

TWO GEOMETRIC LEMMAS FOR \mathcal{S}^{N-1} -VALUED MAPS AND AN APPLICATION TO THE HOMOGENIZATION OF SPIN SYSTEMS*

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Abstract. We prove two geometric lemmas for \mathcal{S}^{N-1} -valued functions that allow to modify sequences of lattice spin functions on a small percentage of nodes during a discrete-to-continuum process so as to have a fixed average. This is used to simplify known formulas for the homogenization of spin systems.

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Dedicated to Enrique Zuazua on the occasion of his 60th birthday.

1. INTRODUCTION

A motivation for the analysis in the present work has been the study of molecular models where particles are interacting through a potential including both orientation and position variables. In particular we have in mind potentials of Gay-Berne type in models of Liquid Crystals [5, 6, 17, 19, 20]. In that context a molecule of a liquid crystal is thought of as an ellipsoid with a preferred axis, whose position is identified with a vector $w \in \mathbb{R}^3$ and whose orientation is a vector $u \in \mathcal{S}^2$. Given α and β two such particles, the interaction energy will depend on their orientations u_α, u_β and the distance vector $\zeta_{\alpha\beta} = w_\beta - w_\alpha$. We will concentrate on some properties on the dependence of the energy on u due to the geometry of \mathcal{S}^2 (more in general, of \mathcal{S}^{N-1}).

We restrict to a lattice model where all particles are considered as occupying the sites of a regular (cubic) lattice in the reference configuration. Note that in this assumption $\zeta_{\alpha\beta} = \beta - \alpha$ can be considered as an additional parameter and not a variable. Otherwise, in general the dependence on $\zeta_{\alpha\beta}$ is thought to be of Lennard-Jones type (for the treatment of such energies, still widely incomplete, we refer to [8, 10, 12]).

We introduce an energy density $G : \mathbb{Z}^m \times \mathbb{Z}^m \times \mathcal{S}^{N-1} \times \mathcal{S}^{N-1} \rightarrow \mathbb{R} \cup \{+\infty\}$, so that

$$G^\xi(\alpha, u, v) = G(\alpha, \alpha + \xi, u, v)$$

represents the free energy of two molecules oriented as u and v , occupying the sites α and $\beta = \alpha + \xi$ in the reference lattice. Note that we have included a dependence on α to allow for a microstructure at the lattice level, but the energy density is meaningful also in the homogeneous case, with G^ξ independent of α . Such energies

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are the basis for the variational analysis of complex multi-scale behaviours of spin systems (see, *e.g.*, [9] for the derivation of energies for liquid crystals, [2] for a study of the XY-model, [14] for a very refined study of the N-clock model, [15, 16] for chirality effects).

In order to understand the collective behaviour of a spin system, we introduce a small scaling parameter $\varepsilon > 0$, so that the description of such a behaviour can be formalized as a limit as $\varepsilon \rightarrow 0$. For each Lipschitz set Ω the discrete set $\mathbb{Z}_\varepsilon(\Omega) := \{\alpha \in \varepsilon\mathbb{Z}^m : (\alpha + [0, \varepsilon]^m) \cap \Omega \neq \emptyset\}$ represents a ‘discretization’ of the set Ω at scale ε . We also let $R > 0$ define a cut-off parameter representing the relevant range of the interactions (which we assume to be finite).

We define the family of scaled functionals

$$E_\varepsilon(u) = \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \leq R}} \sum_{\alpha \in R_\varepsilon^\xi(\Omega)} \varepsilon^m G^\xi \left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha + \varepsilon\xi) \right)$$

with domain functions $u : \mathbb{Z}_\varepsilon(\Omega) \rightarrow \mathcal{S}^{N-1}$, where $R_\varepsilon^\xi(\Omega) := \{\alpha \in \mathbb{Z}_\varepsilon(\Omega) : \alpha, \alpha + \varepsilon\xi \in \Omega\}$.

Extending functions defined on $\mathbb{Z}_\varepsilon(\Omega)$ to piecewise-constant interpolations, we may define a discrete-to-continuum convergence of u_ε to u . The assumptions on G ensure that u takes values in the unit ball. We can then perform an asymptotic analysis using the notation of Γ -convergence (see *e.g.* [7, 8, 13]). Energies as E_ε , but with u taking values in general compact sets K have been previously studied by Alicandro, Cicalese and Gloria (2008) [3], who describe the limit with a two-scale homogenization formula. In the case of \mathcal{S}^{N-1} -valued functions we simplify the homogenization formula reducing to test functions u satisfying a constraint on the average. This is a non-trivial fact since this constraint is non-convex, and its proof is the main technical point of the work.

The key observation is that we can modify the sequences u_ε so that they satisfy an exact condition on their average. We formalize this fact in two geometrical lemmas. The first one is a simple observation that each point in the unit ball in \mathbb{R}^N with $N > 1$ can be written exactly as the average of k vectors in \mathcal{S}^{N-1} for all $k \geq 2$, while the second one allows to modify sequences u_{ε_j} satisfying an asymptotic condition on the discrete average of u_ε with a sequence \tilde{u}_ε satisfying a sharp one and with the same energy E_ε up to a negligible error. This can be done if the asymptotic average of u_ε has modulus strictly less than one. In this case, most of the values of u_ε are not aligned; this allows to use a small percentage of these values to correct the asymptotic average to a sharp one by using the first lemma.

