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TWO GEOMETRIC LEMMAS FOR $\mathcal{S}^{N-1}$-VALUED MAPS AND AN APPLICATION TO THE HOMOGENIZATION OF SPIN SYSTEMS*

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Abstract. We prove two geometric lemmas for $\mathcal{S}^{N-1}$-valued functions that allow to modify sequences of lattice spin functions on a small percentage of nodes during a discrete-to-continuum process so as to have a fixed average. This is used to simplify known formulas for the homogenization of spin systems.

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Dedicated to Enrique Zuazua on the occasion of his 60th birthday.

1. Introduction

A motivation for the analysis in the present work has been the study of molecular models where particles are interacting through a potential including both orientation and position variables. In particular we have in mind potentials of Gay-Berne type in models of Liquid Crystals [5, 6, 17, 19, 20]. In that context a molecule of a liquid crystal is thought of as an ellipsoid with a preferred axis, whose position is identified with a vector $w \in \mathbb{R}^3$ and whose orientation is a vector $u \in \mathcal{S}^2$. Given α and β two such particles, the interaction energy will depend on their orientations u_α, u_β and the distance vector $\zeta_{\alpha\beta} = w_\beta - w_\alpha$. We will concentrate on some properties on the dependence of the energy on u due to the geometry of $\mathcal{S}^2$ (more in general, of $\mathcal{S}^{N-1}$).

We restrict to a lattice model where all particles are considered as occupying the sites of a regular (cubic) lattice in the reference configuration. Note that in this assumption $\zeta_{\alpha\beta} = \beta - \alpha$ can be considered as an additional parameter and not a variable. Otherwise, in general the dependence on $\zeta_{\alpha\beta}$ is thought to be of Lennard-Jones type (for the treatment of such energies, still widely incomplete, we refer to [8, 10, 12]).

We introduce an energy density $G : \mathbb{Z}^m \times \mathbb{Z}^m \times \mathcal{S}^{N-1} \times \mathcal{S}^{N-1} \to \mathbb{R} \cup \{+\infty\}$, so that

$$G^\xi(\alpha, u, v) = G(\alpha, \alpha + \xi, u, v)$$

represents the free energy of two molecules oriented as u and v, occupying the sites α and $\beta = \alpha + \xi$ in the reference lattice. Note that we have included a dependence on α to allow for a microstructure at the lattice level, but the energy density is meaningful also in the homogeneous case, with G^ξ independent of α. Such energies

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are the basis for the variational analysis of complex multi-scale behaviours of spin systems (see, *e.g.*, [9] for the derivation of energies for liquid crystals, [2] for a study of the XY-model, [14] for a very refined study of the N-clock model, [15, 16] for chirality effects).

In order to understand the collective behaviour of a spin system, we introduce a small scaling parameter $\varepsilon > 0$, so that the description of such a behaviour can be formalized as a limit as $\varepsilon \to 0$. For each Lipschitz set Ω the discrete set $\mathbb{Z}_\varepsilon(\Omega) := \{\alpha \in \varepsilon\mathbb{Z}^m : (\alpha + [0,\varepsilon)^m) \cap \Omega \neq \emptyset\}$ represents a 'discretization' of the set Ω at scale ε. We also let $R > 0$ define a cut-off parameter representing the relevant range of the interactions (which we assume to be finite).

We define the family of scaled functionals

$$E_\varepsilon(u) = \sum_{\substack{\xi\in\mathbb{Z}^m\\ |\xi|\le R}} \sum_{\alpha\in R^\xi_\varepsilon(\Omega)} \varepsilon^m G^\xi\left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha+\varepsilon\xi)\right)$$

with domain functions $u : \mathbb{Z}_\varepsilon(\Omega) \to \mathcal{S}^{N-1}$, where $R^\xi_\varepsilon(\Omega) := \{\alpha \in \mathbb{Z}_\varepsilon(\Omega) \ : \ \alpha,\, \alpha+\varepsilon\xi \in \Omega\}$.

Extending functions defined on $\mathbb{Z}_\varepsilon(\Omega)$ to piecewise-constant interpolations, we may define a discrete-to-continuum convergence of u_ε to u. The assumptions on G ensure that u takes values in the unit ball. We can then perform an asymptotic analysis using the notation of Γ-convergence (see *e.g.* [7, 8, 13]). Energies as E_ε, but with u taking values in general compact sets K have been previously studied by Alicandro, Cicalese and Gloria (2008) [3], who describe the limit with a two-scale homogenization formula. In the case of $\mathcal{S}^{N-1}$-valued functions we simplify the homogenization formula reducing to test functions u satisfying a constraint on the average. This is a non-trivial fact since this constraint is non-convex, and its proof is the main technical point of the work.

The key observation is that we can modify the sequences u_ε so that they satisfy an exact condition on their average. We formalize this fact in two geometrical lemmas. The first one is a simple observation that each point in the unit ball in $\mathbb{R}^N$ with $N > 1$ can be written exactly as the average of k vectors in S^{N-1} for all $k \geq 2$, while the second one allows to modify sequences u_{ε_j} satisfying an asymptotic condition on the discrete average of u_ε with a sequence $\widetilde{u}_\varepsilon$ satisfying a sharp one and with the same energy E_ε up to a negligible error. This can be done if the asymptotic average of u_ε has modulus strictly less than one. In this case, most of the values of u_ε are not aligned; this allows to use a small percentage of these values to correct the asymptotic average to a sharp one by using the first lemma.

