q with respect to the inner product (3.2) onto $K = \mathbf{L}^2(\mathcal{G}; C)$ is given by

$$P_K^Q = Q^{-\frac{1}{2}} P_K Q^{\frac{1}{2}} \quad (4.10)$$

where $Q := \text{diag}(Q_e)_{e \in \mathbf{E}}$ is a block diagonal matrix and P_K is the minimizing projector with respect to the standard inner product (4.9).

Proof. By (4.7), $P_K^Q u =: z$ is the unique element in K such that

$$\sum_{e \in \mathbf{E}} \int_0^{\ell_e} Q_e(x) (w_e(x) - z_e(x)) \cdot (\bar{u}_e(x) - \bar{z}_e(x)) dx \leq 0 \quad \text{for all } w \in K. \quad (4.11)$$

As $Q_e(x)$ is symmetric positive definite, it admits a square root $Q_e(x)^{\frac{1}{2}}$ which is still symmetric positive definite (and is of class H^1 as a function of x). Hence (4.11) is equivalent to

$$\sum_{e \in \mathbf{E}} \int_0^{\ell_e} Q_e(x)^{\frac{1}{2}} (w_e(x) - z_e(x)) \cdot Q_e(x)^{\frac{1}{2}} (\bar{u}_e(x) - \bar{z}_e(x)) dx \leq 0 \quad \text{for all } w \in K. \quad (4.12)$$

Since $Q_e^{\frac{1}{2}}$ is bijective on C , by setting $v_e := Q_e^{\frac{1}{2}} u_e$ and $\tilde{w}_e := Q_e^{\frac{1}{2}} w_e$, we may equivalently re-write (4.12) as

$$\Re \langle \tilde{w} - Q^{\frac{1}{2}} z, v - Q^{\frac{1}{2}} z \rangle \leq 0 \quad \text{for all } \tilde{w} \in K.$$

By (4.7) we obtain $Q^{\frac{1}{2}} z = P_K v$ and (4.10) follows. \square

In many applications Q_e , and thus also $Q_e^{\frac{1}{2}}$ and $Q_e^{-\frac{1}{2}}$, are real-valued. In such a case, (4.10) for $C = \mathbb{R}$ and hence $K = L^2(\mathcal{G}; \mathbb{R})$ yields

$$P_K^Q u = Q^{-\frac{1}{2}} \Re \left(Q^{\frac{1}{2}} u \right) = \Re u. \quad (4.13)$$

Proposition 4.7. Under the assumptions of Theorem 4.1, let

$$\Re \xi \in \bigoplus_{v \in \mathbf{V}} Y_v \quad \text{for all } \xi \in \bigoplus_{v \in \mathbf{V}} Y_v \quad (4.14)$$

and the matrix-valued mapping Q_e be real-valued for all $e \in E$. Then the semigroup generated by \mathcal{A} is real if and only if the matrix-valued mappings M_e, N_e are real-valued for all $e \in E$.

Proof. We use Lemma 4.5 for the convex subset of real-valued functions $K = \mathbf{L}^2(\mathcal{G}, \mathbb{R})$ and the projector $P_K^Q u = \Re u$ obtained in (4.13). Also observe that we can without loss of generality assume $\omega = 0$, since reality of a semigroup is invariant under scalar perturbations of its generator. Now, it follows from (4.14) that $P_K^Q u \in D(\mathcal{A})$ whenever $u \in D(\mathcal{A})$. We deduce that reality of the semigroup is equivalent to

$$\Re(\mathcal{A}\Re u, i\Im u) \leq 0 \quad \text{for all } u \in D(\mathcal{A}).$$

By applying the same trick as in the proof of Proposition 2.5 in [58], that is by plugging $-\Re u + i\Im u$ into the above inequality, we obtain

$$(\mathcal{A}\Re u, \Im u) = \sum_{e \in E} \int_0^{\ell_e} \left((Q_e M_e) \frac{d\Re u_e}{dx} + (Q_e N_e) \Re u_e \right) \cdot \Im u_e dx \in \mathbb{R} \quad \text{for all } u \in D(\mathcal{A}),$$

which by a simple localization argument is in turn equivalent to

$$(Q_e M_e) \frac{du_e}{dx} + (Q_e N_e) u_e \text{ is real-valued for all real-valued } u \in D(\mathcal{A}) \text{ and all } e \in E. \quad (4.15)$$

The conclusion then follows from Lemma 4.8 below that yields an equivalent, but easier to check, formulation of (4.15). \square

Lemma 4.8. *Assume the matrix-valued mapping Q_e to be real-valued for all $e \in E$. Then (4.15) holds if and only if the matrix-valued mappings M_e, N_e are real-valued for all $e \in E$.*

Proof. As all Q_e are real-valued, it suffices to show that (4.15) holds if and only if the matrices $Q_e M_e$ and $Q_e N_e$ are real for all $e \in E$. As this second property is clearly sufficient for (4.15) to hold, it suffices to prove the converse implication. For that purpose, fix a real-valued function $\varphi \in \mathcal{D}(\mathbb{R})$ with a support included into $[-1, 1]$ and such that $\varphi'(0) = 1$. Now fix one edge $e \in E$, one point $x_0 \in (0, \ell_e)$ and one $i \in \{1, \dots, k_e\}$. Then for all n large enough, define u_n as follows: $u_{n, e'} = 0$, for all edges $e' \neq e$ and $u_{n, e} = \varphi(n(x - x_0))e_i$, where $e_i = (\delta_{ij})_{j=1}^{k_e}$ is the i -th vector of the canonical basis of \mathbb{R}^{k_e} . The parameter n is chosen large enough so that the support of $u_{n, e}$ is included into $(0, \ell_e)$ so that $u_n \in D(\mathcal{A})$ (and is real valued). Taking this function in (4.15), we find

$$n\varphi'(n(x - x_0))(Q_e M_e)(x)e_i + \varphi(n(x - x_0))(Q_e N_e)(x)e_i \in \mathbb{R}^{k_e} \quad \text{for all } x \in (0, \ell_e).$$

By evaluating this expression at x_0 , dividing by n and letting n goes to infinity, we find that

$$(Q_e M_e)(x_0)e_i \in \mathbb{R}^{k_e}.$$

In other words, $Q_e M_e$ is real-valued. Once this property holds, (4.15) reduces to

$$(Q_e N_e)u_e \text{ is real-valued for all real-valued } u \in D(\mathcal{A}) \text{ and all } e \in E.$$

This directly implies that $Q_e N_e$ is real-valued since $\bigoplus_{e \in E} (\mathcal{D}(0, \ell_e))^{k_e}$ is included into $D(\mathcal{A})$. \square

Before going on let us mention that condition (4.14) can be simplified in the following way.

Lemma 4.9. *The condition (4.14) holds if and only if Y_v , for each $v \in V$, is spanned by vectors with real entries only.*

Proof. Let Y_v be spanned by entry-wise real vectors y_1, \dots, y_n and let $y \in Y_v$. Then there exist $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that $y = \sum_{i=1}^n \alpha_i y_i$. Because all entries of each y_i are real, it follows that $\Re y$ is again a linear combination of these basis vectors,

$$\Re y = \sum_{i=1}^n (\Re \alpha_i) y_i \in Y_v.$$

For the converse first note that (4.14) implies that for any $y \in Y_v$ also $\Re y, \Im y \in Y_v$. If now y_1, \dots, y_n is any basis of Y_v , then the entry-wise real vectors $\Re y_i, \Im y_i, i = 1, \dots, n$, span Y_v . \square

Let us now continue with the study of positivity of the semigroup. Here, we will without loss of generality restrict ourselves to real Hilbert space $L^2(\mathcal{G}, \mathbb{R})$. We first notice that each $Q_e(x)^{\frac{1}{2}}$ is a lattice isomorphism if and only if $Q_e(x)^{\frac{1}{2}}$ and hence $Q_e(x)$ are diagonal.

Lemma 4.10. *Let P be a real $k \times k$ ($k \geq 1$) matrix that is symmetric and positive definite. Then P is a lattice isomorphism, i.e., $P(\mathbb{R}_+^k) = \mathbb{R}_+^k$, if and only if P is diagonal.*

Proof. The diagonal character of P added with its positive definiteness trivially imply that $P(\mathbb{R}_+^k) = \mathbb{R}_+^k$. Hence, we only need to prove the converse implication. For that purpose denote by $p_{ij}, 1 \leq i, j \leq k$ (resp. $q_{ij}, 1 \leq i, j \leq k$) the entries of P (resp. P^{-1}). Now notice that from our assumption directly follows $p_{ij}, q_{ij} \geq 0$, for all i, j . Moreover, for all i, j with $i \neq j$, we have

$$\sum_{\ell=1}^k p_{i\ell} q_{\ell j} = 0,$$

or, equivalently,

$$p_{i\ell} q_{\ell j} = 0 \quad \text{for all } \ell = 1, \dots, k.$$

Taking $\ell = j$, we find that $p_{ij} = 0$ since the diagonal entries of P^{-1} are strictly positive. This shows that P is diagonal as requested. \square

We continue by applying Lemma 4.5 to the convex subset of positive-valued functions $\mathbf{L}^2(\mathcal{G}, \mathbb{R}_+)$. In the following we will adopt the notation

$$\mathbf{1}_{\{u \geq 0\}} := (\mathbf{1}_{\{u_e \geq 0\}})_{e \in E}, \quad \mathbf{1}_{\{u \leq 0\}} := (\mathbf{1}_{\{u_e \leq 0\}})_{e \in E},$$

for the vector-valued characteristic function of the nonnegative and nonpositive support of u , respectively. Here each $\mathbf{1}_{\{u_e \geq 0\}}$ is the diagonal $k_e \times k_e$ matrix associated to $u_e = \left(u_e^{(1)}, \dots, u_e^{(k_e)} \right)^\top$,

$$\mathbf{1}_{\{u_e \geq 0\}} := \text{diag} \left(\mathbf{1}_{\{u_e^{(1)} \geq 0\}}, \dots, \mathbf{1}_{\{u_e^{(k_e)} \geq 0\}} \right).$$