Optimizing on all the functions satisfying the same average condition satisfied by their limit we show that the Γ -limit of the sequence E_ε , for functions $u \in L^\infty(\Omega, B_1^N)$ is a continuum functional

$$E_0(u) = \int_{\Omega} G_{\text{hom}}(u) \, dx,$$

and the function G_{hom} satisfies a homogenization formula

$$G_{\text{hom}}(z) = \lim_{T \rightarrow \infty} \frac{1}{T^m} \inf \left\{ \mathcal{E}_T(u) : \frac{1}{T^m} \sum_{\alpha \in \mathbb{Z}_1(Q_T)} u(\alpha) = z \right\},$$

where $Q_T = (0, T)^m$ and

$$\mathcal{E}_T(u) = \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \leq R}} \sum_{\beta \in R_1^\xi(Q_T)} G^\xi(\beta, u(\beta), u(\beta + \xi)).$$

Note that the constraint in the homogenization formula involves the values of $u(\alpha)$, which belong to the non-convex set \mathcal{S}^{N-1} . This is an improvement with respect to Theorem 5.3 in [3], where the integrand of the limit

is characterized imposing a weaker constraint on the average of u ; namely it is shown that it equals

$$\overline{G}_{\text{hom}}(z) = \lim_{\eta \rightarrow 0^+} \lim_{T \rightarrow \infty} \frac{1}{T^m} \inf \left\{ \mathcal{E}_T(u) : \left| \frac{1}{T^m} \sum_{\alpha \in Z_1(Q_T)} u(\alpha) - z \right| \leq \eta \right\}. \quad (1.1)$$

A formula with a sharp constraint may be useful in higher-order developments, which characterize microstructure, interfaces and singularities.

The plan of the paper is as follows. In Section 2 we introduce the notation for discrete-to-continuum homogenization. In Section 3 we state and prove the geometric lemmas on \mathcal{S}^{N-1} -valued functions. In Section 4 we prove the homogenization formula, and finally in Section 5 we give a proof of the homogenization theorem.

2. NOTATION AND SETTING

Let $m, n \geq 1$, $N \geq 2$ be fixed. We denote by $\{e_1, \dots, e_m\}$ the standard basis of \mathbb{R}^m . Given two vectors $v_1, v_2 \in \mathbb{R}^n$, by (v_1, v_2) we denote their scalar product. If $v \in \mathbb{R}^m$, we use $|v|$ for the usual euclidean norm. \mathcal{S}^{N-1} is the standard unit sphere of \mathbb{R}^N and B_1^N the closed unit ball of \mathbb{R}^N . If $x \in \mathbb{R}$, its integer part is denoted by $[x]$. We also set $Q_T = (0, T)^m$ and $\mathcal{B}(\Omega)$ as the family of all open subsets of Ω . If A is an open bounded set, given a function $u : A \rightarrow \mathbb{R}^N$ we denote its average over A as

$$\langle u \rangle_A = \frac{1}{|A|} \int_A u(x) \, dx.$$

2.1. Discrete functions

Let $\Omega \subset \mathbb{R}^m$ be an open bounded domain with Lipschitz boundary, and let $\varepsilon > 0$ be the spacing parameter of the cubic lattice $\varepsilon\mathbb{Z}^m$. We define the set

$$\mathbb{Z}_\varepsilon(\Omega) := \{\alpha \in \varepsilon\mathbb{Z}^m : (\alpha + [0, \varepsilon]^m) \cap \Omega \neq \emptyset\}$$

and we will consider discrete functions $u : \mathbb{Z}_\varepsilon(\Omega) \rightarrow \mathcal{S}^{N-1}$ defined on the lattice. For $\xi \in \mathbb{Z}^m$, we define

$$R_\varepsilon^\xi(\Omega) := \{\alpha \in \mathbb{Z}_\varepsilon(\Omega) : \alpha, \alpha + \varepsilon\xi \in \Omega\},$$

while the “discrete” average of a function $v : \mathbb{Z}_\varepsilon(A) \rightarrow \mathcal{S}^{N-1}$ over an open bounded domain A will be denoted by

$$\langle v \rangle_A^{d, \varepsilon} = \frac{1}{\#(\mathbb{Z}_\varepsilon(A))} \sum_{\alpha \in \mathbb{Z}_\varepsilon(A)} v(\alpha).$$

2.2. Discrete energies

We assume that the Borel function $G : \mathbb{R}^m \times \mathbb{R}^m \times \mathcal{S}^{N-1} \times \mathcal{S}^{N-1} \rightarrow \mathbb{R}$ satisfies the following conditions

$$\text{(boundedness)} \quad \sup \{ |G(\alpha, \beta, u, v)| : \alpha, \beta \in \mathbb{R}^m, u, v \in \mathcal{S}^{N-1} \} < \infty; \quad (2.1)$$

$$\text{(periodicity)} \quad \text{there exists } l \in \mathbb{N} \text{ such that } G(\cdot, \cdot, u, v) \text{ is } Q_l \text{ periodic}; \quad (2.2)$$

$$\text{(lower semicontinuity)} \quad G \text{ is lower semicontinuous.} \quad (2.3)$$

Given $\xi \in \mathbb{R}^m$, we use the notation

$$G^\xi(\alpha, u, v) = G(\alpha, \alpha + \varepsilon\xi, u, v), \quad (2.4)$$

and define the functionals

$$\sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \leq R}} \sum_{\alpha \in R_\varepsilon^\xi(\Omega)} \varepsilon^m G^\xi \left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha + \varepsilon\xi) \right) \quad (2.5)$$

for $u : \mathbb{Z}_\varepsilon(\Omega) \rightarrow \mathcal{S}^{N-1}$.

