ESAIM: COCV 27 (2021) 52
https://doi.org/10.1051/cocv/2021048

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

ON A CERTAIN COMPACTIFICATION OF AN ARBITRARY SUBSET OF $\mathbb{R}^m$ AND ITS APPLICATIONS TO DIPERNA-MAJDA MEASURES THEORY

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Abstract. We present a constructive proof of the fact, that for any subset $\mathscr{A} \subseteq \mathbb{R}^m$ and a countable family $\mathscr{F}$ of bounded functions $f : \mathscr{A} \to \mathbb{R}$ there exists a compactification $\mathscr{A}' \subset \ell^2$ of $\mathscr{A}$ such that every function $f \in \mathscr{F}$ possesses a continuous extension to a function $\bar{f} : \mathscr{A}' \to \mathbb{R}$. However related to some classical theorems, our result is direct and hence applicable in Calculus of Variations. Our construction is then used to represent limits of weakly convergent sequences $\{f(u^\nu)\}$ *via* methods related to DiPerna-Majda measures. In particular, as our main application, we generalise the Representation Theorem from the Calculus of Variations due to Kałamajska.

Mathematics Subject Classification. 49J45, 49R99, 28A33.

Received June 22, 2020. Accepted April 25, 2021.

1. Introduction

The behaviour of a composition of a weakly convergent sequence with a nonlinear mapping is essentially non-trivial. Indeed, taking $\Omega = (0,1)^n$ and knowing knowing that u^ν converges weakly to u in $L^1(\Omega)$ (denoted by $u^\nu \rightharpoonup u$) we are not able to say much about the weak limit in $L^1(\Omega)$ of the sequence $f(u^\nu)$, where $f : \mathbb{R} \to \mathbb{R}$ is of a linear growth. In particular, we have no reasons to expect that $f(u^\nu) \rightharpoonup f(u)$. Indeed, let us consider

$$\Omega = (0,1) \text{ and } u^\nu(x) = \sum_{j=1}^{\nu} (-1)^j \chi_{(\frac{j-1}{\nu}, \frac{j}{\nu})}.$$

It is easily seen by the classical Riemann-Lebesgue Lemma (see for example [10], Lem. 1.2), that $u^\nu \rightharpoonup 0$ in $L^1(\Omega)$. If we take however $f(\lambda) = |\lambda|$, we immediately observe that $f(u^\nu) \equiv 1$ for every ν and hence $f(u^\nu) \rightharpoonup 1$.

This particular example of an oscillatory sequence is however easily controlled by the use of The Young Theorem (see Thm. 2.10 below). The Theorem shows that, under certain assumptions, satisfied here,

$$f(u^\nu) \rightharpoonup \int_{\mathbb{R}} f(\lambda) \mathrm{d}\nu_x(\mathrm{d}\lambda) \text{ in } L^1(\Omega)$$

Keywords and phrases: DiPerna-Majda measures, compactification, weak-$\star$ convergence.

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for some probabilistic measures ν_x, defined on $\mathbb{R}$ and called Young Measures. In the described case we easily compute that

$$\nu_x = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1},$$

where δ_y is a Dirac measure concentrated in y. As it is shown below, an essential role was played by the condition of equiintegrability, satisfied by the sequence $f(u^\nu)$.

Let us show how the lack of equiintegrability disturbs the weak limit, modifying and extending slightly the example presented in [1]. We consider Ω as above and a sequence

$$v^\nu(x) = \nu\chi_{(\frac{1}{2}-\frac{1}{\nu},\frac{1}{2})}(x) - \nu\chi_{(\frac{1}{2},\frac{1}{2}+\frac{1}{\nu})}(x).$$

The sequence v^ν is not weakly convergent in $L^1(\Omega)$. We can however represent its elements in the form $v^\nu\mathscr{L}$, where $\mathscr{L}$ denotes the Lebesgue measure on Ω. This way, the sequence is bounded in the space of bounded Radon measures on Ω. The classical Banach-Alaoglu Theorem yields that the sequence, up to an extraction of a subsequence, is weakly-$\star$ convergent in this space. In fact, the weak-$\star$ limit of $v^\nu\mathscr{L}$ is 0. Furthermore, we easily compute that the Young Measure generated by v^ν is $\nu_x = \delta_0$. Taking again $f(\lambda) = |\lambda|$ we see however that $f(v^\nu)$ converges weakly-$\star$ to $2\delta_{\frac{1}{2}}$. The sequence v^ν experiences a concentration effect in $x = \frac{1}{2}$ and the effect cannot be controlled solely by the notion of Young Measure.

The control of the aforementioned concentration effects usually requires an addition of a complementary topological space used to measure the behaviour of the sequence at infinity. The concept of introducing families of measures on such an additional space was proposed by DiPerna and Majda [11] and was then efficiently developed [1]. For that purpose, a notion of compactification of Euclidean spaces $\mathbb{R}^m$ gathered a certain attention in modern Calculus of Variations. We recall that a compactification is a dense embedding into a compact space. One of the most natural and direct examples is adding a sphere, which is linked to the so-called recession function in relaxing functionals of linear growth, see for example [6, 8, 9, 13]. Let us recall that for a function $f : \mathbb{R}^m \to \mathbb{R}$, by its recession function we mean

$$f^\infty : S^{m-1} \to \mathbb{R};\ f^\infty(\vartheta) \stackrel{\text{def}}{=\!=} \lim_{t\to+\infty} \frac{f(t\vartheta)}{t}.$$

The existence of the computed limits, as well as the continuity of f^∞, typically has to be assumed. We see that $\mathbb{R}^m$ – the domain of the integrand f – is complemented by a sphere or, in other words, is compactified by a unit ball by adding a sphere to $\mathbb{R}^m$. Furthermore, the compactification is designed in such a way, that $\dfrac{f(u)}{1+|u|}$ may be extended to a continuous function on a closed unit ball. The examples of handling with such a sequence under a violation of these assumptions are given in Section 4.2. It is shown how adding a different object than a sphere can help in particular situation of a 'linearly oscilating' integrand. We see that for practical reasons, we need a precise knowledge of the shape of a compactification of $\mathbb{R}^m$, as well as the continuity of the integrand on the whole compactification.

This way, compactifications appear naturally in the field of DiPerna-Majda measures [18, 22, 25]. Papers dealing with DiPerna-Majda measures at most times assume directly, that some compactification exists and more – it is metric and separable, as these properties are often vital for further results. Of course, there also exist several papers in the field of DiPerna-Majda measures, dealing with the aforementioned compactification by a unit ball, for example [1, 11].

The following problem appears. Given a subset $\mathscr{A} \subseteq \mathbb{R}^m$ and a continuous function $f : \mathscr{A} \to \mathbb{R}$ we need to find a compactification $\mathscr{A}'$ of $\mathscr{A}$ and a dense embedding $\varphi : \mathscr{A} \hookrightarrow \mathscr{A}'$ such that the function $f \circ \varphi^{-1} : \varphi(\mathscr{A}) \to \mathbb{R}$ possesses a continuous extension $\bar{f} : \mathscr{A}' \to \mathbb{R}$. A very natural solution to that problem seems to be the classical Čech-Stone compactification $\beta\mathscr{A}$. Indeed, every continuous function $f : \mathscr{A} \to \mathbb{R}$ possesses an extension to a continuous function $\bar{f} : \beta\mathscr{A} \to \mathbb{R}$. Unfortunately, taking a very simple $\mathscr{A} = \{1 - \frac{1}{n} : n \in \mathbb{N}\} \subset [0, 1]$ we obtain a compactification $\beta\mathscr{A}$ which is non-metrizable, non-second countable and of cardinality $2^{2^{\mathbb{N}}}$ (see [12], Cor. 3.6.12 for details). This shows that a more modest construction is needed to obtain a compactification with metric and visible geometric structure. There are several approaches towards this problem, due to Gelfand and Naimark [14, 15], Engelking [12] or discussed by Keesling [23], which we will review in Section 5.

All the known earlier approaches require only that $\mathscr{A}$ is a subset of an arbitrary Tychonoff's space $\mathbb{X}$ (see Definition 2.2 for the details), which is not necessarily an Euclidean space. They use abstract arguments, which do not benefit from any knowledge of special properties of $\mathbb{X}$. We present a variant of a solution to the problem stated above, as well as an easy and constructive proof in the particular case $\mathbb{X} = \mathbb{R}^m$, which is the first of our main results. The precise formulation is given in Theorem 3.2. What is here also some additional benefit is a direct construction of the desired compact space, as well as of a homeomorphic embedding $\varphi : \mathscr{A} \to \mathscr{A}'$. In our case $\mathscr{A}'$ is proved to be a compact subset of $\mathbb{R}^{m+1}$ for any single function f.

In the next step we can consider a countable family $\mathscr{F}$ of functions $f : \mathscr{A} \to \mathbb{R}$ and construct a compactification $\mathscr{A}' \subset \ell^2$ such that $f \circ \varphi^{-1}$ possesses a continuous extension for every $f \in \mathscr{F}$. In particular, our compact space $\mathscr{A}'$ is metric and separable, as it inherits these properties from ℓ^2. Hence every measure on this set possesses a well-defined support, which is in general not always the case (see [24]). This important feature was not guaranteed by the classical methods, which we explain in Section 5. A precise and direct construction was needed for particular applications in Calculus of Variations we had in mind. An example of a possible application is given in Section 4.2.

To motivate the second of our main results, let us discuss two theorems which are of interest in the field of Calculus of Variations. The first is a variant of Young (DiPerna-Majda) Theorem for discontinuous integrands – Theorem 2.11 due to Kałamajska [18], see also [17, 19–21] for related results. The theorem shows a representation formula for the weak-$\star$ limit for the sequences of compositions $\{f(u^\nu)d\mu\}$, where $f : \mathbb{R}^m \to \mathbb{R}$ is a continuous function on every set $A_i, i = 1, \dots, k$. It is assumed that the sets $\{A_i\}_{i=1}^k$ form a partition of $\mathbb{R}^m$ and every Borel set A_i is compactified by some $\gamma A_i \subset \mathbb{R}^N$ for some N. The representation of the limit, given in (2.1), requires an integration of f with respect to some measure over the remainder $\gamma A_i \setminus \varphi_i(A_i)$, where $\varphi_i : A_i \to \gamma A_i$ is a homeomorphic embedding. In particular, the Theorem requires a knowledge about the shape of the set γA_i, as well as the construction of an embedding φ_i. Without that we are unable to compute the aforementioned limit of $\{f(u^\nu)d\mu\}$. The proof of the Representation Theorem 2.11 exploits a distance function on γA_i, while one of other assertions of the statement uses a support of the certain DiPerna-Majda measure defined on γA_i. For purposes of Representation Theorem 2.11, we need to know the precise shape of the compactification, construction of homeomorphic embeddings φ_i and insure that γA_i are metric spaces. The classical methods are hence not helpful.

Several questions appear around the Representation Theorem from [18]. A first doubt is whether the assumption $\gamma A_i \subseteq \mathbb{R}^N$ decreases the class of integrands f compatible with the Representation Theorem. We recall that in Representation Theorem 2.11 it is required for f to be continuously extendable to a function on γA_i. In Lemma 3.1 we answer that this assumption may be satisfied by any integrand f and a proper γA_i and one may take $N = m + 1$ for $A_i \subset \mathbb{R}^m$.

Another question is whether these methods let us investigate a non-studied class of functionals, namely one of the type $\int_\Omega f(x, u(x))\mathrm{d}x$, *i.e.* such that the integrand depends not only on the values of the function $u(x)$, but also on x. In this case we would look for a weak-$\star$ limit of $\{f(x, u^\nu)\mathrm{d}\mu\}$. To proceed with such tasks we require a compactification $\gamma\mathbb{R}^m$ of $\mathbb{R}^m$ – a target space for functions u^ν – such that every function $i_x : p \mapsto f(x, p) \in \mathbb{R}$ is continuously extendable to function defined on $\gamma\mathbb{R}^m$ and the space $\gamma\mathbb{R}^m$ is independent of x. However, in further analysis it is required that $\gamma\mathbb{R}^m$ is metric, separable space (see [17, 19–22]). Arranging the compactification for every function i_x separately is too naïve for those purposes.

In this paper we present Theorem 4.3 – a generalisation of the Representation Theorem 2.11, dealing with integrands dependent on x, as well as some sufficient conditions for integrand $f = f(x, u)$ to admit a proper compactification. This is the second of our main results.