Recall also, that for any $f \in H^1$ one can write

$$f^+ = \mathbf{1}_{\{f \geq 0\}} f, \quad -f^- = \mathbf{1}_{\{f \leq 0\}} f, \quad \text{and} \quad (f^+)' = \mathbf{1}_{\{f \geq 0\}} f'.$$

Proposition 4.11. *In addition to the assumptions of Theorem 4.1, let M_e, N_e, Q_e be real-valued for all $e \in E$, $Q_e(x)$ be diagonal for $e \in E$ and all $x \in [0, \ell_e]$, and*

$$\xi^+ \in \bigoplus_{v \in V} Y_v \text{ for all } \xi \in \bigoplus_{v \in V} Y_v. \quad (4.16)$$

Then the semigroup generated by \mathcal{A} on $L^2(\mathcal{G}, \mathbb{R})$ is positive if and only if for all $e \in E$ the matrices $M_e(x)$ are diagonal for all $x \in [0, \ell_e]$, and all off-diagonal entries of the matrices $N_e(x)$ are nonnegative for a.e. $x \in [0, \ell_e]$.

Proof. By Lemma 4.10 we may apply Lemma 4.5 to the closed convex subset $K = \mathbf{L}^2(\mathcal{G}, \mathbb{R}_+)$. Therefore, by diagonality and positivity of $Q_e(x)$, the minimizing projector P_K given by (4.10) takes the simpler form

$$P_K^Q u = Q^{-\frac{1}{2}} \left(Q^{\frac{1}{2}} u \right)^+ = u^+. \quad (4.17)$$

Since real scalar perturbations of the generator do not affect positivity of the semigroup, we may assume that $\omega = 0$. As (4.16) yields $P_K u \in D(\mathcal{A})$ for all $u \in D(\mathcal{A})$, the semigroup is thus positive if and only if

$$(\mathcal{A}u^+, (u - u^+)) \leq 0 \quad \text{for all } u \in D(\mathcal{A}).$$

By applying (3.6) and (4.17) we obtain the equivalent condition

$$\sum_{e \in E} \int_0^{\ell_e} Q_e (M_e(u_e^+)' + N_e u_e^+) \cdot (u_e - u_e^+) dx \leq 0 \quad \text{for all } u \in D(\mathcal{A}), \quad (4.18)$$

which we rewrite using the characteristic functions as

$$\sum_{e \in E} \int_0^{\ell_e} (Q_e M_e \mathbf{1}_{\{u_e \geq 0\}} u_e' + Q_e N_e \mathbf{1}_{\{u_e \geq 0\}} u_e) \cdot (\mathbf{1}_{\{u_e \leq 0\}} u_e) dx \leq 0 \quad \text{for all } u \in D(\mathcal{A}). \quad (4.19)$$

The conclusion then follows from Lemma 4.13 below that furnishes an equivalent, but easier to check, formulation of (4.19). \square

Remark 4.12. In the same way as in Lemma 4.9, by replacing the real and imaginary by the positive and negative part, respectively, we can see that (4.16) holds if and only if, for each $v \in V$, Y_v is spanned by vectors with positive entries only.

Let us prove the last step that is still missing in the proof of Proposition 4.11.

Lemma 4.13. *Let $Q_e(x)$ be diagonal for all $e \in E$ and for all $x \in [0, \ell_e]$. Then (4.19) holds if and only if for all $e \in E$ the matrices $M_e(x)$ are diagonal for all $x \in [0, \ell_e]$, and all off-diagonal entries of the matrices $N_e(x)$ are nonnegative for a.e. $x \in [0, \ell_e]$.*

Proof. First observe that since $Q_e(x)$ are all diagonal matrices with strictly positive diagonal elements, $M_e(x)$ is diagonal if and only if $Q_e(x)M_e(x)$ is diagonal and $N_e(x)$ has nonnegative off-diagonal entries if and only if the same holds for $Q_e(x)N_e(x)$. Obviously, these properties imply that

$$\begin{aligned} \sum_{e \in E} \int_0^{\ell_e} (Q_e M_e \mathbf{1}_{\{u_e \geq 0\}} u_e') \cdot (\mathbf{1}_{\{u_e \leq 0\}} u_e) dx &= 0 \quad \text{and} \\ \sum_{e \in E} \int_0^{\ell_e} (Q_e N_e \mathbf{1}_{\{u_e \geq 0\}} u_e) \cdot (\mathbf{1}_{\{u_e \leq 0\}} u_e) dx &\leq 0. \end{aligned}$$

yielding (4.19).

Conversely, assume that (4.19) holds. Introduce function $\varphi: \mathbb{R} \rightarrow \mathbb{R}_+$,

$$\varphi(y) := \begin{cases} y + 1 & \text{if } y \in [-1, 0], \\ 1 - y & \text{if } y \in [0, 1], \\ 0 & \text{else,} \end{cases}$$

and define

$$\psi_- : \mathbb{R} \rightarrow \mathbb{R}_+, y \mapsto \varphi(2y + 1), \quad \psi_+ : \mathbb{R} \rightarrow \mathbb{R}_+, y \mapsto \varphi(2y - 1).$$

Notice that

$$\int_{-1}^1 \varphi'(y) \psi_-(y) dy > 0, \quad \int_{-1}^1 \varphi'(y) \psi_+(y) dy < 0. \quad (4.20)$$

Now fix one edge $\mathbf{e} \in \mathbf{E}$, one point $x_0 \in (0, \ell_{\mathbf{e}})$ and $1 \leq i, j \leq k_{\mathbf{e}}$, $i \neq j$. Then for all n large enough, define $v_{n,\pm}$ as follows: $v_{n,\pm,\mathbf{e}'} = 0$, for all edges $\mathbf{e}' \neq \mathbf{e}$ and $v_{n,\pm,\mathbf{e}} = (v_n^{(1)}, \dots, v_n^{(k_{\mathbf{e}})})^\top$ with only two nonzero entries: $v^{(i)}, v^{(j)}$ defined by

$$v_n^{(i)}(x) = \varphi(n(x - x_0)) \text{ and } v_n^{(j)}(x) = -\psi_{\pm}(n(x - x_0)).$$

The parameter n is chosen large enough so that the support of $v_n^{(i)}, v_n^{(j)}$ is included into $(0, \ell_{\mathbf{e}})$ and $u_{n,\pm} \in D(\mathcal{A})$, as it vanishes in a neighborhood of each vertex. Plugging this test-function into (4.19), dividing the expression by n and letting n goes to infinity, we find that

$$(Q_{\mathbf{e}}(x_0)M_{\mathbf{e}}(x_0))_{ji} \int_{-1}^1 \varphi'(y) \psi_{\pm}(y) dy \geq 0.$$

With the help of (4.20), we deduce that $Q_{\mathbf{e}}(x_0)M_{\mathbf{e}}(x_0)$ is diagonal.

Taking this into account, (4.19) reduces to

$$\sum_{\mathbf{e} \in \mathbf{E}} \int_0^{\ell_{\mathbf{e}}} (Q_{\mathbf{e}}N_{\mathbf{e}}\mathbf{1}_{\{u_{\mathbf{e}} \geq 0\}}u_{\mathbf{e}}) \cdot (\mathbf{1}_{\{u_{\mathbf{e}} \leq 0\}}u_{\mathbf{e}}) dx \leq 0. \quad (4.21)$$

Now fix one edge $\mathbf{e} \in \mathbf{E}$, and $1 \leq i < j \leq k_{\mathbf{e}}$. Define v as follows: $v_{\mathbf{e}'} = 0$, for all edges $\mathbf{e}' \neq \mathbf{e}$ and $v_{\mathbf{e}} = (v^{(1)}, \dots, v^{(k_{\mathbf{e}})})^\top \in \mathcal{D}(0, \ell_{\mathbf{e}})^{k_{\mathbf{e}}}$ with only two nonzero entries: $v_n^{(i)}, v_n^{(j)}$ defined by

$$v^{(i)}(x) = -|\chi(x)| \text{ and } v^{(j)}(x) = |\chi(x)|,$$

with $\chi \in \mathcal{D}(0, \ell_{\mathbf{e}})$. As before this function belongs to $D(\mathcal{A})$, and with this choice of v in (4.21) we find that

$$\int_0^{\ell_{\mathbf{e}}} (Q_{\mathbf{e}}N_{\mathbf{e}})_{ij} \chi^2 dx \geq 0.$$

Since this holds for all $\chi \in \mathcal{D}(0, \ell_{\mathbf{e}})$, by Lemma B.2, we conclude that $(Q_{\mathbf{e}}(x)N_{\mathbf{e}}(x))_{ij} \geq 0$ for all $i \neq j$ and all $x \in [0, \ell_{\mathbf{e}}]$. \square

Remark 4.14. As a simple corollary, if $Q_e(x)$, $M_e(x)$, and $N_e(x)$ are all diagonal, (4.19) is automatically satisfied and we are left with (4.16), a condition depending on the boundary conditions only, a result reminiscent of Proposition 5.1 in [15] in the case of *parabolic* systems.