Optimizing on all the functions satisfying the same average condition satisfied by their limit we show that the Γ-limit of the sequence E_ε, for functions $u \in L^\infty(\Omega, B^N_1)$ is a continuum functional

$$E_0(u) = \int_\Omega G_{\rm hom}(u)\, {\rm d}x,$$

and the function $G_{\rm hom}$ satisfies a homogenization formula

$$G_{\rm hom}(z) = \lim_{T\to\infty} \frac{1}{T^m} \inf\Big\{\mathcal{E}_T(u) : \frac{1}{T^m} \sum_{\alpha\in Z_1(Q_T)} u(\alpha) = z\Big\},$$

where $Q_T = (0,T)^m$ and

$$\mathcal{E}_T(u) = \sum_{\substack{\xi\in\mathbb{Z}^m\\ |\xi|\le R}} \sum_{\beta\in R^\xi_1(Q_T)} G^\xi\left(\beta, u(\beta), u(\beta+\xi)\right).$$

Note that the constraint in the homogenization formula involves the values of $u(\alpha)$, which belong to the non-convex set $\mathcal{S}^{N-1}$. This is an improvement with respect to Theorem 5.3 in [3], where the integrand of the limit is characterized imposing a weaker constraint on the average of u; namely it is shown that it equals

$$\overline{G}_{\rm hom}(z)=\lim_{\eta\to0^+}\lim_{T\to\infty}\frac{1}{T^m}\inf\Big\{\mathcal{E}_T(u):\Big|\frac{1}{T^m}\sum_{\alpha\in Z_1(Q_T)}u(\alpha)-z\Big|\leq\eta\Big\}. \tag{1.1}$$

A formula with a sharp constraint may be useful in higher-order developments, which characterize microstructure, interfaces and singularities.

The plan of the paper is as follows. In Section 2 we introduce the notation for discrete-to-continuum homogenization. In Section 3 we state and prove the geometric lemmas on $\mathcal{S}^{N-1}$-valued functions. In Section 4 we prove the homogenization formula, and finally in Section 5 we give a proof of the homogenization theorem.

2. Notation and setting

Let $m,n\geq1$, $N\geq2$ be fixed. We denote by $\{e_1,\dots,e_m\}$ the standard basis of $\mathbb{R}^m$. Given two vectors $v_1,v_2\in\mathbb{R}^n$, by (v_1,v_2) we denote their scalar product. If $v\in\mathbb{R}^m$, we use $|v|$ for the usual euclidean norm. $\mathcal{S}^{N-1}$ is the standard unit sphere of $\mathbb{R}^N$ and B_1^N the closed unit ball of $\mathbb{R}^N$. If $x\in\mathbb{R}$, its integer part is denoted by $\lfloor x\rfloor$. We also set $Q_T=(0,T)^m$ and $\mathcal{B}(\Omega)$ as the family of all open subsets of Ω. If A is an open bounded set, given a function $u:A\to\mathbb{R}^N$ we denote its average over A as

$$\langle u\rangle_A=\frac{1}{|A|}\int_A u(x)\,\mathrm{d}x.$$

2.1. Discrete functions

Let $\Omega\subset\mathbb{R}^m$ be an open bounded domain with Lipschitz boundary, and let $\varepsilon>0$ be the spacing parameter of the cubic lattice $\varepsilon\mathbb{Z}^m$. We define the set

$$\mathbb{Z}_\varepsilon(\Omega):=\{\alpha\in\varepsilon\mathbb{Z}^m:\ (\alpha+[0,\varepsilon)^m)\cap\Omega\neq\emptyset\}$$

and we will consider discrete functions $u:\mathbb{Z}_\varepsilon(\Omega)\to\mathcal{S}^{N-1}$ defined on the lattice. For $\xi\in\mathbb{Z}^m$, we define

$$R_\varepsilon^\xi(\Omega):=\{\alpha\in\mathbb{Z}_\varepsilon(\Omega)\ :\ \alpha,\,\alpha+\varepsilon\xi\in\Omega\},$$

while the "discrete" average of a function $v:\mathbb{Z}_\varepsilon(A)\to\mathcal{S}^{N-1}$ over an open bounded domain A will be denoted by

$$\langle v\rangle_A^{d,\varepsilon}=\frac{1}{\#(\mathbb{Z}_\varepsilon(A))}\sum_{\alpha\in\mathbb{Z}_\varepsilon(A)}v(\alpha).$$

2.2. Discrete energies

We assume that the Borel function $G:\mathbb{R}^m\times\mathbb{R}^m\times\mathcal{S}^{N-1}\times\mathcal{S}^{N-1}\to\mathbb{R}$ satisfies the following conditions

$$\text{(boundedness)}\ \sup\big\{|G(\alpha,\beta,u,v)|:\alpha,\beta\in\mathbb{R}^m,\,u,v\in\mathcal{S}^{N-1}\big\}<\infty; \tag{2.1}$$

$$\text{(periodicity) there exists } l\in\mathbb{N} \text{ such that } G(\cdot,\cdot,u,v) \text{ is } \ Q_l\ \text{ periodic;} \tag{2.2}$$

$$\text{(lower semicontinuity) } G \text{ is lower semicontinuous.} \tag{2.3}$$

Given $\xi\in\mathbb{R}^m$, we use the notation

$$G^\xi\,(\alpha,u,v)=G(\alpha,\alpha+\varepsilon\xi,u,v), \tag{2.4}$$

and define the functionals

$$\sum_{\substack{\xi\in\mathbb{Z}^m\\ |\xi|\le R}}\ \sum_{\alpha\in R^\xi_\varepsilon(\Omega)} \varepsilon^m G^\xi\left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha+\varepsilon\xi)\right) \tag{2.5}$$

for $u : \mathbb{Z}_\varepsilon(\Omega) \to \mathcal{S}^{N-1}$.