Let us note that Representation Theorem is related to the classical Convergence Theorem from Set-valued Analysis – Theorem 2.12 ([3], Thm. 7.2.1). The Convergence Theorem can be used to describe in terms of inclusions the limits of $f(u^\nu(x))$, where f can be possibly discontinuous. It assumes that u^ν is converging almost everywhere to u and $f(u^\nu)$ is weakly convergent in $L^1(\Omega)$. It is clear from the proof given in p. 271 of [3] that some variants of Convergence Theorem may be deduced from the Representation Theorem from [18]. For more precise information see Remarks in p. 4 of [17], p. 2 of [18], p. 4 of [19]. Letting $u^\nu(x)$ converge strongly to $u(x)$ in Representation Theorem gives us a variant of Convergence Theorem 2.12. This way, contrary to the formulation of Convergence Theorem, we obtain a precise integral formula instead of an inclusion. In Representation Theorem 2.11 only weak convergence is needed. In Theorem 2.11 however we assume some special properties on integrand f, while in Theorem 2.12 such an assumption is not mandatory. It becomes natural to ask for a theorem working in possibly general setting, so that both Representatrion Theorem 2.11 and Convergence Theorem 2.12 become its special cases. Such a generalisation may contribute to both Calculus of Variations and Set-valued Analysis. This is our desired future application of the result.

2. Preliminaries

2.1. Notation

In the paper we will use several notions and notations. For any subset D of a normed vector space by $\operatorname{conv}(D)$ we will mean the convex hull of D and by $\overline{\operatorname{conv}}(D)$ we will mean the closed convex hull, that is the closure of $\operatorname{conv}(D)$. The function f with the domain D and values in T will be denoted by $f : D \to T$. Following [3] and analogously to the previous notation the function F with the domain D and values in 2^T (that is – subsets of T) will be denoted by $F : D \rightrightarrows T$ and referred as a multifunction. The space of continuous, real-valued functions on D will be denoted by $C(D)$. By $C_0(D)$ we will mean the subspace of $C(D)$ consisting of compactly supported functions.

For any topological space T, by $\mathcal{M}(T)$ we will mean the space of finite, signed, Borel measures defined on T. We recall that the variation of a measure $m \in \mathcal{M}(T)$ is a measure on T defined for any subset $S \subset T$ *via*

$$|m|(S) = \sup_{\{v\in C(T), |v|\leq 1\}} \int_S v(x) m(\mathrm{d}x).$$

Let us recall that the space $\mathcal{M}(T)$ is a normed space with the total variation norm, that is $||m|| = |m|(T)$. For any $m \in \mathcal{M}(T)$, by the support of a measure m we will mean the smallest closed set $C \subseteq T$ such that $|m|(T \setminus C) = 0$. The support of a measure m will be denoted by $\operatorname{supp} m$. The subspace of $\mathcal{M}(T)$ consisting of positive, probabilistic measures will be denoted by $\mathcal{P}(T)$.

Let us now recall the standard Lebesgue spaces. Let $m \in \mathcal{M}(T)$ and $p \in [1, +\infty)$. We will say that the function $f : T \to \mathbb{R}$ belongs to $L^p(T, m)$ whenever $\int_T \big(f(x)\big)^p |m|(\mathrm{d}x) < +\infty$ and that f belongs to $L^\infty(T, m)$, whenever there exist a value v such that $|m|(\{x \in T : f(x) > v\}) = 0$. We stress that, even when m is a signed measure, the definitions of spaces and natural norms provided by them are dependent only on the variation measure $|m|$, which is a positive measure.

For every $m \in \mathcal{M}(D)$ and any subset of an Euclidean space D we will say that the mapping $\{\nu_x\} : D \to \mathcal{M}(T), x \mapsto \nu_x$ is weakly-$\star$ measureable with respect to m, whenever for every $v \in C_0(T)$ the mapping $D \to \mathbb{R}, x \mapsto \int_T v(\lambda)\nu_x(\mathrm{d}\lambda)$ is measureable in the usual sense; we will write then $\{\nu_x\} \in \mathcal{M}(D, T, m)$. Whenever measures ν_x belong to $\mathcal{P}(D)$ for almost every x, we will write $\{\nu_x\} \in \mathcal{P}(D, T, m)$; For any sequence of subsets of an Euclidean space D_n, we recall the notions of set limits, that is

$$\limsup D_n \overset{\text{def}}{=\!=} \bigcap_{N\in\mathbb{N}} \bigcup_{n>N} D_n \text{ and } \liminf D_n \overset{\text{def}}{=\!=} \bigcup_{N\in\mathbb{N}} \bigcap_{n>N} D_n.$$

2.2. Basic properties of compactifications

In this section we will present basic notions needed to deal with compactifications.

Definition 2.1. Let $\mathbb{X}$ be a topological space. We say that a topological space $\gamma\mathbb{X}$ is a `compactification of` $\mathbb{X}$, whenever $\gamma\mathbb{X}$ is compact, there exists a homeomorphism $\varphi : \mathbb{X} \to \varphi(\mathbb{X}) \subseteq \gamma\mathbb{X}$ and $\varphi(\mathbb{X})$ is dense in $\gamma\mathbb{X}$.

The sets $\varphi(\mathbb{X})$ and $\mathbb{X}$ are often identified in the literature. In this paper, however, we will directly distinguish between them to strengthen the role of the embedding φ.

Let us now recall the classic definitions of Hausdorff and Tychonoff spaces.

Definition 2.2. Let $\mathbb{X}$ be a topological space. We will say that $\mathbb{X}$ is a `Hausdorff space`, whenever for any pair of points $x \neq x'$, there exist disjoint open sets U, U' such that $x \in U$ and $x' \in U'$.

We will say that $\mathbb{X}$ is `Tychonoff space` whenever for any closed set $F \subseteq \mathbb{X}$ and a point $x \in \mathbb{X} \setminus F$ there exists a continuous function $f : \mathbb{X} \to \mathbb{R}$ such that $\forall_{y \in F} f(y) = 0$ and $f(x) = 1$.

It is clear that any Tychonoff space is Hausdorff, while the converse does not hold (see [12], Exam. 1.5.6). The classes are however equivalent in the category of compact spaces.

Fact 2.3 (Thm. 3.1.9 in [12])**.** Let $\mathbb{X}$ be a compact, Hausdorff space. Then $\mathbb{X}$ is Tychonoff.

Proof. First let us notice that it is sufficient to prove that for every pair of disjoint, closed sets $A, B \subset \mathbb{X}$ there exist disjoint open sets U, V such that $A \subset U, B \subset V$. Indeed, from Tietz Theorem ([12], Thm. 1.5.11) follows at once, that every space satisfying the mentioned property is Tychonoff.

Let us now fix $a \in A, b \in B$ and from Hausdorff property let us find open, disjoint sets $U_{a,b}, V_{a,b}$ such that $a \in U_{a,b}$ and $b \in V_{a,b}$. Executing this procedure for every $a \in A$ gives us a cover of set A by family $\left\{U_{a,b}\right\}_{a \in A}$. As A is compact, we may choose a finite subcover and get $A \subset U_{a_1,b} \cup U_{a_2,b} \cup \ldots \cup U_{a_n,b}$. Define now $U_b \stackrel{\text{def}}{=\!=} U_{a_1,b} \cup U_{a_2,b} \cup \ldots \cup U_{a_n,b}$, $V_b \stackrel{\text{def}}{=\!=} V_{a_1,b} \cap V_{a_2,b} \cap \ldots \cap V_{a_n,b}$. This way we obtain disjoint open sets U_b, V_b such that $A \subset U_b, b \in V_b$. Similarly we may execute this procedure for every $b \in B$ and choose a finite subcover, getting $B \subset V_{b_1} \cup V_{b_2} \cup \ldots V_{b_k}$. Keeping in mind that $A \subset U_b$ for any $b \in B$, we obtain that $A \subset U_{b_1} \cap U_{b_2} \cap \ldots \cap U_{b_k}$. Taking now $U \stackrel{\text{def}}{=\!=} U_{b_1} \cap U_{b_2} \cap \ldots \cap U_{b_k}$ and $V \stackrel{\text{def}}{=\!=} U_{b_1} \cap U_{b_2} \cap \ldots \cap U_{b_k}$ finishes the proof. □

Definition 2.4. Let $\mathbb{X}, \mathbb{Y}$ be topological spaces and $f : \mathbb{X} \to \mathbb{Y}$ – a continuous function. Let $\gamma\mathbb{X}$ be certain compactification of $\mathbb{X}$ defined *via* homeomorphism $\varphi : \mathbb{X} \to \varphi(\mathbb{X}) \subseteq \gamma\mathbb{X}$. We will say that the function f is `admissible` for compactification $\gamma\mathbb{X}$, whenever there exists a continuous function $\bar{f} : \gamma\mathbb{X} \to \mathbb{Y}$ such that $\bar{f}(x) = \left(f \circ \varphi^{-1}\right)(x)$ for every $x \in \varphi(\mathbb{X})$.

From the definition of compactification it follows that, whenever f is admissible, the function $\bar{f}$ is uniquely determined by f. We will often refer to $\bar{f}$ as `extension of` f.

From now on, we will focus our interest in admissibility of real-valued functions, that is – we take $\mathbb{Y} = \mathbb{R}$ in the Definition 2.4. It is easily visible in that case, that a necessary condition for admissibility of such function is its boundedness. Keeping that in mind, let us recall the two useful notions dealing with real-valued functions.

Definition 2.5. Let $\mathscr{F}$ be a set of continuous real-valued functions on a topological space $\mathbb{X}$. We say that $\mathscr{F}$ forms a `ring of continuous functions` whenever the function $z \equiv 0$ belongs to $\mathscr{F}$ and for any $f, g \in \mathscr{F}$ we have $f \pm g, f \cdot g \in \mathscr{F}$.

We will often deal with rings consisting only of bounded functions. Such a ring will be referred as `ring of bounded continuous functions`.

Let us note, that the ring of continuous functions needs not to be not unital, that is – may not possess the '1' element. The examples of such a non-unital ring considered most often are continuous functions on $\mathbb{R}^n$ vanishing at infinity, or compactly supported. This circumstance changes in case of the following notion.

Definition 2.6. Let $\mathscr{F}$ be a ring of continuous functions on a Tychonoff space $\mathbb{X}$. We will say that the ring $\mathscr{F}$ is `complete` whenever

(a) every constant function belongs to $\mathscr{F}$,
(b) for any closed set $F \subseteq \mathbb{X}$ and a point $x \in \mathbb{X} \setminus F$ there exists function $f \in \mathscr{F}$ such that $\forall_{y \in F} f(y) = 0$ and $f(x) = 1$ (in other words – $\mathscr{F}$ separates closed sets from points outside of them),
(c) $\mathscr{F}$ is closed with respect to uniform convergence.

Due to condition (a), every complete ring of functions contains a constant function equal to one – the '1' element in the ring. From that and Kuratowski-Zorn Lemma one may prove that there exist maximal ideals in the ring, which happens to be a crucial feature for purposes of the Engelking's statement, which we discus later. Let us focus on some properties of compacifications and functions, which are admissible for them.

Proposition 2.7 (Properties of the class of admissible functions)**.** *Let $\mathbb{X}$ be Tychonoff space and $\gamma\mathbb{X}$ – its compactification. Let $\mathscr{F}$ be the set of all functions, which are admissible for this compactification. Then $\mathscr{F}$ is a ring of bounded continuous functions. Furthermore, it satisfies conditions (a), (c) from the Definition 2.6. If $\mathbb{X}$ is Hausdorff, also (b) is satisfied.*

Proof. Conditions for the ring, as well as boundedness and (a) from Definition 2.6 do not reuire explanation. To prove (c), let us assume that the sequence f_i converges uniformly on $\mathbb{X}$ to f. We will prove that the extensions of f_i's – $\bar{f}_i$ – form a Cauchy sequence in the space of continuous functions on $\gamma\mathbb{X}$ with the supremum norm. Indeed, as $\varphi(\mathbb{X})$ is dense in $\gamma(\mathbb{X})$, we have

$$\sup_{y \in \gamma(\mathbb{X})} |\bar{f}_n(y) - \bar{f}_k(y)| = \sup_{y \in \varphi(\mathbb{X})} |\bar{f}_n(y) - \bar{f}_k(y)| = \sup_{x \in \mathbb{X}} |f_n(x) - f_k(x)|$$

and from the uniform convergence of f_i's on $\mathbb{X}$ we see that for n, k large enough the right-hand side of the above equality is bounded by ε. We have shown that $\bar{f}_i$ forms a Cauchy sequence, and, as $C(\gamma(\mathbb{X}))$ is complete, this sequence has a continuous limit $\bar{f}$. Checking that $\bar{f}$ is an extension of f is straightforward.