Example 4.15. If Dirichlet conditions are imposed on all endpoints ℓ_e (resp., on all endpoints 0) and $Q_e(0)M_e(0)$ is positive semidefinite (resp. $Q_e(\ell_e)M_e(\ell_e)$ is negative semidefinite) for all e , then the corresponding nonpositive isotropic subspaces Y_v are isomorphic to a direct product of k_v blocks of size k_e that are either zero or \mathbb{C}^{k_e} . The same holds for subspaces Y_v^\perp but with the opposite pattern. Since each edge e has exactly one initial endpoint 0 and one terminal endpoint ℓ_e , we get exactly one corresponding nonzero block ($\equiv \mathbb{C}^{k_e}$) in all the spaces \widetilde{Y}_v^\perp , $v \in V$. Hence,

$$\dim \sum_{v \in V} \widetilde{Y}_v^\perp = \sum_{v \in V} \dim Y_v = \sum_{e \in E} k_e = k.$$

Since also $\gamma_v(u) \in Y_v$ for all $v \in V$, by Theorem 4.1, \mathcal{A} generates a strongly continuous quasi-contractive semigroup.

Example 4.16. The boundary conditions considered in Example 4.15 do *de facto* turn the network into a collection of decoupled intervals. Instead, we now use the setting described in Section 3.2 and impose conservation of mass conditions, which result in a global boundary condition $u(\ell) = Wu(0)$ for some column stochastic block diagonal matrix W with $|E|$ blocks of sizes $k_e \times k_e$. We have

$$\begin{aligned} Y &= \left\{ \left(\alpha_{e_1}, \alpha_{e_2}, \dots, \alpha_{e_{|E|}}, \sum_{e \in E} w_{1e} \alpha_e, \dots, \sum_{e \in E} w_{|E|e} \alpha_e \right)^\top : \alpha_{e_i} \in \mathbb{C}^{k_{e_i}}, e_i \in E \right\} \\ &= \left\{ (\alpha, W\alpha)^\top : \alpha \in \mathbb{C}^k \right\} = \ker \begin{pmatrix} -W & I \end{pmatrix}, \end{aligned}$$

hence condition (3.42) is not satisfied: indeed, 1 is an eigenvalue of W and therefore its associated eigenvector α_0 yields a non trivial element $(\alpha_0, \alpha_0)^\top$ in $Y \cap K$, hence (3.42) cannot hold due to Lemma 3.12. But if M is diagonal or spatially constant and positive definite, then we can use condition (4.5) that in the case of our choice of Y reduces to the invertibility of the matrix W , which agrees with Proposition 2.1 in [40]. Then, under these additional assumptions, \mathcal{A} generates a contractive semigroup provided

$$(-Q(0)M(0) + W^\top Q(\ell)M(\ell)W) \alpha \cdot \bar{\alpha} \leq 0 \quad \text{for all } \alpha \in \mathbb{C}^k.$$

where $Q(0) := \text{diag}(Q_e(0))_{e \in E}$ (resp. $Q(\ell) := \text{diag}(Q_e(\ell_e))_{e \in E}$) and $M(0) := \text{diag}(M_e(0))_{e \in E}$ (resp. $M(\ell) := \text{diag}(M_e(\ell_e))_{e \in E}$).

If the matrices Q_e, M_e are diagonal and real-valued, and if moreover the diagonal and off-diagonal entries of $N_e(x)$ are for all $e \in E$ and a.e. $x \in [0, \ell_e]$ real and nonpositive, respectively, then the semigroup generated by \mathcal{A} is positive. The case of spatially constant, and diagonal matrices with positive entries on $L^1(\mathcal{G})$ corresponds to flows in networks as studied in [42], see also [40] and the references there. There, W is the generalized adjacency matrix of the line graph.

We conclude this section by elaborating on a comparison principle between semigroups. Let A_1, A_2 be two operators, each generating a positive semigroup – say, $(T_1(t))_{t \geq 0}, (T_2(t))_{t \geq 0}$. Then $(T_2(t))_{t \geq 0}$ is said to *dominate* $(T_1(t))_{t \geq 0}$ if

$$T_1(t)f \leq T_2(t)f \quad \text{for all } f \geq 0 \text{ and all } t \geq 0.$$

We can now formulate the following.

Proposition 4.17. *Let two operators A_1, A_2 with domain*

$$\begin{aligned} D(A_1) &:= \{u \in D_{\max} : \gamma(u) \in Y_1\}, \\ D(A_2) &:= \{u \in D_{\max} : \gamma(u) \in Y_2\} \end{aligned}$$

satisfy our standing assumptions as well as the assumptions of Theorem 4.1. Let the semigroups generated by A_1, A_2 , say, $(T_1(t))_{t \geq 0}, (T_2(t))_{t \geq 0}$, be both real and positive. Then $(T_2(t))_{t \geq 0}$ dominates $(T_1(t))_{t \geq 0}$ if Y_1 is an ideal of Y_2 , i.e., $Y_1 \subset Y_2$ and

$$0 \leq \xi \leq \chi \text{ with } \xi \in Y_2 \text{ and } \chi \in Y_1 \text{ implies } \xi \in Y_1.$$

and additionally

$$(A_1 u, v) \leq (A_2 u, v) \quad \text{for all } u, v \in D(A_1) \text{ s.t. } u, v \geq 0.$$

Proof. First of all, observe that domination is not affected if A_1, A_2 are rescaled by the same real scalar ω ; hence, we can without loss of generality assume both A_1, A_2 to be dissipative. The proof can then be performed combining Lemma 4.5 and Theorem 2.24 of [58]. \square

An interesting case arises when A_1, A_2 are defined by means of the same matrices M, Q, N , and only the boundary conditions are different, i.e., $Y_1 \neq Y_2$. This is e.g. the case if Y_1 is the space spanned by $\Pi v_1, \dots, \Pi v_m$, where v_1, \dots, v_m are the vectors spanning Y_2 and the projector Π is a block diagonal operator matrix whose diagonal blocks are a zero matrix (of any size ≥ 1) and an identity matrix (of any size $\leq 2|\mathbf{E}|$). In the case of the transport equation on a network studied in [42], this corresponds to comparing a given network with a new network with additional Dirichlet conditions in some vertices.

We may discuss in a similar way the issue of L^∞ -contractivity, i.e., the invariance of $\mathbf{L}^2(\mathcal{G}; C)$ for $C := [-1, 1]$ under the semigroup generated by \mathcal{A} . We omit the details.

5. EXAMPLES

5.1. Linearized Saint-Venant models

Here we study a system where the linearized Saint-Venant model (2.6) is considered on all the J edges of a network, and hence the unknown $(h, u)^\top$ is replaced by $(h_e, u_e)^\top_{e \in \mathbf{E}}$. While our approach applies to arbitrary networks and variable functions H, V that may differ across the edges, for the sake of simplicity we here restrict to the case of constant (real) coefficients H, V that are independent of the edges, to $H > 0$ (which is physically reasonable), and to a star-shaped network with J edges, for some integer $J \geq 2$, see Figure 2. More precisely, we let $\mathbf{E} := \{e_1, \dots, e_J\}$, and identify each edge e_j with $(0, \ell_j)$ (the parametrization of e_j is determined by the arrow in Fig. 2): v_0 will correspond to the endpoint ℓ_1 for e_1 and to the initial point 0 for all other edges. The external vertex of e_j will be denoted by $v_j, j = 1, \dots, J$.

Now we need to fix the boundary conditions at all vertices. According to our approach they are related to the operators T_v defined in (3.9). In our case we have (see (2.8))

$$Q_e M_e = - \begin{pmatrix} gV & gH \\ gH & HV \end{pmatrix} \equiv: -B.$$

Because the matrix B is independent of e , and also symmetric and invertible due to condition (2.7), we thus have

$$T_{v_1} = B, \text{ and } T_{v_i} = -B \text{ for all } i \geq 2$$

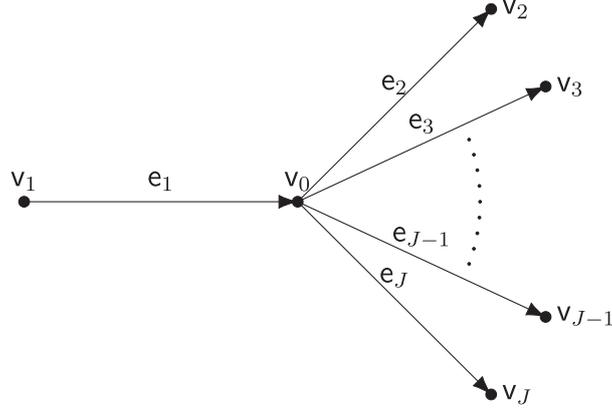


FIGURE 2. A star-shaped network with one incoming and $J - 1$ outgoing edges.

at the external vertices v_i , $i \geq 1$, while at the interior vertex v_0 T_{v_0} is a block diagonal matrix:

$$T_{v_0} = \text{diag}(B, -B, \dots, -B).$$

Now, let us notice that the two eigenvalues λ_{\pm} of B satisfy

$$\lambda_+ \lambda_- = gH(V^2 - gH).$$

Hence, under the subcritical flow condition $gH - V^2 > 0$ (see [6], p. 14), λ_+ and λ_- are of opposite sign. On the contrary, under the supercritical flow condition $gH - V^2 < 0$, λ_+ and λ_- have the same sign; but as $\lambda_+ + \lambda_- = (g + H)V > 0$, in this case they are both strictly positive.

Now, we distinguish between these two flow conditions.

- 1) If $gH - V^2 < 0$, then at the external vertices the only choice for a totally isotropic subspace Y_{v_i} associated with T_{v_i} is $\langle (0, 0)^\top \rangle$. This already yields $2J$ boundary conditions and there is no more freedom to manage the internal vertex v_0 . In other words, under this choice of boundary conditions the associated operator \mathcal{A} cannot generate a group.