2.3. Discrete-to-continuum convergence

In what follows we identify each discrete function u with its piecewise-constant extension \tilde{u} defined by $\tilde{u}(t) = u(\alpha)$ if $t \in \alpha + [0, \varepsilon)^m$. We introduce the sets:

$$\mathcal{A}_\varepsilon(\Omega; \mathcal{S}^{N-1}) := \left\{ \tilde{u} : \mathbb{R}^m \rightarrow \mathcal{S}^{N-1} : \tilde{u}(t) \equiv u(\alpha) \text{ if } t \in \alpha + [0, \varepsilon)^m, \text{ for } \alpha \in \mathbb{Z}_\varepsilon(\Omega) \right\}.$$

If no confusion is possible, we will simply write u instead of \tilde{u} . If $\varepsilon = 1$ we will simply write $\mathcal{A}(\Omega; \mathcal{S}^{N-1})$ in the place of $\mathcal{A}_1(\Omega; \mathcal{S}^{N-1})$.

Up to the identification of each function u with its piecewise-constant extension, we can consider energies $E_\varepsilon : L^\infty(\Omega, \mathcal{S}^{N-1}) \rightarrow \mathbb{R} \cup \{+\infty\}$ of the following form:

$$E_\varepsilon(u; \Omega) = \begin{cases} \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \leq R}} \sum_{\alpha \in R_\varepsilon^\xi(\Omega)} \varepsilon^m G^\xi \left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha + \varepsilon\xi) \right) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega; \mathcal{S}^{N-1}), \\ +\infty & \text{otherwise.} \end{cases} \quad (2.6)$$

Let $\varepsilon_j \rightarrow 0$ and let $\{u_j\}$ be a sequence of functions $u_j : \mathbb{Z}_{\varepsilon_j}(\Omega) \rightarrow \mathcal{S}^{N-1}$. We will say that $\{u_j\}$ converges to a function u if \tilde{u}_j is converging to u weakly* in L^∞ . Then we will say that the functionals defined in (2.5) Γ -converge to E_0 if E_ε defined in (2.6) Γ -converge to E_0 with respect to that convergence.

2.4. The homogenization theorem

We will prove the following discrete-to-continuum homogenization theorem.

Theorem 2.1. *Let E_ε be the energy defined in (2.5) and suppose that (2.1)–(2.3) hold. Then E_ε Γ -converge to the functional*

$$E_0(u) = \int_{\Omega} G_{\text{hom}}(u) \, dx \quad (2.7)$$

defined for functions $u \in L^\infty(\Omega, B_1^N)$. The function G_{hom} is given by the following asymptotic formula

$$G_{\text{hom}}(z) = \lim_{T \rightarrow \infty} \frac{1}{T^m} \inf \left\{ E_1(u; Q_T) : \langle u \rangle_{Q_T}^{d,1} = z \right\}. \quad (2.8)$$

The treatment of the average condition in (2.8) will be performed using a geometric lemma which exploits the geometry of \mathcal{S}^{N-1} , as shown in the next section.

3. TWO GEOMETRIC LEMMAS

In this section we provide two general lemmas. The first one is a simple observation on the characterisation of sums of vectors in \mathcal{S}^{N-1} , while the second one allows to satisfy conditions on the average of discrete functions with values in \mathcal{S}^{N-1} .

Lemma 3.1. *Let u be a vector in the ball B_k^N in \mathbb{R}^N centered in the origin and with radius $k \geq 2$; then u can be written as the sum of k vectors on \mathcal{S}^{N-1} :*

$$u = \sum_{i=1}^k u_i \quad u_i \in \mathcal{S}^{N-1}.$$

Equivalently, given $u \in B_1^N$ and $k \geq 2$, u can be written as the average of k vectors on \mathcal{S}^{N-1} :

$$u = \frac{1}{k} \sum_{i=1}^k u_i \quad u_i \in \mathcal{S}^{N-1}.$$

Proof. We proceed by induction on k .

Let $k = 2$ and let $u \in B_2^N$. The set $(u + \mathcal{S}^{N-1}) \cap \mathcal{S}^{N-1}$ is not empty set. If we choose $v \in (u + \mathcal{S}^{N-1}) \cap \mathcal{S}^{N-1}$ then the first induction step is proven with $u_1 = v$ and $u_2 = (u - v)$.

Suppose that the claim holds for $k - 1$. Let $u \in B_k^N$ and note that the set $(u + \mathcal{S}^{N-1}) \cap B_{d-1}^N$ is not empty. If $v \in (u + \mathcal{S}^{N-1}) \cap B_{d-1}^N$ by the inductive hypothesis we may write $v = u_1 + \dots + u_{k-1}$ with $u_j \in \mathcal{S}^{N-1}$. The claim is then proved by setting $u_k = u - v$. \square

Lemma 3.2. *Let $A \subset \mathbb{R}^m$ be an open bounded set with Lipschitz boundary. Let $\delta_j > 0$ be a spacing parameter and $u_j : \mathbb{Z}_{\delta_j}(A) \rightarrow \mathcal{S}^{N-1}$ be a sequence of discrete function. Suppose that $u_j \rightharpoonup^* u$ in $L^\infty(A, B_1^N)$ and that the average of u on A satisfies $|\langle u \rangle_A| < 1$. Then, for all j there exist \tilde{u}_j such that*

1. *the discrete average $\langle \tilde{u}_j \rangle_A^d := \frac{1}{\#\mathbb{Z}_{\delta_j}(A)} \sum_{i \in \mathbb{Z}_{\delta_j}(A)} \tilde{u}_j(i)$ is equal to $\langle u \rangle_A$;*

2. *the function \tilde{u}_j is obtained by modifying the function u_j in at most $2P_j$ points, with $\frac{P_j}{\#\mathbb{Z}_{\delta_j}(A)} \rightarrow 0$.*

Proof. To simplify the notation we set $\mathbb{Z}_j(A) = \mathbb{Z}_{\delta_j}(A)$ and $u_j^i = u_j(i)$.