For the proof of (b) let us remind that from Fact 2.3 it follows that $\gamma\mathbb{X}$ is Tychonoff, whenever it is Hausdorff. Let us consider now any closed $F \subset \mathbb{X}$ and $x \in \mathbb{X} \setminus F$ and assume that $\gamma\mathbb{X}$ is a compactification of $\mathbb{X}$ *via* homeomorphism $\varphi : \mathbb{X} \to \gamma\mathbb{X}$. As $\varphi(F)$ is closed in $\varphi(\mathbb{X})$, from the definition of inherited (so-called trace) topology it follows that there exists a set $K \subset \gamma\mathbb{X}$ which is closed in $\mathbb{X}$ and its intersection with $\varphi(\mathbb{X})$ is precisely $\varphi(F)$. Hence, $\varphi(x) \notin K$. From the definition of Tychonoff space – there exist then a continuous function g on $\gamma\mathbb{X}$ such that $g \equiv 0$ on K and $g\big(\varphi(x)\big) = 1$. Now the desired admissible function is precisely $f = g_{|\varphi(\mathbb{X})} \circ \varphi$. $\square$

2.3. Engelking's statement on compactifications

We are in position to state the Engelking's Theorem. However based on ideas from Lemma 1 of [15], it was Engelking who formulated the statement this way.

Theorem 2.8 (Engelking in [12], p. 240)**.** *Let $\mathscr{F}$ be a complete ring of bounded continuous functions on Tychonoff space $\mathbb{X}$. Then there exists a topological space $\Sigma\mathbb{X}$, which is a compacification of $\mathbb{X}$ and satisfies*

(a) every function $f \in \mathscr{F}$ is admissible for compactification $\Sigma\mathbb{X}$,
(b) every function, which is admissible for compactification $\Sigma\mathbb{X}$, belongs to $\mathscr{F}$.

Remark 2.9. Theorem 2.8 is commonly quoted and well-known. Up to our knowledge its proof has not been written so far. In [12] it is given as an exercise. The hinted proof of the entire statement requires introducing the topology on the set of ideals of the ring $\mathscr{F}$, it is hence technically complicated and unnecessarily demanding if in our mind only the case $\mathbb{X} \subseteq \mathbb{R}^m$ is needed. The stronger assumption on $\mathbb{X}$ allows a different proof, using essentially less advanced tools and geometrically clear, non-abstract ideas.

Some further information about the other, less direct methods of creating compactifications are given in Section 5.

2.4. The Representation Theorem from [18] and the Convergence Theorem

We begin with the Classical Young Theorem, highly inspired by [4]. We present a variant similar to the one in [16].

Theorem 2.10 (The Classical Young Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded and measureable set with respect to certain Borel measure μ. Assume that $K \subseteq \mathbb{R}^m$ is closed and $u^\nu : \Omega \to \mathbb{R}^m$ is a sequence of functions such that for every open $U \supseteq K$ we have*

$$\lim_{\nu\to+\infty} |\{x \in \Omega : u^\nu(x) \notin U\}| = 0 \textit{ and } \lim_{M\to+\infty} \sup_\nu |\{x \in \Omega : |u^\nu(x)| > M\}| = 0.$$

Then there exists a subsequence (still denoted by $\{u^\nu\}$) and a mapping $\{\nu_x\} \in \mathcal{P}(\Omega, \mathbb{R}^m, \mu)$ such that $\operatorname{supp} \nu_x \subseteq K$ for almost every $x \in \Omega$ and for every Carathéodory function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ such that the sequence $\{f(\cdot, u^\nu(\cdot)\}$ is uniformly integrable we have

$$f\big(\cdot, u^\nu(\cdot)\big) \rightharpoonup < \nu_x, f > \stackrel{def}{=\!=} \int_{\mathbb{R}^m} f(x, \lambda)\nu_x(\mathrm{d}\lambda) \textit{ in } L^1(\Omega, \mu).$$

In order to recall the Representation Theorem from [18], let us form a set of assumptions and notations used in the sequel. We deal with the following assumptions.

(H1) Ω is a open and bounded domain in $\mathbb{R}^n$ equipped with measure μ.
(H2) Sets $A_1, A_2, \ldots, A_k$ form a partition of $\mathbb{R}^m$.
(H3) $g : \mathbb{R}^m \to [0, +\infty)$ is a function satisfying $g_i \stackrel{\text{def}}{=\!=} g_{|A_i} \in C(A_i)$ and $g_i(\lambda) \geq \alpha > 0$ for every $\lambda \in A_i \cap \partial A_i$ and some α.
(H4) For every $i = 1, 2, \ldots, k$ and some N a space $\gamma A_i \subset \mathbb{R}^N$ is a compactification of A_i.
(H5) $\mathscr{F}$ is the class of functions $f : \mathbb{R}^m \to \mathbb{R}$ such that the function $f_i \stackrel{\text{def}}{=\!=} f_{|A_i}/g_i$ is admissible for compactification γA_i for every $i = 1, 2, \ldots, k$.

We present the Representation Theorem due to Kałamajska [18], which will be generalised by Theorem 4.3 with the help of Theorem 3.2.

Theorem 2.11. *[18, Representation Thm. 3.1] Under assumptions (H1-5) let the sequence $\{u^\nu\}$ of μ-measureable functions $u^\nu : \Omega \to \mathbb{R}^m$ satisfy*

(T) $\limsup_\nu \mu\big(\{x \in \Omega : |u^\nu(x)| \geq r\}\big) \xrightarrow{r\to+\infty} 0,$
(B) $\sup_\nu \int_\Omega g(u^\nu)\mu(\mathrm{d}x) < \infty.$

Then there exist

(a) a subsequence of $\{u^\nu\}$, denoted by the same expression,
(b) measures $\bar{m}^i, m^i$ on Ω, such that $\bar{m}^i$ is absolutely continuous with respect to μ and $\operatorname{supp} m^i \subseteq \operatorname{supp} \mu$ for any $i = 1, 2, \ldots, k$,
(c) a family of probability measures $\{\mu_x\}_{x\in\Omega}$ defined on $\mathbb{R}^m$ and such that the function $x \mapsto \mu_x$ is weakly-$\star$ measureable with respect to μ (to abbreviate we just denote it by $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbb{R}^m, \mu)$),
(d) families $\{\bar{\nu}^i_x\} \in \mathcal{P}(\Omega, \partial A_i \cap A_i, \mu)$ and $\{\nu^i_x\} \in \mathcal{P}(\Omega, \gamma A_i \setminus \varphi_i(A_i), m^i)$

such that for every $f \in \mathscr{F}$ the subsequence $\{f(u^\nu(x))\mu(\mathrm{d}x)\}$ converges weakly-$\star$ in the space of signed measures to the signed measure represented by

$$\sum_{i=1}^{k}\Bigg(\int_{\mathrm{int}\,A_i} f(\lambda)\mu_x(\mathrm{d}\lambda)\mu(\mathrm{d}x) + \int_{\partial A_i\cap A_i} f(\lambda)\bar{\nu}^i_x(\mathrm{d}\lambda)\bar{m}^i(\mathrm{d}x) + \int_{\gamma A_i\setminus\varphi_i(A_i)} \tilde{f}_i(\lambda)\nu^i_x(\mathrm{d}\lambda)m^i(\mathrm{d}x)\Bigg). \tag{2.1}$$

Moreover, $\{\mu_x\}_{x\in\Omega}$ is the classical Young Measure generated by the sequence $\{u^\nu\}$, as in Theorem 2.10.

Now we switch our attention to the classical Convergence Theorem from [3], which plays an important role in Set-Valued analysis.

Theorem 2.12. *[3, Thm. 7.2.1] Let $n \in \mathbb{N}$ and $F_n : \mathbb{R}^k \rightrightarrows \mathbb{R}^m$ be such multifunctions, that for every $x \in \mathbb{R}^k$ there exist an open neighbourhood V such that $\bigcup_{n\in\mathbb{N}} F_n(V)$ is bounded. We denote the graph of F_n as G_n. Assume further, that Ω is an open subset of $\mathbb{R}^n$ and $x_j : \Omega \to \mathbb{R}^k, y_j : \Omega \to \mathbb{R}^m$ – measureable functions such that*

a) x_j converges to x almost everywhere;
b) $y_j \in L^1(\Omega, \mathbb{R}^m)$ is weakly convergent in L^1 to y;
c) for almost every $w \in \Omega$ and every U – open neighbourhood of 0 in $\mathbb{R}^k \times \mathbb{R}^m$ there exist K such that $\forall_{k>K}(x_k(w), y_k(w)) \in G_k + U$.

Then for almost every $w \in \Omega$ we have $y(w) \in \overline{\mathrm{conv}} F^\#(x(w))$, where $F^\#$ is the multifunction whose graph is equal to $\limsup G_n$.

3. Compactification of an arbitrary subset of an Euclidean space

3.1. The proof in the chosen case

In this section we present our construction of compactification of an arbitrary subset of $\mathbb{R}^m$ and show some of its properties. In the sequel the construction will be exploited to generalise Representation Theorem 2.11.

We will use the letter $\mathscr{A}$ for an arbitrary subset of $\mathbb{R}^m$.

The key role in our construction is the following, simpler version of Theorem 3.2.

Lemma 3.1 (Finitely many functions case)**.** *Let $\mathscr{A}$ be a subset of $\mathbb{R}^m$ and $f_1, f_2, \dots, f_k : \mathscr{A} \to \mathbb{R}$ be continuous and bounded functions. Then there exists a compactification $\kappa\mathscr{A} \subseteq \mathbb{R}^{m+k}$ such that every f_i can be continuously extended to $\bar{f}_i : \kappa\mathscr{A} \to \mathbb{R}$.*

Proof. We will divide it into steps.

Step 1 We observe that without a loss of generality we may assume, that $\mathscr{A}$ is bounded. Indeed, we may apply diffeomorphism $\phi : \mathscr{A} \to \mathbb{R}^m, \phi(x) \overset{\mathrm{def}}{=\!=} \frac{x}{1+|x|}$, so that a homeomorphic copy of $\mathscr{A}$ is a subset of the open unit ball and $\phi^{-1}(y) = \dfrac{y}{1-|y|}$. It is now sufficient to show the assertion for functions $\tilde{f} \overset{\mathrm{def}}{=\!=} f \circ \phi^{-1}$ defined on the bounded copy of $\mathscr{A}$.

Step 2 Since $\mathscr{A}$ is assumed to be a bounded subset of $\mathbb{R}^m$, the closure of $\mathscr{A}$ is compact. We use now the classical homeomorphism between a domain and a graph of a continuous function. Namely, we set $\varphi :$

$\mathscr{A} \to \mathscr{A} \times \mathbb{R}^k; \varphi(x) \stackrel{\text{def}}{=\!=} \big(x, f_1(x), f_2(x), \dots f_k(x)\big)$ and observe that the image $\varphi(\mathscr{A})$ is homeomorphic to $\mathscr{A}$.

Step 3 We define $\kappa\mathscr{A} \stackrel{\text{def}}{=\!=} \overline{\varphi(\mathscr{A})}$. Note that $\kappa\mathscr{A}$ is indeed compact, as $\mathscr{A}$ was a bounded set in $\mathbb{R}^m$ and f_i were assumed to be bounded functions. Thus the set $\varphi(\mathscr{A})$ is bounded in $\mathbb{R}^{m+k}$ and its closure is compact. Obviously, $\varphi(\mathscr{A})$ is a homeomorphic copy of $\mathscr{A}$ and it is dense in $\kappa\mathscr{A}$. We get that $\kappa\mathscr{A}$ is a compactification of $\mathscr{A}$

Step 4 Now, for any i we set $\bar{f}_i : \kappa(\mathscr{A}) \to \mathbb{R}, \bar{f}_i(x, y_1, y_2, \dots, y_k) \stackrel{\text{def}}{=\!=} y_i$. As defined $\bar{f}_i$ is obviously continuous. Also it satisfies $\bar{f}_{i|\varphi(\mathscr{A})} = f \circ \varphi^{-1}$, which finishes the proof. □

The situation becomes more involved, when the given family of admissible functions is not finite, like in Lemma 3.1, but countable. The following statement is one of our main results in this chapter.

Theorem 3.2 (Countably many functions case)**.** *Let $\mathscr{A}$ be a bounded subset of $\mathbb{R}^m$ and $f_1, f_2 \dots : \mathscr{A} \to \mathbb{R}$ be continuous and bounded, functions. Then there exists a compactification $\kappa\mathscr{A} \subseteq \ell^2$ such that for any index i the function f_i can be continuously extended to $\bar{f}_i : \kappa\mathscr{A} \to \mathbb{R}$.*

Proof. Let us assume that $f_i \not\equiv 0$ for any i. This way we obtain, that for any i,

$$0 < \sup_{x \in \mathscr{A}} |f_i(x)| < +\infty.$$

For simplicity, $\sup_{x\in\mathscr{A}} |f_i(x)|$ we will denote by $\sup|f_i|$.