But we may hope for the generation of a semigroup. Hence as $T_{v_1} = B$ has two positive eigenvalues we surely need to impose that $Y_{v_1} = \{(0, 0)^\top\}$, *i.e.*,

$$h(v_1) = u(v_1) = 0, \tag{5.1}$$

while we are free to impose boundary conditions or not at v_j , for all $j \geq 2$. Since T_{v_0} has 2 positive eigenvalues and $2(J - 1)$ negative eigenvalues, by Lemma C.3, the maximal dimension of a subspace Y_{v_0} of the nonpositive isotropic cone associated with T_{v_0} is $2(J - 1)$. Choosing Y_{v_0} as the subspace associated with the negative eigenvalues leads to a decoupled system and is of less interest. Letting instead

$$Y_{v_0} = \text{span}\{(1, 0, 1, 0, \dots, 1, 0)^\top, (0, 1, 0, 1, \dots, 0, 1)^\top\},$$

we observe that

$$T_{v_0} \xi \cdot \bar{\xi} = (2 - J)B(x, y)^\top \cdot (\bar{x}, \bar{y})^\top \leq 0,$$

for all $\xi \in Y_{v_0}$, *i.e.*, $\xi = x(1, 0, 1, 0, \dots, 1, 0)^\top + y(0, 1, 0, 1, \dots, 0, 1)^\top$ for some $x, y \in \mathbb{C}$, since B is positive definite. This choice corresponds to the continuity of the water depth and the velocity at v_0 , namely

$$h_j(0) = h_1(\ell_1), \quad v_j(0) = v_1(\ell_1) \quad \text{for all } j \geq 2$$

and yields $2(J-1)$ boundary conditions. They are complemented by the two conditions (5.1) at v_1 , and by no conditions at v_i for $i \geq 2$. This leads to $k = 2J$ boundary conditions for which

$$T_v \xi \cdot \bar{\xi} \leq 0, \quad \text{for all } \xi \in Y_v, v \in \mathcal{V}.$$

To conclude the generation of a semigroup by Theorem 4.1 it suffices to notice that (3.19) is valid because $\widetilde{Y}_{v_1}^\perp = \mathbb{C}^2 \times \{(0, 0)\}^{J-1}$ is clearly independent of $\widetilde{Y}_{v_0}^\perp$, while $\widetilde{Y}_{v_j}^\perp$ are trivial for all $j \geq 2$, so (3.21) holds, and

$$\dim Y_{v_0} = 2, \quad \dim Y_{v_1} = 0, \quad \text{and} \quad \dim Y_{v_j} = 2 \text{ for all } j \geq 2.$$

Furthermore, by Lemma 4.8 and Proposition 4.7, the semigroup is real. Finally, by Lemma 4.13 condition (4.19) does not hold, hence the semigroup is not positive.

- 2) If we are in the subcritical case $gH - V^2 > 0$, then the eigenvalue λ_+ (resp. λ_-) is positive (resp. negative). Let us first analyze the possibility to have a group. In that case, by Lemma C.2, at any external vertex v a totally isotropic subspace of q_v is of dimension at most one, while at the interior vertex v_0 it is at most J . Let us present the following example. If U_\pm is the normalized eigenvector of B associated with λ_\pm , then according to (C.8),

$$U = U_- + i\sqrt{\frac{\lambda_-}{\lambda_+}} U_+$$

is an isotropic vector of the sesquilinear form associated with B . Therefore for all $j = 1, \dots, J$, we take Y_{v_j} as the vector space spanned by U . We proceed similarly at v_0 by fixing J isotropic vectors constructed in the proof of Lemma C.2. To have a coupling system, one possibility is the following one. We notice that the eigenvectors of T_{v_0} associated with positive eigenvalues are

$$U_1^+ = (U_+^\top, 0, \dots, 0)^\top$$

for the eigenvalue λ_+ and

$$U_2^+ = (0, U_-^\top, 0, \dots, 0)^\top, \dots, U_J^+ = (0, 0, \dots, 0, U_-^\top)^\top$$

for the eigenvalue $-\lambda_-$. Similarly, the eigenvectors of T_{v_0} associated with negative eigenvalues are

$$U_1^- = (U_-^\top, 0, \dots, 0)^\top$$

for the eigenvalue λ_- and

$$U_2^- = (0, U_+^\top, 0, \dots, 0)^\top, \dots, U_J^- = (0, 0, \dots, 0, U_+^\top)^\top$$

for the eigenvalue $-\lambda_+$. We can now take Y_{v_0} as the vector space spanned by $U_J^- + U_1^+$ and by vectors $U_j^- + U_{j+1}^+$ for all $j = 1, \dots, J-1$. Then $\widetilde{Y}_{v_0}^\perp$ is clearly independent of $\{\widetilde{Y}_{v_i}^\perp : i = 1, \dots, J\}$ which is also an independent set itself, and (3.19) holds since $\sum_{v \in \mathcal{V}} \dim Y_v = 2J$. Furthermore, each Y_v is by construction

a totally isotropic subspace associated with the quadratic form q_v , hence we are in the position to apply Theorem 3.7 and deduce that the associated operator \mathcal{A} generates a group, which is by Corollary 3.9 is unitary if and only if $V = 0$. As before the (forward) semigroup is real and does not preserve positivity. In the subcritical case $gH - V^2 > 0$, examples of boundary conditions leading to a semigroup can be easily built as before.

5.2. Wave type equations

Wave-type equations on graphs have been intensively studied in the literature, let us mention [2, 38, 45, 48, 54, 57]. Here we focus on extending these results to rather general elastic systems modeled as

$$\ddot{u}_e(t, x) = u_e''(t, x) + \alpha_e \dot{u}_e'(t, x) + \beta_e \dot{u}_e(t, x) + \gamma_e u_e'(t, x), \quad t \geq 0, \quad x \in (0, \ell_e), \quad (5.2)$$

where $\alpha_e \in C^1([0, \ell_e])$ and $\beta_e, \gamma_e \in L^\infty(0, \ell_e)$ are real-valued functions. For the sake of simplicity, we hence restrict to stars as in Figure 2, which can be regarded as building blocks of more general networks. It turns out that (5.2) is equivalent to

$$\dot{U}_e = M_e U_e' + N_e U_e,$$

for the vector function $U_e = (u_e', \dot{u}_e)^\top$, where

$$M_e = \begin{pmatrix} 0 & 1 \\ 1 & \alpha_e \end{pmatrix}, \quad N_e = \begin{pmatrix} 0 & 0 \\ \gamma_e & \beta_e \end{pmatrix}.$$

As M_e is symmetric, Assumptions 2.1 are automatically satisfied by choosing Q_e as the identity matrix. As usual, the boundary conditions at the vertices are related to the values of M_e at the endpoints of the edge e , that generically are given by

$$M_e(\mathbf{v}) = \begin{pmatrix} 0 & 1 \\ 1 & \alpha_e(\mathbf{v}) \end{pmatrix},$$

when \mathbf{v} is one of the endpoints of e ; hence $M_e(\mathbf{v})$ has two real eigenvalues of opposite sign,

$$\lambda_{\pm} = \frac{1}{2} \left(\alpha_e(\mathbf{v}) \pm \sqrt{\alpha_e(\mathbf{v})^2 + 4} \right).$$

We are thus in the same situation as in the Saint-Venant model with the subcritical condition: we thus do not give any further details about the choice of boundary conditions at the interior vertices, since all ideas presented there carry over to the present case. Note that for an exterior vertex \mathbf{v} , Neumann condition

$$u_e'(\mathbf{v}) = 0,$$

can be equivalently described by means of the totally isotropic subspace Y_v spanned by $(0, 1)^\top$. In this case, $(0, 1)^\top$ will be an isotropic vector if and only if $\alpha_e(\mathbf{v}) = 0$. On the contrary, Dirichlet boundary condition at an exterior vertex \mathbf{v}

$$u_e(\mathbf{v}) = 0,$$

leads to $\dot{u}_e(\mathbf{v}) = 0$, and corresponds to the choice $Y_{\mathbf{v}}$ spanned by $(1, 0)^\top$. Our approach also allows us to discuss absorbing boundary condition (see [19] for instance))

$$u'_e(\ell_e) = -\kappa \dot{u}_e(\ell_e)$$

with $\kappa \in (0, \infty)$: indeed it then corresponds to the space spanned by $U = (-\kappa, 1)^\top$, hence

$$M_e(\mathbf{v})U \cdot \bar{U} = \alpha_e(\mathbf{v}) - 2\kappa \leq 0,$$

that will be nonpositive as soon as $\kappa \geq \alpha_e(\mathbf{v})/2$.

Let us finally notice that provided the boundary conditions are nice enough to generate a group, it will be unitary group if and only if $\gamma_e = 0$ and $\beta_e = \frac{\alpha'_e}{2}$, for every edge e . In case of a generation of a semigroup, it will be real provided (4.14) holds, while if all γ_e are nonpositive, it will be never positive (this is in particular the case for the wave equation, as $\alpha_e = \beta_e = \gamma_e = 0$).

5.3. Hybrid transport/string equations

Network-like systems described by equations which are partially of diffusive and partially of transport type have been studied by Hussein and one of the authors in [32]; characterization of the right transmission conditions leading to well-posedness has proved a difficult task. In the following we turn to the different but related task of connecting transport and wave equations; this can suggest natural ways of coupling first and second order differential operators.