2.3. Discrete-to-continuum convergence

In what follows we identify each discrete function u with its piecewise-constant extension $\tilde{u}$ defined by $\tilde{u}(t) = u(\alpha)$ if $t \in \alpha + [0, \varepsilon)^m$. We introduce the sets:

$$\mathcal{A}_\varepsilon(\Omega;\mathcal{S}^{N-1}) := \Big\{\tilde{u} : \mathbb{R}^m \to \mathcal{S}^{N-1}\ :\ \tilde{u}(t) \equiv u(\alpha) \text{ if } t \in \alpha + [0,\varepsilon)^m, \text{for } \alpha \in \mathbb{Z}_\varepsilon(\Omega)\Big\}.$$

If no confusion is possible, we will simply write u instead of $\tilde{u}$. If $\varepsilon = 1$ we will simply write $\mathcal{A}(\Omega;\mathcal{S}^{N-1})$ in the place of $\mathcal{A}_1(\Omega;\mathcal{S}^{N-1})$.

Up to the identification of each function u with its piecewise-constant extension, we can consider energies $E_\varepsilon : L^\infty(\Omega, \mathcal{S}^{N-1}) \to \mathbb{R} \cup \{+\infty\}$ of the following form:

$$E_\varepsilon(u;\Omega) = \begin{cases} \displaystyle\sum_{\substack{\xi\in\mathbb{Z}^m\\ |\xi|\le R}}\ \sum_{\alpha\in R^\xi_\varepsilon(\Omega)} \varepsilon^m G^\xi\left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha+\varepsilon\xi)\right) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega;\mathcal{S}^{N-1}),\\ +\infty & \text{otherwise.}\end{cases} \tag{2.6}$$

Let $\varepsilon_j \to 0$ and let $\{u_j\}$ be a sequence of functions $u_j : \mathbb{Z}_{\varepsilon_j}(\Omega) \to \mathcal{S}^{N-1}$. We will say that $\{u_j\}$ *converges to a function* u if $\tilde{u}_j$ is converging to u weakly* in L^∞. Then we will say that the functionals defined in (2.5) Γ-converge to E_0 if E_ε defined in (2.6) Γ-converge to E_0 with respect to that convergence.

2.4. The homogenization theorem

We will prove the following discrete-to-continuum homogenization theorem.

Theorem 2.1. *Let E_ε be the energy defined in (2.5) and suppose that (2.1)–(2.3) hold. Then E_ε Γ-converge to the functional*

$$E_0(u) = \int_\Omega G_{\rm hom}(u)\,{\rm d}x \tag{2.7}$$

defined for functions $u \in L^\infty(\Omega, B^N_1)$. The function $G_{\rm hom}$ is given by the following asymptotic formula

$$G_{\rm hom}(z) = \lim_{T\to\infty}\frac{1}{T^m}\inf\Big\{E_1(u;Q_T) : \langle u\rangle^{d,1}_{Q_T} = z\Big\}. \tag{2.8}$$

The treatment of the average condition in (2.8) will be performed using a geometric lemma which exploits the geometry of $\mathcal{S}^{N-1}$, as shown in the next section.

3. Two geometric lemmas

In this section we provide two general lemmas. The first one is a simple observation on the characterisation of sums of vectors in $\mathcal{S}^{N-1}$, while the second one allows to satisfy conditions on the average of discrete functions with values in $\mathcal{S}^{N-1}$.

Lemma 3.1. *Let u be a vector in the ball B_k^N in $\mathbb{R}^N$ centered in the origin and with radius $k \geq 2$; then u can be written as the sum of k vectors on $\mathcal{S}^{N-1}$:*

$$u = \sum_{i=1}^{k} u_i \qquad u_i \in \mathcal{S}^{N-1}.$$

Equivalently, given $u \in B_1^N$ and $k \geq 2$, u can be written as the average of k vectors on $\mathcal{S}^{N-1}$:

$$u = \frac{1}{k}\sum_{i=1}^{k} u_i \qquad u_i \in \mathcal{S}^{N-1}.$$

Proof. We proceed by induction on k.

Let $k = 2$ and let $u \in B_2^N$. The set $(u + \mathcal{S}^{N-1}) \cap \mathcal{S}^{N-1}$ is not empty set. If we choose $v \in (u + \mathcal{S}^{N-1}) \cap \mathcal{S}^{N-1}$ then the first induction step is proven with $u_1 = v$ and $u_2 = (u - v)$.

Suppose that the claim holds for $k-1$. Let $u \in B_k^N$ and note that the set $(u + \mathcal{S}^{N-1}) \cap B_{d-1}^N$ is not empty. If $v \in (u + \mathcal{S}^{N-1}) \cap B_{d-1}^N$ by the inductive hypothesis we may write $v = u_1 + \cdots + u_{k-1}$ with $u_j \in \mathcal{S}^{N-1}$. The claim is then proved by setting $u_k = u - v$. □

Lemma 3.2. *Let $A \subset \mathbb{R}^m$ be an open bounded set with Lipschitz boundary. Let $\delta_j > 0$ be a spacing parameter and $u_j : \mathbb{Z}_{\delta_j}(A) \to \mathcal{S}^{N-1}$ be a sequence of discrete function. Suppose that $u_j \rightharpoonup^* u$ in $L^\infty(A, B_1^n)$ and that the average of u on A satisfies $|\langle u\rangle_A| < 1$. Then, for all j there exist $\tilde{u}_j$ such that*

1. *the discrete average $\langle \tilde{u}_j\rangle_A^d := \dfrac{1}{\#\mathbb{Z}_{\delta_j}(A)} \displaystyle\sum_{i\in\mathbb{Z}_{\delta_j}(A)} \tilde{u}_j(i)$ is equal to $\langle u\rangle_A$;*
2. *the function $\tilde{u}_j$ is obtained by modifying the function u_j in at most $2P_j$ points, with $\dfrac{P_j}{\#\mathbb{Z}_{\delta_j}(A)} \to 0$.*

Proof. To simplify the notation we set $\mathbb{Z}_j(A) = \mathbb{Z}_{\delta_j}(A)$ and $u_j^i = u_j(i)$.