As previously, we may assume that $\mathscr{A}$ is bounded and thus $||x||_2 \le M$ for every $x \in \mathscr{A}$, where $||x||_2$ stays for the standard Euclidean norm. Let us now define the embedding $\varphi : \mathscr{A} \to \ell^2$ *via*

$$\varphi(x) \stackrel{\text{def}}{=\!=} \big(x, 2^{-1}(\sup|f_1|)^{-1} f_1(x), 2^{-2}(\sup|f_2|)^{-1} f_2(x), \dots, 2^{-j}(\sup|f_j|)^{-1} f_j(x), \dots\big). \tag{3.1}$$

Note that the image of φ is a subset of the set

$$\{(x_i) \in \ell_2 : ||(x_1, x_2, \dots, x_m)||_2 \le M, |x_{m+j}| \le 2^{-j}\},$$

which is compact by the standard argument used for Tychonoff cube. Of course φ^{-1} is continuous, as it is a projection. Obviously φ is bijective. What is non-trivial is the continuity of φ itself.

To that end let us first note that it is enough to check continuity in ℓ^1. Indeed, if for any $x_k \to x$ in $\mathscr{A}$ we will show that $\varphi(x_k) \to \varphi(x)$ in ℓ^1, then such a convergence in ℓ^2 also follows. Indeed, let us take $||x||_1$ as a Manhattan norm, that is $||(x_1, x_2, \dots)||_1 = |x_1| + |x_2| + \dots$. From Hölder inequality, $||\varphi(x_k) - \varphi(x)||_2 \le ||\varphi(x_k) - \varphi(x)||_1 ||(1, 1, \dots)||_\infty = ||\varphi(x_k) - \varphi(x)||_1$. It follows, that it is sufficient to check convergence in ℓ^1, where the calculations are less involving. We remind that the assumption $||x||_2 \le M$ implies by Hölder inequality in $\mathbb{R}^m$, that $||x||_1 \le M\sqrt{m}$, so we may still consider a bounded $\mathscr{A}$.

Our aim now is to show that for arbitrary $x \in \mathscr{A}$ and a sequence $x_k \to x$ we have

$$\sum_{i=1}^{\infty} \frac{|f_i(x_k) - f_i(x)|}{\sup|f_i| 2^i} \to 0 \text{ with } k \to +\infty.$$

For simplicity of notation let us take functions $v_i \stackrel{\text{def}}{=\!=} f_i / \sup|f_i|$. Such defined functions satisfy the condition $|v_i| \le 1$ and we need to check that

$$\Delta_k \stackrel{\text{def}}{=\!=} \sum_{i=1}^{\infty} |v_i(x) - v_i(x_k)| 2^{-i} \to 0 \text{ with } k \to +\infty.$$

Note that $|v_i(x) - v_i(x_k)| \le 2$ and thus the investigated sum is no bigger than 2.

For the proof that $\Delta_k \to 0$ we assume on the contrary that

$$\limsup_{k\to+\infty} \Delta_k = \delta > 0.$$

As δ is finite, let us take N such that

$$\sum_{i>N} 2 \cdot 2^{-i} < \delta/4$$

and then k_0 such big that for $i \le N$ and $k > k_0$ we have

$$|v_i(x) - v_i(x_k)| < \delta/4.$$

Take now any $k > k_0$. We have that

$$\Delta_k = \sum_{i=1}^{N} |v_i(x) - v_i(x_k)| 2^{-i} + \sum_{i>N} |v_i(x) - v_i(x_k)| 2^{-i} < \delta/4 + \delta/4 = \delta/2.$$

This contradicts however the assumption on the $\limsup \Delta_k$, as it holds for every k big enough.

Regarding continuity of φ and its aforementioned properties, we obtain that φ is a homeomorphism $\mathscr{A} \hookrightarrow \ell^2$. We may thus take for $\kappa\mathscr{A}$ the closure of the image of φ and define

$$\bar{f}_i(x, y_1, y_2, \ldots) \stackrel{\text{def}}{=\!=} \sup |f_i| 2^i y_i,$$

which completes the proof. $\square$

Remark 3.3. In the theorem above we chose the compactification to be a subset of the Banach space ℓ^2, because we wanted this target space to be an infinite dimensional variant of an Euclidean space. As the Euclidean metric is analogous to the ℓ^2 norm, the choice seems to be natural. The proof shows however, that one may in fact take ℓ^1 instead of ℓ^2. Furthermore, having in mind that the canonical embedding $\ell^1 \hookrightarrow \ell^p$ is a contraction on the Tychonoff cube for every $p \in [1, +\infty]$, the statement would be correct for every space $\ell^p, p \in [1, +\infty]$.

In fact, in the spirit of the classical Banach-Mazur Theorem [5], the choice of the space ℓ^1 is very natural as well. Let us briefly recall that the Theorem proves, that any separable Banach space is a continuous image of the space ℓ^1.

3.2. Properties of the compactification κ

We will follow by some observations regarding the compactification $\kappa\mathscr{A}$ described in Lemma 3.1 and Theorem 3.2.

Remark 3.4 (Non-minimality of the class of admissible functions)**.** The compactification $\kappa\mathscr{A}$ does not satisfy any natural condition similar to point *b*) of Theorem 2.8. Indeed, let us consider $\mathscr{A} = (0, 2\pi)$ and functions $f_1(x) = \sin x, f_2(x) = \cos x$. Such a set of functions separates points of the space $\mathscr{A}$. Application of the construction described in Lemma 3.1 gives a compact space homeomorphic to the closed interval $[0, 2\pi]$. The function $f : \mathscr{A} \to \mathbb{R}, f(x) = x$ is admissible for $\kappa\mathscr{A}$. This function however does not belong to the complete ring of functions generated by f_1, f_2, as it is immediate to see that any function in that ring will posses equal limits in $x = 0$ and $x = 2\pi$.

Remark 3.5 (Uniformly continuous functions)**.** An easy generalisation of the previous reasoning shows the following. Let us take any at most countable set of real-valued, bounded functions f_i on $\mathscr{A}$ – a bounded subset of

$\mathbb{R}^m$. Let $\kappa\mathscr{A}$ be the compactification generated by these functions *via* method from Theorem 3.2. Let $f : \mathscr{A} \to \mathbb{R}$ be any uniformly continuous and bounded function. Then f is admissible for $\kappa\mathscr{A}$.

Indeed, as f is uniformly continuous on a bounded set $\mathscr{A}$, it possesses limits $\lim_{x\to x_0} f(x)$ for every $x_0 \in \overline{\mathscr{A}}$. Thus we may define $\bar{f}(x_0, y_1, y_2, \ldots) \stackrel{\text{def}}{=\!=} \lim_{x\to x_0} f(x)$.

It is worth noticing that this is not the case for unbounded $\mathscr{A}$. To check this, let us take $\mathscr{A} = [1, +\infty)$ and compacitfication generated by $f(x) = e^{-x}$ *via* method from Lemma 3.1. Applying Step 1. from the proof gives us the new space $\phi(\mathscr{A}) = [\frac{1}{2}, 1)$ and $(f \circ \phi^{-1})(y) = e^{-y/(1-|y|)}$. Again we see that when x approaches infinity (and thus y approaches 1), function f vanishes. Thus, $\kappa\mathscr{A}$ is homeomorphic $[\frac{1}{2}, 1]$. Let us now investigate the function $\sin : \mathscr{A} \to \mathbb{R}$. Obviously $\sin \circ \phi^{-1}$ does not possess a limit at 1, so it is not admissible for $\kappa\mathscr{A}$. The condition of uniform continuity is then not sufficient for admissibility.

4. Generalisation of the Representation Theorem

4.1. Representation Theorem for discontinuous integrands

We will construct here a certain application of Lemma 3.1 and Theorem 3.2 to the theory of measures of DiPerna-Majda. We begin with generalising Representation Theorem established in [18].

It is possibly worth noting that we have proven what was an assumption in Representation Theorem 2.11. The assumption stated that every brick A_i possesses a compactification $\gamma A_i \subset \mathbb{R}^N$ such that a continuous and bounded function f_i/g_i is admissible. Our elementary reasoning presented in Lemma 3.1 shows that for any $A_i \subseteq \mathbb{R}^m$ and continuous and bounded $f : A_i \to \mathbb{R}$ we have the compactification $\kappa A_i \subseteq \mathbb{R}^{m+1}$, for which the function f is admissible. In particular, from the embedding of κA_i into an Euclidean space it follows that the space κA_i is separable and metric. Thus the support of an arbitrary measure on κA_i exists, which is not the case in the general setting: see for example page 68, Example 7.1.3 and page 73, Proposition 7.2.5 points (i) and (iii) in [7] or [24]. Let us stress here that a certain part of the further analysis is based on the behaviour of the support of certain measures. Its existence is essential for the theory.

Our aim is to generalise Representation Theorem 2.11 to a certain class of the integrands of the type $f(x, u)$, in other words such one that is dependent not only of a value of a function u, but also on particular $x \in \Omega$, where the value $u(x)$ is taken. We begin with the following remark.

Remark 4.1. Let us assume that for open and bounded $\Omega \subset \mathbb{R}^n$ we have $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ defined by $f(x, u) = f_1(x) f_2(u)$ and f is bounded. We further assume that f_1 is continuous and bounded, while f_2 – continuous and bounded on certain bricks A_i. In this situation we may choose any $x_0 \in \Omega$ such that $f_1(x_0) \neq 0$ and apply the compactification procedure shown in Lemma 3.1 for function $u \mapsto f(x_0, u)$, obtaining the compactifications κA_i. Let us now note that the for any x the function $u \mapsto f(x, u)$ is admissible for this compactification. Indeed, $f(x, u) = f(x_0, u)\dfrac{f_1(x)}{f_1(x_0)}$, and the statement follows from the fact that the admissible functions form an algebra and $f(x_0, u)$ is admissible.

In fact, the above remark can be generalised. Let us then assume that for an open and bounded $\Omega \subset \mathbb{R}^n$ we have $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$. We further assume that f is bounded, continuous with respect to x and continuous with respect to v on given bricks A_i. Our aim is to construct a compactification of A_i on which the function $f(x, v)$ is continuous for every $x \in \Omega$. For that purpose let us first fix countable, dense subset $\mathcal{O} = \{x_i\}$ in Ω. Let us then take $f_i(u) \stackrel{\text{def}}{=\!=} f(x_i, u)$ and apply Theorem 3.2 for created sequence f_i, obtaining compact space κA_i on which every function f_i is admissible. Let us define then function $\bar{f} : \Omega \times \kappa A_i$ in the following way. For fixed $x \in \Omega$ let us choose a sequence $x_i \in \mathcal{O}$ converging to x. We define

$$\bar{f}(x, u) \stackrel{\text{def}}{=\!=} \lim_{i\to\infty} \bar{f}_i(u).$$

Of course, such a definition is in general not proper, but from the assumption of the continuity of f with respect to x it follows that indeed $\bar{f}(x,u)$ is independent of the choice of the sequence x_i. It is however not enough to establish continuity of $\bar{f}$ on bricks A_i. This will be satisfied under the following assumption:

$$\sup_{u\in A_i} |f(x_n,u)-f(x,u)| \to 0, \text{ as } x_n \to x. \tag{4.1}$$

The assumption reads as: $f(x,\cdot)$ is a uniform limit of functions $f(x_n,\cdot)$ whenever $x_n \to x$. In particular it gives us continuity of $f(x,\cdot)$ for every $x\in\Omega$. As a consequence, we obtain continuity of $\tilde{f}$. Indeed, let us fix (x,u) and take $(x_n,u_n)\to(x,u)$. We have

$$|\bar{f}(x,u)-\tilde{f}(x_n,u_n)| \le |\bar{f}(x,u)-\bar{f}(x,u_n)| + |\bar{f}(x,u_n)-\bar{f}(x_n,u_n)|.$$

Now first term goes to 0 from the continuity of $\bar{f}(x,\cdot)$, while the second vanishes thanks to the property (4.1). Altogether, the following lemma is established.

Lemma 4.2. *Let Ω be an open and bounded set in $\mathbb{R}^n$. Set $A_1,A_2,\dots,A_k$ to be a partition of $\mathbb{R}^m$ and assume that $f:\Omega\times\mathbb{R}^m\to\mathbb{R}$ is a bounded function, continuous on $\Omega\times A_i, i=1,2,\dots,k$ and satisfying (4.1). Then, for every i there exists a compctification κA_i of the set A_i such that the function $f(x,\cdot)$ is admissible for every $x\in\Omega$. Furthermore, the extension $\bar{f}:\Omega\times\kappa A_i$ is continuous.*

Let us also notice that the class of functions $f=f(x,u)$ satisfying (4.1) covers all functions of the type $f(x,u)=f_1(x)f_2(u)$ described by Remark 4.1. This is exactly why the following, generalised version of the Theorem 2.11 uses only observations covered by the more general Lemma 4.2.