Note that, by the weak convergence of u_j ,

$$\eta_j := |\langle u_j \rangle_A^d - \langle u \rangle_A| = o(1) \tag{3.1}$$

as $j \rightarrow +\infty$. We will treat the case that $\eta_j \neq 0$ since otherwise we simply take $\tilde{u}_j = u_j$.

Since $\langle u_j \rangle_A^d \rightarrow \langle u \rangle_A$, by the hypothesis that $|\langle u \rangle_A| < 1$ we may suppose that

$$|\langle u_j \rangle_A^d| \leq 1 - 2b \tag{3.2}$$

for all j , for some $b \in (0, 1/2)$.

Claim: setting $B = b/(4 - 2b)$, for every $i \in \mathbb{Z}_j(A)$ there exist at least $B \#\mathbb{Z}_j(A)$ indices $l \in \mathbb{Z}_j(A)$ such that $(u_j^i, u_j^l) \leq 1 - b$.

Indeed, otherwise there exists at least one index i for which the set

$$\mathcal{A}_b := \{l \in \mathbb{Z}_j(A) : (u_j^i, u_j^l) > 1 - b, l \neq i\} \tag{3.3}$$

is such that $\#\mathcal{A}_b \geq (1 - B)\#\mathbb{Z}_j(A)$ and we have

$$\begin{aligned}
|\langle u_j \rangle_A^d| &\geq (\langle u_j \rangle^d, u_j^i) = \frac{1}{\#\mathbb{Z}_j(A)} \sum_{l \in \mathbb{Z}_j(A)} (u_j^l, u_j^i) \\
&= \frac{1}{\#\mathbb{Z}_j(A)} \sum_{l \in \mathcal{A}_b} (u_j^l, u_j^i) + \frac{1}{\#\mathbb{Z}_j(A)} \sum_{l \in \mathbb{Z}_j(A) \setminus \mathcal{A}_b} (u_j^l, u_j^i) \\
&\geq \frac{1}{\#\mathbb{Z}_j(A)} (\#\mathcal{A}_b(1 - b) - (\#\mathbb{Z}_j(A) - \#\mathcal{A}_b)) \\
&= \frac{1}{\#\mathbb{Z}_j(A)} \left((2 - b)\#\mathcal{A}_b - \#\mathbb{Z}_j(A) \right) \\
&\geq (2 - b)(1 - B) - 1 = 1 - \frac{3}{2}b \\
&> |\langle u_j \rangle_A^d|,
\end{aligned}$$

where we have used (3.2) in the last estimate. We then obtain a contradiction, thus proving the claim.

By the Claim above, there exist $(B/2)\#\mathbb{Z}_j(A)$ pairs of indices (i_s, l_s) with $\{i_s, l_s\} \cap \{i_r, l_r\} = \emptyset$ if $r \neq s$ and

$$(u_j^{i_s}, u_j^{l_s}) \leq 1 - b. \quad (3.4)$$

Since $\eta_j \rightarrow 0$, with fixed $c > 0$ we may suppose that

$$B \#\mathbb{Z}_j(A) > 2 \left\lfloor \frac{\eta_j}{c} \#\mathbb{Z}_j(A) \right\rfloor + 1 \quad (3.5)$$

for all j .

We now set

$$P_j = \left\lfloor \frac{\eta_j}{c} \#\mathbb{Z}_j(A) \right\rfloor + 1, \quad (3.6)$$

so that by (3.5) there exist pairs (i_s, l_s) as above, with $s \in I_j := \{1, \dots, P_j\}$. Note that P_j satisfies the second claim of the theorem.

If for fixed j we define the vector

$$w = \sum_{i \in \mathbb{Z}_j(A)} u_j^i - \#(\mathbb{Z}_j(A)) \langle u \rangle_A - \sum_{s \in I_j} (u_j^{i_s} + u_j^{l_s}),$$

then we have

$$\begin{aligned}
|w| &\leq \#(\mathbb{Z}_j(A)) |\langle u_j \rangle^d - \langle u \rangle_A| + \sum_{s \in I_j} |u_j^{i_s} + u_j^{l_s}| \\
&\leq \#(\mathbb{Z}_j(A)) \eta_j + \sum_{s \in I_j} \sqrt{2 + 2(u_j^{i_s}, u_j^{l_s})} \\
&\leq \#(\mathbb{Z}_j(A)) \eta_j + P_j \sqrt{4 - 2b}.
\end{aligned}$$