Note that, by the weak convergence of u_j,

$$\eta_j := |\langle u_j\rangle^d - \langle u\rangle_A| = o(1) \tag{3.1}$$

as $j \to +\infty$. We will treat the case that $\eta_j \neq 0$ since otherwise we simply take $\tilde{u}_j = u_j$.

Since $\langle u_j\rangle_A^d \to \langle u\rangle_A$, by the hypothesis that $|\langle u\rangle_A| < 1$ we may suppose that

$$|\langle u_j\rangle_A^d| \leq 1 - 2b \tag{3.2}$$

for all j, for some $b \in (0, 1/2)$.

Claim: setting $B = b/(4-2b)$, for every $i \in \mathbb{Z}_j(A)$ there exist at least $B\,\#\mathbb{Z}_j(A)$ indices $l \in \mathbb{Z}_j(A)$ such that $(u_j^i, u_j^l) \leq 1 - b$.

Indeed, otherwise there exists at least one index i for which the set

$$\mathcal{A}_b := \big\{l \in \mathbb{Z}_j(A) : (u_j^i, u_j^l) > 1 - b,\ l \neq i\big\} \tag{3.3}$$

is such that $\#\mathcal{A}_b \geq (1-B)\#\mathbb{Z}_j(A)$ and we have

$$\begin{aligned} |\langle u_j\rangle_A^d| &\geq (\langle u_j\rangle^d, u_j^i) = \frac{1}{\#\mathbb{Z}_j(A)} \sum_{l\in\mathbb{Z}_j(A)} (u_j^l, u_j^i) \\ &= \frac{1}{\#\mathbb{Z}_j(A)} \sum_{l\in\mathcal{A}_b} (u_j^l, u_j^i) + \frac{1}{\#\mathbb{Z}_j(A)} \sum_{l\in\mathbb{Z}_j(A)\setminus\mathcal{A}_b} (u_j^l, u_j^i) \\ &\geq \frac{1}{\#\mathbb{Z}_j(A)} (\#\mathcal{A}_b(1-b) - (\#\mathbb{Z}_j(A) - \#\mathcal{A}_b)) \\ &= \frac{1}{\#\mathbb{Z}_j(A)} \Big((2-b)\#\mathcal{A}_b - \#\mathbb{Z}_j(A)\Big) \\ &\geq (2-b)(1-B) - 1 = 1 - \frac{3}{2}b \\ &> |\langle u_j\rangle_A^d|, \end{aligned}$$

where we have used (3.2) in the last estimate. We then obtain a contradiction, thus proving the claim.

By the Claim above, there exist $(B/2)\#\mathbb{Z}_j(A)$ pairs of indices (i_s, l_s) with $\{i_s, l_s\} \cap \{i_r, l_r\} = \emptyset$ if $r \neq s$ and

$$(u_j^{i_s}, u_j^{l_s}) \leq 1 - b. \tag{3.4}$$

Since $\eta_j \to 0$, with fixed $c > 0$ we may suppose that

$$B\,\#\mathbb{Z}_j(A) > 2\Big\lfloor \frac{\eta_j}{c} \#\mathbb{Z}_j(A) \Big\rfloor + 1 \tag{3.5}$$

for all j.

We now set

$$P_j = \Big\lfloor \frac{\eta_j}{c} \#\mathbb{Z}_j(A) \Big\rfloor + 1, \tag{3.6}$$

so that by (3.5) there exist pairs (i_s, l_s) as above, with $s \in I_j := \{1, \dots, P_j\}$. Note that P_j satisfies the second claim of the theorem.

If for fixed j we define the vector

$$w = \sum_{i\in\mathbb{Z}_j(A)} u_j^i - \#(\mathbb{Z}_j(A))\langle u\rangle_A - \sum_{s\in I_j} (u_j^{i_s} + u_j^{l_s}),$$

then we have

$$\begin{aligned} |w| &\leq \#(\mathbb{Z}_j(A))|\langle u_j\rangle^d - \langle u\rangle_A| + \sum_{s\in I_j} |u_j^{i_s} + u_j^{l_s}| \\ &\leq \#(\mathbb{Z}_j(A))\eta_j + \sum_{s\in I_j} \sqrt{2 + 2(u_j^{i_s}, u_j^{l_s})} \\ &\leq \#(\mathbb{Z}_j(A))\eta_j + P_j\sqrt{4-2b}. \end{aligned}$$