Before stating the theorem, let us introduce one new assumption, that is

(H5') $\mathscr{F}$ is the class of functions $f:\Omega\times\mathbb{R}^m\to\mathbb{R}$ such that the function $f_i(x,\cdot)\overset{\text{def}}{=\!=} f(x,\cdot)_{|\Omega\times A_i}/g_i(\cdot)$ is continuous and bounded on A_i for every $i=1,2,\dots,k$, every $x\in\Omega$ and satisfy (4.1), that is

$$\sup_{u\in A_i} |f(x_n,u)-f(x,u)| \to 0, \text{ as } x_n \to x. \tag{4.2}$$

Theorem 4.3. *Assume (H1-3), (H5') and let $f\in\mathscr{F}$. Then, for every $i=1,2,\dots,k$ there exist κA_i – compact subsets of ℓ_2, which are compactifications of A_i (with embeddings φ_i) and such that $f_i(x,u)/g_i(u)$ is extendable to a continuous function $\bar{f}:\Omega\times\kappa A_i\to\mathbb{R}$.*

Take any sequence $\{u^\nu\}$ of μ-measureable functions $u^\nu:\Omega\to\mathbb{R}^m$ satisfying

(T) $\limsup_\nu \mu\big(\{x\in\Omega : |u^\nu(x)|\ge r\}\big) \xrightarrow{r\to+\infty} 0,$
(B) $\sup_\nu \int_\Omega g(u^\nu)\mu(\mathrm{d}x)<\infty.$

Then there exist

(a) a subsequence of $\{u^\nu\}$, denoted by the same expression,
(b) measures $\bar{m}^i, m^i$ on Ω, such that $\bar{m}^i$ is absolutely continuous with respect to μ and $\operatorname{supp} m^i \subseteq \operatorname{supp}\mu$ for any $i=1,2,\dots,k$.
(c) a family of probability measures $\{\mu_x\}_{x\in\Omega}$ defined on $\mathbb{R}^m$ and such that the function $x\mapsto\mu_x$ is weakly-$\star$ measureable with respect to μ (to abbreviate we just denote it by $\{\mu_x\}_{x\in\Omega}\in\mathcal{P}(\Omega,\mathbb{R}^m,\mu)$).
(d) families $\{\bar{\nu}^i_x\}\in\mathcal{P}(\Omega,\partial A_i\cap A_i,\mu)$ and $\{\nu^i_x\}\in\mathcal{P}(\Omega,\kappa A_i\setminus\varphi_i(A_i),m^i)$.

such that for every function $f \in \mathscr{F}$, *which is admissible for compactification* $\kappa\mathscr{A}_i$, *the measure* $\{f(x, u^\nu(x))\mu(\mathrm{d}x)\}$ *converges weakly-*$\star$ *in the space of signed measures to the signed measure represented by*

$$\sum_{i=1}^{k}\bigg(\int_{intA_i} f(x,\lambda)\mu_x(\mathrm{d}\lambda)\mu(\mathrm{d}x) \\ + \int_{\partial A_i\cap A_i} f(x,\lambda)\bar{\nu}^i_x(\mathrm{d}\lambda)\bar{m}^i(\mathrm{d}x) \\ + \int_{\kappa A_i\setminus\varphi_i(A_i)} \bar{f}_i(x,\lambda)\nu^i_x(\mathrm{d}\lambda)m^i(\mathrm{d}x)\bigg).$$

Moreover, $\{\mu_x\}_{x\in\Omega}$ *is the classical Young Measure generated by the sequence* $\{u^\nu\}$, *as in Theorem 2.10.*

We stress here, that, due to Lemma 3.1, Theorem 4.3 recovers Representation Theorem 2.11, whenever the integrand f does not depend on x.

Remark 4.4. Let us mention, that the condition (B) is trivial when $\mu(\Omega) < +\infty$ and g is bounded. Indeed, in this case

$$\sup_\nu \int_\Omega g(u^\nu)\mu(\mathrm{d}x) \le (\sup g)\mu(\Omega).$$

On the other hand, whenever $g(v) \to +\infty$, as $|v| \to +\infty$, the condition (B) implies (T). To see that, take $M = \sup_\nu \int_\Omega g(u^\nu)\mu(\mathrm{d}x)$ and for an arbitrary ε take such L, that $g(v) > M\varepsilon^{-1}$ whenever $|v| > L$. We see now that

$$\mu\big(\{x\in\Omega : |u^\nu(x)| \ge L\}\big) \le \mu\big(\{x\in\Omega : g(u^\nu(x)) \ge M\varepsilon^{-1}\}\big).$$

As $\mu\big(\{x\in\Omega : g(u^\nu(x)) \ge M\varepsilon^{-1}\}\big)(M\varepsilon^{-1}) \le \int_\Omega g(u^\nu)\mu(\mathrm{d}x)$, we see that

$$\mu\big(\{x\in\Omega : g(u^\nu(x)) \ge M\varepsilon^{-1}\}\big) \le \varepsilon M^{-1}\int_\Omega g(u^\nu)\mu(\mathrm{d}x) \le \varepsilon.$$

Before the proof, we will formulate and prove the following Lemma. It is just a very minor modification of Lemma 3.3 in [18], but we will present the proof for reader's convenience.

Lemma 4.5. *Let* $\Omega \subset \mathbb{R}^n$ *be the compact set equipped with a Radon measure* μ *and* $A \subset \mathbb{R}^m$ *be Borel, compactified by metrizable* γA *with an embedding* φ. *Moreover, let us assume that* $g \in C(A)$ *is non-negative and a sequence* $\{u^\nu\} : \Omega \to \mathbb{R}^m$ *satisfies*

$$\sup_\nu \int_{\{x\in\Omega : u^\nu(x)\in A\}} g(u^\nu)\mu(\mathrm{d}x) < +\infty$$

and generates the Young measure $\{\mu_x\}_{x\in\Omega}$.

Let us define a sequence of measures $\{L^\nu\}$ *on* $\Omega\times\gamma A$ *via the condition, that for any* $F \in C(\Omega\times\gamma A)$ *we have*

$$(F, L^\nu) \stackrel{def}{=\!=} \int_{\{x\in\Omega : u^\nu(x)\in A\}} F(x, \varphi(u^\nu(x))g(u^\nu(x))\mu(\mathrm{d}x). \tag{4.3}$$

Then, up to a choice of certain subsequence (without any change in notation), there exist measures L on $\Omega\times\gamma A$, $\tilde m$ on Ω and $\{\tilde\nu_x\}\in\mathcal{P}(\Omega,\gamma A,\tilde m)$ such that

$$\begin{aligned} L^\nu &\overset{\star}{\rightharpoonup} L \text{ in } \mathcal{M}(\Omega\times\gamma A),\\ (F,L) &= \int_\Omega\int_{\gamma A} F(x,\lambda)\tilde\nu_x(\mathrm{d}\lambda)\tilde m(\mathrm{d}x), \\ \operatorname{supp}\tilde m &\subseteq \operatorname{supp}\mu. \end{aligned} \tag{4.4}$$

Moreover, let $\tilde m = p(x)\mu+\tilde m_s$ be the Radon-Nikodym decomposition of $\tilde m$ with respect to μ. If $\gamma A\setminus\varphi(A)\neq\emptyset$, then $\tilde\nu_x(\gamma A\setminus\varphi(A))=1$ for $\tilde m_s$-almost every x, while for $\gamma A\setminus\varphi(A)=\emptyset$ we have $\tilde m_s\equiv 0$.

If we further take $U\overset{def}{=\!=}\operatorname{int}A$ and $U^0\overset{def}{=\!=}\varphi(U)$ and assume that $f\in C(\Omega\times\gamma A)$ is such that the function $F(x,\lambda)\overset{def}{=\!=} f(x,\varphi(\lambda))g(\lambda)$ is continuous on $\Omega\times A$, as well as satisfies (4.1) on A, vanishes on $\Omega\times\partial A$ and have 0 limits when the second coordinate tends to infinity, then

$$\int_U f(x,\varphi(\lambda))g(\lambda)\mu_x(\mathrm{d}\lambda) = p(x)\int_{U^0} f(x,\lambda)\tilde\nu_x(\mathrm{d}\lambda) \tag{4.5}$$

for μ-almost every $x\in\Omega$, where μ_x is a standard Young measure generated by u^ν.

Proof. As we assumed γA to be metrizable, we see that $C(\Omega\times\gamma A)$ is separable and hence from Banach-Alaouglu Theorem, measures on $(\Omega\times\gamma A)$ are weakly-$\star$ compact. This proves the existence of measure L.

Let us now define $\tilde m$ *via* the condition, that

$$(h,\tilde m) = \int_{\Omega\times\gamma A} h(x)L(\mathrm{d}x,\mathrm{d}\lambda) \text{ for every } h\in C(\Omega).$$

Applying the classical slicing measure argument from [26], we obtain the existence of the family of positive measures $\{\tilde\nu_x\}_{x\in\Omega}\in L^\infty(\Omega,\mathcal{M}(\gamma A),\tilde m)$ such that (4.4) holds.

We will check that for $\tilde m$-almost every x, measures $\tilde\nu_x$ are probabilistic. Take any $l\in C(\Omega)$. Taking $F=l$ in (4.4), we get

$$(l,\tilde m) = \int_\Omega l(x)\big(\int_{\gamma A}\tilde\nu_x(\mathrm{d}\lambda)\big)\tilde m(\mathrm{d}x).$$

We see that $\tilde\nu_x(\gamma A)\tilde m(\mathrm{d}x)=\tilde m(\mathrm{d}x)$ as measures on Ω, hence $\tilde\nu_x$ are probabilistic $\tilde m$-almost everywhere.

We will split the remaining part of the reasoning into two cases. At first we will deal with the situation when $\gamma A\setminus\varphi(A)\neq\emptyset$. The alternative case will be considered separately.

Let us now assume that $\gamma A\setminus\varphi(A)\neq\emptyset$. We consider the function $D(\lambda)=\operatorname{dist}(\lambda,\gamma(A)\setminus\varphi(A))$, *i.e.* the distance from the point $\lambda\in\gamma A$ to the remainder of the compactification. As γA was assumed to be metrizable, the function is well-defined, bounded and continuous. We define $h^\nu(x)\overset{\text{def}}{=\!=} D(\varphi(u^\nu))g(u^\nu)\chi_{\{z\in\Omega:u^\nu(z)\in A\}}(x)$. We will show, that the sequence h^ν is uniformly integrable in $L^1(\Omega,\mu)$. To that end, let us define $A_\epsilon\overset{\text{def}}{=\!=}\{\lambda\in A:\operatorname{dist}(\varphi(\lambda),\gamma A\setminus\varphi(A))<\epsilon\}$, set $M>0$ and observe that

$$\begin{aligned} &\int_{\{x\in\Omega:h^\nu(x)>M\}} h^\nu(x)\mu(\mathrm{d}x)\\ &\quad= \int_{\{x:h^\nu(x)>M,u^\nu(x)\in A_\epsilon\}} h^\nu(x)\mu(\mathrm{d}x) + \int_{\{x:h^\nu(x)>M,u^\nu(x)\in A\setminus A_\epsilon\}} h^\nu(x)\mu(\mathrm{d}x). \end{aligned}$$

The first term of the line below is bounded by $\epsilon\int_{\{x:u^\nu(x)\in A\}} g(u^\nu(x))\mu(\mathrm{d}x)$. To deal with the second, let us observe that $\varphi(A\setminus A_\epsilon)$ is compact. We will show that its complement is open. The complement of $\varphi(A\setminus A_\epsilon)$ equals $\{\lambda\in\gamma A : \mathrm{dist}(\lambda,\gamma A\setminus A)<\epsilon\}$ end hence it is open. The compactness of $\varphi(A\setminus A_\epsilon)$, together with the continuity of $g\circ\varphi^{-1}$ on $\varphi(A\setminus A_\epsilon)$ shows that the second term vanishes, when M is big enough.

From uniform integrability we get that there exist $h\in L^1(\Omega,\mu)$ such that $h^\nu\mu \overset{\star}{\rightharpoonup} h\mu$ in measures. We have, however, that for arbitrary $\psi\in C(\Omega)$

$$\int_\Omega \psi h^\nu\mu(\mathrm{d}x) = (\psi D, L^\nu)\to(\psi D, L) = \int_\Omega \psi h\mu(\mathrm{d}x) = \int_\Omega \psi(x)\int_{\gamma A} D(\lambda)\tilde{\nu}_x(\mathrm{d}\lambda)\tilde{m}(\mathrm{d}x).$$

Using Radon-Nikodym decomposition $h = p(x)\mu + \tilde{m}_s$ and setting $\mathcal{F}(x) \overset{\mathrm{def}}{=\!=} \int_{\gamma A} D(\lambda)\tilde{\nu}_x(\mathrm{d}\lambda)$ we finally get

$$\int_\Omega \psi h\mu(\mathrm{d}x) = \int_\Omega \psi\mathcal{F}p\mu(\mathrm{d}x) + \int_\Omega \psi\mathcal{F}\tilde{m}_s(\mathrm{d}x).$$

Having in mind, that $h^\nu\mu \overset{\star}{\rightharpoonup} h\mu$, we see that the second term vanishes, so $\mathcal{F}(x)=0$ for $\tilde{m}_s$-almost every $x\in\Omega$. As D is strictly positive on $\varphi(A)$, we get that $\tilde{\nu}_x(\varphi(A))=0$ $\tilde{m}_s$-almost everywhere.