Since $\#\mathbb{Z}_j(A)\eta_j < cP_j$ by (3.6), we then have $|w| \leq cP_j + P_j\sqrt{4-2b}$. We finally choose $c > 0$ such that $\sqrt{4-2b} < 2-c$, so that

$$|w| < 2P_j.$$

By Lemma 3.1, applied with $u = -w$ and $k = 2P_j$, there exists a set of $2P_j$ vectors in \mathcal{S}^{N-1} , that we may label as

$$\{\bar{u}_j^{i_s}, \bar{u}_j^{l_s} : s \in I_j\},$$

such that

$$\sum_{s \in I_j} (\bar{u}_j^{i_s} + \bar{u}_j^{l_s}) = -w. \quad (3.7)$$

If we now define \tilde{u}_j by setting

$$\tilde{u}_j^i = \begin{cases} \bar{u}_j^i & \text{if } i \in \{i_s, l_s : s \in I_j\} \\ u_j^i & \text{otherwise,} \end{cases} \quad (3.8)$$

we have

$$\begin{aligned} \langle \tilde{u}_j \rangle_A^d &= \frac{1}{\#\mathbb{Z}_j(A)} \left(\sum_{i \in \mathbb{Z}_j(A)} u_j^i - \sum_{s \in I_j} (u_j^{i_s} + u_j^{l_s}) + \sum_{s \in I_j} (\bar{u}_j^{i_s} + \bar{u}_j^{l_s}) \right) \\ &= \frac{1}{\#\mathbb{Z}_j(A)} \left(\sum_{i \in \mathbb{Z}_j(A)} u_j^i - \sum_{s \in I_j} (u_j^{i_s} + u_j^{l_s}) - w \right) \\ &= \langle u \rangle_A, \end{aligned}$$

and the proof is concluded. \square

Remark 3.3. The assumption $|\langle u \rangle_A| < 1$ in Lemma 3.2 is sharp: if $|\langle u \rangle_A| = 1$, we may have $u_j \xrightarrow{*} u$, such that $u_j \neq u$ and $|\langle u_j \rangle_A^{d, \delta_j}| = 1$ at every point (for example take u and u_j constant vectors in \mathcal{S}^{N-1}). In this case, in order to have $\langle u_j \rangle_A^{d, \delta_j} = \langle u \rangle_A$, we should change the function u_j in every point.

4. THE HOMOGENIZATION FORMULA

In this section we prove that the homogenization formula characterizing G_{hom} in Theorem 2.1 is well defined, and derive some properties of that function.

Proposition 4.1. *Let G be a function satisfying (2.1)–(2.3) and let G^ε be defined as in (2.4). For all $T > 0$ consider an arbitrary $x_T \in \mathbb{R}^m$, then the limit*

$$\lim_{T \rightarrow \infty} \frac{1}{T^m} \inf \left\{ E_1(u; x_T + Q_T) : \langle u \rangle_{x_T + Q_T}^{d, 1} = z \right\} \quad (4.1)$$

exists for all $z \in B_1^N$.

Proof. Let $z \in B_1^N$ be fixed. In the following we will assume G to be 1-periodic (which means that in (2.2) we consider $l = 1$) and $x_T = 0$, since the general case can be derived similarly following arguments already present

for example in [1, 3] and only needing a heavier notation. Let $t > 0$ and consider the function

$$g_t(z) = \frac{1}{t^m} \inf \left\{ E_1(u, \zeta; Q_t) : \langle u \rangle_{Q_t}^{d,1} = z \right\}. \quad (4.2)$$

In the rest of the proof we will drop the dependence on z . Let u_t be a test function for g_t such that

$$\frac{1}{t^m} E_1(u_t; Q_t) \leq g_t + \frac{1}{t}, \quad (4.3)$$

For every $s > t$ we want to prove that $g_s < g_t$ up to a controlled error.

For fixed s, t , we introduce the following notation:

$$I := \left\{ 0, \dots, \left\lfloor \frac{s}{t} \right\rfloor - 1 \right\}^m.$$

We can construct a test functions for g_s as

$$u_s(\beta) = \begin{cases} u_t(\beta - ti) & \text{if } \beta \in ti + Q_t \quad i \in I \\ \bar{u}(\beta) & \text{otherwise,} \end{cases}$$

where \bar{u} is a \mathcal{S}^{N-1} -valued function such that $\langle u_s \rangle_{Q_s}^{d,1} = z$. We can choose such \bar{u} thanks to Lemma 3.1: define

$$\mathbb{Z}(Q_s) = \mathbb{Z}^m \cap Q_s, \quad Q_{s,t} = \left(\bigcup_{i \in I} (ti + Q_t) \cap \mathbb{Z}(Q_s) \right).$$

We want \bar{u} to be such that

$$\sum_{\beta \in \mathbb{Z}(Q_s)} u_s(\beta) = z \#(\mathbb{Z}(Q_s)).$$

Equivalently

$$\sum_{\substack{\beta \in Q_{s,t} \\ \beta \in ti + Q_t}} u_t(\beta - ti) + \sum_{\beta \in \mathbb{Z}(Q_s) \setminus Q_{s,t}} \bar{u}(\beta) = z \#(\mathbb{Z}(Q_s)),$$

which means that

$$\sum_{\beta \in \mathbb{Z}(Q_s) \setminus Q_{s,t}} \bar{u}(\beta) = z \left(\#(\mathbb{Z}(Q_s)) - \#(Q_{s,t}) \right). \quad (4.4)$$

On the left-hand side of (4.4) we are summing $\#(\mathbb{Z}(Q_s)) - \#(Q_{s,t})$ vectors in \mathcal{S}^{N-1} while on the right-hand side we have a vector which belongs to a ball whose radius is at most $\#(\mathbb{Z}(Q_s)) - \#(Q_{s,t})$.

If $|z| < 1$, thanks to Lemma 3.1 we know that it is possible to choose the values of \bar{u} in such a way that the relation (4.4) is satisfied.