The simplest toy model is to consider a scalar wave equation set in an interval $(0, \ell_2)$ and a scalar transport equation in $(0, \ell_1)$ coupled *via* their common endpoint that is assumed to be 0. This means that we consider the system

$$\begin{cases} \ddot{u}(t, x) = u''(t, x), & t \geq 0, x \in (0, \ell_2), \\ \dot{p}(t, x) = -p'(t, x), & t \geq 0, x \in (0, \ell_1), \end{cases} \quad (5.3)$$

with Dirichlet boundary condition at ℓ_2

$$u(t, \ell_2) = 0, \quad (5.4)$$

and the transmission condition at 0

$$u'(t, 0) = \alpha p(t, 0), \quad \dot{u}(t, 0) = \beta p(t, 0), \quad (5.5)$$

where α, β are two real numbers fixed below in order to guarantee well-posedness of our system.

As in Section 5.2, by identifying the interval $(0, \ell_i)$ with e_i , $i = 1, 2$, introducing $u_{e_1} = (u', \dot{u})^\top = (u_{e_1,1}, u_{e_1,2})^\top$, and setting $u_{e_2} = p$, we can transform our system into a first order system of the form (2.1) with

$$M_{e_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and therefore} \quad Q_{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $M_{e_2} = -1$ (hence $Q_{e_2} = 1$).

Notice that our network possesses three vertices, \mathbf{v}_0 , corresponding to 0 (with $E_{\mathbf{v}_0} = \{e_1, e_2\}$), \mathbf{v}_1 , corresponding to the endpoint ℓ_1 of e_1 (with $E_{\mathbf{v}_1} = \{e_1\}$), and \mathbf{v}_2 , corresponding to the endpoint ℓ_2 of e_2 (with $E_{\mathbf{v}_2} = \{e_2\}$). Since there is no boundary condition at \mathbf{v}_1 , we set $Y_{\mathbf{v}_1} = \mathbb{C}^2$, to take into account (5.4),

we set $Y_{v_2} = \mathbb{C} \times \{0\}$, and (5.5) requires to take

$$Y_{v_0} = \text{span}\{(\alpha, \beta, 1)^\top\}.$$

With this notation, we see that

$$\begin{aligned} q_{v_1}(x) &= -|x|^2, & x \in \mathbb{C}^2, \\ q_{v_2}(\xi) &= 0, & \xi \in Y_{v_2}, \end{aligned}$$

while

$$q_{v_0}(\xi) = (1 - 2\alpha\beta)|x_3|^2 \quad \text{for all } \xi = (x_1, x_2, x_3)^\top \in Y_{v_0}.$$

This means that Y_{v_0} is a subspace of the nonpositive isotropic cone associated with q_{v_0} if and only if $2\alpha\beta \geq 1$.

Finally, as

$$\widetilde{Y_{v_1}^\perp} = \{0\} \times \{0\} \times \{0\}, \quad \widetilde{Y_{v_2}^\perp} = \{0\} \times \mathbb{C} \times \{0\}, \quad \widetilde{Y_{v_0}^\perp} = \text{span}\{(1, 0, -\alpha)^\top, (0, 1, -\beta)^\top\},$$

condition (3.19) is fulfilled if $\beta \neq 0$. Therefore, system (5.3) with the boundary/transmission conditions (5.4)–(5.5) is governed by a strongly continuous semigroup if $2\alpha\beta \geq 1$. The semigroup is real but not positive.

A toy model of a transport process sandwiched between two diffusive ones, all three taking place on intervals of length 1, was proposed in Section 2 of [32] (we refer to that paper for an interpretation of such a model in terms of delayed equations and for a possible biologic motivation). If diffusion is replaced by a wave equation, the corresponding hyperbolic system satisfies the Assumptions 2.1 with

$$M := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and } N \equiv 0, \quad \text{hence } Q := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} :$$

where the unknown is

$$\mathbf{u} := (u', \dot{u}, p, v', \dot{v})^\top.$$

Here we have two interior vertices v_1, v_2 : the former corresponds to the endpoint 0 of both $(0, \ell_1)$ and $(0, \ell_2)$, while the latter corresponds to the endpoint 0 of $(0, \ell_3)$ and the endpoint ℓ_1 of $(0, \ell_1)$. Hence imposing conditions (5.4)–(5.5) as before along with Dirichlet boundary condition at ℓ_3

$$v(t, \ell_3) = 0, \tag{5.6}$$

and the transmission condition at v_2

$$v'(t, 0) = \gamma p(t, \ell_1), \quad \dot{v}(t, 0) = \delta p(t, \ell_1), \tag{5.7}$$

with two real numbers γ, δ , one can show well-posedness of this problem if $2\alpha\beta \geq 1$, $2\gamma\delta \geq -1$ and $\gamma \neq 0$. The semigroup is, again, real but not positive.

5.4. Hybrid string/beam equations

Ammari *et al.* [3, 5] have proposed models that consist of several combinations of strings and beams. In particular, in [3] a collection of 1 wave and N beam equations is considered on a star graph. The necessary $4N + 2$ conditions consist of the following;

- $N + 1$ transmission conditions: continuity of all solutions at the star's center along with a Kirchhoff-type condition on the beams' shear forces (third derivative of solutions) and the strings' flux (first derivative);
- $3N + 1$ boundary conditions: zero conditions on the beams' slopes and bending moments (first and second derivatives, respectively) along with closed feedbacks on the beams' shear forces and the strings' fluxes.

This model trivially satisfies Assumptions 2.1 with

$$M_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M_e = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

in the case of the string-like and beam-like edges, respectively. In this case, Theorem 4.1 applies and we deduce that the hyperbolic system is governed by a strongly continuous semigroup. Indeed, arguing as in Section 3.2 we deduce that this semigroup is contractive, hence the energy of solutions is decreasing.

The aim in [3] was to discuss the stabilization of an elastic system: the reason why it makes sense to consider closed feedbacks is that if they are replaced by zero conditions, then in view of the duality between continuity and Kirchhoff conditions, a direct computation shows the assumptions of Corollary 3.9 are satisfied and we conclude that the system is governed by a unitary group.

5.5. The Dirac equation

The 1D Dirac equation is briefly discussed in Section 1.1 of [66]: it was later extended to the case of networks and thoroughly studied by Bolte and his coauthors [9, 10], who also observed that it then takes on each edge the form

$$i\hbar \frac{\partial}{\partial t} \psi = \left(\hbar c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} + mc^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \psi \quad \text{in } (-\infty, \infty) \times (-\infty, \infty)$$

for a \mathbb{C}^2 -valued unknown $\psi = (\psi^{(1)}, \psi^{(2)})$. A parametrization of skew-adjoint realizations on a network has been presented in [9]; we are going to study the more general problem of finding boundary conditions that lead to a group or merely a semigroup, which still yields *forward* well-posedness of the Dirac equation.

To begin with, observe that Assumptions 2.1 are especially satisfied by taking

$$M_e = \begin{pmatrix} 0 & ic \\ -ic & 0 \end{pmatrix}, \quad Q_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad N_e = \begin{pmatrix} -i\frac{mc^2}{\hbar} & 0 \\ 0 & i\frac{mc^2}{\hbar} \end{pmatrix}, \quad e \in E.$$

We hence deduce from Lemma 4.8 that, no matter what the boundary/transmission conditions look like, a semigroup governing the Dirac equation cannot be real, let alone positive.

Let us now study the quadratic form q_v . We observe that by (3.9), T_v is $k_v \times k_v$ block-diagonal matrix whose diagonal blocks equal $\pm M_e$, according to the appropriate value ι_{ve} of the incidence matrix. Hence, if we write

$$\gamma_v(\psi) = (\psi_e^{(1)}(v), \psi_e^{(2)}(v))_{e \in E_v}^\top =: (\xi_e, \eta_e)_{e \in E_v}^\top \in \mathbb{C}^{k_v},$$

then this vector is an isotropic vector for the associated quadratic form q_v if and only if

$$\sum_{e \in E} \iota_{ve} \Im(\xi_e \cdot \bar{\eta}_e) = 0. \quad (5.8)$$

Any set of vector spaces $Y_v \subseteq \mathbb{C}^{k_v}$, consisting of vectors $(\xi_e, \eta_e)_{e \in E_v}^\top$ that satisfy (5.8) possibly induces boundary conditions that determine a realization of \mathcal{A} generating a group; in fact, necessarily a *unitary* group, since (3.28) is clearly satisfied. A well-known example is that of vectors ξ being scalar multipliers of the “characteristic function” $\mathbf{1}_{E_v} := \mathbf{1}_{\{e \in E_v\}}$ and η with the same support E_v and orthogonal to ξ , where $E_v \subset E$ is the set of edges incident with any given vertex v . Recalling the notation ι_{E_v} from Example 3.4, this gives rise to

$$Y_v := \text{span}\{\mathbf{1}_{E_v}\} \oplus \text{span}\{\iota_{E_v}\}^\perp,$$

corresponding to continuity of the first coordinate of ψ across all vertices, and a Kirchhoff condition on the second coordinate, (clearly, swapping the transmission conditions in vertices satisfied by the two coordinates yields again a group generator); in this case, the condition (3.19) need not be satisfied. However, we can apply Theorem 3.11, since this choice of Y_v implies that $T_v^{-1}Y_v^\perp = Y_v$. Another possibility is *e.g.* given by letting $Y_v := \{(\alpha_1, \alpha_1, \dots, \alpha_{|E_v|}, \alpha_{|E_v|})^\top : \alpha \in \mathbb{C}^{k_v}\}$, corresponding to $\psi_e^{(1)}(v) = \psi_e^{(2)}(v)$.