In the second case, that is when $\gamma A\setminus A=\emptyset$, we observe that A is compact and for any $F\in C(\gamma A)$ the sequence $h^\nu(x) \overset{\mathrm{def}}{=\!=} F(\varphi(u^\nu(x))g(u^\nu(x))\chi_{\{x:u^\nu\in A\}}$ is uniformly bounded, hence uniformly integrable. Knowing that $h^\nu\mu \overset{\star}{\rightharpoonup} h\mu$ we calculate again that

$$\int_\Omega \psi h^\nu\mu(\mathrm{d}x) \to \int_\Omega \psi\mathcal{F}p\mu(\mathrm{d}x) + \int_\Omega \psi\mathcal{F}\tilde{m}_s(\mathrm{d}x) = \int_\Omega \psi h\mu(\mathrm{d}x).$$

We see that the second term vanishes. Plugging $F\equiv 1$, having in mind that ν_x are probabilistic we obtain that $\tilde{m}_s=0$.

For the last part of the Lemma assume that $f\in C(\Omega\times\gamma A)$ is such that $F(x,\lambda) \overset{\mathrm{def}}{=\!=} f(x,\varphi(\lambda))g(\lambda)$ belongs to $C(\Omega\times A)$ and satisfies (4.1) on A, vanishes on $\Omega\times\partial A$ and has 0 limits when the second coordinate tends to infinity. In particular, the function F can be extended to a function defined on the whole $\mathbb{R}^m$ and vanishing in the infinity. This lets us apply the classical Young Theorem 2.10 to see that

$$F(x,u^\nu(x)) \rightharpoonup \mathscr{F}(x) \overset{\mathrm{def}}{=\!=} \int_{\mathbb{R}^m} F(x,\lambda)\mu_x(\mathrm{d}\lambda) = \int_{\mathrm{int}\,A} f(x,\varphi(\lambda))g(\lambda)\mu_x(\mathrm{d}\lambda) \text{ in } L^1(\Omega,\mu).$$

We have then $\mathscr{F}\mu = (f,\tilde{\nu}_x)\tilde{m} = (f,\tilde{\nu}_x)p(x)\mu + (f,\tilde{\nu}_x)\tilde{m}_s$. As f vanishes on $\gamma A\setminus\varphi(A)$, from the already proved parts of the lemma it follows that $(f,\tilde{\nu}_x)\tilde{m}_s\equiv 0$. □

Proof of the Theorem 4.3. The existence of appropriate spaces κA_i follows readily from Theorem 3.2 and Lemma 4.2. The remaining part of the proof is a slight modification of the proof of Theorem 3.1 in [18], but we will present it for completeness.

Using additivity of the integral, we may assume that f vanishes on every brick except for one A_i, which will be referred as A. For $F(x,\lambda) \overset{\mathrm{def}}{=\!=} f(x,\varphi^{-1}(\lambda))/g(\varphi^{-1}(\lambda))$ and $u^\nu(x)\in A$ we see that

$$f(x,u^\nu(x)) = (f/g)g = F(x,\varphi(u^\nu(x))g(u^\nu(x)).$$

Hence, from Lemma 4.5, we get

$$f(x,u^\nu(x))\mu(\mathrm{d}x) \overset{\star}{\rightharpoonup} \int_{\kappa A} F(x,\lambda)\tilde{\nu}_x(\mathrm{d}\lambda)\tilde{m}(\mathrm{d}x)$$

$$= \int_{\kappa A} F(x,\lambda)\tilde{\nu}_x(\mathrm{d}\lambda)p(x)\mu(\mathrm{d}x) + \int_{\kappa A} F(x,\lambda)\tilde{\nu}_x(\mathrm{d}\lambda)\tilde{m}_s(\mathrm{d}x).$$

By Lemma 4.5 we know, that the second term is in fact an integral over $\kappa A \setminus \varphi(A)$. Splitting the first integral into two pieces and applying (4.5), we get

$$f(x,u^\nu(x))\mu(\mathrm{d}x) \xrightarrow{\star} \int_{\operatorname{int} A} f(x,\lambda)\mu_x(\mathrm{d}\lambda)\mu(\mathrm{d}x)$$
$$+ \int_{\kappa A\setminus\varphi(\operatorname{int} A)} F(x,\lambda)\tilde{\nu}_x(\mathrm{d}\lambda)p(x)\mu(\mathrm{d}x) + \int_{\kappa A\setminus\varphi(A)} F(x,\lambda)\tilde{\nu}_x(\mathrm{d}\lambda)\tilde{m}_s(\mathrm{d}x) = a+b+c.$$

Noting that $\kappa A \setminus \varphi(\operatorname{int} A) = (\kappa A \setminus \varphi(A)) \cup \varphi(\partial A \cap A)$ lets us write

$$b + c = \int_{\varphi(\partial A\cap A)} F(x,\lambda)\tilde{\nu}_x(\mathrm{d}\lambda)p(x)\mu(\mathrm{d}x) + \int_{\kappa A\setminus\varphi(A)} F(x,\lambda)\tilde{\nu}_x(\mathrm{d}\lambda)\tilde{m}(\mathrm{d}x) = d + e.$$

Consider a function $h(x) \stackrel{\text{def}}{=\!=} \tilde{\nu}_x(\kappa A \setminus \varphi(A))$ and set $\Omega' \stackrel{\text{def}}{=\!=} \{x \in \Omega : h(x) \neq 0\}$. Choose any $y \in \kappa A \setminus \varphi(A)$. Let us define measures $m \stackrel{\text{def}}{=\!=} h\tilde{m}$ and ν_x by the condition, that for any $G \in C(\kappa A \setminus \varphi(A))$ we have

$$(G,\nu_x) = \begin{cases} 1/h(x)\int_{\kappa A\setminus\varphi(A)} G(\lambda)\tilde{\nu}_x(\mathrm{d}\lambda) & \text{for } x \in \Omega', \\ G(y) & \text{for } x \notin \Omega'. \end{cases}$$

Notice that

$$e = \int_{\kappa A\setminus\varphi(A)} F(x,\lambda)\nu_x(\mathrm{d}\lambda)m(\mathrm{d}x).$$

To deal with d, we introduce a function

$$w(x) \stackrel{\text{def}}{=\!=} \int_{\varphi(\partial A\cap A)} 1/g(\varphi^{-1}(\lambda))\tilde{\nu}_x(\mathrm{d}\lambda).$$

Choose an arbitrary $a \in \partial A \cap A$, set $\Omega'' \stackrel{\text{def}}{=\!=} \{x \in \Omega : \tilde{\nu}_x(\varphi(\partial A \cap A)) > 0\}$ and define $\bar{\nu}_x$ by the condition, that for any $G \in C(\partial A \cap A)$ we have

$$(G,\bar{\nu}_x) = \begin{cases} 1/w(x)\int_{\varphi(\partial A\cap A)} G/g(\varphi^{-1}(\lambda))\tilde{\nu}_x(\mathrm{d}\lambda) & \text{for } x \in \Omega'', \\ G(a) & \text{for } x \notin \Omega'' \end{cases}$$

and see that now

$$d = \int_{\partial A\cap A} f(x,\lambda)\bar{\nu}_x(\mathrm{d}\lambda)w(x)p(x)\mu(\mathrm{d}x).$$

Setting $\bar{m} \stackrel{\text{def}}{=\!=} w(x)p(x)\mu$ finishes the proof. □

4.2. Examples

For the first application we deal with the following, very simple situation.

Example 4.6 (Discontinuity on a hyperplane)**.** Let us take an open and bounded set $\Omega \subset \mathbb{R}^n$ equipped with Lebesgue measure $\mathscr{L}^n$, functions $u = (u_1, u_2, \ldots, u_m) : \Omega \to \mathbb{R}^m$ and $f(x,u) = a_1(x)\chi_{(-\infty,0)}(u_1) + a_2(x)\chi_{\{0\}}(u_1) + a_3(x)\chi_{(0,+\infty)}(u_1)$, where a_i are arbitrary continuous and bounded functions on Ω.

Let us set $H \overset{\text{def}}{=\!=} \{u \in \mathbb{R}^m : u_1 = 0\}$ and analogously $H^+ \overset{\text{def}}{=\!=} \{u \in \mathbb{R}^m : u_1 > 0\}$, $H^- \overset{\text{def}}{=\!=} \{u \in \mathbb{R}^m : u_1 < 0\}$. We take bricks $A_1 = H^-, A_2 = H, A_3 = H^+$, and set functions $g_i \equiv 1, i = 1,2,3$. We will explain that the function f satisfies the assumptions of Theorem 4.3. Indeed, condition (H5') is satisfied. Let us take $f_i \overset{\text{def}}{=\!=} f_{|A_i}$ for $i = 1,2,3$. Every function $f_i/g_i = a_i$ is continuous on $\Omega \times A_i$. Furthermore, we easily see that for any sequence $x_n \in \Omega, x_n \to x$ we have $f_i(x_n, u) \equiv a_i(x_n) \to a_i(x) \equiv f_i(x,u)$. As functions a_i are independent of u, the above convergence is the uniform convergence of functions dependent on u, which is exactly what was required in condition (4.1).

Let us know explain the shape of κA_i's in this situation. For brick A_1 we start by a homeomorphism $\phi : u \mapsto \dfrac{u}{1+|u|} = v$, which maps H^- into an open semiball, *i.e.* $B^- \overset{\text{def}}{=\!=} \{v \in \mathbb{R}^m : |v| < 1, v_1 < 0\}$. Our function $f_1 : \Omega \times H^-, f_1(x,u) = a_1(x)$ is now formally transformed to $f_1^b : \Omega \times B^-, f_1^b(x,v) \overset{\text{def}}{=\!=} f_1(x, \phi^{-1}(v)) = a_1(x)$.

We take any countable and dense subset $\mathcal{O} = \{x_1, x_2, \ldots,\}$ of Ω and use procedure described in Lemma 4.2. Since then we obtain a sequence of functions $f_i(v) \overset{\text{def}}{=\!=} f_1(x_i, v) = a_1(x_i)$ and use Theorem 3.2 to construct κA_1. The embedding $\varphi : B^- \to \ell_2$ defined in (3.1) for such $f_i's$ reads as

$$\varphi(v) = (v, 2^{-1}, 2^{-2}, 2^{-3}, \ldots).$$

As we see, the image of φ is actually a semiball $B^- \subset \mathbb{R}^m$ naturally embedded in ℓ_2 *via* $x \mapsto (x, 0, 0, \ldots)$ and then shifted by a vector $(\underbrace{0, \ldots, 0}_{m}, 2^{-1}, 2^{-2}, 2^{-3}, \ldots)$. Its closure – κA_1 – is then a closure of B^- shifted in the same way. It is homeomorphic (and, up to an equivalent disturbance of metric in ℓ_2, isometric) with $\overline{B^-} = \{v \in \mathbb{R}^m : |v| \le 1, v_1 \le 0\}$. For simplicity we may thus take $\kappa A_1 = \overline{B^-}$.

The construction of κA_3 is perfectly analogous. We leave to the reader to check that $\kappa A_3 = \overline{B^+} = \{v \in \mathbb{R}^m : |v| \le 1, v_1 \ge 0\}$. A very similar reasoning also shows that $\kappa A_2 = \overline{B_{m-1}} = \{v \in \mathbb{R}^m : |v| \le 1, v_1 = 0\}$, *i.e.* it is a closed unit ball of dimension $(m-1)$.

Having κA_i's constructed, we may take any tight (*i.e.* satisfying condition (T) in Thm. 4.3) sequence $u_j : \Omega \to \mathbb{R}^m$ and from the Theorem 4.3 it follows that the measure $f(x, u_j)\mathscr{L}^n$ converges weakly-$\star$ to a measure described by

$$\begin{aligned} & a_1(x) \int_{H^-} \mu_x^1(\mathrm{d}v)\mathrm{d}x + a_1(x) \int_{\partial B^- \setminus B^-} \nu_x^1(\mathrm{d}v) m^1(\mathrm{d}x) \\ & + a_2(x) \int_H \bar{\nu}_x^2(\mathrm{d}v)\bar{m}^2(\mathrm{d}x) + a_2(x) \int_{\{v \in \mathbb{R}^m : v_1 = 0, |v| = 1\}} \nu_x^2(\mathrm{d}v) m^2(\mathrm{d}x) \\ & + a_3(x) \int_{H^+} \mu_x^3(\mathrm{d}v)\mathrm{d}x + a_3(x) \int_{\partial B^+ \setminus B^+} \nu_x^3(\mathrm{d}v) m^3(\mathrm{d}x), \end{aligned}$$

where measures $\mu_x^1, \mu_x^3, \bar{\nu}_x^2, \bar{m}^2, \nu_x^1, m^1, \nu_x^2, m^2, \nu_x^3, m^3$ are like in Theorem 4.3.