If $|z| = 1$, we simply set $\bar{u}(\beta) \equiv z$, and again (4.4) is satisfied. Moreover, we observe that

$$R_1^\xi(Q_s) \subseteq \left(\bigcup_{i \in I} R_1^\xi(ti + Q_t) \right) \cup \left(R_1^\xi \left(Q_s \setminus \bigcup_{i \in I} (ti + Q_t) \right) \right) \cup \left(\bigcup_{i \in I} (ti + (\{0, \dots, t + R\}^N \setminus \{0, \dots, t - R\}^N)) \right)$$

and if β belongs to one of the last two set of indices, then $D_1^{\xi} \zeta_s(\beta) = M(\xi/|\xi|)$.
 Recalling now (2.1), for some $\bar{C} > 0$ big enough, we have that

$$\begin{aligned} g_s &\leq \frac{1}{s^m} E_1(u_s; Q_s) \\ &\leq \left\lfloor \frac{s}{t} \right\rfloor^m \frac{1}{s^m} E_1(u_t; Q_t) + \frac{1}{s^m} \bar{C} \left(s^m - \left\lfloor \frac{s}{t} \right\rfloor^m t^m + \left\lfloor \frac{s}{t} \right\rfloor^m ((t+R)^m - (t-R)^m) \right). \end{aligned}$$

Using now (4.3) we get

$$g_s \leq \left\lfloor \frac{s}{t} \right\rfloor^m \frac{t^m}{s^m} \left(g_t + \frac{1}{t} \right) + \frac{1}{s^m} \bar{C} \left(s^m - \left\lfloor \frac{s}{t} \right\rfloor^m t^m + \left\lfloor \frac{s}{t} \right\rfloor^m ((t+R)^m - (t-R)^m) \right).$$

Letting now $s \rightarrow +\infty$ and then $t \rightarrow +\infty$, we have that

$$\limsup_{s \rightarrow +\infty} g_s(z) \leq \liminf_{t \rightarrow +\infty} g_t(z),$$

which concludes the proof. \square

Remark 4.2. Note that for $z \in \mathcal{S}^{N-1}$ the only test function for the minimum problem in (4.1) is the constant z , so that the limit is actually an average over the period with $u = z$.

Proposition 4.3. *The function G_{hom} as defined in (2.8) is convex and lower semicontinuous in B_1^N .*

Proof. We want to show that for every $0 \leq t \leq 1$ and for every $z_1, z_2 \in B_1^N$ it holds:

$$G_{\text{hom}}(tz_1 + (1-t)z_2, M) \leq t G_{\text{hom}}(z_1, M) + (1-t) G_{\text{hom}}(z_2, M). \quad (4.5)$$

Let $k \in \mathbb{N}$ be fixed; having (2.2) in mind, and thanks to Proposition 4.1, it is not restrictive to take $k \in \mathbb{N}$. We define

$$g_k(z) = \frac{1}{k^m} \inf \left\{ E_1(u; Q_k) : \langle u \rangle_{Q_k}^{d,1} = z \right\}. \quad (4.6)$$

In the following we will denote $g_k^1 = g_k(z_1)$, $g_k^2 = g_k(z_2)$.

Let u_k^1 and u_k^2 be functions such that

$$\frac{1}{k^m} E_1(u_k^1; Q_k) \leq g_k^1 + \frac{1}{k}, \quad (4.7)$$

$$\frac{1}{k^m} E_1(u_k^2; Q_k) \leq g_k^2 + \frac{1}{k}. \quad (4.8)$$

Let $h > k$ be such that $h/k \in \mathbb{N}$. Denote $g_h = g_h(tz_1 + (1-t)z_2)$, we define the following test function for g_h :

$$u_h(\beta) = \begin{cases} u_k^1(\beta - ki) & \text{if } \beta \in ki + Q_k \quad i \in \left\{ 0, \dots, \frac{h}{k} - 1 \right\}^{m-1} \times \left\{ 0, \dots, \left\lfloor \frac{h}{k} t \right\rfloor - 1 \right\} \\ u_k^2(\beta - ki) & \text{if } \beta \in ki + Q_k \quad i \in \left\{ 0, \dots, \frac{h}{k} - 1 \right\}^{m-1} \times \left\{ \frac{h}{k} - \left\lfloor \frac{h(1-t)}{k} \right\rfloor, \dots, \frac{h}{k} - 1 \right\} \\ \bar{u}(\beta) & \text{otherwise,} \end{cases}$$

Reasoning as in Proposition 4.1, thanks to Lemma 3.1 we can choose the values of \bar{u} such that

$$\langle u_s \rangle_{Q_s}^{d,1} = tz_1 + (1-t)z_2.$$

By (2.1) and (2.2), for some $\bar{C} > 0$ we get

$$\begin{aligned} g_h &\leq \frac{1}{h^m} E_1(u_h, \zeta_h; Q_h) \\ &\leq \frac{1}{h^m} \left(\frac{h}{k}\right)^{m-1} \left\lfloor \frac{h}{k} t \right\rfloor E_1(u_k^1; Q_k) + \frac{1}{h^m} \left(\frac{h}{k}\right)^{m-1} \left\lfloor \frac{h(1-t)}{k} \right\rfloor E_1(u_k^2; Q_k) \\ &\quad + \frac{1}{h^m} \bar{C} \left(h^m - \left(\frac{h}{k}\right)^{m-1} \left(\left\lfloor \frac{h}{k} t \right\rfloor + \left\lfloor \frac{h(1-t)}{k} \right\rfloor \right) k^m \right) \\ &\quad + \frac{1}{h^m} \bar{C} \left(\frac{h}{k}\right)^m ((k+R)^m - (k-R)^m). \end{aligned}$$

Then, thanks to (4.7) and (4.8), we can rewrite the above relation as

$$\begin{aligned} g_h &\leq \frac{k^m}{h^m} \left(\frac{h}{k}\right)^{m-1} \left\lfloor \frac{h}{k} t \right\rfloor \left(g_k^1 + \frac{1}{k} \right) + \frac{k^m}{h^m} \left(\frac{h}{k}\right)^{m-1} \left\lfloor \frac{h(1-t)}{k} \right\rfloor \left(g_k^2 + \frac{1}{k} \right) \\ &\quad + \frac{1}{h^m} \bar{C} \left(h^m - \left(\frac{h}{k}\right)^{m-1} \left(\left\lfloor \frac{h}{k} t \right\rfloor + \left\lfloor \frac{h(1-t)}{k} \right\rfloor \right) k^m \right) \\ &\quad + \frac{1}{h^m} \bar{C} \left(\frac{h}{k}\right)^m ((k+R)^m - (k-R)^m). \end{aligned}$$

Letting $h \rightarrow +\infty$ and then $k \rightarrow +\infty$, we can conclude the proof of the convexity.