If we turn to the issue of mere contractive well-posedness, then we observe that any non positive isotropic cone consists of vectors such that

$$\sum_{e \in E} \iota_{ve} \Im(\xi_e \bar{\eta}_e) \geq 0; \quad (5.9)$$

accordingly, any Y_v all of whose elements satisfy (5.9) is a candidate for generation of a semigroup. Indeed, by Theorem 4.1 such a choice of Y_v induces a realization of the operator \mathcal{A} that generates a strongly continuous contractive semigroup if additionally (3.19) holds or all elements of $T_v^{-1}Y_v^\perp$ satisfy (5.9). A somewhat trivial example is given by $Y_v := \{0\} \oplus \mathbb{C}^{|E_v|}$, corresponding to decoupled case of Dirichlet boundary conditions imposed at an endpoint of each interval on both coordinates of the unknown; or, more generally (and interestingly) of $Y_v = \{(\iota B \eta, \iota_{E_v} \odot \eta) : \eta \in \mathbb{C}^{k_v}\}$ for some matrix B of size k_v with only purely imaginary eigenvalues (implying that B and B^* are accretive and dissipative) since $Y_v^\perp = \{(\eta, u_{E_v} \odot B^* \eta) : \eta \in \mathbb{C}^{k_v}\}$, where for two vectors $a = (a_e)_{e \in E}$ and $b = (b_e)_{e \in E}$, we recall that $a \odot b$ means the Hadamard product of a and b defined by

$$a \odot b = (a_e b_e)_{e \in E}.$$

5.6. Second sound in networks

So-called “second sound” is an exotic, wave-like phenomenon of heat diffusion that was first proposed by Landau to explain unusual behaviors in ultracold helium. Second sound has ever since been observed in several materials – most recently by Huberman *et al.* [31] also in graphite around cozy $130^\circ K$. As thoroughly discussed in [62], one classical model going back to Lord and Shulman [49] boils down to the linear equations of thermoelasticity

$$\begin{cases} \ddot{z} - \alpha z'' + \beta \theta' & = 0 & \text{in } (0, \ell) \times (0, +\infty), \\ \dot{\theta} + \gamma q' + \delta \dot{z}' & = 0 & \text{in } (0, \ell) \times (0, +\infty), \\ \tau_0 \dot{q} + q + \kappa \theta' & = 0 & \text{in } (0, \ell) \times (0, +\infty), \end{cases} \quad (5.10)$$

where z , θ , and q represent the displacement, the temperature difference to a fixed reference temperature, and the heat flux, respectively, and $\alpha, \beta, \gamma, \delta, \tau_0, \kappa$ are positive constants. Racke has discussed in [62] the asymptotic stability of this system under three classes of boundary conditions:

- (i) $z(0) = z(\ell) = q(0) = q(\ell) = 0$,
- (ii) $z(0) = z(\ell) = \theta(0) = \theta(\ell) = 0$,
- (iii) $\alpha z'(0) = \beta \theta(0)$, $\theta'(0) = 0$, $z(\ell) = \theta(\ell) = 0$,

proving in detail well-posedness in the case of (i) and suggesting to use a similar strategy to study (ii) and (iii). In fact, the boundary conditions (iii) actually represents a dynamic condition for the unknown q at 0, and hence seem to require a subtler analysis: we will consider them along with further hyperbolic systems with dynamic boundary conditions in a forthcoming paper [41]. We rewrite (5.10) as (2.1) by letting $u_e := (z'_e, \dot{z}_e, \theta_e, q_e)$; then the Assumptions 2.1 are satisfied taking

$$M_e := \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & -\beta & 0 \\ 0 & -\delta & 0 & -\gamma \\ 0 & 0 & -\frac{\kappa}{\tau_0} & 0 \end{pmatrix}, \quad Q_e := \begin{pmatrix} \alpha\delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \frac{\beta\gamma\tau_0}{\kappa} \end{pmatrix}, \quad \text{and} \quad N_e := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau_0} \end{pmatrix}.$$

The choice of Q_e is rather natural and indeed a similar term was also used to regularize the inner product by Racke, see (18) of [62]. A direct computation shows that

$$Q_e M_e = \begin{pmatrix} 0 & \alpha\delta & 0 & 0 \\ \alpha\delta & 0 & -\beta\delta & 0 \\ 0 & -\beta\delta & 0 & -\beta\gamma \\ 0 & 0 & -\beta\gamma & 0 \end{pmatrix}$$

with four eigenvalues of the form $\pm\sqrt{\frac{H \pm 2\sqrt{K}}{2}}$, where $H := \alpha^2\delta^2 + \beta^2\delta^2 + \beta^2\gamma^2$ and $K := H^2 - 4\alpha^2\beta^2\gamma^2\delta^2$. Because $H^2 > K$ whenever $\alpha, \beta, \gamma, \delta > 0$, $Q_e M_e$ has two positive and two negative eigenvalues.

This is coherent with both above choices of boundary conditions (in the purely hyperbolic case of $\tau_0 \neq 0$). For the sake of simplicity, let us focus for a while on the case of an individual interval that can be expressed in our formalism taking as Y at each endpoint the spaces

$$\mathbb{C} \times \{0\} \times \mathbb{C} \times \{0\} \quad \text{and} \quad \mathbb{C} \times \{0\} \times \{0\} \times \mathbb{C},$$

respectively. A further possible choice for a subspace of the null isotropic cone is *e.g.*

$$\{0\} \times \mathbb{C} \times \{0\} \times \mathbb{C},$$

corresponding to

$$- z'(0) = z'(\ell) = \theta(0) = \theta(\ell) = 0.$$

If we however regard an interval as a loop (a network with one edge and one vertex), all these boundary conditions turn out to be only special cases of a more general setting. Indeed, a direct computation shows that a necessary condition for the vector

$$\gamma(u) := (z'(0), \dot{z}(0), \theta(0), q(0), z'(\ell), \dot{z}(\ell), \theta(\ell), q(\ell))^T$$

to lie in the null isotropic cone of T defined as in (3.39) is that

$$\Re(Z_1 \cdot \bar{Z}_2 - Q \cdot \bar{\Theta} - Z_2 \cdot \bar{\Theta}) = 0 \quad (5.11)$$

where

$$Z_1 := \begin{pmatrix} -\alpha z'(0) \\ \alpha z'(\ell) \end{pmatrix}, \quad Z_2 := \begin{pmatrix} \delta \dot{z}(0) \\ \delta \dot{z}(\ell) \end{pmatrix}, \quad \Theta := \begin{pmatrix} -\beta \theta(0) \\ \beta \theta(\ell) \end{pmatrix}, \quad Q := \begin{pmatrix} \gamma q(0) \\ \gamma q(\ell) \end{pmatrix}.$$

This is for instance the case if

$$Z_1 \perp Z_2 \quad \text{and} \quad (Z_2 + Q) \perp \Theta;$$

this condition can *e.g.* be enforced by imposing

$$Z_1, \Theta \in \text{span}\{\mathbf{1}_{\mathbb{C}^2}\}^\perp, \quad Z_2, Q \in \text{span}\{\mathbf{1}_{\mathbb{C}^2}\},$$

where $\text{span}\{\mathbf{1}_{\mathbb{C}^2}\}$ is the subspace of \mathbb{C}^2 spanned by the vector $(1, 1)^\top$. This is a hardly surprising choice for the reader familiar with evolution equations on networks which corresponds to periodic-type conditions

$$- \dot{z}(0) = \dot{z}(\ell), \quad z'(0) = z'(\ell), \quad \theta(0) = \theta(\ell), \quad \text{and} \quad q(0) = q(\ell)$$

and in turn to

$$\gamma(u) \in Y := \text{span}\{\mathbf{1}_{\mathbb{C}^2}\} \oplus \text{span}\{\mathbf{1}_{\mathbb{C}^2}\} \oplus \text{span}\{\mathbf{1}_{\mathbb{C}^2}\} \oplus \text{span}\{\mathbf{1}_{\mathbb{C}^2}\}.$$

Indeed, $\dim(Y) = 4$, hence condition (3.42) is satisfied.

This paves the way to the study of second sound on collection of intervals with coupled boundary conditions, an especially interesting issue, as second sound has been conjectured in [31] to take place in graphene – a network: more precisely, hexagonal lattice of carbon atoms –, already at room temperature.

Indeed, one can apply our general theory in order to describe transmission conditions leading to well-posedness; an easy computation shows that the relevant equation is a higher dimensional counterpart of (5.11). An educated guess suggests to study conditions of continuity (across the ramification nodes) on both displacement and temperature, *i.e.*, on z – hence \dot{z} – and θ , along with a Kirchhoff-type condition on z' and q . It is remarkable that this choice does not satisfy (5.11). However, it is not difficult to see that all boundary values that satisfy either

- continuity on z – hence \dot{z} – as well as q , along with
- Kirchhoff-type conditions

$$\sum_{e \in E_v} \iota_{ve} z'_e(v) = 0 \quad \text{and} \quad \sum_{e \in E_v} \iota_{ve} \theta_e(v) = 0$$

on z' and θ ; or else

- continuity on z' and θ , along with
- Kirchhoff-type conditions

$$\sum_{e \in E_v} \iota_{ve} \dot{z}_e(v) = 0 \quad \text{and} \quad \sum_{e \in E_v} \iota_{ve} q_e(v) = 0$$

on z – hence \dot{z} – as well as q define a totally isotropic subspace of the null isotropic cone. (If the vertex v has degree 1, then in both cases the first conditions become void, whereas the second reduce to Dirichlet conditions.) Again, we see that (3.19) is satisfied and conclude that the system is governed by a strongly continuous group on $\mathbf{L}^2(\mathcal{G})$.

All above spaces Y are invariant under taking both the real and the positive part. Furthermore, Q, M, N are real valued and Q, N are diagonal, but M is not, hence by Proposition 4.7 and Proposition 4.11 the semigroup generated by \mathcal{A} with any of these transmission conditions is real but not positive.