Example 4.7 (Single brick case)**.** Let us assume that $f : \Omega \times \mathbb{R}^m$ is continuous and bounded, satisfying (4.1). In this situation we deal with one brick $A_1 = \mathbb{R}^m$ and obtain that, under assumptions of Theorem 4.3, there exist a subsequence of the sequence $f(x, u^\nu)\mathrm{d}x$ converging weakly-$\star$ to

$$\int_{\mathbb{R}^m} f(x,\lambda)\mu_x(\mathrm{d}\lambda)\mathrm{d}x + \int_{\kappa\mathbb{R}^m \setminus \mathbb{R}^m} f(x,\lambda)\nu_x(\mathrm{d}\lambda) m(\mathrm{d}x),$$

reproducing the classic DiPerna-Majda Theorem from Theorem 1 in [11].

Let us illustrate this case by two simple situations presented below.

Example 4.8 (A concentration in an arbitrary point of the remainder)**.** We take $\Omega \stackrel{\text{def}}{=\!=} (0,1)$ and for every function $u \in L^1(\Omega;\mathbb{R})$ define a functional

$$I_f(u) \stackrel{\text{def}}{=\!=} \int_\Omega f(u)\mathrm{d}x, \text{ where } f(u) \stackrel{\text{def}}{=\!=} \frac{1+|u|}{3}(2+\sin u). \tag{4.6}$$

We stress that

$$\frac{1}{3}(1+|u|) \le f(u) \le 1+|u|,$$

so f is of a linear growth. One can easily see however, that the classical recession function

$$f^\infty : \{-1,1\} \to \mathbb{R}, f^\infty(\xi) = \lim_{t\to+\infty} \frac{f(t\xi)}{t}$$

is not well defined. Therefore, for the analysis of the given functional, a more general compactification is needed.

Following the notation of the Theorem 4.3, we take $g(u) = 1+|u|$ and create the compactification $\kappa\mathbb{R}$ as in Lemma 3.1. We use an embedding

$$\varphi : \mathbb{R} \to \mathbb{R}^2, y \mapsto \big(\frac{y}{1+|y|}, \frac{f}{g}(y)\big)$$

and set $\kappa\mathbb{R} \stackrel{\text{def}}{=\!=} \overline{\varphi(\mathbb{R})}$. We easily see that

$$\kappa\mathbb{R} \setminus \varphi(\mathbb{R}) = \big(\{-1\} \cup \{1\}\big) \times [\frac{1}{3}, 1]$$

and the function

$$\tilde{f} : \kappa\mathbb{R} \to \mathbb{R}, \tilde{f}(z_1,z_2) = z_2$$

coincides with $\frac{f}{g}(\varphi^{-1}(z))$ on $\varphi(\mathbb{R})$.

We consider a bounded sequence in $L^1(\Omega)$ given by

$$u^\nu(x) \stackrel{\text{def}}{=\!=} (2\pi\nu+\alpha)\chi_{(\frac{1}{2},\frac{1}{2}+\frac{1}{\nu})}(x),$$

where α is a given constant angle, $\alpha \in [0,2\pi)$ and $\nu \in \mathbb{N}$. Our aim is to give a precise representation formula for the weak-$\star$ limit of the sequence $f(u^\nu)\mathscr{L}$, where f is given in (4.6) and $\mathscr{L}$ is a Lebesgue measure on Ω. Let us take a continuous and bounded $h : \Omega \to \mathbb{R}$ and a function $k : \mathbb{R} \to \mathbb{R}$, which is admissible for the aforementioned compactification $\kappa\mathbb{R}$. We remind that a function k is admissible for the compactification $\kappa\mathbb{R}$ whenever k/g possesses a continuous extension to $\tilde{k} : \kappa\mathbb{R} \to \mathbb{R}$. We compute

$$\begin{aligned}\int_\Omega h(x)k(u^\nu(x))\mathrm{d}x &= \int_{\Omega\setminus(\frac{1}{2},\frac{1}{2}+\frac{1}{\nu})} h(x)k(0)\mathrm{d}x + \int_{1/2}^{1/2+1/\nu} h(x)k(2\pi\nu+\alpha)\mathrm{d}x \\ &= k(0)\int_{\Omega\setminus(\frac{1}{2},\frac{1}{2}+\frac{1}{\nu})} h(x)\mathrm{d}x + \frac{k(2\pi\nu+\alpha)}{1+2\pi\nu+\alpha}\frac{1+2\pi\nu+\alpha}{\nu}\nu\int_{1/2}^{1/2+1/\nu} h(x)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\to k(0)\int_\Omega h(x)\mathrm{d}x+\lim_{\nu\to+\infty}\tilde{k}\Big(\frac{2\pi\nu+\alpha}{1+2\pi\nu+\alpha},\frac{f}{g}(2\pi\nu+\alpha)\Big)2\pi h\left(\frac{1}{2}\right)\\
&=k(0)\int_\Omega h(x)\mathrm{d}x+\tilde{k}\frac{2}{3}\Big(1+\sin\alpha\Big)2\pi h\left(\frac{1}{2}\right)\\
&=\int_\Omega h(x)\Big(\int_{\mathbb{R}}k(\lambda)\delta_0(\mathrm{d}\lambda)\Big)\mathrm{d}x+\int_\Omega h(x)\Big(\int_{\big(\{-1\}\cup\{1\}\big)\times[\frac{1}{3},1]}\tilde{k}(\lambda)\delta_{\frac{2}{3}(1+\sin\alpha)}(\mathrm{d}\lambda)\Big)2\pi\delta_{1/2}(\mathrm{d}x).
\end{aligned}
$$

The computation shows that, in particular,

$$
f(u^\nu)\mathscr{L}\overset{\star}{\rightharpoonup}\Big(\int_{\mathbb{R}}f(\lambda)\delta_0(\mathrm{d}\lambda)\Big)\mathscr{L}+\Big(\int_{\big(\{-1\}\cup\{1\}\big)\times[\frac{1}{3},1]}\tilde{f}(\lambda)\delta_{\frac{2}{3}(1+\sin\alpha)}(\mathrm{d}\lambda)\Big)2\pi\delta_{1/2}(\mathrm{d}x). \tag{4.7}
$$

Moreover, the same formula holds, whenever we put some other admissible function k instead of f. We note that for every point p in the interval $\{1\}\times[1/3,1]$ there exist $\alpha\in[0,2\pi]$ such that the DiPerna-Majda measure $\delta_{(1,\frac{2}{3}+\sin\alpha)}$ concentrates in p. We easily see that, taking $u'^\nu(x)\overset{\text{def}}{=\!=}-2\pi\nu+\alpha\chi_{(\frac{1}{2},\frac{1}{2}+\frac{1}{\nu})}(x)$ the same would hold for points in $\{-1\}\times[1/3,1]$. This shows that every point of $\kappa\mathbb{R}\setminus\varphi(\mathbb{R})$ is mandatory for the formula (4.7) to hold or, in other words, the compactification $\kappa\mathbb{R}$ is minimal for the use of Theorem 4.3 with the integrand f.

Example 4.9 (A concentration spread on the remainder)**.** Let us now consider a slightly different sequence, that is

$$
v^\nu\overset{\text{def}}{=\!=}(4\pi\nu x)\chi_{(\frac{1}{2},\frac{1}{2}+\frac{1}{\nu})}(x),
$$

taking the integrand f, density g, the interval Ω and the compactification $\kappa\mathbb{R}$ as in the previous example. We also remind that $\nu\in\mathbb{N}$. We take h,k as before. Furthermore, for technical reasons, let us assume that that k is strictly positive and $h\big(1/2\big)\neq 0$. We compute

$$
\begin{aligned}
\int_\Omega h(x)k(v^\nu(x))\mathrm{d}x&=\int_{\Omega\setminus(\frac{1}{2},\frac{1}{2}+\frac{1}{\nu})}h(x)k(0)\mathrm{d}x+\int_{1/2}^{1/2+1/\nu}h(x)k(4\pi\nu x)\mathrm{d}x\\
&=\int_{\Omega\setminus(\frac{1}{2},\frac{1}{2}+\frac{1}{\nu})}h(x)k(0)\mathrm{d}x+\frac{1+4\pi\nu\vartheta}{\nu}\nu\int_{1/2}^{1/2+1/\nu}h(x)\frac{k}{g}(4\pi\nu x)\mathrm{d}x,
\end{aligned}
\tag{4.8}
$$

where the last equality follows from the mean value theorem and $\vartheta\in(\frac{1}{2},\frac{1}{2}+\frac{1}{\nu})$. Plugging $\nu x=y$, we get

$$
\nu\int_{1/2}^{1/2+1/\nu}h(x)\frac{k}{g}(4\pi\nu x)\mathrm{d}x=\int_{\frac{\nu}{2}}^{\frac{\nu}{2}+1}h\left(\frac{y}{\nu}\right)\frac{k}{g}(4\pi y)\mathrm{d}y. \tag{4.9}
$$

Having in mind that $\frac{k}{g}(4\pi y)=\tilde{k}\left(\frac{4\pi y}{1+4\pi y},\frac{f}{g}(4\pi y)\right)$, plugging $y=\frac{\nu}{2}+\alpha$ and moving ν to $+\infty$, we see that

$$
\frac{k}{g}(2\pi\nu+4\pi\alpha)\to\tilde{k}\big(1,\frac{f}{g}(4\pi\alpha)\big)
$$

and hence the use of Lebesgue dominated convergence theorem yields

$$
\int_{\frac{\nu}{2}}^{\frac{\nu}{2}+1}h\left(\frac{y}{\nu}\right)\frac{k}{g}(4\pi y)\mathrm{d}y\to\int_0^1 h\left(\frac{1}{2}\right)\tilde{k}\big(1,\frac{f}{g}(4\pi\alpha)\big)\mathrm{d}\alpha. \tag{4.10}
$$

Putting now (4.8), (4.9) nad (4.10) together, we get that

$$f(v^\nu(x))\mathscr{L} \xrightharpoonup{\star} \Big(\int_{\mathbb{R}} f(\lambda)\delta_0(\mathrm{d}\lambda)\Big)\mathscr{L} + \Big(\int_{\{1\}\times[\frac{1}{3},1]} \tilde{f}(\lambda)\mu(\mathrm{d}\lambda)\Big)4\pi\delta_{\frac{1}{2}}(\mathrm{d}x),$$

where μ is such a measure, that

$$\int_0^1 \bar{k}\left(1, \frac{2+\sin 4\pi\alpha}{3}\right)\mathrm{d}\alpha = \int_{\{1\}\times[\frac{1}{3},1]} \bar{k}(\lambda)\mu(\mathrm{d}\lambda)$$

for every function $\bar{k}$ continuous on the interval $\{1\}\times[\frac{1}{3},1]$. Moreover, the same formula holds, whenever we put some other admissible function k instead of f. Loosening the technical assumptions $k>0$ and $h(1/2)\neq 0$ is very easy and left to the reader.

Remark 4.10. Let us note that the two examples above successfully cover all the integrands k, for which the recession function exists and is continuous. In such case, the function $\tilde{k}$ is constant on both $\{-1\}\times[1/3,1]$ and $\{1\}\times[1/3,1]$

Now let us briefly focus on a much more involving situation.

Example 4.11 (The need of a multidimensional compactification)**.** Let us take $\Omega=(0,1)$ and take $f(u) \overset{\text{def}}{=\!=} 1+a(x)|u|$, where $a:\Omega\to\mathbb{R}$ is an arbitrary continuous and bounded function. Following the notation of the Theorem 4.3 we set $g(u)=1+|u|$, so that $(f/g)(x,u)=\dfrac{(1+a(x)|u|)}{1+|u|}$. We easily see that the function (f/g) satisfies the assumption (4.1). Indeed,

$$\sup_{u\in\mathbb{R}}|(f/g)(x,u)-(f/g)(x_n,u)| \le |a(x)-a(x_n)|\frac{|u|}{1+|u|} \to 0, \text{ as } x_n\to x.$$

For every fixed $\bar{x}$, we see however that the shape of the graph of $(f/g)(\bar{x},u)$ relies on the value $a(\bar{x})$ and hence differs, dependently on $\bar{x}$. It seems that we need to treat every x separately, when arranging a compactification $\kappa\mathbb{R}$, exactly like in Lemma 4.2. Therefore, there is no visible way of embedding $\kappa\mathbb{R}$ into $\mathbb{R}^N$ for any N.

Now we move to the more involving reasoning, which generalises ([18], Thm. 4.2) to the situation of the integrand dependent on x, which was not considered so far.