From the convexity and the boundedness of G_{hom} we deduce that it is continuous in the interior of B_1^N . Moreover, by (2.3) it is lower semicontinuous at points on the boundary of B_1^N . \square

Remark 4.4. Note that the function G_{hom} may not be continuous on B_1^N , even if it is convex. It suffices to take a nearest-neighbor energy $G = G^\xi$ independent on α , with $G(e_1, e_1) = 0$ and $G(u, v) = 1$ otherwise. In this case, in particular $G_{\text{hom}}(e_1) = 0$ and $G_{\text{hom}}(z) = 1$ if $z \in \mathcal{S}^{N-1}$, $z \neq e_1$.

5. PROOF OF THE HOMOGENIZATION THEOREM

Thanks to the geometric lemmas in Section 3, we can now easily give a proof of the homogenization theorem. We remark that it will be sufficient to prove a lower bound, since we may resort to the homogenization result of Alicandro, Cicalese and Gloria [3] in order to give an upper bound for the homogenized functional. Indeed, by (1.1) we have $\bar{G}_{\text{hom}} \leq G_{\text{hom}}$, so that functional (2.7) is an upper bound for the homogenized energy. Note that we could directly prove the upper bound using approximation results and constructions starting from the formula for G_{hom} , but this would be essentially a repetition of the arguments in [3].

In order to prove a lower bound we will make use of Lemma 3.2 and of the Fonseca-Müller blow-up technique [11, 18]. Let $\varepsilon_j \rightarrow 0$ be a vanishing sequence of parameters, let $u \in L^\infty(\Omega, B_1^N)$ and let $u_j \rightharpoonup^* u$ with $u \in L^\infty(\Omega, B_1^N)$. We define the measures μ_j by setting

$$\mu_j(A) = \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \leq R}} \sum_{\alpha \in R_{\varepsilon_j}^\xi(\Omega)} \varepsilon_j^m G^\xi \left(\frac{\alpha}{\varepsilon_j}, u_j(\alpha), u_j(\alpha + \varepsilon_j \xi) \right) \delta_{\alpha + \frac{\varepsilon_j}{2} \xi}(A)$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$, where δ_x denotes the usual Dirac delta measure at x . Since the measures are equibounded, by the weak* compactness of measures there exists a limit measure μ on Ω such that, up to subsequences, $\mu_j \rightharpoonup^* \mu$. We consider the Radon-Nikodym decomposition of the limit measure μ with respect to the m -dimensional Lebesgue measure \mathcal{L}^m :

$$\mu = \frac{d\mu}{dx} d\mathcal{L}^m + \mu^s, \quad \mu^s \perp \mathcal{L}^m.$$

Besicovitch Derivation Theorem [4] states that almost every point in Ω with respect to \mathcal{L}^m is a Lebesgue point for μ . So, we may suppose that $x_0 \in \mathbb{Z}_{\varepsilon_j}(\Omega)$ be a Lebesgue point both for u and for μ and let $Q_\rho(x_0) = x_0 + (-\rho/2, \rho/2)^m$. We then have

$$\frac{d\mu}{dx}(x_0) = \lim_{\rho \rightarrow 0^+} \frac{\mu(Q_\rho(x_0))}{\mathcal{L}^m(Q_\rho(x_0))} = \lim_{\rho \rightarrow 0^+} \frac{1}{\rho^m} \mu(Q_\rho(x_0)). \quad (5.1)$$

Recalling that

$$\mu(Q_\rho(x_0)) = \lim_{j \rightarrow +\infty} \mu_j(Q_\rho(x_0)) \quad (5.2)$$

except for a countable set of ρ , by a diagonalization argument on (5.1) and (5.2) we can extract a subsequence $\{\rho_j\}$ such that

$$\frac{d\mu}{dx}(x_0) = \lim_{j \rightarrow +\infty} \frac{1}{\rho_j^m} \mu_j(Q_{\rho_j}(x_0))$$

holds. This means that

$$\frac{d\mu}{dx}(x_0) = \lim_{j \rightarrow +\infty} \left(\frac{\varepsilon_j}{\rho_j} \right)^m \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \leq R}} \sum_{\alpha \in R_{\varepsilon_j}^{\xi}(\Omega)} G^{\xi} \left(\frac{\alpha}{\varepsilon_j}, u_j(\alpha), u_j(\alpha + \varepsilon_j \xi) \right) \delta_{\alpha + \frac{\varepsilon_j}{2} \xi}(Q_{\rho_j}(x_0)). \quad (5.3)$$