Furthermore, \mathcal{A} generates merely a semigroup whenever the space Y defining the boundary conditions is a subspace of the nonpositive isotropic cone of T : this can *e.g.* be enforced by assuming that

$$Z_1 = BZ_2 \quad \text{and} \quad (Z_2 + Q) = -C\Theta$$

for some dissipative matrices B, C , provided Y has the correct dimension. Because

$$Q_e N_e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\beta\gamma}{\kappa} \end{pmatrix}$$

is dissipative and Q_e, M_e are spatially constant, this semigroup is then automatically contractive.

APPENDIX A. HYPERBOLICITY REVISITED

Lemma A.1. *Let $[0, \ell_e] \ni x \mapsto M_e(x) \in M_{k_e}(\mathbb{C})$ be a Lipschitz continuous matrix-valued function such that $M_e(x)$ is invertible for each $x \in [0, \ell_e]$. Then matrix $M_e(x)$ is Lipschitz-diagonalizable, i.e., there exist two Lipschitz continuous matrix-valued functions $[0, \ell_e] \ni x \mapsto S_e(x) \in M_{k_e}(\mathbb{C})$ and $[0, \ell_e] \ni x \mapsto D_e(x) \in M_{k_e}(\mathbb{C})$ such that for all $x \in [0, \ell_e]$ both $S_e(x), D_e(x)$ are invertible, $D_e(x)$ is diagonal and real, and furthermore*

$$M_e(x) = S_e^{-1}(x)D_e(x)S_e(x), \quad (\text{A.1})$$

if only if Assumptions 2.1.(3) holds for some Lipschitz continuous, uniformly positive definite matrix-valued function $[0, \ell_e] \ni x \mapsto Q_e(x) \in M_{k_e}(\mathbb{C})$.

Proof. If the matrix M_e can be diagonalized as above, then we readily check that Assumptions 2.1.(3i) holds with the Hermitian matrices

$$Q_e(x) := S_e^*(x)D_e^{2n}(x)S_e(x), \quad x \in [0, \ell_e],$$

for any $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. This matrix is indeed uniformly positive definite, because for any $\xi \in \mathbb{C}^{k_e}$, one has

$$Q_e(x)\xi \cdot \bar{\xi} = D_e^{2n}(x)Y(x) \cdot \overline{Y(x)}, \quad (\text{A.2})$$

where $Y(x) := S_e(x)\xi$. Since the mappings

$$[0, \ell_e] \rightarrow [0, \infty) : x \rightarrow \|D_e^{-2n}(x)\| \quad \text{and} \quad [0, \ell_e] \rightarrow [0, \infty) : x \rightarrow \|S_e^{-1}(x)\|$$

are continuous and positive, there exists a positive constant α such that

$$0 < \|D_e^{-2n}(x)\| \leq \alpha \quad \text{and} \quad 0 < \|S_e^{-1}(x)\| \leq \alpha \quad \text{for all } x \in [0, \ell_e].$$

These estimates in (A.2) lead to

$$Q_e(x)\xi \cdot \bar{\xi} \geq \alpha^{-1}\|Y(x)\|^2 \geq \alpha^{-3}\|\xi\|^2 \text{ for all } x \in [0, \ell_e].$$

Finally as a composition of Lipschitz continuous mappings, $Q_e(\cdot)$ is Lipschitz continuous, too.

Conversely, let us assume that Assumptions 2.1.(3) holds. First, as each $Q_e(x)$ is Hermitian and positive definite, $Q_e^{\frac{1}{2}}(x)$ is well-defined. Now we notice that Assumptions 2.1.(3i) is equivalent to

$$Q_e^{\frac{1}{2}}(x)M_e(x)Q_e^{-\frac{1}{2}}(x) = Q_e^{-\frac{1}{2}}(x)M_e^*(x)Q_e^{\frac{1}{2}}(x) \text{ for all } x \in [0, \ell_e].$$

Since this right-hand side is the adjoint of the left-hand side, each matrix $Q_e^{\frac{1}{2}}(x)M_e(x)Q_e^{-\frac{1}{2}}(x)$, $x \in [0, \ell_e]$, is Hermitian, hence it is diagonalizable by a family of unitary matrices $S_{0,e}(x)$ and real diagonal matrices $D_e(x)$ such that

$$S_{0,e}(x) \left(Q_e^{\frac{1}{2}}(x)M_e(x)Q_e^{-\frac{1}{2}}(x) \right) S_{0,e}^*(x) = D_e(x) \text{ for all } x \in [0, \ell_e],$$

which yields (A.1) with $S_e(x) := S_{0,e}(x)Q_e^{\frac{1}{2}}(x)$. By assumptions on M_e and Q_e , both $S_e(\cdot), D_e(\cdot)$ are Lipschitz continuous functions and $S_e(x), D_e(x)$ are invertible matrices for all $x \in [0, \ell_e]$. \square

APPENDIX B. THREE VERSIONS OF THE FUNDAMENTAL LEMMA OF CALCULUS OF VARIATIONS

We first prove a density result in the subset of positive integrable functions; we recall the notation in (4.6) and write likewise $\mathcal{D}(0, \ell; \mathbb{R})$ for the set of real-valued test functions.

Lemma B.1. *Let $\ell > 0$. The set*

$$\{\varphi^2 : \varphi \in \mathcal{D}(0, \ell; \mathbb{R})\}$$

is dense in $L^1(0, \ell; \mathbb{R}_+)$.

Proof. Indeed let us fix $u \in L^1(0, \ell; \mathbb{R}_+)$, then \sqrt{u} belongs to $L^2(0, \ell; \mathbb{R})$ and therefore there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of functions in $\mathcal{D}(0, \ell; \mathbb{R})$ such that

$$\varphi_n \rightarrow \sqrt{u} \text{ in } L^2(0, \ell; \mathbb{R}) \text{ as } n \rightarrow \infty. \tag{B.1}$$

Therefore by Cauchy–Schwarz’s inequality we have

$$\begin{aligned} \int_0^\ell |u - \varphi_n^2| \, dx &= \int_0^\ell |(\sqrt{u} - \varphi_n)(\sqrt{u} + \varphi_n)| \, dx \\ &\leq \|\sqrt{u} - \varphi_n\|_{L^2(0, \ell)} \|\sqrt{u} + \varphi_n\|_{L^2(0, \ell)} \\ &\leq \|\sqrt{u} - \varphi_n\|_{L^2(0, \ell)} (\|\sqrt{u}\|_{L^2(0, \ell)} + \|\varphi_n\|_{L^2(0, \ell)}). \end{aligned}$$

By (B.1), we conclude that this right-hand side tends to zero. \square

Now we prove two variants of the fundamental lemma of the calculus of variations (or “du Bois–Reymond’s lemma”).

Lemma B.2. *Let $h \in L^\infty(0, \ell; \mathbb{R})$ satisfy*

$$\int_0^\ell h\varphi^2 dx \geq 0 \quad \text{for all } \varphi \in \mathcal{D}(0, \ell; \mathbb{R}), \quad (\text{B.2})$$

then $h \geq 0$. If in particular

$$\int_0^\ell h\varphi^2 dx = 0 \quad \text{for all } \varphi \in \mathcal{D}(0, \ell; \mathbb{R}), \quad (\text{B.3})$$

then $h = 0$.

Proof. As the second assertion is a direct consequence of the first one, it remains to check the first one. Let $h \in L^\infty(0, \ell; \mathbb{R})$ satisfy (B.2), then as it is in $L^1(0, \ell; \mathbb{R})$, it can be split up as

$$h = h^+ - h^-,$$

where $h^+, h^- \in L^1(0, \ell; \mathbb{R}_+)$ is the positive and negative part of h respectively. According to (B.3) and Lemma B.1 we have

$$\int_0^\ell hh^- dx \geq 0,$$

which in turn leads to

$$\int_0^\ell (h^-)^2 dx = 0,$$

and proves that $h = h^+$ is nonnegative. □

This Lemma allows to prove a matrix-valued version of the fundamental lemma of the calculus of variations.

Lemma B.3. *Let k be a positive integer and let $A \in L^\infty(0, \ell; \mathbb{C}^{k \times k})$ be such that*

$$A^*(x) = A(x) \text{ for a.e. } x \in (0, \ell).$$

If A satisfies

$$\int_0^\ell Au \cdot \bar{u} dx = 0 \quad \text{for all } u \in \mathcal{D}(0, \ell)^k, \quad (\text{B.4})$$

then $A = 0$.

Proof. First we show that the diagonal entries of $A = (A_{i,j})_{1 \leq i, j \leq k}$ are zero. Indeed let us fix $i \in \{1, \dots, k\}$ and in (B.4) take test functions u in the form $u = \varphi e_i$, where $\varphi \in \mathcal{D}(0, \ell; \mathbb{R})$ is arbitrary and e_i is the i th element of the canonical basis of \mathbb{C}^k . Then we get

$$\int_0^\ell A_{ii}\varphi^2 dx = 0 \quad \text{for all } \varphi \in \mathcal{D}(0, \ell; \mathbb{R}),$$

and Lemma B.2 yields $A_{ii} = 0$ because A_{ii} is real-valued.

Let us now manage the off-diagonal entries of A . Fix $i, j \in \{1, \dots, k\}$ with $i < j$. Now we chose two family of test-functions in (B.4):

(1) First take test functions u in the form

$$u = \varphi(e_i + e_j),$$

where $\varphi \in \mathcal{D}(0, \ell; \mathbb{R})$ is arbitrary. Then (B.4) reduces to

$$\int_0^\ell (A_{ij} + A_{ji})\varphi^2 dx = 0 \quad \text{for all } \varphi \in \mathcal{D}(0, \ell; \mathbb{R}).$$

Again $A_{ij} + A_{ji} = 2\Re A_{ij}$ is real-valued because A is hermitian, and Lemma B.2 yields $\Re A_{ij} = 0$.