Since $\#\mathbb{Z}_j(A)\eta_j < c\,P_j$ by (3.6), we then have $|w| \le c\,P_j + P_j\sqrt{4-2b}$. We finally choose $c>0$ such that $\sqrt{4-2b} < 2-c$, so that

$$|w| < 2P_j.$$

By Lemma 3.1, applied with $u=-w$ and $k=2P_j$, there exists a set of $2P_j$ vectors in $\mathcal{S}^{N-1}$, that we may label as

$$\{\overline{u}_j^{i_s}, \overline{u}_j^{l_s} : s \in I_j\},$$

such that

$$\sum_{s\in I_j}(\overline{u}_j^{i_s} + \overline{u}_j^{l_s}) = -w. \tag{3.7}$$

If we now define $\tilde{u}_j$ by setting

$$\tilde{u}_j^i = \begin{cases} \overline{u}_j^i & \text{if } i \in \{i_s, l_s : s \in I_j\} \\ u_j^i & \text{otherwise,} \end{cases} \tag{3.8}$$

we have

$$\begin{aligned} \langle \tilde{u}_j\rangle_A^d &= \frac{1}{\#\mathbb{Z}_j(A)}\Big(\sum_{i\in\mathbb{Z}_j(A)} u_j^i - \sum_{s\in I_j}(u_j^{i_s}+u_j^{l_s}) + \sum_{s\in I_j}(\overline{u}_j^{i_s}+\overline{u}_j^{l_s})\Big) \\ &= \frac{1}{\#\mathbb{Z}_j(A)}\Big(\sum_{i\in\mathbb{Z}_j(A)} u_j^i - \sum_{s\in I_j}(u_j^{i_s}+u_j^{l_s}) - w\Big) \\ &= \langle u\rangle_A, \end{aligned}$$

and the proof is concluded. □

Remark 3.3. The assumption $|\langle u\rangle_A| < 1$ in Lemma 3.2 is sharp: if $|\langle u\rangle_A| = 1$, we may have $u_j \rightharpoonup^* u$, such that $u_j \neq u$ and $|\langle u_j\rangle_A^{d,\delta_j}| = 1$ at every point (for example take u and u_j constant vectors in $\mathcal{S}^{N-1}$). In this case, in order to have $\langle u_j\rangle_A^{d,\delta_j} = \langle u\rangle_A$, we should change the function u_j in every point.

4. The homogenization formula

In this section we prove that the homogenization formula characterizing G_{hom} in Theorem 2.1 is well defined, and derive some properties of that function.

Proposition 4.1. *Let G be a function satisfying (2.1)–(2.3) and let G^ξ be defined as in (2.4). For all $T>0$ consider an arbitrary $x_T \in \mathbb{R}^m$, then the limit*

$$\lim_{T\to\infty}\frac{1}{T^m}\inf\Big\{E_1(u; x_T+Q_T) : \ \langle u\rangle_{x_T+Q_T}^{d,1} = z\Big\} \tag{4.1}$$

exists for all $z \in B_1^N$.

Proof. Let $z\in B_1^N$ be fixed. In the following we will assume G to be 1-periodic (which means that in (2.2) we consider $l=1$) and $x_T=0$, since the general case can be derived similarly following arguments already present for example in [1, 3] and only needing a heavier notation. Let $t > 0$ and consider the function

$$g_t(z) = \frac{1}{t^m} \inf\Big\{ E_1(u, \zeta; Q_t) :\ \langle u \rangle^{d,1}_{Q_t} = z \Big\}. \tag{4.2}$$

In the rest of the proof we will drop the dependence on z. Let u_t be a test function for g_t such that

$$\frac{1}{t^m} E_1(u_t; Q_t) \leq g_t + \frac{1}{t}, \tag{4.3}$$

For every $s > t$ we want to prove that $g_s < g_t$ up to a controlled error.

For fixed s, t, we introduce the following notation:

$$I := \Big\{ 0, \dots, \Big\lfloor \frac{s}{t} \Big\rfloor - 1 \Big\}^m.$$

We can construct a test functions for g_s as

$$u_s(\beta) = \begin{cases} u_t(\beta - ti) & \text{if} \quad \beta \in ti + Q_t \quad i \in I \\ \bar{u}(\beta) & \text{otherwise,} \end{cases}$$

where $\bar{u}$ is a $\mathcal{S}^{N-1}$-valued function such that $\langle u_s \rangle^{d,1}_{Q_s} = z$. We can choose such $\bar{u}$ thanks to Lemma 3.1: define

$$\mathbb{Z}(Q_s) = \mathbb{Z}^m \cap Q_s, \qquad Q_{s,t} = \Big(\bigcup_{i \in I} (ti + Q_t) \cap \mathbb{Z}(Q_s) \Big).$$

We want $\bar{u}$ to be such that

$$\sum_{\beta \in \mathbb{Z}(Q_s)} u_s(\beta) = z \#(\mathbb{Z}(Q_s)).$$

Equivalently

$$\sum_{\substack{\beta \in Q_{s,t} \\ \beta \in ti + Q_t}} u_t(\beta - ti) + \sum_{\beta \in \mathbb{Z}(Q_s) \setminus Q_{s,t}} \bar{u}(\beta) = z \#(\mathbb{Z}(Q_s)),$$

which means that

$$\sum_{\beta \in \mathbb{Z}(Q_s) \setminus Q_{s,t}} \bar{u}(\beta) = z \Big(\#(\mathbb{Z}(Q_s)) - \#(Q_{s,t}) \Big). \tag{4.4}$$

On the left-hand side of (4.4) we are summing $\#(\mathbb{Z}(Q_s)) - \#(Q_{s,t})$ vectors in $\mathcal{S}^{N-1}$ while on the right-hand side we have a vector which belongs to a ball whose radius is at most $\#(\mathbb{Z}(Q_s)) - \#(Q_{s,t})$.

If $|z| < 1$, thanks to Lemma 3.1 we know that it is possible to choose the values of $\bar{u}$ in such a way that the relation (4.4) is satisfied.