Also note that, by the weak* convergence of u_j to u , we have

$$\langle u \rangle_{Q_\rho(x_0)} = \lim_j \langle u_j \rangle_{Q_\rho(x_0)}^{\varepsilon_j} = \lim_j \langle u_j \rangle_{Q_\rho(x_0)}^{d, \varepsilon_j},$$

and that for almost every x_0 we have

$$\lim_{\rho \rightarrow 0^+} \langle u \rangle_{Q_\rho(x_0)} = u(x_0),$$

so that we may assume that

$$\lim_j \langle u_j \rangle_{Q_{\rho_j}(x_0)}^{d, \varepsilon_j} = u(x_0).$$

We can parameterizing functions on a common unit cube, by setting

$$v_j(\gamma) = u_j(x_0 + \rho_j \gamma) \quad \text{and} \quad \delta_j = \frac{\varepsilon_j}{\rho_j}.$$

With this parameterization, (5.3) reads

$$\frac{d\mu}{dx}(x_0) = \lim_{j \rightarrow +\infty} \delta_j^m \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \leq R}} \sum_{\gamma + \frac{x_0}{\rho_j} \in R_{\delta_j}^\xi(\frac{1}{\rho_j}\Omega)} G^\xi \left(\frac{x_0}{\varepsilon_j} + \frac{\gamma}{\delta_j}, v_j(\gamma), v_j(\gamma + \delta_j \xi) \right) \delta_{\gamma - \frac{x_0}{\rho_j} + \frac{\delta_j}{2} \xi}(Q_1(0)).$$

Note that we have $\lim_j \langle v_j \rangle_{Q_1(0)}^{d, \delta_j} = u(x_0)$. We can then apply Lemma 3.2 with v_j in the place of u_j and $A = Q_1(0)$. We obtain a family \tilde{v}_j with

$$\langle \tilde{v}_j \rangle_{Q_1(0)}^{d, \delta_j} = u(x_0), \quad (5.4)$$

and such that $\tilde{v}_j(\gamma) = v_j(\gamma)$ except for a set P_j of γ with $\#P_j = o(\delta_j^{-m})$. From this, the finiteness of the range of interactions and the boundedness of G^ξ , we further rewrite as

$$\frac{d\mu}{dx}(x_0) = \lim_{j \rightarrow +\infty} \delta_j^m \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \leq R}} \sum_{\gamma + \frac{x_0}{\rho_j} \in R_{\delta_j}^\xi(\frac{1}{\rho_j}\Omega)} G^\xi \left(\frac{x_0}{\varepsilon_j} + \frac{\gamma}{\delta_j}, \tilde{v}_j(\gamma), \tilde{v}_j(\gamma + \delta_j \xi) \right) \delta_{\gamma - \frac{x_0}{\rho_j} + \frac{\delta_j}{2} \xi}(Q_1(0)).$$

We now set

$$T_j = \frac{\rho_j}{\varepsilon_j} = \delta_j^{-1}, \quad x_{T_j} = \frac{x_0}{\varepsilon_j},$$

so that

$$\begin{aligned} & \frac{d\mu}{dx}(x_0) \\ &= \lim_{j \rightarrow +\infty} \frac{1}{T_j^m} \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \leq R}} \sum_{\gamma + \frac{x_{T_j}}{T_j} \in R_{T_j^{-1}}^\xi(\frac{1}{\rho_j}\Omega)} G^\xi \left(x_{T_j} + T_j \gamma, \tilde{v}_j(\gamma), \tilde{v}_j \left(\gamma + \frac{\xi}{T_j} \right) \right) \delta_{\gamma - \frac{x_{T_j}}{T_j} + \frac{1}{2T_j} \xi}(Q_1(0)) \\ &= \lim_{j \rightarrow +\infty} \frac{1}{T_j^m} \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \leq R}} \sum_{\eta + x_{T_j} \in R_1^\xi(\frac{1}{\varepsilon_j}\Omega)} G^\xi \left(x_{T_j} + \eta, \tilde{v}_j \left(\frac{\eta}{T_j} \right), \tilde{v}_j \left(\frac{\eta + \xi}{T_j} \right) \right) \delta_{\eta - x_{T_j} + \frac{1}{2} \xi}(Q_{T_j}(0)). \end{aligned}$$

We now set $w_j(\eta) = v_j(\frac{\eta}{T_j})$ and use the boundedness of G^ξ and R to deduce that

$$\frac{d\mu}{dx}(x_0) \geq \lim_{j \rightarrow +\infty} \frac{1}{T_j^m} \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \leq R}} \sum_{\eta + x_{T_j} \in R_1^\xi(Q_{T_j})} G^\xi \left(x_{T_j} + \eta, \tilde{w}_j(\eta), \tilde{w}_j(\eta + \xi) \right) \delta_{\eta - x_{T_j} + \frac{1}{2} \xi}(Q_{T_j}(0)).$$

Note indeed that by considering interactions in $R_1^\xi(Q_{T_j})$ we neglect a contribution of a number of interactions of order $O(T_j^{m-1})$; *i.e.*, an energy contribution of order $O(T_j^{-1})$. Noting that w_j satisfies the constraint

$$\langle \tilde{w}_j \rangle_{x_{T_j} + Q_{T_j}(0)}^{d, 1} = u(x_0),$$

thanks to (5.4), by Proposition 4.1 we finally deduce that

$$\frac{d\mu}{dx}(x_0) \geq \lim_{j \rightarrow \infty} \frac{1}{T_j^m} \inf \left\{ E_1(w; x_{T_j} + Q_{T_j}) : \langle w \rangle_{x_{T_j} + Q_{T_j}}^{d,1} = u(x_0) \right\} = G_{\text{hom}}(u(x_0)).$$

Since this holds for almost all $x_0 \in \Omega$, we have proved the desired lower bound, and hence the convergence result.

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