(2) Second take test functions u in the form

$$u = \varphi(e_i + ie_j),$$

where $\varphi \in \mathcal{D}(0, \ell; \mathbb{R})$ is arbitrary to obtain

$$\int_0^\ell (A_{ij} - A_{ji})\varphi^2 dx = 0 \quad \text{for all } \varphi \in \mathcal{D}(0, \ell; \mathbb{R}).$$

Since $A_{ij} - A_{ji} = 2i\Im A_{ij}$, this means that

$$\int_0^\ell \Im A_{ij}\varphi^2 dx = 0 \quad \text{for all } \varphi \in \mathcal{D}(0, \ell; \mathbb{R}),$$

and therefore $\Im A_{ij} = 0$ due to Lemma B.2. □

APPENDIX C. ON SUBSPACES OF ISOTROPIC CONES ASSOCIATED WITH A QUADRATIC FORM

In this section we fix a positive integer k and a hermitian and invertible matrix $P \in \mathbb{C}^{k \times k}$. Its associated quadratic form q is defined by

$$q(\xi) = P\xi \cdot \bar{\xi}, \quad \xi \in \mathbb{C}^k.$$

Now we introduce some cones associated with q , see Definition 3.1 of [46].

Definition C.1. (1) The *null isotropic cone* associated with the quadratic form q is defined as the set of isotropic vectors associated with q , namely the set of vectors $\xi \in \mathbb{C}^k$ such that

$$q(\xi) = 0. \tag{C.1}$$

A subspace of the null isotropic cone associated with q is called a *totally isotropic subspace* and the *isotropy index* (of the quadratic space associated with q), denoted here by $i(q)$, is the maximum of the dimensions of the totally isotropic subspaces.

(2) The *nonpositive (resp. nonnegative) isotropic cone* associated with the quadratic form q is defined as the set of vectors $\xi \in \mathbb{C}^k$ such that

$$q(\xi) \leq 0 \text{ (resp. } \geq 0). \tag{C.2}$$

From Lemma 1.2 of [53] we know that $i(q) \leq k/2$ but, surprisingly, we could not find in the literature a reference that yields a characterization of $i(q)$. Hence the goal of this appendix is to characterize this isotropic index as well as the maximal dimension of any subspace of nonpositive isotropic cones.

Let $\{\lambda_i\}_{i=1}^k$ be the set of eigenvalues of P , repeated according to their multiplicities and enumerated in an increasing order, and denote by $\{u_i\}_{i=1}^k$ the set of the associated normalized eigenvectors, *i.e.*,

$$Pu_i = \lambda_i u_i \quad \text{and} \quad u_i \cdot \bar{u}_j = \delta_{ij} \quad \text{for all } i \in \{1, \dots, k\}.$$

Denote by k_- (resp. k_+) the number of negative (resp. positive) eigenvalues of P . Without loss of generality, we may assume that $\lambda_i < 0$ for $i \leq k_-$ and $\lambda_i > 0$ for $i > k_-$. We will see that the isotropic index $i(q)$ agrees with $\min\{k_-, k_+\}$.

Lemma C.2. *Any subspace of the null isotropic cone associated with the form q has dimension at most $\kappa := \min\{k_-, k_+\}$. Furthermore if $\kappa \geq 1$, there exist at least 2^κ subspaces of the null anisotropic cone associated with q of dimension κ .*

Proof. If $\kappa = 0$, this means that P is either positive definite or negative definite and therefore the associated isotropic cone is reduced to $\{0\}$. So the only case of interest is the case $\kappa \geq 1$. By symmetry, we can assume that $\kappa = k_-$. So let us now fix a subspace I of the isotropic cone associated with q . Every nonzero $u \in I$ can be written as

$$u = \sum_{i=1}^k \alpha_i u_i, \tag{C.3}$$

for some $\alpha_i \in \mathbb{C}$ which are not all zero. Since (C.1) is equivalent to

$$\sum_{i=k_-+1}^k |\alpha_i|^2 \lambda_i = - \sum_{i=1}^{k_-} |\alpha_i|^2 \lambda_i, \tag{C.4}$$

we find that there exists at least one $i \leq k_-$ such that $\alpha_i \neq 0$.

Assume that $K := \dim I > k_-$ and let $\{U_i\}_{i=1}^K$ be a basis of I . Let us write

$$U_i = \sum_{j=1}^k \alpha_{ij} u_j, \tag{C.5}$$

for some $\alpha_{ij} \in \mathbb{C}$. By the previous remark, for all $i \leq K$, there exists $j \leq k_-$ such that $\alpha_{ij} \neq 0$.

Now we use a sort of Gram–Schmidt procedure: Starting with $i = 1$ and without loss of generality (else we change the enumeration) we can assume that $\alpha_{11} \neq 0$ and, consequently, we have

$$u_1 = \delta_1 \tilde{U}_1 - \sum_{j=2}^k \frac{\alpha_{1j}}{\alpha_{11}} u_j.$$

where we have set $\tilde{U}_1 := U_1$ and $\delta_1 := \frac{1}{\alpha_{11}}$. Plugging this expression into (C.5) with $i = 2$, we find that

$$U_2 = \frac{\alpha_{21}}{\alpha_{11}} U_1 + \sum_{j=2}^k \tilde{\alpha}_{2j} u_j, \tag{C.6}$$

with some $\tilde{\alpha}_{2j} \in \mathbb{C}$. This means that the new vector $\tilde{U}_2 := U_2 - \frac{\alpha_{21}}{\alpha_{11}}U_1$, that is still in I , has at least one coefficient $\tilde{\alpha}_{2j}$ different from zero for $j \in \{2, \dots, k_-\}$. Again, after a possible change of enumeration, we can assume that $\tilde{\alpha}_{22} \neq 0$, hence we have

$$u_2 = \delta_2 \tilde{U}_2 - \sum_{j=3}^k \frac{\alpha_{2j}}{\tilde{\alpha}_{22}} u_j.$$

where $\delta_2 = \frac{1}{\tilde{\alpha}_{22}}$. Note that the new set $\{\tilde{U}_1, \tilde{U}_2\} \cup \{U_i\}_{i=3}^K$ forms a basis of I . By iterating this procedure, after k_- steps, we will find a basis $\{\tilde{U}_i\}_{i=1}^{k_-} \cup \{U_i\}_{i=k_-+1}^K$ of I such that

$$u_i = \delta_i \tilde{U}_i - \sum_{j=k_-+1}^k \beta_{ij} u_j \quad \text{for all } i = 1, \dots, k_-, \quad (\text{C.7})$$

for some $\delta_j \in \mathbb{C} \setminus \{0\}$ and some $\beta_{ij} \in \mathbb{C}$.

By using the expansion (C.5) of U_{k_-+1} and (C.7), we find

$$U_{k_-+1} = \sum_{j=1}^{k_-} \alpha_{ij} \delta_j \tilde{U}_j + \sum_{j=k_-+1}^k \gamma_{ij} u_j,$$

for some $\gamma_{ij} \in \mathbb{C}^*$. We then arrive to a contradiction because on one hand the vector $V := U_{k_-+1} - \sum_{j=1}^{k_-} \alpha_{ij} \delta_j \tilde{U}_j$ is in I , hence $q(V) = 0$, while on the other hand $V \neq 0$ is a linear combination of the u_j 's for $j \geq k_- + 1$, hence $q(V) > 0$.

For the last assertion, if in (C.3), for all $i = 1, \dots, k_-$, we chose

$$\alpha_i = 1 \quad \text{and} \quad \alpha_{i'} = 0 \quad \text{for all } i' \notin \{i, k_- + i\},$$

condition (C.4) will hold if and only if

$$|\alpha_{k_-+i}|^2 = -\frac{\lambda_i}{\lambda_{k_-+i}},$$

or equivalently

$$\alpha_{k_-+i} = \pm i \sqrt{\frac{\lambda_i}{\lambda_{k_-+i}}}.$$

This yields the isotropic vectors

$$U_i^\pm = u_i \pm i \sqrt{\frac{\lambda_i}{\lambda_{k_-+i}}} u_{k_-+i}. \quad (\text{C.8})$$

And, since the U_i^+ 's and the U_i^- 's are linearly independent, we find 2^{k_-} possibilities. \square

A similar assertion holds for subspaces of the nonpositive isotropic cone associated with the quadratic form q – one such subspace is spanned by the first k_- eigenvectors $\{u_i\}_{i=1}^{k_-}$ of P .

Lemma C.3. *Any subspace of the nonpositive isotropic subspace associated with the form q has dimension at most k_- .*

Proof. If $k_- = 0$, this means that P is positive definite and therefore the only possible choice for such a subspace is $\{0\}$. So the only case of interest is the case $k_- \geq 1$. Now the proof is exactly the same as the one of the previous Lemma. Indeed let I be such a subspace and let $u \in I$ different from zero, then it admits the splitting (C.3) with some $\alpha_i \in \mathbb{C}$ not all zeroes. Since the constraint

$$Pu \cdot \bar{u} \leq 0,$$

is equivalent to

$$\sum_{i=k_-+1}^k |\alpha_i|^2 \lambda_i + \sum_{i=1}^{k_-} |\alpha_i|^2 \lambda_i \leq 0, \quad (\text{C.9})$$

we again find that there exists at least one $i \leq k_-$ such that $\alpha_i \neq 0$. The previous argument then leads to $\dim I \leq k_-$. \square

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