If $|z| = 1$, we simply set $\bar{u}(\beta) \equiv z$, and again (4.4) is satisfied. Moreover, we observe that

$$R^\xi_1(Q_s) \subseteq \Big(\bigcup_{i \in I} R^\xi_1(ti + Q_t) \Big) \cup \Big(R^\xi_1 \Big(Q_s \setminus \bigcup_{i \in I} (ti + Q_t) \Big) \Big) \cup \Big(\bigcup_{i \in I} (ti + (\{0, \dots, t + R\}^N \setminus \{0, \dots, t - R\}^N)) \Big)$$

and if β belongs to one of the last two set of indices, then $D_1^\xi \zeta_s(\beta) = M(\xi/|\xi|)$.
Recalling now (2.1), for some $\bar{C} > 0$ big enough, we have that

$$\begin{aligned} g_s &\le \frac{1}{s^m} E_1(u_s; Q_s) \\ &\le \left\lfloor \frac{s}{t} \right\rfloor^m \frac{1}{s^m} E_1(u_t; Q_t) + \frac{1}{s^m} \bar{C} \left(s^m - \left\lfloor \frac{s}{t} \right\rfloor^m t^m + \left\lfloor \frac{s}{t} \right\rfloor^m ((t+R)^m - (t-R)^m) \right). \end{aligned}$$

Using now (4.3) we get

$$g_s \le \left\lfloor \frac{s}{t} \right\rfloor^m \frac{t^m}{s^m} \left(g_t + \frac{1}{t} \right) + \frac{1}{s^m} \bar{C} \left(s^m - \left\lfloor \frac{s}{t} \right\rfloor^m t^m + \left\lfloor \frac{s}{t} \right\rfloor^m ((t+R)^m - (t-R)^m) \right).$$

Letting now $s \to +\infty$ and then $t \to +\infty$, we have that

$$\limsup_{s \to +\infty} g_s(z) \le \liminf_{t \to +\infty} g_t(z),$$

which concludes the proof. □

Remark 4.2. Note that for $z \in \mathcal{S}^{N-1}$ the only test function for the minimum problem in (4.1) is the constant z, so that the limit is actually an average over the period with $u = z$.

Proposition 4.3. *The function $G_{\rm hom}$ as defined in (2.8) is convex and lower semicontinuous in B_1^N.*

Proof. We want to show that for every $0 \le t \le 1$ and for every $z_1, z_2 \in B_1^N$ it holds:

$$G_{\rm hom}(tz_1 + (1-t)z_2, M) \le t\, G_{\rm hom}(z_1, M) + (1-t)\, G_{\rm hom}(z_2, M). \tag{4.5}$$

Let $k \in \mathbb{N}$ be fixed; having (2.2) in mind, and thanks to Proposition 4.1, it is not restrictive to take $k \in l\mathbb{N}$. We define

$$g_k(z) = \frac{1}{k^m} \inf \Big\{ E_1(u; Q_k) : \ \langle u \rangle_{Q_k}^{d,1} = z \Big\}. \tag{4.6}$$

In the following we will denote $g_k^1 = g_k(z_1)$, $g_k^2 = g_k(z_2)$.
Let u_k^1 and u_k^2 be functions such that

$$\frac{1}{k^m} E_1(u_k^1; Q_k) \le g_k^1 + \frac{1}{k}, \tag{4.7}$$

$$\frac{1}{k^m} E_1(u_k^2; Q_k) \le g_k^2 + \frac{1}{k}. \tag{4.8}$$

Let $h > k$ be such that $h/k \in \mathbb{N}$. Denote $g_h = g_h(tz_1 + (1-t)z_2)$, we define the following test function for g_h:

$$u_h(\beta) = \begin{cases} u_k^1(\beta - ki) & \text{if} \quad \beta \in ki + Q_k \quad i \in \left\{0, \dots, \frac{h}{k} - 1\right\}^{m-1} \times \left\{0, \dots, \left\lfloor \frac{h}{k} t \right\rfloor - 1 \right\} \\ u_k^2(\beta - ki) & \text{if} \quad \beta \in ki + Q_k \quad i \in \left\{0, \dots, \frac{h}{k} - 1\right\}^{m-1} \times \left\{ \frac{h}{k} - \left\lfloor \frac{h(1-t)}{k} \right\rfloor, \dots, \frac{h}{k} - 1 \right\} \\ \bar{u}(\beta) & \text{otherwise,} \end{cases}$$

Reasoning as in Proposition 4.1, thanks to Lemma 3.1 we can choose the values of $\bar{u}$ such that

$$\langle u_s\rangle^{d,1}_{Q_s} = tz_1 + (1-t)z_2.$$

By (2.1) and (2.2), for some $\bar{C} > 0$ we get

$$\begin{aligned} g_h &\le \frac{1}{h^m} E_1(u_h, \zeta_h; Q_h) \\ &\le \frac{1}{h^m}\left(\frac{h}{k}\right)^{m-1} \left\lfloor \frac{h}{k} t \right\rfloor E_1(u^1_k; Q_k) + \frac{1}{h^m}\left(\frac{h}{k}\right)^{m-1} \left\lfloor \frac{h(1-t)}{k} \right\rfloor E_1(u^2_k; Q_k) \\ &\quad + \frac{1}{h^m}\bar{C}\left(h^m - \left(\frac{h}{k}\right)^{m-1}\left(\left\lfloor \frac{h}{k}t\right\rfloor + \left\lfloor \frac{h(1-t)}{k}\right\rfloor\right) k^m\right) \\ &\quad + \frac{1}{h^m}\bar{C}\left(\frac{h}{k}\right)^m ((k+R)^m - (k-R)^m). \end{aligned}$$

Then, thanks to (4.7) and (4.8), we can rewrite the above relation as

$$\begin{aligned} g_h \le{}& \frac{k^m}{h^m}\left(\frac{h}{k}\right)^{m-1} \left\lfloor \frac{h}{k} t \right\rfloor \left(g^1_k + \frac{1}{k}\right) + \frac{k^m}{h^m}\left(\frac{h}{k}\right)^{m-1} \left\lfloor \frac{h(1-t)}{k} \right\rfloor \left(g^2_k + \frac{1}{k}\right) \\ &+ \frac{1}{h^m}\bar{C}\left(h^m - \left(\frac{h}{k}\right)^{m-1}\left(\left\lfloor \frac{h}{k}t\right\rfloor + \left\lfloor \frac{h(1-t)}{k}\right\rfloor\right) k^m\right) \\ &+ \frac{1}{h^m}\bar{C}\left(\frac{h}{k}\right)^m ((k+R)^m - (k-R)^m). \end{aligned}$$

Letting $h \to +\infty$ and then $k \to +\infty$, we can conclude the proof of the convexity.