Example 4.12 (Finitely many points of discontinuity)**.** Let us take open and bounded $\Omega\subset\mathbb{R}^n$, equipped with an arbitrary Borel measure μ and an arbitrary bounded function $\hat{f}:\Omega\times\mathbb{R}^m\to\mathbb{R}$ such that $\hat{f}$ is continuous with respect to x. We further require that

$$\hat{f}\in C\big(\Omega\times(\mathbb{R}^m\setminus\{P_1,P_2,\dots,P_k\})\big)$$

and satisfies (4.1). Our aim is to derive a representation formula for the weak-$\star$ limit of $\hat{f}(x,u^\nu)$.

First let us note that, as the number of points P_i is finite, we may find such a radius $r>0$ that balls centred in P_i's and of radius r are disjoint. For such a fixed r, let us define the set $A_0 \overset{\text{def}}{=\!=} \{u\in\mathbb{R}^m : \mathrm{dist}(u,P_i) > r/2 \text{ for every } i\}$. Now the sets $A_0, B(P_i,r), i=1,2,\dots,k$ form an open covering of $\mathbb{R}^m$, to which we may find a subordinate covering of unity – ψ_i. Now, instead of deriving representation formula for an arbitrary $\hat{f}$, we will work with $\psi_i\hat{f}=f$, – a function supported on $\Omega\times A_0$ or $\Omega\times B(P_i,r)$ for one particular $i=1,2,\dots,k$.

Let us begin with the case where f is supported on $\Omega\times A_0$. In this situation it is enough to deal with the previous example, as we may extend the function f by 0 to the whole domain $\Omega\setminus\mathbb{R}^m$.

In the latter case we may decompose $\mathbb{R}^m$ into three bricks, that is $A_1 = \mathbb{R}^m \setminus B(P_i, r)$, $A_2 = B(P_i, r) \setminus \{P_i\}$, $A_3 = \{P_i\}$, where radius r is precisely the same as before. Note that in this case, from the construction of the unit partition ψ_i, we have $f \equiv 0$ on $\Omega \times A_1$, as well as $f \equiv 0$ on the sufficiently small neighbourhood of $\Omega \times \partial B(P_i, r)$.

This observations show that the representation formula contains a null ingredient when dealing with the brick A_1. Also, if we use homeomorphism $\beta : A_2 \to R \overset{\text{def}}{=\!=} B(0, r+1) \setminus B(0,1)$ given by $\beta : u \mapsto u - P_i + \dfrac{u - P_i}{|u - P_i|}$ we see that the function $\check{f} : \Omega \times R \to \mathbb{R}$, $\check{f}(x, v) \overset{\text{def}}{=\!=} f(x, \beta^{-1}(v))$ vanishes on $\partial B(r+1, 0)$.

Therefore, the limit measure given by Theorem 4.3 will have a form $M = M_1 + M_2 + M_3$, where $M_1 = 0$, $M_3 = f(x, P_i)\bar{m}\mathrm{d}x$ and

$$M_2 = \int_{A_2} f(x,v)\mu_x(\mathrm{d}v)\mu(\mathrm{d}x) + \int_{\kappa R \setminus \varphi(R)} f(x,v)\nu_x(\mathrm{d}v)m(\mathrm{d}x),$$

and if we divide the remainder set $\kappa R \setminus \varphi(R)$ into parts

$$R_1 \overset{\text{def}}{=\!=} \{(v_1, v_2, \dots) \in \ell_2 : (v_1, \dots, v_m) \in \partial B(0,1)\},$$

$$R_2 \overset{\text{def}}{=\!=} \{(v_1, v_2, \dots) \in \ell_2 : (v_1, \dots, v_m) \in \partial B(0,r+1)\}$$

we may write

$$M_2 = \int_{A_2} f(x,v)\mu_x(\mathrm{d}v)\mu(\mathrm{d}x) + \int_{R_1} f(x,v)\nu_x(\mathrm{d}v)m(\mathrm{d}x).$$

Altogether, we get that

$$M = \int_{A_2} f(x,v)\mu_x(\mathrm{d}v)\mu(\mathrm{d}x) + \int_{R_1} f(x,v)\nu_x(\mathrm{d}v)m(\mathrm{d}x) + f(x,P_i)\bar{m}\mathrm{d}x.$$

Moreover, as the sequence $\{f(x, u^\nu)\}$ is bounded, we see that the measures $m, \bar{m}$ are absolutely continuous with respect to μ. This results in

$$M = \int_{A_2} f(x,v)\mu_x(\mathrm{d}v)\mu(\mathrm{d}x) + \int_{R_1} f(x,v)\nu_x(\mathrm{d}v)p(x)\mu(\mathrm{d}x) + f(x,P_i)q(x)\mu(\mathrm{d}x).$$

If we take now any function $h = h(v)$ continuous on $\mathbb{R}^m$, supported in $B(P_i, r)$ and $f \equiv 1$ in the neighbourhood of P_i, we see that it is admissible for compactifications κA_i and its extension is constantly equal to 1 on R_1. Hence we have

$$h(u^\nu)\mu \overset{\star}{\rightharpoonup} \int_{B(P_i,r)\setminus\{P_i\}} (h(v)\mu_x(\mathrm{d}v) + p(x) + q(x))\mu(\mathrm{d}x).$$

On the other hand, the Young Theorem 2.10 yields

$$h(u^\nu)\mu(\mathrm{d}x) \overset{\star}{\rightharpoonup} \int_{B(P_i,r)} h(v)\mu_x(\mathrm{d}v)\mu(\mathrm{d}x).$$

Since then, we see that $p(x)+q(x)=\int_{P_i} h(v)\mu_x(\mathrm{d}v)$ and, in general,

$$p(x)+q(x)=\int_{P_i} f(x,v)\mu_x(\mathrm{d}v).$$

5. Information about existing methods

5.1. Engelking theorem

In the book by Engleking [12], chapter 3.12.22(e), page 240, there is given an exercise leading to the formulation of Theorem [12]. The exercise shows that for every complete ring of continuous, bounded functions $\mathscr{F}$ on a Tychonoff's space $\mathbb{X}$ there exists a compactification $\Sigma\mathbb{X}$ such that the class of admissible functions is precisely $\mathscr{F}$. The solution of the exercise is however not given. The statement is well-known and broadly quoted in papers dealing with DiPerna-Majda measures theory. The proof hinted at by Engelking requires introducing topology on the set of ideals of the ring of continuous real valued functions defined on a space $\mathbb{X}$, which is meant to be compactified. From the proof it does not follow whether the resulting compact set can be embedded into any well-understood Banach space. On the other hand, the set $\mathbb{X}$ is only assumed to be Tychonoff regular, which, in some cases, is unnecessarily general for applications.

The idea of the hinted proof is to define space $\Sigma\mathbb{X}$ as the set of all maximal ideals in the ring $\mathscr{F}$. We introduce there the topology by defining its basis. For that purpose for any $f\in\mathscr{F}$, we define U_f – the set of all maximal ideals m in the ring $\mathscr{F}$ such that $f\notin m$. The basis for topology of $\Sigma\mathbb{X}$ is now precisely the family $\{U_f\}_{f\in\mathscr{F}}$.

Let us assume now that the ring $\mathscr{F}$ is separable and the set $\{f_i\}_{i\in\mathbb{N}}$ is dense in $\mathscr{F}$. We know from the Engelking's Theorem 2.8 that $\mathscr{F}$ is the ring of all continuous real-valued functions on $\Sigma\mathbb{X}$. This lets us check easily that

$$d(x,y)\overset{\text{def}}{=\!=}\sum_i \frac{|f_i(x)-f_i(y)|}{2^i\max\limits_{z\in\Sigma\mathbb{X}}|f_i(z)|}$$

is a metric and it induces topology which is equivalent to the topology on $\Sigma\mathbb{X}$. Since then, $\Sigma\mathbb{X}$ is metric.

Let us explain now why the space $\Sigma\mathbb{X}$ may be embedded into the ℓ^2 space. Existence of a countable basis of topology, the so-called second-countablity of the space, is equivalent to metrizability in the class of compact Hausdorff spaces. To see that the second-countability implies metrizability we need first to recall ([12], Thm. 2.3.23). The Theorem says that any second-countable Tychonoff space (so, by fact 2.3, in particular any compact Hausdorff space) may be homeomorphically embedded into Tychonoff cube of a countable weight, that is $[0,1]\times[0,1]\times\ldots$. This space is however homeomorphic to $[-1,1]\times[\frac{-1}{2},\frac{1}{2}]\times[\frac{-1}{2^2},\frac{1}{2^2}]\times\ldots\subset\ell^2$, which is metric. After all, any second-countable compact Hausdorff space is homeomorophic to a subspace of a metric space, hence it is metric.

To see that any metric compact space K is second-countable, it is enough to take, for fixed n, a particular cover of K – $\{B(x,1/n)\}_{x\in K}$ – and choose a finite subcover $\mathcal{U}_n$. Now the the family of open sets chosen in at least one of the subcovers $\mathcal{U}_n$, that is $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ forms a countable basis of topology.

Therefore we see that whenever the ring $\mathscr{F}$ is separable, the space $\Sigma\mathbb{X}$ is metrizable and may be described as a compact subset of Banach space ℓ^2. This fact gives us an information on the topological structure of that space. Nevertheless, the sketched reasoning is not constructive, as we cannot precisely determine the image of the embedding of $\Sigma\mathbb{X}$ into ℓ^2.

5.2. The Gelfand-Naimark Theorem

Another source of knowledge about compactifications, which seems very natural for specialists in analysis, but is not mentioned in the literature around Calculus of Variations, is the classical Gelfand-Naimark Theorem, see Theorem 1 of [15]. The most useful for our purposes variant of the theorem reads as follows ([2], Thm. 1.1.1):

Every commutative $\mathbb{C}^\star$ algebra A with "1" is isometric to the $\mathbb{C}^\star$ algebra of continuous, complex-valued functions on some compact space Φ_A.

The compact set Φ_A is precisely identified as a subset of the dual space to A consisting of such non-zero linear functionals $\phi : A \to \mathbb{C}$ that are also multiplicative (the so-called characters). The topology of Φ_A is an inherited weak-$\star$ topology from the dual space to A.

Let us explain how the statement of the Gelfand-Naimark Theorem contributes to understanding of the problem of compactifications. Take any locally compact and Hausdorff space $\mathbb{X}$. Let us choose any A – a complete ring (see Def. 2.6) of bounded, real-valued functions on $\mathbb{X}$. Obviously, A forms a $\mathbb{C}^\star$ algebra with the identity $\star$ operation. The space Φ_A is then a compactification of $\mathbb{X}$ and the space of continuous functions $C(\Phi_A) = A$. The last equality can be understood in the following manner. The set of continuous functions on $\mathbb{X}$, which can be continuously extended to continuous functions on Φ_A, is precisely A. Surprisingly, a careful analysis, see the proof of Lemma 1 and III on p. 1 in [15] and Satz 2 in [14], shows that the space Φ_A is homeomorphic to the Engelking's compactification of $\mathbb{X}$. Indeed, assigning to every character its kernel is a homeomorphism between Φ_A and $\Sigma\mathbb{X}$. Let us however note that the set Φ_A is identified with a certain subset of the dual space to A, which is a Banach space.

In general case, the compactness of Φ_A holds only with respect to weak-$\star$ topology of the dual of A. Nevertheless, Φ_A happens to be compact in strong, and hence metric, topology whenever A is a countably generated algebra. Furthermore, every continuous function on Φ_A with weak-$\star$ topology is automatically continuous in strong topology (the converse is false). It follows that in case when A is countably generated, Φ_A with metric topology is a compactification of $\mathbb{X}$ such that every function from A possesses a continuous extension to a function on Φ_A. The existence of such a metric compactification was not visible from the Engelking's construction. Unfortunately, the shape of Φ_A is hard to determine.

5.3. The embedding into a long product due to Keesling

The last idea we would like to consider, and seems to be a little noticed, is presented by Keesling in [23]. The author explains there a construction of the compactification analogous to the one by Engelking in more geometric fashion. Let us briefly outline this construction here. Take the set $\mathscr{F}$ of functions on $\mathbb{X}$ (an arbitrary Tychonoff's space), which are expected to be extendable to continuous functions on its compactification. We then use an embedding $i : \Omega \to \prod_{\{f\in\mathscr{F}\}} \mathbb{R}$, $i(x) \stackrel{\text{def}}{=\!=} \big(f(x)\big)_{\{f\in\mathscr{F}\}}$. The compactification is then the closure of the image of i. It is worth mentioning here that injectivity of i is guaranteed by the structure of the set $\mathscr{F}$, while compactness follows from boundedness of every single function $f \in \mathscr{F}$ and Tychonoff's Theorem. Although the construction is essentially less demanding for non-specialists, it does not offer insights concerning the problem of metrizability.

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