From the convexity and the boundedness of $G_{\rm hom}$ we deduce that it is continuous in the interior of B^N_1. Moreover, by (2.3) it is lower semicontinuous at points on the boundary of B^N_1 □

Remark 4.4. Note that the function $G_{\rm hom}$ may not be continuous on B^N_1, even if it is convex. It suffices to take a nearest-neighbor energy $G = G^\xi$ independent on α, with $G(e_1, e_1) = 0$ and $G(u, v) = 1$ otherwise. In this case, in particular $G_{\rm hom}(e_1) = 0$ and $G_{\rm hom}(z) = 1$ if $z \in \mathcal{S}^{N-1}$, $z \neq e_1$.

5. Proof of the homogenization theorem

Thanks to the geometric lemmas in Section 3, we can now easily give a proof of the homogenization theorem. We remark that it will be sufficient to prove a lower bound, since we may resort to the homogenization result of Alicandro, Cicalese and Gloria [3] in order to give an upper bound for the homogenized functional. Indeed, by (1.1) we have $\overline{G}_{\rm hom} \le G_{\rm hom}$, so that functional (2.7) is an upper bound for the homogenized energy. Note that we could directly proof the upper bound using approximation results and constructions starting from the formula for $G_{\rm hom}$, but this would be essentially a repetition of the arguments in [3].

In order to prove a lower bound we will make use of Lemma 3.2 and of the Fonseca-Müller blow-up technique [11, 18]. Let $\varepsilon_j \to 0$ be a vanishing sequence of parameters, let $u \in L^\infty(\Omega, B^N_1)$ and let $u_j \rightharpoonup^* u$ with $u \in L^\infty(\Omega, B^N_1)$. We define the measures μ_j by setting

$$\mu_j(A) = \sum_{\substack{\xi\in\mathbb{Z}^m\\ |\xi|\le R}} \sum_{\alpha\in R^\xi_{\varepsilon_j}(\Omega)} \varepsilon_j^m G^\xi\Big(\frac{\alpha}{\varepsilon_j}, u_j(\alpha), u_j(\alpha+\varepsilon_j\xi)\Big)\delta_{\alpha+\frac{\varepsilon_j}{2}\xi}(A)$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$, where δ_x denotes the usual Dirac delta measure at x. Since the measures are equibounded, by the weak* compactness of measures there exists a limit measure μ on Ω such that, up to subsequences, $\mu_j \rightharpoonup^* \mu$. We consider the Radon-Nikodym decomposition of the limit measure μ with respect to the m-dimensional Lebesgue measure $\mathcal{L}^m$:

$$\mu = \frac{\mathrm{d}\mu}{\mathrm{d}x}\mathrm{d}\mathcal{L}^m + \mu^s, \qquad \mu^s \perp \mathcal{L}^m.$$

Besicovitch Derivation Theorem [4] states that almost every point in Ω with respect to $\mathcal{L}^m$ is a Lebesgue point for μ. So, we may suppose that $x_0 \in \mathbb{Z}_{\varepsilon_j}(\Omega)$ be a Lebesgue point both for u and for μ and let $Q_\rho(x_0) = x_0 + (-\rho/2, \rho/2)^m$. We then have

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x_0) = \lim_{\rho\to 0^+} \frac{\mu(Q_\rho(x_0))}{\mathcal{L}^m(Q_\rho(x_0))} = \lim_{\rho\to 0^+} \frac{1}{\rho^m}\mu(Q_\rho(x_0)). \tag{5.1}$$

Recalling that

$$\mu(Q_\rho(x_0)) = \lim_{j\to+\infty} \mu_j(Q_\rho(x_0)) \tag{5.2}$$

except for a countable set of ρ, by a diagonalization argument on (5.1) and (5.2) we can extract a subsequence $\{\rho_j\}$ such that

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x_0) = \lim_{j\to+\infty} \frac{1}{\rho_j^m}\mu_j(Q_{\rho_j}(x_0))$$

holds. This means that

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x_0) = \lim_{j\to+\infty} \Big(\frac{\varepsilon_j}{\rho_j}\Big)^m \sum_{\substack{\xi\in\mathbb{Z}^m \\ |\xi|\le R}} \sum_{\alpha\in R^\xi_{\varepsilon_j}(\Omega)} G^\xi\Big(\frac{\alpha}{\varepsilon_j}, u_j(\alpha), u_j(\alpha+\varepsilon_j\xi)\Big)\delta_{\alpha+\frac{\varepsilon_j}{2}\xi}(Q_{\rho_j}(x_0)). \tag{5.3}$$

Also note that, by the weak* convergence of u_j to u, we have

$$\langle u\rangle_{Q_\rho(x_0)} = \lim_j \langle u_j\rangle^{\varepsilon_j}_{Q_\rho(x_0)} = \lim_j \langle u_j\rangle^{d,\varepsilon_j}_{Q_\rho(x_0)},$$

and that for almost every x_0 we have

$$\lim_{\rho\to 0^+} \langle u\rangle_{Q_\rho(x_0)} = u(x_0),$$

so that we may assume that

$$\lim_j \langle u_j\rangle^{d,\varepsilon_j}_{Q_{\rho_j}(x_0)} = u(x_0).$$

We can parameterizing functions on a common unit cube, by setting

$$v_j(\gamma) = u_j(x_0 + \rho_j\gamma) \qquad \text{and} \qquad \delta_j = \frac{\varepsilon_j}{\rho_j}.$$

With this parameterization, (5.3) reads

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x_0)=\lim_{j\to+\infty}\delta_j^m\sum_{\substack{\xi\in\mathbb{Z}^m\\|\xi|\le R}}\sum_{\gamma+\frac{x_0}{\rho_j}\in R^\xi_{\delta_j}(\frac{1}{\rho_j}\Omega)}G^\xi\Big(\frac{x_0}{\varepsilon_j}+\frac{\gamma}{\delta_j},v_j(\gamma),v_j(\gamma+\delta_j\xi)\Big)\delta_{\gamma-\frac{x_0}{\rho_j}+\frac{\delta_j}{2}\xi}(Q_1(0)).$$

Note that we have $\lim_j\langle v_j\rangle^{d,\delta_j}_{Q_1(0)}=u(x_0)$. We can then apply Lemma 3.2 with v_j in the place of u_j and $A=Q_1(0)$. We obtain a family $\widetilde{v}_j$ with

$$\langle\widetilde{v}_j\rangle^{d,\delta_j}_{Q_1(0)}=u(x_0),\tag{5.4}$$

and such that $\widetilde{v}_j(\gamma)=v_j(\gamma)$ except for a set P_j of γ with $\#P_j=o(\delta_j^{-m})$. From this, the finiteness of the range of interactions and the boundedness of G^ξ, we further rewrite as

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x_0)=\lim_{j\to+\infty}\delta_j^m\sum_{\substack{\xi\in\mathbb{Z}^m\\|\xi|\le R}}\sum_{\gamma+\frac{x_0}{\rho_j}\in R^\xi_{\delta_j}(\frac{1}{\rho_j}\Omega)}G^\xi\Big(\frac{x_0}{\varepsilon_j}+\frac{\gamma}{\delta_j},\widetilde{v}_j(\gamma),\widetilde{v}_j(\gamma+\delta_j\xi)\Big)\delta_{\gamma-\frac{x_0}{\rho_j}+\frac{\delta_j}{2}\xi}(Q_1(0)).$$

We now set

$$T_j=\frac{\rho_j}{\varepsilon_j}=\delta_j^{-1},\qquad x_{T_j}=\frac{x_0}{\varepsilon_j},$$

so that

$$\begin{aligned}
&\frac{\mathrm{d}\mu}{\mathrm{d}x}(x_0)\\
&=\lim_{j\to+\infty}\frac{1}{T_j^m}\sum_{\substack{\xi\in\mathbb{Z}^m\\|\xi|\le R}}\sum_{\gamma+\frac{x_{T_j}}{T_j}\in R^\xi_{T_j^{-1}}(\frac{1}{\rho_j}\Omega)}G^\xi\Big(x_{T_j}+T_j\gamma,\widetilde{v}_j(\gamma),\widetilde{v}_j\Big(\gamma+\frac{\xi}{T_j}\Big)\Big)\delta_{\gamma-\frac{x_{T_j}}{T_j}+\frac{1}{2T_j}\xi}(Q_1(0))\\
&=\lim_{j\to+\infty}\frac{1}{T_j^m}\sum_{\substack{\xi\in\mathbb{Z}^m\\|\xi|\le R}}\sum_{\eta+x_{T_j}\in R^\xi_1(\frac{1}{\varepsilon_j}\Omega)}G^\xi\Big(x_{T_j}+\eta,\widetilde{v}_j(\frac{\eta}{T_j}),\widetilde{v}_j\Big(\frac{\eta+\xi}{T_j}\Big)\Big)\delta_{\eta-x_{T_j}+\frac{1}{2}\xi}(Q_{T_j}(0)).
\end{aligned}$$

We now set $w_j(\eta)=v_j(\frac{\eta}{T_j})$ and use the boundedness of G^ξ and R to deduce that

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x_0)\ge\lim_{j\to+\infty}\frac{1}{T_j^m}\sum_{\substack{\xi\in\mathbb{Z}^m\\|\xi|\le R}}\sum_{\eta+x_{T_j}\in R^\xi_1(Q_{T_j})}G^\xi\left(x_{T_j}+\eta,\widetilde{w}_j(\eta),\widetilde{v}_j(\eta+\xi)\right)\delta_{\eta-x_{T_j}+\frac{1}{2}\xi}(Q_{T_j}(0)).$$

Note indeed that by considering interactions in $R^\xi_1(Q_{T_j})$ we neglect a contribution of a number of interactions of order $O(T_j^{m-1})$; *i.e.*, an energy contribution of order $O(T_j^{-1})$. Noting that w_j satisfies the constraint

$$\langle\widetilde{w}_j\rangle^{d,1}_{x_{T_j}+Q_{T_j}(0)}=u(x_0),$$

thanks to (5.4), by Proposition 4.1 we finally deduce that

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x_0) \geq \lim_{j\to\infty} \frac{1}{T_j^m} \inf\Big\{E_1(w; x_{T_j} + Q_{T_j}) :\ \langle w\rangle^{d,1}_{x_{T_j}+Q_{T_j}} = u(x_0)\Big\} = G_{\hom}(u(x_0)).$$

Since this holds for almost all $x_0 \in \Omega$, we have proved the desired lower bound, and hence the convergence result.

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